Nodal areas in coherent beams: supplementary material

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This document provides supplementary information to "Nodal areas in coherent beams," <http://dx.doi.org/10.1364/optica.2.000147>. The first section shows the diffraction of a circular nodal area along the direction of propagation. The second sections outlines the derivation of circular nodal areas formed by a focal plane vortex phase element. The third section describes fabrication of the q -plate. The fourth section describes the processing of the experimental data. The fifth section describes the phase of the exit pupil fields surrounding the nodal areas. The final section shows additional numerically calculated phase elements that produce nodal areas of arbitrary shape. © 2015 Optical Society of America

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1. THE DIFFRACTION OF A NODAL AREA

The evolution of the transverse irradiance distribution along the direction of propagation in the vicinity of the nodal area is shown in Fig. S1. The irradiance is exactly zero in the plane perpendicular to the page at $z = 0$ over the region $r < R$ (i.e. $|y/R| < 1$ in Fig. S1).

2. DERIVATION OF CIRCULAR NODAL AREAS FORMED WITH VORTEX PHASE ELEMENTS

The following derivation demonstrates the generation of a circular nodal area and ring of fire at the exit pupil of the $4-f$ imaging system with a vortex phase element located in the intermediate focal plane [1, 2]. We solve for the exit pupil field assuming a circular aperture and a focal plane transmission $t = \exp(im\theta')$, where m is a nonzero even integer. The field directly after the focal plane element due to a plane wave entering an aperture of radius a may be written as

$$U(r', \theta') = \frac{ka^2}{f} \frac{J_1(ka r' / f)}{ka r' / f} e^{im\theta'}, \quad (1)$$

Fig. S1. Field magnitude cross-section along the optical (z) axis in the vicinity of a circular nodal area formed by an $m = 2$ vortex phase element. The x'', y'' plane is located at $z = 0$ and $\lambda = 633$ nm.

where (r', θ') are the polar coordinates in the focal plane, $k = 2\pi/\lambda$, λ is the wavelength, f is the focal length of the lenses, J_1 is the Bessel function of the first kind, and constant phase factors

have been dropped. The exit pupil field is given by the Fourier transform of Eq. 1:

$$U^{(m)}(r'', \theta'') = e^{im\theta''} \frac{ka}{f} \int_0^\infty J_1(kar'/f) J_m(kr''r'/f) dr', \quad (2)$$

where (r'', θ'') are the polar coordinates in the exit pupil plane. For m is nonzero and even,

$$U^{(m)}(r'', \theta'') = \begin{cases} 0 & r'' \leq a \\ \frac{a}{r''} R_n^1\left(\frac{a}{r''}\right) e^{im\theta''} & r'' > a \end{cases}, \quad (3)$$

where R_n^1 are the radial Zernike polynomials and $n = |m| - 1$ [3]. The first few solutions are

$$U^{(\pm 2)}(r'', \theta'') = \frac{a^2}{(r'')^2} e^{\pm i2\theta''}, \quad (4)$$

$$U^{(\pm 4)}(r'', \theta'') = \left(\frac{3a^4}{(r'')^4} - \frac{2a^2}{(r'')^2} \right) e^{\pm i4\theta''}, \quad (5)$$

$$U^{(\pm 6)}(r'', \theta'') = \left(\frac{10a^6}{(r'')^6} - \frac{12a^4}{(r'')^4} + \frac{3a^2}{(r'')^2} \right) e^{\pm i6\theta''}. \quad (6)$$

The optical process described above can also be thought of as a convolution of the circular aperture and the Fourier transform of the vortex phase transmission function:

$$\frac{1}{\lambda f} \text{FT} \{ e^{im\theta'} \} = \frac{(-i)^m k}{f} e^{im\theta''} \int_0^\infty J_m(kr''r'/f) r' dr'. \quad (7)$$

This integral is not directly given in common integral tables, a related integral is [4]

$$\int_0^\infty x \exp(-\alpha x^2) J_\nu(\beta x) dx = \frac{\sqrt{\pi}\beta}{8\alpha^{3/2}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) \right], \quad (8)$$

where I_ν is the modified Bessel function of the first kind, $\text{Re}(\alpha) > 0$, and $\text{Re}(\nu) > -2$. We calculate the result in the limit as α approaches zero:

$$\lim_{\alpha \rightarrow 0} \int_0^\infty x \exp(-\alpha x^2) J_\nu(\beta x) dx = \frac{\nu}{\beta^2}, \quad (9)$$

where $\text{Re}(\nu) > -2$ and $\beta > 0$. Thus, Eq. 7 becomes

$$\frac{1}{\lambda f} \text{FT} \{ e^{im\theta'} \} = \frac{\gamma_m}{(r'')^2} e^{im\theta''}, \quad (10)$$

where $\gamma_m = (-i)^m m f / k$.

3. FABRICATION OF THE Q-PLATE

The q -plate used in this demonstration was prepared by aligning a nematic liquid crystal (LC) material with the orientation angle $\alpha(x', y') = \arctan(by'/ax')$, i.e. $q = 1$ (see Fig. S2). The desired planar alignment of the liquid crystals was induced using a photoalignment technique [5]. We used 0.1% wt. solution of Brilliant Yellow (BY) dye in dimethylformamide (DMF) as the aligning surfactant and Indium-Tin-Oxide (ITO) coated glass substrates

Fig. S2. Transmission of an elliptical q -plate. (a) Phase of the elliptical vortex function $\Phi(x', y')$ with $m = 2$ and $a/b = 2$. Lines of constant phase (dotted) are indicated in steps of $\pi/4$. (b) Zoomed-in image of the fabricated, 1 cm $\times$ 1 cm, elliptical q -plate placed between orthogonal linear polarizers, showing the expected irradiance pattern: $\sin^2(2\alpha)$.

so to have the possibility of applying an external electrical field to the LC film. As part of the fabrication process [6, 7], a polarized UV laser beam was expanded by a set of lenses, sent through a half-wave plate and focused on a sample with a cylindrical lens of 75 mm focal length. Both the waveplate and the sample were attached to rotating mounts connected to a computer through step-motors. By live control of the relative step size of two motorized mounts during exposure it was possible to impress elliptical orientation patterns with angular dependence. Owing to the manufacturing process, a defect 25 μm in diameter occurs at the center of the sample, which is small compared to the dimensions of the q -plate (20 mm $\times$ 15 mm).

4. PROCESSING OF EXPERIMENTAL DATA

The quality of our experimental elliptical dark nodal area was determined as follows. A dark frame was captured immediately before exposing the CCD detector array (SBIG STF-8300). The exposure time was 0.09 seconds and the camera was cooled to

Fig. S3. Line profile across the minor axis direction of the elliptical nodal area. The irradiance data (blue) is averaged over 50 line profiles after the dark-image was subtracted. The bias level (black) is calculated in a fringe region (not shown). We assume that the background signal is constant throughout the image.

0.19°C. The dark current is approximately 0.2 e-/pixel/sec at 0°C and the read noise is $\sim 9.3e^-$ rms [8]. The power inside the ring of fire P_{IN} was calculated by integrating over an elliptical synthetic aperture centered with respect to the ring of fire. The synthetic aperture has half of the area of the entrance aperture of the optical system. The power bias P_B was calculated by integrating over a synthetic aperture of equal size and shape in a fringe region of the image. The total power P_T is the integrated power over the entire image with the bias removed. We define the quality measure as

$$Q = \frac{P_{IN} - P_B}{P_T}. \quad (11)$$

We can calculate the uncertainty in the Q due to detector noise by

$$\Delta Q = \sqrt{P_T^{-2} (\Delta P_{IN}^2 + \Delta P_B^2) + \Delta P_T^2 \left(\frac{P_{IN} - P_B}{P_T^2} \right)^2}, \quad (12)$$

where the uncertainties are denoted by Δ and are reported as standard deviations. The uncertainty in integrated power due to detector noise is given by $\Delta P = \sqrt{\sigma_{read}^2 + \sigma_{dark}^2 + \sigma_{shot}^2}$. The read noise component is $\sqrt{n_{pix} N_{read}^2}$, where n_{pix} is the number of pixels within the synthetic aperture. The noise due to dark current is $\sqrt{i_{dark} n_{pix} t}$, where i_{dark} is the dark current in e-/pixel/sec and t is the exposure time. Shot noise is the square root of the integrated signal in units of photo-electrons. The calculated uncertainty in quality is $\Delta Q = 1.17 \times 10^{-6}$. Thus, assuming only detector noise the quality value can be reported as $Q = (2.418 \pm 0.001) \times 10^{-3}$. However, the majority of the experimental noise arises from imperfect optics and alignment. A line profile through the nodal area is shown in Fig. S3. The estimated bias level is plotted along with the irradiance profile across the minor axis of the ring of fire ellipse (with the dark-image subtracted). The counts are averaged over a 50 pixel width through the center region of the ring of fire.

5. PHASE OF THE EXIT PUPIL FIELDS

Figure S4 includes phases in the exit pupil for the cases shown in Fig. 3 of the Letter. Note that the phase in Fig. S4(b) is

Fig. S4. (a) Amplitude and (b) phase of the exit pupil field in the example shown in Fig. 3(a)-(c) of the Letter. (c)-(d) Same as (a)-(b), but for the example shown in Fig. 3(d)-(f) of the Letter.

Fig. S5. (a) Two disconnected elliptical apertures, with (b) a numerically calculated phase element, yields (c) two elliptical nodal areas. (d)-(f) Same as (a)-(c), but with (d) a starfish-shaped aperture and (f) nodal area.

characterized by an azimuthal vortex phase of topological charge $m = 6$, and radial π phase steps at corresponding locations where zero-valued rings occur. In comparison, Fig. S4(d) has more complicated structure owing to the radial and azimuthal variation in the focal plane phase element (see Fig. 3(e) of the Letter). The exterior field has phase of the form $2\theta'' + v(r'')$, where $v(r'')$ is a real valued function. The phase of the exit pupil field maintains the topological charge of the focal plane element in both cases. We note that there are many (possibly infinite) unique exit pupil fields that contain a nodal area, each with its own diffractive behavior about the exit pupil plane. We emphasize that in the ideal case where the amplitude is zero-valued in the nodal area, the phase is undefined and the phase profile is therefore meaningless.

6. ADDITIONAL NUMERICAL CASES

Numerically calculated phase elements for forming disconnected nodal areas from dual elliptical apertures and a starfish-shaped nodal area from a starfish-shaped aperture are shown in Fig. S5.

To provide an estimate of the computational resources needed for calculating the phase element, we report the timing for the example in Fig. S5(d)-(f). The calculations were performed using a 4096×4096 grid in MATLAB on a desktop computer with 3.6 GHz quad core processor and 8 GB ram. The required contrast (fraction of power inside the nodal area) is used as the stopping condition. With the initial focal plane phase element set to an $m = 2$ vortex, a contrast of 10^{-3} is achieved in 50 seconds (10 iterations) and 10^{-4} is achieved in 94 seconds (19 iterations). Faster computation is possible with initial guesses that provide darker nodal areas than the $m = 2$ vortex.

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