Femtosecond mode locking based on adiabatic excitation of quadratic solitons: supplementary material

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This document provides supplementary information to "Femtosecond mode locking based on adiabatic excitation of quadratic solitons," <http://dx.doi.org/10.1364/optica.2.000667>. We first describe the eigenmodes accessed by our aperiodic QPM structure. We next derive a response function theory to describe the behavior of broadband laser pulses in QPM media, and then give examples for several quasi-phase-matching gratings. Next we describe the numerical simulations used in the main text, including the underlying approximations to obtain a tractable model. Lastly, we discuss the influence of a fanout QPM structure on self-starting characteristics of the laser and how this can be understood in the context of the response function theory. © 2015 Optical Society of America

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1. SECOND-HARMONIC GENERATION: CONSTANT-INTENSITY SOLUTIONS

In this section, we consider solutions of the coupled wave equations for second-harmonic generation (SHG) for which the first and second harmonic waves (FH and SH, respectively) both propagate indefinitely without exchanging energy. We consider single-frequency plane-waves in order to show the key properties of these solutions in the context of self-phase modulation (SPM) and intensity-dependent intracavity losses. The coupled-wave equations (CWEs) for SHG involving plane, continuous waves are given by

$$\frac{dA_1}{dz} = -i \frac{\omega_1 d_{\text{eff}}}{n_1 c} A_2 A_1^* e^{-i\varphi(z)} \quad (\text{S1a})$$

$$\frac{dA_2}{dz} = -i \frac{\omega_1 d_{\text{eff}}}{n_2 c} A_1^2 e^{i\varphi(z)}, \quad (\text{S1b})$$

where A_1 and A_2 are electric field envelope for the FH and SH, respectively, d_{eff} is the effective nonlinear coefficient, n_j are the refractive indices of the two fields, and ω_1 is the angular frequency of the FH. The SH has frequency $\omega_2 = 2\omega_1$. The accumulated phase mismatch is given by $\varphi(z) = \varphi(0) + \int_0^z \Delta k(z') dz'$, for position-dependent phase-mismatch $\Delta k(z) = k_2 - 2k_1 - K_g(z)$, wave-vectors $k_j = n_j \omega_j / c$, and spatially-varying QPM grating k -vector $K_g(z)$.

For further analysis, it is convenient to normalize the CWEs.

We define normalized electric field envelopes a_j according to

$$A_j(z) = \sqrt{\frac{n_1}{n_j} |A_1(0)|^2 + \frac{n_2}{n_j} |A_2(0)|^2} a_j(z), \quad (\text{S2})$$

where we note that $n_1 |A_1|^2 + n_2 |A_2|^2$ is proportional to the total intensity, which is conserved during propagation of the fields. The normalization in Eq. (S2) is useful since it implies that $|a_1(z)|^2 + |a_2(z)|^2 = 1$.

We can also define a nonlinear length, according to

$$L_{NL}^{-1} = \frac{\omega_1 d_{\text{eff}}}{n_1 \sqrt{n_2} c} \sqrt{n_1 |A_1(0)|^2 + n_2 |A_2(0)|^2}, \quad (\text{S3})$$

and use this to define a normalized spatial coordinate $\zeta = z/L_{NL}$ and a normalized phase-mismatch $\delta k = \Delta k L_{NL}$. The accumulated phase-mismatch can now be written as $\varphi(\zeta) = \varphi(0) + \int_0^\zeta \delta k(\zeta') d\zeta'$. With these definitions, Eqs. (S1) can be re-written in normalized form:

$$\frac{da_1}{d\zeta} = -ia_2 a_1^* e^{-i\varphi(\zeta)} \quad (\text{S4a})$$

$$\frac{da_2}{d\zeta} = -ia_1^2 e^{i\varphi(\zeta)}. \quad (\text{S4b})$$

In the case of phase-matched SHG ($\delta k = 0$), $\varphi(\zeta)$ is constant, and the input conditions are $a_2(0) = 0$ and $|a_1(0)| = 1$. The

solution for the SH is then $|a_2(z)|^2 = \tanh(\zeta)^2$, which is the well-known result for phase-matched SHG efficiency.

Here we are concerned with a different regime, specifically cases where no energy is exchanged between the envelopes a_1 and a_2 . If we assume a fixed phase-mismatch δk , Eqs. (S4) support such solutions, and we refer to them as eigenmodes, denoted $a_{j,em}$. As they propagate through the medium, these eigenmodes have constant magnitude, but accumulate phase at a certain rate. Their magnitudes and rate of phase accumulation depend on δk . A useful way to visualize and derive these eigenmodes is via the geometrical description of three-wave mixing introduced in [1], which was used in [2] to analyze chirped QPM media. Using that analysis, with minor modifications to consider SHG rather than three-wave mixing, it can be shown that the eigenmodes satisfy the following equation:

$$|a_{2,em}|^2 = \frac{1}{18} \left(6 + \delta k^2 \mp \delta k \sqrt{12 + \delta k^2} \right). \quad (\text{S5})$$

Since $|a_{1,em}|^2 = 1 - |a_{2,em}|^2$, the fraction of total intensity in the SH is $|a_{2,em}|^2$. The eigenmodes described by Eq. (S5) exist for a certain range of δk : for the “−” case, the range is $\delta k \in [-2, \infty]$; for the “+” case, the range is $\delta k \in [-\infty, +2]$. The $\delta k \rightarrow \pm\infty$ limit corresponds to zero energy in the SH, while the $\delta k = \mp 2$ limit corresponds to zero energy in the FH.

As well as the above amplitude constraint, the relative phases of $a_{j,em}(\zeta)$ are constrained such that

$$a_{1,em}(\zeta)^2 a_{2,em}(\zeta)^* e^{i\varphi(\zeta)} = \frac{1}{54} \left(-\delta k^3 \mp 6\sqrt{12 + \delta k^2} \pm \delta k^2 \sqrt{12 + \delta k^2} \right), \quad (\text{S6})$$

where we emphasize that the right hand side of this equation involves only real and ζ -independent quantities. Thus, the overall phase of one of the waves is not specified, but the relative phase between the waves and their driving polarizations are fixed according to Eq. (S6). In Eqs. (S5) and (S6), the $\mp$ symbols denote the two families of eigenmodes that can exist for each δk . In Fig. 2 of the main text we show the eigenmode corresponding to the upper symbol (i.e. “−” in Eq. (S5)); the other sign corresponds to reversing the sign of δk .

Next, we find the normalized rate of phase accumulation for $a_{j,em}$, denoted γ_1 . Writing $a_{j,em}(\zeta) = |a_{j,em}| \exp(i\Phi_j(\zeta))$ and substituting into Eq. (S4a), we first see that

$$\gamma_1 \equiv -\frac{d\Phi_1}{d\zeta} = \frac{a_{2,em} a_{1,em}^*}{a_{1,em}} e^{-i\varphi(\zeta)} = \frac{a_{2,em} (a_{1,em}^*)^2}{|a_{1,em}|^2} e^{-i\varphi(\zeta)}. \quad (\text{S7})$$

Since $|a_1|^2 = 1 - |a_2|^2$, we can equate the numerator and denominator in Eq. (S7) with Eqs. (S6) and (S5), respectively. Applying these substitutions and simplifying, we find

$$\gamma_1 = -\frac{1 - 3|a_{2,em}|^2}{\delta k}. \quad (\text{S8})$$

Taking $\delta k \rightarrow \infty$ in Eq. (S5), $|a_{2,em}|^2 \rightarrow 1/\delta k^2$. As such, $\gamma_1 \rightarrow -1/\delta k$ in this limit, corresponding to a negative SPM effect when $\delta k \gg 0$. This δk -dependence of $|a_2|^2$ and γ_1 corresponds to that of cascaded quadratic nonlinear interactions, as discussed in the main text.

2. DERIVATION OF THE RESPONSE FUNCTION

In the following two sections, we consider the response of the broadband intracavity FH pulses to arbitrary QPM gratings, to model the subtle dynamical effects that can occur for ultrashort pulses or for QPM gratings having non-ideal domain positions due to fabrication errors. In this section, we first derive the general form of this response function for non-uniform second-order nonlinear media. In section 3, we develop this theory further for the case of QPM media, and show specific QPM grating examples.

To allow for a tractable analytical or semi-analytical model of broadband interactions in arbitrary QPM gratings, we must make some approximations on the allowed nonlinear interactions. Therefore, we first assume that the SHG process only causes a small change in the first-harmonic pulse during propagation, which is a realistic assumption for a low-loss laser with only a moderate soliton phase shift per round-trip.

We now introduce space- and time-dependent envelopes A_j defined in terms of the real-valued electric field E according to

$$\tilde{E}(x, y, z, \omega) u(\omega) = \frac{1}{2} \sum_j \tilde{A}_j(x, y, z, \omega) e^{-ik(\omega)z}, \quad (\text{S9})$$

where tilde denotes a variable expressed in the frequency domain, $u(\omega)$ is the Heaviside step function, $k(\omega)$ is the frequency-dependent wavevector of the material, and the summation is over $j = 1$ (first-harmonic, FH) and $j = 2$ (second-harmonic, SH). The inclusion of $u(\omega)$ constrains the A_j envelopes to be analytic signals with respect to absolute optical frequency ω (zero for $\omega < 0$) [3, 4]. Note that the envelopes A_j propagate unchanged in the absence of any nonlinearity, because we included the full material dispersion in the $\exp(-ik(\omega)z)$ factor in Eq. (S9). After propagating envelopes $\tilde{A}_j$ from $z = 0$ to $z = L$, the output electric field is reconstructed according to Eq. (S9).

The coupled wave equations between these frequency-domain envelopes can also be expressed in the frequency domain, according to

$$\frac{d\tilde{A}_1(\Omega)}{dz} = -i \int_0^\infty \frac{\Omega d(z)}{n(\Omega)c} \tilde{A}_2(\omega) \tilde{A}_1^*(\omega - \Omega) e^{-i\Delta\beta(\omega, \Omega)z} \frac{d\omega}{2\pi} \quad (\text{S10a})$$

$$\frac{d\tilde{A}_2(\omega)}{dz} = -i \int_0^\infty \frac{\omega d(z)}{n(\omega)c} \tilde{A}_1(\Omega) \tilde{A}_1(\omega - \Omega) e^{i\Delta\beta(\omega, \Omega)z} \frac{d\Omega}{2\pi}, \quad (\text{S10b})$$

where the two-variable function $\Delta\beta(\omega, \Omega) \equiv k(\omega) - k(\Omega) - k(\omega - \Omega)$ is the frequency-dependent phase mismatch between the three interacting frequencies: one SH frequency ω , and two FH frequencies Ω and $\omega - \Omega$.

To estimate the output SH A_2 and its influence on the FH A_1 , we can assume that A_1 is only weakly perturbed by the nonlinear process. Under this assumption, we assume a constant A_1 in Eq. (S10b), and then substitute the resulting A_2 into Eq. (S10a) to find the influence on A_1 . The integral of Eq. (S10b), assuming zero input SH, can be written as

$$i\tilde{A}_2(z, \omega) = \int_0^\infty \frac{\omega}{n(\omega)c} \tilde{A}_1(\Omega) \tilde{A}_1(\omega - \Omega) g(-\Delta\beta(\omega, \Omega), z) \frac{d\Omega}{2\pi}, \quad (\text{S11})$$

where the function $g(k, z)$ is a (windowed) spatial Fourier transform of the nonlinear coefficient $d(z)$:

$$g(k, z) = \int_0^z d(z') e^{-ikz'} dz'. \quad (\text{S12})$$

Substituting A_2 into Eq. (S10a) and integrating up to $z = L$ yields

$$\begin{aligned} \tilde{A}_1(L, \Omega) &= \tilde{A}_1(0, \Omega) - i \int_0^\infty \tilde{A}_1(\Omega') \tilde{A}_1(\omega - \Omega') \tilde{A}_1^*(\omega - \Omega) \\ &\times \frac{\Omega \omega}{n(\Omega)n(\omega)c^2} G(-\Delta\beta(\omega, \Omega'), \Delta\beta(\omega, \Omega)) \frac{d\omega d\Omega'}{(2\pi)^2}, \end{aligned} \quad (\text{S13})$$

where the function $G(k_1, k_2)$ is another spatial Fourier transform, defined according to

$$\begin{aligned} G(k_1, k_2) &= -i \int_0^L d(z) g(k_1, z) e^{-ik_2 z} dz \\ &= -i \int_0^L \int_0^z d(z) d(z') e^{-ik_1 z'} e^{-ik_2 z} dz' dz \end{aligned} \quad (\text{S14})$$

For an arbitrary $\Delta\beta(\omega, \Omega)$ function, Eq. (S13) is quite complicated. However, in many cases $\Delta\beta$ can be simplified without much loss in accuracy. For example, if second and higher order dispersion terms at around the center frequency of the FH, ω_1 , can be neglected for the purposes of calculating the nonlinear response, then $\Delta\beta(\omega, \Omega) = k(\omega) - [k(\Omega) + k(\omega - \Omega)]$ simplifies to a single-variable function $\Delta\beta(\omega)$ [5]:

$$\Delta\beta(\omega, \Omega) \approx k(\omega) - 2k(\omega_1) - \frac{\omega - 2\omega_1}{v_g(\omega_1)} \equiv \Delta\beta(\omega) \quad (\text{S15})$$

which depends only on the driven second-harmonic frequency ω , but not on the intermediate frequency variable Ω . With this approximation, we can simplify Eq. (S13), since

$$G(-\Delta\beta(\omega, \Omega'), \Delta\beta(\omega, \Omega)) \approx G(-\Delta\beta(\omega), \Delta\beta(\omega)) \equiv H(\omega). \quad (\text{S16})$$

Applying this approximation to Eq. (S13) yields

$$\begin{aligned} \tilde{A}_1(L, \Omega) &= \tilde{A}_1(0, \Omega) - i \int_0^\infty \tilde{A}_1^*(\omega - \Omega) H(\omega) \frac{\omega}{n(\omega)c} \frac{\Omega}{n(\Omega)c} \\ &\times \left(\int_0^\infty \tilde{A}_1(\Omega') \tilde{A}_1(\omega - \Omega') \frac{d\Omega'}{2\pi} \right) \frac{d\omega}{2\pi}. \end{aligned} \quad (\text{S17})$$

We can note further that the integral in parentheses is a simple convolution, and hence that the final output of the device is given, within the approximations specified, by

$$\begin{aligned} \tilde{A}_1(L, \Omega) &= \tilde{A}_1(0, \Omega) - i \int_0^\infty \tilde{A}_1^*(\omega - \Omega) H(\omega) \frac{\omega}{n(\omega)c} \frac{\Omega}{n(\Omega)c} \\ &\times \mathcal{F} \left[A_1(0, t)^2 \right] (\omega) \frac{d\omega}{2\pi}, \end{aligned} \quad (\text{S18})$$

where $\mathcal{F}[f(t)](\omega)$ denotes the Fourier transform of function $f(t)$ evaluated at (angular) frequency ω ; we use $\mathcal{F}_\omega[f]$ as a shorthand for this expression. Equation (S18) can be simplified further by writing it in terms of Fourier transforms:

$$\begin{aligned} \tilde{A}_1(L, \Omega) &= \tilde{A}_1(0, \Omega) - \\ &i \frac{\Omega}{n(\Omega)c} \mathcal{F}_\Omega \left[A_1^*(t) \mathcal{F}_t^{-1} \left[Q(\omega) \mathcal{F}_\omega \left[A_1^2 \right] \right] \right], \end{aligned} \quad (\text{S19})$$

where the response function $Q(\omega) = H(\omega)\omega/(n(\omega)c)$ allows for both frequency-independent (Kerr-like) and frequency-dependent (Raman-like) contributions to the nonlinear interaction.

To describe this response function in a convenient way, we observe that it is proportional to $G(-\Delta\beta(\omega), \Delta\beta(\omega))$, based on Eq. (S16). It will therefore be useful to introduce a normalized response function according to $\tilde{H}(k) = G(-k, k)/(d_0 L)^2$, where d_0 is the nonlinear coefficient of the material and L is the length. Defining $\tilde{d}(z) = d(z)/d_0$, $\tilde{H}(k)$ is given by

$$\tilde{H}(k) = -\frac{i}{L^2} \int_0^L \int_0^z \tilde{d}(z) \tilde{d}(z') e^{-ik(z-z')} dz' dz, \quad (\text{S20})$$

which is the quantity plotted in section 3, Figs. S1 and S2.

Lastly, it is useful to note that, as well as the above perturbative description, an alternative approach to the problem is via the multiple scale analysis we described in [6]. In this case, the boundary components $a_j^{(1)}$ discussed in that analysis are suppressed by apodization, so that only the leading-order equation needs to be considered (provided the RDC errors are small enough). Interestingly, these boundary terms are also somewhat analogous to those leading to spectral ripples in imperfectly-apodized chirped QPM frequency converters [7]. The advantages of the perturbative approach we have taken in this section is that it is accurate and non-divergent over the whole spectrum and allows for arbitrary $d(z)$ profiles; the disadvantage is the assumption of small changes to the FH and the approximations made on the frequency dependence of $\Delta\beta$. For the conditions relevant to this paper, i.e. for interactions inside a low-loss 1- μm laser cavity, the perturbative approach is very well suited.

3. RESPONSE FUNCTION EXAMPLES

We next consider some specific $d(z)$ profiles and the corresponding normalized response functions $\tilde{H}(k)$. First consider a simple first-order QPM interaction. In this case we substitute $\tilde{d}(z) \approx (2/\pi) \exp(iK_g z)$ and $\tilde{d}(z') \approx (2/\pi) \exp(-iK_g z')$ into Eq. (S20) (+1st Fourier order of the grating driving A_1 ; -1st order driving A_2). With these substitutions, $\tilde{H}(k)$ is given by

$$\begin{aligned} \tilde{H}(k) &\approx -i \left(\frac{2}{\pi L} \right)^2 \int_0^L \int_0^z e^{i\Delta k_1(z-z')} dz dz' \\ &= \left(\frac{2}{\pi} \right)^2 \left(-\frac{1}{\Delta k_1 L} + i \frac{e^{-i\Delta k_1 L} - 1}{\Delta k_1^2 L^2} \right), \end{aligned} \quad (\text{S21})$$

where $\Delta k_1 = k - K_g$ in this expression. This normalized response function is analogous to the one considered in section 3 of [6] for birefringent phase-matching media. The $1/\Delta k_1$ term results in SPM, while the $1/\Delta k_1^2$ term results in losses from the first-harmonic to the second-harmonic.

For the present and much more general case of a non-uniform QPM medium, possibly having small but finite fabrication errors, we must consider the actual domain structure and allow for randomness in the positions of the domain boundaries. We can express the QPM structure according to [8, 9]

$$\tilde{d}(z) = \sum_{m=1}^N (-1)^m \Pi_{z_m-1}^{z_m}(z), \quad (\text{S22})$$

where N is the number of domains, z_m are the domain boundaries, and $\Pi_a^b(z)$ is a window function, equal to 1 for $a \leq z < b$

Fig. S1. Response function $\bar{H}(k)$ for an apodized cascading device, as a function of spatial frequency k . Loss curves: $\text{Im}[\bar{H}(k)]$; phase curves: $\text{Re}[\bar{H}(k)]$. (a) $\bar{H}(k)$ near spatial frequencies corresponding to phase-matched SHG. In both subfigures, the solid and dashed curves correspond to apodized and unapodized QPM grating profiles, respectively. (b) Same curves as in (a), zoomed into regions applicable to a CQN interaction. The loss curves (red) are multiplied by 20 for comparison to the phase curves (blue). For the solid red curve (apodized case), these losses are almost completely suppressed. Far from phase-matching, the phase responses for the two cases are comparable.

and 0 otherwise. This form of $\bar{d}(z)$ can be substituted into Eq. (S20) to find $\bar{H}(k)$. After some algebra, we find

$$\bar{H}(k) = -\frac{i}{(kL)^2} \left[-i(kL) + \sum_{n=1}^N \left(1 - e^{-ikz_n} e^{ikz_{n-1}} \right) + \sum_{n=2}^N \sum_{m=1}^{n-1} (-1)^{n+m} \left(e^{-ikz_n} - e^{-ikz_{n-1}} \right) \left(e^{ikz_m} - e^{ikz_{m-1}} \right) \right]. \quad (\text{S23})$$

This general expression for $\bar{H}(k)$ can be used to study arbitrary QPM designs, including the effectiveness of different apodization profiles, and the influence of random duty cycle (RDC) errors [8, 9].

In Fig. S1, we show $\bar{H}(k)$ evaluated for two example QPM gratings, one apodized and one unapodized. For the former, there is a clear and very strong suppression of the imaginary (loss-related) part of the response function when operating far from phase-matching, as can be seen from Fig. S1(b). The ripples in the phase response are also suppressed. For 1045 nm, the material phase-mismatch is $\Delta k_0 = k_2 - 2k_1 \approx 953.5 \text{ mm}^{-1}$, which for the QPM profile chosen yields $\Delta k \approx 60 \text{ mm}^{-1}$ in the middle region of this device. Since the response function makes no assumptions regarding the pulse structure and is valid both near and far from phase-matching without exhibiting any artificial resonances, we are also able to use Eq. (S19) for efficient numerical simulation of laser dynamics, provided the underlying perturbation approximation still holds.

Fig. S1 shows that the engineered QPM profile fundamentally alters the nonlinear interaction from one which is inherently lossy and modulated (as in birefringently phase-matched media) to an almost pure phase modulation. In real devices, however, the loss-suppression is somewhat hindered by polling fabrication errors [9]. Therefore, we next consider the influence of these devices. For these RDC errors, we assume normally distributed errors in the domain boundaries z_n around their designed (mean) positions z_{n0} . Specifically, we assume $z_n \sim \mathcal{N}(z_{n0}, \sigma_z^2)$. The variance in the length $l_n = z[n] - z[n-1]$

Fig. S2. Influence of random duty cycle (RDC) errors on the response function of the apodized QPM device. (a) Illustration of RDC errors: solid horizontal lines show the positions of the nominal (designed) domain boundaries, denoted $z_0[n]$. Dashed horizontal lines indicate the positions of the actual boundary positions, differing from the ideal ones due to imperfections in the fabrication process [8, 9]. (b) Influence of typical RDC errors on the loss-related part of the response function. The black curve corresponds to a particular example grating with normally-distributed errors; different gratings with the same distribution and same nominal grating design will exhibit different response functions, but the average amplitude and the characteristic frequency over which the response function fluctuates will be comparable. For this example we assume a realistic value of $\sigma_z = 0.2 \text{ } \mu\text{m}$ [9–11]

of each domain, σ_l^2 , is related to the variance of the domain boundary positions, σ_z^2 , by $\sigma_l^2 = 2\sigma_z^2$.

The imaginary (loss-related) part of an example response function, for a particular apodized QPM grating with realistic normally distributed domain errors, is shown in Fig. S2, in comparison with unapodized and apodized gratings having no RDC errors. The RDC-induced changes to the response function vary on a characteristic $\sim \pi/L$ spatial frequency scale, which can be understood via simple Fourier arguments. With improved fabrication quality, i.e. reduced jitter in the QPM domains, the losses associated with this non-ideal response function will be suppressed, since they have an overall scaling with σ_z^2 . Note that in addition to losses, RDC errors also perturb the real (phase-related) part of $\bar{H}(k)$, such that the suppression of ripples in $\text{Re}[\bar{H}(k)]$ evident in Fig. S1(b) (solid blue curve) will no longer be ideal.

4. LASER SIMULATIONS

As well as the theoretical analysis of the preceding sections, we performed full numerical simulations of the mode-locked laser dynamics. In this section, we discuss the numerical model used. The overall approach is to simulate the pulse propagation through each element in the cavity using a general model for nonlinear-optical propagation (generalized nonlinear Schrödinger equation and coupled wave equations), as well as optical pumping, including a longitudinally-varying intensity due to pump absorption.

New models for $\chi^{(2)}$ media have been established in recent years [3, 4], allowing for accurate modelling of cascading-based devices such as supercontinuum generation in ultrabroadband and highly nonlinear regimes. However, for the case of simulating such devices inside a laser cavity, numerical efficiency is particularly important: a very general simulation would re-

quire evolving the pulses with a high spatial resolution along the propagation direction z ; a high resolution of a few fs in time t to resolve the short pulses; a large temporal window (tens of ps) to capture the temporal wings, build-up of the pulse from semiclassical noise, and SESAM relaxation dynamics; thousands of cavity round-trips to allow the pulse to relax to a quasi-steady state at a given population inversion level; hundreds of thousands of cavity round-trips to allow for damping of relaxation oscillations; and resolution of the transverse dimensions of the laser and resulting diffraction effects. Including all of these features simultaneously would be very challenging computationally.

To overcome these issues, we first perform modal overlap integrals to obtain a quasi-1+1D problem (propagation coordinate z and time t), using the z -dependent profile of the FH cavity mode. Secondly, we approximate the single-pass propagation of the FH pulses in the APPLN crystal by the response function description given in section 2. To allow for simultaneous effects such as true $\chi^{(3)}$ nonlinear processes, and to allow for substantial single-pass SPM, we distribute the response function over a few z -steps, rather than a single step as described by Eq. (S19). The corresponding resolution (several hundred μm for our $\sim 2\text{-mm}$ crystal) is in contrast to the few- μm z -steps $\Delta z \ll L_{\text{coh}} = \pi/|\Delta k|$ that would be required to resolve the true evolution of the pulses in a cascading-type interaction, or the sub- μm resolution required for resolving the RDC errors and the actual QPM domain structure.

Thirdly, to reduce the required number of round-trips, we artificially accelerate the population dynamics by a factor $s = 1000$ by simultaneously scaling the emission and absorption cross sections of the quasi-three level medium by s , the total recombination rate by s , and the doping concentration by $1/s$. This scaling preserves the important pulse dynamics (e.g. soliton and gain-bandwidth pulse shaping, nonlinear losses, and SESAM saturation), but it suppresses relaxation oscillations and Q-switching instabilities. These latter effects are suppressed since the simulated cavity is more rapidly damped (rather than the physical laser, which is under-damped due to the long lifetime of the Yb:CALGO gain medium). Therefore this approximate model is appropriate for analysing lasers that reach a steady-state mode-locking regime, in particular where the population inversion is periodic with a period of the cavity round-trip time. However, the sped-up model is inappropriate for non-steady-state operation, for example Q-switched mode-locking.

The response function approximation and artificial acceleration of the population dynamics together provide a speed enhancement of as much as $\sim 10^6$, without significantly altering the soliton-shaping physics relevant to determine modelocking stability, pulse duration, spectral shape, and other laser properties. Thus, our new numerical model is able to simulate almost any modelocked laser configuration with various intracavity nonlinear-optical elements. In future work, we plan to discuss the combined laser and pulse formation dynamics, their quantitative implications for laser design, and absolute calibration of the laser performance using measured cross sections [12], in further detail.

5. ROLE OF FANOUT GRATING IN ACHIEVING SELF-STARTING MODELOCKING

The use of fanout apodized QPM devices has implications for self-starting modelocking. As mentioned in the main text,

we observed that modelocking was easy to obtain with the apodized fanout devices discussed throughout the paper, but achieving modelocking with non-fanout but otherwise comparable devices was significantly more challenging, even when they were apodized. A possible explanation for this observation is in the unfavorable nonlinear-etalon type structure on the real and imaginary parts of the response function: while most of the modulation present in conventional cascading devices is suppressed, there is still some modulation over the spectrum arising from imperfect apodization and from QPM fabrication errors. As we discussed for birefringent phase-matching [6], such etalon-type effects appear to make modelocking more challenging.

The use of fanout QPM structures [13], where the response function varies transversally, offers a solution to this issue. The ideal response (SPM) scales like $1/\Delta k(x, z)$, which varies only slightly across the beam width, and thus the amount of SPM only changes by a small fraction across the FH beam profile. However, the non-idealities in the response function vary much more rapidly, over the characteristic π/L scale in k -space. Because of this more rapid variation, a small transverse slope in grating k -vector from the fanout profile can significantly shift the frequencies of the non-ideal spectral features of the response function for different parts of the beam profile.

We can estimate this effect by considering a more general response function for fanout devices that depends on transverse position x as well as angular frequency ω , of form

$$Q(x, \omega) = Q_0(\omega - \alpha x) \quad (\text{S24})$$

for a nominal transfer function $Q_0(\omega)$, and fanout parameter $\alpha = (d\Delta k_0/d\omega)^{-1}(\partial K_g/\partial x)$ which is approximately constant. Equation (S24) is applicable if the beams are large enough that we can treat each transverse component x as independent (i.e. if diffraction effects are negligible). Each frequency component ω samples a certain range of the nominal transfer function Q_0 , approximately αw_0 for beam waist w_0 . If the non-idealities in Q_0 vary significantly over this frequency range, then each individual frequency component sees a significant variation in its response function $Q(x, \omega)$ across the beam profile. In turn, this spatially-varying part of the response will have reduced modal overlap with the lowest-order Gaussian mode of the laser cavity. This suppression of coupling to the lowest-order spatial mode will in turn lead to a reduced influence on the laser dynamics during pulse build-up, especially if there is an aperture in the cavity that attenuates the higher order spatial modes.

A simple quantity to estimate the suppression effect is obtained by calculating the phase shift introduced by the fanout structure across the generated SH beam:

$$\delta\phi_x(w_0) = \frac{\partial K_g}{\partial x} w_0 L. \quad (\text{S25})$$

When this phase shift significantly exceeds π , we expect a significant shift in the non-idealities of the response function across the beam profile. We can arrive at this constraint by noting that, since the non-idealities in the normalized response function $\tilde{H}(k)$ vary over a characteristic π/L scale, the optical response function $Q(\omega)$ varies over an analogous $\delta\omega = \pi(d\Delta k_0/d\omega)^{-1}L^{-1}$ frequency scale. This $\delta\omega$ can then be compared to the αw_0 scale described above, so that we arrive at $\delta\omega \sim \alpha w_0$, which is the same as $\delta\phi_x(w_0) \sim \pi$ in Eq. (S25).

For the devices used in our experiments, there is a shift in $K_g(x, z)$ of 80 mm^{-1} across the 2-mm transverse aperture of the

QPM grating. Assuming a 2-mm length and a 150- μm beam radius w_0 , we obtain $\delta\phi_x \approx 3.8\pi$, which supports the above explanation of the self-starting characteristics we obtained by using a fanout grating. Note that this same $\delta\phi_x$ condition implies that there is a significant shift in the amplitude and phase profile of the residual second-harmonic exiting the crystal, leading to a spatiotemporally distorted SH beam, as we observe experimentally. Although the output SH is spatiotemporally distorted, within the middle of the device, i.e. region 2 of Fig. 1 (main text), the main SH component follows the spatial and temporal profile of the laser beam $A_1(x, y, z, t)^2$: we showed this behavior for the simplified case of a time-dependent plane wave in Fig. 2(d) of the main text. As such, a reduced output SH beam quality does not imply a reduced homogeneity of the SPM accumulated by the FH laser pulse.

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