SUSY-inspired one-dimensional transformation optics: supplementary material

MOHAMMAD-ALI MIRI,* MATTHIAS HEINRICH, DEMETRIOS N. CHRISTODOULIDES

CREOL, College of Optics and Photonics, University of Central Florida, Orlando, Florida 32816–2700, USA

*Corresponding author: miri@knights.ucf.edu

Published 6 August 2014

This document provides supplementary information to “SUSY-inspired one-dimensional transformation optics,” <http://dx.doi.org/10.1364/optica.1.000089>. The first section includes a detailed derivation of the relations between the reflection/transmission coefficients in superpartner and iso-spectral structures. In Section 2, similar relations are derived for structures with dissimilar backgrounds. The inverse SUSY transformation which allows for adding a bound state to given structure is introduced in Section 3. Finally Section 4 is devoted to the formalism of supersymmetry for TM polarization that is not discussed in the main text. © 2014 Optical Society of America

<http://dx.doi.org/10.1364/optica.1.000089.s001>

S1. Reflection/transmission coefficients of superpartner structures

In order to relate the scattering characteristics of a superpartner to that of its associated fundamental structure, let us first consider an incident plane wave described by $\exp(iXn_0 \cos \theta + iYn_0 \sin \theta)$ impinging on both structures from the left side. The reflected and transmitted waves in the fundamental system are then described by $r \exp(-iXn_0 \cos \theta + iYn_0 \sin \theta)$ and $t \exp(iXn_0 \cos \theta + iYn_0 \sin \theta)$, respectively. Accordingly, the corresponding waves in the superpartner geometry are given by $r_p \exp(-iXn_0 \cos \theta + iYn_0 \sin \theta)$ and $t_p \exp(iXn_0 \cos \theta + iYn_0 \sin \theta)$. By applying the relation $\psi_p \propto (\partial_X + W)\psi$ (that holds between the wave functions of any state pairs in the two structures) to these radiation states, one finds

$$(e^{+iXn_0 \cos \theta} + r_p e^{-iXn_0 \cos \theta}) \propto ((+in_0 \cos \theta + W_-)e^{+iXn_0 \cos \theta} + (-in_0 \cos \theta + W_-)r e^{-iXn_0 \cos \theta}) \quad (S1)$$

for $X \rightarrow -\infty$, and

$$t_p e^{+iXn_0 \cos \theta} \propto (+in_0 \cos \theta + W_+) t e^{+iXn_0 \cos \theta} \quad (S2)$$

for $X \rightarrow +\infty$. In these two equations $W_{\pm}$ denotes the limit of W at $X \rightarrow \pm\infty$ respectively. Taking into account that both equations should have the same proportionality constant,

one can show that the reflection and transmission coefficients are related via:

$$r_p = \frac{W_- - in_0 \cos \theta}{W_- + in_0 \cos \theta} r \quad (S3.a)$$

$$t_p = \frac{W_+ + in_0 \cos \theta}{W_- + in_0 \cos \theta} t \quad (S3.b)$$

Obviously, the reflectivity $R = |r|^2 = |r_p|^2$ as well as the transmittivity $T = 1 - R = |t|^2 = |t_p|^2$ of the superpartner structures is identical. The specific relations for the complex coefficients given in Table 1 in turn follow from $W_+ = -W_-$ in the case of unbroken supersymmetry and $W_+ = W_-$ in the broken supersymmetry regime.

To derive similar expressions for the family of iso-phase structures $\epsilon_f(X; C)$, we label reflected and transmitted waves as $r_f \exp(-iXn_0 \cos \theta + iYn_0 \sin \theta)$ and $t_f \exp(iXn_0 \cos \theta + iYn_0 \sin \theta)$ respectively. One can then relate r_f and t_f to r and t via the transformation $\psi_f \propto (\partial_X - W_f)(\partial_X + W)\psi$. On the other hand, one can write the family of associated superpotentials as

$$W_f(X; C) = W + \partial_X \ln \left(C + \int_{-\infty}^X \psi_0^2(X') dX' \right) \quad (S4)$$

Consequently, W_f and W have the same asymptotic behavior, i.e., $W_{f,\pm} = W_{\pm}$. In the far field $X \rightarrow \pm\infty$, the

transformation simplifies to $\psi_f \propto (\partial_X - W)(\partial_X + W)\psi$. Note that $(\partial_X - W)(\partial_X + W) = (\partial_{XX} + W' - W^2) = H - \alpha$, and therefore $\psi_f \propto (H - \alpha)\psi = (\Omega - \alpha)\psi$. Since the scattered waves depend on Y according to $\exp(iYn_0 \sin \theta)$, the corresponding eigenvalue in the Helmholtz equation is given by $\Omega = n_0^2 \sin^2 \theta$. Hence, $\psi_f \propto (n_0^2 \sin^2 \theta - \alpha)\psi$, and therefore:

$$(e^{+iXn_0 \cos \theta} + r_f e^{-iXn_0 \cos \theta}) \propto (n_0^2 \sin^2 \theta - \alpha)(e^{+iXn_0 \cos \theta} + r_f e^{-iXn_0 \cos \theta}) \quad (S5)$$

for $X \rightarrow -\infty$, and

$$t_f e^{+iXn_0 \cos \theta} \propto (n_0^2 \sin^2 \theta - \alpha) t e^{+iXn_0 \cos \theta} \quad (S6)$$

for $X \rightarrow +\infty$. Given that both equations should be normalized with respect to the same constant, it directly follows that in the iso-phase scenario

$$r_f = r, \quad (S7.a)$$

$$t_f = t. \quad (S7.b)$$

S2. Configurations with dissimilar backgrounds

Even though the formalism of the main text was developed by assuming that the background medium is the same ($n_- = n_+ = n_0$), superpartners obeying Eqs. (2,3) can be generated even if $n_- \neq n_+$ (here $n_{\pm} = \sqrt{\epsilon(X \rightarrow \pm\infty)}$ and). Figure S1(a) shows an example of a step-like distribution in the relative permittivity; the corresponding superpartner ϵ_p is depicted in Fig. S1(b). The superpotential mediating between them no longer has the same absolute value at $X \rightarrow \pm\infty$, but rather is shifted by a certain offset related to the height of the potential step. Neither of the two partner structures does support any guided modes, therefore supersymmetry is necessarily broken. As depicted in Fig. S3.1(c), numerical results obtained by differential transfer matrix method [S2] show that the angle-dependent power reflectivities $R = |r|^2 = |r_p|^2$ and transmittivities $T = 1 - R = |t|^2 \cdot (n_+/n_-)(\cos \theta_+ / \cos \theta_-) = |t_p|^2 \cdot (n_+/n_-)(\cos \theta_+ / \cos \theta_-)$ remain identical for both systems (Fig. S3.1(c)), while the phases do not (Fig. S1(d)). Analytical relations of the scattering coefficients can be found by following analogous steps to those in the previous section. The results are

$$r_p = \frac{W_- - i n_- \cos \theta_-}{W_- + i n_- \cos \theta_-} r \quad (S8.a)$$

$$t_p = \frac{W_+ + i n_+ \cos \theta_+}{W_- + i n_- \cos \theta_-} t, \quad (S8.b)$$

where θ_- and θ_+ represent the incident and transmitted wave angles related through Snell's law $n_- \sin \theta_- = n_+ \sin \theta_+$. It follows that $|r|^2 = |r_p|^2$ and $|t|^2 = |t_p|^2$.

S3. Inverse SUSY transformation

In the unbroken symmetry regime, the conventional SUSY transformation may remove a mode from a given fundamental structure ϵ . In doing so, the total area of the relative permittivity is reduced (see Fig. 2(a,b)). This can be shown easily by noting that in the unbroken supersymmetry regime the two superpartners are related via

$$\epsilon_p - \epsilon = -2W'. \quad (S9)$$

After integrating both sides of this equation we get

$$\int_{-\infty}^{+\infty} \epsilon_p(X) dX - \int_{-\infty}^{+\infty} \epsilon(X) dX = -2(W_+ - W_-). \quad (S10)$$

Since in the unbroken supersymmetry regime $W_- \neq W_+$, the SUSY transformation cannot preserve the total area of the relative permittivity distribution.

On the other hand, one can utilize an inverse SUSY transformation and add a bound state to a given structure ϵ , and in doing so elevate the total area of a given permittivity distribution. We factorize the fundamental Hamiltonian as $H = AB + \alpha$ and define the partner Hamiltonian as $H_{\text{elev}} = BA + \alpha$. Consequently, the two superpartner permittivity distributions can be written as:

$$\epsilon(X) = -W'_{\text{elev}} - W_{\text{elev}}^2 + \alpha \quad (S11.a)$$

$$\epsilon_{\text{elev}}(X) = +W'_{\text{elev}} - W_{\text{elev}}^2 + \alpha \quad (S11.b)$$

Equation (S11.a) can be solved numerically to obtain the superpotential W_{elev} , and from that the partner structure ϵ_{elev} can be constructed through Eq. (S11.b). As expected in this case the partner structure ϵ_{elev} exhibits all the guided mode eigenvalue spectrum of the fundamental structure ϵ , as well as an additional guided mode, which takes the place of its previous ground state. As it turns out, the eigenvalue of this additional state is given by the factorization parameter α . Note that any value $\alpha > \Omega_0$ can be chosen, where Ω_0 represents the ground state eigenvalue of the fundamental structure. Figure S2 illustrates the inverse SUSY transformation for the specific example of the fundamental structure discussed in Fig. 2 of the main text.

S4. SUSY formalism for the TM polarization

Under TM polarization conditions the magnetic field component satisfies the equation:

$$(\partial_{xx} + \partial_{yy} - \partial_x(\ln \epsilon) \partial_x + k_0^2 \epsilon(x)) H_z(x, y) = 0 \quad (S12)$$

In this case again by considering stationary solutions of the form $H_z = \psi(x) e^{i\beta z}$ and after using normalized coordinates $X = k_0 x, Y = k_0 y$ and assuming $\Omega = \beta^2 / k_0^2$ we reach at:

$$(\partial_{XX} - \partial_X(\ln \epsilon) \partial_X + \epsilon) \psi = \Omega \psi \quad (S13)$$

As is, this is not a Schrödinger-like equation, and hence the factorization technique cannot be directly applied. On the other hand, by using the transformation $\psi(X) = \sqrt{\epsilon} f(X)$, this equation can be converted to the desired form:

$$(\partial_{XX} + V_{\text{eff}})f = \Omega f \quad (\text{S14})$$

where V_{eff} is an effective potential that can be expressed in terms of the relative permittivity ϵ as $V_{\text{eff}} = \epsilon - \frac{3}{4}(\epsilon'/\epsilon)^2 + \frac{1}{2}(\epsilon''/\epsilon)$. This relation can also be rewritten as:

$$V_{\text{eff}} = \epsilon + \left(\frac{\epsilon'}{2\epsilon}\right)' - \left(\frac{\epsilon'}{2\epsilon}\right)^2 \quad (\text{S15})$$

Following the SUSY formalism, the two superpartner effective potentials can now be written in terms of the superpotential W via

$$V_{\text{eff}}(X) = +W' - W^2 + \alpha \quad (\text{S16.a})$$

$$V_{\text{eff},p}(X) = -W' - W^2 + \alpha. \quad (\text{S16.b})$$

One can then reconstruct the relative permittivity of the partner structure ϵ_p from its corresponding effective potential $V_{\text{eff},p}$ by numerically solving the nonlinear equation

$$V_{\text{eff},p} = \epsilon_p + \left(\frac{\epsilon'_p}{2\epsilon_p}\right)' - \left(\frac{\epsilon'_p}{2\epsilon_p}\right)^2. \quad (\text{S17})$$

Of importance will be to show that these transformations preserve the reflection/transmission properties of SUSY partners in the TM case. For this reason note that for effective potentials described in Eqs. (S16) we have the intervening relations $f_p \propto (\partial_X + W)f$ and $f \propto (\partial_X - W)f_p$ therefore:

$$\psi_p/\sqrt{\epsilon_p} \propto (\partial_X + W) \psi/\sqrt{\epsilon} \quad (\text{S18.a})$$

$$\psi/\sqrt{\epsilon} \propto (\partial_X - W) \psi_p/\sqrt{\epsilon_p} \quad (\text{S18.b})$$

Obviously presence of the relative permittivities in the denominator makes these last equations different than those of the TE case. However for calculating the reflection/transmission coefficients (see Supplementary section I) only the asymptotic behavior of these equations at $X \rightarrow \pm\infty$ are of our interest. On the other hand, according to Eqs. (S15-S17), the SUSY partners have the same asymptotic behaviors. Therefore one can use the exact same analysis of the Supplementary Section S1 based on Eqs. (S18) to show that the reflection/transmission coefficients of SUSY partners constructed for the TM polarization are related via equations given in Table 1 of the main text.

References

- S1. F. Cooper, A. Khare, and U. Sukhatme, "Supersymmetry and quantum mechanics," Phys. Rep. **251**, 267-385 (1995).
- S2. P. Yeh, "Optical Waves in Layered Media" (Wiley, New York, 1988).

Supplementary figures

Fig. S1 (a) Step-like potential $\epsilon(X) = 2 + \tanh(X/0.3)$ and (b) its superpartner $\epsilon_p(X)$. (c) Despite the potential offset, the intensities of reflected and transmitted waves are identical for both structures, (d) but the phases remain different as expected for the broken SUSY regime.

Fig. S2 Inverse SUSY transformation. (a) Step-index-like fundamental structure $\epsilon(X)$ supporting $N = 3$ guided modes. (b) Its inverse superpartner $\epsilon_{\text{elev}}(X)$, supporting a phase matched mode associated with each of the fundamental structure's states as well as an additional with the desired eigenvalue $\alpha = 2.5$. (c) Superpotential relating the two structures.