A Beginning for the search for a Theory of Everything- Kaluza-Klein Theory

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What you will know from this talk?

- Introduction
- Framework
- Modern Approach
- Closing Remarks

AIM of this talk: To introduce how Kaluza's idea on unifying EM with gravitation shaped into Kaluza-Klein theory

History

1919 (proposed)

Nordström (1914)

[Kaluza, 1921]

Sent his hypothesis to Einstein

⊕

Add-on to Kaluza's hypothesis

(physical touch)

[Klein, 1926]

⇒ **Kaluza-Klein Theory***

On 21 April (1919) Einstein writes [O'Raifeartaigh and Straumann, 1998],

"The idea of achieving [a unified theory] by means of a five-dimensional cylinder world never dawned on me. . . . At first glance I like your idea enormously"

Photos credit: Theodor Kaluza (left) and Oskar Klein (right), photos taken from Science Photo Library (left), Oskar Klein Center (right)
*It was first used in 1933 by Oswald Veblem.

Kaluza's Preliminary Idea

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\delta}\frac{dx^\nu}{d\tau}\frac{dx^\delta}{d\tau} = 0 :\Longrightarrow \text{worldline of a test particle in gravitation field is geodesic}$$

$$H = \frac{1}{2m_0} g^{\mu\nu} p_\mu p_\nu :\Longrightarrow \text{Assuming a Hamiltonian}$$

where m_0 is the rest mass of the particle, p_μ are canonical momenta conjugated to the coordinates x^μ and $\mu, \nu \in \{0, 1, 2, 3\}$. Metric signature $(-+++)$.

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WHY WE CHOOSE HAMILTONIAN?

Using this fact

$$\partial_\mu g^{\nu\delta} = -g^{\nu\rho}(\partial_\mu g_{\rho\sigma}) g^{\sigma\delta}$$

Using Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} \Longrightarrow$$ provides Geodesics equation

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$$= \frac{1}{2m_0}\left[g^{\mu\nu}p_\mu p_\nu - 2\frac{q}{c}A^\mu p_\mu + \left(\frac{q}{c}\right)^2 A^\mu A_\mu\right] \qquad \because A^\mu \equiv g^{\mu\nu}A_\nu$$

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WAIT WAIT!

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$$
\begin{aligned}
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&= \frac{1}{2m_0}\left[g^{\mu\nu} p_\mu p_\nu - 2\frac{q}{c}A^\mu p_\mu + \left(\frac{q}{c}\right)^2 A^\mu A_\mu\right] \qquad \because A^\mu \equiv g^{\mu\nu}A_\nu
\end{aligned}
$$

WAIT WAIT! Assuming geodesic is the projection on the 4D spacetime of a 5D space

$$
\begin{aligned}
H &= \frac{1}{2m_0}\hat{g}^{AB} p_A p_B \qquad :\Longrightarrow \text{Defining Hamiltonian} \qquad : A, B \in \{0,1,2,3,4\} \\
&= \frac{1}{2m_0}\left[\hat{g}^{\mu\nu} p_\mu p_\nu + 2\hat{g}^{\mu 4} p_\mu p_4 + \hat{g}^{44} p_4 p_4\right] \qquad :\Longrightarrow \text{expanding} \sum\sum
\end{aligned}
$$

Kaluza's Idea

Comparing BOTH

$$H = \frac{1}{2m_0}\left[\hat{g}^{\mu\nu} p_\mu p_\nu + 2\hat{g}^{\mu 4} p_\mu p_4 + \hat{g}^{44} p_4 p_4\right]$$

AND

$$H = \frac{1}{2m_0}\left[g^{\mu\nu} p_\mu p_\nu - 2\frac{q}{c}A^\mu p_\mu + \left(\frac{q}{c}\right)^2 A^\mu A_\mu\right]$$

YIELDS

$$\hat{g}^{\mu\nu} = g^{\mu\nu}, \quad \hat{g}^{\mu 4} p_4 = -\frac{q}{c}A^\mu, \quad \hat{g}^{44} p_4 p_4 = \left(\frac{q}{c}\right)^2 A^\mu A_\mu$$

Kaluza's Cylinder Condition

- Components $g^{\mu\nu}$ and A^{μ} are function of x^{ν}
- Thus assumes $\hat{g}^{AB}$ (also $\hat{g}_{AB}$) are function of x^{μ} because new coordinate x^4 does not appear in our new Hamiltonian.

Conclusion,

$\boxed{\text{Metric tensor } \hat{g}^{AB} \text{ does not depend upon fifth dimension } (x^4)}$

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Conclusion,

$$\boxed{\text{Metric tensor } \hat{g}^{AB} \text{ does not depend upon fifth dimension } (x^4)}$$

$\hookrightarrow$ Kaluza's (arbitrary) cylinder condition i.e. $\partial_4 \hat{g}_{AB} = 0 \implies \partial_4 A_{\mu} = 0, \partial_4 g_{\mu\nu} = 0$

- x^4's conjugate momentum p^4 has a constant motion.

Kaluza's Hypothesis Mathematical Construct

$$\begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} \oplus \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{00} & g_{01} & g_{02} & g_{03} \\ g_{00} & g_{01} & g_{02} & g_{03} \\ g_{00} & g_{01} & g_{02} & g_{03} \end{pmatrix} \Leftrightarrow \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & A_0 \\ g_{00} & g_{01} & g_{02} & g_{03} & A_1 \\ g_{00} & g_{01} & g_{02} & g_{03} & A_2 \\ g_{00} & g_{01} & g_{02} & g_{03} & A_3 \\ A_0 & A_1 & A_2 & A_3 & ? \end{pmatrix}$$

$$\text{So, } \hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu 4} \\ \hat{g}_{4\mu} & \hat{g}_{44} \end{pmatrix}$$

New Notations Alert: $\oplus$ defined as in-cooperating EM vector field by GR and $\Leftrightarrow$ defined as in-order to in-cooperate EM with GR, the resulting tensor should be symmetric tensor. I've defined it to make everything in one slide. Please do not take it seriously! ☺

Kaluza's Idea Continued

Thus,

$$\hat{g}^{\mu\nu} = g^{\mu\nu}, \quad \hat{g}^{\mu 4} = -\kappa A^{\mu}, \quad \hat{g}^{44} = \kappa^2 A^{\mu} A_{\mu}$$

- κ is constant such that $\kappa = \frac{q}{p_4 c}$
- Add 1 to $\kappa^2 A^{\mu} A_{\mu}$ such that matrix $\hat{g}^{AB}$ is non-singular i.e. $\det(\hat{g}^{AB}) \neq 0$

$$\hat{g}^{\mu\nu} = g^{\mu\nu}, \quad \hat{g}^{\mu 4} = -\kappa A^{\mu}, \quad \hat{g}^{44} = 1 + \kappa^2 A^{\mu} A_{\mu}$$

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Using the fact that $\hat{g}^{AB}\hat{g}_{AC} = \delta^B_C$, the inverse of $\hat{g}^{AB}$ must be given by

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$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa^2 A_{\mu} A_{\nu}, \quad \hat{g}_{\mu 4} = \kappa A_{\mu}, \quad \hat{g}_{44} = 1$$

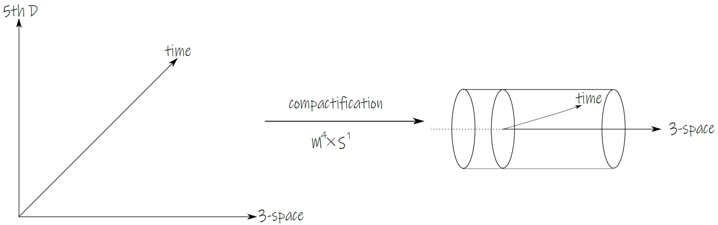
Metric of the 5D space

INSERTING $\hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa^2 A_\mu A_\nu, \quad \hat{g}_{\mu 4} = \kappa A_\mu, \quad \hat{g}_{44} = 1$ IN

$$d\hat{s}^2 = \hat{g}_{AB} dx^A dx^B = \hat{g}_{\mu\nu} dx^\mu dx^\nu + 2\hat{g}_{\mu 4} dx^\mu dx^4 + \hat{g}_{44} dx^4 dx^4$$
$$\therefore d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + (dx^4 + \kappa A_\mu dx^\mu)^2$$

WITH signature $(- + + + +)$

Klein's add-on to Kaluza's Hypothesis



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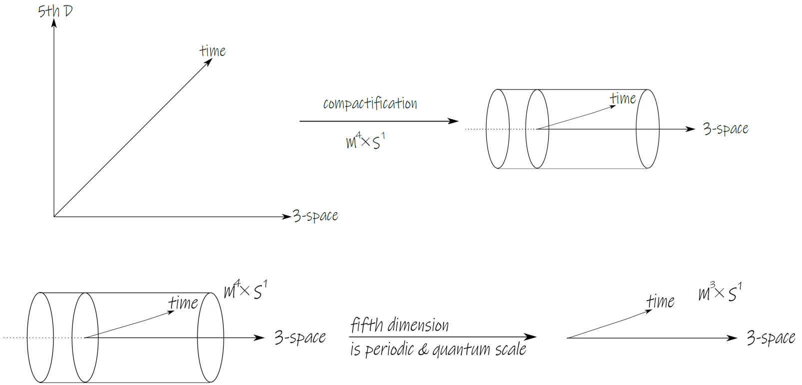

Classically, which is appropriate space $\mathcal{M}^{(5)}$ or $\mathcal{M}^{(4)} \times \mathcal{S}^1$ as ground state?

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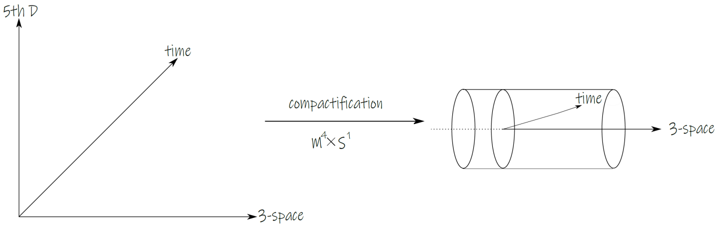

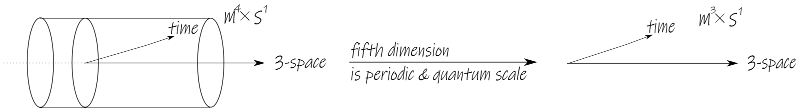

Classically, which is appropriate space $\mathcal{M}^{(5)}$ or $\mathcal{M}^{(4)} \times \mathcal{S}^1$ as ground state? Hard to decide!!! Both has zero energy [Witten, 1981]

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* ∂_4 can be think of killing vector associated with spacetime isometry i.e. $x^4 \mapsto x^4 + \epsilon, \forall \epsilon \in \mathbb{R}$

$$\boxed{\partial_4 \hat{g}_{AB} = 0 \implies (\partial_4 g_{\mu\nu} = 0, \partial_4 A_\mu = 0), \partial_4 \phi = 0}$$

$\hookrightarrow$ Kaluza's refined cylinder condition [Klein, 1926]

Digression

We'll follow a different approach than the Kaluza's original way of using Einstein tensor notation (See [del Castillo, 2019])

Digression

We'll follow a different approach than the Kaluza's original way of using Einstein tensor notation, following the philosophy of Amy's Ice Cream:

Life is uncertain. Eat dessert first.

Digression

Exterior (or Wedge) product

- Start with meaningless symbol $dx^1, \ldots, dx^n$
- Exterior product $\wedge$: $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and $dx^i \wedge dx^i = 0$
- Usual properties of multiplication holds
- Differential forms is co-ordinate independent
- 0-form on $\mathbb{R}^n$ is a function
- 1-form is an expression of the form $\sum_i f_i(x)dx^i$
- 2-form is $\sum_{i,j} f_{ij}(x)dx^i \wedge dx^j$
- k-form is $\sum_I f_I(x)dx^I$ where I is a subset $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ and dx^I is shorthand $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$
- If α is a $k-$form, we say that α has degree k : $\alpha := \sum_I \alpha_I dx^I$.

Digression Contd.

Exterior derivative

- $\alpha := \sum_I \alpha_I dx^I$ so, exterior derivative, $d\alpha := \sum_{I,j} \frac{\partial \alpha_I(x)}{\partial x^j} dx^j \wedge dx^I$
- Holds chain rule
- For example, on $\mathbb{R}^2$, if $\alpha = xydx + e^x dy$ then,

$$\begin{aligned} d\alpha &= ydx \wedge dx + xdy \wedge dx + e^x dx \wedge dy + 0dy \wedge dy \\ &= (e^x - x)dx \wedge dy. \end{aligned}$$

- Simpler case: if f is 0-form then, $df(x) = \sum_j \frac{\partial f(x)}{\partial x^j} dx^j$

General Construction

- Assume $\mathcal{M}^{(5)}$ be (4+1)D Lorentzian manifold with general co-ordinates $\hat{x} := x^A \partial_A$

General Construction

- Assume $\mathcal{M}^{(5)}$ be (4+1)D Lorentzian manifold with general co-ordinates $\hat{x} := x^A \partial_A$; aim to replace by $x_a \hat{e}^a$, $\hat{e}^a$ is 1-form.
- Rewriting Kaluza-Klein condition,
  $\partial_4 \hat{g}_{AB} = 0 \implies (\partial_4 g_{\mu\nu} = 0, \partial_4 A_\mu = 0), \partial_4 \phi = 0$
- Use Clairaut parametrization under the condition that we can foliate our spacetime by spacelike hypersurface, each orthogonal to the Killing vector field ∂_4 i.e. $\mathcal{M}^{(5)} \equiv \mathcal{M}^{(4)} \times \mathcal{M}^{(1)}$. So,

$$d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + (dx^4 + \kappa A_\mu dx^\mu)^2, \quad \text{Set } \kappa = \frac{q}{p_4 c} = 1$$

$$\therefore d\hat{s}^2 = e^{2\alpha\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\phi} (dx^4 + A_\mu dx^\mu)^2$$

α and β are arbitrary constant.

General Construction Contd.

- $\hat{g}_{AB}$ has components in-terms of Clairaut parameterization,

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu, \hat{g}_{\mu 4} = e^{2\beta\phi} A_\mu, \hat{g}_{44} = e^{2\beta\phi}, \beta \neq 0.$$

- Co-ordinate free descriptions:

  - Tetrad basis $\{e^a, e^b\}$ on $\mathcal{M}^{(4)}$: $\eta_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu} \forall a, b \in \{0, 1, 2, 3\}$
  - Fünfbein basis $\{\hat{e}^a, \hat{e}^z\}$ on $\mathcal{M}^{(5)}$: $\hat{\eta}_{ab} = \hat{e}^A{}_a \hat{e}^B{}_b \hat{g}_{AB} \forall a, b \in \{0, \ldots, 4\}$

- By inspection using Tetrad (or Vierbein) and Fünfbein metric: $\hat{e}^a = \exp(\alpha\phi) e^a$ and $\hat{e}^z = \exp(\beta\phi)(dx^4 + A_\mu dx^\mu)$
- z for the basis element pointing in direction associated with the cylinder condition

General Construction Contd.

- To calculate curvature, use Maurer-Cartan structure equations,
  - 5D Torsion two-form: $\hat{T}^a = d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b, a, b \in \{0, \ldots, 4\}$
  - Curvature two-form: $\hat{R}^a{}_b = \hat{\omega}^a{}_b + \hat{\omega}^a{}_c \wedge \hat{\omega}^c{}_b + \hat{\omega}^a{}_z \wedge \hat{e}^z,$

  where $\hat{\omega}^a{}_b$ is Spin connection 1-form i.e. $\hat{\omega}^a{}_b := \hat{\omega}_c{}^a{}_b dx^c$
- Our spacetime is Torsion free, the structure equation reduces to

$$\hat{T}^a = 0 = d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b + \hat{\omega}^a{}_z \wedge \hat{e}^z \tag{1}$$

$$\hat{T}^z = 0 = d\hat{e}^z + \hat{\omega}^z{}_b \wedge \hat{e}^b + \hat{\omega}^z{}_z \wedge \hat{e}^z \tag{2}$$

$$\hat{R}^a{}_b = \hat{\omega}^a{}_b + \hat{\omega}^a{}_c \wedge \hat{\omega}^c{}_b + \hat{\omega}^a{}_z \wedge \hat{e}^z, \tag{3}$$

$$\hat{R}^a{}_z = \hat{\omega}^a{}_z + \hat{\omega}^a{}_c \wedge \hat{\omega}^c{}_z + \hat{\omega}^a{}_z \wedge \hat{e}^z, \tag{4}$$

$$\hat{R}^z{}_z = \hat{\omega}^z{}_z + \hat{\omega}^z{}_c \wedge \hat{\omega}^c{}_z + \hat{\omega}^z{}_z \wedge \hat{e}^z \tag{5}$$

General Construction Contd.

- One form:
  $d\phi = \partial_A \phi dx^A = \partial_\mu \phi dx^\mu + \partial_4 \phi dx^4 = \partial_\mu \phi dx^\mu, \quad \because \partial_4 \phi = 0$
- To calculate Spin connection,

$$
\begin{aligned}
\hat{e}^a &= \exp(\alpha\phi) e^a \\
d\hat{e}^a &= d(\exp(\alpha\phi) e^a) \\
&= d\exp(\alpha\phi) \wedge e^a + \exp(\alpha\phi) de^b \\
&= \alpha \exp(\alpha\phi) d\phi \wedge e^a - \exp(\alpha\phi) \omega^a{}_b \wedge e^b, \quad \text{Using eqn (1) in } \mathcal{M}^{(4)} \\
&= \alpha \exp(\alpha\phi) \partial_\mu \phi (dx^\mu \wedge e^a) - \exp(\alpha\phi) \omega^a{}_b \wedge e^b
\end{aligned}
$$

General Construction Contd.

- Using eqn 1,

$$
\begin{aligned}
\hat{T}^a = 0 &= d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b + \hat{\omega}^a{}_z \wedge \hat{e}^z \\
\text{or, } d\hat{e}^a &= -\hat{\omega}^a{}_b \wedge \hat{e}^b - \hat{\omega}^a{}_z \wedge \hat{e}^z \\
&= -\exp(\alpha\phi)\hat{\omega}^a{}_b \wedge e^b - \exp(\beta\phi)\hat{\omega}^a{}_z \wedge (dx^4 + A_\mu dx^\mu) \\
&= -\exp(\alpha\phi)\hat{\omega}^a{}_b \wedge e^b - \exp(\beta\phi)\hat{\omega}^a{}_z \wedge (dx^4 + \mathcal{A}), \quad \mathcal{A} := A_\mu dx^\mu
\end{aligned}
$$

- Comparing like terms from two ways:

$$
\begin{aligned}
\alpha \exp(\alpha\phi)\partial_\mu\phi(dx^\mu \wedge e^a) &= -\exp(\beta\phi)\hat{\omega}^a{}_z \wedge (dx^4 + \mathcal{A}) \\
\therefore \alpha \exp(\alpha\phi)\partial_\mu\phi(e^a \wedge dx^\mu) &= \exp(\beta\phi)\hat{\omega}^a{}_z \wedge (dx^4 + \mathcal{A}) \qquad (6) \\
\text{and, } \hat{\omega}^a{}_b \wedge e^b &= \omega^a{}_b \wedge e^b \qquad (7)
\end{aligned}
$$

General Construction Contd.

- Similary,

$$
\begin{aligned}
d\hat{e}^z &= d[\exp(\beta\phi)(dx^4 + \mathcal{A})] \\
&= \beta \exp(\beta\phi) d\phi \wedge (dx^4 + \mathcal{A}) + \exp(\beta\phi) d^2 x^4 + \exp \beta\phi d\mathcal{A} \\
&= \beta \exp(\beta\phi) \partial_\mu \phi dx^\mu \wedge (dx^4 + \mathcal{A}) + \exp(\beta\phi)\mathcal{F}, \quad \mathcal{F} := d\mathcal{A}
\end{aligned}
$$

  where $d^2 x^4$ term disappear because $d^2\Phi = 0$ for any differential form Φ and x^4 is zero form.

- Using eqn 2,

$$
\begin{aligned}
\hat{T}^z = 0 &= d\hat{e}^z + \hat{\omega}^z{}_b \wedge \hat{e}^b + \hat{\omega}^z{}_z \wedge \hat{e}^z \\
\text{or, } d\hat{e}^z &= -\exp(\alpha\phi)\hat{\omega}^z{}_b \wedge e^b - \hat{\omega}^z{}_z \wedge \hat{e}^z
\end{aligned}
$$

General Construction Contd.

- Comparing like terms from two ways:

$$\hat{\omega}^z{}_z = \hat{\omega}^{zz} = 0,$$

$$\hat{\omega}^z{}_c = -\beta \exp(-\alpha\phi)\partial^a\phi\hat{e}^z - \frac{1}{2}\mathcal{F}^a{}_b \ \exp\{(\beta - 2\alpha)\phi\}\hat{e}^b,$$

$$\hat{\omega}^{cz} = -\hat{\omega}^{zc} = \beta \exp(-\alpha\phi)\partial_a\phi\hat{e}^z + \frac{1}{2}\mathcal{F}_{ac} \exp[(\beta - 2\alpha)\phi]\hat{e}^c$$

where $\mathcal{F}_{ac}$ is the components of $\mathcal{F}$ in the co-ordinate-free basis. i.e. Electromagnetic tensor. Hint: See this arXiv:1711.09503v1 [gr-qc]

Curvature two-form

- Using all the Machinery we built, in eqn (5):

$$
\begin{aligned}
\hat{R}^z{}_z &= \hat{\omega}^z{}_z + \hat{\omega}^z{}_c \wedge \hat{\omega}^c{}_z + \hat{\omega}^z{}_z \wedge \hat{e}^z \\
\text{or, } \hat{R}^z{}_z &= \hat{\omega}^z{}_c \wedge \hat{\omega}^c{}_z \\
\text{or, } \hat{\eta}_{zz}\hat{R}^{zz} &= \hat{\omega}^z{}_c \wedge \hat{\omega}^{cz}\hat{\eta}_{zz} \\
\text{or, } \hat{R}^{zz} &= \hat{\omega}^z{}_c \wedge \hat{\omega}^{cz} \\
&= -[\beta \exp(-\alpha\phi)\partial^a\phi\hat{e}^z + \frac{1}{2}\mathcal{F}^a{}_b \exp\{(\beta - 2\alpha)\phi\}\hat{e}^b] \wedge \\
&\quad [\beta \exp(-\alpha\phi)\partial_a\phi\hat{e}^z + \frac{1}{2}\mathcal{F}_{ac} \exp\{(\beta - 2\alpha)\phi\}\hat{e}^c] \\
\therefore \hat{R}_{zz} &= \beta^2 \exp(-2\alpha\phi)\Box\phi + \frac{1}{4}\exp\{2(\beta - 2\alpha)\phi\}\mathcal{F}^2
\end{aligned}
$$

where, $\mathcal{F}^2 := \mathcal{F}^{ab}\mathcal{F}_{ab}$, $\Box := \partial_a\partial^a = \partial_\mu\partial^\mu$:= d'Alembert operator or Laplacian operator in 4D.

Curvature two-form Contd.

- Remaining curvature two-forms [Pope, 2003, Miller, 2013]:

$$\hat{R}_{ab} = \exp(-2\alpha\phi)[R_{ab} + (\alpha\beta - 4\alpha^2)\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi] - \frac{1}{2}\exp\{2(\beta - 2\alpha)\phi\}\mathcal{F}_a{}^c\mathcal{F}_{bc}$$

and, $\hat{R}_{az} = \hat{R}_{za} = \frac{1}{2}\exp\{(\beta - \alpha)\phi\}\nabla^b[\exp\{2(\alpha + \beta)\phi\}\mathcal{F}_{ab}]$

where R_{ab} is curvature two-form on $\mathcal{M}^{(4)}, \eta_{ab}$ is 4D Minkowski metric, and ∇^b is covariant derivative on co-ordinate free basis.

5D Ricci scalar

- Trace of the curvature 2-form

$$\hat{R} = \hat{\eta}^{AB}\hat{R}_{AB} = \eta^{ab}\hat{R}_{ab} + \hat{R}_{zz}$$
$$= \eta^{ab}[\exp(-2\alpha\phi)\{R_{ab} + (\alpha\beta - 4\alpha^2)\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi\} -$$
$$\frac{1}{2}\exp\{2(\beta-2\alpha)\phi\}\mathcal{F}_a{}^c\mathcal{F}_{bc}] +$$
$$\beta^2\exp(-2\alpha\phi)\Box\phi + \frac{1}{4}\exp\{2(\beta-2\alpha)\phi\}\mathcal{F}^2$$
$$= \exp(-2\alpha\phi)\{\eta^{ab}R_{ab} + (\alpha\beta - 4\alpha^2)(\partial\phi)^2 - \alpha\eta_a^a\Box\phi\} -$$
$$\frac{1}{2}\exp\{2(\beta-2\alpha)\phi\}\mathcal{F}^2 + \beta^2\exp(-2\alpha\phi)\Box\phi + \frac{1}{4}\exp\{2(\beta-2\alpha)\phi\}\mathcal{F}^2$$
$$\therefore \hat{R} = \exp(-2\alpha\phi)\{R + (\alpha\beta - 4\alpha^2)(\partial\phi)^2 - (\beta^2 - 2\alpha)\Box\phi\}$$
$$-\frac{1}{4}\exp\{2(\beta-2\alpha)\phi\}\mathcal{F}^2 \tag{8}$$

Lagrangian Density

- Easily verify: $\sqrt{-\hat{g}} = \sqrt{-\det(g_{\hat{A}B})} = \exp\{(\beta + 4\alpha)\phi\}\sqrt{-g}$
- Lagrangian density (Setting $\frac{1}{16\pi G} = 1$):

$$\begin{aligned}\mathcal{L} &:= \sqrt{-\hat{g}}\hat{R} \\ &= \sqrt{-g}\exp\{(\beta+2\alpha)\phi\}[R + (\alpha\beta - 4\alpha^2)(\partial\phi)^2 + (\beta^2 - 2\alpha)\Box\phi] - \\ &\quad \frac{1}{4}\sqrt{-g}\exp(3\beta\phi)\mathcal{F}^2\end{aligned}$$

- In-order to include 4D Einstein gravity with its Lagrangian has the form $\sqrt{-g}R$ [Pope, 2003]. i.e. $\beta := -2\alpha$
- Our beautiful Lagrangian becomes

$$\mathcal{L} = \sqrt{-g}[R - 6\alpha^2(\partial\phi)^2 + (4\alpha^2 - 2\alpha)\Box\phi - \frac{1}{4}\exp(-6\alpha\phi)\mathcal{F}^2]$$

Lagrangian Density Contd.

- $-6\alpha^2(\partial\phi)^2$ equivalent to $-\frac{1}{2}(\partial\phi)^2$ is a Klein-Gordon Lagrangian for a massless scalar field. Extract $\alpha = \frac{1}{\sqrt{12}}$.
- Drop $\Box\phi$ term. ($\because$ It gives total derivative in $\mathcal{L}$ which does not contribute to the field equation [Pope, 2003])
- Final version of our beautiful Lagrangian becomes:

$$\mathcal{L} = \sqrt{-g}[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2] \tag{9}$$

- Set $\phi = 0$ retrieves Einstein-Maxwell Lagrangian:

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{4}\mathcal{F}^2)$$

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- But setting $\phi = 0$ is illegal ($\because$ $\hat{g}_{AB}$ will become non-invertible).

Ingredients for Field Equation

- For simplicity, assume 5D spacetime has no boundary
- Remember from General relativity, $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$
- The variation on action is:

$$\begin{aligned}
\delta S_{\text{Kaluza}} &= \int d^5x\delta\mathcal{L} \\
&= \int d^5x\delta\left[\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right)\right] \\
&= \underbrace{\int \sqrt{-g}\delta\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right)d^5x}_{\delta S_1} + \\
&\quad \underbrace{\int\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right)\delta\sqrt{-g}\,d^5x}_{\delta S_2}
\end{aligned}$$

Field Equation on-process

$$\delta S_1 = \int \sqrt{-g}\,\delta\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right) d^5x$$

$$= \int \sqrt{-g}\,\delta\left(g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{4}\exp(-\sqrt{3}\phi)g^{\mu\rho}g^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\right) d^5x$$

$$= \int \sqrt{-g}\left(g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)\partial_\nu\delta\phi\right) d^5x$$

$$-\frac{1}{4}\int \sqrt{-g}\exp(-\sqrt{3}\phi)\left(-\sqrt{3}\mathcal{F}^2\delta\phi + 2\mathcal{F}_{\mu\rho}\mathcal{F}^{\nu\rho}\delta g^{\mu\nu} + 2\mathcal{F}^{\mu\nu}\delta\mathcal{F}_{\mu\nu}\right) d^5x$$

$$\delta S_2 = \int \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right)\delta\sqrt{-g}\,d^5x$$

$$= -\frac{1}{2}\int \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2\right) g_{\alpha\beta}\delta g^{\alpha\beta}\,d^5x$$

Field Equation on-process

- Setting variation of action = 0, this results

$$0 = \int \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} d^5x \tag{10}$$

$$0 = -\frac{1}{2} \int \sqrt{-g} \exp(-\sqrt{3}\phi) \mathcal{F}^{\mu\nu} \delta \mathcal{F}_{\mu\nu} d^5x \tag{11}$$

$$0 = \int \sqrt{-g} \left(\frac{1}{4}\sqrt{3}\mathcal{F}^2 - \frac{1}{2}(\partial^\mu \phi)\partial_\mu \right) \delta\phi d^5x \tag{12}$$

$$\begin{aligned} 0 = \int \sqrt{-g} \Bigg[& R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \left((\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu} \right) - \\ & \frac{1}{2} \left(\mathcal{F}^2_{\mu\nu} - \frac{1}{4} \mathcal{F} g_{\mu\nu} \right) \Bigg] \delta g^{\mu\nu} d^5x \end{aligned} \tag{13}$$

where $\mathcal{F}^2_{\mu\nu} := \mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}$

Serving Field Equation

- Eqn 10 is integral over boundary thus, vanishes.
- Remaining three integrals defined our field equations:

$$0 = \nabla^{\mu}\left(\exp(-\sqrt{3}\phi)\mathcal{F}_{\mu\nu}\right) \tag{14}$$

$$\Box\phi = -\frac{\sqrt{3}}{4}\exp(-\sqrt{3}\phi)\mathcal{F}^2 \tag{15}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}\left(\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{1}{2}\left(\mathcal{F}^2_{\mu\nu} - \frac{1}{4}\mathcal{F}^2 g_{\mu\nu}\right) \tag{16}$$

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- If $\phi = 0, \mathcal{F} = 0$ then, eqn 16 reduces to 4D Einstein pure gravity with no EM field

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Thank You For Listening!
I Invite You To Ask Any Questions You May Have...

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