Introduction to Fisheries Economics

by the use of *Wolfram Language* and *Mathematica*

## Arne Eide

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1. Introduction

Fishing, hunting, and gathering food represent the oldest human ways of life, where people depended on wild food for subsistence. While the importance of hunting and gathering wild food decreased after the introduction of agriculture more than ten thousand years ago, fishing has become increasingly important over the last hundred years as a source of food and wealth for millions of people around the world.

Fishing is an economic activity that produces food for personal consumption, provides access to commercial markets for income, and serves as a form of recreation and pastime. These activities often coexist but can also be viewed as stages in economic and cultural development, beginning with subsistence fishing before evolving into a commercial industry and recreational use of natural resources.

The goal of the earliest stage, subsistence fishing, was to feed one's own household. Early barter economies allowed fish products to be traded, but as a perishable product, there were limits to the tradability of fish. Methods to preserve fish, such as drying, salting, and smoking, were developed, contributing to the growth of fish markets where stored fish could be sold. Infrastructure became a major obstacle, including storage capacity, roads, transport vehicles, as well as organisations, agreements, and security measures for trade investments, all of which were necessary to transport fish to markets in large cities.

Increased demand and trade led to wealth creation, initially in secondary trading and later increasingly among fishers. One consequence of economic growth is that labour becomes more expensive and capital becomes more accessible. Therefore, when possible, labour is substituted by capital. The fishing industry is no exception.

The aim of recreational fishing, presumably the most recent commercial utilisation of fish stock resources, is not primarily to catch fish, as the catch is merely a means to experience fishing. The ultimate goal is the adventure itself, utilising and enjoying the natural environment, immersing in the peaceful surroundings, and engaging in relaxing activities. Recreational fishing has developed into a major industry, encompassing game

fisher accommodations, recreational activities, and the production of fishing tackle and equipment.

Fishing takes on various forms within different societal and economic contexts. This book aims to discuss the economic activity of fishing through formal models while considering the dynamics of utilised fish stocks. The analysis is based on basic microeconomic principles and standard theories of fisheries biology and economics.

1.1   What is a *good* model?

A model is a generic term covering a large range of possible ways of simplifying, emphasising, exaggerating or clarifying complex matter. Graphical models, mathematical models and conceptual model represent different approaches, all aiming to reveal structures, patterns and coherence that are difficult to obtain without a modelling effort. If a model is a simplification of the real world it is of course untrue in the sense that it does not represent the full truth, some elements are missing. That is the whole point of employing models. Hence, a model could never be tested on being true or not, in some sense it is always untrue. The only criteria of the goodness of a model is: Is the model useful?

In figure 1.1 two different graphical models are shown. Which one is most useful depends solely on which problem we seek to solve. None will claim a street map to be at better model than a globe in general, it certainly depends on the problem we will solve. A globe is a better tool if we look at for example distances between large cities in different countries. For this purpose a street map is useless. This does not make a street map a bad model. The close link between the problem we are investigating and the model we use in this investigation is sometimes forgotten. But there is none general purpose model that can be used on all problems, – unfortunately. We have to customise models for the specific problems we want to look into. This also goes to problems within the area in fisheries economics. This book therefore presents a range of different modelling approaches which all may be useful, – given different problems.

Research problems are formulated based on available data and knowledge that we consider relevant. When a problem is formulated, we can develop a method—or model—that we believe can highlight and potentially solve the problem. Figure 1.2 illustrates how the modelling process can be represented as a flow chart. Thus, the flow chart serves as a model of a general modelling process.

We note that a model must be built upon certain assumptions, and these assumptions and model expressions are closely tied to the available knowledge (both empirical and theoretical) and the problem that initiates the process. The model will yield certain results. However, the results of the modelling process do not constitute the solution to the problem; rather, they represent a step in the direction of solving it. The most significant aspect of the process occurs within the green box in Figure 1.2.

When analysing the results of the modelling process, we must consider the properties of the model, the assumptions made, and the pertinent knowledge that we were unable to incorporate into the model. If the modelling endeavour has been successful, we will acquire fresh insights upon completing the initial analysis. This new knowledge may enable us to formulate the research question with greater precision and consequently refine the model. Thus, the modelling process is iterative, allowing us to incorporate new knowledge by thoroughly examining the available information from the outset.

Figure 1.1: Two models utilising geographical maps, a globe to the left and a street map to the right.

The new knowledge obtained through this iterative process may reveal that the chosen model is ineffective for investigating the given problem. This realisation could serve as a starting point for developing a more suitable model to explore the specific problem. Different problems may necessitate alternative modelling approaches.

This book introduces several mathematical fishery models, with the aim of depicting population growth and the economics of fisheries. Disparities in modelling approaches often stem from variations in the problems under investigation.
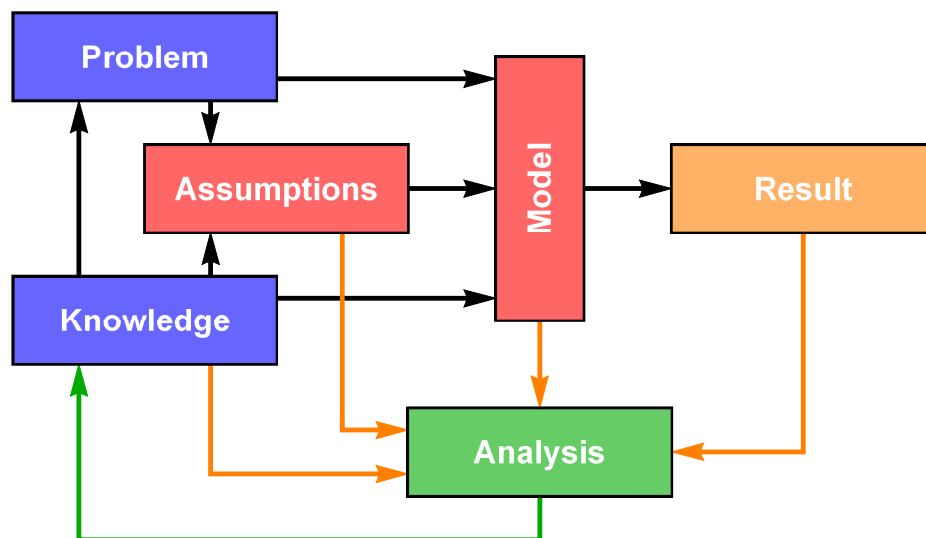
Figure 1.2: Flow chart of the iterative modelling process.

Summary 1.1.1 — What is a good model?. A model that is *useful* for highlighting a
specific problem is *good* for that particular purpose. Another problem might be better
addressed by a different model, suitable for investigating that specific issue. There is no
absolute 'true' model; a good model is one that proves useful for the given problem.

1.2   Interactive use of the codes in the book

In addition to introducing fisheries economics, this book provides simple programming
codes written in the Wolfram Language (*WL*), offering relevant examples of how to utilise
the software *Mathematica* as a modelling tool in fishery economics. Most of the figures
presented in the book are generated using Mathematica, and the programming codes
are provided at the end of the book. To grasp the basics of *Mathematica* and the *WL*,
other resources beyond this textbook should be consulted. Stephen Wolfram's book *An
Elementary Introduction to the Wolfram Language* serves as an excellent starting point.
The examples offered in this book assume some prior familiarity with the fundamentals of
Mathematica and *WL*.

Readers without access to the *Mathematica* software can skip the code sections without
losing the main content of the book. Even if you are unable to execute the codes, you can
still benefit from examining the content of the code sections and the graphical illustrations
provided therein. For readers without access to *Mathematica*, you can test the codes in the
new Wolfram|Alpha Open Code environment (`https://www.open.wolframcloud.com/`)
or using *Mathics*, a free, lightweight alternative to *Mathematica*. In *Mathics*, you should
be able to run most of the *WL* codes presented in the book. *Mathics* is available for free at
`http://mathics.github.io/` and can be run within a web browser.

The motivation behind writing this book is to expand the relatively narrow range of
modelling often found in fisheries economics textbooks. Highly simplified biological models
and basic economic relationships serve as valuable pedagogical tools. However, at times,
they fail to capture essential elements in fisheries dynamics and the interplay between
biological changes and economic behaviour. It's fair to say that fisheries economists have not
had a substantial impact on the development of effective tools in fisheries management[57].
Perhaps alternative modelling approaches are necessary to enable economic reasoning to
contribute toward a deeper understanding and improved management of world fisheries.
The challenge is passed on to the next generation of fisheries economists!

Harvesting technology and population dynamics

2. Fishing effort

2.1 Requirements for fishing

Fishing involves the utilisation of natural aquatic resources, primarily fish stocks. To capture fish – or other seafood products – proper fishing methods must be employed. Certain resources (such as shellfish, crabs, clams, etc.) can be collected from the beach or shallow waters when available, while other resources require more elabourate catching techniques.

Over the thousands of years that humans have relied on aquatic resources for sustenance, numerous methods of capture have been developed. Some have proven to be effective and remain in use, while others have been discarded in favour of better and more efficient alternatives. Today, a wide variety of fishing techniques are employed on a daily basis, including some that have been known since ancient times, and others that have emerged only in recent decades.

Angling is one of the oldest fishing methods known, commonly employed in commercial fisheries worldwide. Hand line and longline fishing are the most significant methods that employ hooks and bait. Different types of fishing nets have also been used since ancient times, including gill nets and drift nets in modern fisheries, as well as more sophisticated net configurations such as purse seines, Danish seines, and trawl gears.
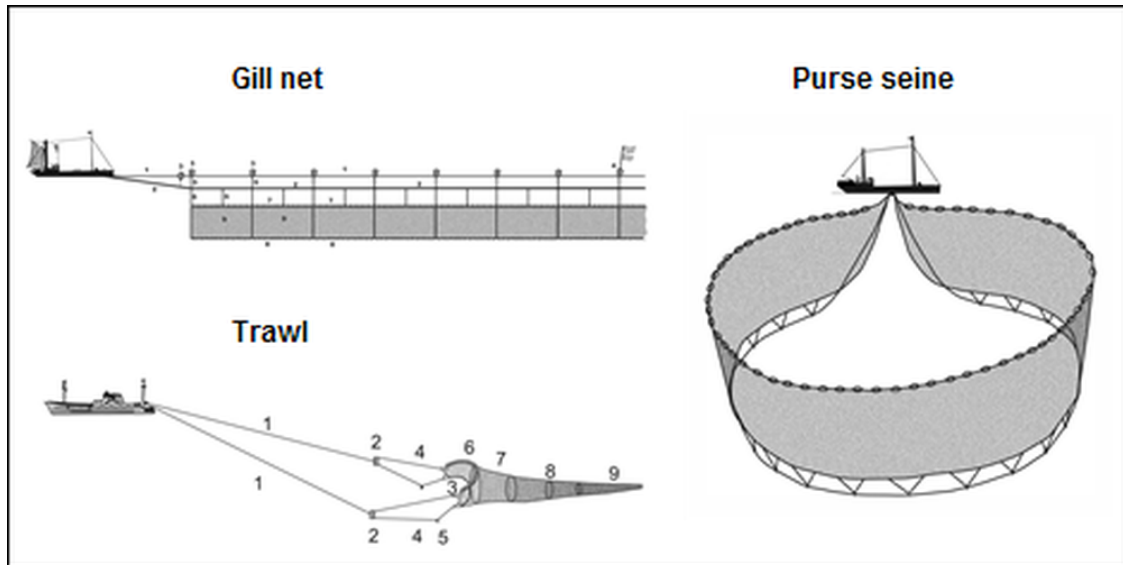
Figure 2.1: Fishing gears commonly used in modern fisheries.
(Source: Wikimedia commons)

2.2   What is fishing effort?

Fishing effort can be regarded as a commodity produced using varying technologies and in different qualities and quantities. As discussed above, there exists a vast array of different methods for capturing fish. The relative efficiency of these methods depends on several variables. In general, fishing techniques can be categorised based on their impact, action, and retention properties in fishing[10].

While certain fishing technologies, such as concealed nets or traps, aim to remain invisible to the fish, others intend to attract fish to the fishing gear through eye-catching and scented bait. The performance of fishing gear varies based on fish densities and gear properties.

Measuring fishing effort is not straightforward. Metrics that prove useful in certain fisheries might be ineffective in others. However, in all cases, fishing effort is measured, like harvest, in various units per unit of time (typically per hour, day, month, or year). In a fishery where the fishing fleet is relatively homogeneous, fishing effort may be measured using terms such as:

  - Number of vessels per unit of time
  - Number of fishing hours, days, or trips per month or year
  - Number of working days per time unit
  - Towing hours (for trawl fisheries) per time unit
  - Total numbers of engine horsepower units per time unit
  - Total number of hooks (for hand line and longline gears) per time unit
  - Total number of nets (for gill net fleets) per time unit
  - Sum of vessels' length
  - Sum of vessels' tonnage
  - or other measures...

In cases involving more diverse fleets, the most effective way to measure a standard fishing effort could involve combining the items listed above. Each case must be evaluated

individually, based on the specific characteristics of the fishery in question. A measurement method is of high quality if all units are measured using the same scale over time.

The challenge of measuring fishing effort within a given fishery should not be underestimated. Different units and dimensions are employed to measure fishing effort across different fisheries, including physical fleet properties (such as differences in vessel size, age, engine power, etc.), home ports, types of operations (seasonal composition, geographical distribution of fishing grounds, etc.), and fishing gears (which can differ between and within seasons).

Fishers must identify profitable combinations of how, where, and when to fish in order to sustain their fishing activity and maximise their objectives. In principle, fisheries management seeks to limit the area of opportunities, which may encourage the fishing fleet to specialise, enabling more efficient utilisation of available opportunities.

> **Summary 2.2.1 — Units to measure fishing effort.** Fishing effort serves as an input parameter in the process of generating fish harvest. Nevertheless, there exists no universally standardised unit for quantifying fishing effort. Upon adopting a particular metric to gauge fishing effort, it is presupposed that this chosen measure serves as a representative indicator encompassing all other elements that collectively constitute the complete fishing effort engaged in fish harvest production.

2.3 Production of fishing effort

Standard economic production theory assumes non-wasteful and economically efficient production. The interpretation of the latter is discussed in Chapter 6. Non-wasteful production represents technological efficiency: Any reduction in input factors will result in reduced harvest production. In the short run, the production technology remains fixed. However, production methods and processes (production technology) evolve over time, and technological breakthroughs could lead to significant shifts in processing technologies. There is substantial empirical evidence suggesting increased production efficiency over time.

As new technologies develop, they may replace previous technologies or coexist with them. In the production of fishing effort, a wide range of different technologies have been developed over time, coexisting in many fisheries. From an economic perspective, we can infer that each technology has sufficient advantages to persist, as each technology proves to be more efficient than any other under specific conditions. We will later revisit some potential implications that the coexistence of different technologies might entail.

In general, we assume that fishing effort (E) is produced through production processes involving varying quantities of labour (L) and capital (K):

$$E = E(L, K) \tag{2.1}$$

Usually, we expect labour and capital to be substitutes. Hence, the same level of effort may be produced using different combinations of labour and capital. A given fishing effort, for example, could be produced using a large quantity of labour and a small quantity of capital, or vice versa.

The variables labour (L) and capital (K) in Equation 2.1 are referred to as *input factors* in the production process. Fishing effort (E) represents the output of the production process.

Figure 2.2 depicts three possible shapes that Equation 2.1 might take for a given quantity Q. A curve describing a constant quantity produced by different combinations of input factors while utilising the same technology is known as an *isoquant*. The red and green isoquants represent two extreme technologies, between which lies the sample space of infinitely many possible isoquants representing other technologies. The red dashed line illustrates a situation where the input factors labour and capital are not substitutes; an increase in one factor does not lead to increased fishing effort unless the other factor is also increased. The green isoquant – the other extreme – is a straight line, illustrating the case of perfect substitution between the two input factors.

When labour and capital are substitutes, the same level of production can be achieved through a small reduction in labour $(-\Delta L)$ accompanied by a corresponding increase in capital (ΔK). The ratio $-\Delta K/\Delta L$ represents the slope of the isoquant, and we have:

$$\lim_{\Delta L \to 0} -\frac{\Delta K}{\Delta L} = -\frac{dK}{dL}$$

From Figure 2.2, we observe that the value of the ratio dK/dL remains constant for the green line, whereas it varies along the blue curve. In the latter case, we can observe that as labour continues to decrease, each unit of labour must be replaced by an increasing amount of capital to maintain production.
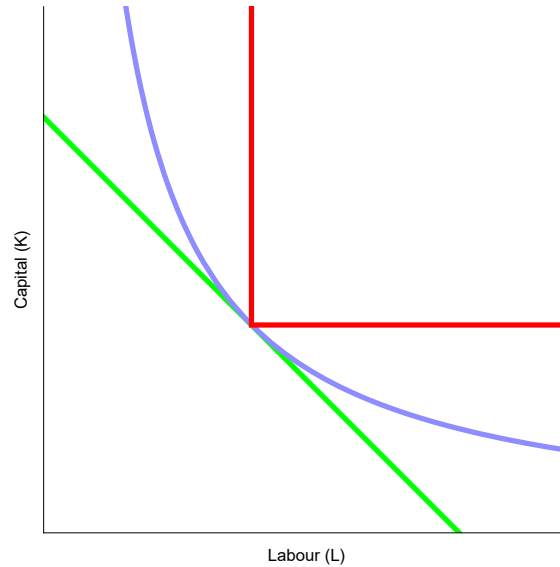
Figure 2.2: Three production processes with different substitution elasticities. The three curves indicate how a constant quantity of fishing effort can be produced using three distinct production technologies, where the substitution elasticity is zero (red curve), between zero and infinity (blue curve), or infinite (green curve).

As discussed in Section 2.2, fishing effort may be measured in various ways. The labour involved during a period of time, for example, could be quantified in terms of the number of hours all fishers spend on fishing. In this case, it may also be relevant to include the time spent on tasks such as preparing for fishing and landing the catch. While all input factors (such as L and K in Equation 2.1) are measured according to units appropriate for each specific factor, all quantities of input factors can be measured in terms of values. The

cost of labour and the cost of capital are measurements of the two input factors converted into monetary values using given unit prices for labour and capital. The cost of labour and capital utilised in production then constitutes the total cost of the resulting fishing effort.

2.4 Introduction to elasticities

Economic theory often employs various marginal measures—average values, derivatives, and elasticities – all of which reflect per unit evaluations. For a function $f(x)$, several per unit evaluations can be conducted:

- The average value of $f(x)$ is $f(x)/x$.
- The marginal value (the derivative with respect to the variable x) is expressed as $f'(x) = df(x)/dx$.
- The percentage change in value resulting from a one percent change in the variable (the elasticity with respect to the variable x) is given by $f'(x)/(f(x)/x)$.

From this list, it can be observed that elasticity is the ratio of the other two measures: the derivative divided by the average value. As a ratio, elasticity is dimensionless, which is particularly advantageous when $f(x)$ can be measured in different units.

An elasticity is a unit-less marginal value that could be seen as a normalised derivative value. If we revisit the discussion of the slopes of the isoquants in Figure 2.2 above, the corresponding elasticity (the elasticity of capital use with respect to the use of labour) of the curves is

$$\lim_{\Delta L \to 0} -\frac{\frac{\Delta K}{K}}{\frac{\Delta L}{L}} = -\frac{\frac{dK}{K}}{\frac{dL}{L}} = -\frac{dK}{dL} \cdot \frac{L}{K}$$

This elasticity indicates the percentage change in capital required to produce the same quantity after a one percent change in the input of labour. The elasticity (like the slopes of the isoquants) must be negative, as a reduction in labour usage must be followed by an increase in capital usage, and a positive shift in labour usage must be offset by reduced capital usage. As shown in the equation above, the elasticity will change for varying valid combinations of labour and capital due to changes in the L/K ratio, even when the derivative (dK/dL) is constant.

In the caption of Figure 2.2, a reference is made to the elasticity of substitution. This is an elasticity often mentioned in production economics. This elasticity also takes into account the prices of the input factors, which will be discussed later.

Earlier, we introduced three different per unit measures. Why do we need three different methods of unit measurement? In principle, one could argue that there are only two distinct principles (average and marginal values), as the third measure (elasticity) combines the other two. The two principles are per unit measures but differ significantly from each other. While an average value considers all unit values, the marginal value only reflects the additional value contributed by the specific unit in question. Code box 2.4.1 illustrates these differences using a simple example.

Code box 2.4.1 — Per unit calculation - the example of distance and speed.
Assume that a person goes for a walk. The walk takes one hour, and the person shifts between two speeds: `sp1` and `sp2`. `sp1` and `sp2` are measured in units of kilometres per hour (km/h). For the first fifteen minutes, the person walks at speed `sp1`, then at speed `sp2` for another half an hour, and thereafter at speed `sp1` for another fifteen minutes. We can calculate the walked distance by multiplying the constant speed by the time spent. The total distance can be expressed as a piecewise function:.

```
In[1]:= distance[x_] := Piecewise[{
        {sp1 x, x <= 1/4},
        {sp1*1/4 + sp2 (x - 1/4), 1/4 < x <= 3/4},
        {sp1*1/4 + sp2 2/4 + sp1*(x - 3/4), x > 3/4}}]
```

Assume $sp1 = 3$ and $sp2 = 7$. When walking at a speed of 3 km/h for thirty minutes, the distance walked should be 1.5 km. Another thirty minutes of walking at a speed of 7 km/h adds 3.5 km, making a total of 5 km altogether.

```
In[2]:= distance[1]
```

```
Out[2]= 5
```

```
In[3]:= Plot[
        distance[x], {x, 0, 1},
        GridLines -> {{1/4, 3/4}, None},
        AxesLabel -> {"Time (hour)", "Distance (km)"}
        ]
```

Out[3]=
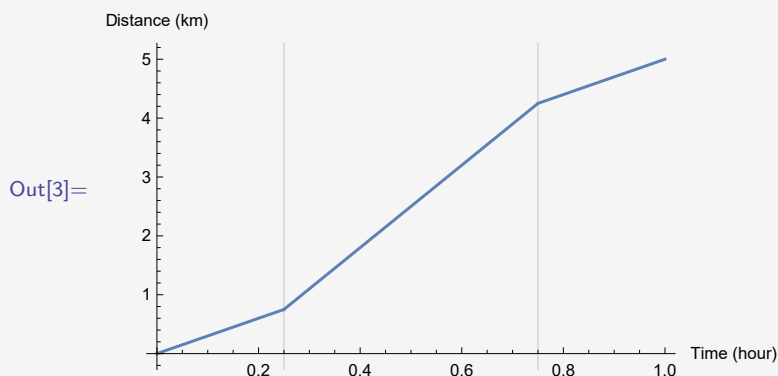
Now, let's examine the per unit measures in the following graphical presentation:

```
In[4]:= Plot[
        {g[x]/x, g'[x], g'[x]/(g[x]/x)}, {x, 0, 1},
        PlotRange   -> {0, Automatic},
        GridLines   -> {{1/4, 3/4}, None},
        AxesLabel   -> {"Time (hour)", "Per unit measure"},
        PlotLegends ->
          Placed[{"Average", "Marginal", "Elasticity"}, Below],
        ]
```

Out[4]=

Per unit measure

Time (hour)

— Average   — Marginal   — Elasticity
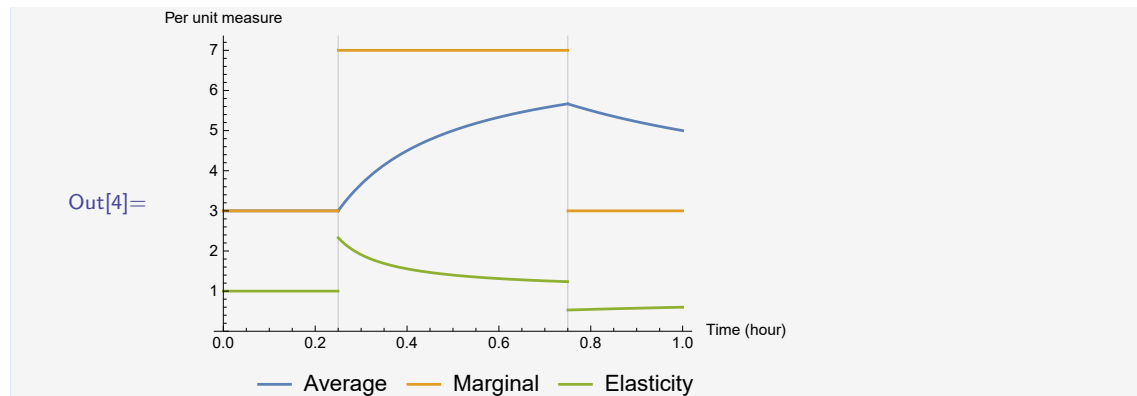
We recognise the speeds `sp1` and `sp2` as the yellow line segments. These are the marginal values, or the time derivatives of distance. In the first quarter, the average speed (blue curve) is identical to the marginal value (since the speed is constant). When the speed increases after 15 minutes, the average value increases toward the marginal value before declining in the last quarter when the speed drops back to 3 km/h.

The elasticity combines the other two; hence, it measures both the immediate change (measured by the marginal value) and the total time and distance. When normalising the marginal value by the total values, the elasticity measures the percentage change in distance for a one percent increase in time. In the first quarter, the percentage change was constant. After increasing the speed, the percentage change in distance for a one percent increase in time also increased, even though the speed was constant. The impact from the first quarter brought the elasticity upwards, but as the first quarter makes up a diminishing part of the total, the elasticity falls towards one percent (as the average value gets closer to the marginal). The last change in speed (after 45 minutes) brings the elasticity below one, slightly climbing.

Summary 2.4.1 — What is an elasticity?.
The elasticity of $f(x)$ equals the percentage change in $f(x)$ when x changes by 1%.

DEMO
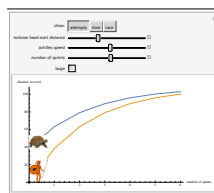
Here is a Mathematica demonstration presenting graphically the race between Achilles and the tortoise.

http://demonstrations.wolfram.com/
ZenosParadoxAchillesAndTheTortoise/

2.5 The Cobb-Douglas function

Equation 2.1 can have different mathematical expressions based on the properties of the production processes. The Cobb-Douglas function is often employed to describe a production process, assuming **unit elasticity of substitution** and an **elasticity of scale** equal to 1. We will revisit these expressions later, after first introducing the Cobb-Douglas production function.

A standard expression for the production of fishing effort (E) as a function of labour (L) and capital (K) in a Cobb-Douglas production process is given by

$$E(L, K) = A \cdot L^\alpha \cdot K^{1-\alpha} \tag{2.2}$$

where A and α are non-negative constants, and $0 \leq \alpha \leq 1$. The implementation of equation 2.2 in *Mathematica* is provided in `In[1]`[1] in Code box 2.5.1 below, where the output elasticities of the two input factors are also shown.

Code box 2.5.1 — Output elasticities in the Cobb-Douglas function.
Defining the Cobb-Douglas function in *Mathematica*:[a]

```
In[1]:= cd[l_, k_] := A * l^α * k^(1 - α)
```

Now find the output elasticity with respect of labour (`l`):

```
In[2]:= D[cd[l, k], l] * l / cd[l, k]
```

```
Out[2]= α
```

and the output elasticity with respect of capital (`k`):

```
In[3]:= D[cd[l, k], k] * k / cd[l, k]
```

```
Out[3]= 1 - α
```

---

[a]Since all internal *Mathematica* commands start with a capital letter we prefer to use lower case letters in our variables, to avoid confusion. In this code the fishing effort symbol (E) in equation 2.2 is replaced by *cd*, indicating that it is a Cobb-Douglas equation.

Capital cost consists of two main components: 1) The market value of the fishing gear and other materials used in the fishing operation (c), and 2) the value of the foregone benefits resulting from allocating the capital to materials for fishing (c_o) rather than elsewhere. The latter is often referred to as *the opportunity cost* of capital.

The cost of labour can be calculated in various ways. If labour is hired, the labour cost includes the wages paid (w) and the opportunity cost of the capital spent on labour (l_o). For an independent fisher spending time on fishing, the labour cost is the income foregone by utilising labour that could have been more optimally allocated elsewhere (c_o).

The three fishing technologies mentioned above (angling, longline, and hand line fishing) use different combinations of labour and capital in their production of fishing effort.

2.6 The CES function

The CES function provides a more general formulation of Equation 2.1. The abbreviation CES stands for *Constant Elasticity of Substitution*, and Figure 2.2 illustrates the two extremes of an infinitely large elasticity of substitution (green) and zero elasticity of substitution (red).

The Cobb-Douglas function is a special case of the CES function, with a constant elasticity of substitution equal to one (unit elasticity of substitution). The blue curve in Figure 2.2 represents a Cobb-Douglas curve, while all three curves are special cases of a CES function.

---

[1]*Mathematica* inputs are numbered as such. Corresponding numbering for output is `Out[1]`.

Code box 2.6.1 — Output elasticities in the CES function.
Defining the CES function in *Mathematica*:

```
In[1]:= ces[l_, k_] := (α l^r + (1 - α) k^r)^(1/r)
```

where r is a parameter related to the elasticity of substitution. The elasticity of substitution is $\eta = \frac{1}{1-r}$. Now, find the output elasticity with respect to labour (l):

```
In[2]:= D[ces[l, k], l] * l / ces[l, k]
```

$$\text{Out[2]=} \quad \frac{l^r \; \alpha}{k^r \; (1-\alpha) \; + \; l^r \; \alpha}$$

and the output elasticity with respect of capital (k):

```
In[3]:= D[ces[l, k], k] * k / ces[l, k]
```

$$\text{Out[3]=} \quad \frac{k^r \; (1-\alpha)}{k^r \; (1-\alpha) \; + \; l^r \; \alpha}$$

Let's create 3D plots for the three cases indicated in figure 2.2, where the elasticity of substitution (η) is equal to ∞, 1, and 0, respectively. In the plots, the input factors, labour and capital, are represented by the two horizontal axes, while the output is measured vertically.

For $\eta = \infty$ (perfect elasticity of substitution):

```
In[4]:= Plot3D[
          ces[l, k] /. {r -> 1, α -> 1/2},
          {k, 0, 1}, {l, 0, 1},
          MeshFunctions -> {#3&},
          Mesh -> {Transpose[
            {Range[5]/6,{#,#,Directive[Thickness[.02], Green],#,#}}
          ] & @ Black}
        ]
```


Out[4]=
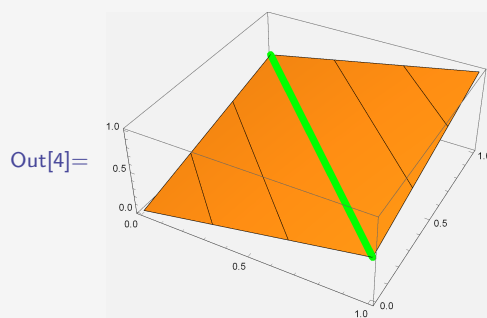
For $\eta = 1$ (the Cobb-Douglas function):

```
In[5]:= Plot3D[
          Evaluate@Limit[ces[l, k]/.{α -> 1/2},{r -> 0}],
          {k, 0, 1}, {l, 0, 1},
          MeshFunctions -> {#3&},
          Mesh -> {Transpose[
            {Range[5]/6,
            {#,#,Directive[Thickness[.02],Lighter@Lighter@Blue],#,#}}
          ] & @ Black}
        ]
```

Out[5]=
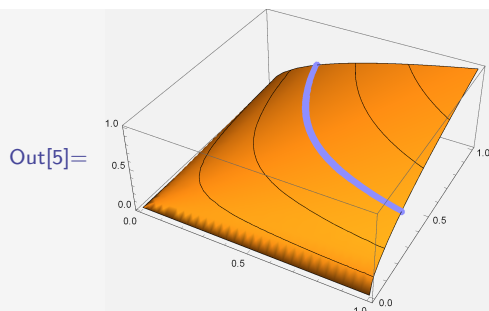
For $\eta \approx 0$ (the Leontief function):

```
In[6]:= Plot3D[
          ces[l, k]/.{r->-1000000000000, α->1/2},
          {k, 0, 1}, {l, 0, 1},
          MeshFunctions -> {#3&},
          Mesh -> {Transpose[
            {Range[5]/6,{#,#,Directive[Thickness[.02], Red],#,#}}
          ] & @ Black}
        ]
```

Out[6]=
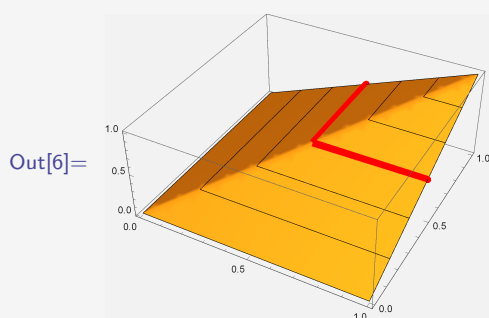
In each of the three plots above, a constant value of output is indicated by colour, according to the colours used in figure 2.2: green ($\eta = \infty$), blue ($\eta = 1$), and red ($\eta = 0$).

2.7   Elasticity of scale

The concept of elasticities is presented in Section 2.4, and output elasticities are derived for the Cobb-Douglas function (Code box 2.5.1) and the CES function (Code box 2.6.1). The interpretation of output elasticities is straightforward: When one of the input factors, l or k, changes (increases or decreases) by one percent, the production output changes accordingly by a percentage equal to the output elasticity of the input factor.

Code box 2.7.1 — Continued from Code box 2.6.1.

Furthermore, in the case of the CES function, the elasticity of scale equals one. In this context, it is convenient to use *Mathematica* to find the solution:

```
In[7]:= mplify[D[ces[l,k],l]*l/ces[l,k] + D[ces[l,k],k]*k/ces[l,k]]
```

Out[7]=

According to the results in Code box 2.5.1, the output elasticity of labour in the Cobb-Douglas function is α, and the output elasticity of capital is $1-\alpha$. If both input factors are

changed simultaneously by one percent, the total change in output equals the sum of the two output elasticities. For the Cobb-Douglas equation, it is easy to see that $\alpha + (1 - \alpha) = 1$. This sum is usually referred to as the elasticity of scale (or returns to scale), indicating the effect that up- and down-scaling have on the produced quantity.

The elasticity of scale is usually assumed to be equal to one, indicating that there are no economies of scale in production. This represents a standard production scenario but need not always be the case. Both the Cobb-Douglas function and the CES function can be adjusted to accommodate scale elasticities different from one. A simple and general formulation of the first function is

$$E(L, K) = A \cdot L^{\alpha} \cdot K^{\beta} \tag{2.3}$$

where the elasticity of scale is $\epsilon = \alpha + \beta$. If both input factors (L and K) are increased/decreased by one percent, the total effort production (E) increases/decreases by ϵ percent. The contribution from the increase or decrease in labour (L) is α, and the contribution from capital (K) is β percent.

In some productions, we find that $\epsilon > 1$, indicating increasing returns to scale, meaning that scaling up the production becomes relatively more efficient. Depending on market factors, this may lead to economies of scale, providing an advantage to large producers compared to small ones.

Exercises

Exercise 2.1 Assume a production of fishing effort according to equation 2.3 where $\alpha = 1$ and $\beta = 1.5$. If labour (L) and capital (K) increase by 10%, how much will the production of fishing effort increase? ■

Exercise 2.2 In the CES function in code box 2.6.1, assume $\eta = 1$. Calculate the output elasticities of labour and capital. ■

3. Fish capture production

In Section 2.1, the necessary requirements for producing fishing effort were discussed. However, it is not sufficient to know how to catch fish; fish stock resources must also be available for harvesting. In economic terms, we would say that the fish stock is an essential input factor in catch production. The other crucial factor in catch production is the fishing aptitude of the fishers, which we will refer to as fishing effort.

Catch production is a process that follows the principle of technologically efficient production discussed in Chapter 2, where some well-known production functions were introduced. In this chapter, we will further discuss how to implement the Cobb-Douglas function (Equation 2.3) to describe fish catch production.

In Chapter 2, we have already discussed how fishing effort (E) is produced by the input factors labour (L) and capital (K). Now, we consider a production process in which fishing effort, as an input factor, contributes to the output, which is fish harvest. Fishing effort alone is not sufficient to produce a harvest of fish; it is also necessary to have access to a fish stock resource. Therefore, the available fish stock is also an input factor in fish harvest production.

In Chapter 5, we will further discuss the properties of the fish stock. Here, we only consider the fish stock in terms of available biomass as one of the two input factors for producing fish catches.

3.1 Stock assessment based on catch statistics

Stock assessment is the foundational core of modern fisheries management. Time series of stock assessments provide information about growth potential, variability, and possible exploitation levels.

The simplest stock assessment method is to compare the amount of catch per unit of fishing

intensity (catch per unit of effort, $CPUE$) between different areas or periods. The highest $CPUE$ is believed to reflect the highest abundance.

However, this method does not estimate the actual number of fish in an area at a specific point in time; it is merely a relative measure used to rank different observations. Nonetheless, to determine whether fish abundance is increasing or decreasing, measuring relative changes is sufficient.

The indices used to measure relative changes (e.g. $CPUE$) are considered as stock estimates. This holds true even for more sophisticated stock assessment methods used in data-rich single-species fisheries, such as the Northeast Arctic cod fishery. In this fishery, high-resolution catch data, annual survey data, and various scientific studies contribute to enriching the stock assessment methodology and refining the processes used to evaluate the stock's condition at any given point in time.

Let's return to the fundamental observation that we can label as the $CPUE$ methodology, where a large catch with a given effort indicates a larger stock abundance than a small catch with the same effort at another point in time. In its simplest form, the $CPUE$ methodology assumes a linear catch function:

$$H(E,X) = q \cdot E \cdot X \tag{3.1}$$

Here, H represents the harvest produced by fishing effort, E, and the stock abundance (measured in biomass), X. q is often referred to as the catchability coefficient, a scaling parameter that also reflects the technological properties of the fishing gear in use.

As seen in Section 2.7, it is easy to observe from Equation 3.1 that catch per unit of effort ($CPUE$) is linear with the stock index $q \cdot X$:

$$CPUE = H(E,X)/E = q \cdot X \tag{3.2}$$

If we had an accurate estimate of the value of q, we could potentially determine the stock biomass in nature, X. In most cases, however, knowing whether the stock is growing or declining is sufficient for making non-critical management decisions. In such cases, the $CPUE$ measure is adequate, assuming that Equation 3.1 holds and that we can accurately measure catch (H) and effort (E). The latter is a significant challenge, which is addressed elsewhere in this document. Here, we will further discuss the methodology, assuming accurate catch and effort observations exist.

Consistently measuring fishing effort is a challenging task that is not easily resolved. There are no standard methods for standardising effort in a heterogeneous fleet at a specific point in time. Additionally, fleet efficiency changes (typically increases) over time, leading to systematic errors when comparing fishing effort measurements across different time periods.

In principle, measuring catch quantities is easier than measuring fishing effort. However, obtaining this information requires a system that can retrieve such data without hidden, manipulated, or illegal catches distorting the data samples.

Code box 3.1.1 — Simple stock assessment by $CPUE$ calculations.

Assume that we have access to time series of catch data and standardised fishing effort over a period of ten years. The catch time series is:

```
In[1]:= catch = {
        2384, 2361, 1586, 1889, 1766, 2456, 1068, 1905, 1425, 1957};
```

The effort was 100 in the first year and increased by 10 every year thereafter. The development of $CPUE$ over time can be observed by plotting catch per unit of effort for each year.

```
In[2]:= ListLinePlot[
        catch/Table[100 + 10*t, {t, 0, 9}],
        Mesh        -> All,
        Frame       -> True,
        PlotTheme   -> "Detailed",
        FrameLabel -> {"Year","CPUE"}
        ]
```

Out[2]=
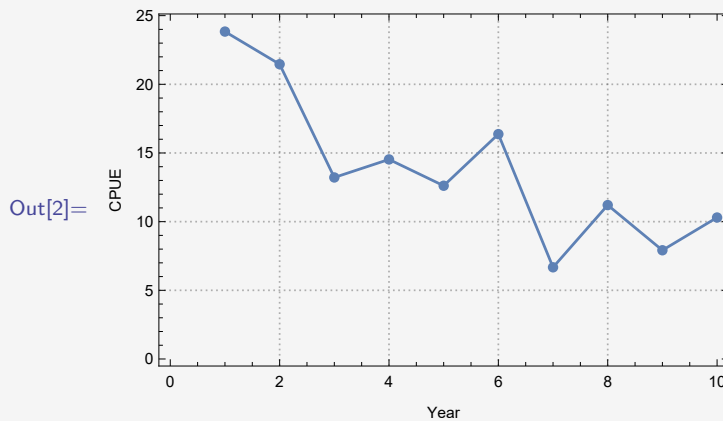
We observe a down-sloping trend in the $CPUE$ measures and assume a linear model

```
In[3]:= model1 = LinearModelFit[
        Transpose[{#, catch/#} &@Table[100 + 10*i, {i, 0, 9}]], x, x]
```

Out[3]= FittedModel[$35.4802 - 0.149425\,x$]

We see that about 67.5% of the variation is explained by the linear model

```
In[4]:= model1["RSquared"]
```

Out[4]= 0.675255

and we retrieve the analysis of variance by

```
In[5]:= model1["ANOVATable"]
```

Out[5]=

	DF	SS	MS	F-Statistic	P-Value
x	1	184.204	184.204	16.6347	0.00354115
Error	8	88.5878	11.0735		
Total	9	272.792			

The results of the linear regression is plotted together with the $CPUE$ observation versus fishing effort.

```
In[6]:=  Show[{
           Plot[model1[e], {e, 0, 240}, PlotStyle -> Dashed],
           ListLinePlot[
             Transpose[{#, catch/#} & @ Table[100 + 10*i, {i,0,9}]],
             Mesh -> All
           ]},
           Frame            -> True,
           PlotRangePadding -> None,
           PlotRange        -> {0, All},
           FrameLabel       -> {"Effort", "CPUE"}
         ]
```

Out[6]=
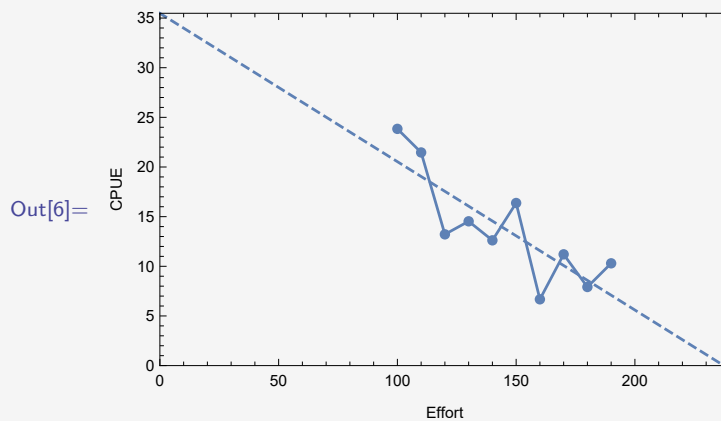
Multiplying the $CPUE$ values above with effort gives the catch, and the linear regression now describes a parabolic curve through the origin.

```
In[7]:=  Show[{
           Plot[model1[e] * e, {e, 0, 240}, PlotStyle -> Dashed],
           ListLinePlot[
             Transpose[{Table[100 + 10*i, {i, 0, 9}], catch}],
             Mesh -> All
           ]},
           Frame            -> True,
           PlotRangePadding -> None,
           PlotRange        -> {0, 2600},
           FrameLabel       -> {"Effort", "Catch"}
         ]
```

Out[7]=
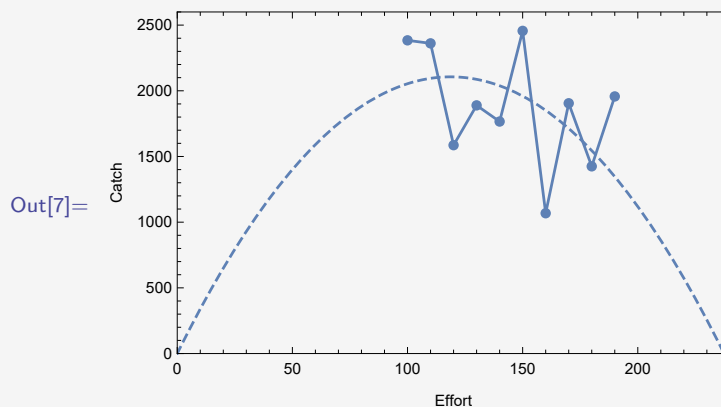
> **Additional remarks 3.1 — How many frogs are needed?.**
> Now, having learned the concept of *catch per unit of effort*, you should be able to solve this small puzzle:
>
> If 37 frogs catch 37 flies in 37 minutes, how many frogs are needed to catch 52 flies in 52 minutes?

3.2 Cobb-Douglas catch production

Catch is produced by two input factors: fishing effort (E) and stock biomass (X), as indicated in Equation 3.1. We now consider the more general Cobb-Douglas production function expressed in Equation 2.3):

$$H(E,X) = q \cdot E^\alpha \cdot X^\beta \tag{3.3}$$

We can see that this expression is equivalent to Equation 3.1 when $\alpha = \beta = 1$.

There may be reasons to expect $\alpha = 1$ and $0 < \beta < 1$[20, 32]. If E is measured in the number of homogeneous boats, we should expect two boats to catch double the amount of fish compared to one boat, assuming a constant stock size (X). Hence, α should equal 1, since $2 \cdot q \cdot E^\alpha \cdot X^\beta = q \cdot (2 \cdot E)^\alpha \cdot X^\beta$ when $\alpha = 1$.

The value of β depends on both the biological properties of the fish stock in question (typically β will be lower for schooling species than for non-schooling species) and the properties of the harvesting technology (for example, gillnetting is expected to have higher β-values than longlining).

Based on similar reasoning as presented above, a better approximation – without introducing more parameters – can be obtained by assuming $\beta = 1/2$:

$$H(E,X) = q \cdot E \sqrt{X} \tag{3.4}$$

3.3 Properties of fishing gears

In Chapter 2 selected fishing gears are listed. Some gears are fishing randomly fish passing through the area (some traps and gill nets). Other gears aim to attract fish to swim into traps or onto hooks where they are captured (fish pots, longline, hand line, etc.), while other fishing gears actively move towards the fish to capture it (trawl, purse seine, Danish seine, etc.).

Gill nets represent gears which in principle aim to be invisible for the fish and therefore catch fish that randomly pass through the nets. In a sea of uniform distribution and random movement of fish (as illustrated in the two left panels of figure 3.1) the expected catch of a fishing gear will be constant, independent of where it is placed in the sea. In the case of gill nets the expected catch in the left panel is one quarter of the expected catch in the right panel, because the density of fish (number of black dots) is four times higher in the centre panel than in the left panel. This is expressed in a *stock-output elasticity* of one ($\beta = 1$) in equation 3.3.

Fishing gears like gill nets or traps have higher stock-output elasticities than other gears. When assuming random movement of fish, β will approach one. With a uniform distribution of fish biomass X and a perfectly invisible gill net (not affecting the fish's decisions on where to move), the gill net will catch a fixed proportion per square meter of net per unit
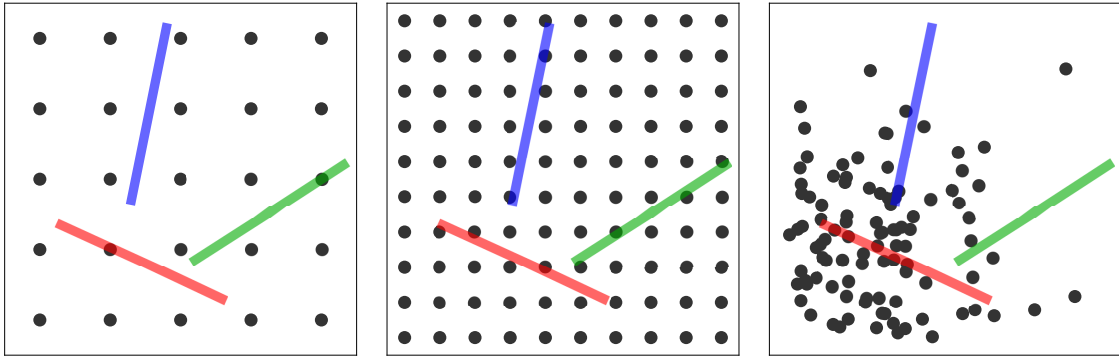
Figure 3.1: Three fishing gears (red, blue and green) placed in a sea of uniform fish distribution (black dots) with low (left panel) and high (centre panel) densities of fish, and in a non-uniformly distributed fish stock in the right panel. The number of dots (indicating fishes) is 25 to the left and 100 in the two other panels.

of time from the stock, depending on the probability of fish passing through the net and the rate of fish movement per unit of time. As seen above, the interpretation of this is that $\beta = 1$.

When these conditions are not met (uniform distribution of fish and invisible fishing gears), typically the β-value will be less than one. The right-hand panel in Figure 3.1 displays a situation of non-uniform fish distribution. Obviously, the red gear is expected to catch more fish than the blue or green gear.

The distribution of fish may be affected by the gear itself, such as when using baited hooks (longline or hand line). Fish will gather around the baited hook, locally increasing fish density. Hence, the chance of hooking a fish becomes higher than indicated by the mean fish density of the area.

Several studies of the Northeast Arctic cod fishery indicate β-values below 1. The findings of Hannesson (1983)[32] (covering the period 1971–78), Flaaten (1987)[22], and Eide et al. (2003)[20] (covering the period 1971–85, as in this study) all suggest β-values between 0.6 and 0.9 for gill nets, while all other fishing gears (trawl, Danish seine, longline, and hand line) have β-values below 0.5.

Most studies assume a priori $\alpha = 1$ for all gears, expecting catch production to be proportional to fishing effort. This seems to be a reasonable assumption, but empirical studies also indicate that the α-value may exceed one[20]. The interpretation of such findings could be higher efficiency when many are participating in fishing. By observing other fishing vessels, the fishers gain better knowledge about fish distribution and can reduce the time spent finding areas with high fish densities. However, the findings may also simply reflect the fact that fishing activity increases when fish availability is high. In that case, the a priori assumption $\alpha = 1$ could be a better suggestion than $\alpha > 1$.

Fishing gear properties, however, are not the only factors affecting the β-value. The spatial distribution of fish also has a significant impact on the stock-output elasticity (β). When fishing randomly with an invisible gear (e.g. the ideal gill net) in a uniformly distributed stock biomass X, then $\beta = 1$. The opposite scenario is a schooling stock, actively targeting the schools where fish density is constant, independent of the size of X (given that $X > 0$). In this idealised case, $\beta = 0$. In that scenario, Equation 3.3) simplifies to $H(E) = q \cdot E^{\alpha}$, where the produced harvest is independent of the stock size X.

Code box 3.3.1 — The most efficient fleet.
Let's consider two homogeneous fleets exploiting the same fish stock. The first fleet employs gill nets, while the second uses longlines. In both cases, the catch per unit of time is described by a Cobb-Douglas function (equation 3.3) where $\alpha = q = 1$. Therefore, the unit catch of effort for each fleet group is described by $H(E, X)/E = X^{\beta}$. We expect the β-value for gill nets to be higher than for longlines. Assuming a uniform distribution and random movement of each fish, as depicted in the two panels to the left in figure 3.1, we set the β-value for gill nets to be equal to 1 and 1/2 for longlines.

```
In[1]:= Plot[
        {x, Sqrt[x]}, {x, 0, 2},
        PlotTheme        -> "Detailed",
        PlotLegends      -> {"Gill net", "Longline"},
        PlotRangePadding -> None
        FrameLabel       -> {"Stock biomass (X)",
                             "Catch per unit of effort"}
    ]
```

Out[1]=
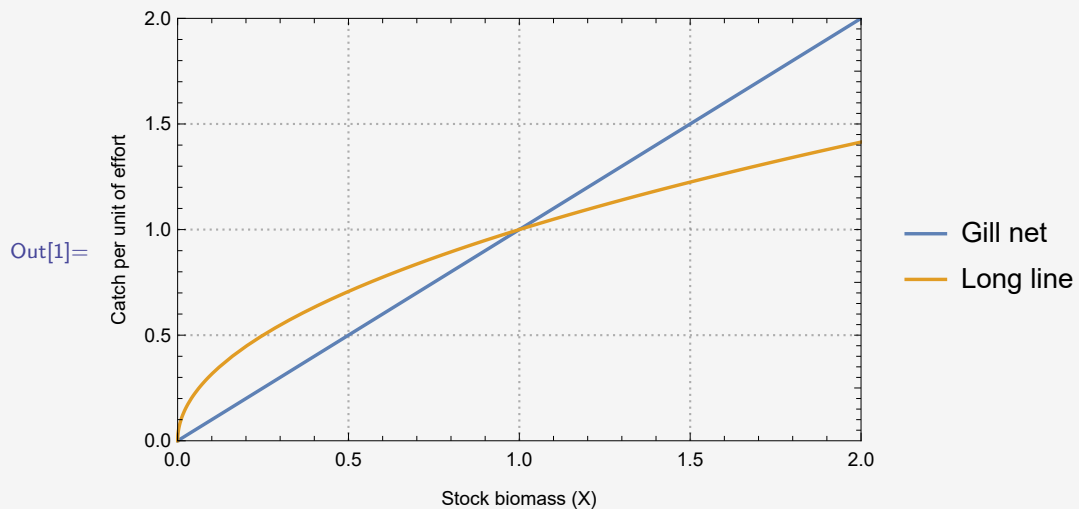
We observe that longlines catch more per unit of effort when $X < 1$, while gill nets catch more when $X > 1$. Building upon the discussion of output elasticities in the Cobb-Douglas equation (refer to code box 2.5.1), we understand that β represents the percentage increase in catch for a one percent increase in stock size. Consequently, when stock biomass increases, the catch by longlines experiences a smaller increment compared to the catch by gill nets. In conclusion, neither of the two fleets is inherently more efficient than the other; efficiency depends on the size of the stock.

The example provided in Code Box 3.3.1 demonstrates how certain gears, in this case longlines, compensate for reduced fish density in the exploited area by luring fish to gather in higher densities around the fishing gear using allurements like bait. However, the advantage of a gill net is that, being invisible and not relying on allurements, it may capture a greater share when the natural fish density increases, whereas the density around a line hook is less dependent on the natural density in the area.

> **Additional remarks 3.2 — Conventional gears and trawl.**
> Gill nets and other invisible traps (without bait) differ from other gears that aim to attract fish. However, they also differ from gears that catch fish by moving around. In fact, the production equation of such gears is closer to that of baited gears than to gill nets and invisible traps. Therefore, the stock-output elasticity (β in equation 3.3) is well below 1 for such gears, while it is closer to 1 for gill nets, assuming a close-to-uniform spatial stock distribution. As a result, the stock-output elasticities of, for example, trawl and longline gears are closer to each other than to the stock-output elasticity of gill nets.
>
> Hence, it may be worth discussing the effectiveness of the common classification of gears into *conventional gears* (most gears except trawls) and *trawls*. From the perspective of production properties, it might be more accurate to differentiate between *active gears* (those that move the gear or aim to move the fish) and *passive gears* (gears that randomly catch what is passing by).
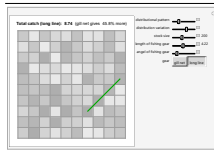
DEMO   Study fishing gear properties under different densities and distributions of fish::

http://demonstrations.wolfram.com/FishingWithLongLineOrGillNet/

3.4   Gear selection

While β expresses the relationship between fish densities and catch, the selective capacities of fishing gears are not captured by this parameter. Different fishing gears have varying abilities to retain fish of different sizes. The design and arrangement of fishing gears can alter their selective properties, making gear selection an important management tool. Gear regulations are among the oldest regulatory measures in fishing.

Figure 3.2 displays three different selection curves. The knife-edge selection is an idealised selection curve that simplifies the problem of gear selection in the models discussed in this book. The idealised curve could be considered as a special case of trawl selection (for simplicity, we refer to trawls even though the curve also has relevance for a number of other gears, such as Danish seine and purse seine). While selection in the idealised curve (actually a piece-wise linear curve) is immediate, there is a gradual transition over a range of fish lengths to achieve 0 to 100% selection in the more realistic curve. The range between L_{25} (25% retained) and L_{75} (75% retained) is often referred to as the *Selection Range (SR)*.

The logistic equation is often assumed to capture the selection pattern of trawls and similar gears. This is the inverse version of the *logit* function $log(x/(1-x))$ that follows a pattern very close to the *probit* function we are familiar with from probability theory and statistics. For the purpose of describing a pattern of selection, it is convenient to express the logistic function as

$$p(L) = \frac{e^{a+bL}}{1 + e^{a+bL}} \tag{3.5}$$
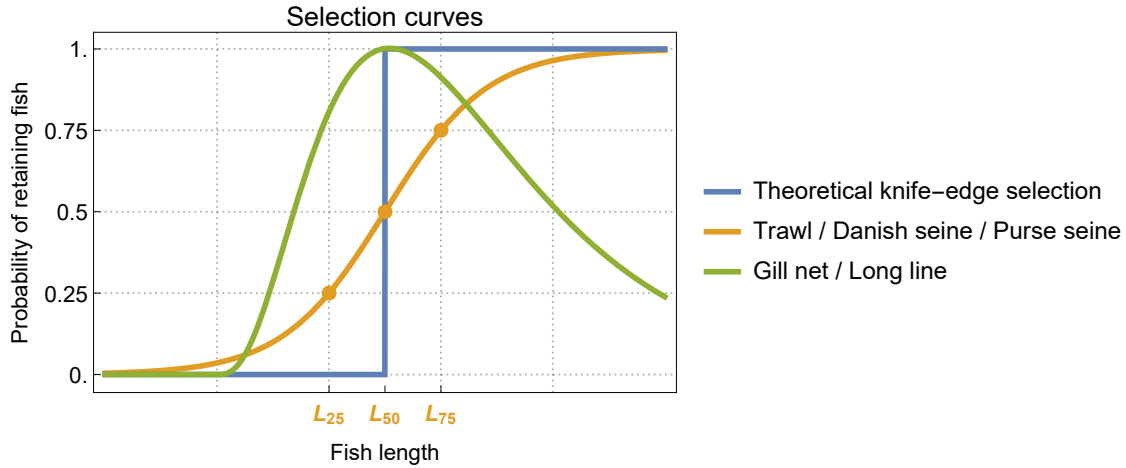
Figure 3.2: The figure illustrates typical selection curves for gill nets and trawl gears. The fish lengths L_{25}, L_{50}, and L_{75} refer to the curve representing trawl (indicated by the three points), reflecting the percentage of exposed fish retained and captured by the gear. However, this reference point is less useful when the selection curve has a declining trend with increasing length, as is the case for gill nets and longlines.

(see also code box 4.2.3 in the next chapter) where a and b are parameters defining L_{50}:

$$L_{50} = -\frac{a}{b} \tag{3.6}$$

SR is defined by the parameter b:

$$SR = \frac{2 \cdot log(3)}{b} \tag{3.7}$$

when we assume SR to cover the 50% between L_{25} and L_{75}. $p(L)$ in equation 3.5 expresses the probability of catch when the fish has length L. From the equations above we see that a and b are determined when L_{50} and SR are given:

$$a = \frac{2 \cdot L_{50} \cdot log(3)}{SR}$$
$$\tag{3.8}$$
$$b = -\frac{a}{L_{50}}$$

Exercises

Exercise 3.1 Provide alternative explanations for why effort-output elasticities (α in equation 3.3) could exceed one. ∎

4. Modelling population dynamics

4.1 Basic principles

A group of individuals of the same species living in a specific geographic area is referred to as a population. Populations may be separated into smaller sub-populations (referred to as stocks) in minor geographic areas. Population dynamics describe how a population or stock grows due to the genetic properties of the population/stock and the environmental constraints within the area where the population/stock is living. In the following, we will not differentiate between populations and stocks.

Population growth is the net effect of individual growth, recruitment, and mortality during a period of time. All these factors depend both on the biological properties of the species and on the physical and biological environment. The size of a population could be measured in terms of the number of individuals (the common measure in the human population) or in terms of weight (the common measure for fish stocks). The latter is often referred to as the stock's biomass, which is the total weight of the stock in nature.

The science of population dynamics originates from the same public discourse that resulted in the disciplines of economics and demographics at the end of the eighteenth century. Several of the mathematical demographic models we will discuss in this chapter originate from that period, at that time focusing on the growth of the human population.
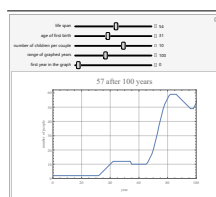
DEMO Study how a human population grows according to reproduction properties and generation time at

http://demonstrations.wolfram.com/
OffspringOfAdamAndEve/

4.2   Classical surplus growth models

Of course, people had been interested in and fascinated by population growth long before
the eighteenth century. One example is the Fibonacci numbers, described by an Italian
mathematician (Leonardo of Pisa Bonacci, about 1170 – about 1250) during the early
years of the thirteenth century (the series, however, has been known since ancient times).

The Fibonacci numbers describe biological growth starting with a newborn pair of rabbits
(1). At the end of the first month, they mate and a new pair is born at the end of the
second month. Then there are two pairs. The first pair gives birth to a new pair every
month (mortality is not considered in this model), while the second pair starts mating like
the first pair after their first month. The Fibonacci series then gives the monthly number
of pairs of rabbits.

Code box 4.2.1 — Fibonacci numbers.

In *Mathematica* you have an internal function listing the Fibonacci numbers:

```
In[1]:= Table[Fibonacci[n], {n, 20}]

Out[1]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,
         610, 987, 1597, 2584, 4181, 6765}
```

```
In[2]:= ListLinePlot[%1, Mesh -> All, PlotRange -> All, Filling -> Axis]
```

Out[2]=
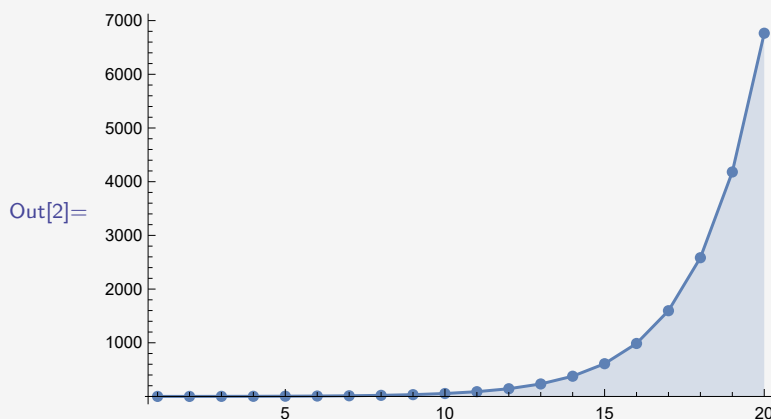
Now have a look at the ratio between two and two consecutive numbers

```
In[3]:= Fibonacci[# + 1]/Fibonacci[#] & /@ Range[20]
```

$$Out[3]= \left\{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \right.$$
$$\left. \frac{987}{610}, \frac{1597}{987}, \frac{2584}{1597}, \frac{4181}{2584}, \frac{6765}{4181}, \frac{10946}{6765} \right\}$$

The numerical values reveal that the ratios soon approach the same value while moving
upwards in the series.

```
In[4]:= N[%]
```

```
Out[4]= {1., 2., 1.5, 1.66667, 1.6, 1.625, 1.61538, 1.61905, 1.61765,
         1.61818, 1.61798, 1.61806, 1.61803, 1.61804, 1.61803, 1.61803,
         1.61803, 1.61803, 1.61803, 1.61803}
```

Let's examine the numerical value of the built-in function `GoldenRatio`, and we obtain the same number.

```
In[5]:= GoldenRatio // N
```

```
Out[5]= 1.61803
```

```
In[6]:= Show[{
           ListLinePlot[%3, PlotRange -> All, Mesh -> All],
           Plot[GoldenRatio, {n, 0, 20}, PlotStyle -> Red]
        }]
```
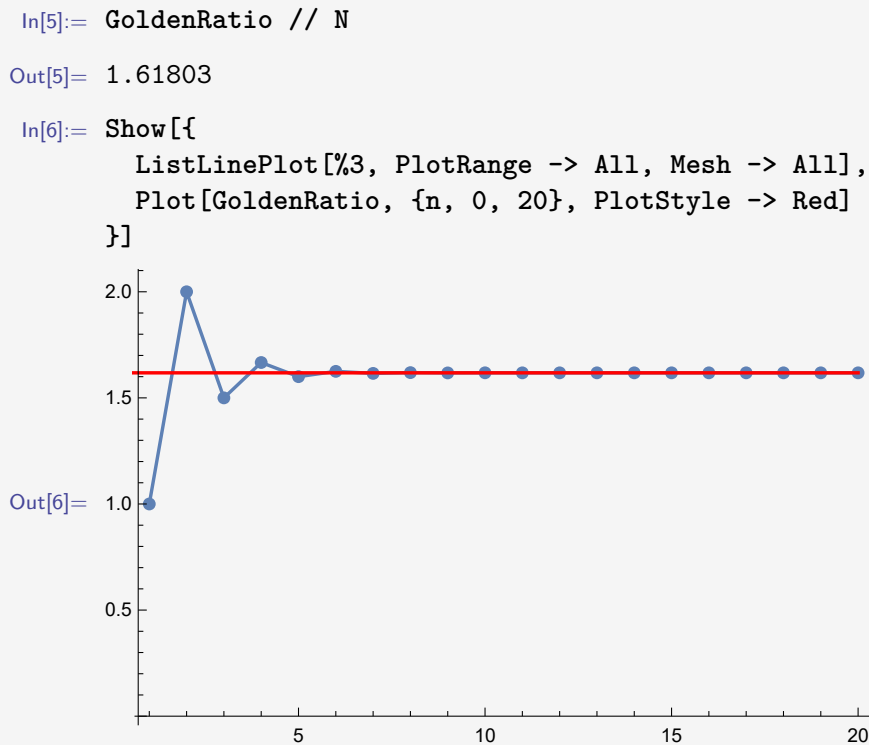

The blue curve (`Out[3]`) coincides with the red curve (the golden ratio, `Out[4]`). Therefore, the golden ratio could be directly obtained from the Fibonacci numbers:

```
In[7]:= Limit[Fibonacci[n + 1]/Fibonacci[n], n -> Infinity]
```

$$Out[7]= \frac{1}{2} \left(1 + \sqrt{5}\right)$$

Let us conclude this session on Fibonacci numbers by exploring one of the numerous examples of how these numbers create beautiful patterns in nature. Here, we utilise the golden ratio to illustrate patterns found in sunflower heads and compare them with a real flower:

```
In[8]:= GraphicsRow[{
          Show[
            Graphics[{
              Lighter@Yellow, AbsolutePointSize[3 + #^(1/3)/2],
              Point[
                Sqrt[#]{Cos[2 Pi # GoldenRatio],Sin[2 Pi # GoldenRatio]}
              ]}
            ] & /@ Range[360],
            Graphics[{Lighter@Yellow, Annulus[{0, 0}, {19, 30}]}]},
            Background -> Darker@Brown,
            PlotRange -> {{-20, 20}, {-20, 20}} ],
          ImageCrop[Import["https://c1.staticflickr.com/2/
            1278/694780262_8874b4f225_b.jpg"], {600, 600}]
        }]
```

Out[8]=
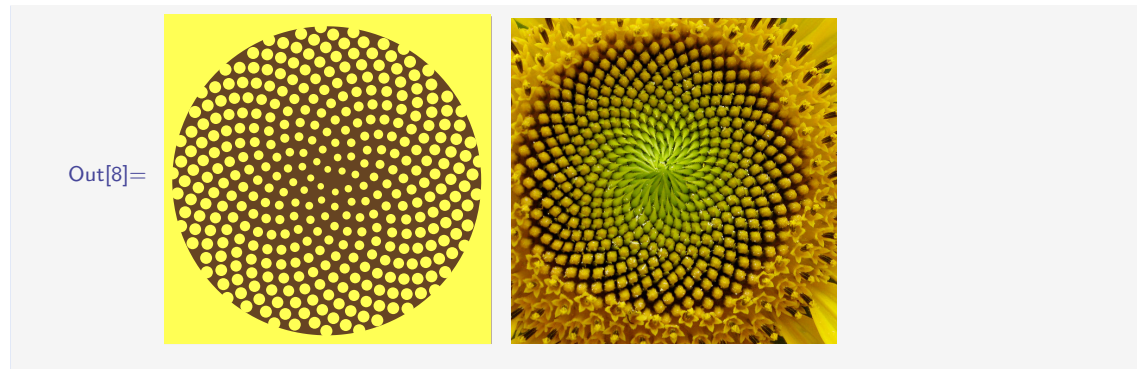
An interesting feature of the series is that each number is equal to the sum of the two previous numbers. At first sight, the series may appear to be rather randomly constructed, but it follows a strict rule where the population of rabbits after only a few months reaches astronomical values. The series is a simple recruitment model and does not include growth in biomass and mortality. However, it demonstrates the exponential power of population growth, which was also discovered in European countries during the eighteenth century.

Code box 4.2.2 — Gompertz' population growth model.
The Gompertz equation 4.1 is solved using *Mathematica* by providing a value for the initial stock biomass (X_0, implemented below as x0). The solution includes a message indicating that other solutions may exist in the general case of this equation.

```
In[1]:= DSolve[
         {x'[t] == -r x[t] Log[x[t]/k], x[0] == x0}, x[t], t
       ][[1, 1]]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[1]= $x[t] \to k \left(\dfrac{x0}{k}\right)^{e^{-r\,t}}$

Plotting equation 4.1 (left below) and its solution (right below) for some given parameter values ($r = 0.5$, $K = 1000$ and $X_0 = 10$)

```
In[2]:= GraphicsRow[{
         Plot[-.5 x Log[x/1000], {x, 0, 1000}],
         Plot[1000 (10/1000)^Exp[-.5 t], {t, 0, 12}]
       }]
```

Out[2]=
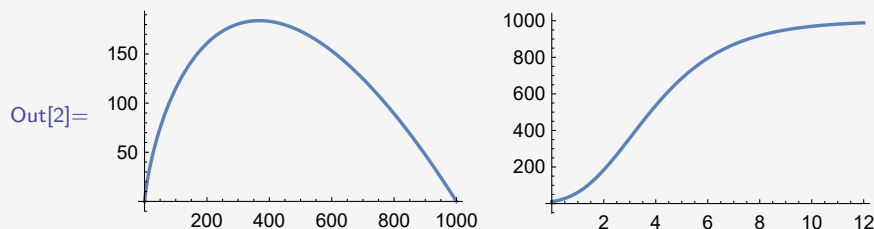
The British scholar Thomas Robert Malthus (1766 - 1834) claimed that the exponential increase in the human population throughout Europe in the eighteenth century would

eventually be repressed at a maximum level (the level of subsistence) where the human population would suffer at the edge of nature's capacity to sustain such a large human population.

Malthus' ideas became very influential but were also criticised by many. The idea that there existed an upper limit for how large a human population could be, and that this limit was related to environmental constraints (e.g. food), was, however, shared by most of the critics.

Benjamin Gompertz (1779 – 1865) published a mathematical demographic model in 1825[28]. When replacing the variable (number of people) with a stock's biomass (X), the Gompertz' growth equation gives the time derivative of the stock biomass:

$$\dot{X}(t) = \frac{dX(t)}{dt} = -r \cdot X(t) \cdot \ln\left(\frac{X(t)}{K}\right) \tag{4.1}$$

where r is a growth rate while K is the environmental carrying capacity level. Since the time derivative is the per unit of time increment in the stock biomass, it provides a straightforward biological interpretation of the speed by which the stock grows towards its natural equilibrium K. As seen from the plot of equation 4.1 in Code box 4.2.2, the point of maximum growth is placed to the left of $K/2$.

Another demographic model, published only a few years after the Gompertz model and with an even greater scientific impact, was suggested by Pierre François Verhulst (1804 - 1849) in 1838 [54]. It is commonly referred to as the logistic growth equation, and in line with the expression above, it may be written as:

$$\dot{X}(t) = r \cdot X(t)\left(1 - \frac{X(t)}{K}\right) \tag{4.2}$$

As seen from equation 4.2 (and illustrated in Code box 4.2.3), the time derivative of the logistic growth describes a parabolic curve. The inflection point of $X(t)$ is therefore found for $X(t) = K/2$. After Raymond Pearl (1879 – 1940) reintroduced the logistic growth curve as a biomass surplus growth model[41], it has become the most common surplus growth model used in mathematical biology and fisheries economics.


Code box 4.2.3 — Verhults population growth model.
The Verhulst equation 4.2 is solved using *Mathematica* by providing a value for the initial stock biomass (X_0, represented below as x0). The solution includes a message indicating that other solutions may exist in the general case of this equation.

```
In[1]:= DSolve[{x'[t]== r x[t](1-x[t]/k), x[0]== x0}, x[t], t][[1, 1]]
```

> Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\text{Out[1]= } x[t] \rightarrow \frac{e^{rt}\ k\ x0}{k - x0 + e^{rt}\ x0}$$

Plotting equation 4.2 (left below) and its solution (right below) for some given parameter values ($r = 0.5$, $K = 1000$ and $X_0 = 10$)

```
In[2]:= GraphicsRow[{
        Plot[.5 x (1 - x/1000), {x, 0, 1000}],
        Plot[(Exp[.5t]1000 * 10)/(1000 - 10 + Exp[.5t]10), {t, 0, 20}]
        }]
```

Out[2]=
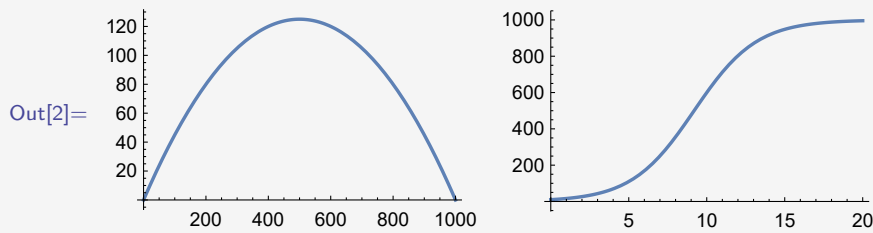
The dynamics of the logistic function have been studied extensively in numerous papers, and the chaotic properties of the function are well-known. The discrete version of the model exhibits dynamic changes, particularly depending on the value of the intrinsic growth parameter r. To investigate how the growth pattern changes according to the value of r, the discrete model is plotted below for three values of r: 0.5, 2.5, and 3.0. The plots illustrate how the stock biomass (X) varies over time (t, the horizontal axis).

```
In[3]:= GraphicsRow[Table[
          ListLinePlot[
            NestList[# + r # (1 - #/1000.) &, 100, 50],
            PlotStyle -> Directive[{Thickness[.01], Red}],
            PlotRange -> {0, 1400}
          ],
          {r, {.5, 2.5, 3}}
        ]]
```

Out[3]=
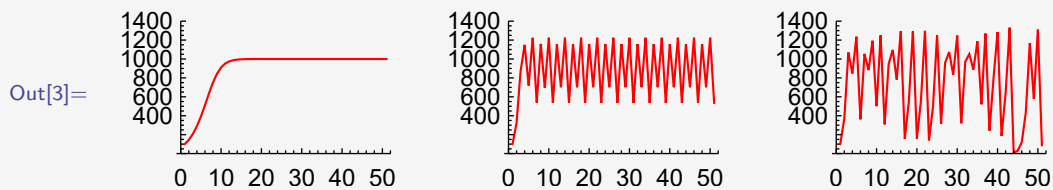
The dynamics described above can be presented using a cobweb diagram, as shown below. Cobweb diagrams display changes over time in $x(t) - x(t+1)$ axis systems. The yellow lines in the diagram correspond to $x(t) = x(t+1)$ (equilibrium).

```
In[4]:= GraphicsRow[Table[
          Show[{
            Plot[{x+r x(1-x/1000.), x}, {x,0,1400}],
            ListLinePlot[(
              nested = NestList[#+r # (1-#/1000.)&, 100, 50];
              nested = Riffle[
                {#, #}& /@ nested,
                Table[Take[RotateLeft[#,i],2], {i, 0, 49}]& @ nested
              ]),
              PlotRange -> All
            ]},
            AspectRatio -> 1,
            PlotStyle   -> Directive[{Thickness[.01], Red}],
            PlotRange   -> {0,All}
          ],
          {r, {.5, 2.5, 3}}
        ]]
```

Out[4]=
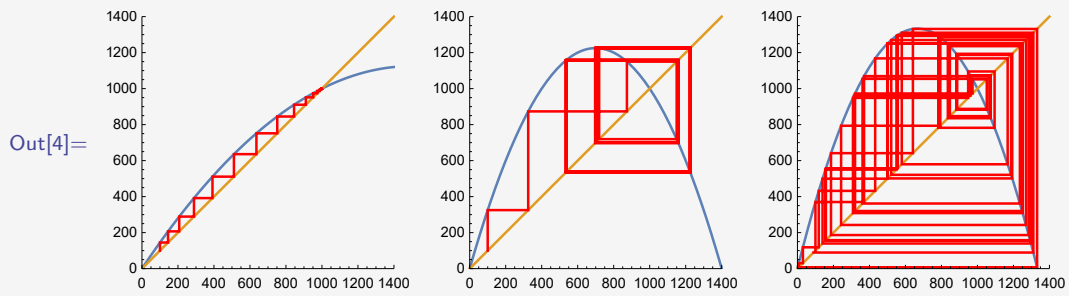
While the left-hand figure reflects smooth growth towards the equilibrium ($X = K$, in this case the value 1000), the middle graph ($r = 2.5$) displays that the equilibrium is an eight-periodic orbit around a stable focus (which, in continuous time, corresponds to a two-period limit cycle), while the right-hand figure displays a chaotic situation.

In 1959, F. J. Richards (1901 -– 1965) published a model in which the two previously presented models (equations 4.1 and 4.2) are special cases, referring to it as *a flexible growth function for empirical use*[44]. Consistent with the previous equations, we express the Richards function as follows:

$$\dot{X}(t) = r \cdot X(t)\left(1 - \left(\frac{X(t)}{K}\right)^{m-1}\right), \tag{4.3}$$

introducing a third parameter (m) that shifts the inflection point of the growth curve upwards or downwards depending on its value. It is easy to observe that equation 4.3 is equivalent to equation 4.2 when $m = 2$. It is also possible to demonstrate that equation 4.3 approaches equation 4.1 as m approaches one. Hence, both the Gompertz growth and the Verhulst growth equations are special cases of the Richards growth equation in 4.3. The graph shown in Code box 4.2.4 illustrates how the m parameter in the Richards growth equation determines whether the growth curve should shift to the left or the right.

Code box 4.2.4 — Surplus growth models.
This session makes use of the package `PopulationGrowth` (freely available at http://site.uit.no/econmult/). If the package is found by *Mathematica* in its file system, it is loaded by the command:

```
In[1]:= Needs["EconMult`PopulationGrowth`"]
```

This lists four surplus production models available in the package:

```
In[2]:= Grid[
          Text /@ {#,
            Notation@SurplusProduction[UseMSY->False, GrowthModel->#],
            SimplifyNotation@SurplusProduction[GrowthModel -> #]
          } & /@ $SurplusProductionModels,
          Frame      -> All,
          Alignment -> Left
        ]
```

Out[2]=

VerhulstSchaefer	$r\,X\left(1-\frac{X}{K}\right)$	$\frac{4\,\text{MSY}\,X\,(K-X)}{K^2}$
GompertzFox	$-r\,X\log\!\left(\frac{X}{K}\right)$	$\frac{e\,\text{MSY}\,X\,(\log(K)-\log(X))}{K}$
RichardsPellaTomlinson	$r\,X\left(1-\left(\frac{X}{K}\right)^{m-1}\right)$	$\dfrac{\text{MSY}\left(X-K\left(\frac{X}{K}\right)^m\right)}{K\left(m^{\frac{1}{1-m}}-m^{\frac{m}{1-m}}\right)}$
QuasiBevertonHolt	$r\,X\left(1-\left(\frac{X}{K}\right)^{\frac{b}{b+1}-1}\right)$	$\dfrac{\left(\frac{1}{b}+1\right)^b (b+1)\,\text{MSY}\left(K\left(\frac{X}{K}\right)^{\frac{b}{b+1}}-X\right)}{K}$

In[3]:=
```
Show[(
  Plot[
    SurplusProduction[
      GrowthModel                         -> RichardsPellaTomlinson,
      CurrentBiomass                      -> x,
      UseMSY                              -> True,
      MaximumSustainableYield             -> 100,
      CatchabilityCoefficient             -> 1,
      BiomassMaximum                      -> 1000,
      RichardsPellaTomlinsonParameter -> #
    ], {x, 0, 1000},
    PlotRange       -> {0, 100},
    Frame           -> True,
    FrameLabel      -> {"Stock size",
      "Natural growth per unit of time"},
    PlotRangePadding -> None
  ] &) /@ Table[Exp[.001 + .1 * i^3] - 1, {i, 0, 4, .5}]
]
```

Out[3]=
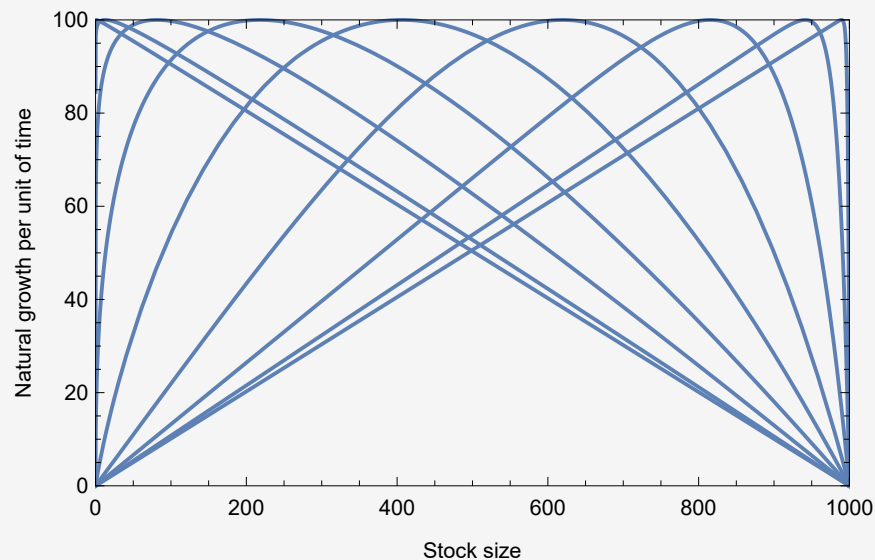
DEMO
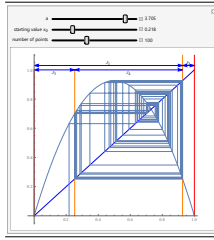
 Investigate the cobweb map of a logistic growth equation:

http://demonstrations.wolfram.com/
AnIntervalEventuallyBoundingTrajectoriesOfTheLogisticMap/

4.3   Depensatory growth

The models introduced in section 4.2 above are all included in the class of compensatory growth models. The term *compensatory* refers to the fact that in these models, the relative growth rate $(\dot{X}(t)/X(t))$ increases as the stock biomass decreases. Biomass losses lead to increasing compensation per unit of biomass.

However, this kind of compensating behavior in the stock may not always be the case in real life. The relative growth rate may increase as the stock biomass decreases up to a certain level, after which the relative growth rate decreases. In some cases, it may even become negative. These scenarios are referred to as critical depensation, defining a critical biomass level below which the stock will go extinct.

A depensatory growth model can be specified based on the logistic equation 4.2 by adding a term that includes a depensation parameter D:

$$\dot{X}(t) = r \cdot X(t)\Big(1 - \frac{X(t)}{K}\Big) \cdot \frac{X(t) - D}{K - D}$$

which results in

$$\dot{X}(t) = \frac{r \cdot X(t) \cdot (K - X(t)) \cdot (X(t) - D)}{K \cdot (K - D)} \tag{4.4}$$

If the depensation parameter D is negative, the depensation is non-critical, while a positive depensation parameter indicates a critically low biomass level. Using *Mathematica*, it is easy to show that when $D = -\infty$, equation 4.2 is restored:

In[1]:= `Limit[(r x (k - x) (x - d))/(k (k - d)), d -> -Infinity]`

Out[1]= $\dfrac{r\ (k\ -\ x)\ x}{k}$

Most of the examples provided in this textbook assume compensatory growth. Shifting to depensatory growth may significantly alter the conclusions of the compensatory growth models. If the stock's ability to strive for increased growth per biomass unit with declining stock size is weakened, the consequences of overfishing may be much more severe.

In the next section, age structure models are introduced. Walters et al. (2008)[56] discuss the relationship between depensatory growth in surplus production models versus age-structured models and claim that depensation is more commonly included in age-structured models.

Code box 4.3.1 — Depensatory growth.

Depensatory growth as expressed in equation 4.4 includes both critical and non-critical depensation levels. The plot of equation 4.4 (below) for different D-values illustrates this. The chosen colours range from non-critical (blue) to critical (red) depensation levels.

```
In[1]:=  Show[
           Plot[
             r x(k-x)(x-d))/(k(k-d)) /. {r->.5, k->1000, d->#},
             {x, 0, 1000},
             PlotStyle -> Hue[.8 - Abs[Log[2 + #/3000]]]
           ] & /@ {-50000, -3000, -1000, -300, 0, 200, 400},
           PlotRange -> All,
           AxesLabel -> {"X", "dX/dt"}
         ]
```
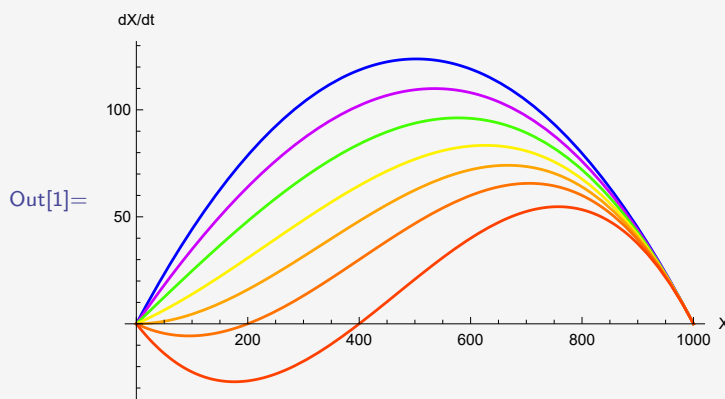

Two of the D-values above are positive and hence define critical biomass levels, corresponding to stock sizes of 200 and 400. The blue curve ($D = -50{,}000$) approaches the logistic growth function (obtained when $D = -\infty$).

The plot below displays the relative growth (or average growth; surplus production per unit of biomass) for the D-values included in the aforementioned plot.

```
In[1]:=  Show[
           Plot[
             r x(k - x)(x - d))/(k(k - d))/x /. {r ->.5, k ->1000, d ->#},
             {x, 0, 1000},
             PlotStyle -> Hue[.8 - Abs[Log[2 + #/3000]]]
           ] & /@ {-50000, -3000, -1000, -300, 0, 200, 400},
           PlotRange -> All,
           AxesLabel -> {"X", "(dX/dt)/X"}
         ]
```
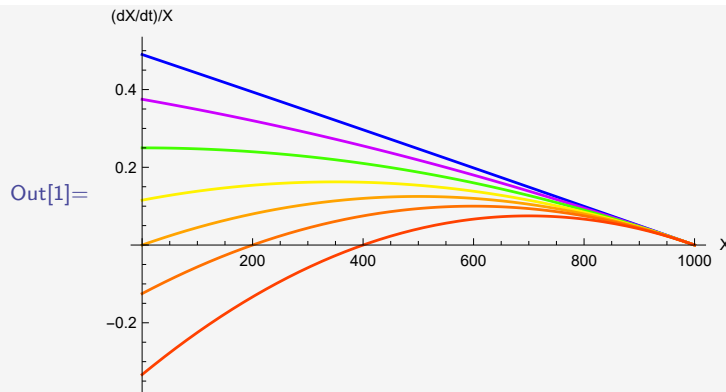
We observe that the two orange or red curves, which represent the cases of critical depensation, become negative at sufficiently low D-values. Other cases also indicate depensation, but not all of them. The blue and magenta coloured curves clearly show an increasing trend as D-values decrease throughout the entire range. This also appears to be the case for the green curve.

The plot below illustrates the marginal values of the relative growth concerning stock biomass X. This plot confirms that the blue, magenta, and green curves indeed belong to the class of compensatory growth models, while the other cases fall into the category of depensatory growth models. We reach this conclusion based on the fact that the first three cases do not enter the positive region of marginal relative growth. Among the depensatory growth cases, the orange (red) curves represent critical depensatory growth, whereas the other two (the yellow curves) represent depensatory growth without any critical biomass values above zero.

```
In[1]:= Show[
          Plot[
            D[r x(1 - x/k)(x - d)/(k - d)/x, x] /.
            {r -> .5, k -> 1000, d -> #, x -> y}, {y, 0, 1000},
            PlotStyle -> Hue[.8 - Abs[Log[2 + #/3000]]]
          ] & /@ {-50000, -3000, -1000, -300, 0, 200, 400},
          PlotRange -> All, AxesOrigin -> {0, 0},
          AxesLabel -> {"X", "-(dX/dt)/X^2"}
        ]
```
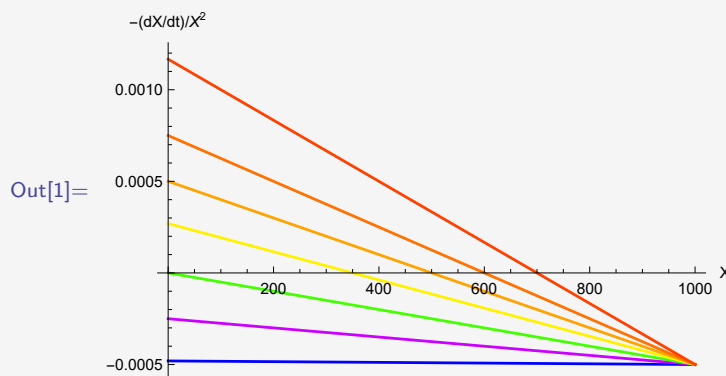

The points where the curves intersect the horizontal axis correspond to the maximum values of the relative growth. While the green curve reaches zero at $D = 0$, the magenta and blue curves do not attain positive values for $D \geq 0$.

Analytically, it is possible to prove that the range of D-values for compensatory growth extends from $-\infty$ to $-K$, non-critical depensatory growth spans from $-K$ to 0, and the critical range lies in the interval of D-values between 0 and K, which represents the maximum biomass level.

4.4 Age structured model

Age-structured population models decompose the population biomass into different age components, each of which is the product of the average individual weight in the age group and the number of individuals. Changes in the age composition of the stock will affect the stock biomass development in ways not necessarily captured by the surplus production models presented in section 4.2.

Age-structured models are often referred to as cohort models, a more general term that allows the stock to be structured in ways other than by year classes, which is the usual structure. For some species, however, structuring by other time intervals (e.g., by month) or more aggregated groups (e.g., mature and immature individuals) is preferable to structuring by year classes.

In 1934, Karl Ludwig von Bertalanffy (1901 – 1972) introduced an individual length growth model for asymptotic growth towards a maximum length (L_∞) with increasing age[6]. Let $L(t)$ be the individual length at age t, k be the length growth rate, and t_0 be the theoretical age of zero length ($L(t_0) = 0$). The von Bertalanffy equation is then given by

$$L(t) = L_\infty\big(1 - e^{-k(t-t_0)}\big) \tag{4.5}$$

Weight growth is closely related to length growth. Let parameter b represent the weight/length relationship, and d be a scaling factor. The individual weight at age t, $W(t)$, is then given by

$$W(t) = d \cdot L(t)^b \tag{4.6}$$

Maximum individual weight is also defined by equation 4.6, $W_\infty = d \cdot L_\infty^b$. Inserting equation 4.5 into equation 4.6 then gives[1]

$$W(t) = W_\infty\big(1 - e^{-k(t-t_0)}\big)^b \tag{4.7}$$

Additional remarks 4.1 — Parameter relations.

Some will emphasise that moving from aggregated surplus production models to cohort models provides clearer biological interpretations of the model parameters.

The growth rate k represents the percentage increment in length in relation to the difference between maximum length and current length. This results in a constant percentage of a diminishing difference, leading to length growth approaching zero as the length approaches L_∞.

$$k = \frac{\dot{L}(t)}{L_\infty - L(t)}$$

b represents the ratio between the percentage growth in weight and the percentage growth of length at the same age. This ratio is assumed to be constant and is usually close to 3, reflecting

---

[1]Taking the time derivative of this function shows that the function is actually a Richards equation (equation 4.3).

the cubic expansion of length.

$$b = \frac{\dot{W}(t)}{W(t)} \Big/ \frac{\dot{L}(t)}{L(t)}$$

d is a simple scaling parameter whose value depends on the units in which weight and length are measured.

$$d = \frac{W_\infty}{L_\infty{}^b}$$

The mortality rate Z represents a constant percentage decline in the number of individuals by age. Based on the relations above, the mortality rate can also be expressed in terms of the percentage increment in weight minus the percentage increment in biomass in a cohort.

$$Z = -\frac{\dot{N}(t)}{N(t)} = \frac{\dot{W}(t)}{W(t)} - \frac{\dot{x}(t)}{x(t)} = b \cdot \frac{\dot{L}(t)}{L(t)} - \frac{\dot{x}(t)}{x(t)}$$

While the individual weight increases with age, the number of individuals in a cohort decreases over time due to natural mortality (predation, age, and diseases). The standard mortality model was first proposed by Baranov in 1918[3]. If R is the initial number of recruits in a cohort (at the age of recruitment, t_R), and the mortality rate is Z, then the number of individuals in the cohort at time t, $N(t)$, is

$$N(t) = R \cdot e^{-Z(t-t_R)} \tag{4.8}$$

The product of equations 4.6 and 4.8 gives the total biomass of the cohort in question:

$$x(t) = N(t) \cdot W(t) = R \cdot W_\infty \cdot \left(1 - e^{-k(t-t_0)}\right)^b \cdot e^{-Z(t-t_R)} \tag{4.9}$$

When assuming constant recruitment (R) and cohort biomass growth as in equation 4.9, the equilibrium biomass of the total stock at time τ is

$$X(\tau) = \int_{t=0}^{t\infty} x_\tau(t)dt = R \cdot W_\infty \int_{t=0}^{t\infty} \left(1 - e^{-k(t-t_0)}\right)^b \cdot e^{-Z(t-t_R)} \cdot dt \tag{4.10}$$

where $x_\tau(t)$ is the biomass of the cohort of age t at time τ, described by equation 4.9. Usually, recruitment is considered a discrete process in time, which should change the integral in equation 4.10 to a sum corresponding to the last part of code box 4.4.1. It follows from equations 4.9 and 4.10 that

$$\dot{X}(\tau) = x(\tau) \tag{4.11}$$

Code box 4.4.1 — Biomass of one cohort and all cohorts.
This section utilises the `PopulationGrowth` package (which is freely available at *http://www.maremacentre.co* If *Mathematica* locates the package within its file system, it can be loaded using the command:

In[1]:= **Needs["EconMult`PopulationGrowth`"]**

The von Bertalanffy equation (equation 4.7) is implemented in the package as:

In[2]:= `IndividualWeight[t] // Notation`

Out[2]//TraditionalForm=

$$W_\infty \left(1 - e^{-k\,(t-t_0)}\right)^b$$

By default, the package assumes the total mortality rate (Z, as in equation 4.8) to be the sum of F and M, which represent the fishing and natural mortality rates, respectively (as seen in equation 5.1). The implementation of the Baranov equation (equation 4.8) is therefore accomplished by:

In[3]:= `IndividNumbers[t] // Notation`

Out[3]//TraditionalForm=

$$R\,e^{-(F+M)\,(t-t_R)}$$

The product of the number of individuals (N) and the individual weight (W) gives the biomass of the cohort (as shown in equation 4.9):

In[4]:= `CohortBiomass[t] // Notation`

Out[4]//TraditionalForm=

$$R\,W_\infty \left(1 - e^{-k\,(t-t_0)}\right)^b\,e^{-(F+M)\,(t-t_R)}$$

Maximum biomass of cohort is found by

In[5]:= `Solve[CohortBiomass'[t] == 0, t] // SimplifyNotation`

> Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[5]//TraditionalForm=

$$\left(t \to \frac{\log\left(\frac{b\,k}{F+M}+1\right)}{k} + t_0\right)$$

This value of t is implemented in the `PopulationGrowth` package as `AgeOfMaxGrowth`, which represents the age at which the cohort reaches its maximum biomass.

In[6]:= `AgeOfMaxGrowth[] // Notation`

Out[6]//TraditionalForm=

$$\frac{\log\left(\frac{b\,k}{F+M}+1\right)}{k} + t_0$$

Maximum biomass of cohort could then be found by

In[7]:= `CohortBiomass[AgeOfMaxGrowth[]] // Notation`

Out[7]//TraditionalForm=

$$R\,W_\infty \left(1 - \frac{1}{\frac{b\,k}{F+M}+1}\right)^b \exp\left(-(F+M)\left(\frac{\log\left(\frac{b\,k}{F+M}+1\right)}{k} - t_R + t_0\right)\right)$$

The maximum biomass is implemented in the package as `MaximumBiomassGrowth`. This test verifies that `MaximumBiomassGrowth` is indeed equivalent to `CohortBiomass[AgeOfMaxGrowth[]]`:

```
In[8]:= MaximumBiomassGrowth[] === CohortBiomass[AgeOfMaxGrowth[]]

Out[8]= True
```

We assume certain numerical values to parameterize the model. The previously mentioned parameters are represented in the package using descriptive names: `InitialAge` (t_0), `WeightLengthRelation` (b), `MaxWeight` (W_∞), `GrowthRate` (k), `MortalityRate` (M, a component of Z), `FishingMortalityRate` (F, a component of Z), `Recruits` (R), and `RecruitmentAge` (t_R). Some parameters will be introduced later: `OldestAge` (age of the oldest cohort in the stock, t_∞) and `CatchAge` (age of the first catch, t_c).

```
In[9]:= values = {
           InitialAge          -> 0,
           WeightLengthRelation -> 3,
           MaxWeight           -> 10,
           GrowthRate          -> .2,
           MortalityRate       -> .2,
           FishingMortalityRate -> F,
           Recruits            -> 1,
           RecruitmentAge      -> 0,
           CatchAge            -> tc,
           OldestAge           -> Infinity};
```

Without fishing, the cohort has its maximum biomass at an age close to seven years.

```
In[10]:= AgeOfMaxGrowth[Sequence @@ values, Fishing -> False]

Out[10]= 6.93147
```

This provides a graphical illustration of how the biomass develops over the lifespan of the cohort, assuming no fishing:

```
In[11]:= Plot[
           CohortBiomass[t, Sequence @@ values, Fishing -> False],
           {t, 0, 20},
           AxesLabel -> {"Biomass of cohort", "Age"}
         ]
```
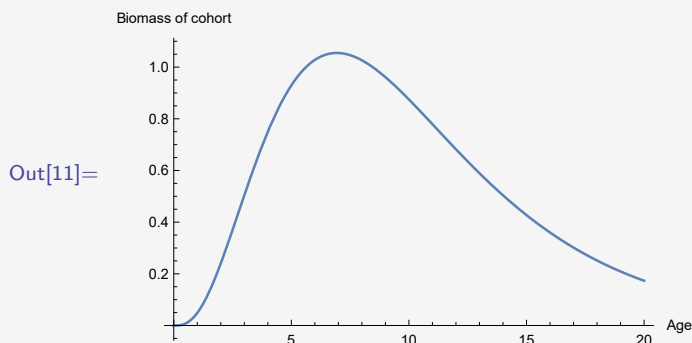

Now, assume that a similar cohort is recruited to the stock every year after the first one:

```
In[12]:= Show[
           Plot[
```

```
        CohortBiomass[
          t - #,
          Sequence @@ values,
          Fishing -> False
        ], {t, 0, 30},
        AxesLabel -> {"Year", "Biomass of cohort"},
        PlotRange -> {0, All}
      ] & /@ Range[0, 30]
    ]
```

Out[12]=
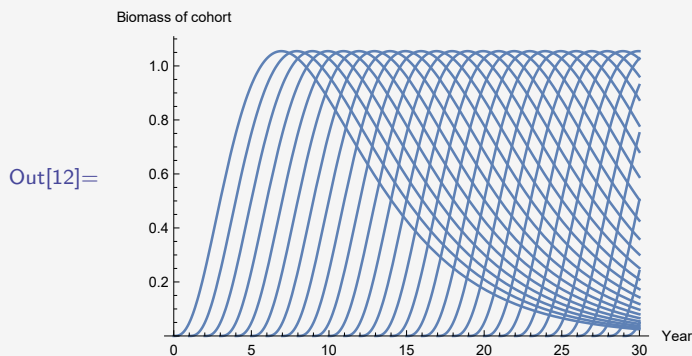
By summing the biomass of all cohorts for each year, we can observe how the stock biomass grows from the first cohort to a fully recruited stock. Negative biomass values are not feasible and are disregarded by utilising the Max function, assuming zero as the lowest possible biomass value:

```
In[13]:= ListLinePlot[
        Total /@ Table[
          Max[
            CohortBiomass[
              t - i, Sequence@@values, Fishing->False
            ], 0
          ], {t, 0, 30}, {i, 0, 30}
        ],
        AxesLabel -> "Year", "Stock biomass"
      ]
```

Out[13]=
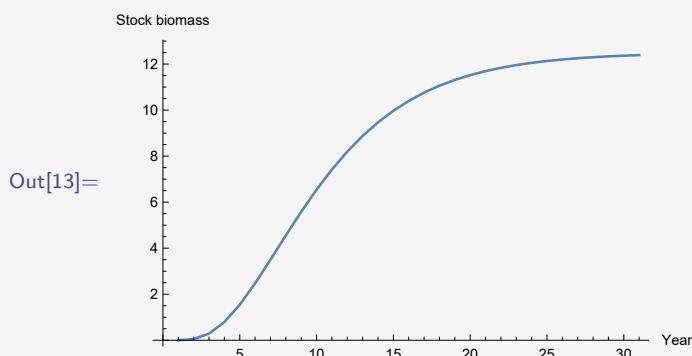
Since this curve is the integral of the cohort biomass curve, the surplus production graph of the age-structured model is obtained by combining the two graphs:

```
In[14]:= ParametricPlot[
```

```
        {PopulationBiomass[t, Sequence @@ values, Fishing -> False],
         CohortBiomass[t, Sequence @@ values, Fishing -> False]},
        {t, 0, 50},
        AspectRatio -> 1/GoldenRatio,
        AxesLabel   -> {"Stock biomass", "Surplus production"}
    ]
```

Out[14]=
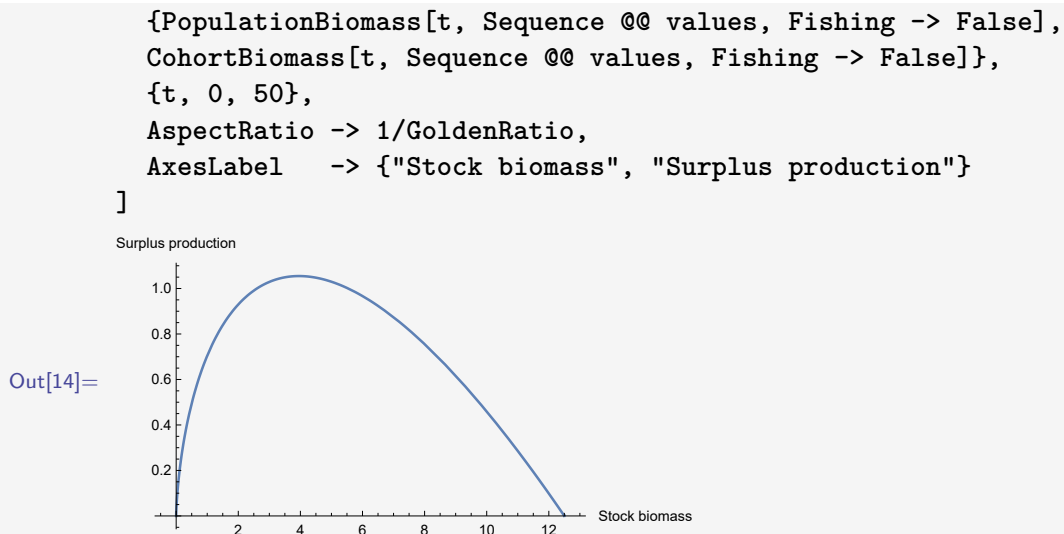
Since the mortality rate and the growth rate in our case are identical, the curve above follows the growth of the "QuasiBevertonHolt" model in the package (stock biomass is denoted as X):

```
In[15]:= SurplusProduction[
        CurrentBiomass -> X,
        GrowthModel     -> "QuasiBevertonHolt",
        Sequence @@ values,
        MaximumSustainableYield -> MaximumBiomassGrowth[
          Sequence @@ values, Fishing -> False
        ],
        BiomassMaximum          -> EquilibriumBiomass[
          Sequence @@ values, Fishing -> False
        ]
    ]
```

Out[15]= $-0.8 \left(1 - \dfrac{1.8803}{X^{1/4}}\right) X$

4.5 Rule based models

Population growth consists of various biological processes, which have been mathematically expressed using different sets of equations in the previous sections of this chapter. In this section, we will take a different, non-mathematical approach to representing the biological processes of population growth. The theoretical foundations of rule-based models were developed 70-80 years ago, but with the introduction of personal computers and the exponential growth of computing power, rule-based approaches became viable alternatives to mathematical models.

Stanislaw Ulam and John von Neumann, both working at the Los Alamos National labouratory in the 1940s, along with Alan Turing, a famous cryptanalyst, were pioneers in the investigation of self-replicating systems, also known as cellular automata. Turing introduced his Turing machines, which pointed towards the development of artificial intelligence, and John Conway became famous for his Game of Life[26, 58], a cellular automaton spatial representation of birth, maintenance, growth, and mortality.

The Game of Life is a simple population growth model that captures essential biological processes, including dynamic spatial distribution patterns. Spatial distributions of biological organisms are crucial for understanding how varying environmental conditions affect growth and mortality. Excluding spatial distributions from population growth models is essentially equivalent to assuming a uniform distribution of biomass. Anyone who has engaged in fishing, whether recreational or professional, knows that uniform fish distributions are not very common.

Cellular automata modelling extends beyond population growth modelling; it is an approach used in various scientific fields. The basic principle is best illustrated through a binary, one-dimensional, nearest-neighbour type of automaton. The term 'automaton' indicates that the current state at a certain point changes automatically according to predefined rules based on the current state of each cell and its neighbours.
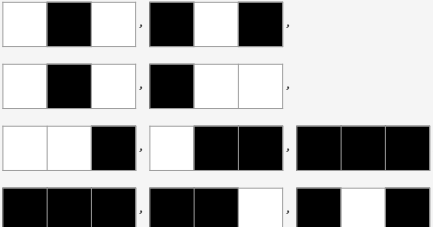
Code box 4.5.1 provides a brief introduction to the fundamental principles of cellular automata modelling related to population growth and development.

Code box 4.5.1 — Introduction to Cellular Automata modelling.
Assume the world to be represented by a row of cells. Each cell has one of two different states: Black or white (1 or 0). In the example below, we assume the world consists of ten cells, and the initial condition is given as a random sequence of black and white cells, illustrated by an array plot.

```
In[1]:= world = RandomInteger[1, 10]
```

```
Out[1]= {1, 0, 1, 0, 0, 1, 1, 1, 1, 0}
```

```
In[2]:= ArrayPlot[{world}, Mesh -> True]
```

Out[2]=

The state of one cell in the next period is determined by the current state of the cell and its two neighbours (one on each side when the range $= 1$, see figure 4.1). The two terminal cells are, in this example, considered to be neighbours.

```
In[3]:= ArrayPlot[
          {#}, Mesh -> True
        ]& /@ (Take[RotateLeft[world, # - 2], 3]& /@ Range[10])
```

Out[3]=

Three cells, each of which may have two different states, can be combined in $2^3 = 8$ ways.

```
In[4]:= Tuples[{0, 1}, 3]
```

```
Out[4]= {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}, {0, 1, 1},
         {1, 0, 0}, {1, 0, 1}, {1, 1, 0}, {1, 1, 1}}
```

Each of the eight combinations may result in a change or no change in the state of the considered cell. This results in $2^8 = 256$ different rules. Rule zero is when all eight combinations lead to white (or empty) cells. The last rule is rule 255, where all combinations lead to black (or filled) cells. In *Mathematica*, we can obtain a graphical presentation of each rule, such as in this example where we are looking at Rule 30:

```
In[5]:= RulePlot[CellularAutomaton[30]]
```

Out[5]=
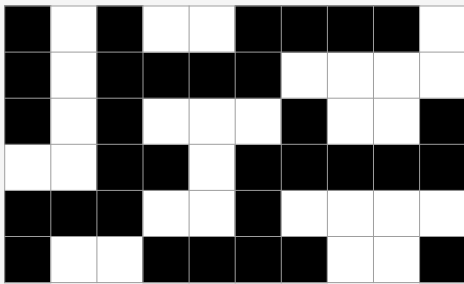
When applying Rule 30 to our cellular automaton, we observe these lines as we progress five steps forward. The initial state of the world is represented by the first line, the second line shows how it changes after one time period based on the rules, and so on:

```
In[6]:= ArrayPlot[CellularAutomaton[30, world, 5], Mesh -> True]
```

Out[6]=

With a limited number of rules, it is straightforward to search for rules that have the potential to describe population growth. If we consider the white cells as representing no biomass and the black cells as filled with a given amount of biomass, we can count the number of black cells. Let's expand the world from 10 to 500 cells and start with 10 black and 490 white cells in period zero, randomly distributed across the 500 cells. We will then calculate forward for 200 periods and count the number of black cells in each period. The selected rules below describe an increase in the number of black cells that approaches a steady-state level, depending on the properties of the rules (Rules 182 and 218 produce five black and three white cells through the rules, while the others have four of each).

```
In[7]:= w0 = RandomSample[Join[Table[0, {490}], Table[1, {10}]], 500];
       ListLinePlot[
         Total /@ CellularAutomaton[#, w0, 200]& /@ #,
         PlotRange   -> {0, 500},
         PlotStyle   -> (Hue[.8-Abs[Log[2+#/300]]]& /@
           {-500, -100, -10, 30, 100, 200}),
         PlotLegends -> #
       ] & @ {30, 86, 110, 124, 182, 218}
```

Out[7]=
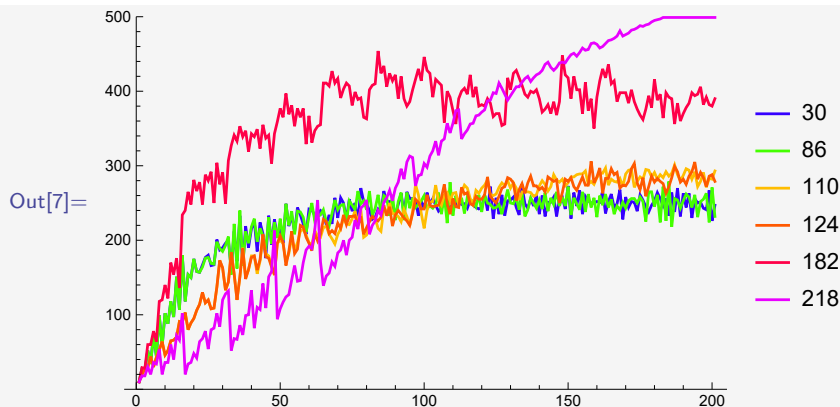
The graphs above do not reflect differences in distributional patterns. We can obtain such differences (at a given period, in this case after 50 periods) in the graph below.

```
In[8]:= Grid[Join[{
        {"", "Initial distribution"},
        {"",
          ArrayPlot[{w0},
            AspectRatio      -> .04,
            PlotRangePadding -> None,
            ImageSize        -> 350
          ]},
          {"Rule", "Distributions after 50 periods"}
        },
        {#, (w1 = CellularAutomaton[#, w0, 200];
          ArrayPlot[{w1[[50]]},
            AspectRatio      -> .04,
            PlotRangePadding -> None,
            ImageSize        -> 350
          ]
        )} & /@ {30, 86, 110, 124, 182, 218}
      ], Alignment -> Center]
```

Out[8]=

The basic rules are simple, yet many of them produce very complex patterns, as also evident from the examples presented in code box 4.5.1. In these examples, we have only considered a neighbourhood of range one, comprising the closest neighbouring cells. However, we can expand this range to include more neighbours, as illustrated in figure 4.1. The figure also demonstrates the differences in neighbouring principles when transitioning from the linear model to higher dimensions.

| von Neumann range=1 | von Neumann range=2 | Moore range=1 | Moore range=2 |
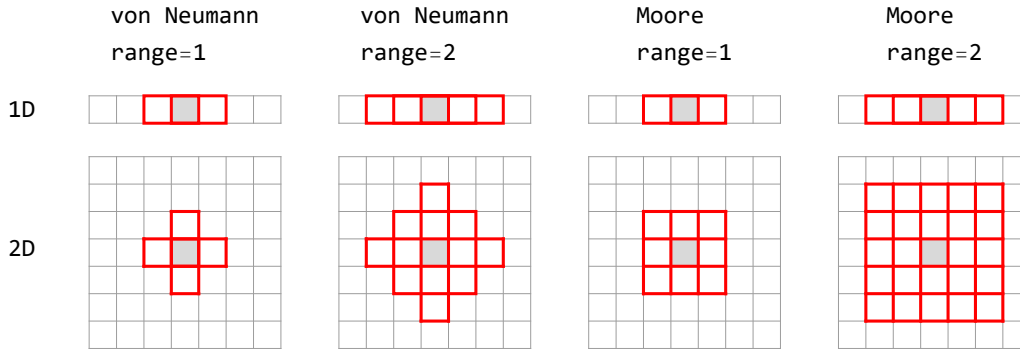
Figure 4.1: The concept of range in 1D and 2D cellular automata is defined according to von Neumann and Moore neighbourhoods. Based on the grey cell, the red lines indicate the neighbours at range one and two in the two dimensions for the two types of neighbourhoods.

Let's continue with simple rules, but now we introduce a state continuum, replacing the discrete states of black, white, or other cells. A continuum could be represented by all real numbers between zero and one. This leads to an infinite number of potential states within the cell. This category of cellular automata is referred to as continuous cellular automata, emphasising the continuous state variable while the time variable remains discrete by definition in cellular automata.

Code box 4.5.2 — Continuous Cellular Automata (CCA).
Continuing from code box 4.5.1 (building on [18])

Now we set up a simple 1D (one-dimensional) CCA model with range $= 1$. In our example, this means that the biomass in one cell is equally distributed into the cell in question and its two neighbouring cells in the next time period. This is achieved using the function:

```
In[9]:= CCAEvolveStep[growth_, list_List, range_] :=
          Map[growth,
            (Sum[RotateLeft[list, i], {i, range}] + list +
            Sum[RotateRight[list, i], {i, range}])/(2 range + 1)
          ]
```

Let us test how the function works using the initial world from input 1 in code box 4.5.1. In this case, we also consider the two terminal cells to be neighbours, indicating a circular world:

```
In[10]:= CCAEvolveStep[growth, world, 1]
```

$$Out[10]= \{growth\left[\tfrac{1}{3}\right], \ growth\left[\tfrac{2}{3}\right], \ growth\left[\tfrac{1}{3}\right], \ growth\left[\tfrac{1}{3}\right], \ growth\left[\tfrac{1}{3}\right],$$
$$growth\left[\tfrac{2}{3}\right], \ growth[1], \ growth[1], \ growth\left[\tfrac{2}{3}\right], \ growth\left[\tfrac{2}{3}\right]\}$$

As we have not yet defined the growth function, only the arguments in the unknown growth function are calculated. These arguments give the total cell biomass after distributing all the initial cell biomasses according to the rule.

A repeated version of `CCAEvolveStep` is obtained by using the `NestList` command in

Mathematica:

```
In[11]:=  CCAEvolveList[growth_, list_List, range_, t_Integer] :=
            NestList[CCAEvolveStep[growth, #, range]&, list, t]
```

Assume linear growth in the biomass within each cell with a growth rate of 50%. If the biomass in one cell exceeds the value 1.0, the fractional value represents the surviving biomass. With this growth function, the world will evolve as shown for the next five periods:

```
In[12]:=  ArrayPlot[
            CCAEvolveList[FractionalPart[# 3/2]&, world, 1, 5],
            Mesh -> True
          ]
```

Out[12]=
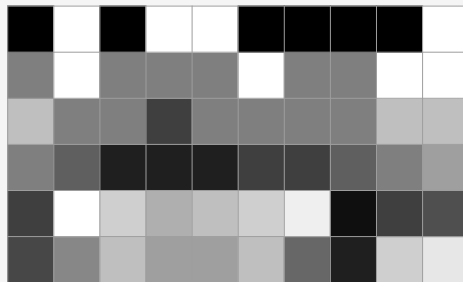
Let us stick to the distributional rule and biological growth presented above, and increase the number of cells from 10 to 500, using w0 from input 7 in code box 4.5.1. Summing up the total biomass of the 500 cells each period results in the following plot:

```
In[13]:=  ListLinePlot[
            Total /@ CCAEvolveList[FractionalPart[# 3/2]&, w0, 1, 100],
            AxesLabel -> {"Period", "Biomass"}
          ]
```

Out[13]=
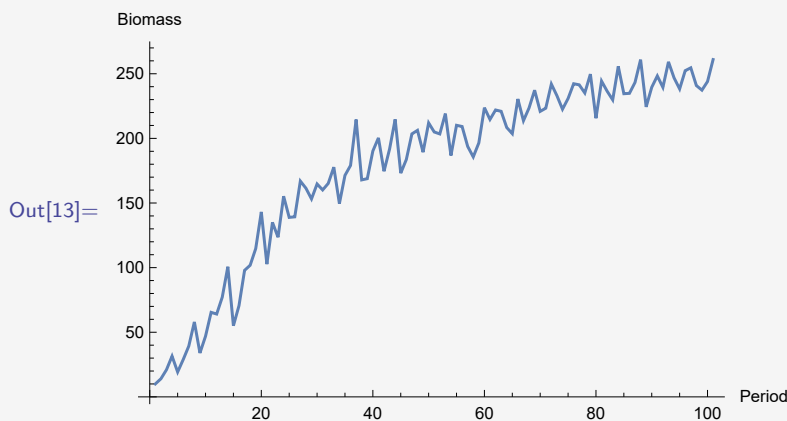
The growth curve above resembles a noisy version of the growth curves we found in code boxes 4.2.2, 4.2.3, and 4.4.1. The growth curve will depict various paths towards the equilibrium level of 250 (half of the 500 cells) when altering the distribution of the initial ten cells filled with biomass.

Let's now introduce 2D models. If we consider a lattice of cells, as shown in figure 4.1, it can represent a graphical area where fish are distributed in patterns based on predefined rules and various characteristics of the cells. Up until now, we have assumed homogeneous

cells, but they can possess different growth properties, which in its simplest form could be implemented by variations in the cells' carrying capacities.

In code box 4.5.3, we introduce the fundamental principles of 2D cellular automata.

Code box 4.5.3 — 2D cellular automata modelling.
Let's begin by creating a 10x10 matrix where 0s and 1s are randomly distributed in different cells. We'll refer to this matrix as "ocean."

```
In[1]:= ocean = Table[RandomInteger[1, 10], {10}]
```
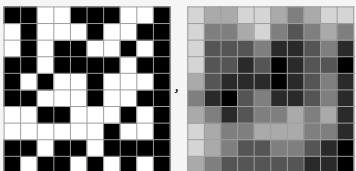
```
Out[1]= {{1, 1, 0, 0, 1, 1, 1, 0, 0, 1}, {0, 1, 0, 0, 0, 1, 0, 0, 1, 1},
        {0, 1, 0, 1, 1, 0, 0, 1, 0, 1}, {1, 1, 0, 1, 1, 1, 1, 0, 1, 1},
        {1, 0, 1, 0, 0, 1, 0, 0, 0, 1}, {1, 1, 0, 0, 0, 1, 0, 0, 1, 1},
        {0, 0, 1, 1, 0, 0, 0, 1, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0, 0, 1},
        {1, 1, 0, 1, 1, 0, 1, 1, 1, 1}, {1, 0, 1, 1, 0, 1, 0, 1, 0, 1}}
```

To implement a rule based on the Moore neighbourhood with a range of 1 (refer to figure 4.1), we need to express the rule using a 3x3 matrix. Let's consider a rule that uniformly distributes the cell content among the nine cells. The rule is as follows:

```
In[2]:= rule = Table[1, {3}, {3}] / 9
```
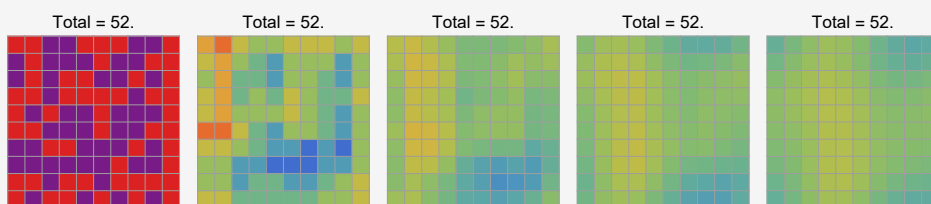
$$Out[2]= \left\{\left\{\left[\frac{1}{9}\right], \left[\frac{1}{9}\right], \left[\frac{1}{9}\right]\right\}, \left\{\left[\frac{1}{9}\right], \left[\frac{1}{9}\right], \left[\frac{1}{9}\right]\right\}, \left\{\left[\frac{1}{9}\right], \left[\frac{1}{9}\right], \left[\frac{1}{9}\right]\right\}\right\}$$

```
In[3]:= ArrayPlot[#, Mesh -> True]& /@ {
        ocean, ListConvolve[rule, ocean, 1, 0]}
```

Out[3]= { , }

The three arguments

```
In[4]:= GraphicsRow[
        ArrayPlot[#,
          Mesh -> True,
          ColorFunction -> "Rainbow",
          PlotLabel -> "Total = "<>ToString @ N @ Total @ Flatten[#],
          PlotRange -> {0, 1}
        ] & /@ NestList[ListConvolve[rule, #, 1] &, ocean, 4]
      ]
```

Out[4]=

In all of the five developmental steps above, the sum of the contents of all cells is 52. The contents of the initial 52 filled cells have been redistributed according to the

rule, assuming the bottom row is a neighbour of the top row and the left column is a neighbour of the right column. This remains consistent regardless of the number of included developmental steps, as shown after 100 steps, for example:

```
In[5]:= Nest[ListConvolve[rule,#,1]&,ocean,100] //Flatten //Total //N
```

```
Out[5]= 52.
```

By adding an argument in the `ListConvolve` command the boarder neighbourhoods are fixed to the given argument, as in this case are empty cells (0):

```
In[6]:= GraphicsRow[
          ArrayPlot[#,
            Mesh        -> True,
            ColorFunction -> "Rainbow",
            PlotLabel -> "Total = "<>ToString @ N @ Total @ Flatten[#],
            PlotRange -> {0, 1}
          ] & /@ NestList[ListConvolve[rule, #, 1, 0] &, ocean, 4]
        ]
```

Out[6]=
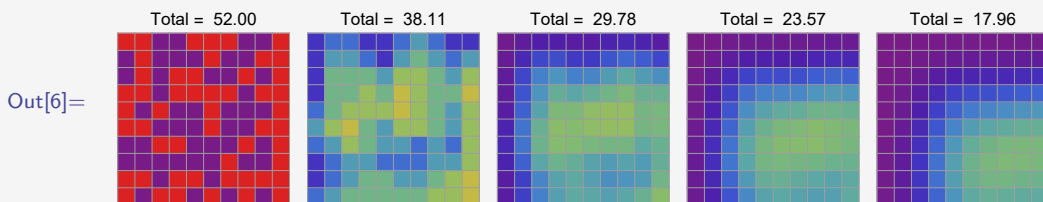
Surrounded by empty cells and allowing the content of the 100 cells (initially with 52 cells filled) to diffuse into the infinite number of surrounding cells, eventually, all cells become empty:

```
In[7]:= Nest[ListConvolve[rule, #, 1, 0] &, ocean, 100
        ] // Flatten // Total // N // Chop
```

```
Out[7]= 0
```

In the case below, the surrounding cells are half filled (1/2), and after some time, all cells become 50% filled:

```
In[8]:= GraphicsRow[
          ArrayPlot[#,
            Mesh        -> True,
            ColorFunction -> "Rainbow",
            PlotLabel -> "Total = "<>ToString@N@Total@Flatten[#],
            PlotRange -> {0, 1}
          ] & /@ NestList[ListConvolve[rule, #, 1, 1/2]&, ocean, 4]
        ]
```

Out[8]=
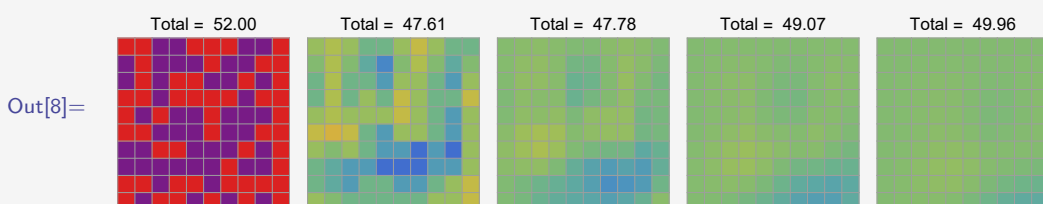
```
In[9]:=  Nest[ListConvolve[rule,#,1,1/2]&,ocean,100]//Flatten//Total//N
```

```
Out[9]= 50.
```

The development could be considered a demonstration of the second law of thermody-namics, reflecting increased entropy as the biomass levels out spatially over time. The Shannon Function H[53] is often employed as a diversity or entropy indicator:

```
In[10]:=  shannonH[l_List] := -Sum[l[[i]]*Log[l[[i]]], {i, Length[l]}]
```

Since

```
In[11]:=  Limit[x * Log[x], x -> 0]
```

```
Out[11]= 0
```

and to accommodate lists which sum up to different totals we improve the code of Function H

```
In[12]:=  shannonH[l_List] :=
            Module[{nl = Select[l/Total[l], # > 0 &]},
              - Sum[nl[[i]] * Log[nl[[i]]], {i, Length[nl]}]
            ]
```

where `nl` is the list normalised to sum up to one. The `shannonH` function then becomes a diversity indicator that reflects the distribution of biomasses in the different cells. Low index values indicate a scattered distribution, while high values indicate homogeneous distribution patterns (high diversity). In our case, this is exemplified by the initial distribution (`ocean`) and equal biomass in all cells (indicated by `x` below):

```
In[13]:=  shannonH[Flatten@ocean]
```

```
Out[13]= Log[52]
```

```
In[14]:=  shannonH[Table[x, 100]]
```

```
Out[14]= Log[100]
```

4.6 Stage structured models

In stage-structured models, the population is divided into different fractions based on age or other measures of development (stages). INTRODUCTION TO STAGE MODELS WITHOUT THE USE OF LINEAR ALGEBRA.

A common method of presenting such information is through the use of the Leslie matrix[37], where the first row of the matrix provides the breeding numbers (b) produced at different stages. The subsequent rows of the square matrix display the survival rates between stages, or the probability of transitioning from stage i to stage $i+1$ (p_i). As the survival rates approach zero, the Leslie matrix simplifies to

$$L = \begin{bmatrix} b_0 & b_1 & \ldots & b_{m-1} & b_m \\ p_0 & 0 & \ldots & 0 & 0 \\ 0 & p_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & p_{m-1} & 0 \end{bmatrix} \tag{4.12}$$

when categorising the population into m stages.

Let's assume that the initial state (at time t) of the population in terms of the number of individuals in each stage is given by the vector $\vec{N}t$,

$$\vec{N}t = \begin{bmatrix} n_{0,t} \\ n_{1,t} \\ n_{2,t} \\ \vdots \\ n_{m,t} \end{bmatrix} \tag{4.13}$$

Then, the number of individuals at time $t+1$ is given by

$$\vec{N}t+1 = L \cdot \vec{N}t$$

or in a more general form when the initial population is given at time zero:

$$\vec{N}t = L^t \cdot \vec{N}0 \tag{4.14}$$

A numerical example is shown in code box 4.6.1, where it is also demonstrated how the equilibrium population is obtained by utilising the properties of the matrix eigenvector.

Code box 4.6.1 — The Leslie Matrix Model.
Defining a general $(m+1) \times (m+1)$ Leslie matrix:

```
In[1]:=  leslie[m_] :=
           Prepend[
             Table[
               If[i != j, 0, p[i]],
               {i, 0, m - 1}, {j, 0, m}
             ],
             b /@ Range[0, m]
           ]
```

A 4×4 Leslie matrix could then for example be obtained:

```
In[2]:= leslie[3] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} b(0) & b(1) & b(2) & b(3) \\ p(0) & 0 & 0 & 0 \\ 0 & p(1) & 0 & 0 \\ 0 & 0 & p(2) & 0 \end{pmatrix}$$

Assume that the first stage of a 4×4 Leslie matrix model covers *juveniles*, followed by *young adults*, *adults*, and *old adults*. A numerical example is provided by the Leslie matrix `vleslie`:

```
In[3]:= vleslie =
           leslie[3] /. {
             b[0] -> 0, b[1] -> .6, b[2] -> 1, b[3] -> 2,
             p[0] -> .5, p[1] -> .7, p[2] -> .5}
```

Out[3]= {{0, 0.6, 1, 2}, {0.5, 0, 0, 0}, {0, 0.7, 0, 0}, {0, 0, 0.5, 0}}

where the first line gives the fecundity (birth rate) of the four stages (starting to the left with the *juveniles*) and the next rows display the survival rates of the first three stages. The equilibrium solution of the model is encapsulated in the eigenvector of the Leslie matrix. We label it `eigenvector`:

```
In[4]:=  eigenvector = Eigenvectors[vleslie][[1]] // Chop
```

Out[4]= {0.844213, 0.422106, 0.295474, 0.147737}

The percentage distribution of the four stages in equilibrium is given by

```
In[5]:=  eigenvector = eigenvector/Total@eigenvector
```

Out[5]= {0.493827, 0.246914, 0.17284, 0.0864198}

The time series of number of individuals of each of the four stages over a period of 15 time units, starting with an initial population where we have 1 *juvenile*, 25 *young adults*, 10 *adults* and 6 *old adults*:

```
In[6]:=  ndevelopment =
            MatrixPower[vleslie, #].{1, 25, 10, 6} & /@ Range[0, 15];
```

Then we find the relative distribution of individuals on the four stages over the time period:

```
In[7]:=  relndevelopment = #/Total@# & /@ ndevelopment;
```

which can be viewed as a list plot of the four stages:

```
In[8]:=  ListLinePlot[
            Transpose@relndevelopment,
            PlotRange  -> All,
            GridLines  -> {None, eigenvector},
            PlotLegends ->
              {"juveniles", "young adults", "adults", "old adults"},
            AxesLabel  -> {"Time", "% of total number"}
         ]
```

Out[8]=
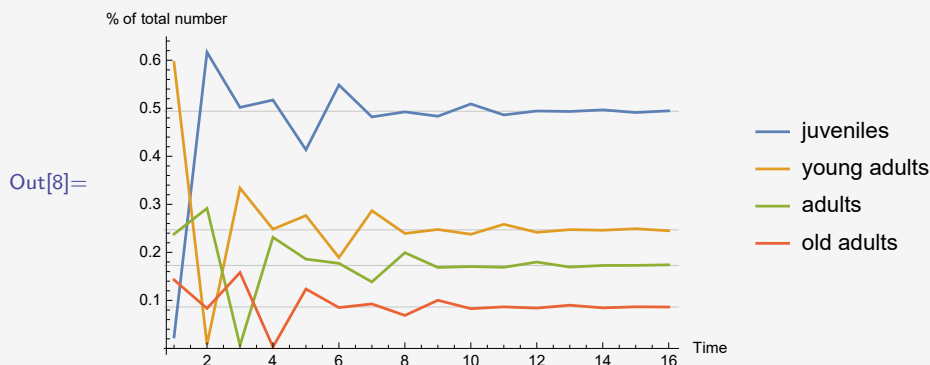
The equilibrium values (`eigenvector`) are indicated as horizontal grid lines in the plot.
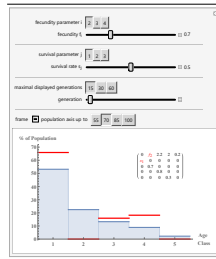
The Leslie matrix can be explored as a
Wolfram Demonstration at

http://demonstrations.wolfram.com/
AgeDistributionsFromALeslieModelForAgeStructuredPopulations

4.7　Multispecies models

In the previous sections, we have explored various methods for modelling the population growth of a single fish stock. In the next chapter, we will introduce a harvest model to determine the net growth of the stock per unit of time. Adding a harvest model to the population dynamics of the fish stock is essentially equivalent to incorporating the impact of predation, with humans acting as predators in this case. Introducing multispecies modelling here can be seen as an anticipation of the exploitation of fish stock resources discussed in Chapter 5.

As implied by its name, multispecies models involve more than one species. Typically, this does not encompass the human exploitation of fish stock resources. However, if it were to be included, it would undoubtedly represent a prey-predator relationship. In essence, there are three main types of multispecies relationships:

- Prey-predator relationships
- Competing species
- Symbiotic relationships (which can also include cannibalism)

One well-known prey-predator model is the Lotka-Volterra model, named after Alfred Lotka and Vito Volterra[55], who independently formulated the model in 1925-1926. Let x represent the prey stock biomass, y the predator stock biomass, a, b, c, and d positive constants, and t the time variable. The model formulation

$$\dot{x}(t) = x(t) \cdot \Big(a - b \cdot y(t)\Big) \quad \dot{y}(t) = -y(t) \cdot \Big(c - d \cdot x(t)\Big) \tag{4.15}$$

describes the population growth per unit of time for both the prey and the predator as functions of x and y.

The Lotka-Volterra model exhibits a stable focus at $x = c/d$ and $y = a/b$. However, as illustrated in Code box 4.7.1, the stable focus (the intersection of the two red isoclines in the phase plot in `Out[2]`) actually represents a focus of limit cycles, the trajectories of which are determined by the initial values of x and y.

Although the practical applicability of the Lotka-Volterra model is limited due to its sensitivity to initial conditions, the model's limit cycle property (see Code box 4.7.1) offers an elegant and straightforward representation of a typical prey-predator relationship. The cycles (shown in `Out[3]` in Code box 4.7.1) illustrate how a large prey stock (blue curve) provides room for growth in the predator stock (yellow curve). As the predator stock's consumption of the prey stock increases, it eventually leads to a decline in the prey stock

biomass. This is followed by a corresponding decrease in the predator stock biomass, which eventually allows the prey stock to recover and initiate the cycle anew at some point in time.


Code box 4.7.1 — The Lotka-Volterra Model.

Assume $a = b = c = d = 1$ in Equation 4.15 and assume an initial predator biomass $x(0) = 0.5$, while the initial prey biomass $(y(t))$ is given three different values: 0.05, 0.3, and 1. The two differential equations (Equations 4.15) are solved numerically using the NDSolve function. We define a function (dsol) in which the initial value of y is a variable:

```
In[1]:= dsol[y0_] :=
          NDSolve[{
            x'[t]  == x[t] (1-y[t]),
            y'[t]  == -y[t] (1-x[t]),
            x[0 ]  == .5,
            y[0]   == y0},
            {x[t], y[t]}, {t, 0, 25}
          ];
```

Then we use the dsol function to plot the trajectories of the $x - y$ dynamics for three different initial values of y. The isoclines of the system are depicted as red lines. The three initial points —, (0.5, 0.05), (0.5, 0.3), and (0.5, 1.0) —, are represented as black points in the plot below. The VectorPlot function indicates the directions and varying strength of the system's dynamics.

```
In[2]:= Show[{
          ParametricPlot[
            Evaluate[{x[t], y[t]} /. dsol[#]], {t, 0, 25}
          ] & /@ {.05, .3, 1},
          VectorPlot[{x(1-y), -y(1-x)},
            {x, 0, 5}, {y, 0, 5},
            VectorScale  -> {.15, .4, (#1 + #2) &},
            VectorPoints -> 12
          ],
          Graphics[{
            Red, Line[{{1, 0}, {1, 5}}], Line[{{0, 1}, {5, 1}}],
            Black, PointSize[.025], Point[{.5, #}] & /@ {.05, .3, 1}
          }]
        }]
```

Out[2]=
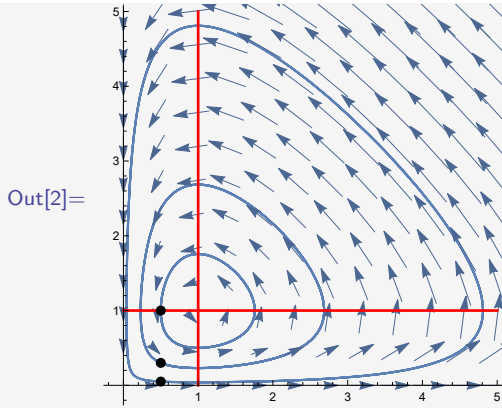
The results can also be presented with respect to the time variable t, plotting the changes in $x(t)$ (blue curve) and $y(t)$ (yellow curve) over a period of 25 time steps:

```
In[3]:= GraphicsRow[
          Plot[Evaluate[{x[t], y[t]} /. dsol[#]], {t, 0, 25},
            PlotRange -> {0, 5},
            PlotLabel -> "x(t) = .5, y(t) = " <> ToString[#]
          ] & /@ {.05, .3, 1}
        ]
```

Out[3]=
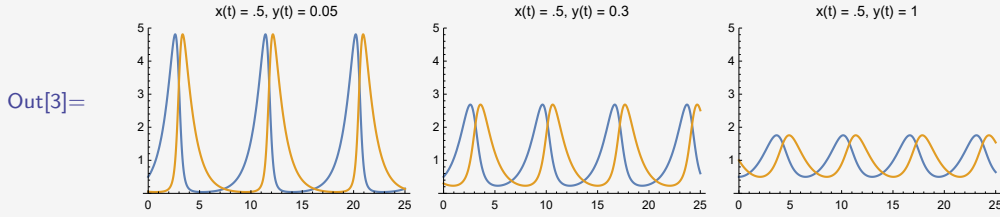
Again: The initial conditions determine the fluctuations in terms of wavelengths and frequencies.

There are various ways to formulate prey-predator relationships beyond the Lotka-Volterra model. As mentioned in the introductory part of this section, a fishery situation bears similarity to a prey-predator scenario. Consider a prey stock that, in the absence of predators, experiences population growth as expressed in Equation 4.2. The presence of predators introduces a negative term to the equation, and we assume that this term is linearly dependent on the biomass of both prey and predator. Furthermore, let's assume that the prey is a vital resource for the predator and that the environmental carrying capacity of the predator is linearly related to the prey stock biomass. This yields the prey-predator model:

$$
\begin{aligned}
\dot{x}(t) &= r \cdot x(t) \cdot \left(1 - \frac{x(t)}{K}\right) - \alpha \cdot x(t) \cdot y(t) \\
\dot{y}(t) &= s \cdot y(t) \cdot \left(1 - \frac{y(t)}{\beta \cdot x(t)}\right)
\end{aligned}
\tag{4.16}
$$

Here, r and s denote the intrinsic growth rates of the prey and predator stocks, while α and β are positive constants. α represents a catchability coefficient for the predator's harvesting of the prey, and β reflects the positive impact of prey availability on predator population growth. The dynamics of Equations 4.16 are illustrated in Figure 4.2. Notably,

observe the differences between the system's isoclines and those of the Lotka-Volterra model in Code box 4.7.1.
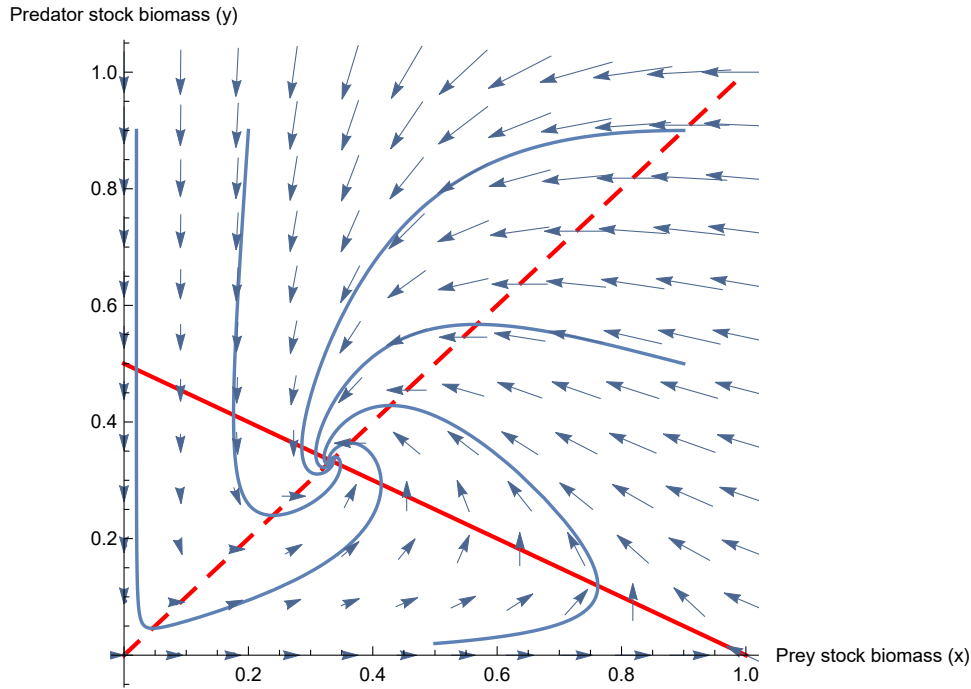
Figure 4.2: Phase plot of the prey-predator model in Equations 4.16 with $r = s = 0.5$ and $K = \alpha = \beta = 1$. The solid red line represents the prey isocline $\dot{x}(t) = 0$, while the dashed red line corresponds to the predator isocline $\dot{y}(t) = 0$. Five trajectories leading to the stable equilibrium (the intersection between the two isoclines) are depicted, each originating from a different initial system condition.

In contrast to the beneficial interaction between prey and predator in a prey-predator relationship, all species experience negative effects from one another in a competitive relationship. The renowned Russian biologist Georgy Gause formulated a model for two competing species[27], which is consistent with the works of Lotka and Volterra. The model is given by:

$$\dot{x}(t) = r \cdot x(t) \cdot \left(1 - \frac{x(t)}{K}\right) - \alpha \cdot x(t) \cdot y(t)$$
$$\dot{y}(t) = s \cdot y(t) \cdot \left(1 - \frac{y(t)}{L}\right) - \beta \cdot x(t) \cdot y(t)$$

(4.17)

Here, r and s represent the intrinsic growth rates of the two competing species, while α, β, K, and L are positive constants. The latter two denote the environmental carrying capacities of the respective species. It's evident that the first equations in both Systems 4.16 and 4.17 are identical, while the second equation in System 4.17 follows the same pattern as the first.

The isoclines of the competing species model may or may not intersect. If they do intersect, it can occur as depicted in Figure 4.3, where the $\dot{y}(t) = 0$ isocline intersects the vertical axis above the $\dot{x}(t) = 0$ isocline (typical case). In such a situation, the intersection between the two isoclines is an unstable equilibrium point. In the opposite scenario (when the $\dot{x}(t) = 0$ isocline intersects the vertical axis above the $\dot{y}(t) = 0$ isocline), the intersection between the

isoclines becomes a stable equilibrium. According to Gause, the latter case only arises if the two species belong to distinct ecological niches[27], implying that they are not directly competing for the same ecological resources. The competitive exclusion principle suggests that two competing species cannot coexist; one will ultimately dominate the ecological niche. Thus, the model exhibits two stable equilibriums (as seen in Figure 4.3), with one excluding species x and the other excluding species y.
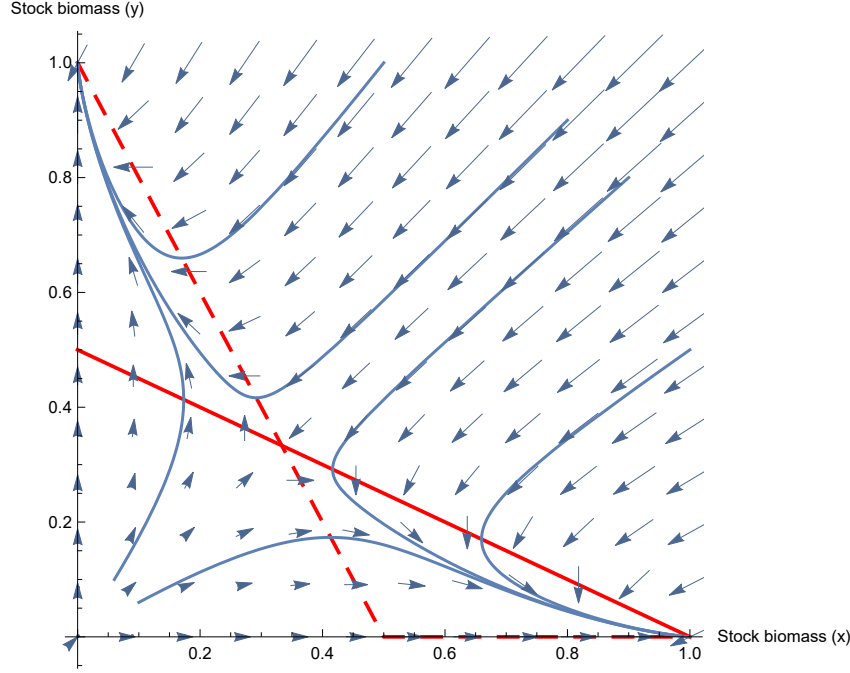

Figure 4.3: Phase plot of the competition model in Equations 4.17 with $r = s = 0.5$ and $K = L = \alpha = \beta = 1$. The solid red line corresponds to the isocline $\dot{x}(t) = 0$, while the dashed red line represents the isocline $\dot{y}(t) = 0$. Six trajectories leading to the two stable equilibriums (at point $(1,0)$ and point $(0,1)$) are shown, each originating from different initial system conditions. The intersection between the two isoclines is an unstable equilibrium point.

Symbiosis, which stands in contrast to competition, represents another possible relationship between two or more species. A simple symbiosis model is given by:

$$
\begin{aligned}
\dot{x}(t) &= r \cdot x(t) \cdot \left(1 - \frac{x(t)}{K + \alpha \cdot y(t)}\right) \\
\dot{y}(t) &= s \cdot y(t) \cdot \left(1 - \frac{y(t)}{L + \beta \cdot x(t)}\right)
\end{aligned}
\tag{4.18}
$$

In this model, r and s (as in Equations 4.16 and 4.17) represent the intrinsic growth rates of the two species. Here, the environmental carrying capacities of x and y are $K + \alpha \cdot y$ and $L + \beta \cdot x$, respectively. Symbiosis and cannibalism models (like any multispecies model) may also exhibit curved isoclines, as demonstrated by Eide and Wikan (2010)[19].
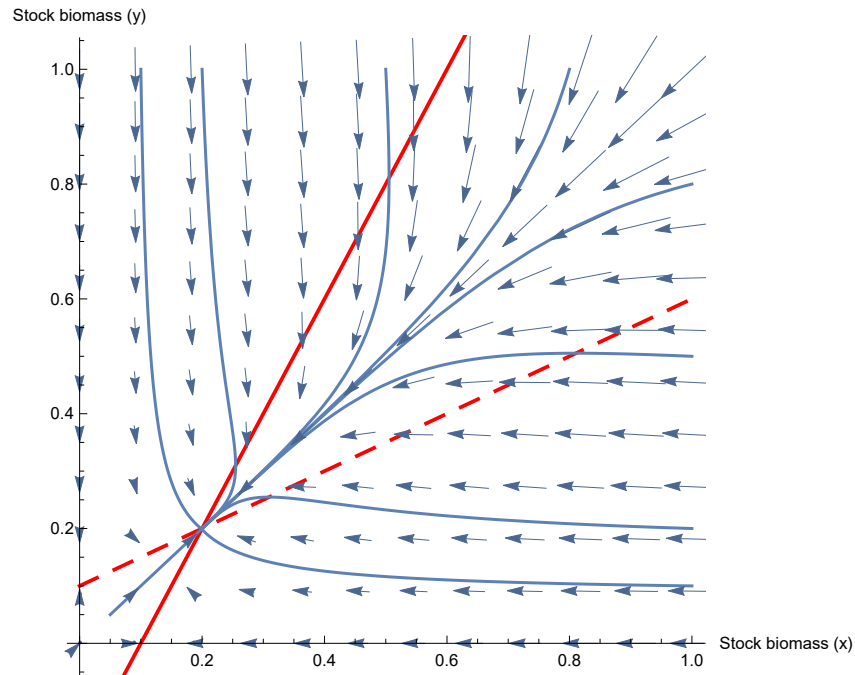
Figure 4.4: Phase plot of the symbiosis model in Equations 4.18 with $r = s = \alpha = \beta = 0.5$ and $K = L = 0.1$. The solid red line corresponds to the isocline $\dot{x}(t) = 0$, while the dashed red line represents the isocline $\dot{y}(t) = 0$. Several trajectories leading to the stable equilibrium at the intersection of the two isoclines are depicted, each originating from different initial system conditions.

DEMO
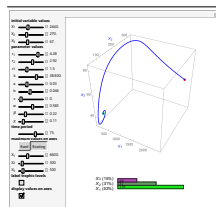
Explore a three-species model as a Wolfram Demonstration here:

http://demonstrations.wolfram.com/EcosystemDynamics/

4.8   Stock assessment

Stock assessments are essential procedures in modern fisheries management, as the state of the stock at any given point in time is the most crucial information for this process. To sustain a fishery, its resource base must be sustained, making the state of the resource base a core issue.

However, stock assessment involves more than just mapping the current stock biomass in the sea. When referring to *the state of the stock* in this context, it also encompasses the future development of the stock at the current level of exploitation as well as at various levels of exploitation.

The primary data sources for conducting fish stock assessments are catch statistics, along with records from trawl (swept-area method) and acoustic surveys. How time series of catch statistics can be utilised to obtain rough stock estimates is described in code box 3.1.1. In this example, a specific population growth model (logistic growth) is assumed, illustrating

the fact that all utilisation of catch data relies on predefined assumptions related to models of stock growth and/or harvest production. Naturally, this also holds true for swept-area surveys; even acoustic stock assessment surveys involve pre-assumed models regarding how acoustic signals relate to stock densities. Therefore, modern technology does not appear capable of solving the fundamental problem described by Slobodkin in Additional remarks 4.2.

> **Additional remarks 4.2 — Is it possible to determine a population's size in nature?.**
> *From "Growth and Regulation of Animal Populations" (page 152), by Lawrence B. Slobodkin*[51]:
> "Unfortunately, one of the most difficult things to determine about any animal population is its size in nature. Derivatives of growth curves and details of courting, mating, psychology, and evolution, which would seem fairly abstruse, are relatively simple to determine; but a complete numerical census requires simultaneous observations of the population over a large area and has been made for very few organisms. Consequently, it would seem that the appropriate procedures for testing the reality of oscillations in population numbers in animals would have to be either theoretical – that is, in terms of biologically realistic models – or experimental."

Exercises

> **Exercise 4.1** Modify the WL code in Code box 4.2.4 (Input 4) and try to reproduce the graphs in Figure 5.2.   ∎

> **Exercise 4.2** Show how to calculate the stable focus of the Lotka-Volterra model.   ∎

5. The concept of equilibrium harvest

The concept of equilibrium harvest is straightforward: Equilibrium harvest is achieved when the surplus biomass production in the stock during a specific period of time is harvested within the same period. However, this simplification poses certain challenges, as the notion of equilibrium itself is a highly theoretical concept.

Surplus production accounts for net production by considering various factors, including *recruitment* (often young fish reaching a size or age suitable for capture from a fishery perspective), *individual growth*, which represents the collective increase in size across all individuals, and *natural mortality*, which denotes the biomass loss due to fish dying naturally during the period.

Clearly, the harvested production during a period cannot precisely match the biomass that constitutes the surplus production during that same period. Additionally, the composition of the harvested biomass matters – whether it comprises newly recruited young fish or older fish.

These issues are disregarded in the standard surplus production models discussed in section 4.2. These models do not account for the age composition of the stock or how recruitment dynamics relate to this composition. A certain biomass at the beginning of a year yields a specific surplus production within that year. Consequently, the issues mentioned above do not affect the surplus production models. However, what about the age structured models presented in section 4.4? Can we maintain the concept of equilibrium harvest in the context of age structured models?

In age structured models, the interpretation of the equilibrium concept revolves around fixed fishing mortality rates for each cohort. The total mortality rate Z is broken down into the natural mortality rate M, which occurs due to predation, age, and diseases, and F, the mortality rate attributed to fishing:

$$Z = M + F \tag{5.1}$$

The harvest obtained from each cohort is then calculated as the product of the fishing mortality rate for the cohort, F_c, and the biomass of the cohort, X_c. This biomass represents the equilibrium biomass of the cohort after maintaining all fishing mortalities constant for an extended period. Consequently, the concept of equilibrium harvest is well-defined within the framework of age structured models.

5.1 Surplus production models

The net growth per unit of time in the stock biomass is determined by the difference between the natural growth of the stock (surplus production $f(X)$) and the harvest ($H(E,X)$):

$$\dot{X}(t) = f(X(t)) - H(E(t), X(t)) \tag{5.2}$$

Upon substituting the surplus production definitions from growth equations 4.1,4.2, and4.3, the corresponding expressions of equation 5.2 become:

$$\dot{X}(t) = -r \cdot X(t) \cdot \ln\left(\frac{X(t)}{K}\right) - H(E(t), X(t)) \tag{5.3}$$

$$\dot{X}(t) = r \cdot X(t)\left(1 - \frac{X(t)}{K}\right) - H(E(t), X(t)) \tag{5.4}$$

$$\dot{X}(t) = r \cdot X(t)\left(1 - \left(\frac{X(t)}{K}\right)^{m-1}\right) - H(E(t), X(t)) \tag{5.5}$$

$H(E,X)$ represents the harvest obtained during a single time step (often a year). Let's assume $H(E,X)$ follows a bi-linear catch equation as given by equation 3.1. When maintaining a constant fishing effort, $E(t) = E$, equilibrium is achieved when $\dot{X}(t) = 0$ for all t. The stock biomass reaches the equilibrium value X, which can be determined in the case of equation 5.4 using the following steps:

$$r \cdot X\left(1 - \frac{X}{K}\right) = q \cdot E \cdot X$$

For $X \neq 0$:

$$r\left(1 - \frac{X}{K}\right) = q \cdot E,$$

and by rearranging the terms, the equilibrium biomass is obtained:

$$X = K\left(1 - \frac{q}{r}E\right) \tag{5.6}$$

We can observe that under the assumption of a logistic growth equation (the Verhulst equation) and a Schaefer production function, the equilibrium biomass forms a linear relationship with any value of the constant fishing effort E.

Code box 5.1.1 — Equilibrium harvest.

The linear relationship in equilibrium between constant fishing effort E and stock biomass X, as expressed in equation 5.6, is illustrated below for certain parameter values ($K = 1$, $r = 1/2$, and $q = 1/2$):

```
In[1]:= Plot[
          x /. Solve[r x (1-x/k) == q ee x && x != 0, x] /. {
            k -> 1, q -> 1/2, r -> 1/2, ee -> e},
          {e, 0, 1},
          PlotStyle -> Red,
          AxesLabel -> {"E", "X"}
        ]
```

Out[1]=
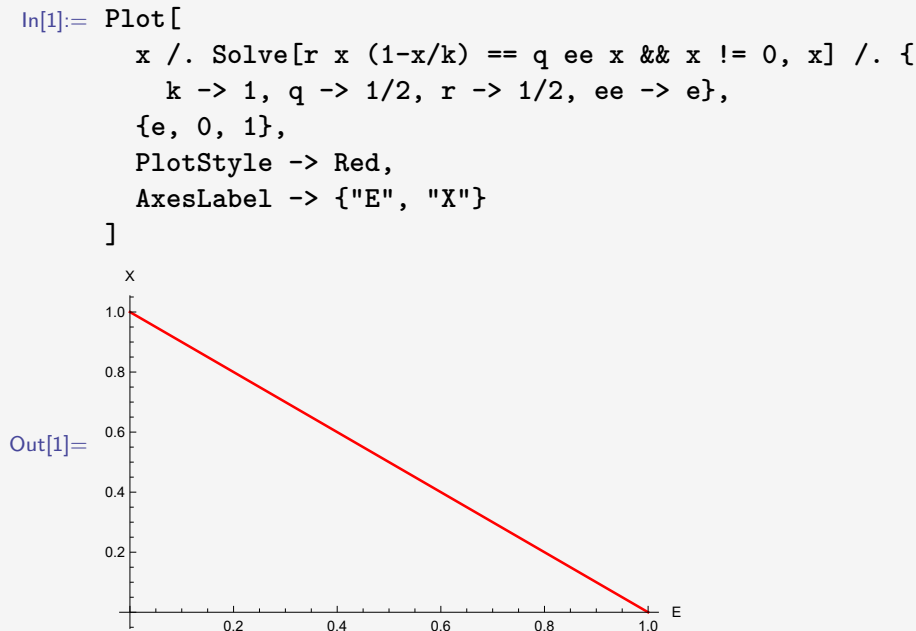
Recall the Cobb-Douglas function plot in Code box 2.6.1 and combine it with a contour plot of the short-term catch equation 3.1 as shown above:

```
In[2]:= Show[
          ContourPlot[
            (q e x /. {k - >1, q - >1/2, r - >1/2}) == #,
            {x, 0, 1}, {e, 0, 1}
          ] & /@ Range[0.01, .6, .04],
          Plot[x /. Solve[r x (1 - x/k) == q ee x && x != 0, x] /. {
              k - >1, q - >1/2, r - >1/2, ee - >e},
            {e, 0, 1},
            PlotStyle -> Red
          ],
          PlotRangePadding -> None,
          FrameLabel      -> {"E", "X"}
        ]
```
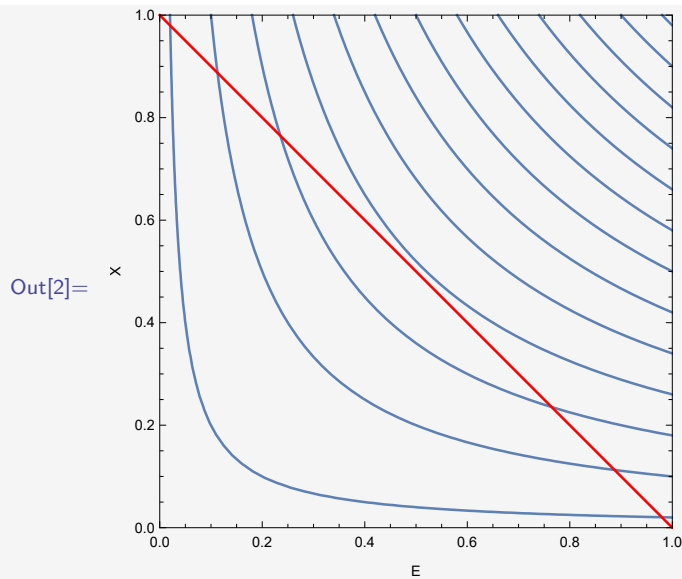
Imagine projecting the red line from the plot above onto the surface of the production equation. The red line would then describe a curve through the three dimensions of E, X, and H, as shown in this plot:

```
In[3]:= ParametricPlot3D[{e, k (1 - q/r e), q e k (1 - q/r e)} /. {
          k -> 1, q -> 1/2, r -> 1/2},
          {e, 0, 1},
          BoxRatios -> {1, 1, 1},
          PlotStyle -> Directive[Thickness[.015], Red],
          AxesLabel -> {"E", "X", "H"}
        ]
```
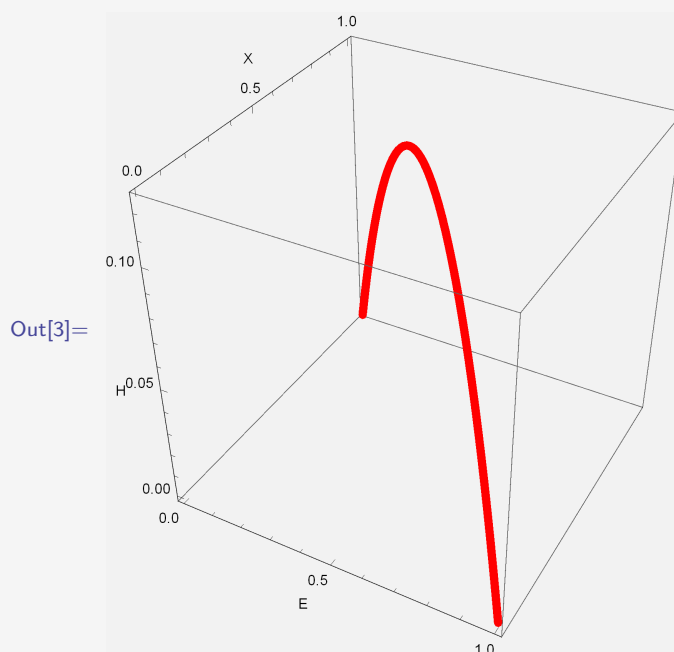

We can also use *Mathematica* to visualise how the long-term relationship (the red curve) fits into the short-term catch equation (the surface below):

```
In[4]:=  Show[{
           Plot3D[q e x /. {k -> 1, q -> 1/2, r -> 1/2},
             {x, 0, 1}, {e, 0, 1},
             MeshFunctions -> {#3 &},
             PlotStyle -> Directive[
                Opacity[0.7], LightBlue, Specularity[White, 50]
             ]
           ],
           ParametricPlot3D[{e, k (1 - q/r e), q e k (1 - q/r e)} /. {
             k - >1, q - >1/2, r -> 1/2},
             {e, 0, 1},
             PlotStyle -> Directive[Thickness[.015], Red]
           ]},
           BoxRatios -> {1, 1, 1},
           AxesLabel -> {"E", "X", "H"},
           ViewPoint -> {2.3, -.6, .8}
         ]
```

Out[4]=
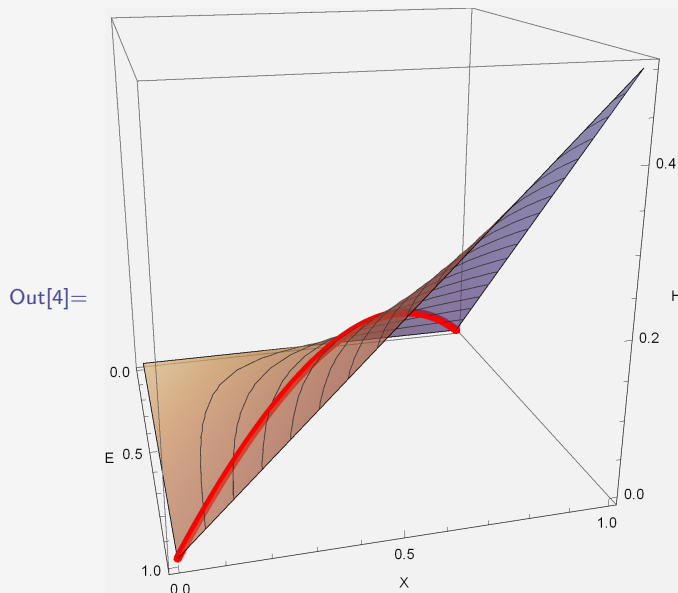
While viewing the red curve from a viewpoint in front of the X-axis (`ViewPoint ->
Infinity, 0, 0`), we see the left graph below. With a viewpoint in front of the E-axis
(`ViewPoint -> 0, -Infinity, 0`), we see the graph on the right below.

```
In[5]:=  GraphicsRow[
           ParametricPlot3D[{e, k (1 - q/r e), q e k (1 - q/r e)} /. {
             k -> 1, q -> 1/2, r -> 1/2},
             {e, 0, 1},
             BoxRatios -> {1, 1, 1/GoldenRatio},
             PlotStyle -> Red,
             ViewPoint -> #,
             AxesLabel -> {"E", "X", "H"}
           ] & /@ {{Infinity, 0, 0}, {0, -Infinity, 0}},
           Spacings -> 50
         ]
```
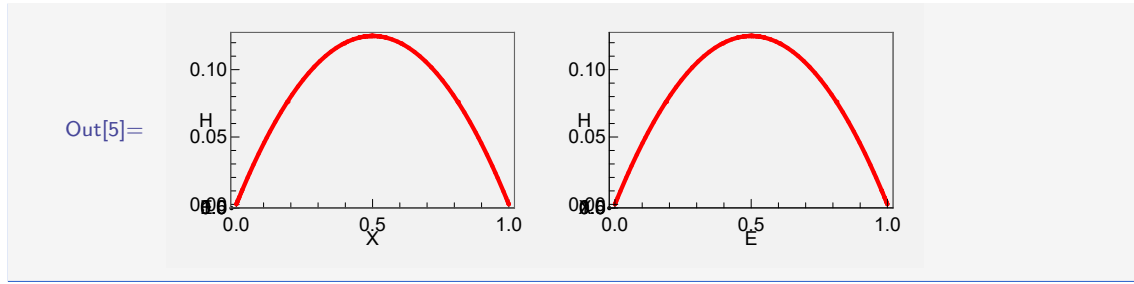
Equation 5.6 represents the isocline for the value zero of the differential equation 5.4 ($\dot{X}(t) = 0$). This isocline divides the $E - X$ plane into two regions: one where the stock biomass increases (below the isocline) and another where the stock biomass decreases (above the isocline), as shown in Figure 5.1.
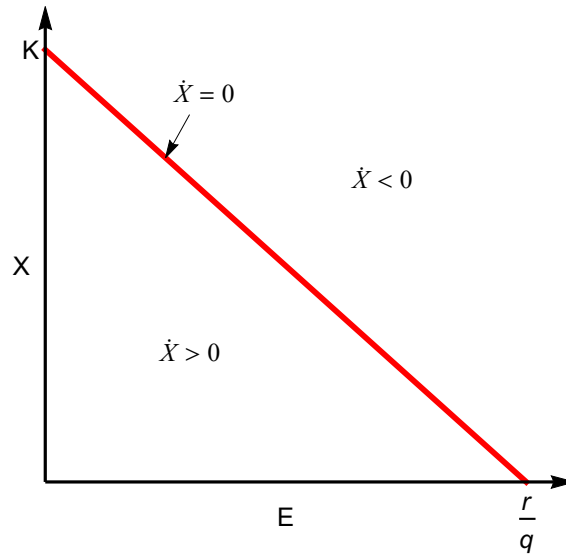
Figure 5.1: The isocline of equation 5.4 is a downward-sloping straight line dividing the plane into two regions: Below the isocline where the stock biomass increases ($\dot{X}(t) > 0$), and above the line where the stock biomass declines ($\dot{X}(t) < 0$).

In Figure 5.1, all potential equilibriums are represented by a line, while all other combinations of X and E lead to either an increase or a decrease in the stock biomass X. Notably, $X = K$ (for $E = 0$) and $X = 0$ (for $E = \frac{r}{q}$, as per equation 5.6) also stand as equilibriums, marking the outer boundaries of the isocline.

Once the equilibrium relationship between fishing effort (E) and stock biomass (X) has been established, equilibrium harvest can be expressed as a function of fishing effort by inserting Equation 5.6 into Equation 3.1:

$$H(E, X(E)) = H(E) = qKE\left(1 - \frac{q}{r}E\right) \tag{5.7}$$

This equation is sometimes referred to as the Schaefer model due to Schaefer's initial utilisation of the model. However, it should not be confused with the short-term catch equation (Equation 3.1), which is also valid beyond the equilibrium.

The concept of Maximum Sustainable Yield (MSY) is determined by maximising Equation 5.7 with respect to effort (E). The maximum is attained when $H'(E) = 0$ and $H''(E) < 0$, which occurs at $E = r/(2q)$. Thus, for Equation 5.7, we have:

$$MSY = H(E = \frac{r}{2q}) = \frac{r \cdot K}{4} \qquad (5.8)$$

5.2 Stock dynamics in discrete time
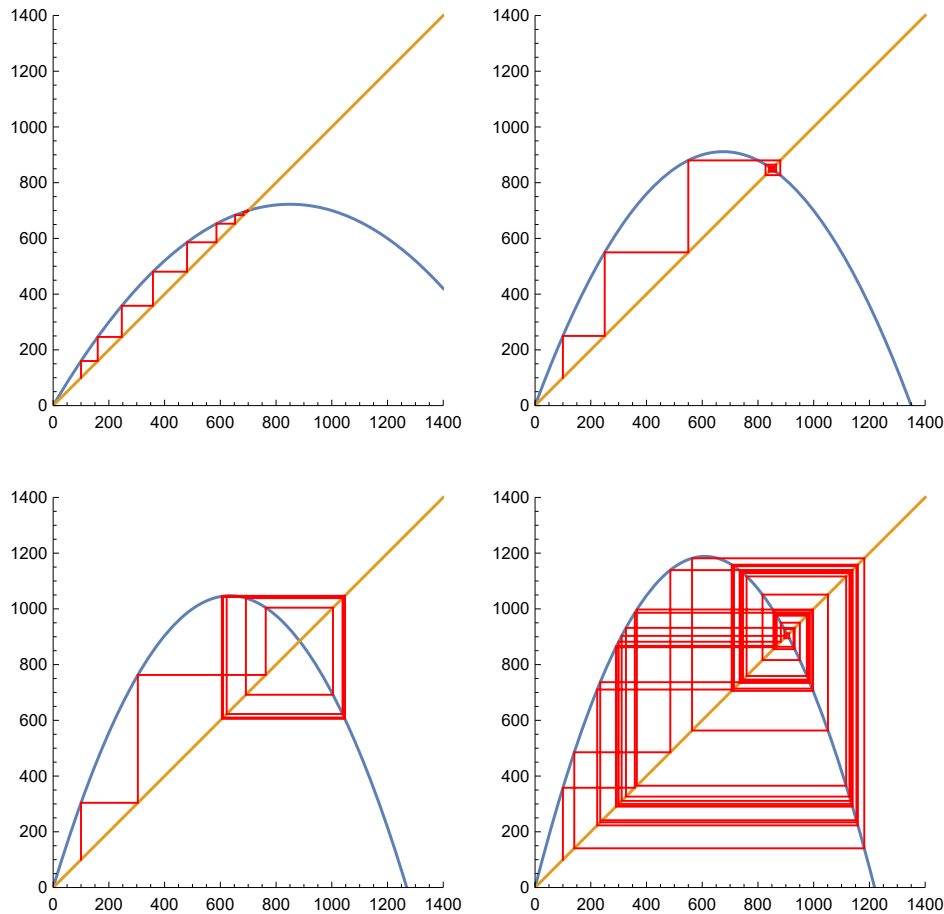
Figure 5.2: The Cobweb diagram (refer to Code box 4.2.3) illustrates how the stock size adjusts over time (t along the horizontal axis and $t+1$ along the vertical axis) based on a constant fishing effort. The discrete-time model (shown by the blue curve) is described by Equation 5.9. The four cases demonstrate how increasing values of the intrinsic growth rate r lead to varying dynamics: stable equilibriums in the top row, transitioning to a limited cycle equilibrium in the lower left figure, and progressing to a chaotic regime in the lower right figure. The yellow lines indicate stock biomass equilibriums, and the red lines depict the temporal dynamics with an initial stock size of 100.

Let's examine the stock dynamics in discrete time under various fixed levels of fishing effort (E), assuming population growth to be described by equations 4.2 and 3.1. In discrete

time, we express this as follows:

$$X(t+1) = X(t) + r \cdot X(t) \cdot \left(1 - \frac{X(t)}{K}\right) - q \cdot E \cdot X(t) \tag{5.9}$$

The first term on the right-hand side represents the current stock biomass, the second term signifies the natural growth per unit of time (Equation 4.2), and the last term represents the biomass removed from the stock due to fishing (Equation 3.1).

Equation 5.9 can be visualised using a Cobweb diagram (Figure 5.2), with $X(t)$ plotted along the horizontal axis and $X(t+1)$ plotted along the vertical axis.
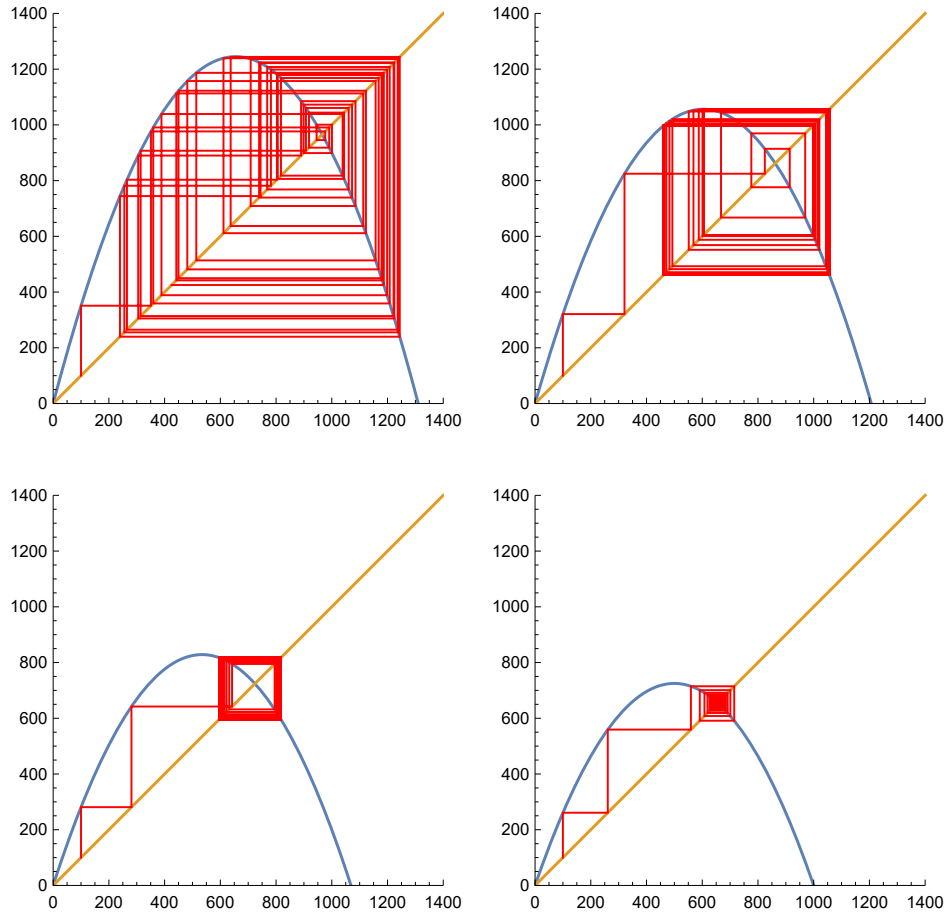
Figure 5.3: Cobweb diagram (refer to Code box 4.2.3) illustrating how the stock size (at time t along the horizontal axis and at time $t+1$ along the vertical axis) adjusts with increasing fishing effort. The discrete-time model (represented by the blue curve) is defined by Equation 5.9, and the four cases demonstrate how higher fishing effort (E) dampens seemingly chaotic behaviour (top left) to stable equilibriums (limited cycles and equilibrium point in the lower right plot). The yellow lines represent stock biomass equilibriums, and the red lines depict the dynamics over time with an initial stock size of 100.

Figure 5.3 illustrates the stabilising effect increasing harvest has on the dynamics pictured in Figure 5.2. In Figure 5.3 the intrinsic growth rate is constant (and high) while the fishing effort increases from top left to lower right.

5.3 Equilibrium harvest and surplus production

We can observe that the given assumptions of a bi-linear catch equation (Equation 3.1) and a parabolic growth equation result in a parabolic equilibrium harvest equation concerning fishing effort (as seen in the top row of Figure 5.4). This outcome arises from the symmetry inherent in parabolic equations and the assumed linearity of the short-term catch equation. The equilibrium catch equation might describe a curve very distinct from the surplus production growth curve, and both depend on the shape of the growth curve and the short-term catch equation.

Nonetheless, there exists a crucial distinction between the two equations depicted in Figure 5.4, namely $f(X)$ and $H(E)$. While the former delineates a path towards the natural equilibrium K, the latter constitutes a collection of equilibriums. Any position on the curve described by $H(E)$ is fixed, and it is inherently impossible, by definition, to assume any change in the system without deviating from the curve (or diverging from the collection of infinitely many equilibriums).
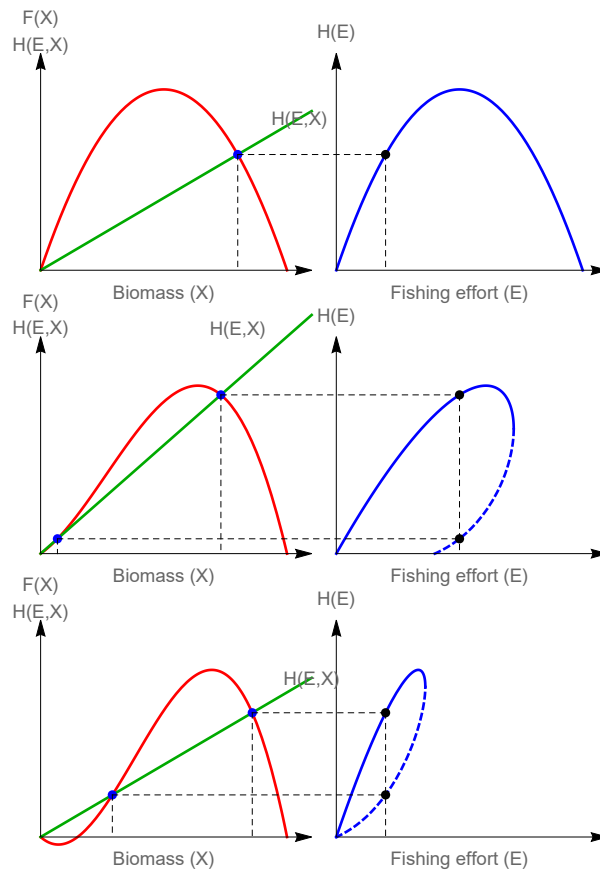
Figure 5.4: Illustration depicting the relationship between biological surplus production (red curves on the left) and equilibrium harvest (blue curves on the right) for three different cases of biological growth. These cases include logistic growth (top row), depensatory growth (middle row), and critical depensatory growth (bottom row, refer to Code box 4.3.1). The latter two cases involve unstable equilibriums on the blue curves, denoted by dashed lines. Across all cases, the path of the blue curves for varying levels of fishing effort (E) is determined by biological growth and a bi-linear harvest equation ($H(E, X)$, represented by green lines on the left).
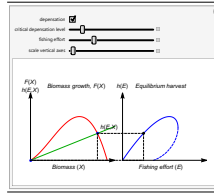
Equilibrium harvest related to different surplus production models:

http://demonstrations.wolfram.com/
SurplusProductionModelsAndEquilibriumHarvest/

5.4 Sustainable yield in age structured models

In age-structured models, the catch needs to be allocated across different age groups, making the catch an integral component of the stock dynamics as it influences age composition and strength. While natural mortality (M in equation 5.1) is often assumed to be constant across all age groups (despite being higher in the earliest life stages, this assumption aligns with a recruitment age assumed to be sufficiently high, rendering it reasonable to assume constant natural mortality), the fishing mortality rate may vary among age groups. Notably significant is the age at which first catch (t_c) occurs, which profoundly impacts the sustainable catch level in a fishery, as expressed by an age-structured model (refer to code box 5.4.1).

Fishing mortality, as indicated by the fishing mortality rate F, represents the output of fishing effort, as expounded in chapter 2. From equation 3.3, it's apparent how catch can be formulated as a function of fishing effort (E) and stock biomass (X). Substituting fishing effort with biological fishing mortality rate simplifies the expression to

$$H(F,X) = F \cdot X \tag{5.10}$$

When $H(F,X) = H(E,X)$ (derived from equation 3.3), we can express fishing mortality as a function of produced fishing effort and current stock biomass:

$$F(E,X) = q \cdot E^{\alpha} \cdot X^{\beta-1} \tag{5.11}$$

When $\alpha = \beta = 1$, this simplifies to $F = q \cdot E$.

The corresponding fishing effort (the inverse of equation 5.11, expressed as a function of fishing mortality rate and stock size) is:

$$E(F,X) = \left(\frac{F \cdot X^{1-\beta}}{q} \right)^{\frac{1}{\alpha}} \tag{5.12}$$

In the code box below, a constant fishing mortality rate (F) is assumed after the age of first catch (t_c) in the stock. However, a variable selective pattern is more realistic in a fishery. The selectivity pattern depends on various factors, primarily the characteristics of the gear (refer to figure 3.2) and the spatial and temporal distributions of year classes and fishing activities.

> **Code box 5.4.1 — Age structured model (Beverton and Holt).**
> Continuing from code box 4.4.1
>
> `TotalCatch` is the yield function in the `PopulationGrowth` package, comprising two variables that define an equilibrium fishery: the fishing mortality rate F and the age of the cohort first recruited to the fishable stock, t_c. Below, we assume sharp selection

at age t_c, at which a constant fishing mortality rate is applied. The total catch has no other analytic expression than the simple product of F (the fishing mortality rate) and X (the stock biomass), without specifying a minimum set of parameter values.

In[16]:= `TotalCatch[] // Notation`

Out[16]//TraditionalForm=

$$F\,X$$

A complex analytical expression (involving β and γ functions) is obtained for `InitialAge -> 0`. A more comprehensible expression is derived by setting the `WeightLengthRelation` to 3 and incorporating an infinite number of cohorts.

In[17]:= `TotalCatch[`
    `OldestAge             -> Infinity,`
    `WeightLengthRelation -> 3`
`] // SimplifyNotation`

Out[17]//TraditionalForm=

$$F\,R\,W_\infty\,e^{M\,(t_R - t_c)}\left(-\frac{3\,e^{k\,(t_0 - t_c)}}{F + k + M} + \frac{3\,e^{2\,k\,(t_0 - t_c)}}{F + 2\,k + M} - \frac{e^{3\,k\,(t_0 - t_c)}}{F + 3\,k + M} + \frac{1}{F + M}\right)$$

A further simplification occurs when the natural mortality rate equals the intrinsic growth rate ($M = k$):

In[18]:= `TotalCatch[`
    `InitialAge            -> 0,`
    `WeightLengthRelation -> 3,`
    `OldestAge             -> Infinity,`
    `MortalityRate         -> GrowthRate,`
    `RecruitmentAge        -> 0`
`] // SimplifyNotation`

Out[18]//TraditionalForm=

$$F\,R\,W_\infty\,e^{-4\,k\,t_c}\left(\frac{3\,e^{k\,t_c}}{F + 3\,k} - \frac{3\,e^{2\,k\,t_c}}{F + 2\,k} + \frac{e^{3\,k\,t_c}}{F + k} - \frac{1}{F + 4\,k}\right)$$

And finally:

In[19]:= `TotalCatch[`
    `CatchAge              -> 0,`
    `OldestAge             -> Infinity,`
    `InitialAge            -> 0,`
    `RecruitmentAge        -> 0,`
    `MaxWeight             -> 1,`
    `MortalityRate         -> GrowthRate,`
    `WeightLengthRelation -> 3`
`] // SimplifyNotation`

Out[19]//TraditionalForm=

$$\frac{6\,F\,k^3\,R}{(F + k)\,(F + 2\,k)\,(F + 3\,k)\,(F + 4\,k)}$$

Let's use the parameter values from input number 9 and plot the surface of the yield function in the $F - t_c$ plane, using the Plot3D function in *Mathematica*:

```
In[20]:= Plot3D[
           TotalCatch[Sequence @@ values],
           {F, 0, 1}, {tc, 0, 20},
           PlotPoints    -> 35,
           PlotRange     -> All,
           MeshFunctions -> {#3 &},
           Mesh          -> 10,
           PlotLabel     -> "Sustainable Yield per recruit (kg)",
           AxesLabel     -> {
             "Fishing mortality rate (F)", "Selection age t_c", ""}
           ]
```
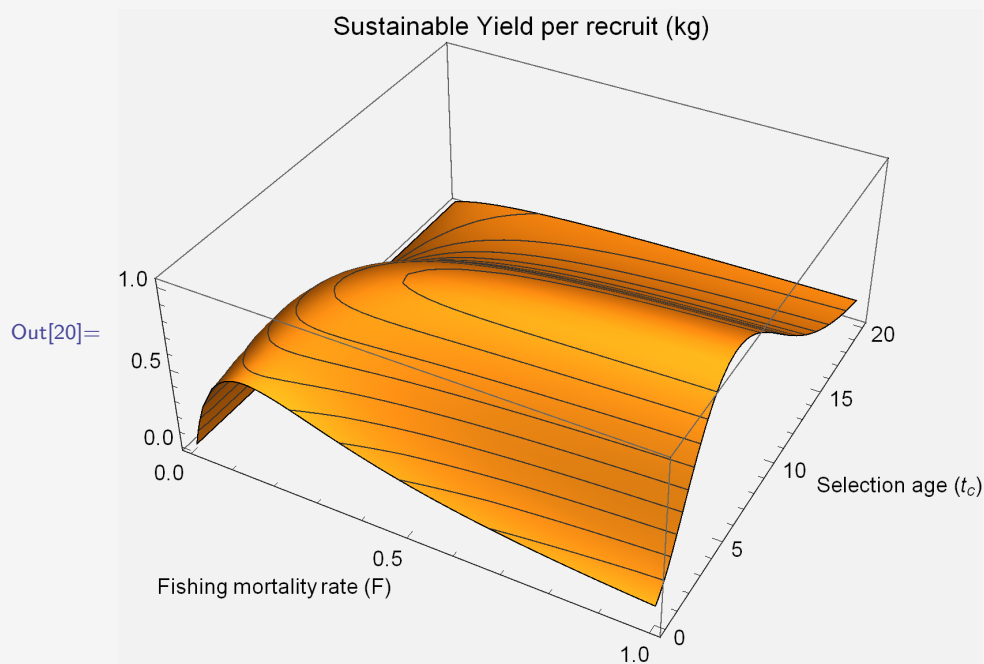

This is the function shown in the plot above:

```
In[21]:= TotalCatch[Sequence @@ values]
```

$$\text{Out[21]= } -10\ e^{-0.2\ tc}\ F\ \left(-\frac{1}{0.2 + F} + \frac{3\ e^{-0.2\ tc}}{0.4 + F} - \frac{3\ e^{-0.4\ tc}}{0.6 + F} + \frac{e^{-0.6\ tc}}{0.8 + F} \right)$$

Maximum Sustainable Yield for any given value of t_c defines a curve in the $F - t_c$ area.

```
In[22]:= MSYCurve = {F /. FindMaximum[
           TotalCatch[
             FishingMortalityRate -> F,
             CatchAge             -> #,
             Sequence @@ values,
             BiomassIncluded      -> Fishable
           ],
           {F, .01},
           PrecisionGoal -> 5,
```

```
              AccuracyGoal  -> 5,
              Method        -> "PrincipalAxis"
           ][[2]], #} & /@ Range[0, 6, .2];
```

In the same way, the Eumetric curve is defined by maximising harvest at any given
fishing mortality rate value.

```
In[23]:= EumetricCurve = {#[[1]], tc /. #[[2]]} & /@ (
        {#, FindMaximum[
           TotalCatch[
             FishingMortalityRate -> #,
             Sequence @@ values,
             BiomassIncluded      -> Fishable
           ],
           {tc, 3},
           PrecisionGoal -> 5,
           AccuracyGoal  -> 5
        ][[2]]} & /@ Join[
           (.001 + .005*(# - 1)) & /@ Range[81],
           (.4 + .2*#) & /@ Range[20]
        ]);
```

By using the ContourPlot function, it is straightforward to produce isopleth diagrams,
which are heavily used by Beverton and Holt[8]. We can draw the MSY curve and the
Eumetric curve in these diagrams.

```
In[24]:= Show[{
        ContourPlot[
          TotalCatch[Sequence @@ values],
            {F, 0, 1}, {tc, 0, 10},
            PlotPoints     -> 50,
            ContourShading -> False,
            ContourStyle   -> Thick,
            Contours       -> 10
          ],
          ListLinePlot[{EumetricCurve, MSYCurve},
            PlotStyle   -> {Directive[Red, Thick],
              Directive[Darker@Green, Thick]},
            PlotLegends -> {"Eumetric curve", "MSY curve"}
          ]
        },
        FrameLabel       -> {"F", "t_c"},
        PlotRangePadding -> None
      ]
```
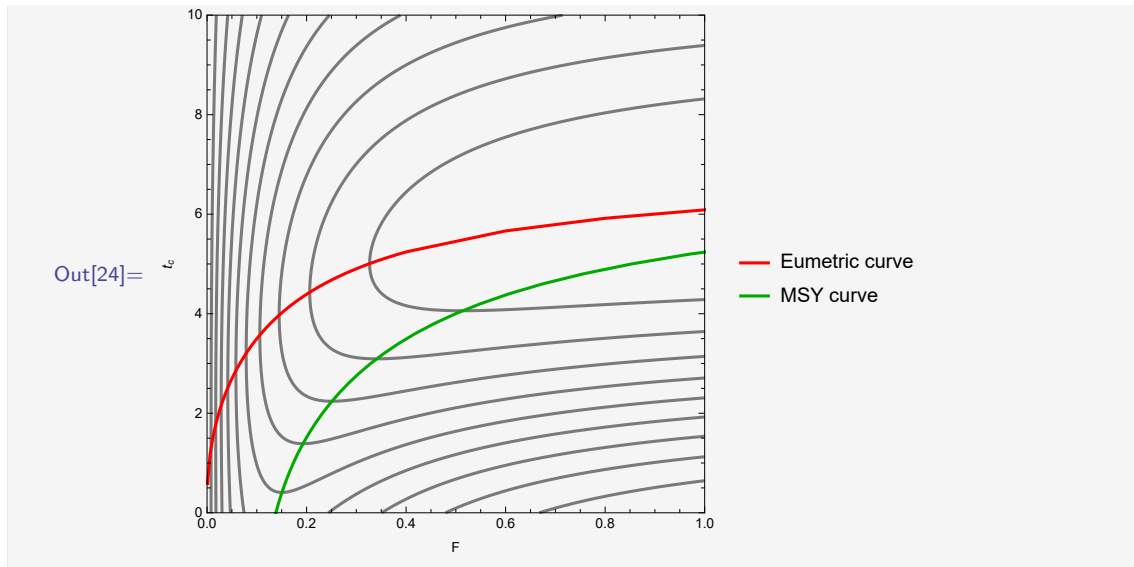
Since the age-structured model presented in Code Box 5.4.1 involves two control variables – the fishing mortality rate F and t_c, the age at which recruitment to the exploited fraction of the stock occurs – the equilibrium harvest depicted in Output 17 of Code Box 5.4.1 could be maximised by selecting specific values of t_c or by choosing certain values of F, or vice versa. The dual problems involve minimising the values of F or t_c while maintaining a given equilibrium harvest.

The trajectory achieved by minimising the fishing mortality rate F for a given equilibrium harvest is referred to as the *eumetric* curve by Beverton and Holt (see Code Box 5.4.1)[7]. We will explore the economic interpretation of the eumetric curve in another chapter. The other curve displayed in Output 24 of Code Box 5.4.1 is the *MSY* curve. The global maximum of sustainable yields in the age-structured model is attained at an infinitely high fishing mortality rate, targeting the fish at the age when the biomass of one cohort is maximised (refer to Input and Output 10 in Code Box 4.4.1). Maximising the yield for all other potential t_c values delineates an *MSY* curve across the $F - t_c$ plane, as illustrated in Output 24 of Code Box 5.4.1.

The Maximum Sustainable Yield (MSY), which in Equation 5.8 is a single point, is depicted as a path in the $F - t_c$ plane within the age-structured model.

DEMO
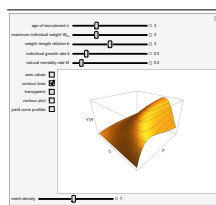
Investigate the Beverton and Holt model under different parametrisations at

http://demonstrations.wolfram.com/
BevertonAndHoltsYieldPerRecruitModel/

5.5   Virtual Population Analysis (VPA)

If obtained catch can be separated into year classes (for example, based on otolith readings), and an age-structured model is assumed, the Virtual Population Analysis (VPA) [31] is a

relatively transparent accounting system that can be used to assess the abundance over time of each year class. These year-classes are referred to as *cohorts*, constituting the total stock (or *legion* when using Roman military terminology). This method is founded on consistency and fixed natural mortality rates. A related, though less transparent method, the eXtended Survivor Analysis (XSA), which employs information from other data sources to provide improved tuning procedures, clearly yields more reliable estimates of recent stock history [49].

VPA is a backward-calculating procedure that traces the history of each cohort using some simple assumptions. VPA becomes increasingly accurate when fishing mortality (F in Equation 5.1) constitutes a larger proportion of the total mortality (Z in Equation 5.1). The fundamental concept is that the number of individuals (N) in a given year (t) at age a—$N_{a,t}$—grows according to the function:

$$\frac{dN_{a,t}}{dt} = -Z \cdot N_{a,t} = -(M + F) \cdot N_{a,t} \tag{5.13}$$

This differential equation has a simple solution:

$$N_{a+1,t+1} = N_{a,t} \cdot e^{-Z_{a,t}} \tag{5.14}$$

where $Z_{a,t}$ represents the mortality the given cohort faced in year y.

The number of individuals within each cohort is unknown, and the only information available is the number of individuals of different ages that were caught, $C_{a,t}$. However, we have the following connection between the two:

$$\frac{dC_{a,t}}{dt} = -F \cdot N_{a,t} \tag{5.15}$$

Now, assuming a fixed fishing mortality rate, $F = F_{a,t}$, the differential equation yields the solution:

$$C_{a,t} = N_{a,t} \cdot \frac{F_{a,t}}{Z_{a,t}} \cdot \left(1 - e^{-Z_{a,t}}\right) \tag{5.16}$$

Equation 5.16 constitutes the core function of VPA, and its application is illustrated in Code box 5.5.1 below. The back-calculation procedure is also depicted in Figure 5.5, where the colored cells could represent registered catches of different cohorts in various years. Each year, a new cohort is added to the stock, and the cohorts are identified by the year of their recruitment to the stock. Assuming that all cohorts experience the same fishing mortality rate in a given year, the relationship among the number of individuals within each cohort will reflect the relative abundance of each cohort in that specific year.

The concept involves retroactively calculating the number of individuals in the cohort from one year ago, before the cohort experienced the natural and fishing mortalities of the previous year. The estimated cohort abundance for recent years is strongly influenced by the initial assumption about the fishing mortality rate, whereas the results become more consistent as one delves further into the historical data.

The extended version of VPA (XSA) proved to be an intermediary stage toward a more statistical approach, the State-Space Fish Stock Assessment Model (SAM) [30], which is now widely adopted within the ICES area and other regions for conducting cohort analysis. The SAM methodology yields significantly more dependable outcomes for recent years, which hold the utmost importance in formulating appropriate management recommendations.
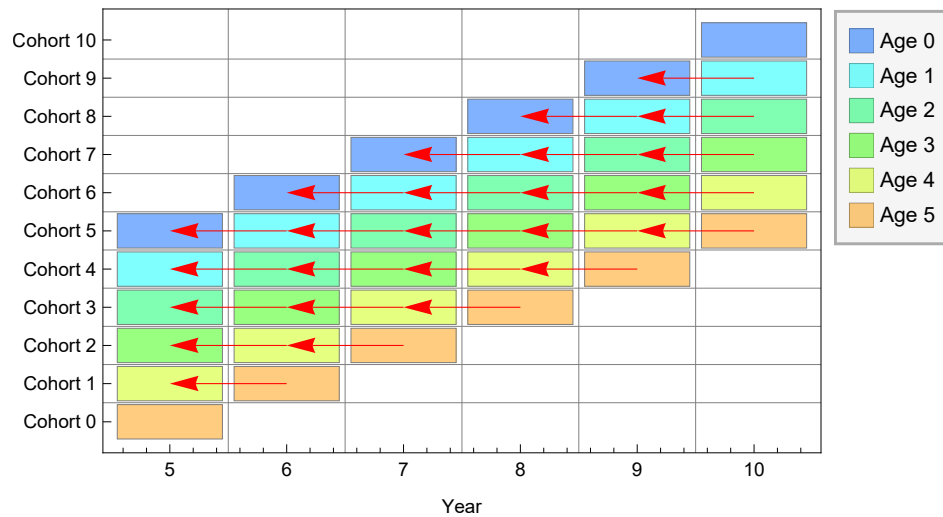
Figure 5.5: The figure illustrates how different year-classes appear at different ages in different years. Each of the cohorts constituting the catches over time could be followed through their life spans, as depicted by the red arrows.

Code box 5.5.1 — Virtual Population Analysis.

In this example we assume catches of six different age groups over a period of ten years

```
In[1]:= c = 6; y = 10;
```

Catch data is obtained randomly

```
In[2]:= SeedRandom[1234];
        data = Transpose[
          Table[Table[RandomInteger[{1, 1000}], {y}], {c}]];
        catch = Transpose[
          Table[
            Join[PadLeft[{0}, Length[data] - i], data[[i]],
            PadLeft[{0}, i - 1]],
            {i, Length[data]}
          ];
```

We can table the random catches by cohorts over time:

```
In[5]:=    TableForm[catch, TableAlignments -> Right]
```

Out[5]=									
0	0	0	0	0	0	0	0	0	89
0	0	0	0	0	0	0	0	532	919
0	0	0	0	0	0	0	816	270	574
0	0	0	0	0	0	638	744	46	906
0	0	0	0	0	134	48	534	492	70
0	0	0	0	530	871	950	694	860	755
0	0	0	769	689	709	543	547	184	0
0	0	642	239	81	150	41	79	0	0
0	663	757	901	707	984	578	0	0	0
898	531	713	588	684	312	0	0	0	0
390	48	207	821	922	0	0	0	0	0

445	652	948	830	0	0	0	0	0	0
970	404	18	0	0	0	0	0	0	0
879	649	0	0	0	0	0	0	0	0
842	0	0	0	0	0	0	0	0	0

The problem now is to back-calculate possible total numbers within each cohort over time. In order to do that we need an estimate of the natural mortality rate (M) of the stock and guess the fishing mortality rate (F) of the last year. We call the first m and the latter fT:

```
In[6]:= m = .2; fT = .3;
```

and finally we perform the back-calculation in order to find the matrix of individual numbers (here n) within each cohort:

```
In[7]:= n = Transpose@PadLeft[0, Reverse@Dimensions[catch]];
        Table[n[[Length[n]-i+1, i]] =
          catch[[Length[n]-i+1, i]]/(fT/(fT + m)*(1-Exp[-fT - m])),
        {i, y}];
        Table[n[[i, y]] =
          catch[[i, y]]/(fT/(fT + m)*(1 - Exp[-fT - m])), {i, c}
        ];
        Quiet@Table[
         cc = catch[[y - i + c - j, j]];
         xx = n[[y - i + c - j, j + 1]];
         temp = Reduce[cc == (1 - m/Log[x/xx]) (x - xx), x, Reals];
         n[[y - i + c - j, j]] =
           If[NumberQ[Last@temp], Last@temp, x /. ToRules[temp // Last]
         ],
        {j, y - 1, 1, -1}, {i, c - 1, 1, -1}];
```

The number of individuals in the stock at any time (horizontally), for each cohort (horizontal lines) at $M = 0.2$, when assuming the last fishing mortality rate to be $F = 0.3$, is found to equal

```
In[11]:= TableForm[n // Round, TableAlignments -> Right]
```

| Out[11]=	0	0	0	0	0	0	0	0	0	377
	0	0	0	0	0	0	0	0	5340	3893
	0	0	0	0	0	0	0	4888	3267	2431
	0	0	0	0	0	0	8772	6606	4738	3838
	0	0	0	0	0	2720	2106	1681	897	297
	0	0	0	0	15507	12217	9217	6690	4852	3198
	0	0	0	7473	5425	3821	2490	1551	779	0
	0	0	2469	1445	968	719	454	335	0	0
	0	13706	10623	8015	5750	4071	2448	0	0	0
	9313	6815	5101	3534	2364	1322	0	0	0	0
	1173	8795	7158	5673	3905	0	0	0	0	0
	9329	7236	5336	3516	0	0	0	0	0	0
	1697	527	76	0	0	0	0	0	0	0
	4323	2749	0	0	0	0	0	0	0	0
	3567	0	0	0	0	0	0	0	0	0

Finally, the fishing mortality rate matrix is calculated on the same format

```
In[13]:= f = Transpose@Append[Transpose[
          Table[If[n[[j, i]] > 0 && n[[j, i + 1]] > 0,
            Log[(Take[n[[j]], {i, i + 1}] /. {List -> Divide})] - m, 0
            ],
            {j, Length@catch},
            {i, Length@Transpose@catch - 1}
          ]
        ],
        PadRight[Table[fT, {c}], Length[catch]]
        ];
       Table[f[[Length[f] - i + 1, i]] = fT, {i, y}];
```

We have now found the fishing mortality rates corresponding to the numbers obtained above. The fishing mortality rate matrix is:

```
In[15]:= TableForm[
          PaddedForm[#, 4, 3] & /@ # & /@ f,
          TableAlignments -> Right
        ]
```

0	0	0	0	0	0	0	0	0	0.300
0	0	0	0	0	0	0	0	0.116	0.300
0	0	0	0	0	0	0	0.203	0.095	0.300
0	0	0	0	0	0	0.084	0.132	0.011	0.300
0	0	0	0	0	0.056	0.025	0.428	0.907	0.300
0	0	0	0	0.038	0.082	0.120	0.121	0.217	0.300
0	0	0	0.120	0.151	0.228	0.274	0.488	0.300	0
0	0	0.336	0.201	0.097	0.260	0.105	0.300	0	0
0	0.055	0.082	0.132	0.145	0.308	0.300	0	0	0
0.112	0.090	0.167	0.202	0.382	0.300	0	0	0	0
0.039	0.006	0.032	0.173	0.300	0	0	0	0	0
0.054	0.105	0.217	0.300	0	0	0	0	0	0
0.970	1.733	0.300	0	0	0	0	0	0	0
0.253	0.300	0	0	0	0	0	0	0	0
0.300	0	0	0	0	0	0	0	0	0

Out[15]= (see table above)

Note that the first guess of fishing mortality rate ($fT = 0.3$) is applied not only on all cohorts the last year, but also on the final catch of each cohort over time.

DEMO
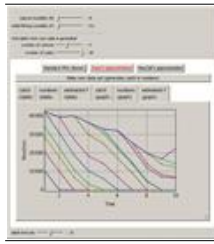
Explore different approaches to VPA

http://demonstrations.wolfram.com/
preview.html?draft/80065/000065/VirtualPopulationAnalysisVPA/

5.6 Cellular automata modelling

DEMO
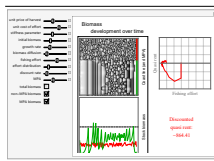
Explore a 1D cellular automata model in which Marine
Protected Area (MPA) is implemented as a regulatory measure at

http://demonstrations.wolfram.com/
CellularAutomataModelOfAnMPAFishery/

5.7 Stage structured fishery models

DEMO
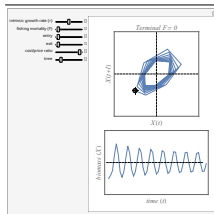
Explore the discrete Ricker population model with delayed
recruitment at

http://demonstrations.wolfram.com/
BioeconomicsOfADiscreteRickerModelWithDelayedRecruitment/

Exercises

Exercise 5.1 At what stock size is MSY obtained in the case of Equation 5.7? ▪

II

Fisheries economics

6. The economics of catch production

Fishing is an economic activity. So far, our focus has been on fishing technology and the necessary factors for harvest production. The production functions we have discussed all assume technological efficiency, meaning that no input factors are wasted. These production functions provide the minimum quantities of production factors necessary to achieve a given output quantity from a technological perspective. Now, we will attempt to identify which combinations of minimum factor levels are also economically efficient.

6.1 The Economics of Effort Production

While we have assumed technological efficiency in the production functions described earlier, we will now introduce the concept of economic efficiency. There are two approaches to achieving economic efficiency in production: minimising the cost of a given production or maximising production within a given budget. In the box below, we provide the *Mathematica* code following the latter principle.

To study economic efficiency in production, we need an expression for the cost of production. We assume the production function in equation 2.3 and consider constant market prices for the two input factors, labour (l) and capital (k). Let the price of labour be w (wage) and the price of capital be i (interest rate). The total cost of utilising labour and capital in the production process is then

$$C(L,K) = w \cdot L + i \cdot K \tag{6.1}$$

As seen from the calculation in Code Box 6.1.1, we find that both technologically and economically efficient Cobb-Douglas production is achieved when the following condition is met:

$$\frac{L}{K} = -\frac{i \cdot \alpha}{w \cdot \beta} \tag{6.2}$$

We observe that the ratio between the two input factors changes when there is a change in one of the prices. An increase in wages (w) leads to a reduction in the use of labour (L), which is then substituted by capital (K) in order to maintain the previous production level. Additionally, we note that the output elasticities of labour and capital influence how labour is substituted for capital.

Code box 6.1.1 — Economic efficiency in production.
The Cobb-Douglas function with constant elasticity of scale equal $\alpha + \beta$:

```
In[1]:= cd[l_, k_] := A * l^α * k^β
```

The cost of production,w being the cost of labour and i the cost of capital:

```
In[2]:= c[l_, k_] := w*l + i*k
```

The problem is to maximise the production within a given budget restriction R. We formulate the Lagrange equation for the problem of constrained maximisation:

```
In[3]:= lagrange[l_, k_] := cd[l, k] - λ (R - c[l, k])

In[4]:= Sequence@@Solve[
          ((λ /. Solve[D[lagrange[l, k], l] == 0, λ][[1]]) ==
            (λ /. Solve[D[lagrange[l, k], k] == 0, λ][[1]])) /. {
          l -> k * x}, x
        ][[1]] /. {x -> l/k}
```

$$Out[4]= \quad \frac{l}{k} \;\rightarrow\; -\frac{i}{w}\,\frac{\alpha}{\beta}$$

Code box 6.1.2 — Economic efficient expansion paths in production.
We start as previously by defining the Cobb-Douglas function:

```
In[1]:= cd[l_, k_] := A * l^α * k^β
```

The cost equation (equation 6.1):

```
In[2]:= c[l_, k_] := w*l + i*k
```

Cost efficient input of capital as a function of labour input is found from expression 6.3:

```
In[3]:= k[l_] := l * w * β / (i * α)
```

Now we plot the directions of the expansion paths for different values of α and β. The elasticity of scale ($\epsilon = \alpha + \beta$)in first three plots equal one, while the last plot shows a case where $\epsilon = \alpha + \beta = 1 + 1 = 2$:

```
In[4]:= GraphicsGrid[
          Partition[
            Show[{
              ContourPlot[cd[l, k] /. {
                  A -> 1, α -> #,  β -> If[# < 1, 1 - #, 1]},
                {l, 0, 1}, {k, 0, 1},
                ContourShading -> None,
                Contours -> {.02, .1, .2, .4, .6, .8}
```

```
        ],
        Plot[k[l] /. {
            α -> #, β -> If[# < 1,1 - #, 1], w -> 1, i -> 1},
          {l, 0, 1},
          PlotStyle -> Red
        ]},
        PlotLabel -> "α = " <> ToString[#] <> ",
          β = " <>  ToString[If[# < 1,1 - #, 1]],
        FrameLabel -> {"Labour (L)", "Capital (K)"},
        FrameTicks -> None
      ] & /@ {.3, .5, .7, 1}, 2
    ],
    Spacings -> {0, 20}
  ]
```

Out[4]=
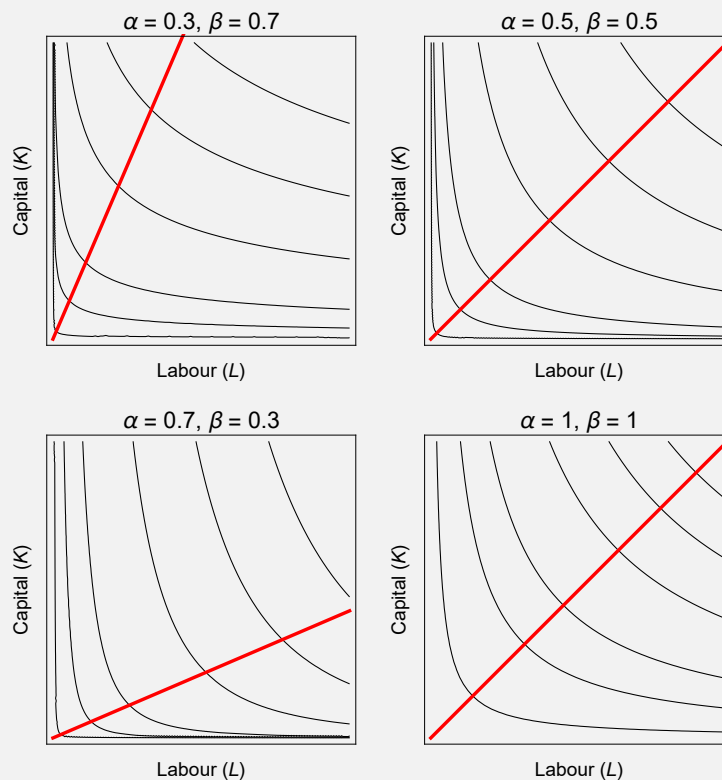
Note that the red expansion path in the case of $\alpha = \beta = 0.5$ is similar to the case when $\alpha = \beta = 1$. From expression 6.2 it is easy to see that this indeed must be true. The production level of the latter case increases, however, at the rate 2:1 compared with the first case.

From expression 6.2, we can observe that the cost-efficient combination of input factors remains constant for given prices (i and w) and given output elasticities (α and β). Therefore, the expansion path described by the optimal combination as production levels vary is linear, as illustrated in Code Box 6.1.2 below.

Expression 6.2 can be rearranged into an equation that explains the efficient quantity of

capital (K) in terms of labour usage (L).

$$K(L) = \frac{w \cdot \beta}{i \cdot \alpha} \cdot L \tag{6.3}$$

As described in Section 2.6, the elasticity of substitution of the Cobb-Douglas function is one. In a typical production process, the elasticity of scale is also expected to be equal to one, as shown in equation 2.2. The plots in Code Box 6.1.2 also illustrate the case of $\epsilon > 1$, representing situations where economies of scale exist.

Code box 6.1.3 — Unit cost of production.

Let's continue from Code box 6.1.2. Let's further assume that $\beta = 1 - \alpha$, as in equation 2.2. The unit cost of production is equation 6.1 divided by equation 2.2 when including equation 6.3. When specifying basic assumptions, we find:

```
In[5]:= Simplify[
          c[l, k[l]] / cd[l, k[l]] /. {β -> 1 - α},
          Assumptions -> {
            Element[{l, w, i, A, α}, Reals],
              w > 0, i > 0, l > 0, A > 0, 0 < α < 1}
          ]
```

$$\text{Out[5]}= \;\; -\frac{i\,(i\,\alpha)^{-\alpha}\,(w - w\,\alpha)^{\alpha}}{A\,(-1 + \alpha)}$$

From this result, we can conclude that for a Cobb-Douglas production process with an elasticity of scale equal to one, the unit cost of output is constant in an economically efficient production.

As seen below, this is not necessarily the case when the elasticity of scale is different from one:

```
In[6]:= Simplify[
          c[l, k[l]] / cd[l, k[l]],
          Assumptions -> {
            Element[{l, w, i, A, α, β}, Reals],
              w > 0, i > 0, l > 0, A > 0, 0 < α < 1, 0 < β < 1}
          ]
```

$$\text{Out[6]}= \;\; \frac{l^{1-\alpha}\,w\,(i\,\alpha)^{\beta}\,(l\,w\,\beta)^{-\beta}\,(\alpha + \beta)}{A\,\alpha}$$

In the special case of $\alpha = \beta = 1/2$ the unit price of production is found to be

```
In[7]:= Simplify[
          c[l, k[l]] / cd[l, k[l]] /. {α -> 1/2, β -> 1/2},
          Assumptions -> {
            Element[{l, w, i, A}, Reals], w > 0, i > 0, l > 0, A > 0}
          ]
```

$$\text{Out[7]}= \;\; \frac{2\,\sqrt{i\,w}}{A}$$

6.2 Total cost and revenue

As shown in Code Box 6.1.3, assuming a constant unit cost of effort production may be a reasonable assumption. Let's assume a constant unit cost of effort denoted as a. Furthermore, assume that the unit cost includes all expenses, including the opportunity costs of labour and capital. Therefore, the cost includes a normal profit. For instance, a *normal profit* is achieved when the income covers the cost. The total cost (TC) of producing fishing effort E is then given by:

$$TC(E) = a \cdot E \tag{6.4}$$

Additionally, let's consider a constant unit price of harvest denoted as p, and assume an equilibrium catch equation, such as Equation 5.7. The total revenue in the fishery (TR) is then given by:

$$TR(E) = p \cdot H(E) \tag{6.5}$$

Since a normal profit is accounted for in the unit cost of effort (a), the total economic rent in the fishery (R), often referred to as abnormal or super-normal rent, can be calculated as the difference between TR and TC:

$$R(E) = TR(E) - TC(E) \tag{6.6}$$
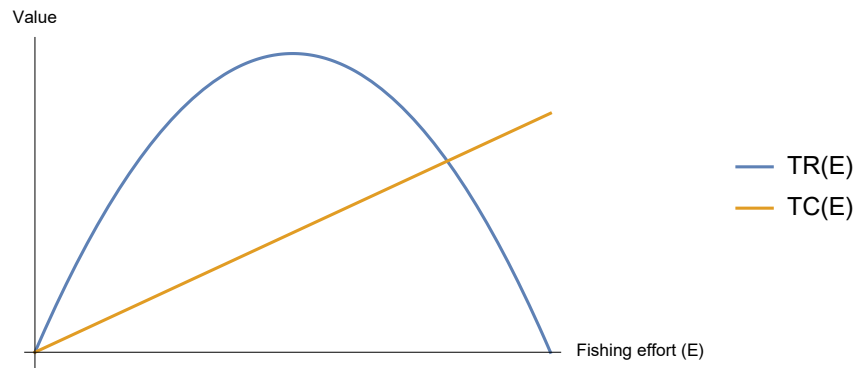
Figure 6.1: The figure illustrates a possible relationship between TC (Equation 6.4) and TR (Equation 6.5).

6.3 Economic rent in fisheries

According to *The New Palgrave Dictionary of Economics*, "*Rent* is the payment for the use of a resource, whether it be land, labour, equipment, ideas, or even money"[1]. The rent for labour is called *wages*, and the payment for money is called *interest*. Payment for a resource whose availability is insensitive to the size of the payment is called *economic rent*, if the insensitivity is permanent, or *quasi-rent* if the insensitivity is temporary.

The British economist Alfred Marshall (1842 - 1924) made several attempts to clarify different terms used to characterize different types of rent[38] and introduced the term *quasi-rent*. For the early classical economists, *rent* was synonymous with *land rent*. It soon became evident that this definition was too narrow, and a number of new terms were introduced, such as *differential rent* and *scarcity rent*. Even after Marshall's contributions,

there is still a lot of confusion related to the use of terms referring to different types of rent. While some economists still adhere to the original definitions, others employ more modern definitions of the terms.

In the context of this book, we will label all payments beyond what may be considered a *normal profit* as *economic rent*. Economic rent may have different origins, and these types of rents may be referred to by specific terms, all of which fall within the category of economic rent. Hence, economic rent is a broad term that includes all kinds of abnormal profits, such as *resource rent*, *differential rent*, *quasi rent*, and *monopoly rent*. This list illustrates some of the confusion related to the labelling of rent. For example, resource rent may be collected as monopoly rent, while differential rent primarily occurs in the short run, as does quasi rent.

The Ricardian land rent, named after the British economist David Ricardo (1772 – 1823), is in modern economics referred to by the more general term *resource rent*. *Resource rent* comprises abnormal profits that originate from all kinds of natural resources, including fish stock resources.

6.4   The model predicts a homogeneous fleet

We assume that each decision maker (vessel owner) follows economic rational behaviour, maximising their individual profit. Furthermore, assume that each vessel has a cost pattern with the standard properties we know from microeconomic production theory. The marginal cost of producing fishing effort (the red curves in Figure 6.2) decreases at low production levels and reaches a minimum level, after which the marginal cost increases with increased production. In this context, we regard a normal profit as an opportunity cost included in the cost equation.

We also assume that all vessel owners are price takers and that the fishing effort produced by one unit is not sufficient to affect the stock size. Consistent with equations 3.1 and 6.5, the revenue of vessel (i) during a given time period is expressed by a constant stock biomass (X), and the only variable is the fishing effort produced by the vessel (u_i):

$$tr_i = p \cdot q_i \cdot X \cdot u_i \tag{6.7}$$

Here, q_i is the vessel-specific catchability coefficient, and p is the constant unit price of harvest. We use the symbol u here to identify the fishing effort produced by one unit (vessel or fisher), rather than e, to avoid confusion with the mathematical constant e (Euler's constant). The total fishing effort is the sum of all unit efforts. For n units, we have:

$$E = \sum_{i=1}^{n} u_i \tag{6.8}$$

Now, we introduce a more general cost equation than the one assumed in equation 6.17, denoted as tc_i, which includes the opportunity costs of labour and capital. The profit-maximising vessel will choose the fishing effort where the marginal revenue equals the marginal cost:

$$tc_i'(u_i) = tr_i'(u_i) = p \cdot q_i \cdot X \tag{6.9}$$

Here, X is the equilibrium biomass, given the biological properties of the resource and the sum of fishing effort produced by all vessels. The expression on the right side of equation 6.9 represents the marginal revenue, indicated by the blue lines in Figure 6.2.
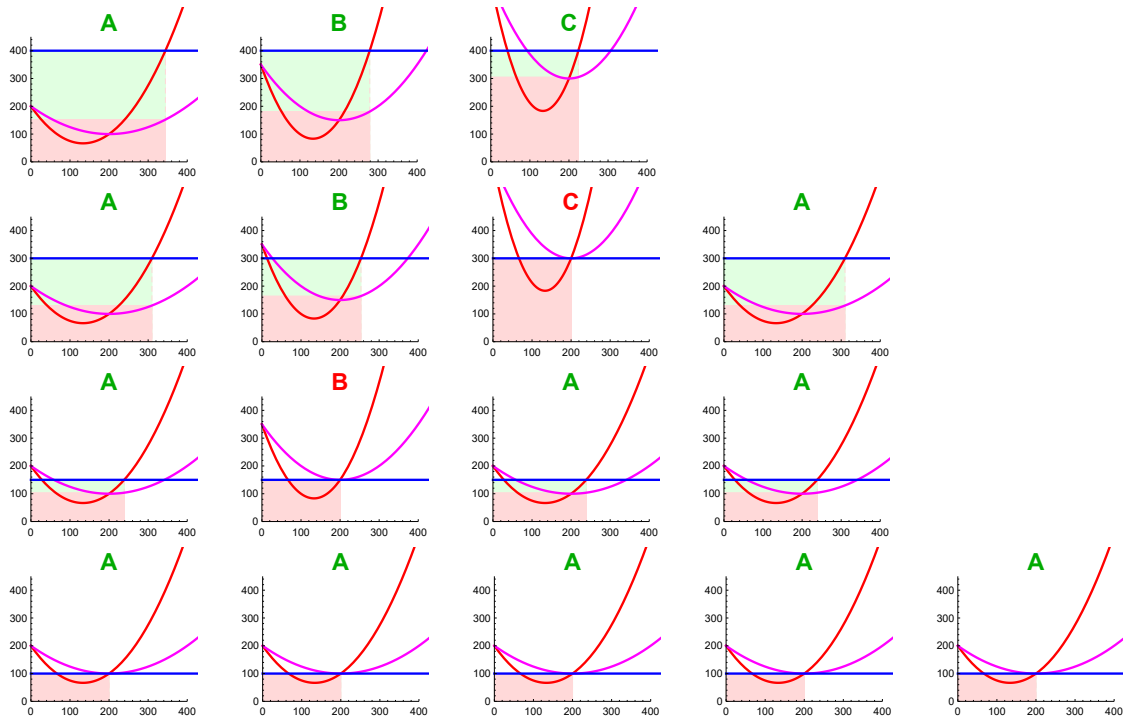
Figure 6.2: This figure indicates a possible development in an open-access fishery from a diverse fleet (vessels A, B, and C in the top line) towards a homogeneous fleet in the bottom line, where all vessels are identical (vessel type A is the most cost-efficient). The blue lines represent marginal revenues, the red curves represent marginal costs, and the magenta curves represent average costs. The light red areas indicate the total cost of production (including a normal profit), while the light green areas represent economic rent. As the production of fishing effort increases, the marginal revenue declines. The four lines represent a step-by-step development, where the less cost-efficient vessels (types B and C) leave the fishery when they are not able to earn a normal profit.

Fishing occurs when the equality 6.9 is satisfied, and $tr'_i(u_i) \geq tc_i/u_i$. In an open-access fishery, equilibrium is ultimately achieved when $tc'_i(u_i) = tc_i/u_i = p \cdot q_i \cdot X_\infty$, where only the most cost-efficient type of vessel (vessel A in Figure 6.2) remains. In this situation, all vessels earn a normal profit, while all economic rent (green areas in Figure 6.2) has disappeared.
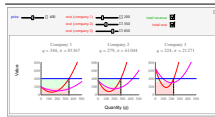
DEMO Study Intra-marginal rent in the Wolfram Demonstration at

http://demonstrations.wolfram.com/IntramarginalRent/

6.5 Pure open access equilibrium

An open and unregulated fishery is often referred to as a *pure open-access fishery*, distinguishing between purely unregulated fisheries and fisheries that are open to all (open-access fisheries) but are under given, common regulations such as gear use, area restrictions, etc.

See subsection 10.1 for a more precise definition.

In perfect markets (read more about perfect markets in section 7.2), the allocation of labour and capital will stabilise where the payoff for each normalised input factor in production is equivalent in all placements. Hence, no additional gain can be obtained by moving labour and capital elsewhere. The obtained profit is referred to as normal profit, which is included in the total cost (TC) defined in Section 6.2.

In a pure open-access fishery, we assume that the single decision makers are each unit of effort (for example, each boat or each fisher). The number of decision units depends on the profitability of the fishery and the speed of entrance to and exit from the fishery. We assume that all decision makers have the same opportunity costs of labour and capital; in other words, there is a given normal profit that is constant for all. We also assume that this normal profit remains constant over time, as we aim to relate the open-access development over time to the equilibrium model described in Section 6.2.

Equilibrium profit beyond the normal profit (abnormal profit) is given by Equation 6.6 as a function of fishing effort. When considering non-equilibrium cases, it is convenient to express the abnormal profit as a function of harvest. Equation 3.1 expresses harvest as a function of two independent variables: effort (E) and stock size (X). This equation is therefore also valid outside the case of equilibrium harvest (short-term harvest). According to Equation 3.1, fishing effort is expressed as

$$E = \frac{H}{q \cdot X} \tag{6.10}$$

Inserting Expression 6.10 into Equation 6.4 gives the short-term total cost as a function of the current stock size (X) and harvest (H):

$$TC(X,H) = \frac{a}{q \cdot X} \cdot H \tag{6.11}$$

The first term on the right-hand side is the unit cost of harvest, $c(X)$:

$$c(X) = \frac{a}{q \cdot X} \tag{6.12}$$

Since the constant price p is the unit income per harvest, the short-term total economic rent (R) can now be expressed as a function of harvest (H):

$$R(X,H) = \Big(p - c(X)\Big) \cdot H \tag{6.13}$$

From Equations 5.2, we have

$$\dot{X} = f(X) - H(E,X)$$

where $f(X)$ represents any population growth functions. In equilibrium ($\dot{X} = 0$), the total economic rent is given by Equation 6.13:

$$R(X) = \Big(p - c(X)\Big) \cdot f(X) \tag{6.14}$$

corresponding to Equation 6.6.

Code box 6.5.1 — Pure open access equilibrium.

Equation 5.7 is obtained directly from Equations 2.1 and 3.1 through these steps

```
In[1]:= f[x_] := r x (1 - x/k)
```

```
In[2]:= h[x_, e_] := q x e
```

```
In[3]:= x[e_] := x /. Solve[f[x] == h[x, e], x][[2]]
```

```
In[4]:= h[e_] := h[x[e], e]
```

```
In[5]:= Plot[
          Evaluate[h[e] /. {r -> 1, k -> 1, q -> 1}],
          {e, 0, 1},
          AxesLabel -> {"Fishing effort (E)", "Equilibrium harvest (H)"}
        ]
```

Out[5]=
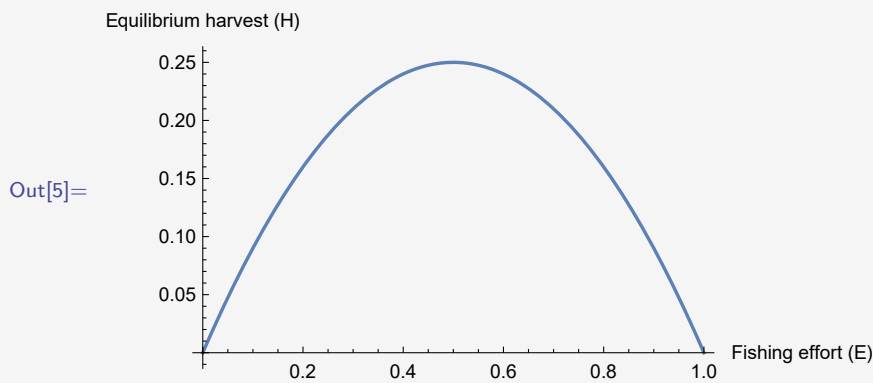
Equilibrium cost and revenue are

```
In[6]:= tc[e_] := a e
```

```
In[7]:= tr[e_] := p h[e]
```

Following the definition of a pure open access fishery (average revenue equals marginal cost) the open access stock biomass (X_∞) is found to be given by the price/cost ratio:

```
In[8]:= xoa = x /. First@Solve[p h[e, x]/e == tc'[e], x]
```

$$\text{Out[8]}= \quad \frac{a}{p\ q}$$

This finding makes it straight forward to identify the open access fishing effort (E_∞):

```
In[9]:= eoa = e /. First@Solve[tr[e]/e == tc'[e], e]
```

$$\text{Out[9]}= \quad -\frac{(a - k\ p\ q)\ r}{k\ p\ q^2}$$

Following the definition of a pure open access fishery (average revenue equals marginal cost)

```
In[10]:= hoa = q eoa xoa
```

$$\text{Out[10]}= \quad -\frac{a\ (a - k\ p\ q)\ r}{k\ p^2\ q^2}$$

In a perfect market, we expect all participants to receive a normal profit (as described by Equation 6.6) after maximising their individual profit. Since each fishing unit (u) utilises

the same biological resource (X) as all other units (a total of n units), the stock size is given. The first-order condition for a maximum is that marginal revenue equals marginal cost, using unit fishing effort e_i as the control variable:

$$p \cdot X(E) \cdot q_i = a_i \tag{6.15}$$

under condition 6.8, assuming that each unit has an individual catchability coefficient (q) and a unit cost of effort (a). From Equation 6.15, we can express the equilibrium stock level by the cost-price ratio:

$$X = X(E) = \frac{a_i}{p \cdot q_i}, \quad i = x \in \mathbb{Z} | 1 \leq x \leq n \tag{6.16}$$

When all units receive a normal profit, according to Equation 6.6, we also have:

$$p \cdot X_\infty \sum_{i=1}^{n} q_i u_i = \sum_{i=1}^{n} a_i u_i \tag{6.17}$$

Since all vessels face the same stock level ($X_\infty = X(E_\infty)$, where E_∞ is the open access equilibrium fishing effort, see Code box 6.5.1 for calculation details), according to Equation 6.16, the relationship a_i/q_i has to be constant for all i. Let's denote this constant ratio as ρ:

$$\rho = \frac{a_i}{q_i}$$

which, when inserted into Equation 6.17, gives the equilibrium expression for X_∞:

$$X_\infty = \frac{\rho}{p} \tag{6.18}$$

We can conclude that the fishing units may differ in scale but not in fishing efficiency. As standardised units, they are all identical in the equilibrium solution. In principle, they may differ when thrown out of equilibrium (due to external factors leading to a decline or increase in stock size), which would then change the profits of different units in different ways. Those earning less than a normal profit will leave the fishery in the perfect market, while those earning a profit beyond the normal level (positive economic rent) will stay and, if possible, increase their fishing effort. A new equilibrium solution is found when all units again obtain normal profits. Ultimately, a global equilibrium solution is found for a fleet that has now become homogeneous.

By definition, fishing efficiency (technological properties) and economic performance (economic properties) are identical within the homogeneous fleet. Hence, $q_i = q$ and $a_i = a$ for all i, and Equation 6.15 is generalised to:

$$p \cdot q \cdot X(E_\infty) = a \tag{6.19}$$

which, also in accordance with Equation 6.16, expresses the open access equilibrium stock biomass:

$$X_\infty = X(E_\infty) = \frac{a}{p \cdot q} \tag{6.20}$$

Equation 6.19 expresses the general definition of open access equilibrium; when the average revenue of the fleet ($p \cdot q \cdot X(E)$) equals the fleet's marginal cost (a).

Note that Equation 6.20 implies that the open access stock level is determined by the cost-price ratio (a/p) and that the biological parameters (r and K) are not included in the expression. If the cost of fishing increases, a larger stock is needed to earn a normal profit. Similarly, an increased price makes it possible to earn a normal profit at a lower stock level. Also, note that as long as $a > 0$, then $X_\infty > 0$. This means that in our model, the stock will never go extinct as long as there are costs associated with fishing.

The unit cost of harvest is given by Equation 6.12 as a function of stock biomass X. By inserting the expression for X_∞ above (Equation 6.20), we see that the unit cost of harvest equals the unit price p, confirming that the open access equilibrium indeed gives a normal profit (which is embedded in the unit cost of effort, a). The path towards the equilibrium solution, however, may include both positive and negative economic rent. A further analysis of this is provided in Code box 6.6.1.

The open access fishing effort (E_∞) is found by inserting Equation 6.20 into Equation 3.1, which is equivalent to Equation 5.7 in equilibrium. Then, solve it for E_∞ (when $E_\infty > 0$):

$$q \cdot \frac{a}{q \cdot p} \cdot E_\infty = q \cdot K \cdot E_\infty \left(1 - \frac{q}{r} E_\infty\right)$$

$$E_\infty = \frac{r}{q} - \frac{a \cdot r}{p \cdot q^2 \cdot K} \tag{6.21}$$

6.6 Open access dynamics

In the section above, we discussed the equilibrium solution of open access to a fishery given perfect markets, biological growth as a function of stock biomass and fishing effort, and a catch function which is linear in the same variables (stock biomass and fishing effort). This section deals with the dynamics leading to the equilibrium solution.

As indicated in Section 4.7, fishing, in terms of modelling, represents the predator in a prey-predator relationship. Over time, the stock size (X) and the fishing effort (E) vary according to the biological, technological, and economic properties of the system. Equation 4.2 describes X and constitutes the first element of the system. How do we determine the changes in the other time-variable, E?

According to the reasoning in Section 7.4, the fishing effort is expected to increase when the profit exceeds the normal level and decrease when the profit is below the normal profit. Let's further assume a larger change when the difference from the normal profit is larger. The unit profit of harvest (Equation 6.12) indicates the levels of positive and negative differences. In addition, we include a stiffness parameter, γ, which determines how fast the changes in fishing effort take place. When the γ value is high, the increase/decrease in effort is large, while it is low for small values of γ. The open access dynamics is then described by:

$$\dot{X}(t) = r \cdot X(t) \cdot \left(1 - \frac{X(t)}{K}\right) - q \cdot E(t) \cdot X(t) \quad \dot{E}(t) = \gamma \cdot \left(p - \frac{a}{q \cdot X(t)}\right) \tag{6.22}$$

The phase plot of Equations 6.22 is illustrated in Figure 6.3 for a fixed γ value ($\gamma = 0.5$). In Code box 6.6.1, the numerical example is further analysed for different γ values. The phase plot provides three different illustrations of the system dynamics: isoclines, a vector field, and time paths. The system has a stable equilibrium at the intersection of the two isoclines.

Code box 6.6.1 — Open access dynamics.

We define a function `dsolve` to find the numerical solutions of the differential Equations 6.22 for given initial values of the variables (X and E) and the stiffness parameter (γ):

```
In[1]:= dsolve[{e0_, x0_, γ_}] := NDSolve[{
          x'[t] == r x[t] (1 - x[t]/k) - q  x[t] e[t],
          e'[t] == γ * (p - a/(q x[t])),
          x[0]  == x0,
          e[0]  == e0
          } /. values,
          {x[t], e[t]}, {t, 0, 50}
        ]
```

Apart from the stiffness parameter (which is a variable in the `dsolve` function) we assume the same parameter values as in Figure 6.3. We place the parameter values in the list `values`:

```
In[2]:= values = {k -> 1, r -> 1, q -> 1.5, a -> .15, p -> .5};
```

We will explore how the stiffness parameter influences the path towards the equilibrium solution, assuming an initial unexploited stock, $X = 1$, and a low fishing effort, $E = 0.01$. We investigate three cases, $\gamma = (0.1, 0.3, 0.6)$. For each case we produce two graphs of the path over time, first placed into the $(TR, TC) - E$ axes system and the second total economic rent (R) over time.

```
In[3]:= GraphicsGrid[Transpose[{
          Show[{
            Plot[
              Evaluate[{, p*q*k*e(1 - q e/r), a e}/.values], {e, 0, 1}],
            ParametricPlot[
              Evaluate[{e[t], p*q*e[t]*x[t]} /. values /.
                dsolve[{.01, 1, #}]], {t, 0, 50}
              ]},
            PlotRange        -> {{0, .85}, {0, .24}},
            PlotRangePadding -> None,
            Frame            -> {{True, False}, {True, False}},
            FrameLabel       -> {"Fishing effort", "TR, TC, OA path"},
            PlotLabel        -> "γ = " <> ToString[#]],
          Plot[
            Evaluate[(p*q*e[t]*x[t] - a e[t]) /. values /.
              dsolve[{.01, 1, #}]], {t, 0, 50},
            PlotRange    -> {-.05, .16}, Filling -> Axis,
            FillingStyle -> {Directive[Red, Opacity[.5]],
              Directive[Green, Opacity[.5]]},
            Frame        -> {{True, False}, {True, False}},
            FrameLabel   -> {"Time", "R"},
            PlotLabel    -> ""
          ]} & /@ {.1, .3, .6}]
        ]
```

Out[3]=
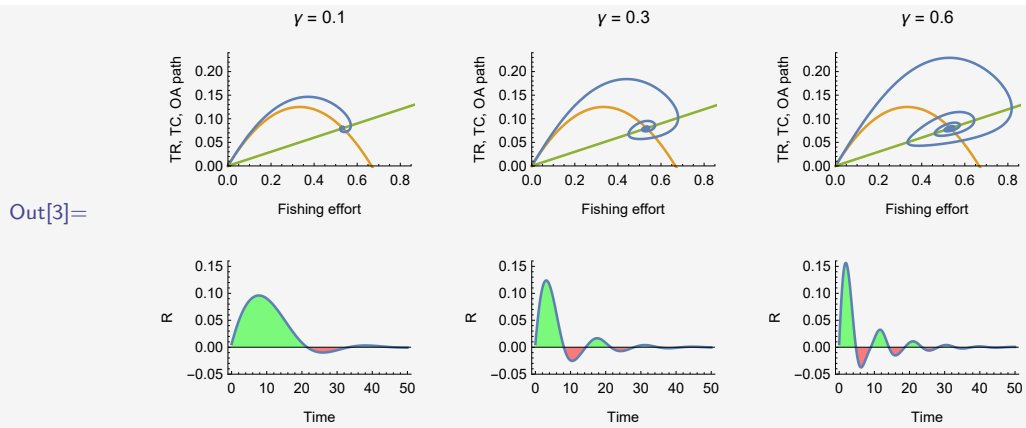
Positive economic rent is coloured green while negative economic rent is coloured red in the plots above. All three cases converges to the equilibrium solution within the investigated fifty periods.

Which of the three cases above yields the highest economic rent? Visually the not discounted flow of economic rent over time seems to be largest in the first case (to the left). Let us include more γ values and the case of discounting at an discount rate of ten percent.

```
In[4]:= gammalist = Table[.001*i^2, {i, 1, 10, .5}];
        Rlist = Flatten@N[
          Integrate[(p*q*e[t]*x[t] - a e[t]), {t, 0, 50}] /. values /.
            dsolve[{.01, 1, #}] & /@ gammalist];
        ListLinePlot[{
          Transpose[{gammalist, Rlist}],
          Transpose[{gammalist, Rlist *
            Table[1/1.05^t, {t, 1, Length[gammalist]}]}]},
          Mesh        -> All, PlotRange -> {0, All},
          AxesLabel   -> {"γ", "∫ R dγ"}, Filling -> Axis,
          PlotLegends -> {"Sum of R over 50 periods",
            "Sum of discounted R over 50 periods"}
        ]
```

Out[4]=
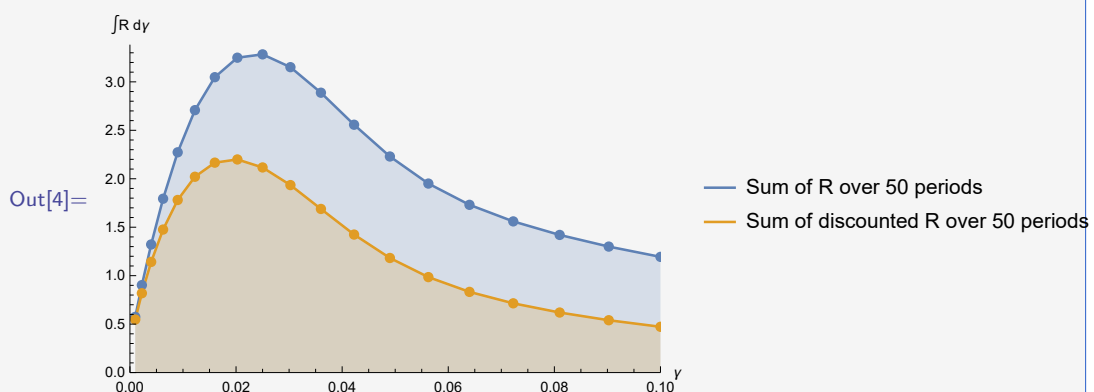
Both the two curves show maximum solutions at low γ levels, $\gamma < 0.03$, the discounted case below the other. In all cases, however, the sum is positive, reflecting the gain (also in an open access fishery) of starting a fishery on an unexploited stock.
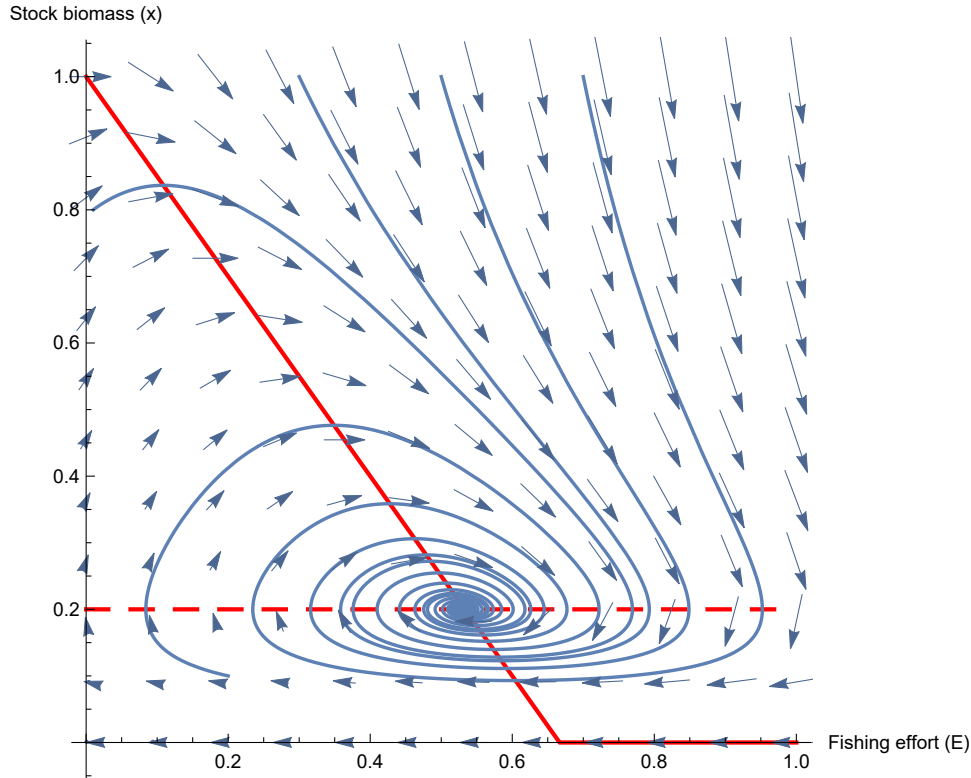
Figure 6.3: Phase plot of a pure open access fishery, as described by Equations 6.22, with the following parameter values: $K = r = 1$, $q = 1.5$, $a = 0.15$, $p = \gamma = 0.5$. Here, γ is the stiffness parameter that measures the speed at which the fleet readjusts outside of equilibrium. In the plot, the red dashed isocline represents the constant stock level $(X = a/(q \cdot p))$ at which the fleet earns a normal profit. The solid red isocline indicates the equilibrium levels of stock biomass at varying fishing effort, similar to Figure 5.1. The solid blue curves depict paths over time toward the equilibrium point from different initial positions. The vector field illustrates the strength and direction of the dynamics in different areas.

DEMO
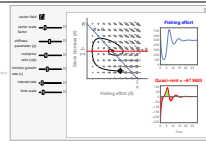
Quasi rent occurs in open access dynamics and can be studied in the WL demonstration at

http://demonstrations.wolfram.com/
QuasiRentInOpenAccessFisheries/

6.7   Maximising economic rent

The solution of maximising rent in equilibrium (Maximum Economic Yield, MEY) is straight forward when having established Equation 6.6. Scott Gordon[29] first published these results in 1954. The first and second order conditions of a maximum are

$$
\begin{aligned}
R'(E) &= TR'(E) - TC'(E) = MR(E) - MC(E) = 0 \\
R''(E) &< 0
\end{aligned}
\tag{6.23}
$$

MR is the marginal revenue with respect of fishing effort (E) and MC is the corresponding marginal cost of effort. The first order condition of a maximum tells that these two should equal each other, $MR = MC$. As illustrated in Figure 6.4 this occurs at the fishing effort E_{MEY}.
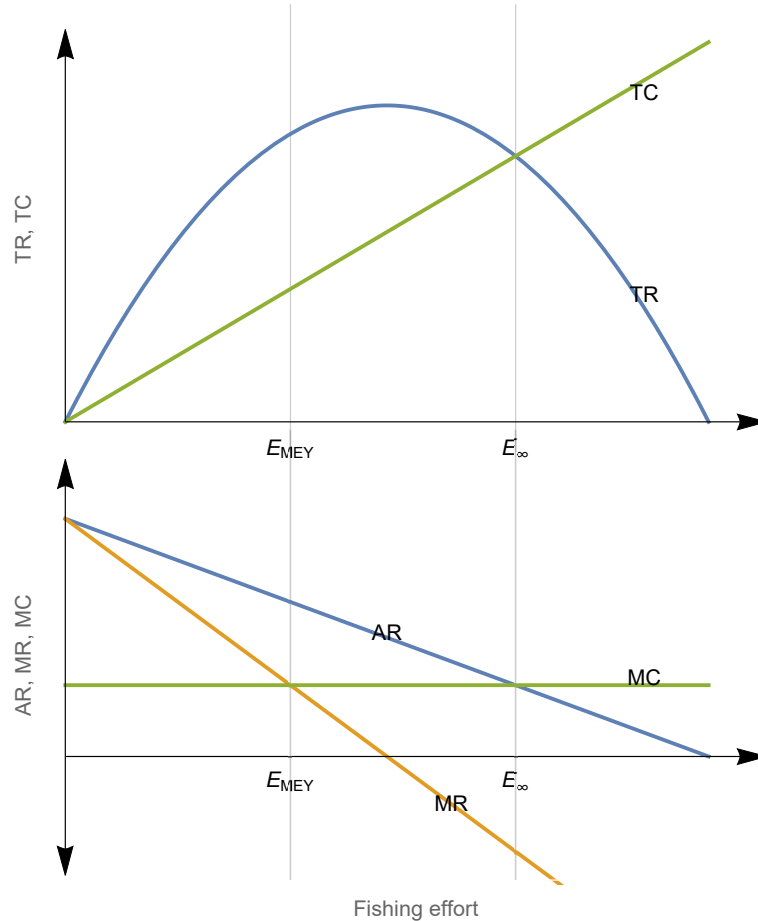
Figure 6.4: The classical presentation of the Gordon-Schaefer model is shown above, with aggregated revenues (TR, blue curve) and costs (TC, green line) on top, and below, the marginal picture includes average revenue (AR, blue line), marginal revenue (MR, orange line), and marginal cost (MC, green line). The equilibrium solutions for open access (E_{∞}) and maximum sustainable yield (E_{MEY}) are indicated by the two vertical lines.

The idea of Maximum Economic Rent (MEY) was established by Scott Gordon and Anthony Scott in the mid 1950ies and several economists believed it to soon replace the biological concept of Maximum Sustainable Yield (MSY) which is equilibrium harvest maximisation without any references to market values.

Additional remarks 6.1 — Talk by Anthony Scott (Bergen, 1993).
"But we should recognise now, I think, that we had been guilty of a confusing application of normative economics. All that fuss about MSY vs other targets should have stayed in the classroom, on the blackboard, for graduate students. The world had need for more of the predictions of economics, not their prescriptions, as Scott Gordon had showed. I now think we had little more business lecturing the fishing world on what they should maximise than we

would have had reproving a merchant for maximising sales or a farmer for maximising his fleet of shiny new equipment. Our comparative advantage lies elsewhere.

In any case some of us were just plain wrong-headed in our obsession with MSY. We had little or no data with which to measure the real-world difference between various optimum catches and stocks. We knew nothing about the regulatory and compliance costs attached to each optimum. We took little or no account of uncertainty, instability and irreversibility. We did not understand that the theory of the second best tended to undermine our faith in knife-edge piecemeal partial optimisation.

We should pause to consider the moral of that chapter in the development of fishery economics. It is still embarrassing to hear a biologist of that generation say apologetically, "of course you economists believe in something entirely different". It was not that we should not have lent our talents to policy formation. Nor should we ignore welfare economics. It is that before we pooh-pooh any particular goal, or instrument for that matter, we should use our economics to predict how it will work and who it will benefit. You may not think this a very surprising conclusion in this day and age. Perhaps it is not. Nevertheless, ignoring it, we wasted a lot of everyone's time. Our reliance on normative economics distracted us from prediction and from measurement activities where we could have done a lot of good."

DEMO
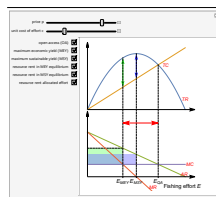
The Gordon-Schaefer model can be explored as a Wolfram Demonstration at

http://demonstrations.wolfram.com/TheGordonSchaeferModel/

6.8   Maximising present value of flow of rent over time

In his textbook on fisheries economics and management, Ola Flaaten presents a problem that involves maximising the present value of fishing over time, using a simple example of investment[23]. This example is framed as a cost-benefit problem, specifically the decision of whether to invest or not. The investment under consideration is to either fish one unit less in the current time period and establish a new equilibrium in the next period, or continue fishing at the current equilibrium catch level (assuming an initial equilibrium fishery).

The core of the problem is to compare the gain from fishing one unit less in the current time period and transitioning to a new equilibrium in the next period (the investment), with the loss incurred by the reduced catch in the current period. The latter loss can be simply expressed as the unit profit, $p - c(X_0)$, where X_0 represents the initial equilibrium stock size (refer to Equation 6.13). On the other hand, the gain involves calculating the discounted (see Code box 6.8.1) flow of the persistent difference in profits between the initial equilibrium condition and the new equilibrium over time. In discrete time, this

summation can be written as:

$$\sum_{t=1}^{\infty} \frac{1}{(1+i)^t} \Big((p - c(X_1)) \cdot f(X_1) - (p - c(X_0)) \cdot f(X_0) \Big)$$
$$= \Big((p - c(X_1)) \cdot f(X_1) - (p - c(X_0)) \cdot f(X_0) \Big)/i \qquad (6.24)$$
$$= \Big(R(X_1) - R(X_0) \Big)/i = \frac{\Delta R(X_0)}{i}$$

In this context, $f(X)$ represents the natural growth in stock as defined in Equation 5.2, R signifies the total economic rent described by Equation 6.14, and $\Delta R(X_0)$ represents the difference in total economic rent in equilibrium resulting from investing one unit of harvest at time $t = 0$. The investment problem is framed as a cost-benefit dilemma: Is it advantageous to invest an amount of $p - c(X_0)$ (representing the immediate loss from fishing one unit less) when the potential gain is $\Delta R(X_0)/i$? Clearly, the answer is affirmative if:

$$\frac{\Delta R(X_0)}{i} > p - c(X_0)$$

Conversely, the investment is deemed disadvantageous if:

$$\frac{\Delta R(X_0)}{i} < p - c(X_0)$$

When the immediate loss incurred by fishing one unit less today equals the long-term discounted gain over all future periods from fishing one unit less, the investor becomes indifferent between choosing to invest or not.

Code box 6.8.1 — Discounting.

Discounting is a method of reflecting intertemporal preferences in terms of values. Since most of us prefer receiving a given value (for example, 100€) sooner (for example, now) rather than later (for example, ten years from now), there will exist a value to be received in the future (in our case, more than 100€ in ten years from now) that corresponds to receiving the value now.

If we assume that the value diminishes by $100i$ percent per year when postponed into the future, the *present value* (PV) of 100€ received ten years from now is

$$PV = \frac{100}{(1+i)^{10}}$$

Or in general, let x be the value in t years from now representing the present value of today.

$$PV = \frac{x}{(1+i)^t}$$

Returning to our initial problem, to find the x-value corresponding to a present value equal 100, we have

```
In[1]:= Solve[100 == x*(1 + i)^-10, x]
```

```
Out[1]= {{x → 100(1 + i)^10}}
```

We see that for each positive value of `i` there is a corresponding value of `x` larger than 100. Consider the case of 10% discounting (`i = 0.1`)

```
In[2]:= % /. {i -> .1}
```

```
Out[2]= {{x → 259.374}}
```

This tells us that given an annual discount rate of ten percent we are indifferent between receiving 100€ today or about 260€ in ten years from now. With a discount rate of ten percent the present value of 260€ in ten years from now is 100€; or: The *future value* ten years into the future of 100€ today is 260€.

So far we have consider cases where we calculate year by year. Such step by step calculations are referred to as *discrete time* calculations. The discount term in such calculations are $(1+i)^t$ or $(1+i)^{-t}$ (depending on if we are calculating future or present values).

In continuous time the corresponding discount term is $e^{\delta t}$ or $e^{-\delta t}$, again depending on if we are considering future or present values. We have used two different symbols for discrete time and continuous time discount rates, i and δ, respectively. Let us investigate how the two relate to each other. First find the limit value of the time derivative of the discrete time expression and let time go towards zero

```
In[3]:= Limit[D[(1 + i)^-t, t], t -> 0]
```

```
Out[3]= -Log[1 + i]
```

Then, let us take the time derivative of the continuous time discount term and find the limit when time approaches zero

```
In[4]:= Limit[D[Exp[-δ t], t], t -> 0]
```

```
Out[4]= -δ
```

If the two marginal expressions above should equal each other, we have

```
In[5]:= Simplify[Solve[-Log[1 + i] == -δ, i], Assumptions -> δ >= 0]
```

```
Out[5]= {{i → - 1 + e^δ}}
```

As seen in the plots below the difference between the two increases by the value of the discount rate. However, within fairly realistic ranges of the discount rate (as indicated in the last figure) the deviation between discrete and continuous time discounting is small.

```
In[6]:= Plot[{δ, -1 + Exp[δ]}, {δ, 0, 1},
          PlotTheme        -> "Detailed",
          AspectRatio      -> 1.8,
          PlotRangePadding -> None
        ]
```

Out[6]=
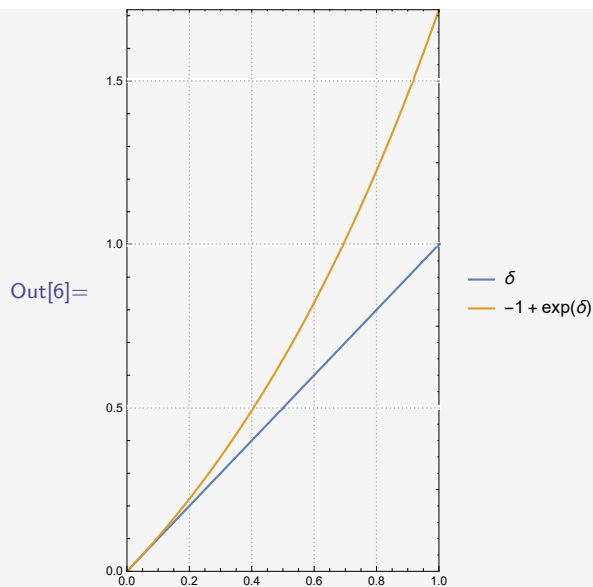
In[7]:= **Show[%,**
   PlotRange   -> {{0, .2}, {0, .2}},
   AspectRatio -> 1
]

Out[7]=
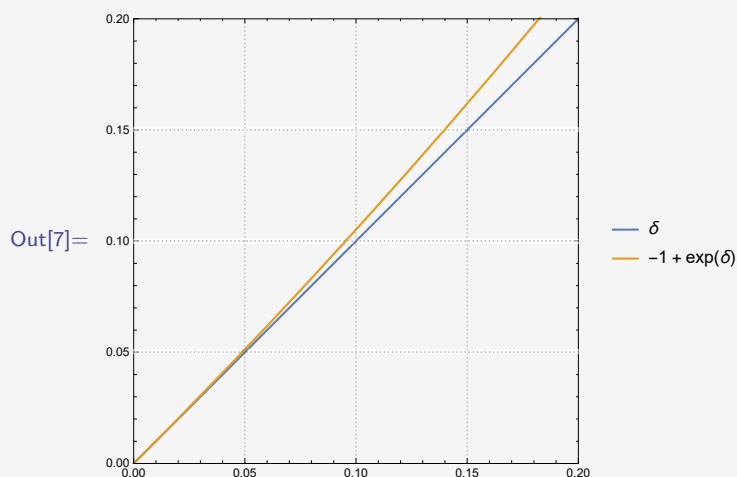
Let us now consider a discounted flow of a fixed value over time. In the discrete time case we have a infinite geometric series

$$\sum_{t=1}^{\infty} \frac{1}{(1+i)^t} = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^n} + \cdots$$

when the fixed value is 1 and the discount rate is i ($i > 0$), summing from $t = 1$ to $t = \infty$. *Mathematica* easily finds the sum

In[8]:= **Sum[(1 + i)^-t, {t, 1, Infinity}]**

Out[8]= $\dfrac{1}{i}$

The corresponding value when using continuous time discounting is the integral of the continuous time discount term, $e^{-\delta t}$, which gives

```
In[9]:= Integrate[Exp[-δ t], {t, 0, Infinity}, Assumptions -> {δ > 0}]

Out[9]=  1
         ─
         δ
```

Note that the expression above is the integral from $t = 0$ (not from $t = 1$ as above) to $t = \infty$.

It is straightforward to convert the discrete-time investment solution to continuous time. The solution of Equation 6.24 is the derivative of R with respect to X, and the investment is $R(X)/f(X)$. Indifference between investing or not is when

$$\frac{R'(X)}{\delta} = \frac{R(X)}{f(X)} \tag{6.25}$$

where δ is the continuous-time discount rate. Figure 6.5 shows how three reference points are defined by this expression. When the right-hand side of Equation 6.25 equals zero (when $p = c(X)$), no economic rent is earned in equilibrium. This is the bioeconomic equilibrium. The left-hand side of the same equation is zero when the marginal total economic rent is zero. Since the second-order derivative of R is negative, R is maximised at this point. $R'(X) = 0$ for $X = X_{MEY}$.
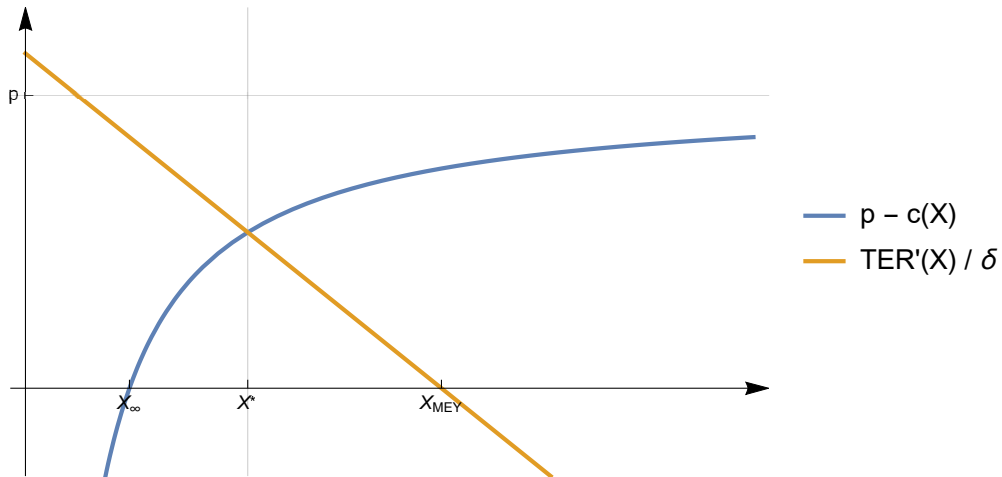
Figure 6.5: Graphical illustration of Equation 6.25, showing the point that maximises present value (at $X = X^*$) as the intersection between the right-hand expression of the equation, $p - c(X)$, and the left-hand expression, $R'(X)/\delta$.

Inserting $R(X)$ (Equation 6.14) and calculating gives the following equilibrium condition of present value maximisation:

$$f'(X) - \frac{c'(X) \cdot f(X)}{p - c(X)} = \delta \tag{6.26}$$

This condition is often referred to as the Golden Rule. The Golden Rule is confirmed by employing the Euler equation on the problem. The Euler equation for equations of the type $g(t, x, \dot{x})$ (which are concave in $(x, \dot{x})$) gives the first-order condition of a maximum of

$$\int_{t0}^{t1} g(t, x, \dot{x}), \mathrm{d}t$$

to be given by

$$\frac{\partial g(t,x,\dot{x})}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial g(t,x,\dot{x})}{\partial \dot{x}}\right) \tag{6.27}$$

In the case of our fisheries model (Equation 6.14), we want to maximise the present value (PV) of the flow of annually discounted rent:

$$PV = \int_0^\infty pv(t,X,\dot{X}), \mathrm{d}t \tag{6.28}$$

While assuming a discount rate equal to δ, pv is

$$pv(t,X,\dot{X}) = \Big(p - c(X)\Big) \cdot \Big(f(X) - \dot{X}\Big) \cdot e^{-\delta t} \tag{6.29}$$

We find the left-hand side of Expression 6.27 to be

$$\frac{\partial pv(t,X,\dot{X})}{\partial X} = \Big(f'(X) \cdot \Big(p - c(X)\Big) - c'(X) \cdot \Big(f(X) - \dot{X}\Big)\Big) \cdot e^{-\delta t} \tag{6.30}$$

Similarly, the right-hand side of Expression 6.27 is found by taking the $\dot{X}$- and time-derivative of Equation 6.29:

$$\frac{\partial}{\partial t}\left(\frac{\partial pv(t,X,\dot{X})}{\partial \dot{X}}\right) = \delta \cdot \Big(p - c(X)\Big) \cdot e^{-\delta t} \tag{6.31}$$

Since PV is maximised when Equation 6.30 equals Equation 6.31, we get this equilibrium ($\dot{X} = 0$) condition for maximum:

$$\Big(f'(X) \cdot \Big(p - c(X)\Big) - c'(X) \cdot (f(X)\Big) \cdot e^{-\delta t} = \delta \cdot \Big(p - c(X)\Big) \cdot e^{-\delta t}$$

$$f'(X) - \frac{c'(X) \cdot f(X)}{p - c(X)} = \delta$$

As we see, Equation 6.26 is then formally confirmed.

The Golden Rule (Equation 6.26) provides a useful interpretation of the optimal solution. The right-hand side, the discount rate δ, is the payoff received in the best alternative placement of the natural capital, while the left-hand side gives the payoff received by natural growth, adjusted for net extraction cost. If the cost is zero ($c(X) = 0$), also $c'(X) = 0$, hence $f'(X) = \delta$ (see Figure 6.6). In this case, the optimal equilibrium is obtained at the stock size where the marginal biological growth, $f'(X)$, equals the discount rate. Since $f'(X) > 0$ only when $X < X_{MSY}$, the optimal solution when $c(X) = 0$ and $\delta > 0$ will always result in a biologically overexploited resource.

For $c(X) > 0$ (the normal case), the value of the left-hand side of Equation 6.26 will increase compared with the situation above, since $c'(X) < 0$ (see Equation 6.12). In this case, negative values of $f'(X)$ are possible solutions (which means that $X > X_{MSY}$), depending on the price-cost ratio. The impact the cost equation ($c(X)$) has, therefore, suggests larger stocks, while the growth properties of the stock (particularly when assuming compensation growth) favour a lower stock size. The final result becomes a trade-off between the two forces, as expressed in Equation 6.26.
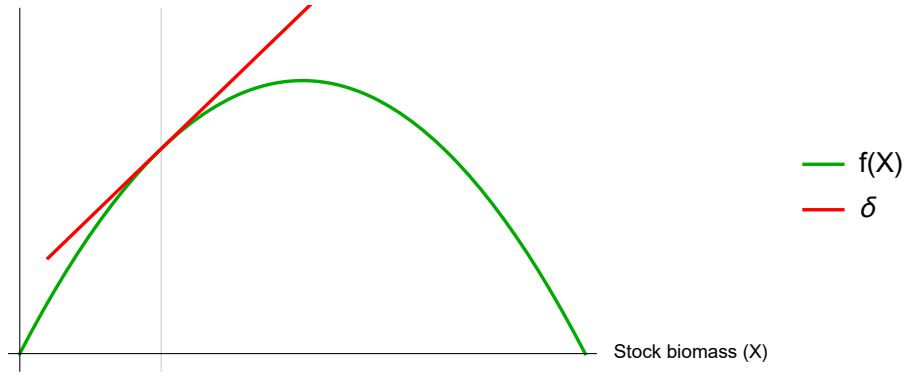
Figure 6.6: In the case of no cost $(c(X) = 0)$, the optimal equilibrium is obtained at the point where $f'(X) = \delta$. Here the discount rate (δ) equals the slope of the red line.

DEMO
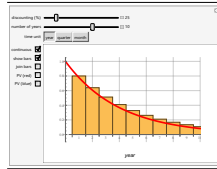
Investigate the differences between discrete and continuous time discounting at

http://demonstrations.wolfram.com/
ContinuousAndDiscreteTimeDiscounting/

6.9   Optimal control theory

In Section 6.8, we found the condition referred to as the Golden Rule. This condition must be satisfied while identifying the equilibrium that maximises the sum of discounted economic rent over time. However, the methods we have discussed so far do not explain how to obtain this equilibrium state in an optimal way. How could the optimal path toward the equilibrium be found?

In 1959, Lev Pontryagin (1908 – 1988) published a paper presenting what was later known as the *maximum principle*[15], and in 1979, Colin Clark and Gordon Munro applied the new theory to a simplified case of a fishery[12], corresponding to the problem described above. From Equation 5.2, we can substitute $\dot{X}(t)$ with $E(t)$ or $H(t)$; both fulfill the requirements of being *control variables* of the maximisation problem. Let us consider the case of Equation 6.28. The catch (H) is the control variable of the problem, and the stock size (X) is the *state variable*. The problem is to maximise PV within the time interval $t \in [0, T]$.

$$\max[PV] = \max\left[\int_0^T pv(t, X(t), H(t)), \mathrm{d}t\right] \tag{6.32}$$

Inserting Equation 6.29, we get

$$\max[PV] = \max\left[\int_0^T (p - c(X(t))) \cdot H(t) \cdot e^{-\delta t}, \mathrm{d}t\right] \tag{6.33}$$

The expression is maximised subject to the stock's biological growth constraint (Equation 5.2):

$$\dot{X} = f(X) - H$$

Equation 6.33 and the constraint above are expressed in a *Hamiltonian equation*, representing a conventional way of solving this type of problems:

$$\mathcal{H} = (p - c(X)) \cdot H \cdot e^{-\delta t} + \lambda(t)\,(f(X) - H) \tag{6.34}$$

The optimal catch, H, is the catch maximising the Hamiltonian for all $t \in [0, T]$. We assume $H \in [0, H_{max}]$.

Similar to a Lagrange equation, the constraint of the problem (in our case, the stock constraint) is represented by the last term where λ appears. While λ is a constant in the Lagrange equation, it is a function of time, t, in the Hamiltonian, since our constraint is now expressed as a differential equation. $\lambda(t)$ is often referred to as the adjoint or costate variable[12].

If H does not constrain the maximisation problem, we have the following necessary conditions of a maximum:

$$\frac{\partial \mathcal{H}}{\partial H} = 0 \tag{6.35}$$

$$\frac{\partial \mathcal{H}}{\partial X} = -\frac{d\lambda}{dt} \tag{6.36}$$

All pairs of $H(t)$ and $X(t)$ that satisfy these two conditions for $t \in [0, T]$ describe the optimal path towards $H(T)$ and $X(T)$.

Code box 6.9.1 — The Maximum Principle.
The Hamiltonian is given as a function of the state variable X and the control variable H in equation 6.34:

```
In[1]:= hamiltonian[x_, h_] := (p - c[x]) h Exp[-δ t] + λ[t](f[x] - h)
```

According to condition 6.35 The derivative of `hamiltonian` with respect of the control variable `h` equals zero. We solve the equation for `λ[t]`:

```
In[2]:= lamda[tt_, xx_] := (λ[t] /. Solve[
            D[hamiltonian[x, h], h] == 0, λ[t]
        ][[1]]) /. {t -> tt, x -> xx}
```

Then we have the definition

```
In[3]:= lamda[t, x]
```
$$\text{Out[3]}= \; e^{-t\,\delta}\;(p - c[x])$$

The economic interpretation of this term is straight forward and easily understood when looking at the expression: The shadow price of one unit of harvest at time `t` is the discounted unit economic rent. The time derivative of the shadow price is

```
In[4]:= D[lamda[t, x[t]], t] // FullSimplify
```
$$\text{Out[4]}= \; -e^{-t\,\delta}\;(p\,\delta - \delta\,c[x[t]] + c'[x[t]]\,x'[t])$$

According to condition 6.36 the derivative of the Hamiltonian with respect of the state variable equals minus the time derivative of λ. We use this condition to find an expression for λ:

```
In[5]:= Solve[
          -D[hamiltonian[x[t], h], x[t]] == D[lamda[t, x[t]], t], λ[t]
        ][[1]] // FullSimplify
```

$$\text{Out[5]= } \left\{ \pmb{\lambda}[\texttt{t}] \rightarrow \frac{e^{-t\,\pmb{\delta}}(\texttt{p}\ \pmb{\delta}\ -\ \pmb{\delta}\ \texttt{c}[\texttt{x}[\texttt{t}]]\ +\ \texttt{c'}[\texttt{x}[\texttt{t}]](\texttt{h}\ +\ \texttt{x'}[\texttt{t}]))}{\texttt{f'}[\texttt{x}[\texttt{t}]]} \right\}$$

We now have two expressions of λ that should be equal:

```
In[6]:= Solve[(λ[t] /. %) == lamda[t, x[t]], δ][[1]] // FullSimplify
```

$$\text{Out[6]= } \left\{ \pmb{\delta} \rightarrow \texttt{f'}[\texttt{x}[\texttt{t}]]\ -\ \frac{\texttt{c'}[\texttt{x}[\texttt{t}]]\ (\texttt{h}\ +\ \texttt{x'}[\texttt{t}])}{\texttt{p}\ -\ \texttt{c}[\texttt{x}[\texttt{t}]]} \right\}$$

The solution is found to be equal to the Golden Rule (equation 6.26) since
`(h + x'[t]) = f[x]` (as given by equation 5.2). Hence, the only pair of `x` and `h` fulfilling
the requirement given above is the equilibrium solution referred to as the Golden Rule
(optimal equilibrium). Consequently a *bang-bang-solution* represents the optimal path
towards the optimal equilibrium (X^*, H^*):

$$H_{opt} = \begin{cases} 0 & \text{for} \quad X < X^* \\ H_{max} & \text{for} \quad X > X^* \\ H^* & \text{for} \quad X = X^* \end{cases}$$

given that $H \in [0, H_{max}]$ and $0 < H^* < H_{max}$.

6.10   Why do we observe fleet diversity?

Given the reasoning above, we should only find one type of vessel, namely the most
cost-efficient type, in all fisheries that have been open for all to participate. As open access
has been the normal situation until quite recently in most fisheries, we should expect to
find homogeneous fleets in most places. However, this does not seem to be the case. So,
why do we find diverse fleets when the theory seems to predict the opposite?

This question may prove to be quite essential for our understanding of the equilibrium
model and its limitations. Since a diverse fleet structure seems to be the situation in most
fisheries, we arrive at two possible conclusions: 1) Something in the equilibrium models we
have been discussing is incorrect, or 2) Equilibrium is a theoretical concept that is never
achieved in real-world fisheries.

The first conclusion perhaps cannot be entirely rejected. As discussed in section 1.1, models
cannot be true, but they can be useful or not. Let us therefore consider the other conclusion.
Is it possible to argue that the equilibrium concept and the models based on it are still
useful, even when an equilibrium situation is never established?

MORE TO COME

Exercises

Exercise 6.1 Why does the production increase more rapidly in the lower-right figure in
Code box 6.1.2 than in the upper-right figure?   ■

Exercise 6.2 What is the unit cost of a product following a Cobb-Douglas production process and economically efficient use of input factors when the output elasticities equal one? ∎

Exercise 6.3 Prove that output eight (`Out[8]`) in Code box 6.8.1 is correct. ∎

7. Fisheries and markets

Commercial fisheries need marketplaces to sell the catches at prices that can at least cover the cost of fishing. As discussed in Chapter 1, there are market challenges related to the sale of perishable goods, such as fresh fish. The development of preservation methods has contributed to increasing the seafood trade. Seafood production and trade today have become a major global industry.

7.1 The global seafood market

Seafood production and trade have undergone significant changes since the Second World War. In 1950, the year the Food and Agriculture Organisation of the United Nations (FAO) started registering global fish harvests, the world catches totaled 16.5 million tons (see figures 7.1 and 7.2). Global catches increased year by year until the mid-1980s, and since then, they have stabilised at around 70 to 80 million tons per year.
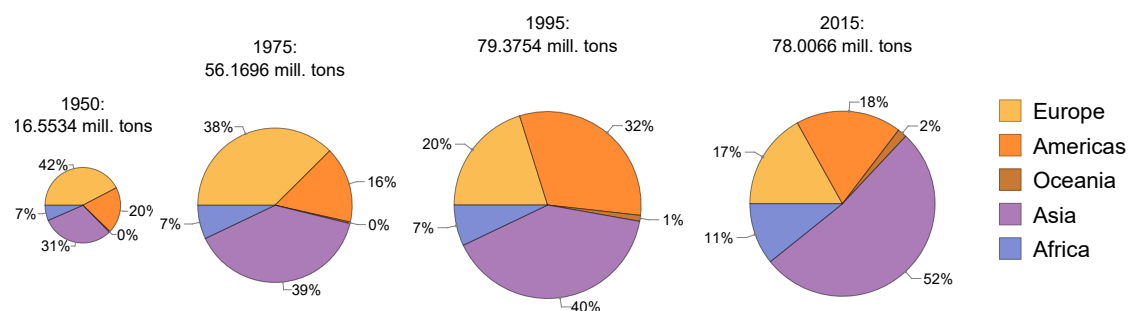

Figure 7.1: Global fish catches in 1950, 1975, 1995, and 2015 retrieved from the FAO online database (http://www.fao.org/fishery/topic/16140/en), separated by continents of the fishing nations.

Global catches have remained stable for over thirty years, but the distribution of catches shows interesting patterns over time. While Europe and the Americas (South and North America) accounted for two-thirds of the catches in the 1950s, their share decreased to about one-third by 2015, even though the total global catch was five times larger than in 1950 (see figure 7.1). This is also reflected in figure 7.2, which depicts the global catches distributed across three levels of economic development. In 1950, approximately three-quarters of the catches were from economically developed countries, while the share in 2015 was about one-quarter. Additionally, we can observe from figure 7.2 that the changes primarily occurred after 1975, coinciding with the introduction of Exclusive Economic Zones (see chapter 9).
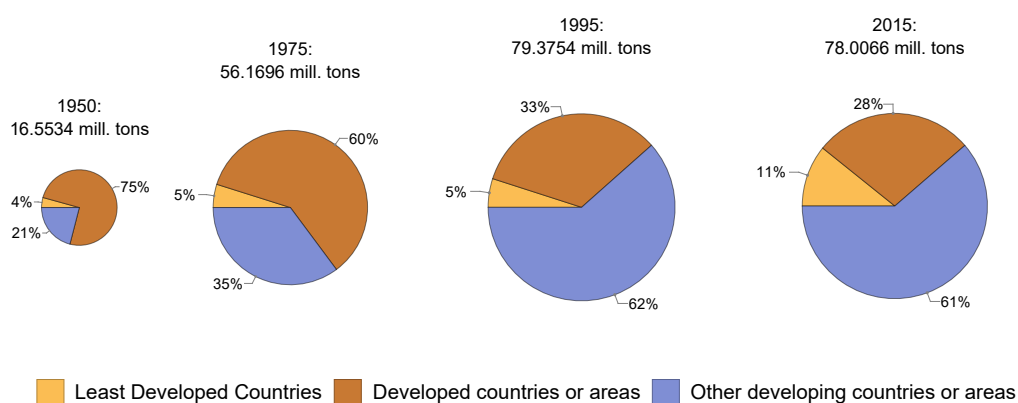
Figure 7.2: Global fish catches in 1950, 1975, 1995, and 2015 retrieved from the FAO online database at http://www.fao.org/fishery/topic/16140/en, separated by the economic state of the fishing nations.

Today, seafood (including aquaculture production) is the single most important commodity offered by developing countries in the global market[24], and it is becoming increasingly important for the least developed economies. Compared to other typical commodities traded from developing countries (agricultural products like fruit, tobacco, cocoa, etc.), seafood product prices are generally kept relatively high in international markets compared to other products originating from developing countries. This could reflect the fact that international demand has increased more than the corresponding supply of such products. It should be noted that figures 7.1 and 7.2 reflect total global catches, not trade. However, most of the global fish catches are traded in international markets.
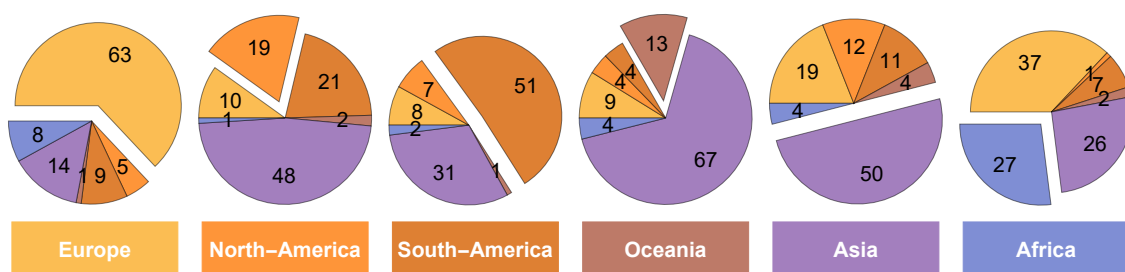
Figure 7.3: Global fish and fish products imports by continent in 2016 in terms of value, retrieved from FAO's SOFIA report of 2018 (pages 58–59). The imported shares of total imports for each continent are indicated by numbers, and imports from their own continent are also included.

Figure 7.3 shows the distribution of values of imported fish and fish products by continent. Sixty-three percent of imported fish in Oceania and 48% in North America come from Asia, while European fish products make up 19% of Asian imports and 37% of African imports. Note that only in Europe, Asia, and South America does internal trade equal or surpass import from other continents. However, internal trade within a continent is closely linked to the number of countries on that continent.
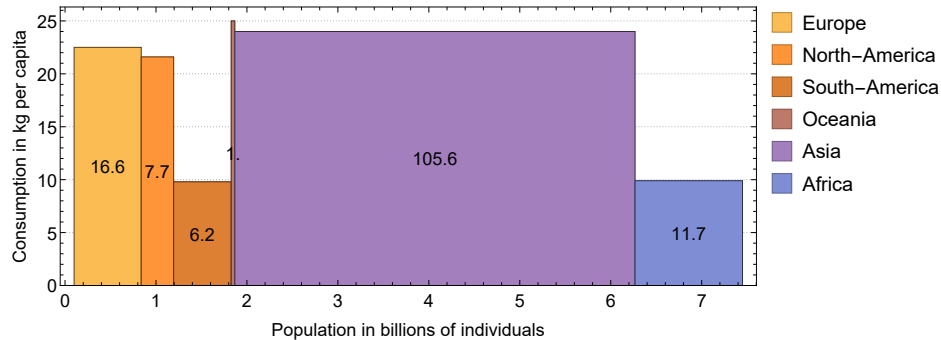
Figure 7.4: Global fish consumption by continent in 2016, showing population size along the horizontal axis and fish consumption per capita along the vertical axis. Data is from FAO's SOFIA report of 2018 (Table 18 on page 72). Total fish consumption for each continent is shown in the centre of each box in million tons.
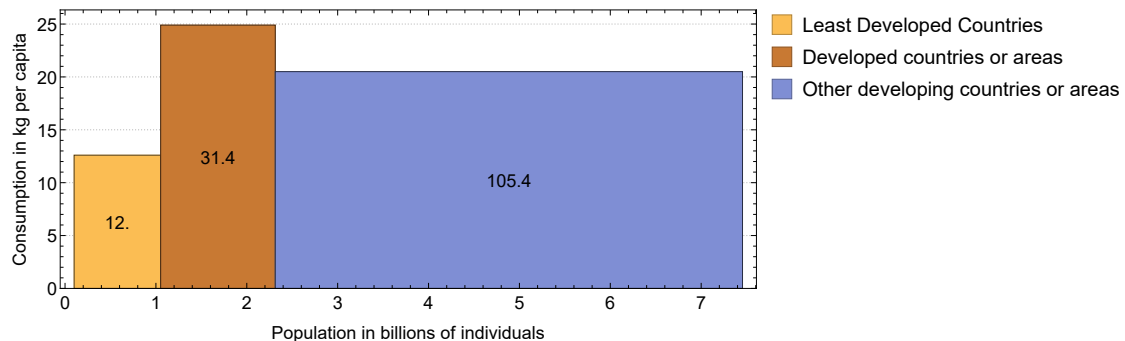
Figure 7.5: Global fish consumption by economy in 2016, showing population size along the horizontal axis and fish consumption per capita along the vertical axis. Data is from FAO's SOFIA report of 2018 (Table 18 on page 72). Total fish consumption for each economy is shown in the center of each box in million tons.

Figures 7.4 and 7.5 show global fish product consumption in 2016, distributed by continent and by economy. On average, an individual in South America and Africa consumes about 10 kilograms per year, while consumption on other continents is more than double that. Figure 7.5 reveals significant differences depending on the economic state, which also explains the modest fish consumption in South America and Africa. Only the small population in Oceania consumes somewhat more per individual than the Asian population. Approximately half of the fish consumption in 2016 came from aquaculture production.

7.2 Market failures

In a perfect market, the price is determined where the demand meets the supply of the good, all share the same information, and no parties have market power, which is the

ability to influence price formation alone. This idealised situation is rare; however, on the other hand, often the price formation will lead to prices close to those we would find in perfect markets. One should think about the market solution as the result of a negotiation between consumers (demand) and suppliers where the correct price is found when no party could increase or decrease the price without losing.
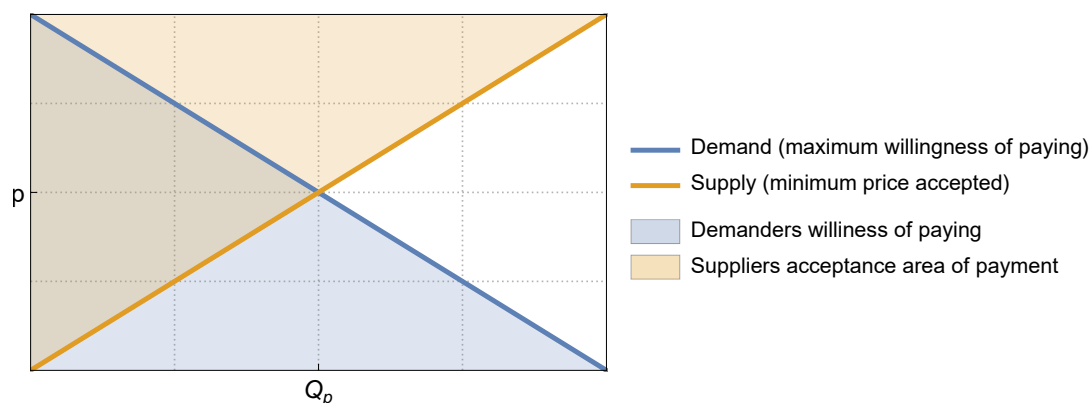
Figure 7.6: The market cross, the intersection between demand and supply, is really one point, (Q_p, p), at the intersection between two areas: The area of acceptance from the supply side and the area of willingness to pay from the demand side. The horizontal axis gives the quantity produced, and the vertical axis gives the unit price of the good.

Market failure is caused by factors disturbing the market solution displayed in figure 7.6. External factors (externalities), reflected in costs not paid for by the producer, hinder perfect market solutions from being obtained. Perfect market solutions may also be altered by governmental regulations where the authorities seek to avoid free market solutions.

In some of the Nordic countries, the government has introduced monopolies on the supply side, deliberately aiming to reduce the quantities sold (e.g. liquor monopolies) or provide the supply side with monopoly profits (as in the case of taxi market regulations, believing that a profit beyond the normal level will contribute to making the service safer for the public).

In other cases, market failure is unintended, as in the case of polluting industries not including the damage caused by pollution in the cost of production. When some of the cost of production is omitted by the producer (by forwarding it to society), it is not reflected in the supply curve (figure 7.6), and the price becomes too low and the quantity too high, compared to including the external cost. The market failure here causes too high production, while in the case of governmentally introduced monopolies, production is less than in perfect markets.

The ex-vessel market of fish (when the catch is transferred from fisher to fish buyer) may also be imperfect. As you can read from Additional Remark 7.1, the fishers (or rather the fishers' sales organisation) can dictate — within certain constraints — the ex-vessel price of fish in Norway. One of the motivations for this is to avoid fish buyers taking advantage of their potential market power. The aim is not to provide the fishers with more market power than the buyers but to level this out. Both parties then have to share the responsibility of finding an efficient price.

Additional remarks 7.1 — The Norwegian *Raw Fish Act*.

Fishers usually do not sell fish directly to consumers. When fish is landed, it normally goes to a fish buyer, and there may be several additional stages until the final product reaches the consumer. The product may undergo processing in various ways before it is finally consumed. However, in modern fish trade, the product could end up in a store on a different continent after a relatively short time. Nevertheless, the price obtained by the fishers may still appear to be rather disconnected from the final price the consumer has to pay. The value chain from catch to consumption is an interesting study in its own right, but it falls outside the scope of this book. Our interest is in the harvest and related subjects, including the price the fisher obtains when selling the fish in a port. We refer to this market as *the ex-vessel market*.

The nature of fishing makes it difficult to organise fishers to operate cohesively in the ex-vessel market. Traditionally, the fisher has therefore been the weaker party in price negotiations with raw fish buyers. When the fish is fresh from the sea and a price has to be negotiated, the buyer is often in a stronger position than the fisher, particularly if there is competition among many fishers needing to sell their fish before it spoils and only a few or even just one buyer (*monopsony*). However, the negotiation may also be influenced by possible commitments the buyers may have to their customers. There are a number of factors affecting the market relationship between fisher and buyer, and in a modern fishing society, the range of possible actions is controlled by laws and regulations.

In order to provide fishers in Norway with stronger bargaining power in the ex-vessel market, the so-called *Raw Fish Act* was passed in 1938, giving the fishers' cooperative sales organisation the right to decide prices and conditions in this market[35].

Buyers had built up their market power and position within the fishing industry based on privileges from the King. After the *Raw Fish Act*, fishers could decide on ex-vessel prices and had to take responsibility for creating a healthy ex-vessel market. If the sales organisation on behalf of the fishers set the price too high, many buyers would not be able to buy, and the catches could not be sold. If the price was set too low, the fishers would lose. It was necessary to have good market information and to be flexible enough to account for geographical and other differences in every single market operation along the coast.

The Norwegian *Raw Fish Act* replaced one market failure with another. It is easy to show that situations may arise where both fishers and buyers have an advantage in omitting the rules and making illegal arrangements to circumvent the regulated system. For this reason, problems of incorrect species registration, manipulated weight measurements, etc., have been, and still are, real issues in regulated ex-vessel fish markets.

There is a inherent market failure in fishing due to multiple individual producers (fishers) targeting the same stock. Each fisher must contend with the impact on the stock caused by the fishing activities of all fishers. Consequently, there are external costs associated with fishing activity, costs that must be borne by all fishers.

Essentially, this constitutes the problem of open access fishing, as discussed in Section 6.5. The market failure arising from external costs would be eliminated in the case of sole ownership of the resource. In this scenario, the owner would maximise profits without interference from other parties, resulting in the indirect control of the stock biomass X in Equation 6.13 through the owner's decisions.

7.3   The problem of a sole owner

A resource owner functions as a monopolist in the sense of having exclusive access to the stock resource, but does not necessarily possess monopoly power within the fish market. If the resource owner is a price taker (lacking market power in the fish market), profit maximisation for the owner will align with the concept of resource rent maximisation outlined in Section 6.7, as originally proposed by Scott Gordon in 1954[29].

Let us consider the scenario of a sole owner with market power, signifying the ability to influence the market price of the harvest by controlling the harvest size, H. This modification transforms Equation 6.14 into:

$$R(X,H) = \Big(p(H) - c(X)\Big) \cdot H \tag{7.1}$$

In equilibrium in the long run, both the catch H and the stock size X are determined by the fishing effort, E, of the sole owner. Suppose the market power of the sole owner is reflected in this inverse demand function:

$$p(H) = p_0 - s \cdot H \tag{7.2}$$

Here, the two parameters (p_0 and s) are positive constants, resulting in a downward-sloping price function in relation to harvest. Substituting Equation 7.2 into Equation 7.1 and replacing H with $f(X)$ (assuming equilibrium, $\dot{X} = 0$ in Equation 5.2), $f(X)$ represents the natural per-period growth in the stock.

$$R(X) = \Big(p_0 - s \cdot f(X) - c(X)\Big) \cdot f(X) \tag{7.3}$$

Equation 7.3 describes the net revenue that the sole owner aims to maximise. The catch H serves as the control variable for the sole owner, so we differentiate Equation 7.3 with respect to H and set it to zero:

$$\frac{dR(X)}{dX} = 0$$
$$\Big(p_0 - 2 \cdot s \cdot f(X)\Big) \cdot f'(X) - c(X) \cdot f'(X) = 0 \tag{7.4}$$

The term $2 \cdot s \cdot f(X)$ in Equation 7.4 distinguishes it from the situation discussed in Section 6.5. Assuming $c(X) = a/(q \cdot X)$ (as in Equation 6.12) and a logistic growth function ($f(X) = r \cdot X \cdot (1 - X/K)$, the first term of Equation 5.4), we can represent Equation 7.4 as:

$$\frac{r}{qK^2} \cdot \Big(aK + q(K - 2X)\Big(2rsX^2 + K(p_0 - 2rsX)\Big)\Big) = 0 \tag{7.5}$$

This equation is cubic in terms of X, yielding three roots. In cases where the sole owner has market power ($s > 0$), they reduce production to leverage this advantage. Even if $s = 0$ and the owner is a price taker, they still decrease fishing effort compared to an open access fishery (as discussed in Section 6.5) to capitalise on potential resource rent.

Equations 6.4 and 6.5 explain total cost (TC) and total revenue (TR) in equilibrium with the variable E (fishing effort). Since there exists a unique relationship between E and X

expressed in Equation 5.6, TC and TR can be expressed as functions of X, as done above. We have:

$$TC(X) = c(X) \cdot f(X) \tag{7.6}$$

and

$$TR(X) = p(X) \cdot f(X) \tag{7.7}$$

assuming an equilibrium price for any stock biomass level (where $p(X) = p_0 - s \cdot f(X)$ as previously assumed).
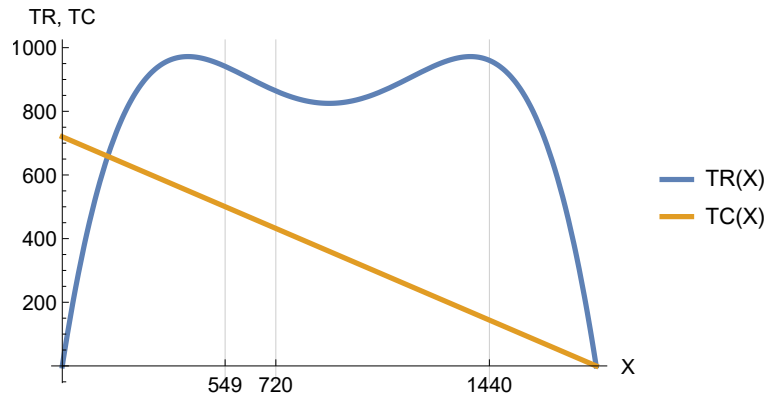
Figure 7.7: Graphical representation of equations 7.6 and 7.7, assuming the price to follow a linear function of harvest (equation 7.2) and a logistic biological growth. Parameter values used are: $r = 1/3$, $q = 1/24$, $K = 1800$, $a = 90$, $p_0 = 18$ and $s = 1/12$. The three solutions of the cubic function is found for $X = 549$ (local maximum of resource rent), $X = 720$ (local minimum of resource rent) and $X = 1440$ (global maximum of resource rent).
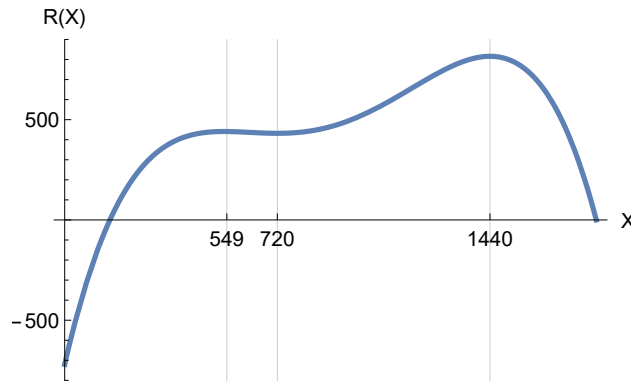
Figure 7.8: Graphical representation of equation 7.3, assuming the price to follow a linear function of harvest (equation 7.2) and a logistic biological growth as in figure 7.7. Parameter values used are as in figure 7.7 and minimum and maximum values of net revenue are indicated.

In Figure 7.7, the blue curve represents total revenue, and the yellow line depicts total cost of fishing. Figure 7.8 presents these curves and displays net revenue (Equation 7.3) for the same parameter values. Net revenue becomes positive when the sole owner covers fishing costs and begins to earn additional rent. The supply offered by the sole owner is influenced

by the cost to be covered. As cost varies with production (harvest), supply changes based on harvest size. Equilibrium situations only are considered, so the supply curve we identify is a collection of equilibriums. In equilibrium, $\dot{X} = 0$, and from Equation 5.2, we have $H(X) = f(X)$, which can be solved for X. Assuming logistic growth, we obtain two roots:

$$X = \frac{rK \pm \sqrt{K} \cdot \sqrt{r(rK - 4H)}}{2r} \tag{7.8}$$

Substituting H for X in Equation 7.6 yields total costs expressed with respect to H. The derivative of the new expression gives the marginal cost per unit of harvest:

$$\frac{dTC}{dH} = \pm \frac{ar}{q\sqrt{K} \cdot \sqrt{r(rK - 4H)}}$$

Given that $rK - 4H$ cannot be negative (as $MSY = rK/4$ according to Equation 5.8), and equilibrium catches cannot exceed MSY, negative marginal cost is excluded. The positive case results in:

$$TC'(H) = \frac{ar}{q\sqrt{K} \cdot \sqrt{r(rK - 4H)}} \tag{7.9}$$
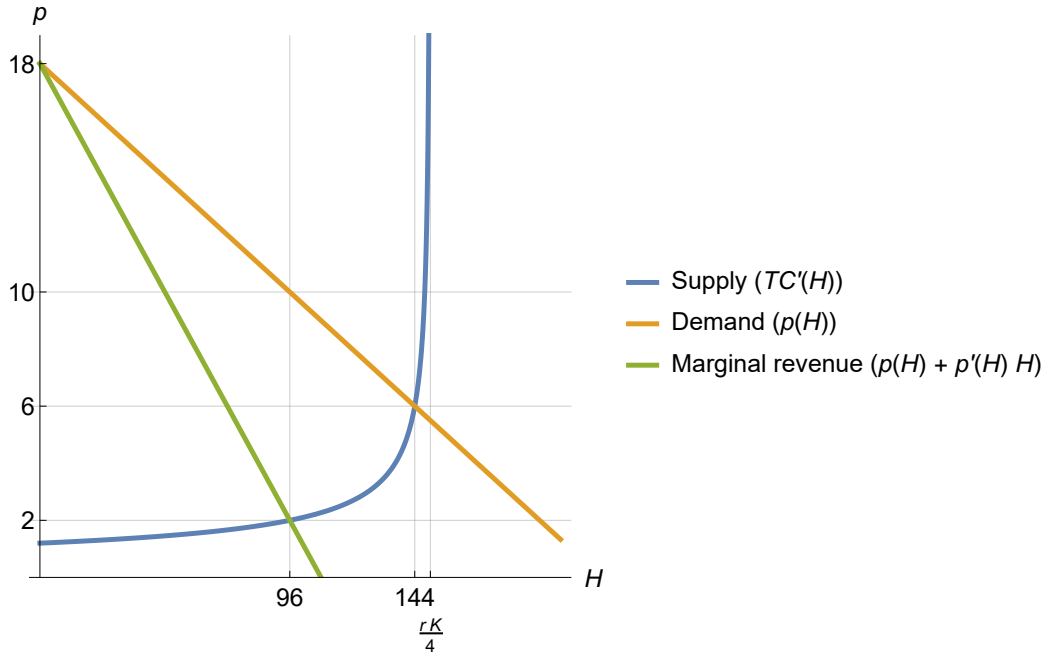
Figure 7.9: Supply and demand in a fishery carried out by a sole owner. The supply curve is equation 7.9, and the (inverse) demand curve is equation 7.2. On basis of the latter the marginal revenue is calculated and the intersection between this and the supply curve gives the harvest by the sole owner. The price is found by the demand curve. The values used for the calculations are given in figure 7.7.

In Figure 7.9, the blue curve represents Equation 7.9. As the curve approaches the value of MSY on the horizontal axis, the price (vertical axis) tends toward infinity. For the illustrated case, where a (cost per unit of effort) is 90, the blue curve signifies full cost recovery and maximised economic rent at each point. The sole owner encounters demand, visualised by the inverse demand curve (yellow line). The revenue in the fishery is the product of the yellow line and the horizontal axis value (catch quantity). The green line represents the marginal revenue with respect to harvest. To maximise profit, the sole owner settles where the marginal cost (supply curve) equals marginal revenue (green curve). In the depicted scenario, the harvest is 96, the price is 10, and the marginal cost of the last unit produced is 2.
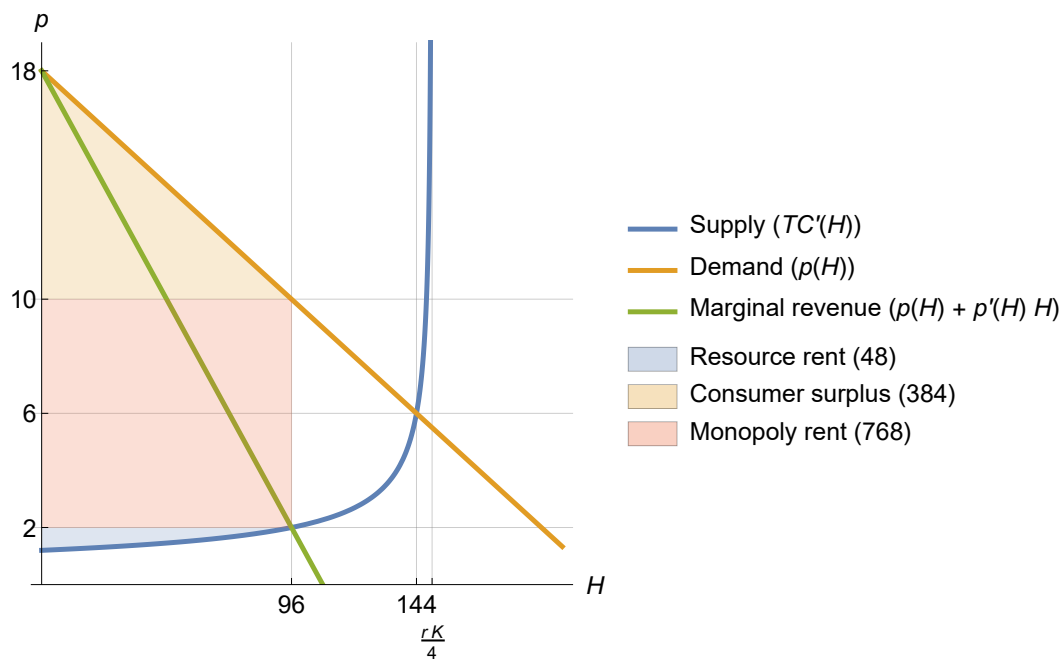
Figure 7.10: In this figures three areas in figure 7.9 are highlighted: The producer and consumers surpluses, in addition to the monopoly profit obtained by the sole owner. The legends of the three areas also include values indicating the size of each area (price times quantity).

In Figure 7.10, specific areas in Figure 7.9 are highlighted, illustrating the social benefits of sole owner production. The blue area is labelled as *Resource rent*, not *Producer surplus*, since the surplus stems from resource properties rather than production technology. The monopoly rent originates from the sole owner's market power.

7.4 The backward bending supply curve

In Section 7.3, we explored the scenario of a sole owner, a situation that is highly uncommon in the utilisation of fish stock resources in the sea. We previously referred to a pure open access fishery as the natural equilibrium or the equilibrium state of an unregulated fishery. In the case of open access to the fish stock resource, the supply curve of the fishing industry will differ from the one discussed earlier.

In 1986, Parzival Copes published a paper[13] in which he presented the well-known Gordon-Schaefer model[29] (discussed in Section 6.7) in terms of supply and demand. A

backward-bending supply curve is a direct outcome of decreasing equilibrium catches as the extent of biological overfishing intensifies. The derivation of the backward-bending supply curve as an inevitable result of the model is demonstrated in Code Box 7.4.1.

Code box 7.4.1 — The backward bending supply curve.
Based on the results presented in Code box 6.5.1 on page 101, we can determine the inverse supply function for harvest (`h`) by solving the open access equilibrium harvest (`hoa`) for `p`.

```
In[11]:= OAsolution =
           FullSimplify[
             Solve[h == hoa, p],
             Element[{a, q, r, k, h}, Reals] && a > 0 && q > 0
           ]
```

$$\text{Out[11]= } \left\{\left\{p \to \frac{a\ (k\ r - \sqrt{k\ r\ (-4\ h + k\ r)})}{2\ h\ k\ q}\right\},\right.$$
$$\left.\left\{p \to \frac{a\ (k\ r + \sqrt{k\ r\ (-4\ h + k\ r)})}{2\ h\ k\ q}\right\}\right\}$$

The two roots indicate that there are two different prices for the same harvest on the supply curve. This is illustrated in the backward-bending supply curve below, using the parameter values shown in figure 7.7 and the following figures. The two roots are represented by two different colors on the curve: blue for the first root and yellow for the second root.

```
In[12]:= Plot[
           Evaluate[
             p /. OAsolution /. {r->1/3, k->1800, q->1/24, a->90}
           ],
           {h, 0, 155},
           AspectRatio -> 1,
           PlotRange   -> {0, 11},
           AxesLabel   -> {"Harvest", "Price"}
         ]
```
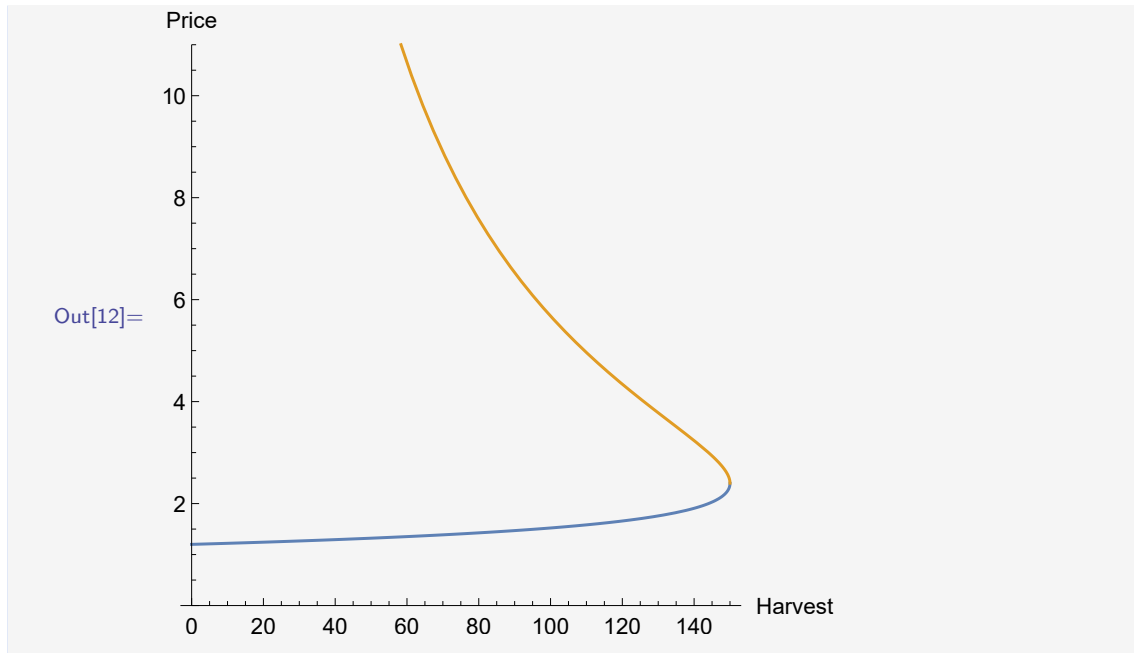
We see from code box 7.4.1 that the inverse supply curve in an open access fishery is given by

$$p(H) = \frac{a\left(rK \pm \sqrt{rK(rK - 4H)}\right)}{2qKH} \tag{7.10}$$

The two supply curves, the sole owners supply (equation 7.9) and the open access supply (equation 7.10) are displayed together in figure 7.11. The figure shows that higher quantities are offered in open access than from a sole owner at a given price when the stock is not biologically overfished in open access. After biological overfishing, the harvest offered from an open access fishery declines and soon becomes lower than the supply from a sole owner, which never reach the area of biological overfishing.

The intersection between the two curves in figure 7.11 occurs when the same catch is yield by the two types of supply, the open access at a biologically overfished stock and the sole owner a biologically sound stock where the surplus growth still i rising with increased stock size. In case of the first the average unit cost (including a normal profit) is exactly covered by the price (as is the situation for all points at the yellow curve), while a resource rent is earned in case of the sole owner. (Remember that we are referring to the case of Maximum Economic Yield (MEY) as the sole owner case.) The resource rent is obtained by measuring the area above the blue curve up to the horizontal line of any given market price, corresponding to the light blue area in figure 7.10.
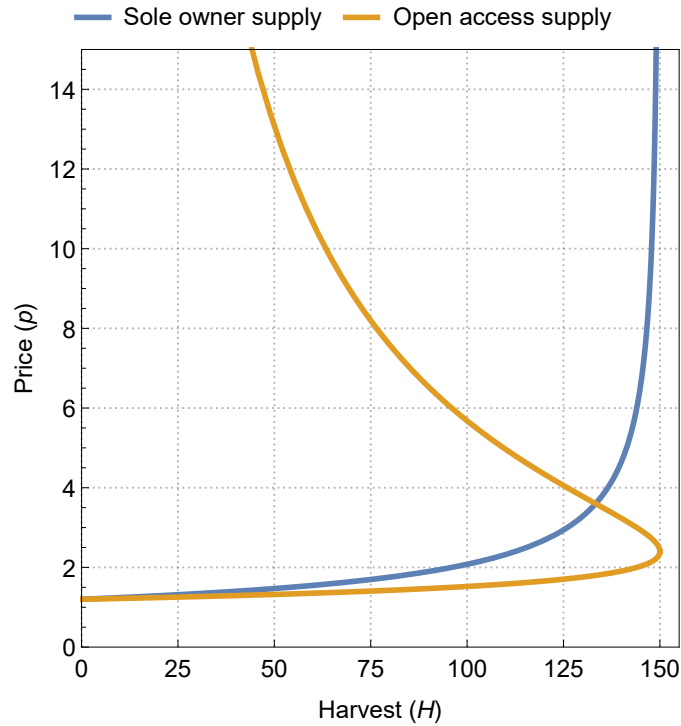
Figure 7.11: The supply curves from figure 7.9 (blue) and code box 7.4.1 (yellow). Parameter values are given in the caption text of figure 7.7.

DEMO
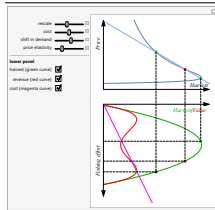
The concept presented in Code box 7.4.1 could be further investigated as a Wolfram Demonstration at

http://demonstrations.wolfram.com/
TheBackwardBendingSupplyFunctionInFisheries/

7.5   Social optimal supply

The golden rule of the social optimum (Equation 6.26) offers a straightforward approach to determine the supply curve while considering the cost of time. We solve Equation 6.26 with respect to p and find:

$$p(X) = \frac{a(rX - \delta K)}{qX(2rX + K(\delta - r))} \tag{7.11}$$

We can substitute for X from Equation 7.8 and obtain the two solutions of equation p (the backward-bending part of the curve is one of the two solutions, as demonstrated in Code

Box 7.4.1) with harvest (H) as the variable, corresponding to Equation 7.10:

$$p_1(H) = \frac{ar\left(-\sqrt{r(rK-4H)}+(r+2\delta)\sqrt{K}\right)}{q\sqrt{K}\left(\sqrt{r(rK-4H)}-r\sqrt{K}\right)\left(\sqrt{r(rK-4H)}-\delta\sqrt{K}\right)}$$

(7.12)

$$p_2(H) = \frac{ar\left(-\sqrt{r(rK-4H)}+(r+2\delta)\sqrt{K}\right)}{q\sqrt{K}\left(\sqrt{r(rK-4H)}+r\sqrt{K}\right)\left(\sqrt{r(rK-4H)}+\delta\sqrt{K}\right)}$$

These two solutions are depicted as an interconnected green curve in Figure 7.12. For the chosen parameter values, the social optimum yields a backward-bending supply curve. However, this outcome depends on the value of the interest rate. In Figure 7.12, the interest rate is relatively high (15%), which, in this case, results in a backward-bending profile.
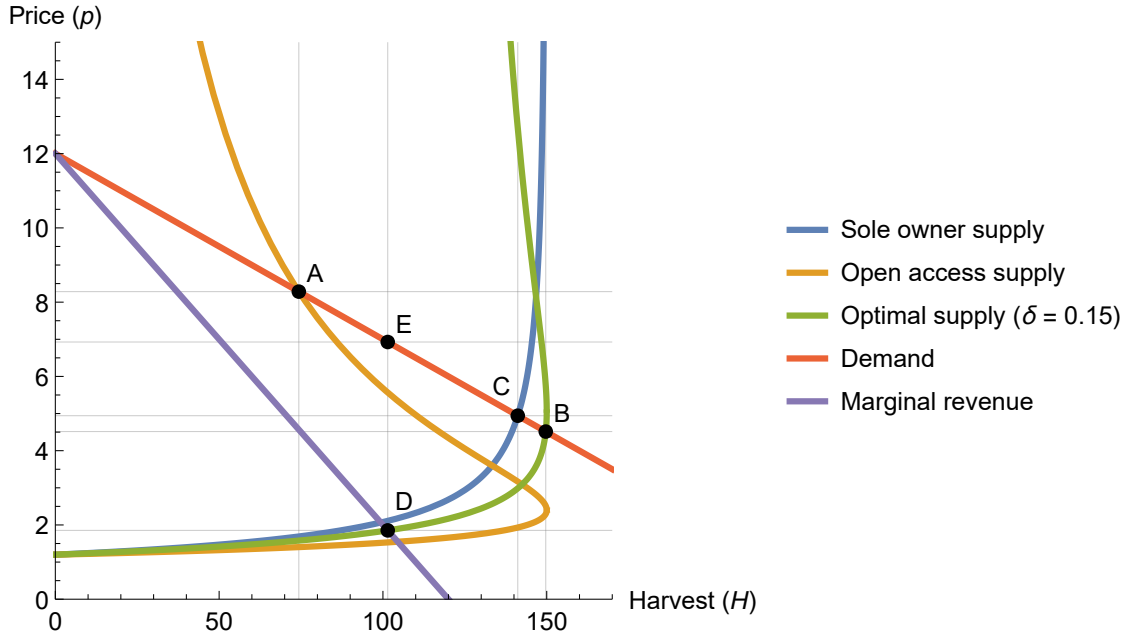
Figure 7.12: The supply curves from figure 7.11, the social optimal supply curve after inserting equation 7.8 into equation 7.11 and the demand curve $p(H) = 12 - H/20$. Other parameter values are given in the caption text of figure 7.7. A, B and C indicates the intersections between the demand curve and the supply curves.

From Equation 7.12, we observe that the optimal supply curve exhibits linearity in the unit cost of effort (a) and the inverse catchability coefficient ($1/q$). The curve's shape is determined by the biological parameters (r and K) and the discount rate (δ). When δ equals zero, the optimal supply aligns with the sole owner's supply, whereas it tends towards the open access supply as δ approaches infinity. This observation is consistent with the discussion related to Figure 6.5 above.

The three marked intersections with the demand curve (labelled as points A, B, and C) in Figure 7.12 should be interpreted differently. Point A evidently represents the open access equilibrium, assuming a perfect ex-vessel market. In this case, a relatively low

harvest volume and high price are evident. Similarly, the social optimum is situated at point B, where social benefits are maximised with a low price and a substantial harvest quantity. Point C is unlikely to result from a sole owner's decisions, as discussed in relation to Figure 7.10 above. However, this discussion needs to be extended after introducing discounting. Even a sole owner should adopt a long-term perspective, extending beyond the present day. As a result, the sole owner's viewpoint shifts from the static *MEY* perspective to encompass the consideration of the optimal supply curve. Nonetheless, the sole owner will still employ market power to reduce harvest quantities and increase prices, similar to the scenario depicted in Figure 7.10. Consequently, Point C becomes a theoretical construct, and in equilibrium, the monopoly will settle at point E, where the optimal supply intersects with the marginal revenue (point D).

Exercises

Exercise 7.1 When fishers determine the ex-vessel price, will it contribute to increasing or decreasing the catch level in open access fisheries? Or will it not affect the catch level? Discuss these questions under given conditions. ∎

Exercise 7.2 Show that the three values of X found in figure 7.7 are actually the three roots of equation 7.5. ∎

Exercise 7.3 Find the total social economic surplus illustrated in figure 7.10 and demonstrate how the values are calculated. ∎

Exercise 7.4 Using equations 7.9 and 7.12, show that the supply of the optimal case indeed equals the *MEY* (the sole owner's) when $\delta = 0$. ∎

8. Economic growth

Fishing activities utilise biological resources to produce products for subsistence use or trade in the market. We refer to these biological resources as natural capital.

8.1 Capitalization

In Chapter 2, we discussed how commodities are produced using input factors such as labour and capital. This chapter presents the neoclassical economic theory of how capital is generated through the production of commodities.

When labour and capital become integrated into commodities through the production processes described in Chapter 2, some of these commodities are consumed. This includes both final consumption and the use of produced commodities in the production of other products. For instance, outputs from the lumber industry serve as input factors in the production of houses, furniture, paper, and many other products.

However, not everything is immediately consumed. Commodities can also be saved for later consumption or investment. These fractions of produced commodities efficiently become part of the capital stock.

8.2 Neoclassical Economic Growth Theory

The Solow-Swan model describes an economic growth model within the framework of neoclassical economics. The following discusses the model set up by Solow[52].

Let's assume there is only one commodity, denoted as Y, which is the output of a production process utilising labour (L) and capital (K) as inputs:

$$Y(L, K) = f(L, K) \tag{8.1}$$

The commodity is partially consumed, and the rest is saved and invested. If the constant saved fraction is represented by s, then $s \cdot Y$ is added to the capital stock K. In continuous time, this can be expressed as

$$\dot{K} = s \cdot Y \tag{8.2}$$

The labour population is assumed to grow exponentially at the rate of n. Therefore, the labour at time t is given by

$$L(t) = L_0 \cdot e^{n \cdot t} \tag{8.3}$$

Since both capital and labour grow, it is interesting to explore the ratio between the two. Let's denote this ratio as γ:

$$\gamma(t) = \frac{K(t)}{L(t)} \tag{8.4}$$

Solving for $K(t)$ and substituting the value from Equation 8.3 yields

$$K(t) = \gamma(t) \cdot L(t) = \gamma(t) \cdot L_0 \cdot e^{n \cdot t} \tag{8.5}$$

Differentiating this equation with respect to time results in

$$\dot{K}(t) = \dot{\gamma}(t) \cdot L_0 \cdot e^{n \cdot t} + n \cdot \gamma(t) \cdot L_0 \cdot e^{n \cdot t} \tag{8.6}$$

Equating this to Equation 8.2 yields

$$\left(\dot{\gamma}(t) + n \cdot \gamma(t) \right) L_0 \cdot e^{n \cdot t} = s \cdot f(L, K) \tag{8.7}$$

Assuming constant returns to scale, we have

$$\dot{\gamma}(t) + n \cdot \gamma(t) = s \cdot f(L, 1) \tag{8.8}$$

and further

$$\dot{\gamma}(t) = s \cdot f(L, 1) - n \cdot \gamma(t) \tag{8.9}$$

By definition, $n = \dot{L}/L$, and since $\gamma = K/L$, we have

$$\frac{\dot{\gamma}(t)}{\gamma(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n \tag{8.10}$$

$$\dot{\gamma}(t) = \gamma(t) \cdot \frac{\dot{K}(t)}{K(t)} - n(t) \cdot \gamma(t) = \gamma(t) \cdot \left(\frac{s \cdot f(L(t), K(t))}{K(t)} - n(t) \right) \tag{8.11}$$

This corresponds to Equation 8.6. When this expression equals zero, the capital-to-labour ratio (γ) remains constant, and the capital grows at the same rate as the population (n):

$$n = \frac{s \cdot f(L, K)}{K} \tag{8.12}$$

8.3 Technological development and growth in wealth

In reality, the capital-to-labour ratio (γ) tends to increase over time, primarily due to technological advancements and a more efficient utilisation of natural resources. The aggregation of growing amounts of capital accelerates the process of technological development, and as a result, $\dot{\gamma}(t)$ may become an increasing function of time.

This progression progressively renders labour more available for new areas of production as the population expands. This, in turn, leads to the accumulation of capital and increased compensation for labour, thereby fostering economic growth. The distribution of this increasing wealth becomes a matter of political concern.

Technological development brings about various effects. Fishing activities shift from being subsistence-based to market-oriented due to advancements in fishing technologies and the rising affluence among fish consumers, enabling them to pay fishers. However, technological development also imposes greater pressure on exploited natural resources, necessitating protective measures against overexploitation.

Hence, effective fisheries management is influenced by both technological advancement and chosen political objectives. During the initial stages of economic development, when fishing activities have minimal impact on stock development, management goals might focus on fostering fishing and developing market infrastructure. As economic development progresses, fishing activities must eventually be regulated and constrained to maintain sustainability. This constitutes the subject of the upcoming third section of this book.

Exercises

Exercise 8.1 Nothing yet. ∎

Fisheries management

9. Overarching fisheries policy

The Food and Agriculture Organisation of the United Nations (FAO) has formulated a *Code of Conduct for Responsible Fisheries*. While the code provides management recommendations, it is not binding for the UN's member nations. Nonetheless, certain parts of the code are based on international agreements that most nations have already approved and accepted. Long-term sustainable use of marine living resources is grounded in the principle of maximum sustainable yield (MSY, see section 5.1), and the entirety of the stock should be taken into account when defining a management regime to attain MSY.

However, it's important to note that while MSY has been established as a general goal for managing all fish stocks, alternative management targets do exist. It's worth mentioning that MSY lacks an economic interpretation and does not guarantee an economically viable fishery. Nevertheless, in practical fisheries policy, the challenge lies in reducing fishing effort from an open access scenario. As highlighted by Anthony Scott (highlight 6.1 on page 107), both biologists and economists can find common ground on this issue.

Ultimately, marine living resources within a nation's national *Exclusive Economic Zones* (EEZ), as defined in the United Nations *Law of the Sea*, belong to that respective EEZ country. Management actions concerning straddling stocks, which spend time in different EEZs, must be agreed upon by all involved nations. This poses a common challenge, as seen with numerous Barents Sea stocks that traverse both Russian and Norwegian EEZs.

High sea resources that do not fall within any country's EEZ are considered *common property resources*. All nations possess the right to exploit such resources, but they also have the responsibility to cooperate with other countries engaged in the same exploitation. This cooperation aims to conserve and manage the resources through suitable international organisations.

The current fisheries policy can be categorised into three stages or decision levels. The primary level involves setting the overall objectives for utilising marine fish stock resources.

The second level pertains to management: What measures need to be implemented to achieve these objectives? This may encompass various strategies, including the pace of development. Additionally, it addresses the challenge of maintaining a desired state after the objectives are met. This level is discussed in greater depth in the following two chapters. The third and final level is somewhat interwoven with the other two levels and deals with the distribution of the benefits derived from utilising fish stock resources. This topic is partially explored in the subsequent two chapters. Here, the primary focus will be on discussing the overarching first level.

9.1   Basis for Management Decisions

The equilibrium of an unregulated fishery is outlined in section 6.1. While theoretically an open access fishery may align with FAO's code of conduct, it is unlikely to be realised. The Law of the Sea stipulates that coastal nations have specific responsibilities to ensure proper conservation and management of living resources within their *Exclusive Economic Zones* (EEZs). If open access fisheries cannot fulfil these responsibilities, additional regulatory measures may be necessary to achieve the defined objectives. Determining how to utilise resources is essentially a political matter, requiring decisions from governing nations within the framework of international agreements. Each government must also account for any international treaties their country may have ratified and the potential resource-sharing arrangements with other nations.

9.2   Potential Political Objectives

Suppose a fish stock resource is found solely within the Exclusive Economic Zone (EEZ) of a single country. In principle, that country may decide to use the resource in various ways: to provide sustenance to the population, to capture the economic value of the resource in the short or long term, or to leave the resource untapped.

Let's consider the first objective listed above: ***Utilising the fish resource to provide sustenance for the population.*** One interpretation of this is to maximise sustainable catch, making MSY the long-term target (refer to the main illustration in figure 9.1). This objective aligns with international regulations and is consistent with the regulatory framework of international organisations like ICES..

Alternatively, the coastal state might prioritise ***using the fish stock resources to generate employment along the coast***. If the relationship between employment and the production of fishing effort conforms to what is assumed in code box 6.1.2 on page 95, sustainable employment is maximised when equilibrium fishing effort is at its highest. According to the bioeconomic model presented in section 6.7, this is achieved in a pure open access fishery, as depicted in figure 9.1. Employment could be further increased through subsidies, potentially depleting the resource. Previous discussions have shown that the open access equilibrium theoretically aligns with achieving a Maximum Sustainable Yield (MSY). In that case, both maximum food production and maximum employment can be realised simultaneously.

The final objective, ***maximising economic rent from the fishery***, is discussed in section 6.7 and illustrated by E_{MEY} in figure 9.1. It's evident that this equilibrium conflicts with the two preceding objectives, as it always occurs at fishing effort levels below both the MSY level and the open access equilibrium.
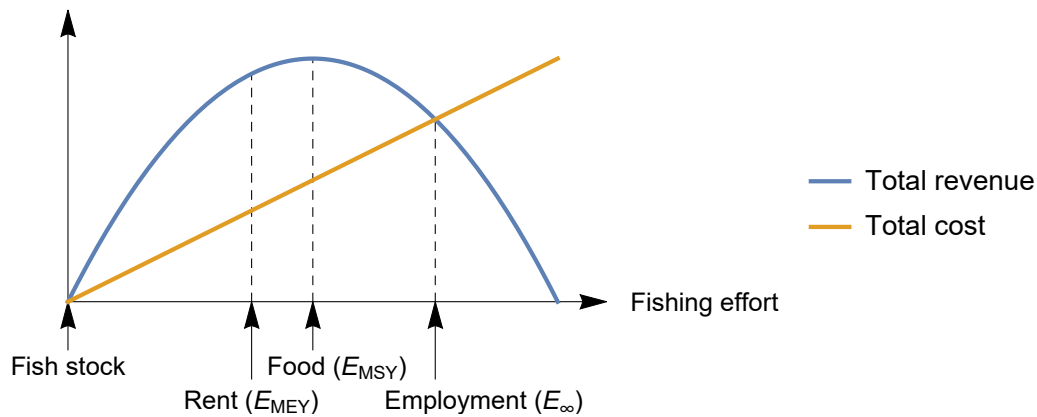
Figure 9.1: Four possible political objectives with the utilisation of a fish stock resource; maximising stock size, rent, food production or employment.

However, maximising the wealth gained from fishing activities may also entail considering the cost of time. Thus, the relevant objective should be to maximize the discounted flow of rent over time. In this scenario, fishing effort will increase, contingent on the applicable discount rate. A theoretically infinite discount rate would coincide with a pure open access fishery.

Numerous other potential objectives could be articulated, such as using the stock to feed other marine resources (sea mammals or fish stocks of higher commercial value, etc.). What is common among all the objectives discussed in this section is that they can be expressed in terms of an equilibrium solution. Given perfect information at any given time, it should be possible to identify an equilibrium solution that represents the specific objective. However, since the equilibrium solution is influenced by stock-specific parameters and typically economic factors, changes in these factors over time will lead to changes in the equilibrium solution. In the real world, equilibrium solutions serve as theoretical goals for governments, involving the utilization of various management methods. Yet, there are also significant political decisions to be made regarding how, and at what rate, one should approach the theoretical goal and equilibrium effort.

Once the goal is established and management is required, a series of new decisions arise. How can the goal be achieved? Which measures should be employed, and how rapidly should progress be made toward the goal? These are questions that are partially addressed in section 9.4, but first, the overarching long-term goal must be clarified.

9.3 Conflicting Objectives

In political discussions concerning the utilization of national natural resources, such as marine fish stocks, multiple goals are typically introduced. As demonstrated earlier, conflicts can arise among different objectives. For instance, maximising employment often conflicts with maximising rent from the fishery.

The issue of conflicting objectives is genuine within the political realm. Politicians aim for positive outcomes across the board, making it challenging to prioritize one specific goal (e.g., employment) over another (e.g., economic gain). The intricacies of fisheries

dynamics may also contribute to concealing such conflicts or, at the very least, complicating a full understanding of the nature of the conflict. It's important to note that politicians are not necessarily fisheries experts. Nevertheless, determining the political objectives for resource utilization is a political responsibility. Thus, it's crucial for politicians to receive well-informed and qualified input before making decisions, and addressing potential conflicting objectives is a vital aspect that should be clarified before decisions are made.

9.4 Pathways Toward a Long-Term Goal

When depleting a stock to levels below the current level, the process may either take time or occur immediately. When rebuilding a stock, the natural growth of the stock represents the fastest way to achieve that. Any fishing during the rebuilding period will impede the restoration process. The process of depleting a stock is exemplified in code box 6.6.1, which showcases an unexploited stock subject to an open access fishery at three different effort levels or entry dynamics. The speed at which such processes occur is an important political decision of significant economic significance.
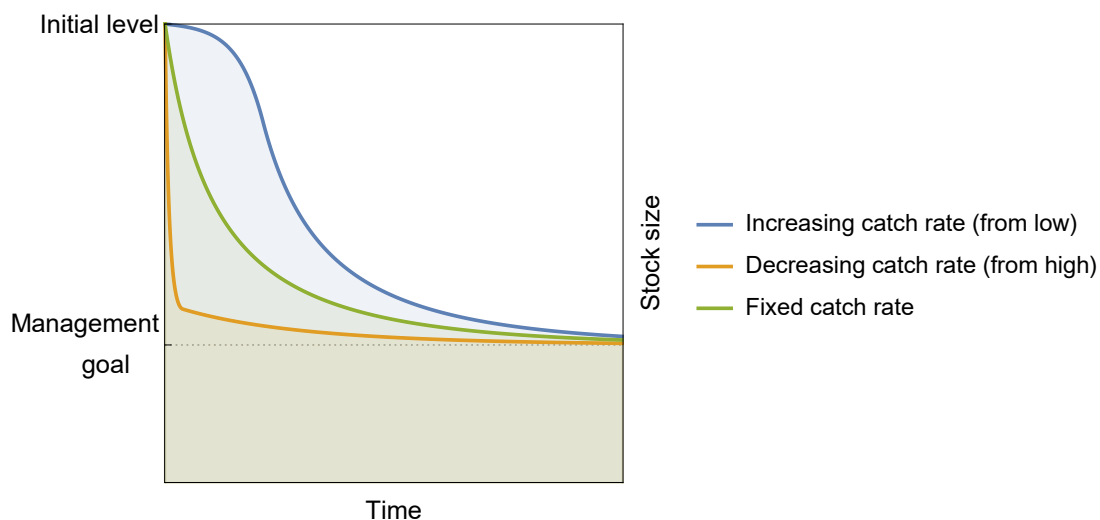
Figure 9.2: Three potential time pathways toward a fixed management goal.

In a simple model, we may find straightforward solutions, such as the bang-bang solution identified in code box 6.9.1. However, as the problem becomes more realistic, we must factor in waiting costs and other elements not encompassed by the simple solution. The three possible solutions depicted in figure 9.2 yield notably distinct economic outcomes over time, and the final goal may be achieved at different points in time. Although the equilibrium solutions are identical, each solution demonstrates different characteristics when accounting for the period preceding equilibrium attainment.

As previously emphasized, real-world fisheries are never truly in equilibrium. Consequently, fisheries dynamics outside equilibrium warrant significant attention. Nevertheless, equilibrium solutions remain crucial and serve as nearly essential reference points for gaining economic and biological insight into fishery dynamics. As demonstrated in chapter 6, fisheries dynamics are governed by attributes reflected in theoretical equilibrium solutions.

9.5 Allocation of Fishing Rights

Open access fisheries can be regulated through spatial and temporal regulations, primarily using general technical rules. Among the earliest constraints applied to modern fishing practices is mesh size regulation. These comprehensive regulations cover all fishing activities without granting specific rights to any individuals, and they still maintain the characteristics of an open access fishery, even if it is no longer a pure open access one.

Technical regulations can improve stock conditions even while the fishery remains open to all participants. The natural progression in modern fisheries management involves constraining the fleet or setting limits on the total catch. Restricting the fleet can be achieved by imposing a license fee for all vessels seeking participation. This effectively closes the commons, requiring an entry fee for all fishers. Similarly, establishing a total quota also closes the commons, as fishing activities must cease once the quota limit is reached.

Limiting fleet activity through entry fees has demonstrated limited effectiveness in conservation when used without additional regulations. Total quota setting may prove effective for stock conservation but can lead to rent dissipation through a phenomenon often referred to as *Olympic fishing* (see section 10.6). With quotas as a finite resource, participants compete to secure the largest share. Consequently, rent is depleted through over-investment in large engines and fast boats. Economically, this situation in the long run closely resembles an open access fishery in terms of profitability.

Introducing property rights can mitigate or entirely eliminate the market failures highlighted earlier. Property rights can be established by restricting fishing rights (reducing the fishing fleet) or limiting the catch amount for each unit (individual catch quotas). For these property rights limitations to be effective, the commons must be closed, granting rights to some while denying them to others. Determining who is permitted to fish and who is denied access to previously communal natural resources is a political decision, often a delicate one. These decisions may involve numerous other political objectives beyond stock conservation and economic benefits. Factors such as alternative employment opportunities, settlement patterns, historical rights, and more can play pivotal roles in determining who is granted property rights.

Property rights can be assigned for various durations, whether short or long term, or as permanent property rights. In cases of the latter (and also for shorter periods), property rights may be transferable, establishing a market for fishing rights (see section 10.7). In principle, this approach should promote economically efficient fishing practices by selecting the most efficient units. This can reduce the influence of other political factors, as discussed earlier.

Exercises

Exercise 9.1 In section 9.2, it is assumed that an increase in fishing effort production also increases employment. Will that always be the case? ∎

10. Regulations and management means

Historically, fisheries management has varied based on changing economic and political conditions in different time periods. According to the saga of Norwegian King Eystein I Magnusson in 1120 AD, cabins were constructed for fishers in the Lofoten region to stimulate fishing activity and increase tax revenue[11]. Until recently, fisheries management focused on promoting fishing activity rather than restricting it, and this was the norm in fishing nations.

To support the fishing industry with better information on fish abundance and new fishing opportunities, marine research institutes were established. As early as 1816, the Norwegian *Lofoten Act* divided the fishing area south of the Lofoten islands into blocks assigned to different fishing villages on the islands[17]. The intention behind this regulation was not to limit fishing but rather to enhance it by reducing the likelihood of gear collisions. However, it soon became clear that marine fish resources were not limitless, and certain fishing activities needed to be controlled. The implementation of technical regulations for fishing gear marked the beginning of modern fisheries management.

Interest in marine resources also spurred fundamental research in new areas, both in oceanography and population dynamics. The establishment of *the International Council for the Exploration of the Sea* (ICES) in 1902 signaled a new era of marine research, focusing on fluctuations in stock availability and studying recruitment variations and oceanography's influence on fish stocks[34]. The emphasis on fish stock recruitment led to initiatives such as cod stocking programs in Norway and Canada in the early 20th century.

Following World War II, concerns grew about large-scale fisheries, and the issue of overfishing became apparent. Seminal scientific publications during the 1950s played a pivotal role in altering perspectives on fishing and its potential impact on exploited resources[29, 46, 47].

Fisheries regulations can be categorised along two dimensions: 1) Indirect or direct regulations, and 2) Control of input or output. These dimensions create a four-field table, as depicted in Table 10.1.

Table 10.1: *Different management means positioned along the two dimensions: Direct-Indirect and Input-Output control.*

	Indirect regulations	Direct regulations
Input control	Tax on effort	Infrastructure Technical regulations Closed season Marine protected area (MPA) Entrance fee Limited entry
Output control	Tax on harvest	Total allowable catch (TAC) Individual quotas Transferable quotas

However, is it enough to simply pick one of the four fields and choose a management measure (if there are multiple options) to effectively address the management problem? Unfortunately, the practical challenges of fisheries management are more intricate. For this reason, various management strategies have been developed. But does adopting a management approach mean the problem is solved?

Regrettably, the management problem cannot be universally defined to suit all fisheries. As demonstrated in Chapter 9, it primarily hinges on political objectives for utilising marine resources, as well as the current state of the fish stock and the fishing fleet. Often, the assumption is made that fishing effort is excessive. In such cases, the core concern becomes reducing fishing activity. However, should this reduction entail removing specific vessels (which ones?) or should all vessels decrease their fishing effort? How should this reduction be enforced? The control aspect is also contingent on the chosen management approach, and various management methods incur distinct costs in terms of implementation and control. Moreover, if the management efforts prove successful in improving the situation – for instance, reducing fishing effort to a desired level – can we expect the selected management approach to be equally effective in sustaining this improvement?

All of these issues necessitate careful consideration when selecting a management approach. This topic is further explored in the subsequent sections of this chapter. We begin our exploration with the management approaches outlined in Table 10.1, commencing chronologically with the initial attempts to regulate fishing activity. Consequently, we move to the red cell in Table 10.1 and commence with a discussion of infrastructure's role.

10.1   Ownership

Why is it necessary to manage living marine natural resources? Typically, fish stock resources are considered common goods, shared among the people residing within a specific region (country) with legal access to these resources. The issue of common ownership among multiple nations sharing the resource is not addressed here.

Economic goods are often categorised into four main types based on fundamental characteristics (see Table 10.2). Fish stock resources typically exhibit *non-excludability*, meaning they are accessible to all individuals, and they are *rivalrous*, implying that their use affects

the availability of the resource. Public goods, such as *clean air* and *sunshine*, are also accessible to all, but their utilisation doesn't diminish their availability. Some other public goods, like *schools* and *public parks*, may seem unaffected by their level of use within a given context.

Table 10.2: *Various types of goods are categorised based on their properties. The red colour indicates that excludable rights require protection from a government or organisation, while the green colour indicates that open access to the goods is possible, requiring no regulation. The orange colour indicates that these types of goods may require management.*

	Excludable	**Non-excludable**
Rivalrous	Private goods	Common pool goods
Non-rivalrous	Club goods	Public goods

Therefore, we distinguish between two categories: *common goods* and *public goods*. Private goods, such as *food*, *clothes*, and club goods (*private parks*, *cable television*, etc.), have defined ownership that excludes others from consuming them. Private goods are also rivalrous (for instance, if you eat *a meal*, you cannot give it away for someone else to eat the same afterwards), similar to fish stocks. On the other hand, club goods (within a given context) are non-rivalrous.

Excludable goods (represented in the red cells of table 10.2) are generally protected by laws, and the existence of such goods is a primary reason why organisations like states and nations are necessary. A state establishes laws and employs the necessary means to uphold law and order (e.g., police or military forces). These laws can also extend to govern the use of non-excludable common goods, regulating their use under open access. In this way, *pure open access* is substituted by *regulated open access.*

The distinction between the two rows in table 10.2 reflects an inherent property of the good that cannot be altered by regulations. However, the difference between the two columns can be subject to regulations. The common pool resource could, for instance, be designated to an individual or an organisation as sole owner property. In doing so, it becomes a private good that requires protection against use by others. With this protection in place, the sole owner can maximise sustainable profit by utilising the resource[29]. If there exists a global market of similar products produced by a multitude of other producers, the concerned resource owner can maximise profit in accordance with a socioeconomic optimum.

10.2   Infrastructure

Fish products are perishable goods, and the time it takes from being caught to consumption is critical. Adequate infrastructure is essential, especially when the fishing grounds are distant from the consumption sites. Access to ice and modern freezing technology are types of infrastructure that significantly extend the time before the product spoils and becomes unfit for human consumption. Roads and other forms of infrastructure connect fishers, buyers, and consumers. Additionally, as mentioned earlier, access to fishers' cabins, ports,

fishing boats, and gear should also be included when listing important fisheries-related infrastructure.

In today's society, we often take such infrastructure for granted, at least in the wealthier parts of the world. During the early Middle Ages in Europe, fishing occurred near major cities, often situated close to rivers. However, rivers soon could not supply the growing cities with fish products, necessitating the import of fish from more distant areas. Dried cod from the northern regions proved to be a perfect substitute. Despite being caught months or even years earlier, the cod dried in the cold climate of northern Norway and Iceland were preserved well enough to withstand the summer heat in southern Europe. Archaeological studies have revealed an increase in marine fish bones from Nordic waters in Western Europe since the beginning of the tenth century[4]. Following the introduction of dried cod in Europe, the use of salt became prevalent in the Nordic countries. This ushered in another fish boom in Europe, with salted herring barrels becoming the new form of currency, following the era of dried cod. The infrastructure of this trade was complex, yet politically and economically significant. Powerful nations vied for control over the trade, which had been dominated by the Hanseatic League in the post-Viking period. Subsequently, Britain gained control over the cod trade[16].

It is evident that improved infrastructure is not intended to reduce fishing activity. On the contrary, a minimum level of infrastructure is a prerequisite for commercial fishing to occur. Even when the distance between fishing grounds and consumption sites is short, necessary infrastructure is required to facilitate the transportation of fish from fishers to consumers, often involving multiple intermediaries. Impeding this process would have a detrimental impact on the fish trade and, consequently, the profitability of fishing.

10.3 Technical Regulations

Technical regulations are concerned with the design and utilization of vessels and fishing gear. Minimum allowable net sizes in various fisheries are common examples of technical regulations. Similarly, regulations dictate how gear should be constructed and used, and they may also specify maximum vessel sizes or loading capacities. The underlying principle of all technical regulations is to make fishing activity less efficient. Consequently, such regulations ultimately result in an increased cost per unit of catch. From an economic standpoint, it is therefore important to compare the increased costs with the presumed economic benefits brought about by the regulation. These benefits might include an enlarged stock size and/or a more favorable age distribution within the stock.

While the outcomes of technical regulations may lead to a less efficient fishery, the objectives of these regulations are typically phrased somewhat differently. The primary concern motivating technical regulations is often to safeguard juveniles and young individuals, thereby ensuring stock recruitment. The aim may also be to protect spawners, but other forms of regulations are typically better suited for achieving this goal, such as closed-season regulations or marine protected areas (MPAs) (see subsection 10.4).

Within the framework of a simple surplus production model (section 6.5), the benefits of such technical regulations are not particularly convincing. According to equation 6.20, the open access stock level is determined by the unit price of harvest (p), the unit cost of effort (a), and the catchability coefficient (q). Let's consider a fishery where mesh size regulation is effective. In this case, the regulation will lead to a reduction in the value of q, contributing to a corresponding increase in the equilibrium open access stock size as
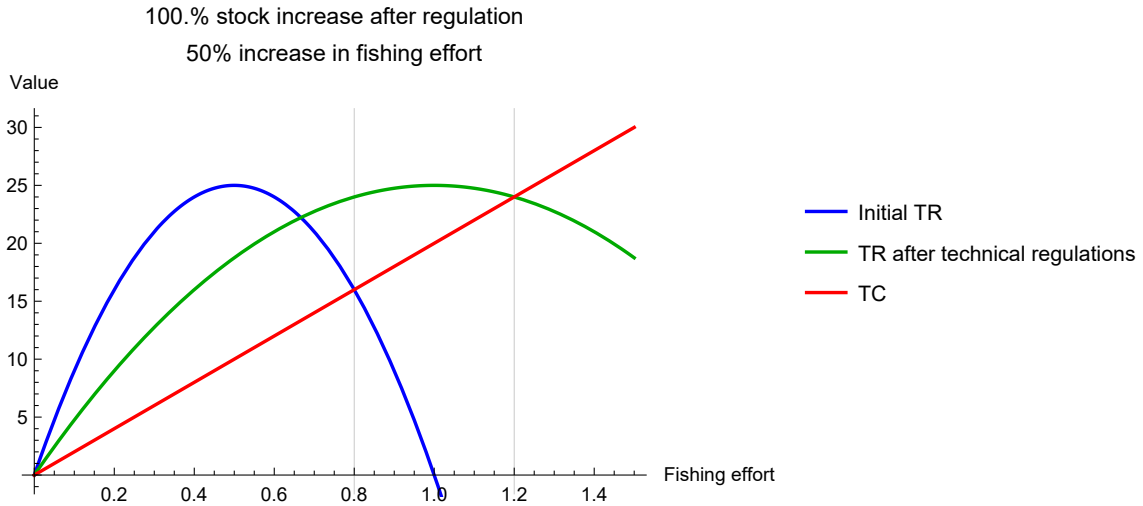
Figure 10.1: Equations 6.4 (TC) and 6.5 (TR) when the parameter values are $K = 100$, $r = 1$, $a = 20$, $p = 1$, $q = 1$ and $s = 0.5$. The blue and green curves show the equilibriums before and after the introduction of mesh size regulation.
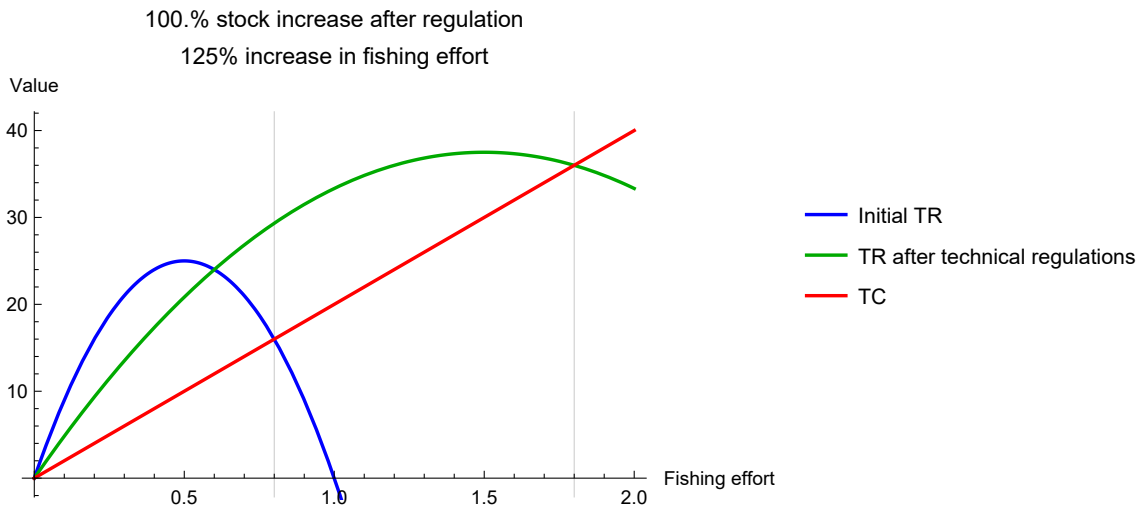

Figure 10.2: Equations 6.4 (TC) and 6.5 (TR) when the parameter values for K, a, p and q are as in figure 10.1, while the value of r increases from 1 to 1.5 as a result of the mesh size regulation.

per equation 6.20. Hence, technical regulation can be expressed by slightly reformulating Equation 3.1:

$$H_s(E, X) = s \cdot q \cdot E \cdot X \tag{10.1}$$

Here, s represents the q-reducing impact of technical regulation, aimed at achieving selective fishing; $0 \leq s < 1$. Remarkably, the equation suggests that if $s = 0.5$, reducing catchability by 50%, the equilibrium stock size in an open access fishery increases by 100%!

Figure 10.1 illustrates the aforementioned example, focusing on cases where there is no other biological feedback from mesh size regulation besides the stock increase resulting from higher fishing costs. Two conclusions can be drawn from this discussion: 1) If a profitable fishery remains possible after implementing technical regulations, the fishing

effort will increase in an open access fishery. 2) If the regulations also positively impact stock growth (as represented by an increased r), the potential economic rent may be lower, equal to, or higher than before the regulations, depending on the extent to which growth is enhanced by the technical regulations.

In this subsection, we have focused on an open access fishery. As discussed in subsection 6.5, we differentiate between pure open access (an unregulated fishery) and open access fisheries subject to specific constraints, such as technical regulations.

10.4   Closures in Time and/or Space

Regulations involving closed seasons, closed areas, or combinations of both are commonly implemented in many fisheries. These types of regulations share certain similarities with the technical regulations discussed earlier. Instead of imposing restrictions on gear use and vessel technology, they impose limitations on where and when fishing is permitted. However, similar to technical regulations, there are numerous ways to define such constraints.

At its simplest, a closed area regulation designates certain geographic regions as marine sanctuaries where fishing is prohibited at all times. More recently, the concept of *marine protected areas* (MPAs) has gained prominence. An MPA could be a marine sanctuary, but it could also encompass areas with specific restrictions on fishing times, allowable gear types, and so forth. Consequently, MPAs offer a more flexible approach to area regulations.

The fundamental idea behind MPA regulations is to conserve fish stocks by safeguarding particular areas from specific fishing activities. It is essential to identify suitable areas for this purpose, which might include spawning grounds, vulnerable habitats, and similar locations. MPAs have received a significant boost from strategic objectives outlined by the Convention on Biological Diversity (CBD), whereby member states have committed to protecting at least 10% of marine habitats in terms of area. As such, MPAs aim to address a broader perspective, focusing on nature preservation and biodiversity. However, within the context of fisheries, we will concentrate on how MPAs impact fishing.

Can we make general statements about the effects of MPAs on fisheries? Numerous studies have been conducted, with results strongly tied to the model assumptions used. Surprisingly, many publications employ non-spatial modelling techniques to explore potential impacts of MPAs on fisheries (such as constructing two biomass components within a standard surplus production model), while others adopt cellular automata methods to model the spatial distribution of fish stocks[18, 40, 48, 50]. One common observation is that MPAs tend to have an immediate negative effect on fisheries. However, the long-term effects depend on various factors, including fish migration patterns (including source-sink models), MPA size, placement, and lifespan, as well as indirect ecosystem effects.

The concept of closed season regulation entails temporarily closing an area during specific time periods. This approach is often used to protect spawners during their spawning season or to conserve juvenile populations. However, when combining bycatch and closed area regulations, complexity can increase significantly, potentially resulting in high monitoring costs and the risk of losing valuable catches.

A notable example of complex closed area regulations is seen in the Barents Sea shrimp trawl fisheries. Technical gear regulations include size sorting grid devices to reduce fish bycatch. However, such devices may not sort out juvenile fish, leading to additional catch regulations for juveniles. The current approach involves temporarily closing areas where

the catch contains more than 8 cod juveniles, 20 haddock juveniles, 3 Greenland haddock juveniles, or 3 redfish juveniles per 10 kilograms of shrimp. When an area is closed, it remains off-limits until the juvenile numbers fall below critical levels[43]. Consequently, test catches are required to assess juvenile content, while shrimp fisheries need to find alternative fishing grounds until the area can be reopened. This example underscores the complexity of area regulations involving multiple species.

10.5 Limited Entry

A common method of closing the commons involves limiting fishing rights to an exclusive group of fishers or vessels. In many cases, this is achieved by introducing a fee that grants fishing rights to anyone who pays (see subsection 10.8). Another approach is to close the commons by including all existing fishers in the fishery and prohibiting new entrants.

A more sophisticated approach to closing the commons involves limiting fishing rights to a specific group of fishers based on predefined criteria. These criteria might include historical rights (including those of indigenous groups), regional priorities, or other specific reasons for prioritising certain groups (such as historically disadvantaged groups or others).

Let's consider a scenario where fishing rights are distributed to reduce the current fishing effort, E_0. Assuming E_0 is measured in vessel years, and fishing rights are allocated to vessels such that $E_1 < E_0$, where E_1 is the total number of vessels granted fishing rights. Refer to figure 10.3 and begin at the intersection of the red line and the blue curve, representing an initial fishing effort of E_0. The government limits new entrants to the fishery, removing $E_0 - E_1$ vessels and shifting to E_R as indicated by arrow A. Rights holders can now enjoy improved economic conditions, and in the long run, the resource rent stabilises at C on the graph. Fishing vessels utilise the excess profit to invest in equipment and vessel enhancements to increase efficiency. The augmented costs per vessel are depicted by a steeper cost equation (the dashed green line), leading to a new equilibrium at E_2, where all additional rent is allocated to making each vessel more efficient. The increased fishing costs (D) push the total cost above the initial E_0 level.

We observe that there are both immediate effects of directly controlling effort by limiting entry and other less predictable effects stemming from changes in economic conditions for the rights holders. As soon as a positive resource rent becomes available, vessel owners have the opportunity to enhance their competitive positions and increase their vessel's fishing capacity. However, without further control, one can expect that all the gained resource rent will be squandered over time.

Limiting the number of vessels may seem like a straightforward strategy. The initial challenge of determining which vessels to exclude could be resolved, for instance, by compensating fishing vessels to exit. Those with the lowest profitability should, in principle, be the first to leave. The number of vessels that should exit the fishery depends on the regulatory objectives (see chapter 9), and in principle, this number could also be identified. However, even after achieving this, technological and economic developments continue. A portion of the profit obtained by licensed vessels will be invested in improving vessels and fishing gear, altering the fishing effort without changing the vessel count. If the number of vessels is fixed and denoted as V, the fishing effort increases over time according to the
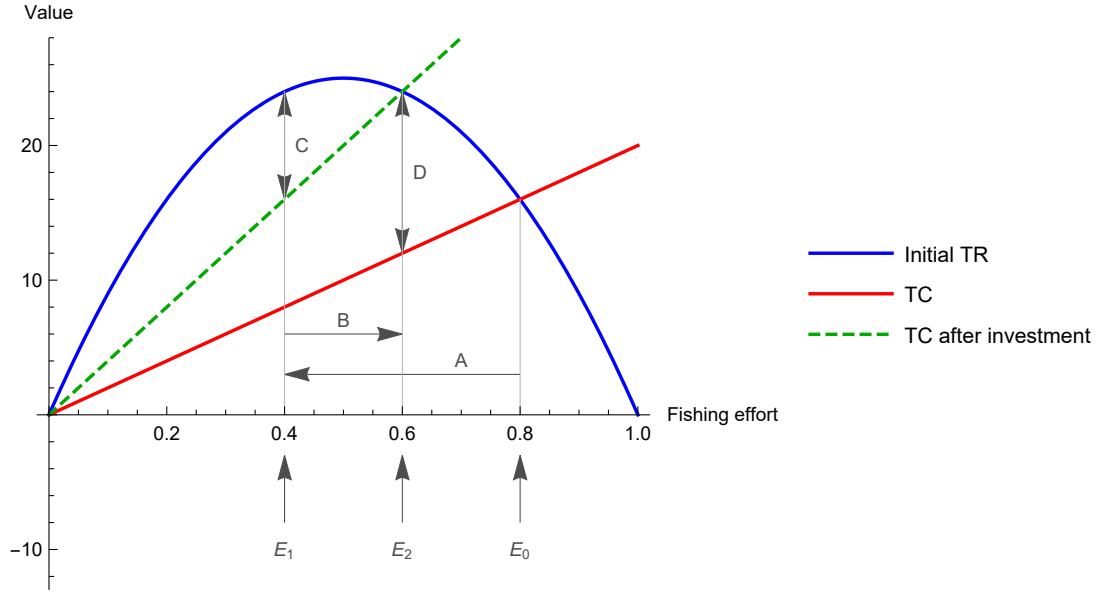
Figure 10.3: Equations 6.4 (TC) and 6.5 (TR) when the parameter values for K, a, p and q are as in figure 10.1.

function:

$$E(t) = \psi(t) \cdot V \tag{10.2}$$

Here, $\psi(t)$ describes the relative development of fishing effort efficiency ($\psi(0) = 1$ and $\psi(t) \geq 1$ for $t > 0$). Equation 10.2 represents the standardized effort of the fleet at time t, standardized by the effort at $t = 0$. From figure 10.3, if $V = E_1$, according to equation 10.2, $E(t) > E_1$ when $t > 0$ and $\psi(t) > 1$. In the long run, assuming that the unit cost of standardized effort decreases over time, a scenario can emerge where $E(t) > E_0$ when t becomes sufficiently large.

Standardization of effort poses an inevitable challenge in limited entry regulations. This standardization not only pertains to temporal standardization but also extends to standardization across various types of fishing boats and gear within a heterogeneous fleet.

10.6 Quota Regulation

Quota regulation has become one of the most frequently used measures for regulating modern Western fisheries. The principle is simple yet highly flexible, and quota regulations can be implemented in numerous ways. Generally, quota regulations are found in intricate combinations with other regulatory measures, but at its core, it involves setting an upper limit (total quota) on what the fleet can harvest over a specified period (usually a year).

This regulation is often referred to as an *Olympic fishery* because it triggers a race to catch as much as possible before the total quota is reached. To manage this, vessels need to install powerful engines to reach fishing grounds rapidly, and they must return to port quickly to unload the catch and prepare for subsequent trips if quota remains. Fishing operations prioritise quantity over quality. The potential economic rent that could have resulted from limiting the catch below open access levels is instead lost in over-investments and fleet overcapacity.

The Olympic fishery challenge can be addressed by allocating the total quota to fishing units, fishers, or vessels, often referred to as *individual quotas*. With each vessel having a share of the total quota, there is no need for competition to secure a share. Individual vessels can fish their allocated quota as it suits them, considering other seasonal opportunities, price fluctuations, and more.

Individual quotas can take various forms, from non-transferable vessel quotas with time limits to transferable quota shares without time constraints, as discussed below.

10.7 Individual Transferable Quotas (ITQ)

Common ownership of fish stock resources leads to rent dissipation and resource overuse. The issue is rooted in the absence of ownership and, as Gordon illustrated[29], can be solved by allocating the resource to a single owner (a monopoly). However, several factors discourage this solution unless a state is the owner, and all fishing activities occur within a state-controlled enterprise encompassing both resource and processing, as in a communist state. Within a democratic political system, such solutions are deemed inefficient and undesirable.

Given an initial distribution of limited, transferable fishing rights (ITQs), a market for these rights will emerge. Holders of fishing rights will be willing to sell their rights at certain prices, and those seeking rights will have a willingness to pay. When the willingness to pay exceeds the compensation price of one owner, this likely reflects variations in fishing efficiency and cost between sellers and buyers. Consequently, the quota may be obtained more efficiently (at a higher price and/or lower cost) in a free market. ITQs serve as a tool to optimise the fishery, provided an optimal quota is set.

Some argue that ITQs also offer the benefit of the quota market providing information to help managers identify the optimal quota[2]. However, this argument is based on highly strict and conflicting assumptions. The first assumption involves a homogeneous fleet, as seen in the simple Gordon-Schaefer model (refer to figure 6.4). In this case, willingness to pay aligns with the compensation price of all property owners, and the free market solution is already established. Consequently, cost is minimised, the price is fixed, and no transactions occur.

When relaxing the assumption of a homogeneous fleet, the argument becomes considerably more intricate. If the fleet is heterogeneous, it may be demonstrated that ITQs may not necessarily contribute to making fishing activities more efficient[33]. Despite this knowledge, which has been available for decades, ITQs have been introduced in multiple countries as a means to enhance the efficiency of the fishing industry.

The advantage of ITQ regulation lies in the distribution of property rights based on supply and demand. However, external factors, such as access to financial institutions and existing debts, could exert more influence on the final outcome than differences in costs and efficiency.

10.8 Taxation

Economists often propose taxation as an efficient means of internalising external costs associated with economic activities. Taxation is also a relevant regulatory tool in the

context of fisheries and was initially suggested in the 1950s[29]. In production activities, there are essentially two methods of implementing tax regulations: through taxation of input factors of production or through taxing the final product. In fisheries, negative taxes or subsidies have been more common than positive taxes, both on inputs and outputs.
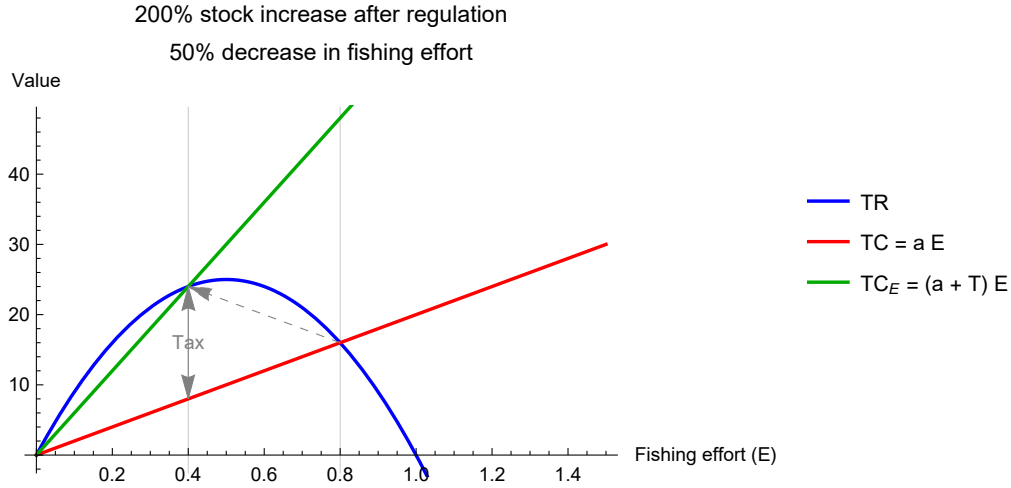

Figure 10.4: Equations 6.4 (TC) and 6.5 (TR) when the parameter values for K, a, p and q are as in figure 10.1, while in this case the value of r increases from 1 to 2 as a result of the mesh size regulation.

$$TC_E = (a + T)E \tag{10.3}$$

Exercises

Exercise 10.1 In subsection 10.3, it is mentioned that if $s = 0.5$, the stock size is increased by 100%. Explain why and demonstrate it mathematically, assuming logistic growth of the fish stock. ∎

Exercise 10.2 At the end of subsection 10.4, a complex bycatch regulation in the Barents Sea shrimp fishery is described. How would you evaluate such regulations? ∎

11. Complexity and vagueness

Fishing is an economic activity that can have external effects; the fishing activities of one entity can influence the opportunities of others. When externalities reach a certain level, it may become necessary to restrict fishing activities through technical and other regulations, as discussed in chapter 10. The regulatory methods discussed in chapter 10 share a commonality: they target a single species in the fishery. While there might be some bycatch, this situation still holds true for many fisheries in Europe and the Americas. However, over the past 50 to 60 years, the majority of the global harvest has shifted from these continents to Asia, making Asia the dominant region in world fisheries (figure 7.1). Most of the Asian fisheries are multi-species fisheries. How can such fisheries be effectively regulated?

In addition to the challenge of regulating multi-species fisheries, there are other reasons to explore alternatives to single-species regulations. There is a growing focus on the impact a single species can have on the entire ecosystem. The management of a single species fishery may lead to external effects on other parts of the ecosystem and, consequently, on other fisheries as well. While basic principles of ecosystem dynamics are understood, this field is characterized by significant knowledge gaps. The complexity of these systems encompasses both temporal and spatial dynamics on all levels. Managing such systems must rely on limited insights into both the dynamics and the state of the system. As a result, alternative management approaches are required beyond those discussed in chapter 10.

11.1 Uncertainty

In theory, it is possible to assume perfect knowledge, as we often do in economic models. However, even with perfect knowledge, there are areas of uncertainty where no certainties exist. Interestingly, such uncertainty is a crucial aspect of human communication and languages. An illustrious example is the Sorites paradox[21], also known as the paradox of a heap. The term *heap* is vague, which is precisely why it is a useful term. When you

have a heap of grains and start removing grains one by one, at what point (number of grains left in the heap) does it cease to be considered a heap (figure 11.1)? No one can definitively answer, as a heap is not defined by a specific number of members. Nonetheless, we comprehend a heap as a concrete and practical term to employ.
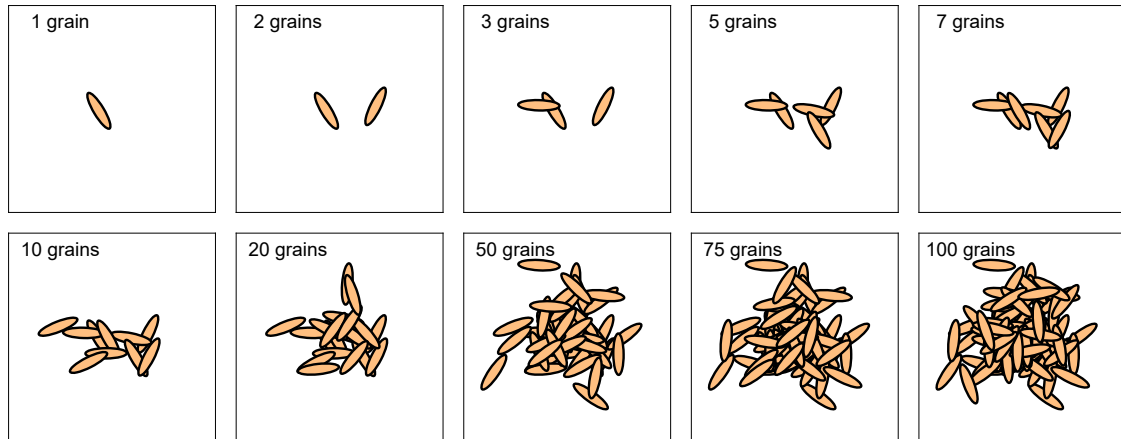

Figure 11.1: At which grain is the heap no longer a heap?

Another form of vagueness that may be more readily understood and accepted is vagueness related to measurements. Precise measurement is essential in a modern society, and most decisions involve some form of measurement. The fundamental purpose of measuring is to enable comparisons. To compare two things, events, or situations, we need to employ the same type of measurement for both. In this manner, it is, in principle, possible to compare entities that may not even be precisely measurable by devising or defining a standard measurement methodology.

11.2 Complexity and Precision

Figure 11.2 depicts the sample space for knowledge-based decisions. The blue shaded region indicates the domain within which we can operate, while the white area remains inaccessible. In other words, precise analyses with high complexity are beyond reach, yet we can conduct precise analyses of simple matters or imprecise, intuitive analyses of complex issues. This figure draws inspiration from the concepts introduced by Lotfi A. Zadeh (1921 – 2017)[59], who introduced fuzzy sets to accommodate solutions for problems that are too intricate for exact analysis. In chapters 9 and 10, we employed a simple model and conducted highly precise analyses. According to figure 11.2, this positions us in the magenta region (*Precise analysis*). As we progress along the horizontal axis to the right, precision must be sacrificed as complexity increases.

Can decisions be made when we lack precise knowledge about the current state and even have limited insight into future states and the potential impacts of different actions? Indeed, this is a common occurrence in real life. We rely on available knowledge, experience, and beliefs to guide our decisions. In essence, we employ *heuristics*.
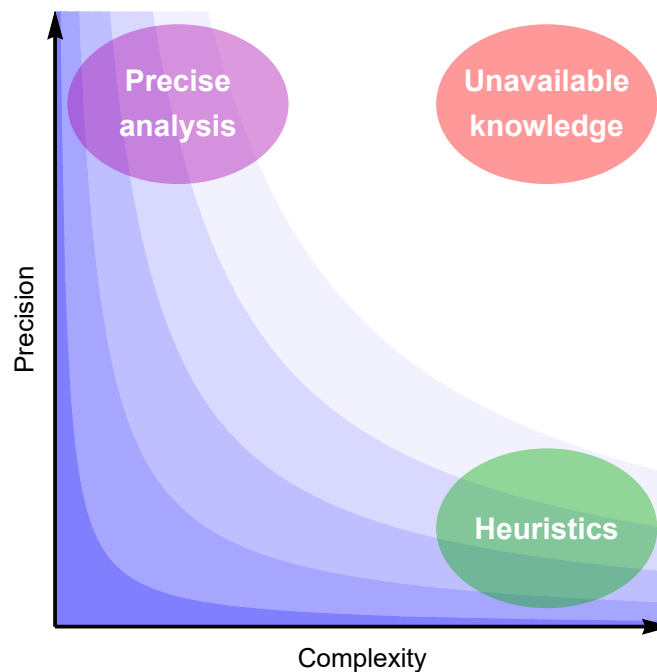
Figure 11.2: Sample space of analyses within the dimensions of *complexity* and *precision*. It is assumed to be an inverse relation between the two dimensions and the blue colour indicates the available space of reasoning; the more blue, the easier to handle.

11.3 Expert Systems

In our everyday lives, we employ numerous active and immediate responses to situations that demand such actions. Simple and commonplace examples include catching a ball in midair or swiftly grabbing a falling glass. How do we execute these responses? While the question is not new, there is no unanimous answer to it. Theories exist that attempt to explain how we track a ball's trajectory through the air and catch it before it lands[39]. Models have been developed to predict how a baseball outfielder might run to catch a ball. However, as noted by several researchers, these explanations seem overly simplistic[9]. The mechanics behind how we successfully strike moving objects remain not fully comprehended. Nonetheless, we demonstrate this skill ourselves, and it's even possible to create robots capable of hitting moving objects[5]. How is this phenomenon feasible?

The fascinating aspect is that despite our incomplete understanding of how we perform such tasks (or how other living creatures do the same), we can still devise sets of rules that yield the same outcomes when programming robots to follow these rules. This approach is known as the *heuristic methodology*. Traditional abstract analyses, based on established knowledge such as Newton's laws of motion, frequently fall short when addressing challenges like designing running or baseball-playing robots. The problems are too intricate to be tackled analytically. In contrast, straightforward if-then rules based on heuristic knowledge can be processed swiftly, enabling rapid iteration of these rules. This iterative process continually adjusts the robot's behaviour until the intended task is accomplished.

Let's now place the provided example above within the knowledge pyramid depicted on the left side of Figure 11.3. All the observations we've gathered regarding catching balls and other airborne objects form the dataset, which we refer to as *data*. After processing this data, we can explain what we've observed, resulting in *information*. The available
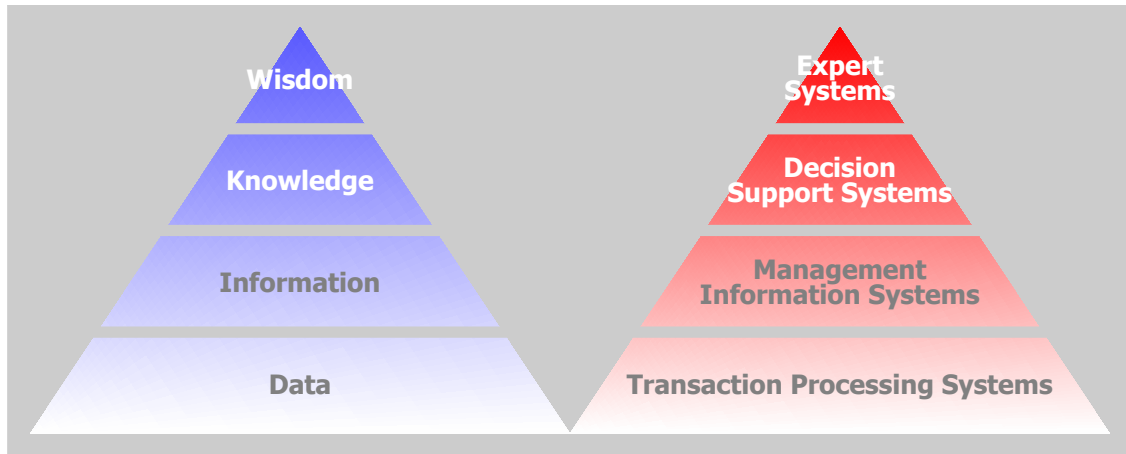
Figure 11.3: The knowledge-pyramid, or the data–information–knowledge–wisdom hierarchy, to the left, and the corresponding Expert System to the right[45].

information prompts questions about how to elucidate our observations, and this leads to the transformation of information into *knowledge* regarding the mechanisms behind the observed events. Going beyond knowledge, *wisdom* incorporates values such as social, ethical, and aesthetic considerations to effectively apply the acquired knowledge. Wisdom represents a deeper understanding of knowledge.

The right side of Figure 11.3 illustrates how the knowledge pyramid translates into the challenge of designing a robot capable of catching a ball. The objectives and comprehensive insight are encapsulated within the *Expert* level, which enables the formulation of necessary *Decisions* to be executed. These decisions are guided by rules supported by *Information* about the status of all relevant variables at any given time. This information is derived from the *Transactions* that constitute the technical blueprint of the robot's design.

We've now progressed from the pinnacle and moved downward in the pyramid, symbolising the construction of the robot. One might raise the valid objection that the task of managing fisheries greatly differs from building a ball-catching robot. Nevertheless, there are remarkable parallels when translating the robot problem back to the original issue: Can we artificially replicate this action of catching a ball without comprehending how a human accomplishes it? In the context of fisheries management, the question becomes: Can we exploit a portion of the ecosystem without complete knowledge of how its components interact, while minimising the risk of harming other elements?

The objective of fisheries management is to regulate fishing activities according to political objectives based on existing knowledge of biological and economic dynamic systems and their interplay. As discussed in Section 11.8, we do not possess a complete understanding of how this system operates and how our fishing impacts it. The concept of Harvest Control Rules (HCR) aims to apply simple principles of common sense to minimize the risk of overfishing, depletion of stocks, and negative ecosystem consequences resulting from our fishing activities.

Harvest Control Rules are founded on heuristic knowledge expressed through a set of rules, which provides a way to structure the knowledge deemed important for the management process. While this knowledge is often not easily translatable into algorithms and mathematical models, it can be embedded in straightforward heuristic rules. Figure 11.3 illustrates the problem by situating knowledge closer to wisdom than to data, signifying

the deeper insight embodied by knowledge compared to data, though complete wisdom on how to structure the data correctly remains elusive. Each step in the knowledge pyramid represents an increased capacity to structure primary observations. Information is essentially data in a structured form, while knowledge is actionable information[45].

11.4 Set Theory

Traditional set theory, developed by German mathematician Georg Cantor (1845 – 1918), categorises elements into classes or categories. A set A containing integers between 0 and ten would encompass the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9. Any other integer or number would not be part of set A. Set A contains a finite number of elements. However, sets can also consist of infinitely many elements. For instance, set B could encompass all even numbers. This includes 2, 4, 6, and 8 from set A, as well as 10, 12, 14, and so on, extending infinitely. We consider 2, 4, 6, and 8 to be the intersection of sets A and B ($A \cap B$), while 1, 3, 5, 7, 9, and all even numbers form the union of the two sets ($A \cup B$). These concepts are illustrated in Figure 11.4.
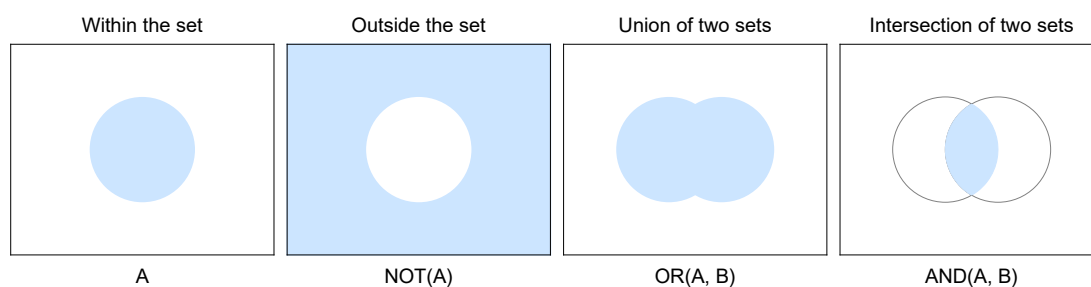

Figure 11.4: Different concepts included in set theory. To the left the light blue disk illustrates a closed set. The next figure shows the area not included in the first set, followed by the union of two sets. Finally the figure to the right shows the intersection between two sets.

Let's name all the members in set A that are not part of set B as the complement of A and B ($A \setminus B$). Therefore, the complement $A \setminus B$ consists of the integers 1, 3, 5, 7, and 9; all the odd numbers contained in set A.

Set theory offers a precise and logical approach to classifying different items. While this precision suits numbers admirably, set theory also aims to classify other objects. For instance, you might have a set of *pears* and another set of *apples*. A third set could encompass all *fruit*. The sets of *pears* and *apples* form subsets of the set of *fruits*.

Sets of *fruits*, *pears*, and *apples* can be considered fairly precise categories or classes, even though some of us might struggle with categorising an apple that looks like a pear, and vice versa. However, categorising the "fruit" category presents a greater challenge. For example, where should bananas (botanically classified as berries) be placed?

Let's delve into a more intricate classification: Imagine creating a set of *green apples* and another set of *red apples*. Both of these sets would be subsets of *apples*. However, the union of these two subsets would only encompass a portion of the set of *apples*. Many apples

exhibit both green and red (and possibly yellow, orange, etc.) colours. In cases where a red apple has a small green spot, should it be considered part of the subset of red apples?

Computer scientist and electrical engineer Lotfi Zadeh (1921 – 2017) introduced the concept of partial membership in sets during the 1960s. For red and green apples, the degree of membership in the two sets could, for instance, be determined by the percentage of red and green colour on the apple peel.
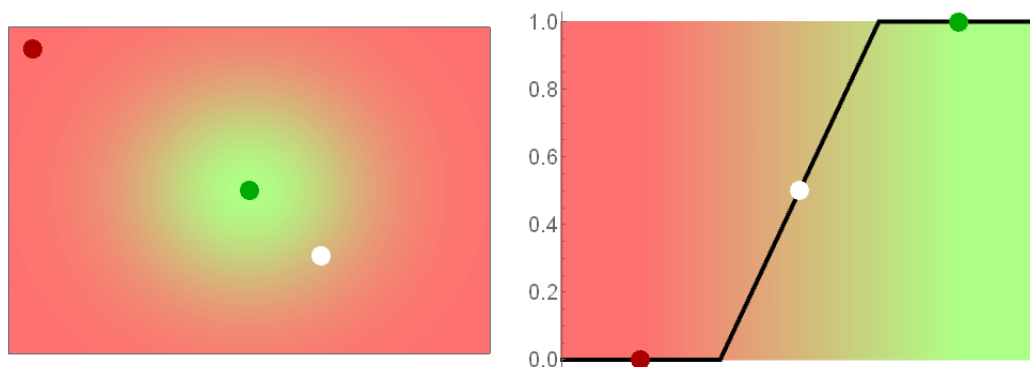

Figure 11.5: The diagrams depict partial memberships in the two sets of red and green apples. On the left, there are three points. The red point is positioned within a red area encompassing a green cloud. The green point is fully a member of the green cloud (or fuzzy set), and the white point is situated within the fuzzy region between red and green. In the right-hand panel, the broken black line and the vertical scale illustrate degrees of membership in the set of green apples. In this context, the white point (equivalent to the white point in the left panel) possesses a membership degree of 0.5 (50%) in the set of green apples.

However, one might question whether the percentage of red and green color provides a highly precise classification. In figure 11.5, we observe only red and green colors, and the membership degrees in the subsets of red and green apples appear to be precise. Yet, colors are mixtures where one label (color name) transitions into others. While some might argue that colors can still be expressed precisely mathematically, our perception of colors is undoubtedly individual. We share color terms (red, green, etc.) but not necessarily the perception. People often differ in how they apply terms to describe colors.

Other terms that encompass varying degrees of vagueness include *soft* and *hard*, *young* and *old*, *short* and *tall*, *wide* and *narrow*, and so forth. These terms don't necessarily need to be strictly opposite; the challenge often lies in the gray area between the two extremes.

11.5 Fuzzy Sets

The extension of traditional set theory that permits the inclusion of fuzziness in properties, such as the greenness of an apple (the black broken line in the right panel of figure 11.5), also enables the fuzzification of Sorites paradox (figure 11.1).

To comprehend the concept of fuzzy terms and their useful application, let's focus on the terms *young*, *middle-aged*, and *old* as an illustrative example of fuzziness. These three

terms aren't precisely defined in terms of specific age intervals (years), and the fuzziness intensifies when we add adjectives like *almost*, *slightly*, *moderately*, *very*, etc. (e.g., *very young* or *almost old*). In daily conversations, it's even possible to be *a little bit young* and *a little bit old* simultaneously. Rather than causing confusion, such constructions appear to offer valuable information and deeper insight.

Let's return to the problem presented in figure 11.1 and examine whether we can address the issue using fuzzy sets. Identifying fuzzy sets is a process termed *Fuzzification*. We introduce the fuzzy set of heaps, which allows for partial memberships within the set. We need to agree on which cases in figure 11.1 are not heaps and when something definitely qualifies as a heap. We propose that seven grains constitute a heap, while one grain does not. However, what about the cases between two, three, four, five, and six grains? We assign them partial membership, as depicted in figure 11.6. Thus, two grains possess a membership degree of around 17% in the heap set. Note that the corresponding relation between partial memberships in the set of green apples is indicated by the black line on the right side of figure 11.5.
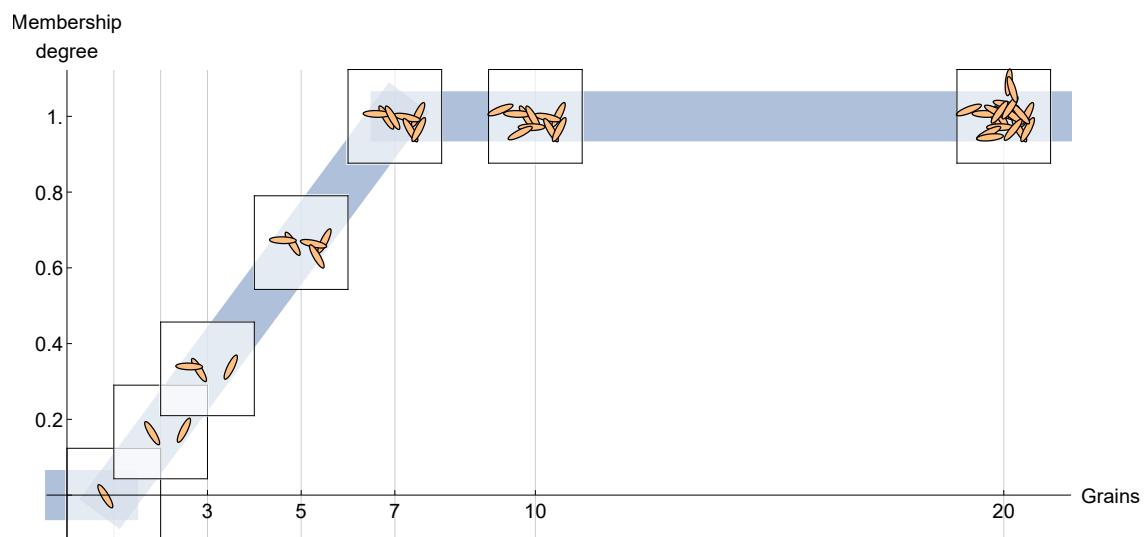

Figure 11.6: According to this classification, one grain is not categorised as a heap, while two grains possess a relatively low membership degree in the set of heaps. The issue of the heap is initially introduced in figure 11.1 at the beginning of this chapter. The dashed blue line represents *the membership function* of grains within the set of heaps.

Code Box 11.5.1 provides an additional simple example of fuzzy sets, this time encompassing three distinct categories: the sets of *young*, *middle-aged*, and *old* people. These terms – *young*, *middle-aged*, and *old* – are all imprecise. We use them not because we lack more precise terms (we could use age in years or groupings from one definite age value to another) but because the imprecision of these terms makes them more useful. You don't need to know someone's exact age to label them as young or old. This is determined based on experience, appearance, gait, and past observations, even without knowledge of their age. These factors are what we refer to as *heuristics*.

Code box 11.5.1 — An example of fuzzy sets.

Assume that we categorise the individual ages in a population in terms of *young*, *middle aged* and *old*. We assume that persons younger than 15 years are 100% young, e.g. they are 100% members in the set of young people. Correspondingly we assume them to have no membership in this set when they reach an age of 35 years. The given ages are of course objects of discussion. If you disagree you only shift the ages with the preferred ones. Therefore we make piece-wise linear functions in which these values may be altered. When assuming linear functions although there are many other possibilities. We express the function `young` by joining three elements:

```
In[1]:=  young[age_, min_: 15, max_: 35] :=
            Piecewise[{
             {1, age <= min},
             {-(age - max)/(max - min), min < age < max},
             {0, age >= max}
            }]
```

When it comes to middle aged people we have to bring in more ages. We assume zero membership up to the age of 20 and after the age of 50. 100% membership degree in the set of middle aged people we place between 25 and 35 years. Again, these values are to be discussed. Different age groups may for example have different perceptions of where to place the critical border values. The piece-wise linear function describing membership degrees in the fuzzy set of middle aged then have five elements:

```
In[2]:=  middle[age_, min_: 20, mid1_: 25, mid2_: 35, max_: 50] :=
            Piecewise[{
             {0, age <= min},
             {1 + (age - mid1)/(mid1 - min), min < age < mid1},
             {1, mid1 < age < mid2},
             {-(age - max)/(max - mid2), mid2 < age < max},
             {0, age >= max}
            }]
```

We assume no memberships in the fuzzy set of old people for persons younger than 25 years and 100% membership when older than 60 years and construct the piece-wise linear function `old` corresponding to the two above:

```
In[3]:=  old[age_, min_: 25, max_: 60] :=
            Piecewise[{
             {0, age <= min},
             {1 + (age - max)/(max - min), min < age < max},
             {1, age >= max}
            }]
```

A plot displays the three fuzzy sets. We see that the green line (function `old`) follows a similar path as the black line in the right panel of figure 11.5. While that figure only includes one fuzzy set (the fuzzy set of green apples), our figure includes three fuzzy sets.

```
In[4]:=  Plot[{young[age], middle[age], old[age]}, {age, 0, 70},
            PlotStyle        -> AbsoluteThickness[4],
            PlotTheme        -> "Detailed",
```

```
              PlotRangePadding -> {None, Automatic},
              FrameLabel        -> {"Age (years)", "Membership degree"},
              BaseStyle         -> 12
          ]
```

Out[4]=
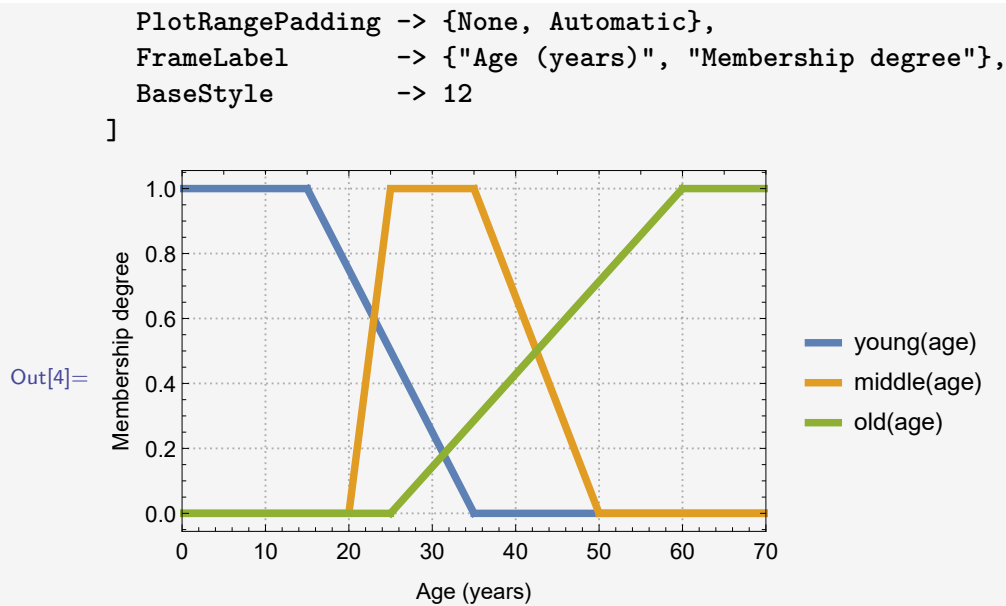
The intersections between the young and middle aged and between the middle aged and old are found by

```
In[5]:= youngmiddle =
        age /. FindMinimum[(young[age] - middle[age])^2, {age, 0},
        Method -> "PrincipalAxis"][[2]]
```

Out[5]= 23.

```
In[6]:= middleold =
        age /. FindMinimum[(middle[age] - old[age])^2, {age, 70},
        Method -> "PrincipalAxis"][[2]]
```

Out[6]= 42.5

This means that when employing the principle of highest membership degree, people younger than 23 years are *young*, people between 23 and 42.5 years are *middle aged* and people older than 42.5 years are *old*, according to our defuzzification procedure.

The statement *he is old* may be interpreted by one person as indicating a probability of more than 50% that he is over 50 years old, while another (potentially older) person might interpret the statement as implying a 50% probability that the man is over 60 years old. However, this statement pertains to a realm of perception and communication that is not adequately represented by probabilistic reasoning. There is essentially no feasible way to apply probabilistic theory and quantify the probabilities associated with statements like *most young men are healthy*[61]. Nevertheless, statements of this nature (referred to as perception-based information by Zadeh) constitute a substantial portion of meaningful and necessary conversations.

We can introduce as many fuzzy sets as we wish; as long as we are able to distinguish between different age categories, adding sets is possible. This also helps to highlight the distinction between probability and degrees of set membership. When summing up all membership degrees for each age, any number within the range of 0 to the number of sets in the sample can result. This concept is crucial for understanding what fuzzy sets convey and how we can proceed with defuzzification after fuzzification. In code box 11.5.1, we

have fuzzified the age patterns based on age and established membership degrees in fuzzy sets according to the measured ages. Defuzzification can be achieved in various ways, such as the principle of *Highest Membership Degree*. In this approach, individuals (age groups) are labelled *young* below the intersection of *young* and *middle aged* in the figure within code box 11.5.1. Between this intersection and the point of intersection between *middle aged* and *old*, the label becomes *middle aged*, while to the right of the last intersection, the label is *old*.

11.6   Fuzzy categories in fisheries

Undeveloped, *Developing*, *Fully exploited*, *Overfished*, and *Collapsed* are terms commonly employed to characterise the state of a fishery. Froese and Kesner-Reyes[25] have attempted to define these fuzzy terms using distinct values, and Daan et al.[14] provide an insightful analysis of the simple algorithm proposed by Froese and Kesner-Reyes. They suggest that catches below 10% of the maximum historical catch in a fishery are labelled *Undeveloped* if they occur before the maximum catch, and as *Collapsed* if they occur after. Similarly, catches ranging from 10% to 50% of the maximum catch are labelled *Developing* if they occur before the maximum catch, and *Overfished* if they occur after. For all other cases (catches exceeding 50% of the maximum catch), the fisheries are termed *Fully exploited*.

Code box 11.6.1 — Froese/Kesner-Reyes method on uniformly distributed catch data.
In order to make it possible to replicate the exact random sequences employed here, we initiate a seed random process:

```
In[1]:= SeedRandom[1234];
```

Let us assume 500 independent fisheries and look at a sequence of 100 periods (years):

```
In[2]:= species = 500;
        years   = 100;
```

We now define uniformly distributed random catch data between 0 and 1 each year for all species:

```
In[4]:= random = RandomReal[1, species, years];
```

We express each catch in each fishery in terms of percentage of maximum catch (`eval1`) and find the year of this maximum value (`eval2`):

```
In[5]:= eval1 = #/Max[#] & /@ random;
        eval2 = Flatten[Position[#, 1.] & /@ eval1];
```

We define an empty list to store the results (`res`) and finally we employ the algoritm of Froese and Kesner-Reys to produce the plot over time:

```
In[7]:= res = {};

In[8]:= Do[
          AppendTo[res,
           If[i < eval2[[#]],
             If[eval1[[#, i]] <= .1, 1,
               If[eval1[[#, i]] < .5, 2, 3]
```

```
          ],
          If[eval1[[#, i]] <= .1, 5,
           If[eval1[[#, i]] < .5, 4, 3]
          ]
        ]
       ],
      {i, years}
     ] & /@ Range[species];

  ArrayPlot[
    Transpose[
      Sort[#, #1 > #2 &] & /@ Transpose[Partition[res, years]]
    ],
    ColorRules       -> Table[i -> ColorData[97][i], {i, 5}],
    PlotRange        -> 5,
    AspectRatio      -> 1/GoldenRatio,
    BaseStyle        -> 12,
    PlotRangePadding -> None,
    FrameLabel       -> {"Share of stocks", "Year"},
    PlotLegends      -> Placed[
      {"Undeveloped", "Developing", "Fully exploited",
       "Overfished", "Collapsed"}, Right
    ],
    FrameTicks       -> {
      {{{species, "0%"}, {3 species/4, "25%"}, {species/2, "50%"},
       {species/4, "75%"}, {1, "100%"}}, None},
      {Automatic, None}
    }
  ]
```


Daan et al. demonstrate that this consistently leads to patterns as depicted in code box 11.6.1 when catches are uniformly and randomly distributed between two fixed values. The diagram presented in the code box is a result of 500 random fisheries observed over a span of 100 years, where the interpretation of the fisheries in terms of exploitation categories does not align with the uniform distribution of catches. It can be demonstrated that the pattern illustrated in the figure within code box 11.6.1 remains unaffected by the number of fisheries and the duration of years included; only the granularity of the pattern

changes. In figure 11.7, we observe that while the granularity changes, the pattern remains consistent even when we reduce the number of years from 100 to 25.
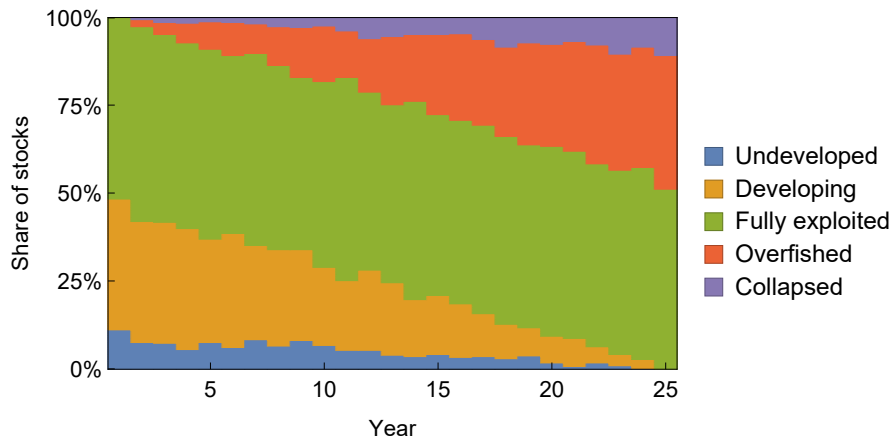
Figure 11.7: The figure corresponds to the one displayed in code box 11.6.1, only here the period is 25 years, not 100 as in the code box.

Is it apparent from the algorithm by Froese and Kesner-Reyes that approximately 50% of uniformly distributed catch data must fall within the category *Fully exploited*, while *Undeveloped* and *Developing* both converge to zero by the end of the period, and *Overfished* and *Collapsed* begin at zero, as illustrated in figure 11.7? Nevertheless, the algorithm has exerted a certain influence and continues to be cited and utilised in many publications, even after its explanation by Daan et al. in 2011[14], such as by Kleisner et al. in 2013[36].

The issue partly stems from the simplicity of the algorithm, but also from the fact that the vague categories are directly translated into precise numbers rather than being fuzzified through the incorporation of partial membership degrees. Such utilisation of vague terms can lead to potential problems, as the categories may not fully capture the nuanced reality.

11.7 Fuzzy logic

Once we have comprehended and defined the concept of fuzzy sets, we can begin to employ them as tools for reasoning. While Boolean logic is familiar, let's explore how this logic can be extended or adapted into fuzzy logic. The fundamental components of Boolean logic are depicted in figure 11.4, where the combination of two sets is denoted as $AND(A, B)$, equivalent to $Min(A, B)$ in fuzzy logic. In Boolean logic, the intersection of two sets is represented by $OR(A, B)$, which corresponds to $Max(A, B)$ in fuzzy logic. These operations, Max and Min in this context, are commonly referred to as *the Zadeh operators*.

> **Additional remarks 11.1 — Fuzzy logic represents a way to structure data, information and knowledge in a non-algorithmic manner.**
> "Fuzzy logic provides a natural framework for the management of uncertainty in expert systems because –in contrast to traditional logical systems –its main purpose is to provide a systematic basis for representing and inferring from imprecise rather than precise knowledge. In effect, in fuzzy logic everything is allowed to be – but need not be – a matter of degree."
> *Zadeh*[60]

Comparable statements can be easily identified within the realm of fisheries management, for instance, *a fish stock with a substantial spawning biomass and a low fishing mortality rate is indicative of its health.* Modern *Harvest Control Rules* (HCRs) often inadvertently incorporate a basic form of *Fuzzy Logic Control. Expert Systems* have been employed to identify relevant factors (indicators) for categorising the stock or ecosystem as healthy or unhealthy. However, these indicators are not entirely perception-based. The chosen indicators (such as spawning biomass and fishing mortality rate) primarily consist of model outputs rather than direct observations.

11.8 Harvest Control Rules (HCR)

During the process of harvesting a fish stock, a portion of the stock's biomass is extracted, leading to a reduction in the overall stock biomass in the ocean. Natural growth mechanisms attempt to counterbalance this loss, as discussed in preceding chapters. Nevertheless, the fish stock is an integral component of a larger ecosystem that may also experience repercussions, which we may only vaguely comprehend.

A few years subsequent to the 1992 UN Rio Declaration, FAO introduced new guidelines encompassing the precautionary approach to fisheries and species introduction. Currently, a substantial number of fishing nations and fishery management organisations have adopted these guidelines[42]. The guidelines advocate the integration of Harvest Control Rules (HCRs) into fisheries management plans as a means of enacting precautionary principles. The progression that emerged following the Convention on Biological Diversity (CBD) culminated in 2000 with the endorsement of the Ecosystem Approach (EA) as an overarching strategy for natural resource management. A comprehensive explanation of how these novel principles are intended to be implemented within the framework of ICES (International Council for the Exploration of the Sea) is provided in Anon. (2006). The ensuing discourse on potential further developments of HCRs presupposes this implementation as its point of departure.

Exercises

Exercise 11.1 Nothing yet ∎

IV

References, Code and Index

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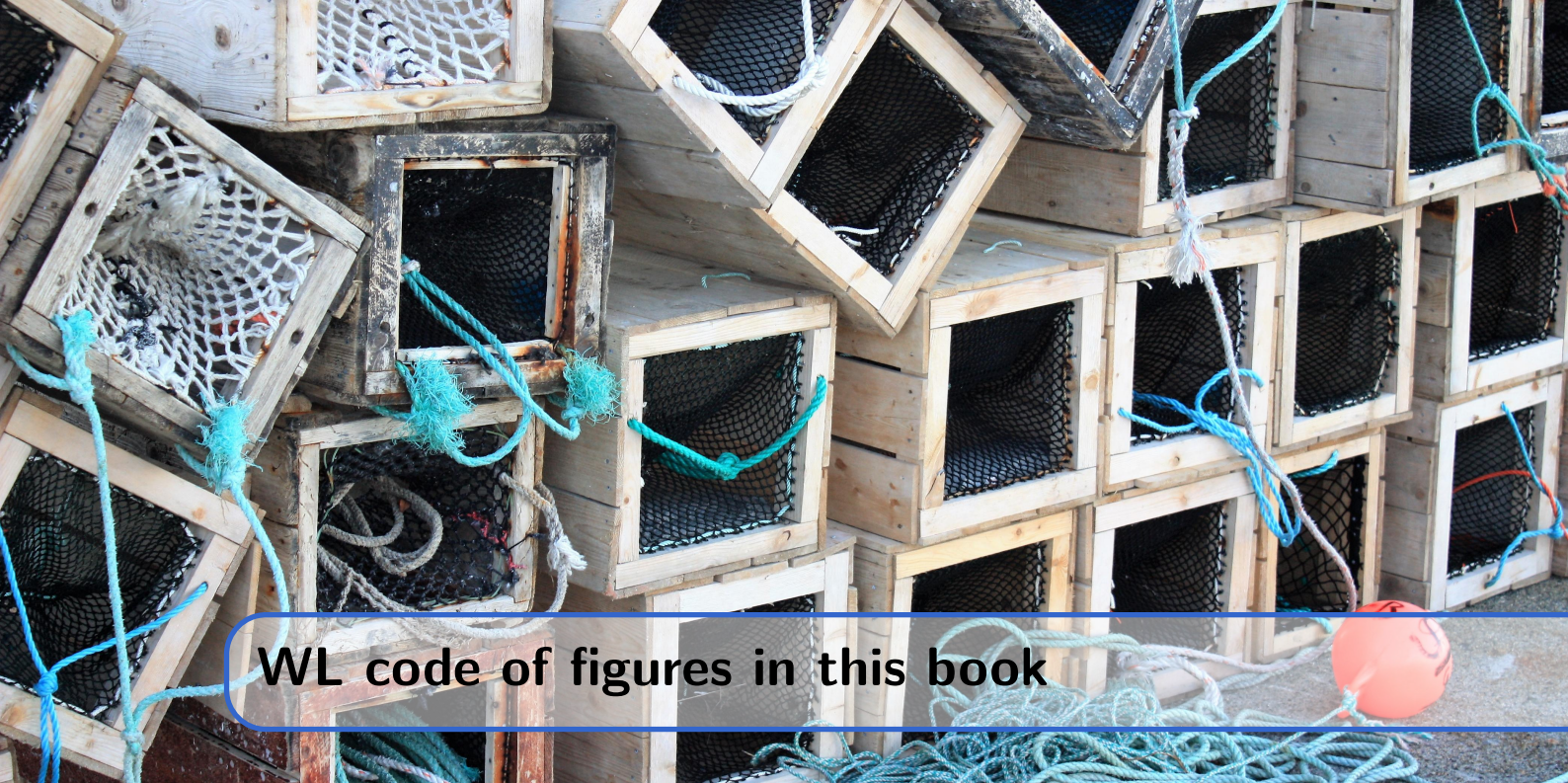
WL code of figures in this book

Figure 1.1 – Maps as models (page 11)

```
In[1]:= GraphicsRow[{
          GeoGraphics[{
            GeoStyling["OutlineMap"],
            Polygon[Entity["Country", "World"]]
            },
            GeoProjection   -> {"Orthographic", "Centering" -> GeoPosition[{0, 60}]},
            GeoRange         -> {All, All},
            GeoRangePadding -> Full
          ],
          GeoGraphics[GeoRange -> {{69.675, 69.697}, {18.93, 19.}}]
        }]
```

Figure 1.2 – Flow chart of a modelling process (page 11)

```
In[1]:= Graphics[{
          EdgeForm[Thick],
          AbsoluteThickness[3],
          Arrowheads[Large],
          Arrow[{{1.75, 7}, {1.75, 9}}],
          Arrow[{{3.25, 7}, {3.25, 7.5}}],
          Arrow[{{3.25, 9}, {3.25, 8.5}}],
          Arrow[{{5.5, 8}, {6.5, 8}}],
          Arrow[{{4, 9.25}, {6.5, 9.25}}],
          Arrow[{{4, 6.75}, {6.5, 6.75}}],
          Arrow[{{7.5, 8}, {8.5, 8}}],
          Orange,
          Arrow[{{7, 6.5}, {7, 5.5}}],
          Arrow[{{10, 7.5}, {10, 5}, {8.5, 5}}],
          Arrow[{{4.75, 7.5}, {4.75, 5.25}, {5.5, 5.25}}],
```

```
        Arrow[{{3.25, 6}, {3.25, 4.75}, {5.5, 4.75}}],
        Darker@Green,
        Arrow[{{7, 4.5}, {7, 4}, {1.75, 4}, {1.75, 6}}],
        Opacity[.6],
        Blue,
        Rectangle[{1, 9}, {4, 10}],
        Rectangle[{1, 6}, {4, 7}],
        Red,
        Rectangle[{2.5, 7.5}, {5.5, 8.5}],
        Rectangle[{6.5, 6.5}, {7.5, 9.5}],
        Orange,
        Rectangle[{8.5, 7.5}, {11.5, 8.5}],
        Darker@Green,
        Rectangle[{5.5, 4.5}, {8.5, 5.5}],
        White,
        Opacity[1],
        Style[Text["Problem", {2.5, 9.5}], 20, Bold],
        Style[Text["Knowledge", {2.5, 6.5}], 20, Bold],
        Style[Text["Assumptions", {4, 8}], 20, Bold],
        Style[Rotate[Text["Model", {7, 8}], Pi/2], 20, Bold],
        Style[Text["Result", {10, 8}], 20, Bold],
        Style[Text["Analysis", {7, 5}], 20, Bold]
        },
        ImageSize -> 600
    ]
```

Figure 2.1 – Fishing gears (page 16)

```
In[1]:= Grid[{{
        Column[{
          Style["\n\t\tGill net", 12, FontFamily -> "Arial", Bold],
          Show[Import[
            "https://upload.wikimedia.org/wikipedia/commons/3/3b/
              Heringsfleet.gif"], ImageSize -> 270
          ]
        }],
        Column[{
          Style["\n\tPurse seine", 12, FontFamily -> "Arial", Bold],
          Import["https://upload.wikimedia.org/wikipedia/commons/6/65/
            Purse_Sein_-_Ringwade.jpg"]
        }]
      }, {
        Column[{
          Style["\n\t\tTrawl", 12, FontFamily -> "Arial", Bold],
          Show[Import[
            "https://upload.wikimedia.org/wikipedia/commons/e/e5/
              Pelagisches_Schleppnetz.jpg"], ImageSize -> 270
          ]
        }],
        SpanFromAbove
      }},
      Alignment -> Top,
      Frame     -> True
      ]
```

Figure 2.2 – Production theory (page 18)

```
In[1]:= Show[{
          Plot[.63 - x, {x, 0, .63},
            PlotStyle -> {Thickness[.01], Green}, Axes -> None
          ],
          Graphics[{
            Thickness[.01], Red,
            Line[{{0.315, .315}, {.315, .8}}],
            Line[{{0.315, .315}, {.8, .315}}],
            Black, Thin, Arrow[{{0, 0}, {0, .8}}],
            Arrow[{{0, 0}, {.8, 0}}],
            Text["Labour (L)", {.65, -.04}],
            Text[Rotate["Capital (K)", Pi/2], {-.04, .65}]
          }],
          ContourPlot[x1 * x2, {x1, 0, .8}, {x2, 0, .8},
            Contours        -> {.1},
            ContourShading -> None,
            ContourStyle   -> {Thickness[.01], Lighter@Lighter@Blue}
          ]},
          Frame        -> False,
          PlotRange    -> {{-.05, All}, {-.05, All}},
          AspectRatio -> 1
        ]
```

Figure 3.1 – Gear properties (page 32)

```
In[1]:= line[start_, angle_, length_] :=
          Line[{start, {length*Sin[angle], length*Cos[angle]} + start}];

In[2]:= SeedRandom[123];
        GraphicsRow[{
          Show[{
            Graphics[{AbsolutePointSize[7], GrayLevel[.2],
              Point /@ Table[{i, j}, {i, 1.5, 10, 2}, {j, 1.5, 10, 2}],
              Red, AbsoluteThickness[5], Opacity[.6], line[{2.1, 4.2}, 2, 5],
              Blue, line[{4.1, 4.9}, .2, 5],
              Darker@Green, line[{5.9, 3.2}, 1, 5]
              }]
            },
            PlotRange  -> {{.5, 10.5}, {.5, 10.5}},
            Frame      -> True,
            FrameTicks -> None],
          Show[{
            Graphics[{AbsolutePointSize[7], GrayLevel[.2],
              Point /@ Table[{i, j}, {i, 10}, {j, 10}],
              Red, AbsoluteThickness[5], Opacity[.6], line[{2.1, 4.2}, 2, 5],
              Blue, line[{4.1, 4.9}, .2, 5],
              Darker@Green, line[{5.9, 3.2}, 1, 5]
              }]
            },
            PlotRange  -> {{.5, 10.5}, {.5, 10.5}},
            Frame      -> True,
            FrameTicks -> None],
          Show[{
```

```
        Graphics[{AbsolutePointSize[7], GrayLevel[.2],
          Point /@ Table[RandomReal[{1, RandomReal[{4, 10}]}, 2], {100}],
          Red, AbsoluteThickness[5], Opacity[.6], line[{2.1, 4.2}, 2, 5],
          Blue, line[{4.1, 4.9}, .2, 5],
          Darker@Green, line[{5.9, 3.2}, 1, 5]
          }]
      },
      PlotRange  -> {{.5, 10.5}, {.5, 10.5}},
      Frame      -> True,
      FrameTicks -> None]
    }]
```

Figure 3.2 – Gear selection (page 35)

```
In[1]:= Needs["EconMult`PopulationGrowth`"]

In[2]:= Legended[
      Show[{
        Plot[
          CatchSelectionFunction[t,
            CatchSelection -> Logistic,
            CatchAge       -> 5,
            FirstCatchAge  -> 3
          ],
          {t, 0, 10},
          PlotStyle   -> Directive[ColorData[97, 2], AbsoluteThickness[3]],
          PlotTheme   -> "Detailed",
          PlotLegends -> None,
          GridLines   -> {Automatic, Range[0, 1, .25]}
        ],
        Plot[CatchSelectionFunction[t, CatchAge -> 5], {t, 0, 10},
          PlotStyle -> AbsoluteThickness[3]
        ],
        Graphics[{
          ColorData[97, 2], AbsolutePointSize[8],
          Point /@ {{4, .25}, {5, .5}, {6, .75}}
        }],
        Plot[
          Max[0,
            CohortBiomass[t - 2,
              InitialAge           -> 0,
              WeightLengthRelation -> 3,
              MaxWeight            -> 9.5,
              GrowthRate           -> .45,
              MortalityRate        -> .45,
              Recruits             -> 1,
              RecruitmentAge       -> 0,
              OldestAge            -> ∞,
              Fishing              -> False
            ]
          ],
          {t, 0, 10},
          PlotStyle -> Directive[ColorData[97, 3], AbsoluteThickness[3]]
        ] },
        PlotLabel -> "Selection curves",
```

```
                        FrameLabel -> {"Fish length", "Probability of retaining fish"},
                        FrameTicks -> {{Range[0, 1, .25], None}, {{
                            {4, Style["L_{25}", ColorData[97, 2], Bold]},
                            {5, Style["L_{50}", ColorData[97, 2], Bold]},
                            {6, Style["L_{75}", ColorData[97, 2], Bold]}
                          }, None}},
                        BaseStyle -> 12
                    ],
                    LineLegend[
                      Directive[
                        ColorData[97, #],
                        AbsoluteThickness[3]
                      ] & /@ Range[3],
                      {"Theoretical knife-edge selection",
                      "Trawl / Danish seine / Purse seine",
                      "Gill net / Longline"}
                    ]
                ]
```

Figure 4.1 – Cellular automata (page 57)

```
In[1]:= grid[r_, type_: "neumann"] := Module[{tt = type},
          Show[{
            ArrayPlot[Table[0, {7}, {7}], Mesh -> True, PlotRange -> 1],
            Graphics[{LightGray, Rectangle[{3, 3}, {4, 4}]}],
              Switch[tt, "neumann",
                Graphics[{Thick, Red, Line[{{3 - r, 3 + 1}, {3 + r + 1, 3 + 1}}],
                  Line[{{3 - r, 3}, {3 + r + 1, 3}}],
                  Line[{{3 - r, 3}, {3 - r, 3 + 1}}],
                  Line[{{3 + r + 1, 3}, {3 + r + 1, 3 + 1}}],
                  Line[{{3, 3 - r}, {3, 3 + r + 1}}],
                  Line[{{3 + 1, 3 - r}, {3 + 1, 3 + r + 1}}],
                  Line[{{3, 3 - r}, {3 + 1, 3 - r}}],
                  Line[{{3, 3 + r + 1}, {3 + 1, 3 + r + 1}}],
                  EdgeForm[{Thick, Red}], Opacity[0],
                  Rectangle[{3, 3} - r + 1, {3, 3} + r]
                }],
                "moore",
                Graphics[{Thick, Red,
                Table[
                  Line[{{3 - r, 3 - r + i - 1}, {3 + r + 1, 3 - r + i - 1}}],
                  {i, 2 r + 2}],
                Table[
                  Line[{{3 - r + i - 1, 3 - r}, {3 - r + i - 1, 3 + r + 1}}],
                  {i, 2 r + 2}]
                }]
              ]
          }]];
```

```
In[2]:= row[r_] := Show[{
        ArrayPlot[{{0, 0, 0, .15, 0, 0, 0}},
          PlotRange -> {0, 1}, Mesh -> True
        ],
        Graphics[{
          Transparent, EdgeForm[Directive[Thickness[.015], Red]],
          Rectangle[{4 - #, 0}, {3 + #, 1}] & /@ Range[r + 1]
        }]
        }, ImageSize -> 130
        ];
```

```
In[3]:= Grid[{
        {"", "von Neumann\nrange=1", "von Neumann\nrange=2",
        "Moore\nrange=1", "Moore\nrange=2"},
        {"1D", row[1], row[2], row[1], row[2]},
        Prepend[Show[#, ImageSize -> 130] & /@ {
          grid[1, "neumann"], grid[2, "neumann"], grid[1, "moore"], grid[2, "moore"]
          }, "2D"]
        },
        Alignment -> Table[Center, {4}, {2}],
        Spacings  -> {2, 1},
        BaseStyle -> 16
        ]
```

Figure 4.2 − **Prey-predator** (page 67)

```
In[1]:= dsolve1[{x0_, y0_}] :=
        NDSolve[{
          x'[t] == .5 x[t] (1 - x[t]) - x[t] y[t],
          y'[t] == .5 y[t] (1 - y[t]/ x[t]),
          x[0] == x0, y[0] == y0},
          {x[t], y[t]}, {t, 0, 50}
        ];
```

```
In[2]:= Show[{
        Plot[{y /. NSolve[.5 x (1 - x) - x y == 0, y],
          y /. NSolve[.5 y (1 - y/(x)) == 0, y][[2]]},
          {x, 0, 1},
          PlotStyle -> {Directive[Red, Thick], Directive[Red, Dashing[.03], Thick]}
        ],
        ParametricPlot[
          Evaluate[{x[t], y[t]} /. dsolve1[#]], {t, 0, 25}
        ] & /@ {{.9, .5}, {.02, .9}, {.9, .9}, {.2, .9}, {.5, .02}},
        VectorPlot[{.5 x (1 - x) - x y, .5 y (1 - y/ x)}, {x, .00001, 1}, {y, .00001, 1},
          VectorPoints -> 12,
          VectorScale  -> {.1, .4, (#1 + #2) &}
        ]},
        AxesOrigin  -> {0, 0},
        PlotRange   -> {{0, 1}, {0, 1}},
        AspectRatio -> 1,
        AxesLabel   -> {"Prey stock biomass (x)", "Predator stock biomass (y)"},
        ImageSize   -> 450
        ]
```

Figure 4.3 – Competing species (page 68)

```
In[1]:= dsolve2[{x0_, y0_}] :=
          NDSolve[{
            x'[t] == .5 x[t] (1 - x[t]) - x[t] y[t],
            y'[t] == .5 y[t] (1 - y[t]) - x[t] y[t],
            x[0] == x0,
            y[0] == y0},
            {x[t], y[t]}, {t, 0, 50}
          ];

In[2]:= Show[{
          Plot[{y /. NSolve[.5 x (1 - x) - x y == 0, y],
            y /. NSolve[.5 y (1 - y) - x y == 0, y][[2]]},
            {x, 0, 1},
            PlotStyle -> {Directive[Red, Thick], Directive[Red, Dashing[.03], Thick]}
          ],
          ParametricPlot[
            Evaluate[{x[t], y[t]} /. dsolve2[#]], {t, 0, 25}] & /@
              {{.9, .8}, {.1, .06}, {.8, .9}, {.06, .1}, {.5, 1}, {1, .5}},
          VectorPlot[
            {.5 x (1 - x) - x y, .5 y (1 - y) - x y},
            {x, .00001, 1}, {y, .00001, 1},
            VectorPoints -> 12,
            VectorScale  -> {.1, .4, (#1 + #2) &}
          ]},
          AxesOrigin  -> {0, 0},
          PlotRange   -> {{0, 1}, {0, 1}},
          AspectRatio -> 1,
          AxesLabel   -> {"Stock biomass (x)", "Stock biomass (y)"},
          ImageSize   -> 450
        ]
```

Figure 4.4 – Symbiosis (page 69)

```
In[1]:= dsolve3[{x0_, y0_}] :=
          NDSolve[{
            x'[t] == .5 x[t] (1 - x[t]/(.1 + .5 y[t])),
            y'[t] == .5 y[t] (1 - y[t]/(.1 + .5 x[t])),
            x[0] == x0,
            y[0] == y0},
            {x[t], y[t]}, {t, 0, 50}
          ];
```

```
In[2]:= Show[{
         Plot[{y /. NSolve[.5 x (1 - x/(.1 + .5 y)) == 0, y],
           y /. NSolve[.5 y (1 - y/(.1 + .5 x)) == 0, y][[2]]},
           {x, 0, 1},
           PlotStyle -> {Directive[Red, Thick], Directive[Red, Dashing[.03], Thick]}
         ],
         ParametricPlot[
           Evaluate[{x[t], y[t]} /. dsolve3[#]], {t, 0, 25}] & /@
             {{.1, 1}, {1, .1}, {.05, .05}, {.2, 1}, {1, .2},
             {1, .5}, {.5, 1}, {1, .8}, {.8, 1}},
         VectorPlot[
           {.5 x (1 - x/(.1 + .5 y)), .5 y (1 - y/(.1 + .5 x))},
           {x, .00001, 1}, {y, .00001, 1},
           VectorPoints -> 12,
           VectorScale  -> {.1, .4, (#1 + #2) &}
         ]},
         AxesOrigin  -> {0, 0},
         PlotRange   -> {{0, 1}, {0, 1}},
         AspectRatio -> 1,
         AxesLabel   -> {"Stock biomass (x)", "Stock biomass (y)"},
         ImageSize   -> 450
       ]
```

Figure 5.1 – Isoclines (page 76)

```
In[1]:= Show[{
         Plot[1 - x, {x, 0, 1},
           AspectRatio -> 1,
           Ticks -> None,
           Axes -> False,
           PlotStyle -> Directive[Thickness[.01], Red]
         ],
         Graphics[{
           Black, Thick, Arrow[{{0, 0}, {0, 1.1}}], Arrow[{{0, 0}, {1.1, 0}}],
           Text["E", {.5, -.08}], Text["X", {-.05, .5}], Text["K", {-.03, 1}],
           Text["r/q", {1, -.1}],
           Style[Text["ẋ > 0", {.3, .3}], FontFamily -> "Times"],
           Style[Text["ẋ < 0", {.7, .7}], FontFamily -> "Times"],
           Style[Text["ẋ = 0", {.33, .9}], FontFamily -> "Times"],
           Arrowheads[.03], Thin, Arrow[{{.3, .85}, {.25, .75}}]
         }]},
         PlotRange -> All,
         BaseStyle -> 16
       ]
```

Figure 5.2 – Cobweb diagram (1) (page 77)

```
In[1]:= GraphicsGrid[
        Partition[
         Table[
          Show[{
            Plot[{x + r x (1 - x/1000.) - .3*x, x}, {x, 0, 1400}],
            ListLinePlot[(
              nested = NestList[# + r # (1 - #/1000.) - .3*# &, 100, 50];
              nested = Riffle[{#, #} & /@ nested,
                Table[Take[RotateLeft[#, i], 2], {i, 0, 49}] & @ nested]),
              PlotStyle -> Directive[{Thickness[.005], Red}],
              PlotRange -> All
            ]},
            AspectRatio      -> 1,
            PlotRangePadding -> None,
            PlotRange        -> {0, All}
          ],
         {r, {1, 2, 2.6, 3.2}}
         ], 2
        ],
        ImageSize -> 500
      ]
```

Figure 5.3 – Cobweb diagram (2) (page 78)

```
In[1]:= GraphicsGrid[
        Partition[
         Table[
          Show[{
            Plot[{x + 2.9 x (1 - x/1000.) - e*x, x}, {x, 0, 1400}],
            ListLinePlot[(
              nested = NestList[# + 2.9 # (1 - #/1000.) - e*# &, 100, 50];
              nested = Riffle[{#, #} & /@ nested,
                Table[Take[RotateLeft[#, i], 2], {i, 0, 49}] & @ nested]),
              PlotStyle -> Directive[{Thickness[.005], Red}],
              PlotRange -> All
            ]},
            AspectRatio      -> 1,
            PlotRangePadding -> None,
            PlotRange        -> {0, All}
          ],
         {e, {.1, .4, .8, 1}}
         ], 2
        ],
        ImageSize -> 500
        ]
```

Figure 5.4 – Surplus harvest (page 79)

```
In[1]:= q[a_] := If[a, .5, 1, 1]

In[2]:= fig[aa_, scale_: 1, z_: - .3, a_: False] :=
        Module[{
          test = (y /. Drop[Quiet[Solve[y*(1 - y) (y - z)^Boole[a] == q[a] aa y, y]], 1])},
          Show[{
            Plot[(y + 1.2)*(1 - (y + 1.2)) (y - z + 1.2)^Boole[a],
              {y, -1.2, -0.2},
              PlotStyle -> {Red, Thick}
            ],
            Plot[Evaluate @ (q[a] e x /.
              Drop[Quiet[Solve[x (1 - x)(x - z)^Boole[a] == q[a] e x, x]], 1] /. {z->.1}),
              {e, 0, 1},
              PlotRange -> {0, .4/scale},
              PlotStyle -> Drop[{{Blue, Thick, Dashed}, {Blue, Thick}}, 1 - Boole[a]]
            ],
            If[aa > 0,
              Plot[If[q[a] aa * (y + 1.2) < .3, q[a] aa * (y + 1.2)],
                {y, -1.2, -.1 },
                PlotStyle -> {Darker[Green], Thick}
              ],
              Graphics[{}]
            ],
            Graphics[{
              Arrowheads[Medium],
              Arrow[{{-1.2, 0}, {-0.1, 0}}],
              Arrow[{{-1.2, 0}, {-1.2, 0.3/scale}}],
              Arrow[{{0, 0}, {1.1, 0}}],
              Arrow[{{0, 0}, {0, .3/scale}}]
            }],
            Graphics[{
              PointSize[0.015], Blue,
              If[a,
                If[Count[Im[test], 0] == 2,
                  {Point[{# - 1.2, # (1 - #) (# - z)}] & /@ test,
                  Black,
                  If[MemberQ[Positive[test], False],
                    Point[{aa, Last[Sort[test]](1-Last[Sort[test]])(Last[Sort[test]]-z)}],
                    Point[{aa, # (1 - #) (# - z)}] & /@ test
                  ],
                  Thin, Dashed,
                  Line[{{# - 1.2, (1 - #) * # (# - z)}, {# - 1.2, 0}}] & /@ test,
                  Line[{{aa, (1 - test[[2]]) * test[[2]] (test[[2]] - z)}, {aa, 0}}] },
                  {}
                ],
                {Point[{-aa - 0.2, (1 - aa)*aa}],
                Black, Point[{aa, (1 - aa)*aa}],
                Thin, Dashed,
                Line[{{-aa - 0.2, (1 - aa)*aa}, {-aa - 0.2, 0}}],
                Line[{{aa, (1 - aa)*aa}, {aa, 0}}]}
              ]}
            ],
            If[aa > 0 && ! a,
              Graphics[{
```

```
              Thin, Dashed,
              Line[{{-aa - 0.2, (1 - aa)*aa}, {aa, (1 - aa)*aa}}],
              Text[Style["H(E,X)", 14, GrayLevel[.4]],
                {0.05 - aa, If[.05 - aa < -.5, .3, (1 - aa)*aa + 0.05]}]
            }],
            Graphics[{}]
          ],
          If[aa > 0 && a && Count[Im[test], 0] == 2,
            Graphics[{
              Thin, Dashed,
              Line[{{# - 1.2, (1 - #)*#*(# - z)}, {aa, (1 - #)*#*(# - z)}}] & /@ test,
              Text[Style["H(E,X)", 14, GrayLevel[.4]],
                {Min[0.1 - aa, -.05], Min[.3, aa q[a] + .01]}]
            }],
            Graphics[{}]
          ],
          Graphics[{
            Text[Style["Biomass (X)", 14, GrayLevel[.4]], {-0.7, -0.03/scale}],
            Text[Style["Fishing effort (E)", 14, GrayLevel[.4]], {.5, -0.03/scale}],
            Text[Style["f(X)\nH(E,X)", 14, GrayLevel[.4]], {-1.2, .33/scale}],
            Text[Style["H(E)", 14, GrayLevel[.4]], {0, .33/scale}]
          }]
        },
        Axes        -> False,
        PlotRange   -> {{-1.3, 1.1}, {-0.06/scale, .4/scale}},
        AspectRatio -> .6,
        ImageSize   -> 500,
        ImagePadding -> 0
      ]
    ]

In[3]:= GraphicsColumn[{
      fig[.2],
      fig[.5, 1.2, -.2, True],
      fig[.2, 2, .15, True]
      },
      Spacings -> -70
    ]
```

Figure 5.5 – Virtual Population Analysis (page 86)

```
In[1]:= Grid[{{
        Show[
          Graphics[{
            Opacity[.5], EdgeForm[Directive[Gray, Thin]],
            Table[{Hue[#/10],
              Rectangle[{i + 3.55, i - 1.45 + #}, {4.45, # - .55} +  i]},
              {i, 6}] & /@ Range[6],
            Opacity[1], Red,
            Table[Arrow[{{5, #} + i, {4, #} + i}], {i, 5}] & /@ Range[5]
          }],
          AspectRatio -> 1/GoldenRatio,
          Frame      -> True,
          FrameLabel -> {"Year", ""},
          FrameTicks  -> {{Table[{i, "Cohort " <> ToString[i - 1]}, {i, 11}], None},
                          {Automatic, None}},
          GridLines   -> {Table[i + 4.5, {i, 5}], Table[i + .5, {i, 10}]},
          ImageSize   -> 400
        ],
        SwatchLegend[
          Reverse[Directive[Hue[#/10], Opacity[.5]] & /@ Range[6]],
          Style["Age " <> ToString[#], 10] & /@ Range[0, 5],
          LegendMarkerSize -> 12,
          LegendFunction   -> "Panel"
        ]}},
        Alignment -> Top
      ]
```

Figure 6.1 – Total revenue and total cost (page 97)

```
In[1]:= f[x_] := r x (1 - x/k)

In[2]:= h[x_, e_] := q x e

In[3]:= x[e_] := x /. Solve[f[x] == h[x, e], x][[2]]

In[4]:= h[e_] := h[x[e], e]

In[5]:= Plot[
        Evaluate[{p h[e], a e} /.
          {r -> 1, k -> 1, q -> 1, a -> 1/5, p -> 1}],
        {e, 0, 1},
        AxesLabel   -> {"Fishing effort (E)", "Value"},
        PlotLegends -> {"TR(E)", "TC(E)"},
        Ticks       -> None
      ]
```

Figure 6.2 – Economic rent of fleet units (page 99)

```
In[1]:= cc[q_] := b q + c q^2 - c q^3 /400

In[2]:= qeq[p_, rule_] :=
        Module[{temp},
          temp = Select[q /. Solve[p == cc'[q] /. rule, q], Positive];
          temp = If[Length[temp] > 0, Last[temp], -.00001];
          If[temp != -.00001,
            temp = If[(cc'[temp] /. rule) >= (cc[temp]/temp /. rule), temp, 0.000001]
          ];
          temp
        ]

In[3]:= makerow[p_, names_, vars_] :=
        Module[{count = 1},
          GraphicsGrid[{
            Show[
              Graphics[{
                LightGreen, Rectangle[{0, p}, {qeq[p, #], 0}],
                LightRed, Rectangle[{0, 0}, {qeq[p, #], cc[qeq[p, #]]/qeq[p, #]} /. #],
                Thin, Dashed,
                Evaluate[Line[{{qeq[p, #], p}, {qeq[p, #], 0}} /. #]]}
              ],
              Plot[{cc'[q] /. #, cc[q]/q /. #, p}, {q, 0.001, 600},
                PlotRange -> {0, 800},
                PlotStyle -> {{Thick, Red}, {Thick, Magenta}, {Thick, Blue}}
              ],
              Axes        -> True,
              AxesOrigin  -> {0, 0},
              PlotRange   -> {{0, 400}, {0, 450}},
              AspectRatio -> .8,
              ImageSize   -> 180,
              PlotLabel   -> names[[count++]]] & /@ vars}
          ]
        ]

In[4]:= vg[v_] := Style[ToString[v], Bold, Darker@Green, 20]

In[5]:= vr[v_] := Style[ToString[v], Bold, Red, 20]

In[6]:= groupA = {b -> 200, c -> -1};
        groupB = {b -> 350, c -> -2};
        groupC = {b -> 650, c -> -3.5};

In[7]:= GraphicsColumn[{
          makerow[400, {vg[A], vg[B], vg[C]}, {groupA, groupB, groupC}],
          makerow[300, {vg[A], vg[B], vr[C], vg[A]}, {groupA, groupB, groupC, groupA}],
          makerow[150, {vg[A], vr[B], vg[A], vg[A]}, {groupA, groupB, groupA, groupA}],
          makerow[100, {vg[A], vg[A], vg[A], vg[A], vg[A]},
            {groupA, groupA, groupA, groupA, groupA}]
          },
          Alignment        -> Left,
          PlotRangePadding -> None,
          Spacings         -> {0, -10}
        ]
```

Figure 6.3 – Open access fishery (page 106)

```
In[1]:= dsolve[{e0_, x0_}] :=
          NDSolve[{
            x'[t] == 1 x[t] (1 - x[t]) - 1.5  x[t] e[t],
            e'[t] == .5  (.5 - .2/(2 x[t])),
            x[0] == x0,
            e[0] == e0},
            {x[t], e[t]}, {t, 0, 50}
          ];

In[2]:= Show[{
          Plot[{x /. NSolve[1 x (1 - x) - 1.5  x e == 0, x][[2]],
            x /. NSolve[.5  (.5 - .2/(2 x)) == 0, x]},
            {e, 0, 1},
            PlotStyle -> {Directive[Red, Thick], Directive[Red, Dashing[.03], Thick]}
          ],
          ParametricPlot[
            Evaluate[{e[t], x[t]} /. dsolve[#]], {t, 0, 50}
          ] & /@ {{.2, .1}, {.3, 1}, {.5, 1}, {.7, 1}, {.01, .8}},
          VectorPlot[{.5 (.5 - .2/(2 x)), 1 x (1 - x) - 1.5  x e},
            {e, .00001, 1}, {x, .00001, 1},
            VectorPoints -> 12,
            VectorScale  -> {.1, .4, (#1 + #2) &}
          ]},
          AxesOrigin  -> {0, 0},
          PlotRange   -> {{0, 1}, {0, 1}},
          AspectRatio -> 1,
          AxesLabel   -> {"Fishing effort (E)", "Stock biomass (x)"},
          ImageSize   -> 450
        ]
```

Figure 6.4 – The Gordon-Schafer model (page 107)

```
In[1]:= GraphicsColumn[{
          Show[{
            Plot[{ x (1 - x), , .3 x}, {x, 0, 1}],
            Graphics[{
              Text["TR", {.9, .1}],
              Text["TC", {.9, .26}],
              Arrow[{{0, 0}, {1.09, 0}}],
              Arrow[{{0, 0}, {0, .31}}]
            }]},
            PlotRange       -> {{0, 1.09}, {-.005, .31}},
            GridLines       -> {{.35, .7}, None},
            PlotRangePadding -> .02,
            Frame           -> {{True, False}, {True, False}},
            FrameStyle      -> White,
            FrameTicks      -> None,
            FrameTicksStyle -> Directive[{Thickness[0], GrayLevel[.4]}],
            FrameLabel      -> (Style[#, GrayLevel[.4]] & /@ {"", "TR, TC"})
          ],
          Show[{
            Plot[{(1 - x), Evaluate@D[x (1 - x), x], .3}, {x, 0, 1}],
            Graphics[{
```

```
                    Text["AR", {.5, .52}],
                    Text["MR", {.6, -.2}],
                    Text["MC", {.9, .33}],
                    Text["E_MEY", {.35, -.1}],
                    Text["E_MEY", {.35, 1.3}],
                    Text["E_∞", {.7, -.1}],
                    Text["E_∞", {.7, 1.3}],
                    Arrow[{{0, 0}, {1.09, 0}}],
                    Arrow[{{0, 0}, {0, 1.25}}],
                    Arrow[{{0, 0}, {0, -.5}}]
                }]},
            PlotRange        -> {{0, 1.09}, {-.5, 1.35}},
            GridLines        -> {{.35, .7}, None},
            PlotRangePadding -> .02,
            Frame            -> {{True, False}, {True, False}},
            FrameStyle       -> White,
            FrameTicksStyle  -> Directive[{Thickness[0], GrayLevel[.4]}],
            FrameTicks       -> None,
            AxesStyle        -> Transparent,
            FrameLabel       -> (Style[#,GrayLevel[.4]]& /@ {"Fishing effort","AR, MR, MC"})
        ]},
        Spacings -> -30
    ]
```

Figure 6.5 – **Maximising present value** (page 112)

```
In[1]:= f[x_] := r x (1 - x/k)


In[2]:= c[x_] := a/(q x)


In[3]:= values = {p -> 10, a -> 100, q -> .07, k -> 1000, r -> 1};


In[4]:= Plot[
        Evaluate[{(p - c[x]), D[(p - c[xx]) f[xx], xx]} /. values /. {xx -> x}],
        {x, 0, 1000},
        PlotRange -> {-3, 13},
        Ticks -> {{
          {Evaluate[xx /. NSolve[p == c[xx] /. values, xx][[1]]], "X_∞"},
          {Evaluate[xx /. NSolve[D[(p - c[xx]) f[xx], xx] == 0 /. values, xx][[1]]],
            "X_MEY"},
          {Evaluate[xx /. NSolve[D[(p - c[xx]) f[xx], xx] == p - c[xx] /. values, xx][[1]]],
            "X*"}}, {{p /. values, "p"}
          }},
         PlotStyle -> Thick,
         GridLines -> {
           {Evaluate[xx /.
             NSolve[D[(p - c[xx]) f[xx], xx] == p - c[xx] /. values, xx][[1]]]},
           {p /. values}},
         PlotLegends -> {"p - c(X)", "R'(X) / delta"},
         AxesStyle -> Arrowheads[{0.0, 0.03}]
        ]
```

Figure 6.6 – Optimal solution without cost (page 114)

```
In[1]:= Show[{
         Plot[x (1 - x), {x, 0, 1}, PlotStyle -> Darker@Green],
         Plot[(#^2 + (1 - 2 #) x), {x, #/5, 1},
           PlotStyle   -> Red,
           PlotLegends -> LineLegend[{Darker@Green, Red}, {"f(X)", δ}]
         ]},
         PlotRange     -> {0, .3},
         GridLines     -> {{#}, None},
         Ticks         -> None,
         AxesLabel     -> {"Stock biomass (X)"}
       ] & @ .25
```

Figure 7.1 – World catches on continents (page 119)

```
In[1]:= continentcatchdata = {
         <|"year" -> 1950, "Europe" -> 7.013624, "Americas" -> 3.292381,
          "Oceania" -> 0.06043, "Asia" -> 5.104072, "Africa" -> 1.082887|>,
         <|"year" -> 1975, "Europe" -> 21.095164, "Americas" -> 8.953045,
           "Oceania" -> 0.184765, "Asia" -> 21.958926, "Africa" -> 3.977749|>,
         <|"year" -> 1995, "Europe" -> 16.002349, "Americas" -> 25.091366,
           "Oceania" -> 0.815906, "Asia" -> 31.846388, "Africa" -> 5.619419|>,
         <|"year" -> 2015, "Europe" -> 13.184799, "Americas" -> 14.404689,
           "Oceania" -> 1.269387, "Asia" -> 40.746424, "Africa" -> 8.401275|>};

In[2]:= Row@Join[Table[
         PieChart[
           Rest@Values[Select[continentcatchdata, #["year"] === year &][[1]]],
           ImageSize    -> Sqrt[Total@Rest@Values[
             Select[continentcatchdata, #["year"] === year &][[1]]]*1000.]/Sqrt[Pi],
           PlotLabel    -> ToString[year] <> ":\n" <> ToString[
             Total@Rest@Values[Select[continentcatchdata, #["year"] === year &][[1]]]
             ] <> " mill. tons",
           ChartLabels -> Placed[ToString[#] <> "%" & /@ Round[100*Rest@Values[
             Select[continentcatchdata, #["year"] === year &][[1]]]/Total@Rest@Values[
               Select[continentcatchdata, #["year"] === year &][[1]]]], "RadialCallout"],
           PlotRange  -> All,
           BaseStyle  -> 8
         ],
         {year, First /@ Values /@ continentcatchdata}
         ],
         {SwatchLegend[
           Lighter[System`PlotThemeDump`$ThemeDefaultGradient, .2],
           Rest@Keys[continentcatchdata[[1]]],
           LegendMarkerSize -> 12,
           LabelStyle       -> {FontSize -> 12}
         ]}
       ]
```

Figure 7.2 – World catches on economies (page 120)

```
In[1]:= economiccatchdata = {
        <|"year" -> 1950, "Least Developed Countries" ->  0.71071,
          "Developed countries or areas"             -> 12.36886,
          "Other developing countries or areas"       ->  3.473824|>,
        <|"year" -> 1975, "Least Developed Countries" ->  2.734242,
          "Developed countries or areas"             -> 33.6766,
          "Other developing countries or areas"       -> 19.758807|>,
        <|"year" -> 1995, "Least Developed Countries" ->  3.940279,
          "Developed countries or areas"             -> 26.578154,
          "Other developing countries or areas"       -> 48.856995|>,
        <|"year" -> 2015, "Least Developed Countries" ->  8.400506,
          "Developed countries or areas"             -> 21.746714,
          "Other developing countries or areas"       -> 47.859354|>};
```

```
In[2]:= Grid[{Table[
        PieChart[
          Rest@Values[Select[economiccatchdata, #["year"] === year &][[1]]],
          ImageSize  -> Sqrt[Total@Rest@Values[
            Select[economiccatchdata, #["year"] === year &][[1]]]*1000.]/Sqrt[Pi],
          PlotLabel  -> ToString[year] <> ":\n" <> ToString[
            Total@Rest@Values[Select[economiccatchdata, #["year"] === year &][[1]]]
            ] <> " mill. tons",
          ChartLabels -> Placed[ToString[#] <> "%" & /@ Round[100*Rest@Values[
            Select[economiccatchdata, #["year"] === year &][[1]]]/Total@Rest@Values[
              Select[economiccatchdata, #["year"] === year &][[1]]]], "RadialCallout"],
          PlotRange  -> All,
          BaseStyle  -> 8],
        {year, First /@ Values /@ economiccatchdata}
        ],
        {SwatchLegend[
          Lighter[System`PlotThemeDump`$ThemeDefaultGradient, .2][[{1, 3, 5}]],
          Rest@Keys[economiccatchdata[[1]]],
          LegendMarkerSize -> 15,
          LabelStyle       -> {FontSize -> 11},
          LegendLayout     -> {"Row", 1}
          ],
        SpanFromLeft, SpanFromLeft, SpanFromLeft}
        }]
```

Figure 7.3 – Import of fish on continent (page 120)

```
In[1]:=  importdata = {
         <|"continent" -> "Europe", "Europe" -> 63, "North-America" -> 5,
         "South-America" -> 9, "Oceania" -> 1, "Asia" -> 14, "Africa" -> 8|>,
         <|"continent" -> "North-America", "Europe" -> 10,"North-America" -> 19,
         "South-America" -> 21, "Oceania" -> 2, "Asia" -> 48, "Africa" -> 1|>,
         <|"continent" -> "South-America", "Europe" -> 8, "North-America" -> 7,
         "South-America" -> 51, "Oceania" -> 1, "Asia" -> 31, "Africa" -> 2|>,
         <|"continent" -> "Oceania", "Europe" -> 9, "North-America" -> 4,
         "South-America" -> 4, "Oceania" -> 13, "Asia" -> 68, "Africa" -> 4|>,
         <|"continent" -> "Asia", "Europe" -> 19, "North-America" -> 12,
         "South-America" -> 11, "Oceania" -> 4, "Asia" -> 50, "Africa" -> 4|>,
         <|"continent" -> "Africa", "Europe" -> 37, "North-America" -> 1,
         "South-America" -> 7, "Oceania" -> 2, "Asia" -> 26, "Africa" -> 27|>};


In[2]:=  col = {
         RGBColor[0.982864, 0.7431472, 0.3262672],
         RGBColor[0.9965728, 0.58382944, 0.22525344],
         RGBColor[0.8704, 0.50272, 0.2],
         RGBColor[0.74058432, 0.47904, 0.40899168],
         RGBColor[0.64020864, 0.49888, 0.74416944],
         RGBColor[0.4992, 0.5552, 0.8309304]
         };


In[3]:=  explode[pc_, i_] := ReplacePart[pc, Position[pc, False][[i]] -> True]
         (* Thanks to Simon Woods *)


In[4]:=  i = 1;
         GraphicsGrid[{
           Table[
             Grid[{
               {PieChart[
                   Rest@Values[Select[importdata, #["continent"] === cont &][[1]]],
                   ChartLabels -> Placed[ToString[#] & /@ Rest@Values[
                                       Select[importdata,#["continent"]===cont &][[1]]],
                                     "RadialCenter"
                                   ],
                   PlotRange  -> All,
                   ImageSize  -> 200,
                   BaseStyle  -> 22
                 ]~explode~i},
               {Graphics[{
                   EdgeForm -> Black, col[[i++]],
                   Rectangle[{-3, -1}, {3, 1}], White,
                   Style[Text[cont, 0, 0], 22, Bold]}]]}
             },
             Alignment -> Center],
           {cont, First /@ Values /@ importdata}]
           },
           Alignment -> Bottom,
           Spacings  -> -220,
           ImageSize -> 1500
         ]
```

Figure 7.4 – Fish consumption on continents (page 121)

```
In[1]:=  col = {
          RGBColor[0.982864, 0.7431472, 0.3262672],
          RGBColor[0.9965728, 0.58382944, 0.22525344],
          RGBColor[0.8704, 0.50272, 0.2],
          RGBColor[0.74058432, 0.47904, 0.40899168],
          RGBColor[0.64020864, 0.49888, 0.74416944],
          RGBColor[0.4992, 0.5552, 0.8309304]
        };
```

```
In[2]:=  consumptiondata = <|
          "Europe"        -> {16.6, 22.5},
          "North-America" -> {7.7, 21.6},
          "South-America" -> {6.2, 9.8},
          "Oceania"       -> {1.0, 25.0},
          "Asia"          -> {105.6, 24.0},
          "Africa"        -> {11.7, 9.9}
        |>;
```

```
In[3]:=  Grid[{
          {RectangleChart[{
            Labeled[{#[[2, 1]]/#[[2, 2]], #[[2, 2]]}, #[[2, 1]], Center] & /@
              Normal[consumptiondata]},
            BarSpacing  -> None,
            AspectRatio -> .4,
            ImageSize   -> 750,
            PlotTheme   -> "Detailed",
            AxesOrigin  -> {.2, 0},
            BaseStyle   -> 18,
            FrameLabel  -> {"Population in billions of individuals",
              "Consumption in kg per capita"}
          ],
          SwatchLegend[
            col, Keys[consumptiondata],
            LegendMarkerSize -> 20,
            LabelStyle       -> {FontSize -> 20}
          ]}},
          Alignment -> Top
        ]
```

Figure 7.5 – Fish consumption on economies (page 121)

```
In[1]:=  consumptioneconomy = <|
          "Least Developed Countries"            -> {12.0, 12.6},
          "Developed countries or areas"         -> {31.4, 24.9},
          "Other developing countries or areas" -> {105.4, 20.5}
        |>;
```

```
In[2]:=  Grid[{
        {RectangleChart[{
          Labeled[{#[[2, 1]]/#[[2, 2]], #[[2, 2]]}, #[[2, 1]], Center] & /@
            Normal[consumptioneconomy]},
          BarSpacing  -> None,
          AspectRatio -> .4,
          ImageSize   -> 750,
          PlotTheme   -> "Detailed",
          AxesOrigin  -> {.2, 0},
          BaseStyle   -> 18,
          PlotRange   -> {0, 25.02},
          FrameLabel  -> {"Population in billions of individuals",
            "Consumption in kg per capita"}
        ],
        SwatchLegend[
          Lighter[System`PlotThemeDump`$ThemeDefaultGradient, .2][[{1, 3, 5}]],
          Keys[consumptioneconomy],
          LegendMarkerSize -> 20,
          LabelStyle       -> {FontSize -> 20}
        ]}},
        Alignment -> Top
      ]
```

Figure 7.6 – Perfect market (page 122)

```
In[1]:=  Plot[{4 - p, p}, {p, 0, 4},
        Filling -> {1 -> Bottom, 2 -> Top},
        PlotTheme        -> "Detailed",
        PlotStyle        -> AbsoluteThickness[3],
        PlotRangePadding -> None,
        FrameTicks       -> {{{{2, "p"}}, {}}, {{{2, "Q_p"}}, {}}},
        GridLines        -> {Automatic, Range[4]},
        BaseStyle        -> 15,
        PlotLegends      -> {
          Placed[LineLegend[
            Directive @@@ Thread[{ColorData[97, "ColorList"][[;; 2]],
            AbsoluteThickness@3}],
            {"Demand (maximum willingness of paying)",
            "Supply (minimum price accepted)"},
            LegendMarkerSize -> {30, 3}], Right
          ],
          Placed[SwatchLegend[{
            Directive[{Opacity[.3], ColorData[97, "ColorList"][[1]]}],
            Directive[{Opacity[.3], ColorData[97, "ColorList"][[2]]}]
            }, {
            "Demanders williness of paying",
            "Suppliers acceptance area of payment"
            },
            LegendMarkerSize -> {30, 15}
          ], Right]
          }
        ]
```

Figure 7.7 – Sole owner market (page 125)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  R[x_] := (p0 - s f[x] - c(x))*f[x]

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;

In[5]:=  Plot[
           Evaluate[{(p0 - s f[x]) f[x], c(x) f[x]} /. values],
           {x, 0, 1800},
           PlotLegends -> {"TR(X)", "TC(X)"},
           PlotStyle   -> AbsoluteThickness[3],
           GridLines   -> {x /. Chop[N@Solve[R'[x] == 0, x] /. values], None},
           Ticks       -> {Round@x /. Chop[N@Solve[R'[x] == 0, x] /. values], Automatic},
           BaseStyle   -> 12,
           AxesLabel   -> {"X", "TR, TC"}
         ]
```

Figure 7.8 – Sole owner net revenue (page 125)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  R[x_] := (p0 - s f[x] - c(x))*f[x]

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;

In[5]:=  Plot[
           Evaluate[{(p0 - s f[x]) f[x] - c(x) f[x]} /. values],
           {x, 0, 1800},
           PlotStyle -> AbsoluteThickness[3],
           GridLines -> {x /. Chop[N@Solve[R'[x] == 0, x] /. values], None},
           Ticks     -> {Round@x /. Chop[N@Solve[R'[x] == 0, x] /. values], Automatic},
           BaseStyle -> 12,
           AxesLabel -> {"X", "R(X)"}
         ]
```

Figure 7.9 – Sole owner market 2 (page 126)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  xsol = Solve[h == f[x], x];

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;
```

```
In[5]:=  Plot[
           Evaluate[{
             D[c[x]*f[x] /. xsol[[2]], h], 18 - h/12,
             D[(18 - h/12) h /. xsol[[2]], h]} /. values
           ],
           {h, 0, 200},
           PlotRange   -> {0, 19},
           LabelStyle  -> Directive[FontSize -> 15],
           AxesLabel   -> {"H", "p"},
           GridLines   -> {{144, 96, 150}, {2, 6, 10}},
           PlotStyle   -> AbsoluteThickness[3],
           Ticks       -> {{96, 144, 150, "\n  r K "}}, {2, 6, 10, 18},
                                            4
           AspectRatio -> 1,
           PlotLegends -> {
             "Supply (TC'(H))","Demand (p(H))",
             "Marginal revenue (p(H) + p'(H) H)"
             }
         ]
```

Figure 7.10 – Sole owner market 3 (page 127)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  xsol = Solve[h == f[x], x];

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;

In[5]:=  Show[{
           Plot[
             Evaluate[{
               D[c[x]*f[x] /. xsol[[2]], h], 18 - h/12,
               D[(18 - h/12) h /. xsol[[2]], h]} /. values
             ],
             {h, 0, 200},
             PlotRange   -> {0, 19},
             LabelStyle  -> Directive[FontSize -> 15],
             AxesLabel   -> {"H", "p"},
             GridLines   -> {{144, 96, 150}, {2, 6, 10}},
             PlotStyle   -> AbsoluteThickness[3],
             Ticks       -> {{96, 144, {150, "\n  r K "}}, {2, 6, 10, 18}},
                                              4
             AspectRatio -> 1,
             PlotLegends -> {
               Placed[
                 LineLegend[Directive@@@Thread[{ColorData[97, "ColorList"][[;; 3]],
                   AbsoluteThickness@3}], {"Supply "},
                   LegendMarkerSize -> {30, 3}
                 ],
                 Right
               ],
               Placed[
                 SwatchLegend[Directive[{Opacity[.3], ColorData[97][#]}] & /@ {1, 2, 4},
                   {"Producer surplus (48)",
```

```
                "Consumer surplus (384)",
                "Monopoly profit (768)"},
              LegendMarkerSize -> {30, 15}
            ],
            Right
          ]},
       Filling      -> {
         1 -> {2, {Directive[Opacity[.2], ColorData[97][1]], Transparent}},
         2 -> {10, {Transparent, Directive[Opacity[.2], ColorData[97][2]]}}
       }],
       Graphics[{
         EdgeForm[None], Opacity[.2], ColorData[97][4],
         Rectangle[{0, 2}, {96, 10}]
       }]
     }]
```

Figure 7.11 – Supply curves (page 130)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  xsol = Solve[h == f[x], x];

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;

In[5]:=  Plot[{
           Evaluate[D[c[x]*f[x] /. xsol[[2]], h] /. values],
           p /. Solve[x == (x /. Solve[(c[x] == p), x]), p][[1]] /. xsol /. values},
           {h, 0, 150},
           AspectRatio    -> 1,
           PlotRange       -> {{0, 155}, {0, 15}},
           PlotLegends     -> Placed[{"Sole owner supply","Open access supply" },Top],
           LabelStyle      -> Directive[FontSize -> 13],
           FrameLabel      -> {"Harvest (H)", "Price (p)"},
           PlotTheme       -> "Detailed",
           PlotRangePadding -> None,
           PlotStyle       -> AbsoluteThickness[3],
           AspectRatio     -> 1,
           GridLines       -> {Range[0, 150, 25], Automatic},
           FrameTicks      -> {{Automatic, None}, {Range[0, 150, 25], None}}
         ]
```

Figure 7.12 – Three supply curves (page 131)

```
In[1]:=  f[x_] := r x (1 - x/k)

In[2]:=  c[x_] := a/(q x)

In[3]:=  xsol = Solve[h == f[x], x];

In[4]:=  values = r -> 1/3, p0 -> 18, s -> 1/12, k -> 1800, q -> 1/24, a -> 90;
```

```
In[5]:=  h1 = h /. Solve[((p /. Solve[x == (x /.
              Solve[(c[x] == p), x]), p][[1]] /.
              xsol[[1]]) == 12 - h/20) /. values, h
            ][[1]];
         p1 = p /. Solve[x == (x /. Solve[(c[x] == p), x]), p][[1]] /.
            xsol[[1]] /. values /. {h -> h1};
```

```
In[7]:=  h2 = h /. Solve[(D[c[x]*f[x] /. xsol[[2]], h] == 12 - h/20) /. values, h][[1]];
         p2 = D[c[x]*f[x] /. xsol[[2]], h] /. values /. {h -> h2};
```

```
In[9]:=  h3 = h /. Solve[((p /.
              Solve[f'[x] - c'[x] f[x]/(p - c[x]) == δ, p][[1]] /.
              xsol[[2]] /. {δ -> 15/100}) == 12 - h/20) /. values, h
            ][[1]];
         p3 = p /. Solve[f'[x] - c'[x] f[x]/(p - c[x]) == δ, p][[1]] /.
            xsol[[2]] /. {δ -> 15/100} /. values /. {h -> h3};
```

```
In[11]:=  h4 = h /. Solve[((p /.
              Solve[f'[x] - c'[x] f[x]/(p - c[x]) == δ, p][[1]] /.
              xsol[[2]] /. {δ -> 15/100}) == D[(12 - h/20) h, h]) /. values, h
            ][[1]];
         p4 = 12 - h4/20;
         p5 = p /. Solve[f'[x] - c'[x] f[x]/(p - c[x]) == δ, p][[1]] /.
            xsol[[2]] /. {δ -> 15/100} /. values /. {h -> h4};
```

```
In[14]:=  opt = Plot[
           p /. Solve[f'[x] - c'[x] f[x]/(p - c[x]) == δ, p][[1]] /.
            xsol /. values /. {δ -> .15}, {h, 0, 155},
            PlotStyle -> {AbsoluteThickness[3], ColorData[97][3]}
           ];
```

```
In[15]:=  Show[{
            Plot[{
              Evaluate[D[c[x]*f[x] /. xsol[[2]], h] /. values],
              p /. Solve[x == (x /.
                Solve[(c[x] == p), x]), p][[1]] /. xsol /. values, , 12 - h/20},
              {h, 0, 150},
              AspectRatio       -> 1,
              PlotRange         -> {{0, 170}, {0, 15}},
              LabelStyle        -> Directive[FontSize -> 12],
              AxesLabel         -> {"Harvest (H)", "Price (p)"},
              PlotRangePadding  -> None,
              PlotStyle         -> AbsoluteThickness[3],
              AspectRatio       -> 1,
              GridLines         -> {{h1, h2, h3, h4}, {p1, p2, p3, p4, p5}},
              PlotLegends       -> {"Sole owner supply",
                                    "Open access supply" ,
                                    "Optimal supply (δ = 0.15)",
                                    "Demand",
                                    "Marginal revenue" }
            ],
            opt /. Line[x_] :> Line@Split[x, Norm[# - #2] < 10 &],
            Graphics[{AbsolutePointSize[7],
              Point /@ {{h1, p1}, {h2, p2}, {h3, p3}, {h4, p4}, {h4, p5}},
              Style[Text["A", {h1, p1} + {5, .5}], 12],
              Style[Text["B", {h3, p3} + {5, .5}], 12],
              Style[Text["C", {h2, p2} + {-5, .8}], 12],
              Style[Text["D", {h4, p5} + {5, .8}], 12],
              Style[Text["E", {h4, p4} + {5, .5}], 12]
            }]
          ]
```

Figure 9.1 – Political objectives (page 141)

```
In[1]:=  Show[{
           Graphics[{
             Arrow[{{0, 0}, {1.1, 0}}],
             Arrow[{{0, 0}, {0, 30}}],
             Arrow[{{0, -5}, {0, 0}}],
             Arrow[{{.75/2, -8}, {.75/2, 0}}],
             Arrow[{{.5, -5}, {.5, 0}}],
             Arrow[{{.75, -8}, {.75, 0}}],
             Dashed,
             Line[{{.75/2, 0}, {.75/2, 23.4375`}}],
             Line[{{.5, 0}, {.5, 25}}],
             Line[{{.75, 0}, {.75, 18.75}}],
             Style[Text["Fishing effort", {1.3, 0}], 12],
             Style[Text["Fish stock", {0, -6.5}], 12],
             Style[Text["Rent (E_{MEY})", {.75/2, -10}], 12],
             Style[Text["Food (E_{MSY})", {.55, -6.5}], 12],
             Style[Text["Employment (E_∞)", {.85, -10}], 12]
           }],
           Plot[
             {q k e (1 - q/r e) /. {q -> 1, r -> 1, k -> 100}, 25*e},
             {e, 0, 1},
             PlotStyle   -> Thick,
```

```
          PlotLegends -> Placed[{"Total revenue", "Total cost"}, Right],
          PlotRange   -> {0, Automatic}
        ]},
        PlotRange     -> All,
        AspectRatio   -> 1/1.6
      ]
```

Figure 9.2 – **Time paths toward management goal** (page 142)

```
In[1]:= x[x0_] := (f = Min[f*1.02, .7]; x0 + x0 (1 - x0)/100 - (f/100)*x0);
       y[y0_] := (e = Max[e*.94, .7]; y0 + y0 (1 - y0)/100 - (e/100)*y0);
       z[z0_] := z0 + z0 (1 - z0)/100 - (.7/100)*z0;

In[4]:= e = 8; f = .01;
       ListLinePlot[{
         NestList[x, 1, 10*100],
         NestList[y, 1, 10*100],
         NestList[z, 1, 10*100]
         },
         PlotRange        -> {0, 1},
         FillingStyle     -> Opacity[.1],
         Filling          -> Axis,
         AxesOrigin       -> {0, -.5},
         Frame            -> True,
         PlotTheme        -> "Detailed",
         PlotLegends      -> {"Increasing catch rate (from low)",
           "Decreasing catch rate   (from high)", "Fixed catch rate"},
         PlotStyle        -> Thick,
         PlotRangePadding -> None,
         GridLines        -> {None, {.3}},
         FrameTicks       -> {
           {{{.3, "Management\n      goal"},
           {1, "Initial level"}}, None}, None},
         AspectRatio      -> 1,
         BaseStyle        -> 14,
         FrameLabel       -> {{None, "Stock size"}, {"Time", None}}
       ]
```

Figure 10.1 – **Technical regulations (1)** (page 149)

```
In[1]:= Legended[Show[{
           Plot[q k e (1 - q/r e) /. {q -> #, r -> 1, k -> 100}, {e, 0, 1.5},
             PlotStyle -> Switch[#, 1, Blue, .5, Darker@Green]
           ] & /@ {1, .5},
           Plot[20*e, {e, 0, 1.5}, PlotStyle -> Red]},
           PlotRange -> {0, All},
           GridLines -> {(e /. FindRoot[20 e == q k e (1 - q/r e) /.
             {q -> #, r -> 1, k -> 100}, {e, 1}]) & /@ {1, .5}, None},
           PlotLabel -> ToString[
             ((a/(q p) /. {p -> 1, q -> 1/2, a -> 20})/(a/(q p) /.
               {p -> 1, q -> 1, a -> 20}) - 1) * 100] <>
               "% stock increase after regulation\n" <>
               ToString[100*(1.2 - .8)/.8] <> "% increase in fishing effort",
           AxesLabel -> {"Fishing effort", "Value"}
```

```
    ],
    LineLegend[{Blue, Darker@Green, Red},
      {"Initial TR", "TR after technical regulations", "TC"}]
  ]
```

Figure 10.2 – Technical regulations (2) (page 149)

```
In[1]:= Legended[
      Show[{
        Plot[q k e (1 - q/r e) /. {q -> #, r -> 1/#, k -> 100}, {e, 0, 1.5},
          PlotStyle -> Switch[#, 1, Blue, .5, Darker@Green]
        ] ] & /@ {1, .5},
        Plot[20*e, {e, 0, 1.5}, PlotStyle -> Red]},
        PlotRange -> {0, All},
        GridLines -> {(e /. FindRoot[20 e == q k e (1 - q/r e) /.
          {q -> #, r -> 1/#, k -> 100}, {e, 1}]) & /@ {1, .5}, None},
        PlotLabel -> ToString[
          ((a/(q p) /. {p -> 1, q -> 1/2, a -> 20})/(a/(q p) /.
            {p -> 1, q -> 1, a -> 20}) - 1) * 100] <>
            "% stock increase after regulation\n" <>
            ToString[100*(2.4 - .8)/.8] <> "% increase in fishing effort",
        AxesLabel -> {"Fishing effort", "Value"}
      ],
      LineLegend[
        {Blue, Darker@Green, Red},
        {"Initial TR", "TR after technical regulations", "TC"}]
    ]
```

Figure 10.4 – Taxation (1) (page 154)

```
In[1]:= Show[{
      Plot[{1/x, 1/x, 1/x, 1/x}, {x, 0, 5},
        PlotRange    -> {{-1, 4}, {-1, 4}},
        AspectRatio  -> .8,
        AxesOrigin   -> {-.5, -.7},
        Ticks        -> None,
        PlotStyle    -> (Directive[Thickness[#],Blue,Opacity[.1]]& /@ {.32,.21,.12,.05}),
        BaseStyle    -> 12,
        AxesLabel    -> {"Complexity", "Precision"},
        AxesStyle    -> Arrowheads[{0.0, 0.05}]
      ],
      Graphics[{Dashed,
        {Opacity[.2], Red, Disk[#, .7], Opacity[1], Black,
          Text[Style["Precise\nanalysis", 12, Bold], #]}& @{.31, 1/.31},
        {Opacity[.2], Green, Disk[#, .7], Opacity[1], Black,
          Text[Style["Fuzzy\nsets", 12, Bold], #]}& @{3.2, 1/3.2}
      }]
    }]
```

Figure 11.1 – Sorites paradox (page 156)

```
In[1]:= GraphicsGrid[
          Partition[
            Graphics[{
              Blend[{Orange, White}],
              EdgeForm[{Black, AbsoluteThickness[1.5]}],
              SeedRandom[1234];
              Table[
                Rotate[
                  Disk[RandomVariate[NormalDistribution[0, 5], 2], {1, 4}],
                  RandomReal[{0, 2}]*Pi
                ],
                {#}
              ],
              Black,
              Style[Text[ToString[#] <> If[# === 1, " grain", " grains"], {-10, 17}], 12]
              },
              Frame      -> True,
              FrameTicks -> None,
              PlotRange  -> {{-20, 20}, {-20, 20}}
            ] & /@ {1, 2, 3, 5, 7, 10, 20, 50, 75, 100},
            5
          ],
          ImageSize  -> 700,
          Spacings   -> {30, 30}
        ]
```

Figure 11.2 – Complexity (page 157)

```
In[1]:= Show[{
          ContourPlot[x^(1/2)*y^(1/2), {x, 0, 1}, {y, 0, 1},
            ContourStyle  -> Opacity[0],
            ColorFunction -> (Blend[{Blue, White}, # + .5] &),
            Contours      -> {.1, .2, .3, .4, .5, .6},
            Frame         -> None
          ],
          Graphics[{
            AbsoluteThickness[2],
            Arrow[{{0, 0}, {1., 0}}],
            Arrow[{{0, 0}, {0, 1.}}],
            Dashed,
            {Opacity[.4], Darker@Magenta, Disk[#, {.18, .13}], Opacity[1],
              White, Text[Style["Precise\nanalysis", 16, Bold], #]} &@{.2, .85},
            {Opacity[.4], Darker@Green, Disk[#, {.18, .13}], Opacity[1],
              White, Text[Style["Heuristics", 16, Bold], #]} &@{.8, .16},
            {Opacity[.4], Red, Disk[#, {.18, .13}], Opacity[1],
              White, Text[Style["Unavailable\nknowledge", 16, Bold], #]} &@{.8, .85},
            Style[Text["Complexity", {.5, -.05}], 14],
            Style[Text[Rotate["Precision", 90 Degree], {-.05, .5}], 14] }
          ]},
          PlotRange -> All
        ]
```

Figure 11.3 – Expert system (page 158)

```
In[1]:= Show[
        Graphics[{
          GrayLevel[.8],
          Rectangle[{0, 0}, {4, 2}],
          Polygon[
            {{0, 0}, {2, 0}, {1, 2}},
            VertexColors -> {White, White, Lighter@Blue}
          ],
          Polygon[
            {{2, 0}, {4, 0}, {3, 2}},
            VertexColors -> {White, White, Red}
          ],
          Thickness[.01],
          Line[{{0, .5}, {4, .5}}],
          Line[{{0, 1}, {4, 1}}],
          Line[{{0, 1.5}, {4, 1.5}}],
          GrayLevel[1],
          Text["Wisdom", {1, 1.75}],
          Text[
            Style["Expert\nSystems", LineSpacing -> {.8, 0}],
            {3, 1.75}
          ],
          GrayLevel[1],
          Text["Knowledge", {1, 1.25}],
          Text[
            Style["Decision\nSupport Systems", LineSpacing -> {.8, 0}],
            {3, 1.25}
          ],
          GrayLevel[.5],
          Text["Information", {1, 0.75}],
          Text[
            Style["Management\nInformation Systems", LineSpacing -> {.8,0}],
            {3,0.75}
          ],
          GrayLevel[.5],
          Text["Data", {1, 0.25}],
          Text["Transaction Processing Systems", {3, 0.25}] }
        ],
        AspectRatio -> .4,
        BaseStyle   -> {FontFamily -> "Open Sans Extrabold",
                        FontWeight -> Bold, 16},
        Background  -> GrayLevel[.8],
        ImageSize   -> 680
      ]
```

Figure 11.4 – Set theory (page 159)

```
In[1]:= GraphicsRow[{
        Graphics[
          {White, Rectangle[{0, .2}, {2, 1.8}]}, RGBColor[.8, .9, 1], Disk[{1, 1}, .4]},
          Frame           -> True,
          FrameTicks      -> None,
          PlotRangePadding -> None,
          PlotLabel       -> "Within the set"
        ],
        Graphics[
          {RGBColor[.8, .9, 1], Rectangle[{0, .2}, {2, 1.8}], White, Disk[{1, 1}, .4]},
          Frame           -> True,
          FrameTicks      -> None,
          PlotRangePadding -> None,
          PlotLabel       -> "Not within the set"
        ],
        Graphics[
          {White, Rectangle[{0, .2}, {2, 1.8}],
          RGBColor[.8, .9, 1], Disk[{.8, 1}, .4], Disk[{1.2, 1}, .4]},
          Frame           -> True,
          FrameTicks      -> None,
          PlotRangePadding -> None,
          PlotLabel       -> "Union of two sets"
        ],
        Show[{
          Graphics[
            {White, Rectangle[{0, .2}, {2, 1.8}],
            EdgeForm[{Thin, Opacity[.5], Black}], Disk[{.8, 1}, .4], Disk[{1.2, 1}, .4]},
            Frame      -> True,
            FrameTicks -> None
          ],
          Region[
            RegionIntersection[Disk[{.8, 1}, .4], Disk[{1.2, 1}, .4]],
            BaseStyle -> RGBColor[.8, .9, 1]
          ]},
          PlotRangePadding -> None,
          PlotLabel       -> "Intersection of two sets"
        ]},
        Spacings  -> -20,
        ImageSize -> 600
      ]
```

Figure 11.5 – Fuzzy sets (page 160)

```
In[1]:= fuzzyset1 =
        Show[{
          ContourPlot[Exp[-x^2 - y^2], {x, -2, 2}, {y, -2, 2},
            ContourStyle  -> None, Contours -> 100,
            ColorFunction -> (Blend[{RGBColor[1,.3,.3], RGBColor[.6,1,.4]}, #] &),
            Frame         -> True,
            FrameTicks    -> None,
            PlotRange     -> {{-2, 2}, {-1.5, 1.5}},
            AspectRatio   -> 0.9 * 3/4
          ],
          Graphics[
            {PointSize[.03], Blue, Point[{.6, -.6}],
            Darker@Green, Point[{0, 0}], White, Point[{-1.8, 1.3}]
          }]
        }];

In[2]:= fuzzyset2 =
        Show[{
          Plot[.5, {t, .5, 2.5},
            ColorFunction -> (Blend[{RGBColor[1,.3,.3], RGBColor[.6,1,.4]},#]&),
            PlotStyle     -> Thickness[.71]
          ],
          Graphics[{Thickness[.01], Black, Line[{{0, 0}, {1, 0}, {2, 1}, {3, 1}}],
            PointSize[.03], Blue, Point[{1.5, .5}],
            Darker@Green, Point[{2.5, 1}], White, Point[{.5, .01}]
          }]
          },
          PlotRangePadding -> None,
          PlotRange        -> {-.03, 1.03},
          AxesOrigin       -> {0, 0}, Ticks -> {None, Automatic},
          AspectRatio      -> 3/4
        ];

In[3]:= fuzzyset = GraphicsRow[{fuzzyset1, fuzzyset2}]
```

Figure 11.6 – Fuzzy set of a heap (page 161)

```
In[1]:= heaplist = {
         Graphics[{
           Blend[{Orange, White}],
           EdgeForm[{Black, AbsoluteThickness[1]}],
           SeedRandom[1234];
           Table[
             Rotate[Disk[
               RandomVariate[NormalDistribution[0, 5], 2], {1, 4}],
               RandomReal[{0, 2}]*Pi
             ],
             {#}
           ]},
           Frame       -> True,
           FrameTicks -> None,
           ImageSize  -> 50,
           PlotRange  -> {{-14, 14}, {-14, 14}},
           Background -> Directive[Opacity[.7], White]
         ],
         {#, If[# > 5, 1, If[# < 1, 0, 1 + (-5 + #)/4]]}
       } & /@ {1, 2, 3, 5, 7, 10, 20, 50, 75, 100};

In[2]:= fuzzyplot =
         Plot[
           Piecewise[{
             {0, x <= 1},
             {1 + (-5 + x)/4, 1 < x <= 5},
             {1, x > 5}}
           ],
           {x, 0, 50},
           PlotStyle -> Directive[Opacity[.5], AbsoluteThickness[40]]
         ];

In[3]:= Show[
         Graphics[{
           First[fuzzyplot],
           Inset[Sequence @@ #, {0, 0}, Scaled[.2]
           ] & /@ heaplist },
           AbsoluteOptions[fuzzyplot]
         ],
         PlotRange   -> {{0, 21}, {-.06, 1.06}},
         AspectRatio -> .46,
         ImageSize   -> 900,
         Ticks       -> {{3, 5, 7, 10, 20}, Range[0, 1, .2]},
         BaseStyle   -> 16,
         GridLines   -> {{1, 2, 3, 5, 7, 10, 20}, None},
         AxesLabel   -> {"Grains", "Membership\ndegree"}
       ]
```

Figure 11.7 – Fuzzy terms (page 166)

```
In[1]:= SeedRandom[1234];

In[2]:= species = 500;
        years   = 25;

In[4]:= random = RandomReal[1, species, years];

In[5]:= eval1 = #/Max[#] & /@ random;
        eval2 = Flatten[Position[#, 1.] & /@ eval1];

In[7]:= res = {};

In[8]:= Do[
          AppendTo[res,
           If[i < eval2[[#]],
            If[eval1[[#, i]] <= .1, 1,
             If[eval1[[#, i]] < .5, 2, 3]
             ],
            If[eval1[[#, i]] <= .1, 5,
             If[eval1[[#, i]] < .5, 4, 3]
             ]
            ]
           ],
          {i, years}
          ] & /@ Range[species];

        ArrayPlot[
          Transpose[
            Sort[#, #1 > #2 &] & /@ Transpose[Partition[res, years]]
          ],
          ColorRules       -> Table[i -> ColorData[97][i], {i, 5}],
          PlotRange        -> 5,
          AspectRatio      -> 1/GoldenRatio,
          BaseStyle        -> 12,
          PlotRangePadding -> None,
          FrameLabel       -> {"Share of stocks", "Year"},
          PlotLegends      -> Placed[
            {"Undeveloped", "Developing", "Fully exploited",
             "Overfished", "Collapsed"}, Right
          ],
          FrameTicks       -> {
            {{{species, "0%"}, {3 species/4, "25%"}, {species/2, "50%"},
             {species/4, "75%"}, {1, "100%"}}, None},
            {Automatic, None}
          }
        ]
```

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