Topological phase transitions in superradiance lattices: supplementary material

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This document provides supplementary information to “Topological phase transitions in superradiance lattices,” <http://dx.doi.org/10.1364/optica.2.000712>. The effective Hamiltonian and the Chern numbers of the topological superradiance lattices are calculated and related to the superradiance contrast between two timed Dicke states. The coupled wave equations are derived for the numerical simulation of probe pulse propagation and diffraction. We also show the competition between local direct transitions and global topological effects. © 2015 Optical Society of America

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1. EFFECTIVE HAMILTONIAN

The three-level atoms we use to construct the two-dimensional (2D) superradiance lattice (SL) have a ground state $|g\rangle$, an excited state $|e\rangle$ and a third state $|m\rangle$. The optical fields that couple $|e\rangle$ and $|m\rangle$ have three modes with wave vectors $\mathbf{k}_1 = -k_c\hat{x}$, $\mathbf{k}_2 = k_c(\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y})$ and $\mathbf{k}_3 = k_c(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y})$. The interaction Hamiltonian with rotating-wave approximation is

$$H_c = -\hbar \sum_{l=1}^3 \sum_{j=1}^N \kappa_l a_l e^{i\mathbf{k}_l \cdot \mathbf{r}_j} |e_j\rangle \langle m_j| + H.c., \quad (\text{S.1})$$

where κ_l and a_l are the vacuum coupling strength and annihilation operator of the l th mode, respectively. N is the number of atoms and $\mathbf{r}_j$ is the position of the j th atom.

Initially the atomic ensemble is in the ground state $|G\rangle = |g_1, g_2, \dots, g_N\rangle$. A weak probe field with wave vector $\mathbf{k}_p$ can prepare the atomic ensemble in the timed Dicke state (TDS)

$$|e_{\mathbf{k}_p}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{i\mathbf{k}_p \cdot \mathbf{r}_j} |g_1, g_2, \dots, e_j, \dots, g_N\rangle \quad (\text{S.2})$$

Fig. S.1. Timed Dicke states and the associated photon numbers in an SL. The numbers associated with each TDS are the photon number difference from those of state $|e_{\mathbf{k}_p}\rangle$.

by the collective absorption of a single photon. The combinational quantum states of the atoms and three coupling modes can be written as $|e_{\mathbf{k}_p}, n_1, n_2, n_3\rangle$, where n_l ($l = 1, 2, 3$) is the photon number in mode $\mathbf{k}_l$. This state is coupled to $|m_{\mathbf{k}_p-\mathbf{k}_2}, n_1, n_2+1, n_3\rangle$ with coupling strength $-\hbar\kappa_2\sqrt{n_2+1}$. The state $|m_{\mathbf{k}_p-\mathbf{k}_2}\rangle$ is defined the same as in equation (S.2) with e_j replaced by m_j and $\mathbf{k}_p$ replaced by $\mathbf{k}_p - \mathbf{k}_2$. Similar coupling also exists for the other two modes. $|m_{\mathbf{k}_p-\mathbf{k}_2}, n_1, n_2+1, n_3\rangle$ is in turn coupled to $|e_{\mathbf{k}_p+\mathbf{k}_1-\mathbf{k}_2}, n_1-1, n_2+1, n_3\rangle$ or $|e_{\mathbf{k}_p+\mathbf{k}_3-\mathbf{k}_2}, n_1, n_2+1, n_3-1\rangle$ via excitation by modes $\mathbf{k}_1$ or $\mathbf{k}_3$. These states form a honeycomb lattice with discrete momentum coordinates, as shown in Fig. S.1, called the superradiance lattice [1]. The two sublattices of the SL correspond to TDS for $|e\rangle$ and $|m\rangle$. We denote the coupling field frequency as ν_c and the transition frequency between $|e\rangle$ and $|m\rangle$ as ω_{em} . The energy difference between the two sublattice is $\Delta_c = \omega_{em} - \nu_c$. We set the zero energy at the middle of the energies of the two sublattices. Then the energies of the $|e\rangle$ and $|m\rangle$ sublattices are $\Delta_c/2$ and $-\Delta_c/2$, respectively.

Although the coupling strengths in the SL are site-dependent, if we use coherent coupling fields with large average photon numbers $\langle n_l \rangle \gg 1$, the Rabi frequency of the l th mode can be approximated as the classical Rabi frequency $\Omega_l = \kappa_l \sqrt{\langle n_l \rangle}$. The SL with quantized photon numbers has edges when either of the three coupling field photon numbers reduces to zero. The total lattice has a triangular boundary. However, since the TDS have finite life time, which we suppose to be τ on average, once we create an excitation $|e_{\mathbf{k}_p}\rangle$ in the SL, the average distance this excitation can travel is in the order of $\Omega_l \tau$. In this paper we have $\Omega_l \tau \ll \langle n_l \rangle$ and the edges can never be reached. Interesting edge effects might exist in the few-photon limit and it will be discussed elsewhere. Since the photon numbers are correlated to the momenta $\mathbf{k}$'s in the TDS, we can simplify the notations by dropping the photon numbers. The Hamiltonian of these TDS is

$$H = \frac{\hbar\Delta_c}{2} \sum_{\mathbf{k}} (|e_{\mathbf{k}}\rangle\langle e_{\mathbf{k}}| - |m_{\mathbf{k}-\mathbf{k}_1}\rangle\langle m_{\mathbf{k}-\mathbf{k}_1}|) - \sum_{\mathbf{k}} \sum_{l=1}^3 \hbar\Omega_l [|e_{\mathbf{k}}\rangle\langle m_{\mathbf{k}-\mathbf{k}_l}| + h.c.], \quad (\text{S.3})$$

where $\mathbf{k} = \mathbf{k}_p + r(\mathbf{k}_2 - \mathbf{k}_1) + s(\mathbf{k}_3 - \mathbf{k}_2)$ with integers r and s .

The next-nearest-neighbour (NNN) hopping terms are induced by the periodic modulation of the three coupling field Rabi frequencies,

$$\Omega_l = \Omega_s + 2\Omega_d \cos(\nu_d t - \mathbf{k}_d^l \cdot \mathbf{r} + \phi_l), \quad (\text{S.4})$$

where Ω_s and Ω_d are the static and dynamic components of the Rabi frequencies. ν_d is the modulation frequency. ϕ_l is the modulation phase of the l th field. $\mathbf{k}_d^l$ is the modulation wavevector. We choose the atomic ensemble much smaller than $1/|\mathbf{k}_d^l|$ such that the position dependence of the phase $-\mathbf{k}_d^l \cdot \mathbf{r}$ can be neglected. We set $-\mathbf{k}_d^l \cdot \mathbf{r} = 0$ for simplicity. On the other hand, the size of the atomic ensemble shall be much larger than $1/\nu_c$ and the total number of atoms $N \gg 1$ such that

$$\langle e_{\mathbf{k}'} | e_{\mathbf{k}} \rangle = \frac{1}{N} \sum_{j=1}^N e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}_j} \approx \delta_{\mathbf{k}\mathbf{k}'}. \quad (\text{S.5})$$

We assume the atoms be randomly distributed and their number is large enough to cover all possible points in the real space Brillouin Zone. The SL can be regarded as infinite.

We expand the Hamiltonian into static, positive- and negative-frequency components,

$$H = H_0 + H_{+1}e^{i\nu_d t} + H_{-1}e^{-i\nu_d t}, \quad (\text{S.6})$$

where

$$H_0 = \frac{\hbar\Delta_c}{2} \sum_{\mathbf{k}} (|e_{\mathbf{k}}\rangle\langle e_{\mathbf{k}}| - |m_{\mathbf{k}-\mathbf{k}_1}\rangle\langle m_{\mathbf{k}-\mathbf{k}_1}|) - \hbar\Omega_s \sum_{\mathbf{k}} \sum_{l=1}^3 [|e_{\mathbf{k}}\rangle\langle m_{\mathbf{k}-\mathbf{k}_l}| + h.c.], \quad (\text{S.7})$$

$$H_{+1} = -\hbar\Omega_d \sum_{\mathbf{k}} \sum_{l=1}^3 e^{i\phi_l} [|e_{\mathbf{k}}\rangle\langle m_{\mathbf{k}-\mathbf{k}_l}| + h.c.], \quad (\text{S.8})$$

$$H_{-1} = -\hbar\Omega_d \sum_{\mathbf{k}} \sum_{l=1}^3 e^{-i\phi_l} [|e_{\mathbf{k}}\rangle\langle m_{\mathbf{k}-\mathbf{k}_l}| + h.c.]. \quad (\text{S.9})$$

Note that $H_{\pm 1}$ are not Hermitian themselves, but H_{+1} is the Hermitian conjugate of H_{-1} . The phase factors $e^{\pm i\phi_l}$ in $H_{\pm 1}$ play the crucial role for the complex NNN hopping terms in the Haldane model.

The dynamics of the system is a Floquet problem [2, 3]. Based on the Floquet theorem, the wave function can be written as

$$|\Psi\rangle = e^{-i\epsilon t/\hbar} |\psi(t)\rangle, \quad (\text{S.10})$$

where ϵ is the quasi-eigenenergy and

$$|\psi(t)\rangle = \sum_n e^{i\nu_d t n} |\psi_n\rangle, \quad (\text{S.11})$$

is a periodic wave function with period $2\pi/\nu_d$. Substituting equations (S.6)-(S.11) to the Schrödinger equation, $i\hbar\partial|\Psi\rangle/\partial t = H|\Psi\rangle$, we obtain

$$i\hbar \frac{\partial}{\partial t} e^{-i\epsilon t/\hbar} \sum_n e^{i\nu_d t n} |\psi_n\rangle = (H_0 + H_{+1}e^{i\nu_d t} + H_{-1}e^{-i\nu_d t}) e^{-i\epsilon t/\hbar} \sum_n e^{i\nu_d t n} |\psi_n\rangle. \quad (\text{S.12})$$

The terms with the same time evolution phase factors should be equal on both sides. We therefore have

$$(\epsilon - \hbar\nu_d n) |\psi_n\rangle = H_0 |\psi_n\rangle + H_{+1} |\psi_{n-1}\rangle + H_{-1} |\psi_{n+1}\rangle. \quad (\text{S.13})$$

The quasi-eigenenergy can be obtained by diagonalizing the above Hamiltonian.

For the sake of simplicity, we assume the separation between the Floquet sidebands is much larger than the bandwidth, $\nu_d \gg \Omega_{s,d}, \Delta_c$, where the perturbation theory can be applied [4, 5]. When the probe field is near resonance, $\Delta_p = \omega_{eg} - \nu_p \ll \Omega_s$ where ω_{eg} is the transition frequency between $|e\rangle$ and $|g\rangle$ and ν_p is the probe field frequency, only the Floquet band with $n = 0$ in Eq.(S.13) is relevant. For states with eigenfrequencies near $\epsilon = 0$, the effective Hamiltonian can be obtained by standard second-order perturbation as

$$\epsilon |\psi_0\rangle = H_{\text{eff}} |\psi_0\rangle, \quad (\text{S.14})$$

where

$$H_{\text{eff}} = H_0 + H', \quad (\text{S.15})$$

with NNN Hamiltonian

$$H' = \frac{1}{\hbar\nu_d} (H_{+1}H_{-1} - H_{-1}H_{+1}). \quad (\text{S.16})$$

Fig. S.2. Complex next-nearest-neighbor hopping induced by Rabi frequency modulation. (a) The Floquet sidebands of the SL and the NNN transition Ω_{13} (dashed arrow) induced by the interference between two second-order inter-sideband transitions (solid arrow). Phases are labelled for each step. (b) The effective NNN transitions in a unit cell. The NNN hopping Ω_{31} (red arrow), Ω_{23} (blue dot arrow) and Ω_{12} (green dash dot arrow) enclose a nonzero effective magnetic flux.

Explicitly,

$$H' = \sum_{\mathbf{k}} \sum_{l \neq j=1}^3 \hbar \Omega_{lj} \left(|e_{\mathbf{k}+\mathbf{k}_l-\mathbf{k}_j}\rangle \langle e_{\mathbf{k}}| + |m_{\mathbf{k}-\mathbf{k}_l}\rangle \langle m_{\mathbf{k}-\mathbf{k}_l+\mathbf{k}_j}| \right), \quad (\text{S.17})$$

where $\Omega_{lj} = \Omega' [e^{i(\phi_l-\phi_j)} - e^{i(\phi_j-\phi_l)}] = 2i\Omega' \sin(\phi_l - \phi_j)$ with $\Omega' = \Omega_d^2/v_d$. The crucial factor $i = \sqrt{-1}$ comes from the quantum interference between the two pathways shown in Fig. S.2 (a). The loop transitions via NNN hopping accumulate nonzero phases due to these complex factors, as shown in Fig. S.2 (b).

The effective Hamiltonian is greatly simplified in the real-space representation. We denote the real-space basis states as

$$\begin{aligned} |e_{\mathbf{r}_j}\rangle &= |g_1, g_2, \dots, e_j, \dots, g_N\rangle, \\ |m_{\mathbf{r}_j}\rangle &= |g_1, g_2, \dots, m_j, \dots, g_N\rangle, \end{aligned} \quad (\text{S.18})$$

The effective Hamiltonian can be written as

$$H_{\text{eff}} = \sum_j \mathbf{h}(\mathbf{r}_j) \cdot \boldsymbol{\sigma}_j \quad (\text{S.19})$$

where the effective magnetic field $\mathbf{h}(\mathbf{r}_j) = (h_x, h_y, h_z)$ with

$$h_x = -\hbar \Omega_s \sum_{l=1}^3 \cos(\mathbf{r}_j \cdot \mathbf{k}_l), \quad (\text{S.20})$$

$$h_y = \hbar \Omega_s \sum_{l=1}^3 \sin(\mathbf{r}_j \cdot \mathbf{k}_l), \quad (\text{S.21})$$

$$h_z = \frac{\hbar \Delta_c}{2} + 2i\hbar \sum_{l=1}^3 \Omega_{(l+1)l} \sin[\mathbf{r}_j \cdot (\mathbf{k}_{l+1} - \mathbf{k}_l)], \quad (\text{S.22})$$

and the pseudo spin $\boldsymbol{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ with the Pauli matrices for the j th atom defined as $\sigma_j^x = |e_{\mathbf{r}_j}\rangle \langle m_{\mathbf{r}_j}| + |m_{\mathbf{r}_j}\rangle \langle e_{\mathbf{r}_j}|$, $\sigma_j^y = -i|e_{\mathbf{r}_j}\rangle \langle m_{\mathbf{r}_j}| + i|m_{\mathbf{r}_j}\rangle \langle e_{\mathbf{r}_j}|$ and $\sigma_j^z = |e_{\mathbf{r}_j}\rangle \langle e_{\mathbf{r}_j}| - |m_{\mathbf{r}_j}\rangle \langle m_{\mathbf{r}_j}|$.

Fig. S.3. Effective magnetic fields in a Brillouin zone for topological and trivial SL's. (a) A topological SL with $\Delta_c = 0$. (b) A trivial SL with $\Delta_c = 36\Omega'$. The arrows show the vector $h_x\hat{x} + h_y\hat{y}$ and the colours show h_z . The x and y axis are in unit of k_c^{-1} . The phase of the coupling field $\mathbf{k}_l$ is $\phi_l = (l-1)4\pi/3$ and $\alpha = 3\sqrt{3}/2$. Note that the K/K' points at the boundaries of the first Brillouin zone are equivalent.

2. BERRY CONNECTION AND BERRY CURVATURE

The topological properties of the wavefunctions are determined by the Berry connection $\mathbf{A}$ and Berry curvature $\mathbf{B}$,

$$\mathbf{A} = i\langle \psi | \nabla | \psi \rangle, \quad (\text{S.23})$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (\text{S.24})$$

where ∇ and $\nabla \times$ are the gradient and curl operators with respect to $\mathbf{r}$.

Similar to a spin-1/2 in a magnetic field $\mathbf{h}$, the eigenenergies of the SL eigenstates of the effective Hamiltonian in equation (S.19) are

$$\epsilon_{\pm} = \pm h \quad (\text{S.25})$$

where $\pm$ are for the upper and lower bands and h is the magnitude of $\mathbf{h}$. The eigen wavefunction in the upper band can be written as

$$|\psi_+(\mathbf{r})\rangle = \frac{1}{\sqrt{2h(h+h_z)}} \begin{pmatrix} h + h_z \\ h_x + ih_y \end{pmatrix}. \quad (\text{S.26})$$

This wavefunction is well defined except for the south pole of the Bloch sphere where $\mathbf{h} = (0, 0, h_z)$ with $h_z = -h$. The eigen wavefunction near the south pole can be written as

$$|\psi'_+(\mathbf{r})\rangle = e^{i\phi(\mathbf{r})}|\psi_+(\mathbf{r})\rangle = \frac{1}{\sqrt{2h(h-h_z)}} \begin{pmatrix} h_x - ih_y \\ h - h_z \end{pmatrix}, \quad (\text{S.27})$$

where the gauge transformation

$$e^{i\phi(\mathbf{r})} = \frac{h_x - ih_y}{|h_x - ih_y|}. \quad (\text{S.28})$$

This way, the whole Bloch sphere is fully covered by two gauges [6].

The Berry connection in the upper band (except for the south pole of the Bloch sphere) is

$$\begin{aligned} \mathbf{A}_+ &= i\langle\psi_+|\nabla|\psi_+\rangle \\ &= -\frac{1}{2h(h+h_z)}(h_x\nabla h_y - h_y\nabla h_x), \end{aligned} \quad (\text{S.29})$$

and near the south pole we can use

$$\mathbf{A}'_+ = \mathbf{A}_+ - \nabla\phi(\mathbf{r}). \quad (\text{S.30})$$

The Berry curvature is

$$\begin{aligned} \mathbf{B}_+ &= \nabla \times \mathbf{A}_+ \\ &= -2\hat{z}\text{Im}\langle\partial_x\psi_+|\partial_y\psi_+\rangle \\ &= -\frac{\hat{z}}{2h^3}\epsilon_{abc}h_a\partial_xh_b\partial_yh_c, \end{aligned} \quad (\text{S.31})$$

where ϵ_{abc} with $a, b, c \in x, y, z$ is the Levi-Civita symbol. The Chern number is defined as the total Berry curvature in the whole first Brillouin zone

$$\begin{aligned} C &= \frac{1}{2\pi} \oint_{\text{BZ}} \mathbf{B}_+ \cdot d\mathbf{S} \\ &= -\frac{1}{4\pi} \oint_{\text{BZ}} \frac{1}{h^3} \epsilon_{abc} h_a \partial_x h_b \partial_y h_c dx dy, \end{aligned} \quad (\text{S.32})$$

which counts the winding number of the effective magnetic field $\mathbf{h}$ wrapping around the Bloch sphere in the whole Brillouin zone [7, 8].

In Fig.S.3, we plot $\mathbf{h}$ in the Brillouin zone for a topological nontrivial SL where the band gap is opened by the NNN hopping. If we go from the K point $\mathbf{r}_+ = -\frac{4\sqrt{3}\pi}{9k_c}\hat{y}$ to the K' point $\mathbf{r}_- = \frac{4\sqrt{3}\pi}{9k_c}\hat{y}$, $\mathbf{h}$ moves from the north pole to the south pole of the Bloch sphere. The Chern number is one. If the band gap is opened by the on-site offset, Δ_c , h_z is a constant in the Brillouin zone. $\mathbf{h}$ can only cover a patch on the Bloch sphere and the Chern number is zero.

The two bands of H_{eff} have the smallest gap at the K and K' points $\mathbf{r}_{\pm}$, where

$$\sum_{l=1}^3 e^{i\mathbf{r}_{\pm} \cdot \mathbf{k}_l} = 0, \quad (\text{S.33})$$

and hence $h_x = h_y = 0$. At these symmetry points of the Brillouin zone the Hamiltonian is diagonal,

$$\mathcal{H}(\mathbf{r}_{\pm}) = [\frac{\hbar\Delta_c}{2} \pm 2\sqrt{3}\Omega'\hbar\alpha]\sigma_z, \quad (\text{S.34})$$

where $\alpha = -\sum_{l=1}^3 \sin(\phi_{l+1} - \phi_l)$.

Fig. S.4. The eigenstates at K and K' points for topological and trivial SL's. (a) For a topological SL, the eigenstate $|e\rangle$ at $\mathbf{r}_+$ (i.e., K point) is in the upper band while it is in the lower band at $\mathbf{r}_-$ (i.e., K' point). (b) For a trivial SL, $|e\rangle$ is the eigenstate in the upper band at both $\mathbf{r}_+$ and $\mathbf{r}_-$. Red (Blue) lines denote the bands with more components in $|e\rangle$ ($|m\rangle$).

Let us consider the specific case that $\Delta_c = 0$, $\phi_l = (l - 1)4\pi/3$, and thus $\alpha = 3\sqrt{3}/2$. In this case $\mathcal{H}(\mathbf{r}_{\pm}) = \pm 9\hbar\Omega'\sigma_z$ has opposite signs at $\mathbf{r}_{\pm}$, as shown in Fig.S.3 (a). It is obvious that at $\mathbf{r}_+$, the eigenstate in the upperband with eigenenergy $9\hbar\Omega'$ is $|e\rangle$, while at $\mathbf{r}_-$, $|e\rangle$ is the eigenstate in the lower band, as shown in Fig.S.4 (a).

Near $\mathbf{r}_+$, we can use equation (S.26) to describe the wavefunction. However, equation (S.26) cannot describe all the wavefunctions in the upper band. It has a singularity at $\mathbf{r}_-$ where the magnetic field $\mathbf{h}$ points to the south pole. We can remove the singularity by tuning Δ_c . When $\Delta_c > 4\sqrt{3}\Omega'\alpha$, the effective magnetic field $\mathbf{h}$ does not experience the whole Bloch sphere [see Fig.S.3 (b)]. In this case, h_z in equation (S.34) has the same sign for $\mathbf{r}_{\pm}$, $|e\rangle$ is in the upper band at both $\mathbf{r}_{\pm}$, as shown in Fig.S.4 (b). The wavefunction can be described by the same gauge. Thus the single-valued Berry connection $\mathbf{A}$ has no singularities on a closed surface and the Chern number is zero. Generally, the Chern number of the upper band can be written as

$$C = \frac{1}{2} \left[\text{sign}(\Delta_c + 4\sqrt{3}\Omega'\alpha) - \text{sign}(\Delta_c - 4\sqrt{3}\Omega'\alpha) \right]. \quad (\text{S.35})$$

Specifically, when $\Delta_c = 0$, $C = \text{sign}(\alpha)$.

3. DYNAMIC EVOLUTION

We can solve the dynamics of the atomic ensemble in real space with the total Hamiltonian including the probe field,

$$\begin{aligned} H_t &= \sum_{j=1}^N (\langle h_z - \hbar\tilde{\nu}_p | e_{\mathbf{r}_j} \rangle \langle e_{\mathbf{r}_j} | - \langle h_z + \hbar\tilde{\nu}_p | m_{\mathbf{r}_j} \rangle \langle m_{\mathbf{r}_j} | \\ &\quad + [(h_x - ih_y) | e_{\mathbf{r}_j} \rangle \langle m_{\mathbf{r}_j} | + h.c.] \\ &\quad - [\hbar\Omega_p e^{i\mathbf{k}_p \cdot \mathbf{r}_j} | e_{\mathbf{r}_j} \rangle \langle G | + h.c.], \end{aligned} \quad (\text{S.36})$$

with $\tilde{\nu}_p = \Delta_c/2 - \Delta_p$ being the probe detuning with respect to the middle of the band gap and Ω_p the probe field Rabi frequency. The wavefunction in real space

$$|\Psi\rangle = c_G|G\rangle + \sum_{j=1}^N c_e(\mathbf{r}_j)|e_{\mathbf{r}_j}\rangle + c_m(\mathbf{r}_j)|m_{\mathbf{r}_j}\rangle. \quad (\text{S.37})$$

The dynamic equations of the probability amplitudes are

$$\begin{aligned} \dot{c}_e(\mathbf{r}_j) &= \left[-\frac{i}{\hbar}(h_z - \hbar\tilde{\nu}_p) - \gamma_e \right] c_e(\mathbf{r}_j) \\ &\quad - \frac{i}{\hbar}(h_x - ih_y)c_m(\mathbf{r}_j) + i\Omega_p e^{i\mathbf{k}_p \cdot \mathbf{r}_j} c_G, \end{aligned} \quad (\text{S.38})$$

$$\begin{aligned} \dot{c}_m(\mathbf{r}_j) = & \left[\frac{i}{\hbar}(h_z + \hbar\tilde{v}_p) - \gamma_m \right] c_m(\mathbf{r}_j) \\ & - \frac{i}{\hbar}(h_x + i\hbar\gamma_e)c_e(\mathbf{r}_j). \end{aligned} \quad (\text{S.39})$$

where $\gamma_{e/m}$ is the decoherence rate of $|e/m\rangle$ states.

In the limit of weak probe field $\Omega_p \ll \gamma_e$, we have $c_G \approx 1$. In the steady state, $\dot{c}_e(\mathbf{r}_j) = \dot{c}_m(\mathbf{r}_j) = 0$. From Eq.(S.39), we obtain

$$c_m(\mathbf{r}_j) = \frac{h_x + i\hbar\gamma_e}{h_z + \hbar\tilde{v}_p + i\hbar\gamma_m} c_e(\mathbf{r}_j). \quad (\text{S.40})$$

Substituting Eq.(S.40) in Eq.(S.38), we obtain

$$c_e(\mathbf{r}_j) = \frac{\hbar\Omega_p e^{i\mathbf{k}_p \cdot \mathbf{r}_j}}{h_z - \hbar\tilde{v}_p - i\hbar\gamma_e + \frac{\hbar^2 h_x^2 + \hbar^2 h_y^2}{h_z + \hbar\tilde{v}_p + i\hbar\gamma_m}}. \quad (\text{S.41})$$

In the SL coordinates, the wavefunction can be written as

$$|\Psi\rangle = c_G|G\rangle + \sum_{\mathbf{k}} c_{\mathbf{k}}|e_{\mathbf{k}}\rangle + c_{\mathbf{k}-\mathbf{k}_1}|m_{\mathbf{k}-\mathbf{k}_1}\rangle. \quad (\text{S.42})$$

where the probability amplitude

$$c_{\mathbf{k}} = \langle e_{\mathbf{k}}|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-i\mathbf{k} \cdot \mathbf{r}_j} c_e(\mathbf{r}_j). \quad (\text{S.43})$$

For uniformly distributed atoms, in the limit $N \gg \Omega_s/\gamma_e$, we can assume all points in the Brillouin zone are occupied by atoms, so the summation can be written as integration in the first Brillouin zone,

$$c_{\mathbf{k}} = \frac{\sqrt{N}}{S} \oint_{\text{BZ}} c_e(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}, \quad (\text{S.44})$$

where S is the area of the first Brillouin zone.

4. DIFFRACTION CONTRAST AND TOPOLOGY

In this section, we quantitatively compare the contrast between $|c_{\mathbf{k}_+}|^2$ and $|c_{\mathbf{k}_-}|^2$ in topological and trivial SL's. We assume $\tilde{v}_p = 0$ and $\gamma_m = 0$. Then Eq.(S.41) becomes

$$c_e(\mathbf{r}) = \frac{h_z}{h_z - i\hbar\gamma_e} \hbar\Omega_p e^{i\mathbf{k}_p \cdot \mathbf{r}}. \quad (\text{S.45})$$

We assume $\Omega', \Delta_c \ll \Omega_s$ and thus $c_e(\mathbf{r})$ is highly centred at K and K' points, where we have $h_x = h_y = 0$ and

$$c_e(\mathbf{r}) = \frac{\hbar\Omega_p}{h_z - i\hbar\gamma_e} e^{i\mathbf{k}_p \cdot \mathbf{r}}. \quad (\text{S.46})$$

According to Eq.(S.44), the probability amplitude $c_{\mathbf{k}}$ can be approximately calculated by the integration of $c_e(\mathbf{r})$ in small areas near $\mathbf{r}_{\pm}$,

$$\begin{aligned} c_{\mathbf{k}} \approx & \frac{\sqrt{N}\hbar\Omega_p}{S} [e^{i(\mathbf{k}_p - \mathbf{k}) \cdot \mathbf{r}_+} \iint_{\mathbf{r}_+} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy \\ & + e^{i(\mathbf{k}_p - \mathbf{k}) \cdot \mathbf{r}_-} \iint_{\mathbf{r}_-} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy], \end{aligned} \quad (\text{S.47})$$

where the phase factors have been taken out of the integration since they do not change appreciably in those small areas.

For a topological SL where $\Omega' \neq 0$ and $\Delta_c = 0$, h_z has opposite signs at $\mathbf{r}_{\pm}$. We denote $\iint_{\mathbf{r}_+} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy = p + iq$, then $\iint_{\mathbf{r}_-} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy = -p + iq$. Thus

$$c_{\mathbf{k}} \approx 2i \frac{\sqrt{N}\hbar\Omega_p}{S} \{p \sin[(\mathbf{k}_p - \mathbf{k}) \cdot \mathbf{r}_+] + q \cos[(\mathbf{k}_p - \mathbf{k}) \cdot \mathbf{r}_+]\}. \quad (\text{S.48})$$

For $\mathbf{k}_p = -\mathbf{k}_1$, $\mathbf{k}_+ = -\mathbf{k}_3$ and $\mathbf{k}_- = -\mathbf{k}_2$, we have

$$\eta \equiv \frac{|c_{\mathbf{k}_+}|^2 - |c_{\mathbf{k}_-}|^2}{|c_{\mathbf{k}_+}|^2 + |c_{\mathbf{k}_-}|^2} \approx \frac{2\sqrt{3}pq}{3p^2 + q^2}. \quad (\text{S.49})$$

For a trivial SL where $\Omega' = 0$ and $\Delta_c \neq 0$, h_z has the same sign at $\mathbf{r}_{\pm}$. We have $\iint_{\mathbf{r}_+} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy = \iint_{\mathbf{r}_-} (h_z - i\hbar\gamma_e)^{-1} d\mathbf{r} dy = p + iq$ and

$$c_{\mathbf{k}} \approx 2 \frac{\sqrt{N}\hbar\Omega_p}{S} (p + iq) \cos[(\mathbf{k}_p - \mathbf{k}) \cdot \mathbf{r}_+]. \quad (\text{S.50})$$

Therefore, $c_{\mathbf{k}_+} = c_{\mathbf{k}_-}$ and $\eta = 0$.

For a trivial SL with $\Omega' \neq 0$ and $\Delta_c \neq 0$, it is difficult to analytically calculate the results. Nevertheless, as shown in Fig.3 (b) in the main text, η reduces to a small value in the $C = 0$ phases especially for large Ω' .

5. COUPLED-WAVE EQUATIONS

In this section, we calculate the propagation of a weak probe pulse. We set $\mathbf{k}_p = -\mathbf{k}_1$, $\mathbf{k}_+ = -\mathbf{k}_3$ and $\mathbf{k}_- = -\mathbf{k}_2$. There will be directional emission in $\mathbf{k}_{\pm}$ by TDS $|e_{\mathbf{k}_{\pm}}\rangle$. The emitted photons in $\mathbf{k}_{\pm}$ interact with the atomic ensemble the same way as the probe field $\mathbf{k}_p$, creating TDS and resulting in directional emission in the other two modes. For example, emission at $\mathbf{k}_+$ excites the atom ensemble and leads to emission at $\mathbf{k}_-$ and $\mathbf{k}_p$. Therefore, the three optical fields in $\mathbf{k}_{p,\pm}$ are coupled via the atoms. In the following, we derive the coupling equations.

The expectation value of the dipole moment of the j th atom at position $\mathbf{r}_j$ is

$$\begin{aligned} \langle \Psi | \hat{\mu}_j | \Psi \rangle &= \sum_{\mathbf{k}} c_{\mathbf{k}} \langle G | \hat{\mu}_j | e_{\mathbf{k}} \rangle + c.c. \\ &= \frac{\mu}{\sqrt{N}} \sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}_j} + c.c., \end{aligned} \quad (\text{S.51})$$

where $\hat{\mu}_j$ is the dipole operator of the j th atom and $\mu = \langle g_j | \hat{\mu}_j | e_j \rangle$. Since the atoms are homogeneously distributed, the polarization density as a function of positions is,

$$P(\mathbf{r}) = \frac{N}{V} \frac{\mu}{\sqrt{N}} \sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} + c.c., \quad (\text{S.52})$$

where V is the volume of the atomic ensemble. The probe field Rabi frequency is $\Omega_p = \mu E_p / \hbar$ with E_p being the probe field strength. The polarization density contains the Fourier components in $\mathbf{k}_{\pm}$, $P_{\pm,p} = \epsilon_0 \chi_{\pm} E_p$ with $\chi_{\pm} = \sqrt{N} \mu c_{\mathbf{k}_{\pm}} / V \epsilon_0 E_p$ and ϵ_0 is the permittivity in vacuum. The notation $P_{\pm,p}$ means the $\mathbf{k}_{\pm}$ Fourier component of polarization density generated by optical field in $\mathbf{k}_p$. Once modes $\mathbf{k}_{\pm}$ are excited and the corresponding fields $E_{\pm}$ are generated, they also polarize the atoms. For $\phi_1 = 0$, $\phi_2 = 4\pi/3$ and $\phi_3 = 2\pi/3$, the three fields have a cyclic relation between each other. We have $P_{p,+} = \epsilon_0 \chi_- E_+$, $P_{-,+} = \epsilon_0 \chi_+ E_+$, $P_{p,-} = \epsilon_0 \chi_- E_-$ and $P_{+,-} = \epsilon_0 \chi_+ E_-$.

We assume the quasi-static approximation in which the atoms are assumed in steady states, i.e., $c_{\mathbf{k}}$ is given by equation

Fig. S.5. The contrast of the NNN coupling strengths. The contrast $(|\Omega_{13}| - |\Omega_{12}|)/(|\Omega_{13}| + |\Omega_{12}|)$ where Ω_{13} (Ω_{12}) is the NNN hopping coefficient from $|e_{\mathbf{k}_p}\rangle$ to $|e_{\mathbf{k}_+}\rangle$ ($|e_{\mathbf{k}_-}\rangle$). The dot lines are zero points.

(S.44). This approximation is justified when the pulse duration is much longer than the decoherence time $1/\gamma_e$. We also use the slowly-varying-envelope approximation. The fields in the three relevant modes $\mathbf{k}_j$ with $j = p, +$ and $-$ are denoted as $E_j(\mathbf{r})e^{-i\nu_p t + i\mathbf{k}_j \cdot \mathbf{r}}$ where $|\mathbf{k}_j| = \nu_p/c \equiv k_p$. The coupled-wave Maxwell equations for the three modes are

$$\left(\hat{\mathbf{k}}_p \cdot \nabla + \frac{1}{v_g} \frac{\partial}{\partial t}\right) E_p = \frac{ik_p}{2} (\chi_0 E_p + \chi_- E_+ + \chi_+ E_-), \quad (\text{S.53})$$

$$\left(\hat{\mathbf{k}}_+ \cdot \nabla + \frac{1}{v_g} \frac{\partial}{\partial t}\right) E_+ = \frac{ik_p}{2} (\chi_0 E_+ + \chi_- E_- + \chi_+ E_p), \quad (\text{S.54})$$

$$\left(\hat{\mathbf{k}}_- \cdot \nabla + \frac{1}{v_g} \frac{\partial}{\partial t}\right) E_- = \frac{ik_p}{2} (\chi_0 E_- + \chi_- E_p + \chi_+ E_+), \quad (\text{S.55})$$

where $\hat{\mathbf{k}}_{p/+/-}$ are the unit vectors in the directions of $\mathbf{k}_{p/+/-}$ and v_g is the group velocity of the pulses.

6. LOCAL AND GLOBAL EFFECTS

In the main text, we use the chiral topological current to explain the consistence between η and the topological order. The topological current is a global effect for which we must treat the lattice as a whole. Since the three superradiant TDS $|e_{\mathbf{k}_{p,\pm}}\rangle$ are next nearest neighbours in the lattice, the local effect induced by the transitions inside a unit cell is also important. The direct NNN transition strength from $|e_{\mathbf{k}_p}\rangle$ to $|e_{\mathbf{k}_+}\rangle$ or $|e_{\mathbf{k}_-}\rangle$ is Ω_{13} or Ω_{12} , respectively. If the dynamics is dominated by these direct transitions, we expect the contrast of the probability η should be similar to $(|\Omega_{13}| - |\Omega_{12}|)/(|\Omega_{13}| + |\Omega_{12}|)$, which is plotted in Fig. S.5.

We plot the contrast η in Fig. S.6 for different Ω'/γ_m . The larger Ω' is, the closer η is to Fig. S.5 and the more important is the local effect, as shown in Fig. S.6 (a) and (b). For $\Omega' \leq \gamma_m$, the effect of the direct NNN transitions is suppressed by the decoherence rate γ_m . The global topological effect dominates for most areas and η is closer to Fig. 2 (a) in the main text, as shown Fig. S.6 (c) and (d).

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Fig. S.6. The global and local effects. Diffraction contrast η for an SL with (a) $\Omega' = 1$, (b) $\Omega' = 0.01$, (c) $\Omega' = 0.001$ and (d) $\Omega' = 0.0001$. $\phi_1 = 0$. The dot lines are zero points. $\Omega_s = 3$. $\Delta_p = \Delta_c = 0$. $\gamma_e = 1$ and $\gamma_m = 0.001$.

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