An Alternative Integral Representation of the Dirac Delta

Jan A. Eidissen Posada

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The integral representation of the Dirac delta is the most usual way of representing this function in the mathematics and physics literature. Many physical disciplines, including quantum mechanics, quantum field theory, signal theory, optics or astronomy, have been successfully formulated with it. Given that it is an improper function, the larger framework of distribution theory was developed to put it on a par with other more well-behaved representations, such as the unit impulse function or the Lorentzian and Gaussian representations, and to enable rigorous use of it. The framework is mathematically impeccable, but imposes restrictive conditions on the convolving test function. We review these requirements and propose an alternative integral representation that is well-behaved, and satisfies the sifting property for any continuous convolving function by reinterpreting the Dirac delta as the limit of a sequence of functions and by repositioning limits under integral signs to exploit the fact that they vanish everywhere but in an arbitrarily small vicinity of the origin. The same procedure is used to prove the sampling property for the other three representations mentioned above.

In physical theories, the one-dimensional integral representation of the Dirac delta distribution is usually written as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk = \lim_{N\to\infty}\frac{\sin Nx}{\pi x}\ ;\ N \in \Re \tag{1}$$

It is a well-known fact that this representation is not properly defined along the real axis, and we are usually discouraged from attempting to evaluate it as a plain vanilla real-valued function. In fact, if we attempt to compute its value at x = 1, for example, we arrive at the puzzling result

$$\delta(1) = \frac{1}{\pi}\lim_{N\to\infty}\sin N\ ;\ N \in \Re \tag{2}$$

Because the limit in (2) does not exist. Indeed, $\delta(x)$ only adopts some well-defined, while divergent values, as $x \to 0$. This ambiguity is circumvented by distribution theory, where the Dirac delta is usually written as follows:

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx \equiv \lim_{N\to\infty}\int_{-\infty}^{+\infty}\frac{\sin Nx}{\pi x}\,f(x)\,dx \tag{3}$$

Within this framework, *f(x)* represents a so-called test function. Contingent upon whether or not a distribution is considered to be tempered, the requirements imposed on test functions are more or less stringent. As regards other representations of the Dirac delta distribution, such as the impulse function or the Lorentzian and Gaussian representations, the treatment is analogous:

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx \equiv \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\delta_\varepsilon(x)\,f(x)\,dx\,;\quad \delta_\varepsilon(x) \equiv \begin{cases}\dfrac{1}{2\varepsilon}\ ; & -\varepsilon < x < \varepsilon\\ 0\ \ ; & |x| > \varepsilon\end{cases} \tag{4}$$

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx \equiv \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\frac{\varepsilon}{\pi\left(x^2+\varepsilon^2\right)}\,f(x)\,dx \tag{5}$$

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx \equiv \lim_{N\to\infty}\int_{-\infty}^{+\infty}\frac{N}{\sqrt{\pi}}\,e^{-N^2x^2}\,f(x)\,dx \tag{6}$$

The Dirac delta distribution has the following relevant property:

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx = f(0) \tag{7}$$

Of course, equation (7) trivially implies

$$\int_{-\infty}^{+\infty}\delta(x)\,dx = 1 \tag{8}$$

Equations (7) and (8) can be combined to write the well-known sifting property:

$$\delta(x)\,f(x) = \delta(x)\,f(0) \tag{9}$$

Within the framework of distribution theory, equation (7) is fulfilled in the assumption that *f(x)* is continuously differentiable and the function and its derivatives fall off faster than any power for large x. It can instead be assumed that *f(x)* decreases "sufficiently rapid", is compactly supported, infinitely differentiable, or some other similar and invariably strong requirement. These requirements are due to the intrinsic definition of limits outside integral signs. As a matter of fact, each of the above representations necessarily has a different convergence requirement. In the case of the impulse function, provided that *f(x)* is continuous around the origin, equation (7) is immediately satisfied because

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\delta_\varepsilon(x)\,f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{+\varepsilon}\frac{f(x)}{2\varepsilon}\,dx \tag{10}$$

And since *f(x)* is continuous in an arbitrarily small vicinity of the origin, the mean value theorem can be invoked to prove that, for some $c \in [-\varepsilon, +\varepsilon]$, the last integral in (10) equals $f(c)$, and $f(c) \to f(0)\,,\ \forall c \in [-\varepsilon, +\varepsilon]$ as $\varepsilon \to 0^+$. Hence

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{+\varepsilon}\frac{f(x)}{2\varepsilon}\,dx = \lim_{\varepsilon\to 0^+} f(c) = f(0) \tag{11}$$

In the Lorentzian case, however, *f(x)* must not grow faster than x^2 as $x \to \pm\infty$. Taking, for instance, $f(x) = x^4$, we readily find

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\frac{\varepsilon\,x^4}{\pi\left(x^2+\varepsilon^2\right)}\,dx \equiv \lim_{\varepsilon\to 0^+}\int_{-1/\varepsilon}^{+1/\varepsilon}\frac{\varepsilon\,x^4}{\pi\left(x^2+\varepsilon^2\right)}\,dx =$$

$$= \lim_{\varepsilon\to 0^+}\left\{\frac{\varepsilon\,x^3}{3\pi} - \frac{\varepsilon^3}{\pi}\left[x - \varepsilon\,tg^{-1}\left(\frac{x}{\varepsilon}\right)\right]\right\}\Bigg|_{-1/\varepsilon}^{+1/\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{2}{3\pi\varepsilon^2} \tag{12}$$

And the limit in (12) diverges. In the Gaussian case, the convergence requirement for *f(x)* as $x \to \pm\infty$ is less stringent, but it still depends on the decrease rate of the Gaussian function. If $f(x) = e^{x^4}$, for example, equation (7) is not fulfilled in the Gaussian case, because the limit rapidly diverges, owing to the fact that limits are positioned outside integral signs.

When dealing with the integral representation of the Dirac delta, the situation becomes most restrictive. In fact, it is the pathological behavior of (1) that requires imposing the most stringent conditions on the test functions so that equation (7) can be satisfied for any given representation of the Dirac delta distribution. The treatment is mathematically sound, but involves taking too much care with our choice of test functions. And there are still further disadvantages: in quantum field theory and theoretical physics, where a restrictive mathematical framework is undesirable, we are often confronted with expressions like

$$\left(\frac{1}{2\pi}\right)^4\int d^4p\,f(p)\int d^4x\,e^{i(p-q)x} =$$

$$= \int d^4p\,f(p)\prod_{\alpha=0}^{3}\lim_{N_\alpha\to+\infty}\frac{\sin N_\alpha(p_\alpha - q_\alpha)}{\pi^4(p_\alpha - q_\alpha)} \tag{13}$$

And we find it convenient to express (13) as

$$\left(\frac{1}{2\pi}\right)^4 \int d^4p\, f(p) \int d^4x\, e^{i(p-q)x} = \int d^4p\, f(p)\, \delta^4(p-q) \tag{14}$$

Then, assuming that sufficient convergence conditions are fulfilled by $f(p)$, we can obviously write

$$\int d^4p\, f(p)\, \delta^4(p-q) = f(q) \tag{15}$$

However, the right-hand side of (14) should be written as follows within the framework of distribution theory:

$$\int d^4p\, f(p)\, \delta^4(p-q) \equiv \lim_{N_\alpha \to +\infty} \int d^4p\, f(p) \prod_{\alpha=0}^{3} \frac{\sin N_\alpha (p_\alpha - q_\alpha)}{\pi^4 (p_\alpha - q_\alpha)} \tag{16}$$

After comparing (13) with (16), we note that the limit has jumped outside the integral sign, and the process of interchanging limits and integral signs requires invoking the dominated convergence theorem, which imposes additional restrictions on functionals.

From a physical point of view, it would be very advantageous to be able to prove equation (7) by merely assuming that *f* is continuous around the origin (most physically relevant fields may more easily meet this weak requirement), without having to impose additional conditions on the convolving field or function or invoke the dominated convergence theorem. This can be done with three of the four aforementioned representations, by reinterpreting the Dirac delta as the limit of a sequence of functions, which is tantamount to repositioning limits under the integral sign as follows:

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx \equiv \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \delta_\varepsilon(x) f(x)\, dx \quad ; \quad \delta_\varepsilon(x) \equiv \begin{cases} \dfrac{1}{2\varepsilon} \ ; & -\varepsilon < x < \varepsilon \\ 0 \ ; & |x| > \varepsilon \end{cases} \tag{17}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx \equiv \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(x^2 + \varepsilon^2\right)} f(x)\, dx \tag{18}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx \equiv \int_{-\infty}^{+\infty} \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^2 x^2} f(x)\, dx \tag{19}$$

In the case of (18), for example, if we calculate the value of $\delta(x)$ at $x = \varepsilon$ and $x = \sqrt{\varepsilon}$, we find

$$\delta(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(\varepsilon^2 + \varepsilon^2\right)} = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\varepsilon} \tag{20}$$

$$\delta(\sqrt{\varepsilon}) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(\varepsilon + \varepsilon^2\right)} = \frac{1}{\pi} \tag{21}$$

Meanwhile, for $x = a,\, a \neq 0,\, a \in \Re$ (where $a \neq 0$ implies $a \not\to 0$), we clearly obtain

$$\delta(a) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(a^2 + \varepsilon^2\right)} \approx \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi a^2} = 0 \tag{22}$$

And for $x = \varepsilon^\alpha, \alpha \in \Re, 0 < \alpha < 1/2$ (note that the case α = 0 is implied by (22)) we find

$$\delta(\varepsilon^\alpha) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(\varepsilon^{2\alpha} + \varepsilon^2\right)} \approx \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi \varepsilon^{2\alpha}} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^{1-2\alpha}}{\pi} = 0,$$
$$\forall \alpha \in \Re,\ 0 < \alpha < 1/2 \tag{23}$$

Likewise, in the Gaussian case, we can calculate values such as

$$\delta(1/N) = \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-1} \tag{24}$$

$$\delta(1/N^2) = \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-1/N^2} = \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} \tag{25}$$

While for $x = N^{-\beta}$, $\beta \in \Re$, $0 < \beta < 1$, we readily find

$$\delta\left(N^{-\beta}\right) = \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^{2-2\beta}} = 0,\ \forall \beta \in \Re,\ 0 < \beta < 1 \tag{26}$$

And for $x = a,\, a \neq 0,\, a \in \Re$, the limit is undoubtedly null:

$$\delta(a) = \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^2 a^2} = 0 \tag{27}$$

Thanks to these results, equation (7) can be easily proven for (18) and (19). Starting with the Lorentzian representation and taking into account that, according to (22) and (23), it takes null values at $(-\infty, -\varepsilon^\alpha] \cup [\varepsilon^\alpha, +\infty)$, we can perform the $x = \varepsilon y$ change of variable and recall that $\lim_{\varepsilon \to 0^+} f(\varepsilon y) = f(0)$, $\forall y \in [-\varepsilon^{\alpha-1}, +\varepsilon^{\alpha-1}], \forall \alpha \in \Re, 0 < \alpha < 1/2$ (since *f(x)* is continuous around the origin) to prove

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(x^2 + \varepsilon^2\right)} f(x)\, dx = \int_{-\varepsilon^\alpha}^{+\varepsilon^\alpha} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi\left(x^2 + \varepsilon^2\right)} f(x)\, dx =$$

$$= \frac{1}{\pi} \int_{-\varepsilon^{\alpha-1}}^{+\varepsilon^{\alpha-1}} \lim_{\varepsilon \to 0^+} f(\varepsilon y) \frac{dy}{y^2 + 1} = \frac{f(0)}{\pi} \int_{-\varepsilon^{\alpha-1}}^{+\varepsilon^{\alpha-1}} \lim_{\varepsilon \to 0^+} \frac{dy}{y^2 + 1} =$$

$$= \frac{f(0)}{\pi} \int_{-\varepsilon^\alpha}^{+\varepsilon^\alpha} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{x^2 + \varepsilon^2}\, dx = \frac{f(0)}{\pi} \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{x^2 + \varepsilon^2}\, dx =$$

$$= \frac{f(0)}{\pi} \int_{-\infty}^{+\infty} \frac{dy}{y^2 + 1} = \frac{f(0)}{\pi}\, tg^{-1} x \Big|_{-\infty}^{+\infty} = f(0) \tag{28}$$

In the case of (19), the proof is analogous. Assuming that *f(x)* is continuous around the origin and given that, according to (26) and (27), the Gaussian representation vanishes at $(-\infty, -N^{-\beta}] \cup [N^{-\beta}, +\infty)$, we can perform the $N x = y$ change of variable and take advantage of the fact that $\lim_{N \to \infty} f(y/N) = f(0), \forall y \in (-N^{1-\beta}, +N^{1-\beta}), \forall \beta \in \Re, 0 < \beta < 1,$ to prove (7):

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{+\infty} \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^2 x^2} f(x)\, dx = \int_{-N^{-\beta}}^{+N^{-\beta}} \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^2 x^2} f(x)\, dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-N^{1-\beta}}^{+N^{1-\beta}} \lim_{N \to \infty} f(y/N)\, e^{-y^2} dy = \frac{f(0)}{\sqrt{\pi}} \int_{-N^{1-\beta}}^{+N^{1-\beta}} \lim_{N \to \infty} e^{-y^2} dy =$$

$$= \frac{f(0)}{\sqrt{\pi}} \int_{-N^{-\beta}}^{+N^{-\beta}} \lim_{N \to \infty} \frac{N}{\sqrt{\pi}} e^{-N^2 x^2} dx = \frac{f(0)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \lim_{N \to \infty} N e^{-N^2 x^2}\ dx =$$

$$= \frac{f(0)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2}\ dy = f(0) \tag{29}$$

As regards the impulse function in (17), taking into account that $\lim_{\varepsilon \to 0^+} f(\varepsilon y) = f(0), \forall y \in [-1, +1],$ the proof is immediate:

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0^+} \delta_\varepsilon(x) f(x)\, dx = \int_{-\varepsilon}^{+\varepsilon} \lim_{\varepsilon \to 0^+} \frac{f(x)}{2\varepsilon}\, dx =$$

$$= \int_{-1}^{+1} \lim_{\varepsilon \to 0^+} \frac{f(\varepsilon y)}{2}\, dy = f(0) \tag{30}$$

If we attempt to perform the same calculation with the integral representation of the Dirac delta function, we are impeded by the fact that (1) does not vanish outside an arbitrarily small vicinity of the origin:

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, dx \int_{-\infty}^{+\infty} e^{ikx} dk = \frac{1}{\pi} \int_{-\infty}^{+\infty} \lim_{N \to \infty} \frac{\sin Nx}{x} f(x)\, dx =$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\lim_{N\to\infty} f(y/N)\,\frac{\sin y}{y}\,dy \tag{31}$$

Although the integral of $\sin y/\pi y$ over $(-\infty,+\infty)$ equals one, the assumption that *f(x)* is continuous around the origin does not imply that $\lim_{N\to\infty} f(y/N)=f(0),\quad \forall y\in(-\infty,+\infty)$ and (7) cannot be proved. If we evaluate this limit at *y = N*, for example, we obtain $f(1)$ instead of $f(0)$.

Note that the Fourier inversion theorem cannot be invoked to prove (1) for the integral representation, because *f(k)* = 1 is not in $\mathcal{L}^1(\mathbb{R})$, even though *sin kx* and *cos kx* are suitable test functions and therefore:

$$\int_{-\infty}^{+\infty}\delta(x)\,e^{-ikx}\,dx=1 \tag{32}$$

Similarly, the well-known trick of inserting a convergence factor into (1) for proving that the integral and Lorentzian representations are equivalent is arguable. Although this equivalence can be apparently proven as follows:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk \overset{?}{=} \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} e^{ikx-|k|\varepsilon}dk \overset{?}{=} \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi\left(x^2+\varepsilon^2\right)} \tag{33}$$

In doing so, we are overlooking the presence of infinite integration limits. Since improper integrals are typically defined by Cauchy principal values, in this case we should write

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk \equiv \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{+R} e^{ikx}dk \tag{34}$$

If we insert a convergence factor into equation (34) and perform the integration in k, we are confronted with the limit of $e^{(ix\pm\varepsilon)R}$ as $R\to\pm\infty$ and $\varepsilon\to 0+$, which is ill-defined.

However, we can find an alternative integral representation of the Dirac delta that fulfills equation (7) for any continuous function and may alternatively be used in physical theories. Our choice is:

$$\delta(x)\equiv\frac{2}{\pi}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin(Nx/2)}{x}\right)^2=\frac{1}{2\pi}\lim_{N\to\infty}\frac{1}{N}\left[\int_{-N/2}^{+N/2} e^{ikx}dk\right]^2 \tag{35}$$

In fact, we can use (35) to prove (7) following the same procedure as in previous cases. First of all, let us note that

$$\delta(2/N)=\frac{\sin^2 1}{2\pi}\lim_{N\to\infty} N \tag{36}$$

$$\delta(2/\sqrt{N})\equiv\frac{1}{2\pi}\lim_{N\to\infty}\sin^2\sqrt{N} \tag{37}$$

$$\delta(2N^{-\alpha})=\frac{1}{2\pi}\lim_{N\to\infty}\frac{\sin^2 N^{1-\alpha}}{N^{1-2\alpha}}=0\,,\ \forall\alpha\in\Re\,,\ 0<\alpha<1/2 \tag{38}$$

$$\delta(a)=\frac{2}{\pi}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin\left(Na/2\right)}{a}\right)^2=0\,,\ \forall\alpha\in\Re\,,\ a\neq 0 \tag{39}$$

By virtue of (38) and (39), this representation of the Dirac delta vanishes at $(-\infty,-N^{-\alpha}]\cup[N^{-\alpha},+\infty),\ \forall\alpha\in\Re\,,\ 0<\alpha<1/2$. Considering this before performing the $Nx/2=y$ change of variable, while recalling that $\lim_{N\to\infty} f(2y/N)=f(0),\ \forall y\in[-N^{1-\alpha},+N^{1-\alpha}],\ \forall\alpha\in\Re\,,0<\alpha<1/2$, for any function *f(x)* that is continuous around the origin, we can easily prove (7):

$$\int_{-\infty}^{+\infty}\delta(x)\,f(x)\,dx=\frac{2}{\pi}\int_{-\infty}^{+\infty}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin(Nx/2)}{x}\right)^2 f(x)\,dx=$$

$$=\frac{2}{\pi}\int_{-N^{-\alpha}}^{+N^{-\alpha}}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin(Nx/2)}{x}\right)^2 f(x)\,dx=$$

$$=\frac{1}{\pi}\int_{-N^{1-\alpha}/2}^{+N^{1-\alpha}/2}\lim_{N\to\infty} f(2y/N)\left(\frac{\sin y}{y}\right)^2 dy=\frac{f(0)}{\pi}\int_{-N^{1-\alpha}/2}^{+N^{1-\alpha}/2}\lim_{N\to\infty}\left(\frac{\sin y}{y}\right)^2 dy=$$

$$=\frac{2f(0)}{\pi}\int_{-N^{-\alpha}}^{+N^{-\alpha}}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin(Nx/2)}{x}\right)^2 dx=$$

$$=\frac{2f(0)}{\pi}\int_{-\infty}^{+\infty}\lim_{N\to\infty}\frac{1}{N}\left(\frac{\sin(Nx/2)}{x}\right)^2 dx=\frac{f(0)}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin y}{y}\right)^2 dy=f(0) \tag{40}$$

In this calculation, we have reintroduced the $Nx/2=y$ change of variable after extracting $f(0)$ from the integral to take advantage of the fact that (35) vanishes at $(-\infty,-N^{-\alpha}]\cup[N^{-\alpha},+\infty),\ \forall\alpha\in\Re\,,\ 0<\alpha<1/2$ and that we can thus extend the integration limits to $(-\infty,+\infty)$ before inserting the same change of variable once more to express (35) as a parameter independent integral, whose result is well-known.

Note that the same method has been used to prove (7) for all four representations. In fact, it can be applied to prove equation (7) for other sequences of functions that are null everywhere except in an arbitrarily small vicinity of the origin, where they take positive, divergent values.

Provided that *f(x)* is continuous in an arbitrarily small region of x = y, for any x, y∈ℜ, equation (7) can be easily generalized to the more general sampling property:

$$\int_{-\infty}^{+\infty}\delta(x-y)\,f(x)\,dx=f(y) \tag{41}$$

Denoting $V=L_1\,L_2\,L_3$, where the 1, 2, 3 subscripts stand for the Cartesian coordinates x, y, z, and $T\equiv L_0$, the three and four-dimensional representations of (35) in momentum space can be expressed as

$$\delta^{(3)}(\vec{k})=\left(\frac{2}{\pi}\right)^3\lim_{V\to\infty}\frac{1}{V}\prod_{i=1}^{3}\left(\frac{\sin(L_i k_i/2)}{k_i}\right)^2=$$

$$=\left(\frac{1}{2\pi}\right)^3\lim_{V\to\infty}\frac{1}{V}\prod_{i=1}^{3}\left[\int_{-L_i/2}^{+L_i/2} e^{ik_i x_i}dx^i\right]^2 \tag{42}$$

And

$$\delta^{(4)}(k)=\left(\frac{2}{\pi}\right)^4\lim_{V,T\to\infty}\frac{1}{VT}\prod_{i=0}^{3}\left(\frac{\sin(L_i k_i/2)}{k_i}\right)^2=$$

$$=\left(\frac{1}{2\pi}\right)^4\lim_{V,T\to\infty}\frac{1}{VT}\prod_{i=0}^{3}\left[\int_{-L_i/2}^{+L_i/2} e^{ik_i x_i}dx^i\right]^2 \tag{43}$$

Where L_1, L_2, L_3 are assumed to tend separately to infinity. Furthermore, we can use the following notation:

$$\delta^{(3)}(\vec{k})\equiv\lim_{V\to\infty}\delta_V^{(3)}(\vec{k})\ \ ;\ \ \delta^{(4)}(k)\equiv\lim_{V,T\to\infty}\delta_V^{(3)}(\vec{k})\cdot\delta_T(k_0)\equiv\lim_{V,T\to\infty}\delta_{VT}^{(4)}(k) \tag{44}$$

We can also write:

$$\left(\delta_V^{(3)}(\vec{k})\right)^{1/2}=\left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\sqrt{V}}\prod_{i=1}^{3}\left[\int_{-L_i/2}^{+L_i/2} e^{ik_i x_i}dx^i\right] \tag{45}$$

$$\left(\delta_{VT}^{(4)}(k)\right)^{1/2}=\left(\delta_V^{(3)}(\vec{k})\,\delta_T(k_0)\right)^{1/2}\equiv\left(\frac{1}{2\pi}\right)^2\frac{1}{\sqrt{VT}}\prod_{i=0}^{3}\left[\int_{-L_i/2}^{+L_i/2} e^{ik_i x_i}dx^i\right] \tag{46}$$

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