Summary of Gauge Theory

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Abelian Gauge Field

- <u>Chosen framework</u>: Electrodynamics

$$\mathcal{L}_{QED} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

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- Requirement 1: Gauge covariant derivative

$$D_\mu := \partial_\mu + ieA_\mu(x)$$

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- Finally: Gauge invariant QED Lagrangian

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Types of representation (reps)

- Trivial reps: For $a \in \mathfrak{g}$ lie algebra associated with Lie group G (i.e. tangent space) and $h \in G$ (i.e. Lie group manifold)

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Matter fields transforming in the trivial reps would not interact with the gauge fields

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Matter fields can also transforms in the fundamental and their conjugates in the conjugate fundamental reps

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For SU(N), $U^{-1} = U^\dagger$. For SO(N), $U^{-1} = U^T$

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- Finally: Lagrangian of Yang-Mills coupled with matter fields

$$\mathcal{L}_{\text{YM+Matter}} = \bar{\psi}(x)(i\not{D} - m)\psi(x) - \frac{1}{2}\mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}]$$

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- Mathematically, their transformation rules are described by representation.
- Fields belongs to a representation space while the representation of the lie group describes large no. of gauge transformations.
- Representation of lie algebra not only describes infinitesimal gauge transformation but also the coupling of fields to the gauge potentials.
- In non-abelian gauge theory, there is no classical point of view.

Thank You For Listening!
I Invite You To Ask Any Questions You May Have...