High-throughput sensing of freely flowing particles with optomechanofluidics: supplementary material

KEWEN HAN[†], JUNHWAN KIM, AND GAURAV BAHL^{*}

Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Illinois 61801, USA

^{*}Corresponding author: bahl@illinois.edu

[†]khan56@illinois.edu

Published 2 June 2016

This document provides supplementary information to "High-throughput sensing of freely flowing particles with optomechanofluidics," <http://dx.doi.org/10.1364/optica.3.000585>. We provide experimental details and additional discussion. © 2016 Optical Society of America

<http://dx.doi.org/10.1364/optica.3.000585.s001>

1. EXPERIMENTAL METHODS FOR PARTICLE TRACKING

Center-frequency tracking (shown in Fig. 3a, main manuscript) is performed through Lorentzian fitting of the instantaneous vibrational power spectrum (Figure S1). Such post-processing allows us to see details of the frequency perturbation during particle transits. Figure S2 shows the center-frequency trace matched with particle longitudinal position at various times. The phonon mode linewidth can also be extracted from the Lorentzian fit, allowing us to probe the additional dissipation induced by particles.

Particle radial position is determined using the projected images from both cameras. For each view the 2D symmetry line for the OMFR image is first determined (Figure S3a, middle yellow line). A two dimensional Gaussian fit to the greyscale intensity plot of the each particle image is performed, and the apex of the Gaussian fit provides the apparent position of the particle center (Figure S3b). Under far-field assumption, the actual location of the particle can easily be triangulated from the projections on each camera (Figure S4). We note that the determination of the particle radial position is subject to uncertainty both in determining the 3D OMFR central axis, and in determining the apparent position of the particle center (Figure S3).

Fig. S1. Finding the center frequency and linewidth of the spectrum using Lorentzian curve fitting – Instantaneous vibrational power spectra (blue dots) are fitted using a Lorentzian function (red curve). The center frequency (CF) of the oscillation is the instantaneous phonon mode frequency, while the -3 dB bandwidth is the phonon mode linewidth.

Fig. S2. Phonon mode frequency shift measured while particles are flowing through the OMFR – Particles flowing through the OMFR are observed simultaneously by two cameras. We show here the images from both cameras matched in time domain with the phonon mode frequency shift. The observed frequency shift depends on both the longitudinal and radial position of the particle. The frequency shifts at the apexes are collected for analyzing the effect of particle radial location.

Fig. S3. Image processing method – (a) To determine the 2D OMFR image symmetry line, two edge lines (blue, orange) are first determined with a linear fit edge search. The slope and intercept of the symmetry line (yellow) is then determined by averaging the two edge fits. Since the OMFR image is affected by illumination intensity, angle, and focal plane position, we estimate uncertainty for the edge determination of about $2.5 \mu\text{m}$. Therefore the uncertainty for the each 2D symmetry line determination is about $2.5 \mu\text{m}$ as well, and the 3D OMFR central axis is located in the circle of roughly $2.5 \mu\text{m}$ radius. The particle position determination is therefore subject to $\pm 2.5 \mu\text{m}$ error due to the central axis offset. (b) Color images from each camera are converted to a greyscale intensity plot. This false color image represents darkest regions in purple and lightest regions in yellow. A 2D Gaussian fit is applied, with the peak representing the particle image center. The full width at half maximum (FWHM) of the fit in both x and y direction are indicated by the black bars. For each particle, the larger x-axis FWHM of the particle image fit from two cameras is used as the approximate uncertainty of the particle radial location in Manuscript Fig. 4c.

Fig. S4. Determining true particle position – The particle position relative to the OMFR central axis can be determined by triangulating the apparent particle locations from the two cameras, while accounting for refractive index.

2. MODEL FOR ESTIMATING THE FREQUENCY PERTURBATION

For an inviscid fluid contained in an acoustic resonant chamber having a rigid wall, the angular resonant phonon frequency, Ω , can be found by solving the Helmholtz equation for the pressure field, p , of the fluid [1]:

$$\left(\frac{\Omega}{c_l}\right)^2 = \frac{\int_{V_c} |\nabla p|^2 dV}{\int_{V_c} |p|^2 dV}, \quad (1)$$

where subscript l stands for the fluid, the c_l is the speed of sound of the fluid, and the integrations are performed over the entire cavity fluid volume (V_c). This equation can be rewritten in the form:

$$\frac{\frac{1}{2} \int_{V_c} \frac{|\nabla p|^2}{\rho_l \Omega^2} dV}{\frac{1}{2} \int_{V_c} \frac{|p|^2}{\rho_l c_l^2} dV} = \frac{K_{max}}{V_{max}} = 1, \quad (2)$$

where ρ_l is the density of fluid. The numerator is equal to the maximum kinetic energy in the pressure field (K_{max}), and the denominator is equal to the maximum potential energy in the pressure field (V_{max}). Both K_{max} and V_{max} are equal to the total stored energy in the system therefore the ratio is always equal to one.

In the OMFR system, however, the resonator shell does participate in the phonon mode. Thus we will have to take interaction between the shell and the fluid into account while solving the Helmholtz equation, which manifests itself by the form of a surface integral over the fluid-shell boundary. As we will show below, the integral remains unchanged if the perturbation of the particle is so small such that this integral term is essentially constant. Inspired by previous work [2], we find that for the two slightly different cavities, i.e., the sample-free cavity (case 1) and the sample-loaded cavity (case 2), we can rewrite the Helmholtz equation as

$$-\kappa_1 \Omega_1^2 p_1 = \vec{\nabla} \cdot \left(\frac{1}{\rho_1} \vec{\nabla} p_1 \right), \quad (3)$$

$$-\kappa_2 \Omega_2^2 p_2 = \vec{\nabla} \cdot \left(\frac{1}{\rho_2} \vec{\nabla} p_2 \right). \quad (4)$$

Here, ρ is the density of the material and κ represents the compressibility of the material. In the OMFR system, the introduction of the sample particle perturbs the resonance properties of the system, changing the pressure distribution within the OMFR from p_1 to p_2 . In case 1, $\rho_1 = \rho_l$ and $\kappa_1 = \kappa_l = 1/\rho_l c_l^2$ everywhere within the cavity. In case 2, density and compressibility are modified to $\rho_2 = \rho_s$ and $\kappa_2 = \kappa_s$ only at the sample location. Here subscript s indicates sample particle. ρ_s is the density of the particle. κ_s is the compressibility of the particle, given by $1/\rho_s(c_p^2 - \frac{4}{3}c_s^2)$, where c_p and c_s are the P-wave and S-wave (acoustic) velocity in the particle material, respectively. After multiplying Eqn. 3 by p_2 and Eqn. 4 by p_1 and integrating both equations over the entire cavity volume within the OMFR shell (V_c), we get

$$-\int_{V_c} \kappa_1 \Omega_1^2 p_1 p_2 dV = \int_{V_c} \vec{\nabla} \cdot \left(\frac{1}{\rho_1} \vec{\nabla} p_1 \right) p_2 dV, \quad (5)$$

$$-\int_{V_c} \kappa_2 \Omega_2^2 p_1 p_2 dV = \int_{V_c} \vec{\nabla} \cdot \left(\frac{1}{\rho_2} \vec{\nabla} p_2 \right) p_1 dV. \quad (6)$$

Applying divergence theorem to the right hand side of Eqn. 5 and Eqn. 6, we get

$$-\int_{V_c} \kappa_1 \Omega_1^2 p_1 p_2 dV = \int_{S_c} \frac{p_2}{\rho_1} (\vec{\nabla} p_1 \cdot \vec{n}) dS - \int_{V_c} \frac{1}{\rho_1} \vec{\nabla} p_1 \cdot \vec{\nabla} p_2 dV, \quad (7)$$

$$-\int_{V_c} \kappa_2 \Omega_2^2 p_1 p_2 dV = \int_{S_c} \frac{p_1}{\rho_2} (\vec{\nabla} p_2 \cdot \vec{n}) dS - \int_{V_c} \frac{1}{\rho_2} \vec{\nabla} p_1 \cdot \vec{\nabla} p_2 dV, \quad (8)$$

where S_c represents the surface of the cavity volume within the OMFR shell, and $\vec{n}$ is the outward-pointing unit normal vector seen from inside the fluid domain. Now if we further assume the perturbation due to sample loading is very limited such that $p_1 \approx p_2 \approx p$, the surface integral terms in Eqn. 7 and Eqn. 8 are both equal to $\int_{S_c} p/\rho_l (\vec{\nabla} p \cdot \vec{n}) dS$. Subtracting Eqn. 7 from Eqn. 8, we find

$$\int_{V_c} \kappa_1 \Omega_1^2 |p|^2 dV - \int_{V_c} \kappa_2 \Omega_2^2 |p|^2 dV = \int_{V_c} \frac{1}{\rho_1} |\vec{\nabla} p|^2 dV - \int_{V_c} \frac{1}{\rho_2} |\vec{\nabla} p|^2 dV. \quad (9)$$

Since the surface integrals cancel, we can see that the effect of the shell can be neglected if the pressure field change is small. Recall that $\kappa_1 = \kappa_2 = \kappa_l$ and $\rho_1 = \rho_2 = \rho_l$ in the fluid fraction. We thus can rewrite Eqn. 9 with a little rearrangement as

$$\kappa_l (\Omega_1^2 - \Omega_2^2) \int_{V_c} |p|^2 dV + (\kappa_l - \kappa_s) \Omega_2^2 \int_{V_s} |p|^2 dV = \left(\frac{1}{\rho_l} - \frac{1}{\rho_s} \right) \int_{V_s} |\vec{\nabla} p|^2 dV, \quad (10)$$

where V_s represents the sample volume. Finally, by defining $\Omega_2 = \Omega' = 2\pi f'$ and $\Omega_1 = \Omega_o = 2\pi f_o$, we obtain the perturbed resonance frequency

$$f' = f_o \sqrt{\frac{1 - \frac{\rho_s - \rho_l}{\rho_s} B}{1 + \frac{\kappa_s - \kappa_l}{\kappa_l} A}}, \quad (11)$$

$$\text{where } A = \frac{\int_{V_s} |p|^2 dV}{\int_{V_c} |p|^2 dV} \text{ and } B = \frac{\int_{V_s} |\vec{\nabla} p|^2 dV}{k_l^2 \int_{V_c} |p|^2 dV},$$

where $k_l = 2\pi f_o/c_l$ is the unperturbed wavenumber associated with phonons in the fluid fraction of the OMFR. It is easy to see that A and B are proportional to acoustic potential and kinetic energy associated with the particle volume, respectively.

For the 11 μm polystyrene particle, $\rho_s = 1.5 \text{ g/cm}^3$, $c_p = 2350 \text{ m/s}$, and $c_s = 1120 \text{ m/s}$ [3]. We can thus calculate $(\rho_s - \rho_l)/\rho_s = 0.0476$ and $(\kappa_s - \kappa_l)/\kappa_l = -0.457$. This allows us to estimate the frequency perturbation for a given vibrational mode through Eqn. 11, by computing values of A and B . Example plots are presented in Figure S5a. The perturbed frequency clearly depends on the position of the particle, since both A and B are fixed functions of position related to the pressure mode shape p .

We also see from Eqn. 11 that both the density contrast, $(\rho_s - \rho_l)/\rho_s$, and compressibility contrast, $(\kappa_s - \kappa_l)/\kappa_l$, will

modify the resonant frequency. If we are provided the density of a sample particle, it is straightforward to compute the compressibility of the particle if the vibrational mode shape is known. For example, for a 11 μm polystyrene particle at 14 μm radial offset from the OMFR center, we measured the phonon frequency shift to be 14.09 kHz. Given the density of the polystyrene, we can calculate the compressibility contrast to be -0.471 . Thus, the polystyrene particles have estimated compressibility of $2.41 \times 10^{-10} \text{ Pa}^{-1}$.

Residual errors in our current prediction model come from multiple sources. First, we note that since neither the precise geometry of the OMFR, nor the accurate material properties of the OMFR are known, the calculation of A and B are only based on an estimated FEM model (see Methods). Second, we have assumed that the pressure field remains unchanged upon addition of the particle. This assumption is not accurate, especially when the particle has an acoustic scattering effect that cannot be ignored. This can occur when the compressibility contrast is large, or when the particle size is similar to or larger than the acoustic wavelength. As we can see from Figure S5b, when the particle is relatively large and sound-hard, the acoustic scattering from the particle can modify the phonon modeshape severely. A better calibration of the system could be performed using particles with known material properties in the future.

Fig. S5. (a) The phonon frequency shift is affected by the position-dependent A and B parameters for the phonon mode (Eqn. 11). (b) The phonon mode can be modified significantly by sound-hard spherical particles (dark blue circle) differently at various radial positions.

3. SENSING LIMIT

In order to estimate the sensing limit of a 55 μm OMFR used in this work, the Allan deviation of the center frequency is measured over 180 seconds (Figure S6). No feedback control or stabilization is applied. The device is operated in lab air with ambient environmental fluctuations causing the measured frequency fluctuations.

For small particles, the frequency perturbation is generally proportional to the volume V_s of the particle (Eqn. 11). For the 6 μm particles used in this work, our model predicts a fairly high frequency perturbation of about 40 kHz when the particle is at the OMFR center. On the other hand, our experiments indicate a trend towards 15 kHz - 40 kHz frequency shift when the particle is at the OMFR center (Manuscript Figure 4). We thus use a conservative value of 20 kHz for this estimation. We can

now convert the Allan deviation noise measurement to a Noise-Equivalent Particle Diameter (NEPD) as shown in Figure S6.

Fig. S6. Sensitivity estimations of the OMFR system – The Noise-Equivalent Particle Diameter (NEPD) defines the minimum sensing noise floor of an OMFR. The data presented here are for a 55 μm diameter OMFR (30.3 MHz center frequency) in the form of an Allan deviation plot with x-axis representing averaging time. We estimate a minimum sensing level of 840 nm with averaging time of 2 seconds. A similarly sized 31 MHz OMFR was demonstrated previously in [4] operating in the above-threshold optomechanical oscillation regime. Using their Allan deviation data (see Fig. 3b in that work) we estimate an improved minimum NEPD of 660 nm at 0.2 seconds averaging time (red dashed line) for that mode of operation.

We note also that the data in the present work is gathered when the phonon mode optomechanical oscillation is still in the sub-threshold regime (i.e. ‘cold cavity’), with very low optical pump power. Temperature variations dominate the frequency fluctuations here, resulting in the long averaging time of 2.03 seconds before the minimum NEPD of 840 nm is achieved. An OMFR with similar frequency was reported previously in the above-threshold optomechanical oscillation regime [4], with a much lower Allan deviation floor. We can thus estimate that a minimum NEPD of 660 nm could be achieved when operated in this above threshold regime over only 0.2 seconds averaging time. Feedback stabilization of the oscillator, and environmental controls on temperature and pressure can improve these noise floors significantly.

REFERENCES

1. E. Leung, “Resonance frequency shift of an acoustic chamber containing a rigid sphere,” *The Journal of the Acoustical Society of America* **72**, 615–620 (1982).
2. S. Wang, J. Zhao, Z. Li, J. M. Harris, and Y. Quan, “Differential Acoustic Resonance Spectroscopy for the acoustic measurement of small and irregular samples in the low frequency range,” *Journal of Geophysical Research* **117**, B06203 (2012).
3. W. M. Haynes, *CRC Handbook of Chemistry and Physics*, 96th Edition (CRC Press, 2015).
4. K. H. Kim, G. Bahl, W. Lee, J. Liu, M. Tomes, X. Fan, and T. Carmon, “Cavity optomechanics on a microfluidic resonator with water and viscous liquids,” *Light: Science & Applications* **2**, e110 (2013).