Supporting Information: Classical and Quantum Shortcuts to Adiabaticity in a Tilted Piston

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Because the classical function $\eta(q,p) = \mathrm{sign}(p)$ is non-analytic, the matrix representation of its quantal counterpart $\hat{\eta}$ cannot be obtained by a procedure like the one used in Sec. 5. Here we instead construct the matrix representation of η by equating its classical and quantum auto-correlation functions.

Consider a quantum particle in a box with a flat base ($s = 0$) and hard walls at $q = 0$ and $q = L$, described by the Hamiltonian $\hat{H}' = \hat{p}^2/2m + \Theta(\hat{q}; 0, L)$. Following Ref. [1], we write the quantum auto-correlation function of $\hat{\eta}$, for the eigenstate $|\alpha\rangle$, as

$$\begin{aligned} C_\alpha(\tau) &= \langle\alpha|\hat{\eta}\exp\left(\frac{i\hat{H}'\tau}{\hbar}\right)\hat{\eta}\exp\left(-\frac{i\hat{H}'\tau}{\hbar}\right)|\alpha\rangle \\ &= \sum_\beta |\tilde{\eta}_{\alpha\beta}|^2 \exp\left[\frac{i(E_\beta - E_\alpha)\tau}{\hbar}\right], \end{aligned} \tag{S1}$$

where $\tilde{\eta}_{\alpha\beta} = \langle\alpha|\hat{\eta}|\beta\rangle$, and E_α is the energy corresponding to the eigenstate $|\alpha\rangle$. The Fourier transform of the auto-correlation function is

$$\mathbb{C}_\alpha(\omega) = \sum_\beta |\tilde{\eta}_{\alpha\beta}|^2 \delta(\omega - \omega_{\alpha\beta}), \tag{S2}$$

where

$$\omega_{\alpha\beta} \equiv \frac{E_\beta - E_\alpha}{\hbar}. \tag{S3}$$

For a classical particle evolving under the equivalent Hamiltonian, $\eta = \mathrm{sign}(p)$ is a square wave pulse with unit amplitude over a time period around the energy shell. The functions $\eta_0^E(t)$ and $\eta_\tau^E(t)$ describe the dependence of η on time for a particle of energy E that starts from $L = 0$ at times $t = 0$ and $t = -\tau$ respectively, as depicted in Fig.1. The classical auto-correlation function, $C_E(\tau) = (1/T)\int_0^T dt\, \eta_0^E(t)\eta_\tau^E(t)$, is a triangular wave given by

$$C_E(\tau) = \begin{cases} \frac{T-4\tau}{T} &, \quad 0 \le \tau \le \frac{T}{2} \\ \frac{4\tau-3T}{T} &, \quad \frac{T}{2} \le \tau \le T \end{cases}, \tag{S4}$$

shown in Fig.1. The Fourier transform of $C_E(\tau)$ is

$$\mathbb{C}_E(\omega) = \sum_{\text{odd } \gamma=-\infty}^{\infty} \frac{4}{\pi^2\gamma^2}\delta(\omega - \omega_\gamma), \tag{S5}$$

where

$$\omega_\gamma = \frac{2\pi\gamma}{T}. \tag{S6}$$

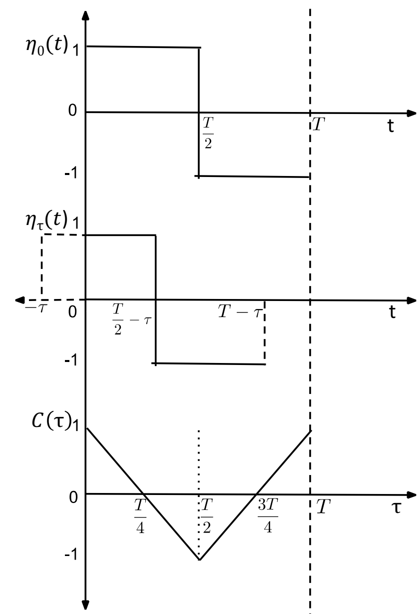

FIG. 1: The function $\eta_0(t)$ plotted over one time period of oscillation is a square wave (top figure). The function $\eta_\tau(t)$ is obtained by shifting this square wave leftward by an amount τ (middle figure). The autocorrelation function $C(\tau)$ is the product of these square wave pulses, integrated over one period, yielding a triangular wave (bottom figure).

The correspondence principle suggests that the functions $\mathbb{C}_\alpha(\omega)$ and $\mathbb{C}_E(\omega)$ ought to be equal, in the semiclassical limit, when $E_\alpha = E$. To compare these functions, we first note that for one dimensional systems, the classical action $J(E) = \oint_E p \cdot dq$ satisfies

$$\frac{dJ}{dE} = T. \tag{S7}$$

For neighboring energy levels $|\alpha\rangle$ and $|\alpha+1\rangle$, the energy spacing is

$$dE = E_{\alpha+1} - E_\alpha = \hbar\omega_{\alpha,\alpha+1}, \tag{S8}$$

and the action spacing is given by the Bohr-Sommerfeld quantization condition:

$$dJ = 2\pi\hbar. \tag{S9}$$

From Eqs.(S7) - (S9) we obtain $\omega_{\alpha,\alpha+1} = 2\pi/T$, which generalizes to

$$\omega_{\alpha\beta} = \frac{2\pi(\beta-\alpha)}{T}, \tag{S10}$$

provided α and β are not too far apart.

Comparing Eqs.(S6) and (S10) we confirm that the delta-functions in Eqs.(S2) and (S5) appear at the same frequencies, and by equating the coefficients of these delta-functions we obtain

$$|\tilde{\eta}_{\alpha\beta}| = \begin{cases} \frac{2}{|\alpha-\beta|\pi} & \alpha-\beta = odd \\ 0 & \alpha-\beta = even \end{cases}. \tag{S11}$$

To ensure that the operator $\hat{\eta}$ is Hermitian (as it represents a physical observable), we impose the condition $\tilde{\eta}_{\alpha\beta} = \tilde{\eta}^*_{\beta\alpha}$, which then implies

$$\tilde{\eta}_{\alpha\beta} = \begin{cases} \pm\frac{2i}{(\alpha-\beta)\pi} & \alpha-\beta = odd \\ 0 & \alpha-\beta = even \end{cases} \tag{S12}$$

Finally to determine the sign in Eq.(S12), the ground state eigenfunction of $\hat{H}'(t)$ was boosted by a momentum $p = \pi k/L$, where $k \in \mathbb{Z}$, which results in the wave packet $\psi(q) = \sqrt{\frac{2}{L}}\sin(\frac{\pi q}{L})\exp(\frac{i\pi k q}{L})$. By demanding that $\langle\psi|\hat{\eta}|\psi\rangle \to 1$ for $k \gg 1$ and $\langle\psi|\hat{\eta}|\psi\rangle \to -1$ for $k \ll -1$, a series of straightforward calculations yields

$$\tilde{\eta}_{\alpha\beta} = \begin{cases} \frac{2i}{(\beta-\alpha)\pi} & \alpha-\beta = odd \\ 0 & \alpha-\beta = even \end{cases} \tag{S13}$$

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[1] Feingold, M.; Peres, A. Distribution of Matrix Elements of Chaotic Systems. *Phys. Rev. A* **1986**, *34*, 591-595.