Single-exposure optical focusing inside scattering media using binarized time-reversed adapted perturbation: supplementary material

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This document provides supplementary information to “Single-exposure optical focusing inside scattering media using binarized time-reversed adapted perturbation,” <http://dx.doi.org/10.1364/optica.2.000869>. It includes details about system configuration, simulation method of TRAP and *b*-TRAP focusing, procedure of real-time focusing, and theoretical framework of deep flow measurements. It also extends the discussion about the degrees of freedom, diffraction efficiency, and peak-to-background ratio of *b*-TRAP focusing. © 2015 Optical Society of America

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1. SUPPLEMENTARY METHODS

A. System setup

The system is schematically shown in Supplementary Fig. S1. A Q-switched 532 nm laser (Customized, Elforlight Inc., UK) produced 6 ns, 0.6 mJ pulses with a repetition rate tunable from 500 Hz to 1000 Hz. The coherence length of the laser was measured to be > 0.5 m, with sporadic fringe instability due to mode hopping. The pulse train entered a Mach-Zehnder interferometer through a polarizing beamsplitter (PBS) with a splitting ratio adjustable via a half-wave plate (HWP). Two acousto-optic modulators (AOMs, AFM-502-A1, IntraAction, USA) were cascaded in the sample arm (which is slightly different from the schematic shown in Fig. 1, but the function is unchanged) to provide tunable optical frequency shifts. The frequency-shifted light illuminated a scattering sample, with the scattered light collected on the other side in a transmission configuration. The $f = 35$ mm plano-convex lens used to collect the scattered light imaged the back side of the scattering sample to the surface of an amplitude-only SLM (HED 6001, Holoeye, Germany) having 8 bit graylevel and 1920×1080 resolution. An optical polarizer was used in conjunction with the SLM for best amplitude modulation contrast. The SLM surface was conjugated to the image plane of an sCMOS camera (pco.edge, PCO AG, Germany) by a camera lens (85mm f/3.5G AF-S DX ED VR, Nikon, Japan) with 1:1 digital pixel matching. The light in the other arm went through another HWP to rotate its polarization by 90°. The light was further split into a reference and a reading beam by a 50:50 beam splitting plate. The reference beam was attenuated

ten times by a neutral density filter and properly delayed to minimize mode-hopping-induced fringe instability. The reference and reading beams were carefully aligned to be coaxial after being combined by a 90 (reflection):10 (transmission) BS. They were collimated and expanded by an afocal system to 1 inch in diameter before being merged with the sample beam via a 50:50 BS. Switching the sample and reading beams “on” and “off” was

Fig. S1. System schematic. AOM, acousto-optic modulator; BS, beam splitter; CMOS, complementary metal-oxide semiconductor camera; PBS, polarizing beam splitter; sCMOS, scientific CMOS; SLM, spatial light modulator.

controlled by two optical shutters (Uniblitz LS3, Vincent Associates, USA), which, together with the two AOMs and the sCMOS camera, shared the same time-base with the laser Q-switch. During hologram measurement, shutter 1 was open while shutter 2 was closed, resulting in an interference pattern formed by the sample and reference beams. After two intensity patterns were captured by the sCMOS camera and subsequently processed, a TRAP hologram was calculated, binarized, and displayed on the SLM. In the time-reversed focusing mode, shutter 1 was closed and shutter 2 was open, allowing a strong reading beam to reconstruct the TRAP light. To improve the focal PBR, the strong retro-reflected zero-order component was effectively blocked by a black dot (printed on a transparency) placed at the focus of the $f = 35$ mm light-collecting lens. The zero-order block converted binary amplitude to binary phase modulation, see Supplementary Discussion C for details. Inside the scattering sample, the light intensity distribution on the focal plane (target plane) was mirrored to the image plane of a CMOS camera (FireflyMV, Point Gray, Canada) by a BS if real-time monitoring of the focus was needed.

B. Simulation of TRAP focusing

The electric field distribution on the input plane is denoted by $E_{in}(t)$, where t is time. The two-dimensional electric field distribution is represented by a 50-by-50 matrix with elements obeying the circular random Gaussian distribution. This input field matrix can be converted to an input vector $\mathbf{E}_{in}(t)$ through linear indexing. An idealized absorptive moving target, 1-by-1 pixel in size, is represented by setting the corresponding matrix element to 0. The electric field distribution on the detection plane is found by $\mathbf{E}_S(t) = T\mathbf{E}_{in}(t)$, the subscript “S” denotes the sample beam, and T is a 2500-by-2500 circular Gaussian random matrix representing transmission through the scattering medium and propagation to the detection plane. The detected intensity distribution is $\mathbf{I}(t) = |\mathbf{E}_S(t) + \mathbf{E}_R(t)|^2$, where $\mathbf{E}_R$ is a planar reference beam with constant complex entries.

In all cases, intensity measurements are performed at a fixed interval (Δt) determined by the camera frame rate. Accordingly, the variable “ t ” is replaced by a subscript index i to identify the temporal sequence. We assume that between measurements, both the input field $\mathbf{E}_{in}$ and the transmission matrix T decorrelate. Such decorrelation is simulated by adding a random vector (or matrix) whose elements are, on average, K_d times smaller than those of the original one.

Phase-modulation-based TRAP focusing (Fig. 2(a)) measures 4 intensity maps to calculate a single field distribution. During Δt , the target moves (downward) 6 pixels and the decorrelation parameter $K_d = 5$. The phase of $\mathbf{E}_{R,i}$ cycles through $0, \pi/2, \pi, 3\pi/2$ to facilitate the field calculation: $\mathbf{E}_{S,4} = (\mathbf{I}_1 - \mathbf{I}_3) + i(\mathbf{I}_2 - \mathbf{I}_4)$, $\mathbf{E}_{S,8} = (\mathbf{I}_5 - \mathbf{I}_7) + i(\mathbf{I}_6 - \mathbf{I}_8)$. The TRAP phase map is accordingly obtained from $\phi_{\text{TRAP}} = -\arg[\mathbf{E}_{S,8} - \mathbf{E}_{S,4}]$. The time-reversed field distribution on the input plane is calculated by $\mathbf{E}_{in,TR} = T_8^T \exp[i\phi_{\text{TRAP}}]$, where the superscript “T” denotes matrix transposition.

For double-exposure b -TRAP focusing (Fig. 2(b)), the target moves 4 pixels during Δt and $K_d = 5$. The phase of $\mathbf{E}_{R,i}$ is kept constant for $i = 1, 2$ and the two intensity maps are calculated based on

$\mathbf{I}_i = |\mathbf{E}_{S,i} + \mathbf{E}_{R,i}|^2$ ($i = 1, 2$). The TRAP hologram is obtained from $\mathbf{H}_{de} = \mathbf{I}_2 - \mathbf{I}_1$ (where “de” denotes double exposure). The time-reversed field distribution on the input plane is obtained from $\mathbf{E}_{in,TR} = T_2^T (\mathbf{H}_{de} - \overline{\mathbf{H}}_{de})$, where the mean $\overline{\mathbf{H}}_{de}$ is subtracted due to the blockage of the zero-order diffraction.

For single-exposure b -TRAP focusing (Fig. 2(c)), the target moves 1 pixel during Δt and $K_d = 50$. The smaller target moving distance and medium decorrelation are due to a faster system response (in this situation, Δt is determined by the laser repetition rate rather than the camera frame rate). A phase shift of π (rad) is introduced between $\mathbf{E}_{R,i}$ ($i = 1, 2$), giving rise to an automatic interferometric subtraction within a single camera exposure. The summation of the two intensity distributions, calculated from $\mathbf{I}_i = |\mathbf{E}_{S,i} + \mathbf{E}_{R,i}|^2$ ($i = 1, 2$), generates a hologram $\mathbf{H}_{se} = \mathbf{I}_1 + \mathbf{I}_2$ (the subscript “se” denotes single exposure). The time-reversed field distribution on the input plane is obtained from $\mathbf{E}_{in,TR} = T_2^T (\mathbf{H}_{se} - \overline{\mathbf{H}}_{se})$, where the mean $\overline{\mathbf{H}}_{se}$ is subtracted to account for the zero-order blockage.

The time-reversed intensity distribution on the input plane is calculated from $\mathbf{I}_{in,TR} = |\mathbf{E}_{in,TR}|^2$, which is converted from linear indexing back to matrix representation and further interpolated on a 250-by-250 grid for denser presentation.

C. Real-time focusing procedure

The timing diagram used in the real-time, double-exposure b -TRAP focusing experiment is schematically shown in Fig. S2. In focusing cycle i , the trigger of the sCMOS camera encompasses a single laser shot to record image (interferogram) $i+1$. Image i recorded in the preceding cycle is then subtracted from the current image and digitally processed to generate a binary hologram. As soon as the hologram becomes available, it is displayed onto an amplitude-only SLM. Once the hologram is completely displayed and stabilized, shutter 1 is closed (i.e., the sample beam is blocked) and shutter 2 opened (i.e., reading beam is enabled), and the CMOS camera takes a snapshot of the focal light intensity distribution. The shutters are then flipped, and the SLM is reset (i.e., showing a bright image with all pixels set to 1) to finish the i -th focusing cycle. Labeled in the figure are the execution times for the above steps, amounting to a repetition rate of ~ 7 Hz. The step of SLM reset can be eliminated by using an alternative digital optical phase conjugation (DOPC) configuration, in which the hologram displayed on the SLM is not detected by the sCMOS camera [1]. This will increase the repetition rate to ~ 11 Hz.

D. Relationship between flow speed and focal PBR

In the simplest scenario, consider a single particle moving inside a scattering medium, as shown in Fig. S3. The electric field distribution on the target plane is represented by vectors $\mathbf{E}_{in,i}$, with $i = 1, 2$ denoting the target positions. If the medium’s decorrelation is negligible, the output fields on the detection plane can be found by multiplying the transmission matrix T with $\mathbf{E}_{in,i}$, i.e., $\mathbf{E}_{S,i} = T\mathbf{E}_{in,i}$. The power of the reconstructed TRAP field is proportional to the squared norm of the field difference:

$$P_{\text{TRAP}} \propto \|\mathbf{E}_{S,1} - \mathbf{E}_{S,2}\|^2 = \eta_p \|\mathbf{E}_{in,1} - \mathbf{E}_{in,2}\|^2.$$

Fig. S2. Timing diagram for real-time, moving target tracking with double-exposure b-TRAP focusing.

In the above equation, η_p is the power transmission coefficient from the target plane to the detection surface (and the SLM plane). The non-zero elements of the differential field vector $\mathbf{E}_{in,1} - \mathbf{E}_{in,2}$ obey the Gaussian distribution

$$p_{R,I} = (2\pi\sigma^2)^{-1} \exp\left[-(R^2 + I^2)/2\sigma^2\right],$$

(where σ is the standard deviation, and R, I are the real and imaginary parts, respectively), and only reside in the shaded area in Fig. S3(a). It implies that

$$P_{\text{TRAP}} \propto 2\eta_p n_{in} A\sigma^2, \quad (\text{S1})$$

where n_{in} is the number density of the input mode and A is the area of the shaded region. In reaching Eq. (S1), we assume that the speckle size on the target plane is much smaller than the dimension of the particle. Because n_{in} , η_p , and σ^2 are constants, and the TRAP focusing power is linearly related to A , a circular particle with a diameter D and displacement d thus leads to a normalized TRAP focusing power of

$$P_{\text{TRAP, norm}} = 1 - \left\{ 2 \cos^{-1}\left(\frac{d}{D}\right) - \sin\left[2 \cos^{-1}\left(\frac{d}{D}\right)\right] \right\} / \pi. \quad (\text{S2})$$

$P_{\text{TRAP, norm}}$ is plotted against d in Fig. S3b. It can be shown that Eq. (S2) holds when multiple particles with identical shape move together at the same speed.

To verify the above theory, we simulated $\|\mathbf{E}_{s,1} - \mathbf{E}_{s,2}\|^2$ at $d = D/4, D/2, 3D/4$ and D , by assigning 2500-by-1 and 2500-by-2500 circular Gaussian random matrices to $\mathbf{E}_{in}$ and T , respectively. $\mathbf{E}_{in}$ was reshaped to 50-by-50 to form a 2D mesh, with its corresponding elements set to zero according to the position of an absorptive circular object. The modified matrices were unfolded to 2500-by-1 to generate $\mathbf{E}_{in,i}$ ($i = 1, 2$), and $\mathbf{E}_{s,i} = T\mathbf{E}_{in,i}$. The corresponding focal light intensity distribution $\mathbf{I}_{in,TR}$ was measured experimentally with a CMOS camera (Firefly MV, Point Gray, Canada) positioned on the target plane. The displacement d was controlled by adjusting the flow rate at a fixed laser pulse interval. The above results are co-plotted in Fig. 6(b) and (c). It is

Fig. S3. Displacement-encoded TRAP focusing. (a) A particle moves through a background speckle on the target plane. (b) Normalized TRAP power and field correlation ($\text{corr}[\mathbf{E}_{s,1}, \mathbf{E}_{s,2}]$) plotted against normalized displacement d/D .

evident that an intensified differential field $\|\mathbf{E}_{s,1} - \mathbf{E}_{s,2}\|$ leads to stronger focusing. The TRAP power enhancement is due to a decreased correlation of the scattered fields, mathematically expressed as

$$\text{corr}[\mathbf{E}_{s,1}, \mathbf{E}_{s,2}] = \frac{\langle \mathbf{E}_{s,1}, \mathbf{E}_{s,2} \rangle}{\langle \mathbf{E}_{s,1}, \mathbf{E}_{s,1} \rangle},$$

where “corr” and the angle bracket denote correlation coefficient and inner product, respectively. By noting that $\mathbf{E}_{s,1} = \mathbf{E}_{s,a} + \mathbf{E}_{s,b}$ and $\mathbf{E}_{s,2} = \mathbf{E}_{s,b} + \mathbf{E}_{s,c}$ (a, b, c correspond to the regions labeled in Fig. S3(a)) and $\langle \mathbf{E}_{s,x}, \mathbf{E}_{s,y} \rangle = 0$ if $x \neq y$ ($x, y = a, b, c$), it is evident that

$$P_{\text{TRAP, norm}} = 1 - \text{corr}[\mathbf{E}_{s,1}, \mathbf{E}_{s,2}].$$

$\text{corr}[\mathbf{E}_{s,1}, \mathbf{E}_{s,2}]$ is plotted in Fig. S3(b) and Fig. 6(d) as dashed lines; the discrete stars in Fig. 6(d) are numerical solutions.

2. SUPPLEMENTARY DISCUSSION

A. Degrees of freedom

As discussed in the main text, for any practical implementation of TRAP focusing, it is essential to measure the scattered field with a single laser shot. This can be achieved by digital off-axis holography (as employed in TRACK focusing [2]) or on-axis amplitude holography (as in b-TRAP focusing).

In the off-axis scheme, the sample beam is combined with an oblique reference beam on the SLM surface, as shown in Fig. S4(a). The total field on the SLM is [3]

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_s + \mathbf{E}_R \exp(-jk_\theta x). \quad (\text{S3})$$

In the above equation, $\mathbf{E}_s$ and $\mathbf{E}_R$ are the complex amplitudes of the sample and reference beams, and k_θ is the wavevector of the reference beam along x . The field distribution in Eq. (S3) gives rise to an intensity distribution of $\mathbf{I}_{\text{tot}} = \mathbf{E}_{\text{tot}} \mathbf{E}_{\text{tot}}^*$, whose Fourier transform is:

$$\begin{aligned} \tilde{\mathbf{I}}_{\text{tot}} &= \tilde{\mathbf{E}}_s \star \tilde{\mathbf{E}}_s + |\tilde{\mathbf{E}}_R|^2 \\ &+ \tilde{\mathbf{E}}_R^* \tilde{\mathbf{E}}_s(k_x, k_y) \otimes \delta(k_x - k_\theta, k_y) \\ &+ \tilde{\mathbf{E}}_R \tilde{\mathbf{E}}_s^*(-k_x, -k_y) \otimes \delta(k_x + k_\theta, k_y) \end{aligned} \quad (\text{S4})$$

Fig. S4. Two hologram recording schemes and their Fourier spectra. (a) Off-axis configuration, and (b) Fourier spectrum of the recorded field in (a). (c) On-axis configuration, and (d) Fourier spectrum of the recorded field in (c).

where $\star$ and $\otimes$ denote 2D autocorrelation and convolution, respectively, and $\delta(\bullet)$ is the Dirac delta function. Variables with a tilde are complex amplitudes in the Fourier domain. The Fourier spectrum expressed by Eq. (S4) is illustrated in Fig. S4(b). The big circle centered at the origin shows the position of $\tilde{\mathbf{E}}_s \star \tilde{\mathbf{E}}_s$, the adjacent smaller circle represents the term with $\delta(k_x - k_\theta, k_y)$, which carries the signal. Because of autocorrelation, the radius of the big circle is twice the signal bandwidth B_s (equivalent to the radius of the small circle). Accordingly, to avoid aliasing, it is required that $k_\theta > 3B_s$. Moreover, the intensity pattern is sampled by a pixelated detector at a spatial frequency k_s , producing shifted copies in the Fourier spectrum, as shown by the dashed circles. As illustrated in Fig. S4(b), to further avoid aliasing due to sampling, the following condition must be satisfied:

$$k_s > 8B_s. \quad (\text{S5})$$

In contrast, the situations for *b*-TRAP focusing are shown in Fig. S4(c), (d), where it is unnecessary to explicitly measure $\mathbf{E}_s$ and aliasing from $\tilde{\mathbf{E}}_s \star \tilde{\mathbf{E}}_s$ can be tolerated and treated as a linearly superimposed background. We thus only consider $\mathbf{E}_s$ in the spectral diagram (Fig. S4(d)) and find:

$$k_s > 2B_s. \quad (\text{S6})$$

Comparing Eqs (S5) and (S6), *b*-TRAP focusing has 4 times more spatial bandwidth than that of the off-axis approach (at a fixed k_s), leading to a DOF increase of 16 times.

B. Diffraction efficiency

The diffraction efficiency of the SLM is defined as the ratio between the optical power converted to the desired wavefront and the total input power. Here we compare the efficiencies of digital amplitude- and phase-only SLMs. For both types of devices, the total efficiency can be written as

$$\eta_{\text{tot}} = \eta_R \eta_{\text{PL}} \eta_o.$$

In the above equation, η_R and η_{PL} denote the efficiencies associated with reflection and the diffraction due to pixelation. Both types of devices have similar η_R and η_{PL} . η_o is an original efficiency to be measured, which can be very different for amplitude- and phase-based SLMs.

For amplitude-only SLMs, the hologram can be treated as a superposition of gratings with various spatial frequencies. By assuming that the intensity of the speckle pattern be evenly distributed over the entire bandwidth B_s , η_o can be found by averaging the diffraction efficiencies at different spatial frequencies. In principle, an amplitude grating in the form of

$$t_A(x, y) = \frac{1}{2} [1 + \cos(k_x x)] \text{rect}\left(\frac{x}{2w}\right) \text{rect}\left(\frac{y}{2h}\right) \quad (\text{S7})$$

diffracts into the +1 order $1/16 = 6.25\%$ of the total input power [4], which was measured to be $\sim 3.8\%$, slightly lower than the theoretical value. The same system was used to measure the efficiency of a phase-only modulator (Pluto, Holoeye, Germany), the average efficiency is found to be $\sim 34.6\%$. Given the above results, we estimated that the diffraction efficiency of the amplitude-only approach is $(34.6\%)/(3.8\%) \approx 10$ times less than that of the phase-only approach.

If hologram binarization and zero-order blockage are considered simultaneously, the above result stays unchanged. With hologram binarization, the peak-to-background ratio obtained using amplitude-only SLM for binary-phase modulation is $\sim 40\%$ of that attainable using a phase-only approach (see the next Section for details), while the background intensity of amplitude-only modulation is $\sim 25\%$ of that in the phase-only modulation (in binary amplitude modulation, half of the pixels are closed, reducing the background by half; the zero-order block further decreases the background by half). The above analysis implies that the peak focus intensity in binary amplitude-only modulation is 10% of that in phase-only modulation.

C. Peak-to-background ratio (PBR)

Denoting E_k as the electric field amplitude of the k -th mode on the focusing plane, the expected value of its intensity can be written as [5]:

$$\langle I_k \rangle = \langle E_k E_k^* \rangle = N \langle |t|^2 \rangle + N(N-1) \langle |t|^2 \rangle \langle \cos \varphi \rangle^2 \quad (\text{S7})$$

In the above equation, N is the total number of mode, $\langle \bullet \rangle$ denotes ensemble average, and t is the element-wise transmission coefficient which, for partial phase conjugation, obeys the complex circular Gaussian distribution [5]. In Eq. (S7), φ is the angle of the transmission coefficient which has different probability density for in-focus and background modes. For a background mode, φ is evenly distributed within $[-\pi, \pi]$, nullifying the second term on the right-hand-side (RHS) of Eq. (S7). Since $|t|^2$ obeys the exponential distribution, $\langle |t|^2 \rangle = 2\sigma^2$, where σ^2 is the variance of $\text{Re}[t]$. Thus, the average background intensity becomes $2N\sigma^2$, as expected from incoherent addition. For the in-focus (target) mode, binary phase modulation restricts the range of φ to $[-\pi/2, \pi/2]$ by flipping the sign of any mode whose phase is in the complementary range. In this case, $\langle \cos \varphi \rangle^2 = 4/\pi^2$. Considering that $|t|$ obeys the Rayleigh distribution, $\langle |t|^2 \rangle = \pi\sigma^2/2$, we find that the expected focus intensity is $2N\sigma^2 + 2N(N-1)\sigma^2/\pi$. As a result, we conclude that the ideal PBR for binary phase modulation is

$$\text{PBR}_{\text{ideal}} = 1 + \frac{N-1}{\pi}, \quad (\text{S8})$$

which doubles the PBR of binary amplitude modulation [5].

The binary hologram of *b*-TRAP focusing is calculated from intensity maps:

$$I_{DE} = |E_{S,2}|^2 - |E_{S,1}|^2 + 2|\Delta E_S|E_R \cos \varphi \quad (\text{S9})$$

for the double-exposure *b*-TRAP focusing and

$$I_{SE} = |E_{S,1}|^2 + |E_{S,2}|^2 - \langle |E_{S,1}|^2 + |E_{S,2}|^2 \rangle + 2|\Delta E_S|E_R \cos \varphi \quad (\text{S10})$$

for the single-exposure *b*-TRAP focusing. In both Eqs. (S9) and (S10), E_R is a scalar constant representing the amplitude of a planar reference beam whose phase is set to zero. $E_{S,1}$ and $E_{S,2}$ denote the sample beam's amplitudes before and after perturbation of the target, and $\Delta E_S = E_{S,2} - E_{S,1}$. In reaching Eq. (S10), a constant mean $\langle |E_{S,1}|^2 + |E_{S,2}|^2 \rangle + 2E_R$ is removed as the binarization threshold. Ideally, a binary hologram is formed by setting a pixel value to +1 if $2|\Delta E_S|E_R \cos \varphi \geq 0$, or to -1 if $2|\Delta E_S|E_R \cos \varphi < 0$. This ensures that the phases of all modes are distributed within $[-\pi/2, \pi/2]$ at the focus, yielding a PBR defined by Eq. (S8).

However, the existence of the other terms (all terms except the cosine terms) on the RHS of Eqs. (S9) and (S10) will induce binarization error and subsequently results in sign flip (+1 to -1 and vice versa). Consequently, the phase distribution at the focal point is no longer evenly distributed within $[-\pi/2, \pi/2]$, instead, the probability density function (pdf) becomes

$$p(\varphi) = \begin{cases} \frac{1}{\pi(1+v)}, & \varphi \in [-\pi, -\pi/2) \cup [\pi/2, \pi) \\ \frac{v}{\pi(1+v)}, & \varphi \in [-\pi/2, \pi/2) \end{cases} \quad (\text{S11})$$

where v is a ratio defined by $N/(N-N')$ with N' being the number of sign-flipped mode. Using Eq. (S11), we can re-evaluate the PBR to get:

$$PBR = 1 + \frac{N-1}{\pi} \left(1 - \frac{2N'}{N} \right)^2. \quad (\text{S12})$$

To evaluate the PBR in Eq. (S12), the value of N' is needed. We will discuss how to calculate N' for double- and single-exposure *b*-TRAP focusing below.

For double exposure *b*-TRAP focusing, denoting $\alpha = |E_{S,2}|^2 - |E_{S,1}|^2$, $\beta = 2|\Delta E_S|E_R \cos \varphi$, the total probability of error can be expressed as:

$$P_e = P(\alpha\beta < 0)P(|\alpha| > |\beta|) = \frac{1}{2}P_{\alpha\beta}, \quad (\text{S13})$$

where we use $P_{\alpha\beta}$ to represent $P(|\alpha| > |\beta|)$ for simplicity. By inserting $\Delta E_S = E_{S,2} - E_{S,1}$ into the expression for α and assuming $|E_{S,1}| \approx |E_{S,2}| \gg |\Delta E_S|$, we can easily prove that

$$\alpha \approx 2|E_{S,1}||\Delta E_S| \cos(\phi), \quad (\text{S14})$$

where ϕ is the phase difference between $E_{S,1}$ and ΔE_S . According to Eq. (S14), it is straightforward to write

$$P_{\alpha\beta} = P(X + Y > 0), \quad (\text{S15})$$

in which

$$X = |E_{S,1}||\cos \phi|, \quad (\text{S16})$$

$$Y = -E_R|\cos \phi|. \quad (\text{S17})$$

Here, the sample beam amplitude $E_{S,1}$ obeys the complex circular Gaussian distribution, $\text{Var}[\text{Re}(E_{S,1})] = \text{Var}[\text{Im}(E_{S,1})] = \sigma_s^2$, ϕ and φ are evenly distributed between $[-\pi, \pi]$. Consequently, it can be shown that the pdf of X is:

$$p_X(x) = \sqrt{\frac{2}{\pi\sigma_s^2}} e^{-\frac{x^2}{2\sigma_s^2}}, x \in [0, +\infty), \quad (\text{S18})$$

which is a half-normal distribution, and the pdf of Y is:

$$p_Y(x) = \frac{2}{\pi E_R \sqrt{1 - (x/E_R)^2}}, x \in [-E_R, 0]. \quad (\text{S19})$$

According to Eqs. (S15), (S18), (S19), $P_{\alpha\beta}$ can be calculated from

$$P_{\alpha\beta} = \int_0^\infty dz [p_X \otimes p_Y](z), \quad (\text{S20})$$

where the symbol “ $\otimes$ ” denotes convolution. After mathematical manipulations, the probability $P_{\alpha\beta}$ can be further expressed as

$$P_{\alpha\beta} = \frac{2}{\pi} \int_{-1}^0 dx \frac{\text{erfc}(-x/(\sqrt{2}r))}{\sqrt{1-x^2}}. \quad (\text{S21})$$

In Eq. (S21), r is the ratio between the sample and reference intensities: $r = I_s/I_R = 2\sigma_s^2/E_R^2$, and $\text{erfc}(\cdot)$ is the complimentary error function. From Eq. (S21), we conclude that the error probability of double-exposure *b*-TRAP focusing is solely dependent on the beam intensity ratio r .

For single-exposure *b*-TRAP focusing, following the same procedure and taking advantage of $|E_{S,1}|^2 + |E_{S,2}|^2 \approx 2|E_{S,1}|^2$, we get

$$X = ||E_{S,1}|^2 - \langle |E_{S,1}|^2 \rangle|, \quad (\text{S22})$$

$$Y = -E_R|\Delta E_S||\cos \phi|. \quad (\text{S23})$$

$$p_X(x) = \begin{cases} \frac{1}{2\sigma_s^2} \left[\exp\left(-\frac{2\sigma_s^2-x}{2\sigma_s^2}\right) + \exp\left(-\frac{2\sigma_s^2+x}{2\sigma_s^2}\right) \right], & x \in [0, 2\sigma_s^2) \\ \frac{1}{2\sigma_s^2} \exp\left(-\frac{2\sigma_s^2+x}{2\sigma_s^2}\right), & x \in [2\sigma_s^2, \infty) \end{cases} \quad (\text{S24})$$

$$p_Y(x) = \sqrt{\frac{2}{\pi E_R^2 \sigma_\Delta^2}} e^{-\frac{x^2}{2E_R^2 \sigma_\Delta^2}}, x \in (-\infty, 0]. \quad (\text{S25})$$

In the above equations, σ_Δ denotes the standard deviation of $\text{Re}[\Delta E_S]$. The probability $P_{\alpha\beta}$ can be calculated by Eq. (S20) and is expressed after mathematical manipulations as:

$$P_{\alpha\beta} = \frac{\xi + \varsigma}{\sqrt{2\pi\sigma_s^2\sigma_\Delta^2} E_R}. \quad (\text{S26})$$

In Eq. (S26), the parameters ξ and ς are defined as

$$\xi = \int_0^{2\sigma_s^2} dz \int_z^{2\sigma_s^2} dx \exp \left\{ - \left[\frac{2\sigma_s^2 - x}{2\sigma_s^2} + \frac{(z-x)^2}{2E_R^2\sigma_\Delta^2} \right] \right\}, \quad (\text{S27})$$

and

$$\varsigma = \int_0^\infty dz \int_z^\infty dx \exp \left\{ - \left[\frac{2\sigma_s^2 + x}{2\sigma_s^2} + \frac{(z-x)^2}{2E_R^2\sigma_\Delta^2} \right] \right\}. \quad (\text{S28})$$

Eqs. (S27) and (S28) can be integrated to yield closed-form expressions, despite being very complicated. One can instead numerically evaluate Eqs. (S27) and (S28) to calculate $P_{\alpha\beta}$. From Eqs. (S26)–(S28), we conclude that the error probability for single-exposure *b*-TRAP focusing not only depends on the beam intensity ratio r , but also on the ratio between the sample and differential sample beam (e.g., ΔE_s) intensities, represented by $2\sigma_s^2$ and $2\sigma_\Delta^2$, respectively.

By inserting Eq. (S13) into Eq. (S12), the expression for the PBR of *b*-TRAP focusing can be readily found to be

$$PBR = 1 + \frac{N-1}{\pi} (1 - P_{\alpha\beta})^2, \quad (\text{S29})$$

in which the probability $P_{\alpha\beta}$ can be evaluated by Eq. (S21) for double-exposure, or by Eqs. (S26)–(S28) for single-exposure focusing schemes.

It should be noted that the term $(1 - P_{\alpha\beta})^2$ in Eq. (S29) overestimates the PBR reduction when $P_{\alpha\beta}$ is small. This is because in deriving Eq. (S12) (and subsequently Eq. (S29)), we assumed that the modulus $|t|$ and angle φ are independent. However, when $P_{\alpha\beta}$ is small, the sign-flipped modes (with phase distributed outside $[-\pi/2, \pi/2]$) tend to have smaller modulus, because these modes are more prone to thresholding error caused by contamination from the squared terms in Eqs. (S9) and (S10). A rigorous account of this problem is very complicated and is beyond our interest; below we use a simple and heuristic method to yield an improved solution.

From the above analysis, to calculate the focal intensity when $P_{\alpha\beta}$ approaches zero, neglecting the contribution of the N' sign-flipped modes, rather than forcing them to destructively interfere as in Eq. (S29), is a more accurate approximation. Following this concept, by substituting $N-N'$ for N in the second term on the RHS of Eq. (S7), we find the formula for PBR when $P_{\alpha\beta}$ approaches zero:

$$PBR \approx 1 + \frac{N}{\pi} \left(1 - \frac{1}{2} P_{\alpha\beta} \right)^2, P_{\alpha\beta} \ll 1. \quad (\text{S30})$$

Comparing Eq. (S29) and Eq. (S30), it is found that $P_{\alpha\beta}$ is scaled by a factor f : $f = 1/2$ when $P_{\alpha\beta} \rightarrow 0$, $f = 1$ when $P_{\alpha\beta} \rightarrow 1$. In the simplest case, f can be fitted using a linear function: $f = 1/2(P_{\alpha\beta} + 1)$, which leads to the improved formula for the PBR:

$$PBR \approx 1 + \frac{N}{\pi} \left(1 - \frac{P_{\alpha\beta}(P_{\alpha\beta} + 1)}{2} \right)^2. \quad (\text{S31})$$

Numerical simulations were performed to verify the above theory, and the results are shown in Fig. 3 in the main text. We set $N = 100$ in the simulation, and the size of the transmission matrix (TM) was $N \times N$. All elements of the TM were independently and identically distributed obeying the complex circular Gaussian distribution. The input was assumed to be a plane wave by setting all N input elements to unity, and TRAP perturbation was absorptive where one input element was set to zero. This resulted in $\sigma_s^2 = N\sigma_\Delta^2$. The beam intensity ratio $2\sigma_s^2/E_R^2$ was increased from 10^{-5} to 10^2 in 40 steps by changing E_R . At each intensity ratio, 900 simulations were performed for double- and single-exposure *b*-TRAP focusing, then focal PBRs were calculated and averaged (shown in discrete dots for double-exposure, and triangles for single-exposure schemes in Fig. 3). The corresponding theoretical curves were co-plotted implementing Eq. (S31), where $P_{\alpha\beta}$ was evaluated by Eq. (S21) for double-exposure, and by Eqs. (S26)–(S28) for single-exposure focusing schemes. From the plot, we concluded that the agreement between the model and simulation was good, despite small discrepancies at the turning points due to imperfect selection of the scaling factor f .

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