Generalised spectral dimensions in non-perturbative quantum gravity

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Based on *Classical and Quantum Gravity* **40** (2023) 125003 [arXiv:2207.05117]

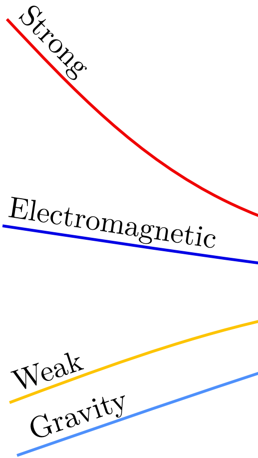
In-Collaboration with Marcus Reitz, Dániel Németh, Andrzej Görlich, Jakub Gizbert-Studnicki

Differential Geometry Seminars 2023
School of Mathematical Sciences – The University of Adelaide
12 May 2023

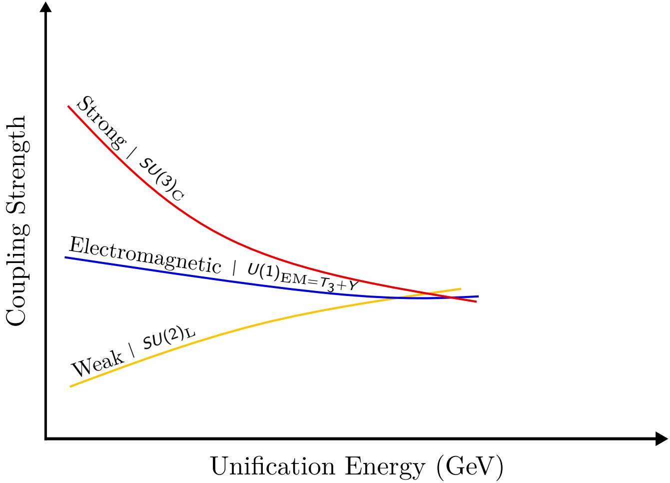
What you will see in this talk?

- **Part 1**: The Motivation
- **Part 2**: The Model
- **Part 3**: The Observable
- **Part 4**: The Result

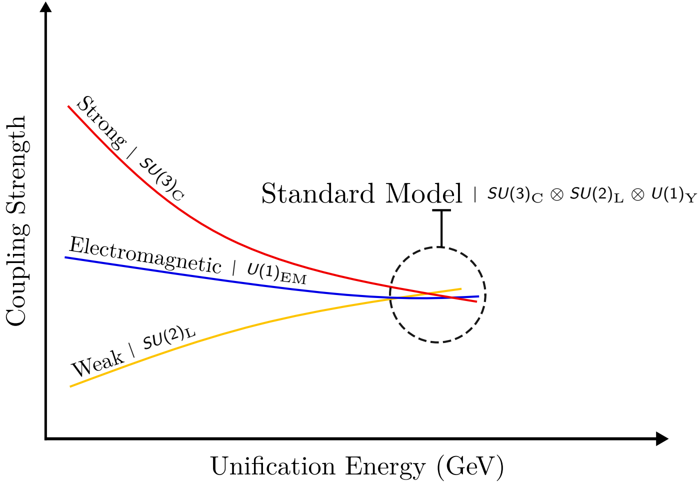
Part 1 Why Quantum Gravity?



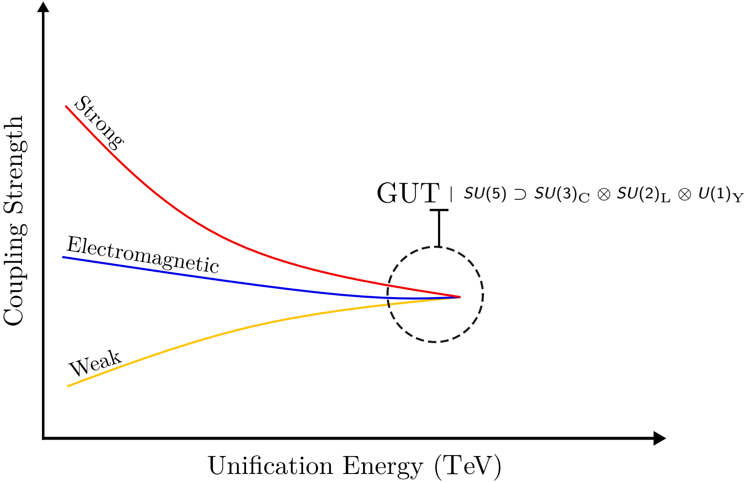
So far we know in the Theoretical High-Energy Physics



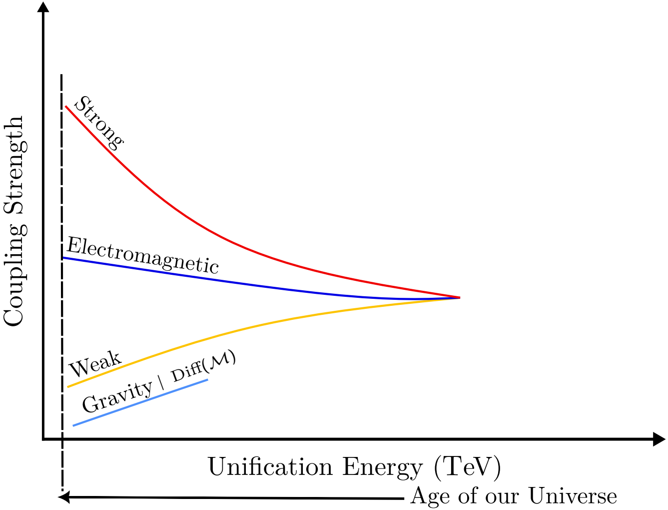
Well known Periodic Table for Physicists



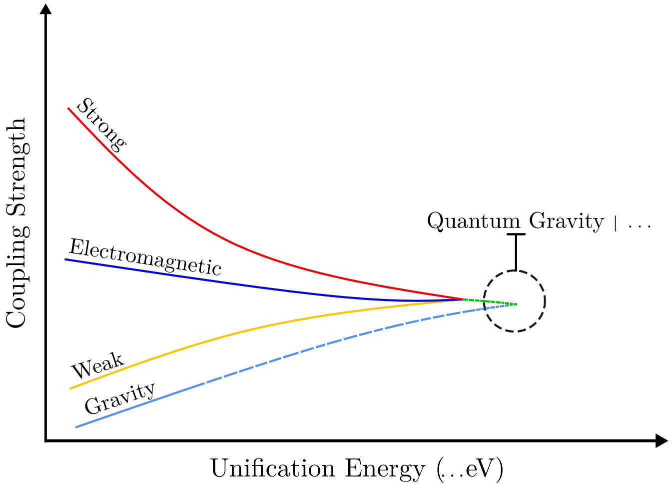
Beyond Standard Model . . .



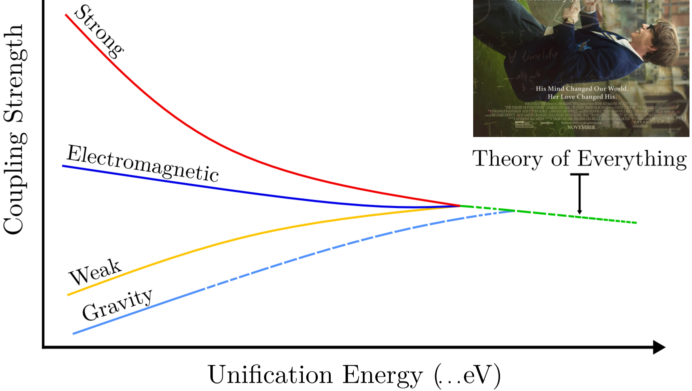
Our Present Cosmos

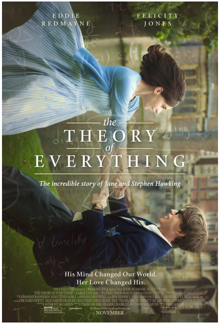


Does this mean Gravity has Quantum Sector?

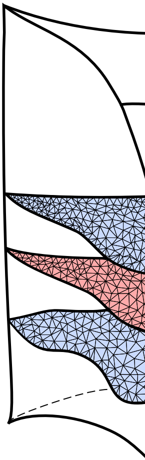


Is this too far?





Part 2 Causal Dynamical Triangulations



Non-perturbative Quantum Mechanics [Feynman, Rev. Mod. Phys. 20, 367 (1948)]

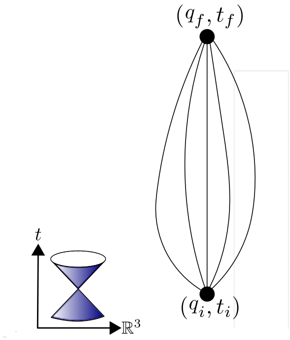


Figure: (1+1)-dimensional Quantum paths

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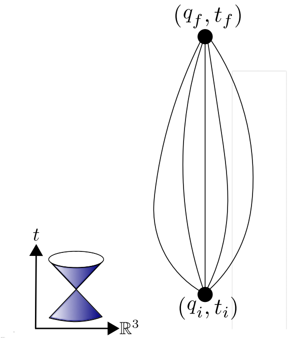

Figure: (1+1)-dimensional Quantum paths

Probability amplitude in the configuration of paths,

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q_i}^{q_f} \mathcal{D}[q] \; e^{iS[q]/\hbar}$$

where

$$S[x] = \int_{t_i}^{t_f} \mathcal{L} \, \mathrm{d}t$$

is the classical action. The observable $\mathcal{O}$ is

$$\langle q_f, t_f | \hat{\mathcal{O}} | q_i, t_i \rangle = \int_{q_i}^{q_f} \mathcal{D}[q] \; \mathcal{O} e^{iS[q]/\hbar}$$

Non-perturbative Quantum Gravity [Misner, Rev. Mod. Phys. 29, 497 (1957)]

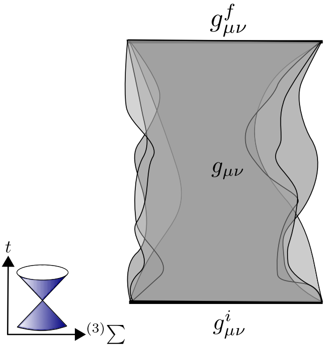

Figure: (3+1)-dimensional Quantum spacetimes

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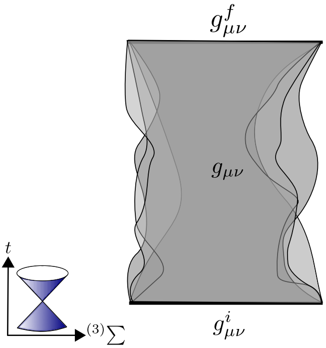

Figure: (3+1)-dimensional Quantum spacetimes

Probability amplitude (formal) in the configuration of (pure) spacetime geometries,

$$\langle g^f_{\mu\nu} | g^i_{\mu\nu} \rangle := \sum_{\mathcal{M} \in \mathrm{Top.}} \int_{\mathcal{M}} \mathcal{D}[g_{\mu\nu}] \; e^{iS[g_{\mu\nu}]/\hbar}$$

where

$$S[g_{\mu\nu}] = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{GHY}}[g_{\mu\nu}]$$

such that

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{-g}(R - 2\Lambda),$$

$$S_{\mathrm{GHY}}[g_{\mu\nu}]^{\dagger\dagger} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \mathrm{d}^3 y \, \sqrt{h} K$$

†† Gibbons & Hawking, Phys. Rev. D 15, 2752 (1977)

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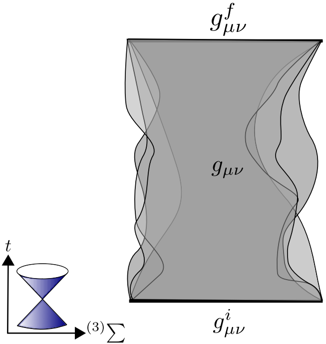

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- For simplicity, assume $(\mathcal{M}, g_{\mu\nu})$ closed with fixed topology.

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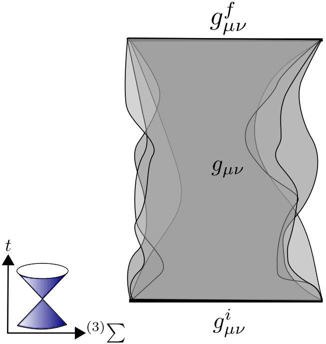

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The observable $\mathcal{O}$ (formal) is

$$\langle g^f_{\mu\nu} | \hat{\mathcal{O}} | g^i_{\mu\nu} \rangle := \int_{\mathcal{M}} \mathcal{D}[g_{\mu\nu}] \; \mathcal{O}[g_{\mu\nu}] e^{iS[g_{\mu\nu}]/\hbar}$$

Regge Calculus [Regge, Nuovo Cim 19, 558–571 (1961)]

- $S_{\mathrm{EH}}[g_{\mu\nu}]$ is sum of total curvature $\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}R$ and total volume $\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}$.

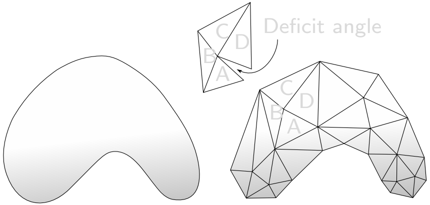

Figure: Illustration† of Regge Calculus in Lorentzian manifold. Left: Manifold $\mathcal{M}$. Right: Simplicial Manifold $\mathcal{T}$ (Piece-wise linear geometry)

† Inspired from Misner et al., Gravitation (1973)

- Curvature is located at σ_0^{d-2}.

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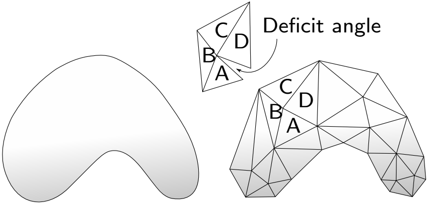

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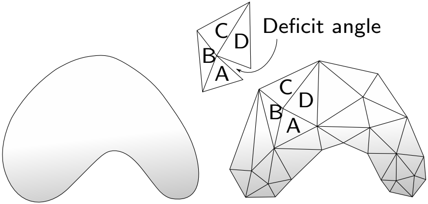

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- Deficit angle $\epsilon_{\sigma_0^{d-2}}$ defines the curvature at σ_0^{d-2} with dihedral angle $\theta := \theta(\sigma_i^d, \sigma_0^{d-2})$. i.e.

$$\epsilon_{\sigma_0^{d-2}} = \left(2\pi - \sum_{\sigma^d \ni \sigma_0^{d-2}} \theta(\sigma_i^d, \sigma_0^{d-2}) \right) e^{i\phi(\sigma_0^{d-2})} \in \mathbb{C}$$

where phase $\phi(\sigma_0^{d-2}) = \begin{cases} 0 & \text{if } \sigma_0^{d-2} \text{ is timelike} \\ -\pi/2 & \text{if } \sigma_0^{d-2} \text{ is spacelike} \end{cases}$

Causal Dynamical Triangulations (CDT) [Ambjørn et al., Nucl.Phys. B610 347-382 (2001)]

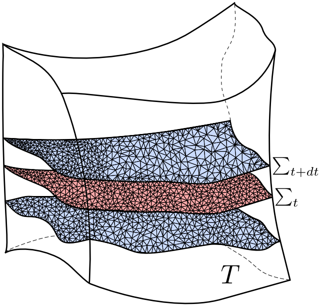

Figure: CDT spacetime T foliated by a family of space-like (Cauchy) hypersurfaces $(\sum_t)_{t\in\mathbb{Z}}$. Similar like ADM formalism in classical GR.

- 4D CDT spacetime T is diffeomorphic to $\mathcal{M} = R \times \sum$
- Given the lattice cutoff a, introduce asymmetry between space and time (α, a new coupling in the model):

$$\text{if } V_{\sigma^1_{\text{spacelike}}} = a \text{ then } (V_{\sigma^1_{\text{timelike}}})^2 = -\alpha a^2 \mid \alpha > 0.$$

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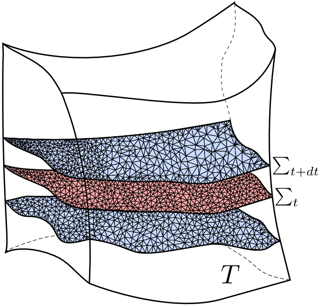

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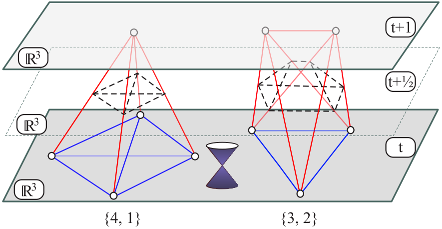

Figure: Visualization of fundamental building blocks in 4D CDT. Dotted line illustrates the cross-section of σ^4. Blue σ^1 is spacelike. Red σ^1 is timelike.

- Using Regge calculus and topological identities gives (CDT) Regge action:

$$S_R^{(L)}[T] = -(\kappa_0 + 6\Delta)\ N_0 + \kappa_4\ (N_{\{4,1\}} + N_{\{3,2\}}) + \Delta\ N_{\{4,1\}}$$

where L mean in Lorentzian spacetime, κ_0 is (inverse) bare gravitational constant, κ_4 is bare cosmological constant, and Δ is asymmetry parameter.

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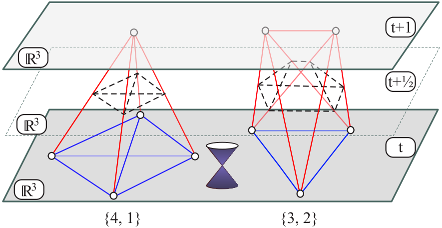

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CDT Monte-Carlo Simulation

[CDT Handbook: Ambjørn et al., Phys. Rep. 519 127-210 (2012)]

- Perform analytic continuation of time to Euclidean (E) signature (i.e. Wick rotation), results into

$$dt^{(L)} \mapsto dt^{(E)} = i\ dt^{(L)} \implies (V_{\sigma^1_{\text{timelike}}})^2 = -\alpha a^2 \mapsto \tilde{\alpha} a^2 \mid \tilde{\alpha} > 0$$

- Upon the constraint (in 4D) $\tilde{\alpha} > \sqrt{\frac{7}{12}}$ gives $S_R^{(L)}[T] = i\ S_R^{(E)}[T]$ so

$$\langle g^f_{\mu\nu} | g^i_{\mu\nu} \rangle \equiv \mathcal{Z} := \sum_T \frac{1}{C_T} e^{-S_R^{(E)}[T]} \quad \text{(The Partition Function)}$$

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Renormalization Group Flow in CDT [Ambjørn, Gizbert-Studnicki et al., EPJC 77 152 (2017)]

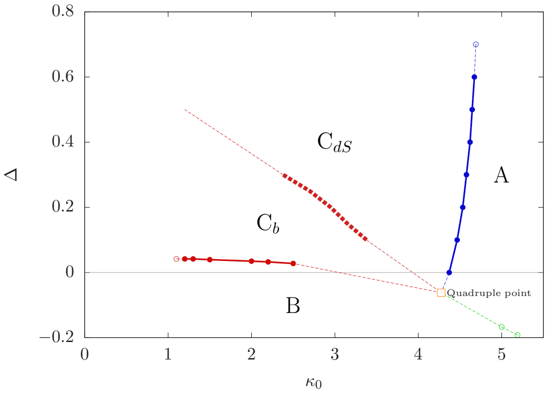

Figure: CDT phase diagram in terms of the bare couplings κ_0 & Δ, with $T^4 = S^1 \times S^1 \times S^1 \times S^1$, and after fine tuning κ_4 so that triangulation fluctuate around target volume.

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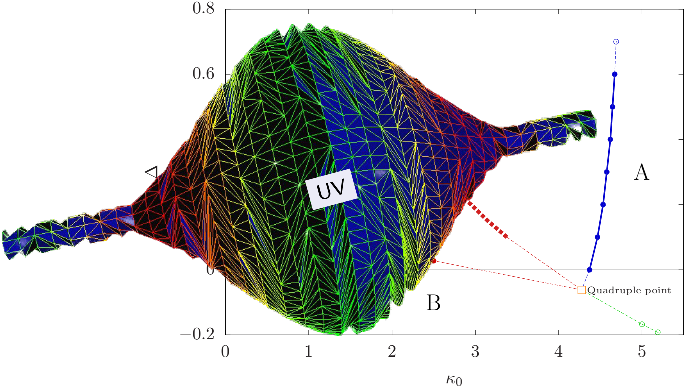

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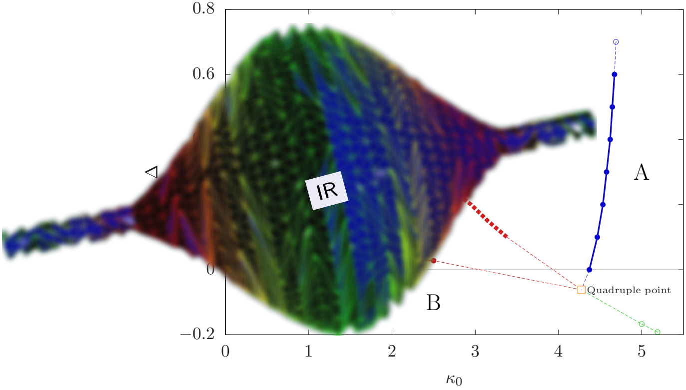

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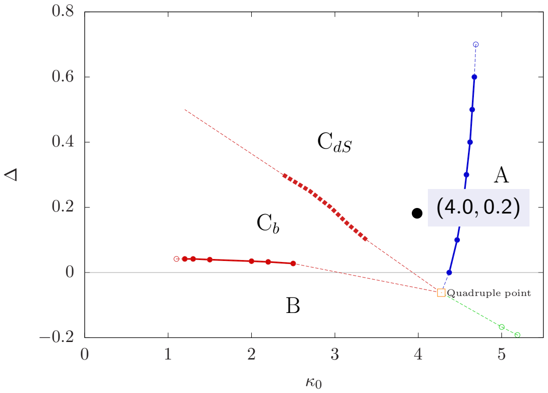

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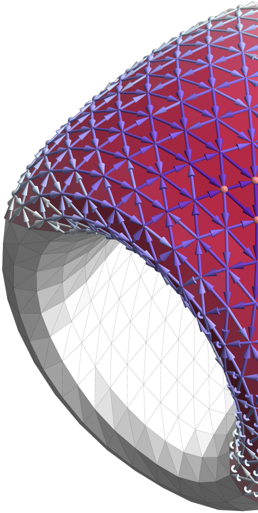
Part 3 Generalised Spectral Dimension



Continuum tensor diffusion

[Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)] [Gilkey - Asymptotic Formulae in Spectral Geometry (2003)]

Let $(\mathcal{M}, g)$ be a closed, connected, smooth oriented, n-dimensional Riemannian manifold. Let $\Lambda^k T^*\mathcal{M}$ is k-exterior (co-tangent) bundle over $\mathcal{M}$. So, the space of smooth k-forms $A^k(\mathcal{M})$ is the section of $\Lambda^k T^*\mathcal{M}$.

Definition. Smooth Differential k-form ω on $\mathcal{M}$ is a linear map

$$\omega : \mathfrak{X}_k(\mathcal{M}) \mapsto C^\infty(A^k(\mathcal{M}))$$

where $\mathfrak{X}_k(\mathcal{M})$ is the Lie-algebra of all smooth tensor field of type $(0, k)$.

Definition. Exterior Algebra $A(\mathcal{M})$ of smooth differential forms on $\mathcal{M}$ is defined as the grading

$$A(\mathcal{M}) := \bigoplus_{k=0}^{n} A^k(\mathcal{M}).$$

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$$A(\mathcal{M}) := \bigoplus_{k=0}^{n} A^k(\mathcal{M}).$$

Continuum tensor diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Definition. Hodge-de Rham operator (or the Laplace-Beltrami operator on k-forms, or the k-Laplacians), an elliptic and self-adjoint operator which acts on $A^k(\mathcal{M})$, denoted by

$$\Delta^{(k)} : A(\mathcal{M}) \ni A^k(\mathcal{M}) \mapsto A^k(\mathcal{M}) \in A(\mathcal{M})$$

is defined by

$$\Delta^{(k)} : A^k(\mathcal{M}) \ni \omega \mapsto \Delta^{(k)}\omega := d^{k-1} \circ \delta^k(\omega) + \delta^{k+1} \circ d^k(\omega) \in A^k(\mathcal{M}).$$

Note. $\Delta^{(0)}$ is the Laplace-Beltrami operator Δ.

Remark. $\langle \Delta^{(k)}\omega, \omega \rangle \geq 0 \ \forall \ \omega \in L^2 A^k(\mathcal{M})$, and has (purely) point spectrum.

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Continuum tensor diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Definition. Heat Bundle over the base manifold $\mathcal{M} \times \mathcal{M} \times \mathbb{R}^+$ consists of a bundle manifold $\Lambda^k T^*\mathcal{M} \otimes \Lambda^k T^*\mathcal{M} \times \mathbb{R}^+$ with a surjective bundle projection

$$\pi : \Lambda^k T^*\mathcal{M} \otimes \Lambda^k T^*\mathcal{M} \times \mathbb{R}^+ \mapsto \mathcal{M} \times \mathcal{M} \times \mathbb{R}^+$$

such that at point $(x, y; \tau) \in \mathcal{M} \times \mathcal{M} \times \mathbb{R}^+$, the fiber $\pi^{-1}(x, y; \tau) = \Lambda^k T_x^*\mathcal{M} \otimes \Lambda^k T_y^*\mathcal{M} \times \mathbb{R}^+$.

Definition. Heat k-kernel is a smooth section $K^{(k)}$ of a heat bundle such that at point $(x, y; \tau) \in \mathcal{M} \times \mathcal{M} \times \mathbb{R}^+$ gives (double) (k,k)-form $K^{(k)}(x, y; \tau) \in \pi^{-1}(x, y; \tau)$ and satisfy:

$$\left(\Delta^{(k)} + \partial_\tau\right) K^{(k)} = 0$$
$$\implies K^{(k)}(x, y; \tau) = \exp\left(-\tau\Delta^{(k)}\right) K^{(k)}(x, y; \tau = 0)$$

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Continuum tensor diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Remark. $\forall y \in \mathcal{M}, \tau \in (0, +\infty)$, and with Borel probability measure $\mu_g(x)$ then,

$$\int_{\mathcal{M}} K^{(k)}(x, y; \tau)\, \mathrm{d}\mu_g(x) = 1.$$

i.e. total probability is preserved over time.

Definition. Heat Trace $\tilde{K}^{(k)}(\tau)$ is defined as

$$\tilde{K}^{(k)}(\tau) := \int_{\mathcal{M}} K^{(k)}(x, x; \tau)\, \mathrm{d}\mu_g(x) = \sum_i e^{-\tau \lambda_i^{(k)}} \equiv \mathrm{Tr}\left(e^{-\tau \Delta^{(k)}}\right)$$

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Definition. <u>(Average) Return Probability</u> $P^{(k)}(\tau)$, defined as

$$P^{(k)}(\tau) := \frac{\tilde{K}^{(k)}(\tau)}{Vol(\mathcal{M}, g)}$$

where $\mathrm{Vol}(\mathcal{M}, g) := \int_{\mathcal{M}} \mathrm{d}\mu_g(x)$.

Continuum tensor diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Remark. Heat trace $\tilde{K}^{(k)}$ on $(\mathcal{M}, g)$ has the following asymptotic expansion as $\tau \searrow 0$:

$$\tilde{K}^{(k)}(\tau) \sim \tau^{-\frac{n}{2}} \sum_{i=0}^{\infty} B_i(\Delta) \tau^i$$

where $B_i(\Delta)$ are integrated C^∞ functions of the metric g, called heat k-coefficients.

Remark. Dimension of $\mathcal{M}$ can be extracted from the asymptotic expansion

$$n = -2 \frac{\mathrm{d} \log\left(P^{(k)}(\tau)\right)}{\mathrm{d} \log(\tau)} \quad \text{as } \tau \searrow 0.$$

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Definition (Physically Inspired). (Running) Generalised Spectral Dimension $D^{(k)}(\tau)$ is defined by dropping $\tau \searrow 0$ condition such that

$$D^{(k)}(\tau) := -2\frac{\mathrm{d}\log\left(P^{(k)}(\tau)\right)}{\mathrm{d}\log(\tau)}.$$

Discrete Tensor Diffusion [Desbrun, Hirani, Leok, Marsden - Discrete Exterior Calculus (2005)]

Let $\sigma^k = [\sigma_1^0, \ldots, \sigma_{k+1}^0]$ be an oriented simplex, with $k \geq 1$.

Definition. The orientation of σ gives each of the $(k-1)$-dimensional faces an induced orientation. For $k > 1$, if i is even the induced orientation of the face $\{\sigma_1^0, \ldots, \not\sigma_i^0, \ldots, \sigma_{k+1}^0\}$ is the same as the orientation of the oriented simplex $[\sigma_1^0, \ldots, \not\sigma_i^0, \ldots, \sigma_{k+1}^0]$ where $\not\sigma_i^0$ is ignored. Otherwise, it is the opposite one.

Example. For $k = 2$,

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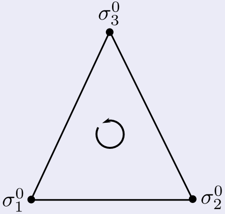

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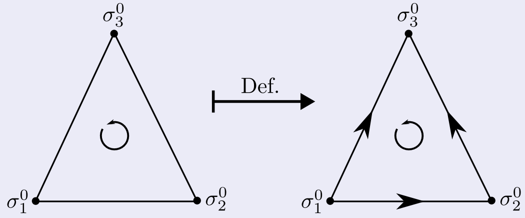

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Note. We chose: $\sigma_1^0 < \ldots < \sigma_{k+1}^0$.

Discrete Tensor Diffusion [Friedman - Computing Betti Numbers via Combinatorial Laplacians (1998)]

Definition. Incidence Matrix I_k^{k+1} between the k-dimensional sub-simplices σ^k and σ^{k+1} in a simplicial complex T is defined as

$$(I_k^{k+1})_{ij} = \begin{cases} 1 & \text{if } \sigma_i^k \prec \sigma_j^{k+1} \quad \text{with same orientation,} \\ -1 & \text{if } \sigma_i^k \prec \sigma_j^{k+1} \quad \text{with opposite orientation,} \\ 0 & \text{if } \sigma_i^k \not\prec \sigma_j^{k+1}. \end{cases}$$

Definition. Combinatorial k-Laplacian $L^{(k)}$ is a discrete counterpart of $\Delta^{(k)}$, defined on a space of discrete k-form fields $A^k(T)$, is a real symmetric (sparse) matrix of rank $\#\sigma^k \text{in}\, T$, given by

$$L^{(k)} = L_+^{(k)} + L_-^{(k)}, \quad L_+^{(k)} = (I_k^{k+1})^T I_k^{k+1}, \quad L_-^{(k)} = I_{k-1}^k (I_{k-1}^k)^T.$$

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Discrete tensor diffusion [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

Given a continuum heat k-kernel with continuous diffusion time τ,

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\begin{aligned}
K^{(k)}(x, y; \tau) &= \exp\left(-\tau \Delta^{(k)}\right) K^{(k)}(x, y; 0) \\
&\approx \exp\left(-sL^{(k)}\right)^{t} K^{(k)}(x, y; 0) \\
&\quad \text{where } t = \tau / s, s \in \mathbb{R}, t \in \mathbb{Z}^{+}, K^{(k)} \in A^{k}(T) \\
&= (\mathbb{I} - sL^{(k)} + \mathcal{O}(\tau^{2}))^{t} K^{(k)}(x, y; 0)
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Definition. Discrete heat k-kernel $K^{(k)}(x, y; \tau) \in A^{k}(T)$ is defined as a matrix

$$K^{(k)}(x, y; \tau + s) = (\mathbb{I} - sL^{(k)}) K^{(k)}(x, y; \tau)$$

where $\tau = ts, s \in \mathbb{R}, t \in \mathbb{Z}^{+}$, and $sL^{(k)}$ is a normalized k-Laplacian such that $\sum_{x} K^{(k)}(x, y; \tau) = 1 \; \forall \; y$.

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Proposition. For any $0 \leq k \leq n$, $0 < s < 2/\lambda_{\max}$ where $\lambda_{\max} \in \mathrm{Spec}(L^{(k)})$ such that it preserves $\sum_x K^{(k)}(x, y; \tau) = 1 \ \forall\ y$.

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Note. We chose $s = 1/\lambda_{\max}$.

Discrete tensor diffusion [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

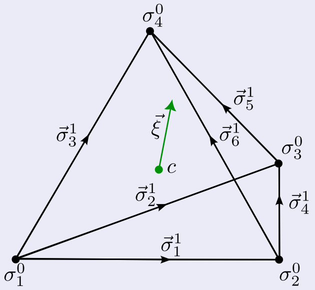
Definition. Discrete k-form tensor field is a surjective function $\sharp : \Omega^k(T) \to \mathfrak{X}^k(T)$,

$$(\alpha^k)^\sharp(\sigma_i^n) = \sum_{\sigma_j^k \prec \sigma_i^n} \alpha^k(\sigma_j^k)\vec{\sigma}_j^k,$$

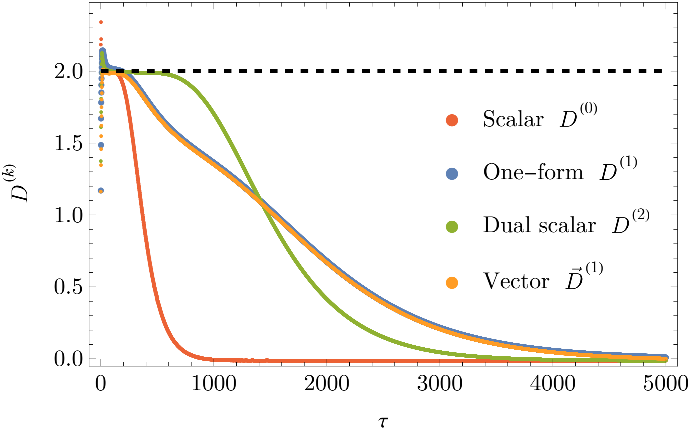
where $(\alpha^k)^\sharp(\sigma_i^n)$ is the value (in $\otimes^k \mathbb{R}^n$) of the discrete vector field $X^k = (\alpha^k)^\sharp$ in the simplex σ_i^n, $\alpha^k(\sigma_j^k)$ is the value (in $\mathbb{R}$) of the discrete k-form evaluated on the simplex σ_j^k, and $\vec{\sigma}_j^k$ is the k-tensor associated to the simplex σ_j^k.

Discrete tensor diffusion [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

Example. For $k = 3$, given a one-form field α^1, the corresponding vector field evaluated on σ^3 is given by $\vec{\xi} = \alpha^\sharp(\sigma^3) = \sum_{i=1}^{6} \alpha(\sigma_i^1)\vec{\sigma}_i^1$.



3 Lessons from flat 2-torus [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]



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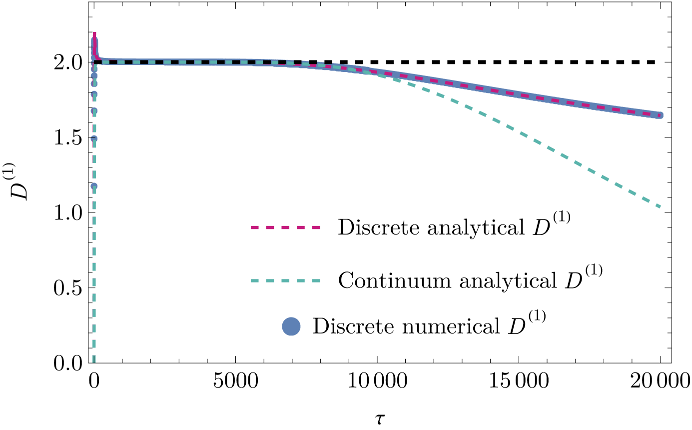
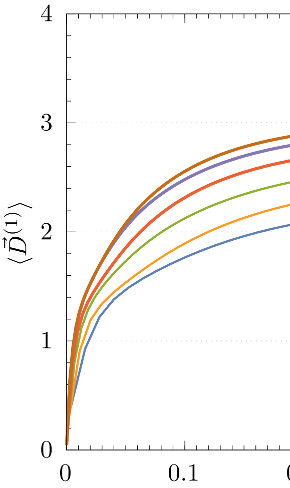


Part 4 Results



3D Spatial slice of 4D CDT [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

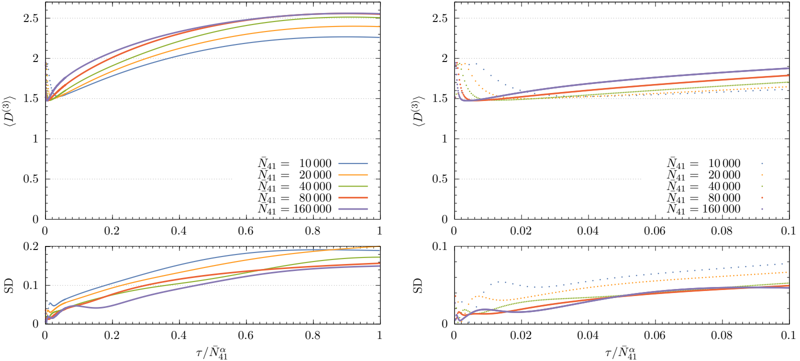


Figure: Upper right shows $\langle D^{(3)} \rangle$ for small diffusion times.

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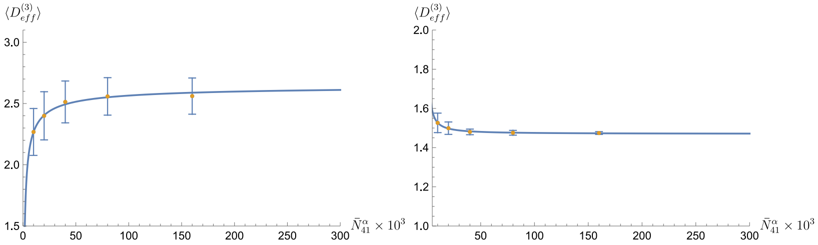

Figure: Power law fit of the form $a + b \cdot \bar{N}^{c}_{41}$. For large diffusion time (left), extrapolate an IR effective dimension $\langle D^{(3)}_{eff} \rangle = 2.66 \pm 0.21$ in the infinite volume limit. For short diffusion time (right), extrapolate a UV effective dimension $\langle D^{(3)}_{eff} \rangle = 1.47 \pm 0.01$.

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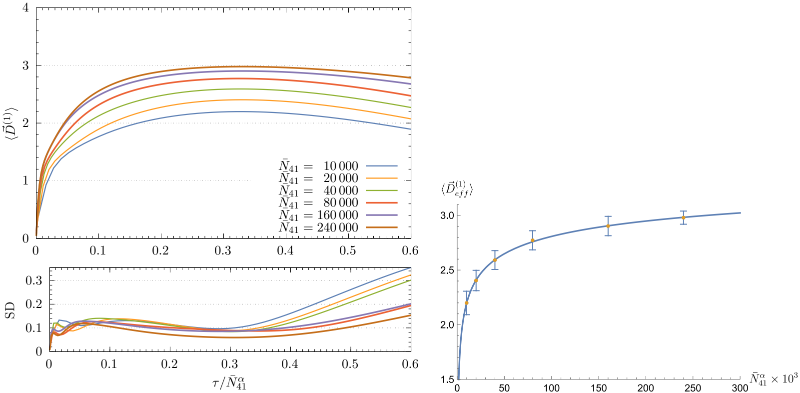

Figure: An effective dimension $\langle \vec{D}^{(1)}_{eff} \rangle = 4.03 \pm 0.45$ in the infinite volume limit.

Summary [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

Type	Effective Spectral Dimension
Scalar $\langle D^{(0)} \rangle$	~ 4
One-form $\langle D^{(1)} \rangle$	4.98 ± 0.36
Vector $\langle \vec{D}^{(1)} \rangle$	4.03 ± 0.45

Type	UV	IR
Two-form $\langle D^{(2)} \rangle$	1.44 ± 0.23	2.30 ± 0.64
Tensor $\langle \vec{D}^{(2)} \rangle$	1.27 ± 0.26	2.03 ± 0.11
Dual scalar $\langle D^{(3)} \rangle$	1.47 ± 0.01	2.66 ± 0.21

Note: All measurements done in 3D spacelike hypersurfaces of 4D CDT.

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Dual scalar $\langle D^{(3)} \rangle$	1.47 ± 0.01	2.66 ± 0.21

Note: All measurements done in 3D spacelike hypersurfaces of 4D CDT.

Thank You For Listening!
I Invite You To Ask Any Questions You May Have...
Interested to Learn CDT: PhysicsLog.com/CDT

NATIONAL SCIENCE CENTRE
POLAND

Research financed by the National Science Centre, Poland, under grant no. 2019/33/B/ST2/00589

ISQG Presents

Basics of Quantum Gravity

16 May 2023 to 16 November 2023

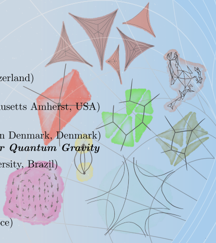
Online via Zoom

The Basics of Quantum Gravity online school is a series of lectures on the key concepts and techniques involved in the research of Quantum Gravity. It is dedicated to young researchers who are tackling this great challenge in theoretical physics. The goal is to bridge several paths and approaches to quantum gravity and provide young researchers with the tools to work on all of these frameworks and utilize various formalisms.

The launching lecture for this event will take place on May 16th. The event will run throughout Europe's spring and summer of 2023, featuring lectures on String Theory, Loop Gravity, the perturbative and non-perturbative renormalization flow of General Relativity, and quantum information in Gravity. Additional lectures on other topics will be planned for the fall and winter.

Lecture Series

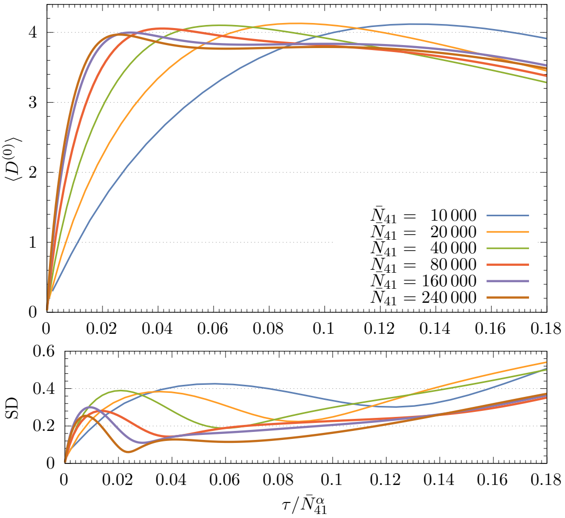
- Djordje MINIC (Virginia Tech, USA)
  The Challenges of Quantum Gravity
- Flaminia GIACOMINI (ETH Zürich, Switzerland)
  The Quantum of Quantum Gravity
- John DONOGHUE (University of Massachusetts Amherst, USA)
  Perturbative Quantum Gravity
- Astrid EICHHORN (University of Southern Denmark, Denmark)
  Non-perturbative Renormalization for Quantum Gravity
- Horatiu NASTASE (São Paulo State University, Brazil)
  String Theory
- Hal HAGGARD (Bard College, USA)
  Loop Quantum Gravity
- Andrea PUHM (École Polytechnique, France)
  AdS/CFT and Holography



Organizing Committee

• Johanna Borissova • Alicia Castro • Guilherme Franzmann • Etera Livine
• Luciano Petruzziello • Damodar Rajbhandari • V H Satheeshkumar

For more information about the school, please visit https://indico.cern.ch/e/BasicQG and scan the QR code to redirect to the registration page.



$\langle D^{(0)} \rangle$
SD
$\tau/\bar{N}_{41}^{\alpha}$
$\bar{N}_{41} = 10\,000$
$\bar{N}_{41} = 20\,000$
$\bar{N}_{41} = 40\,000$
$\bar{N}_{41} = 80\,000$
$\bar{N}_{41} = 160\,000$
$\bar{N}_{41} = 240\,000$

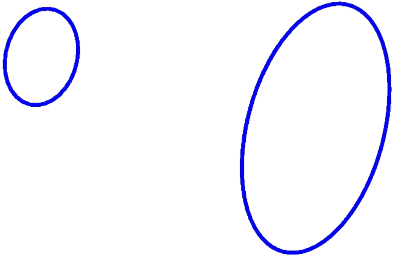
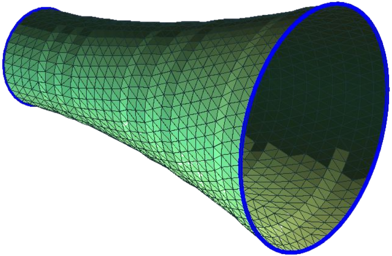


Figure: Illustration of (1+1) dimensional CDT Simulation . Credit: Andrzej Görlich



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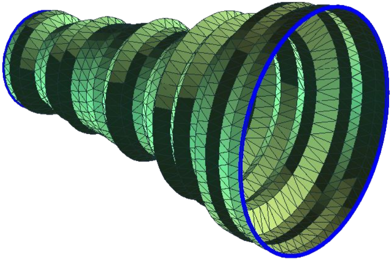

Figure: Illustration of (1+1) dimensional CDT Simulation when it's thermalized. Credit: Andrzej Görlich

Renormalization Group Flow in CDT [Ambjørn et al., Phys. Rep. 519 127-210 (2012)]

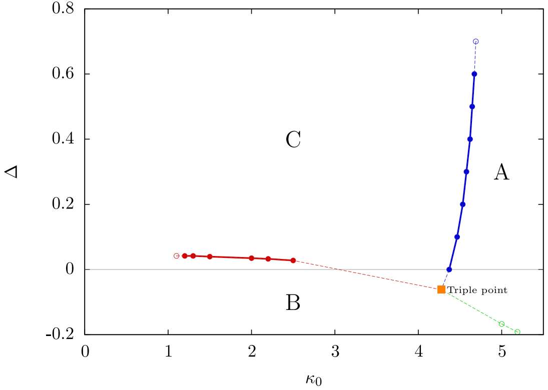

Figure: CDT phase diagram in terms of the bare couplings κ_0 & Δ, with $T^4 = S^1 \times S^1 \times S^1 \times S^1$, and after fine tuning κ_4 so that triangulation fluctuate around target volume

Renormalization Group Flow in CDT [Ambjørn et al., Phys. Rep. 519 127-210 (2012)]

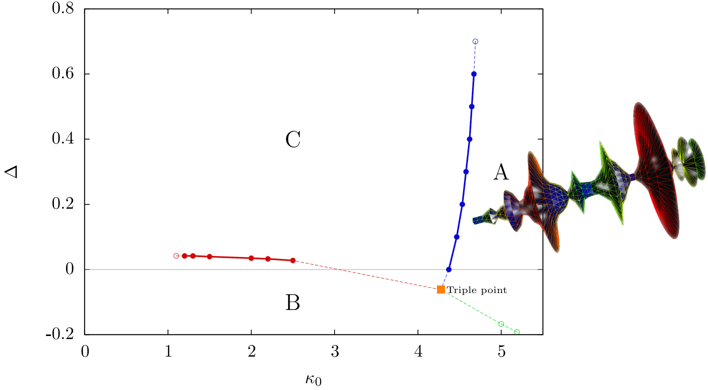

Figure: CDT phase diagram in terms of the bare couplings κ_0 & Δ, with $T^4 = S^1 \times S^1 \times S^1 \times S^1$, with phases: branched polymer phase (A)

Renormalization Group Flow in CDT [Ambjørn et al., Phys. Rep. 519 127-210 (2012)]

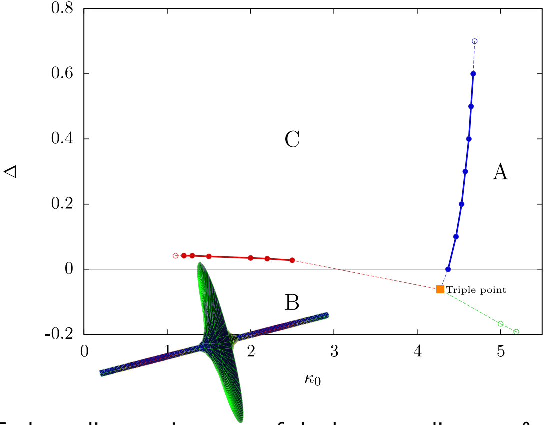

Figure: CDT phase diagram in terms of the bare couplings κ_0 & Δ, with $T^4 = S^1 \times S^1 \times S^1 \times S^1$, with phases: branched polymer phase (A), collapsed phase (B)

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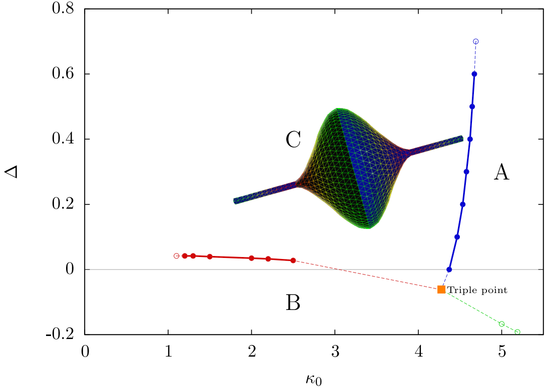
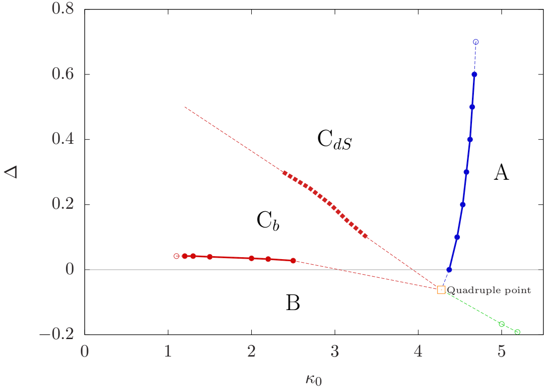


Figure: CDT phase diagram in terms of the bare couplings κ_0 & Δ, with $T^4 = S^1 \times S^1 \times S^1 \times S^1$, with phases: branched polymer phase (A), collapsed phase (B), the de Sitter (semi-classical) phase (C). Illustration Credit: Andrzej Görlich

Renormalization Group Flow in CDT [Ambjørn, Gizbert-Studnicki et al., EPJC 77 152 (2017)]



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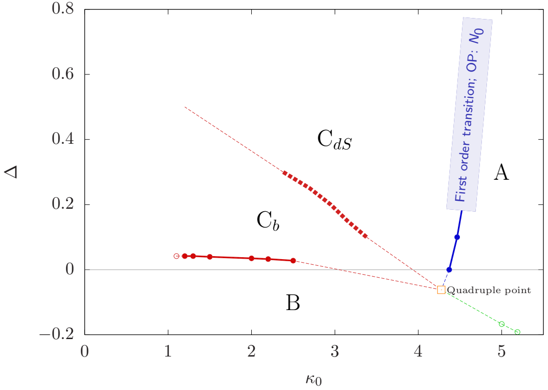

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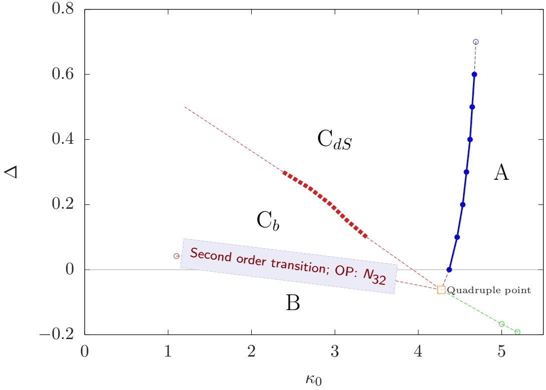

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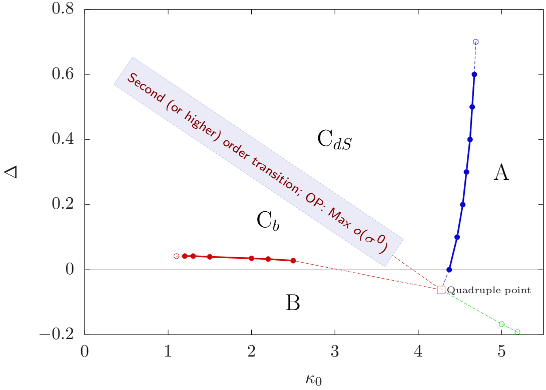

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Continuum scalar diffusion - A Revision

[Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)] [Grigor'yan - Heat Kernel and Analysis on Manifolds (2009)] [Rosenberg - The Laplacian on a Riemannian Manifold (1997)] [Chavel - Eigenvalues in Riemannian Geometry (1984)]

Definition. Laplace-Beltrami operator, a second-order elliptic self-adjoint partial differential operator which acts on $C^\infty(\mathcal{M})$, denoted by

$$\Delta : C^\infty(\mathcal{M}) \ni f \mapsto \Delta f \in C^\infty(\mathcal{M}),$$

defined by

$$\Delta f := -\operatorname{div}(\operatorname{grad}(f))$$

Remark. Let $\mathcal{M}$ is compact. $\operatorname{Spec}^{(0)}(\mathcal{M}, g)$ of Δ is (in most cases) discrete sequence of real numbers

$$0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \uparrow +\infty$$

. i.e. $\langle \Delta f, f \rangle \geq 0 \; \forall \; f \in L^2 C^\infty(\mathcal{M})$.

Continuum scalar diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

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$$\Delta f := -\operatorname{div}(\operatorname{grad}(f))$$

Remark. Let $\mathcal{M}$ be a closed. $\mathrm{Spec}^{(0)}(\mathcal{M}, g)$ of Δ is (purely) point spectrum

$$0 = \lambda_0 < \underbrace{\lambda_1}_{m_{\lambda_1}\text{times}} < \underbrace{\lambda_2}_{m_{\lambda_2}\text{times}} < \ldots \uparrow +\infty$$

where each eigenvalue $\lambda_j,\ j \in \mathbb{N}$ of Δ is repeated according to its (finite) multiplicity m_{λ_j}, and accumulates only at infinity.
Note. $\dim(\ker \Delta) = 1$ (zero modes).

Continuum scalar diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Definition. Heat Kernel (propagator) on closed n-dimensional $(\mathcal{M}, g)$ is a continuous function

$$K(x, y; \tau) : \mathcal{M} \times \mathcal{M} \times (0, +\infty) \mapsto \mathbb{R}$$

that satisfies:

1. K is $C^2(\mathcal{M})$ in x and y, and $C^1(\mathcal{M})$ in diffusion time τ.
2. $\left(\Delta + \frac{\partial}{\partial \tau}\right) K(x, y; \tau) = 0 \implies K(x, y; \tau) = \exp(-\tau\Delta) K(x, y; 0)$
3. $K(x, y; \tau) = K(y, x; \tau)$.
4. $\lim\limits_{\tau \downarrow 0+} K(\cdot, y; \tau) = \delta^{(n)}(\cdot - y) \; \forall \; y \in \mathcal{M}$, where $\delta^{(n)}$ is the Dirac delta function.
5. For any compactly supported function f on $\mathcal{M}$,

$$\lim_{\tau \downarrow 0+} \int_{\mathcal{M}} K(x, y; \tau) f(x) \, \mathrm{d}\mu_g(x) = f(y)$$

for every $y \in \mathcal{M}$, where $\mu_g(x)$ is a Borel probability measure.

Continuum scalar diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Remark. $\forall y \in \mathcal{M}, \tau \in (0, +\infty)$ then,

$$\int_{\mathcal{M}} K(x, y; \tau)\, \mathrm{d}\mu_g(x) = 1.$$

i.e. total probability is preserved over time.

Property. Heat Semi-group:

$$K(x, y; \tau_1 + \tau_2) = \int_{\mathcal{M}} K(x, z; \tau_1) K(z, y; \tau_2)\, \mathrm{d}\mu_g(z)\,.$$

Definition. Heat Trace $\tilde{K}(\tau)$ is spectral invariant and defined as

$$\tilde{K}(\tau) := \int_{\mathcal{M}} K(x, x; \tau)\, \mathrm{d}\mu_g(x) = \sum_i e^{-\tau\lambda_i} \equiv \mathrm{Tr}\big(e^{-\tau\Delta}\big)$$

Note. $\dim(\ker \Delta) = \lim_{\tau \to +\infty} \tilde{K}(\tau)$.

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Continuum scalar diffusion

[Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

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Definition. (Average) Return Probability $P(\tau)$, defined as

$$P(\tau) := \frac{\tilde{K}(t)}{\mathrm{Vol}(\mathcal{M}, g)} = \frac{\sum_i e^{-\tau\lambda_i}}{\mathrm{Vol}(\mathcal{M}, g)}$$

where $\mathrm{Vol}(\mathcal{M}, g) := \int_{\mathcal{M}} \mathrm{d}\mu_g(x)$.

Continuum scalar diffusion [Craioveanu et al. - Old and New Aspects in Spectral Geometry (2001)]

Remark. If $\mathcal{M} \equiv \mathbb{R}^n$ then, $K : \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty) \mapsto \mathbb{R}$, defined by $K(x, y; \tau) = (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4\tau}}$ is Gauss-Weierstrass kernel, and $\tilde{K}(\tau) \sim (\tau)^{-\frac{n}{2}}$.

Remark. Heat trace $\tilde{K}$ on $(\mathcal{M}, g)$ has the following asymptotic expansion as $\tau \searrow 0$:

$$\tilde{K}(\tau) \sim \tau^{-\frac{n}{2}} \sum_{i=0}^{\infty} A_i(\Delta) \tau^i$$

where $A_i(\Delta)$ are integrated C^∞ functions of the metric g, called heat coefficients.

Remark. Dimension of $\mathcal{M}$ can be extracted from the asymptotic expansion

$$n = -2 \frac{\mathrm{d} \log(P(\tau))}{\mathrm{d} \log(\tau)} \quad \text{as } \tau \searrow 0.$$

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Continuum scalar diffusion [Reitz, Németh, Rajbhandari, Görlich, Gizbert-Studnicki - CQG IOPScience (2023)]

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Remark. Dimension of $\mathcal{M}$ can be extracted from the asymptotic expansion

$$n = -2 \frac{\mathrm{d} \log(P(\tau))}{\mathrm{d} \log(\tau)} \quad \text{as } \tau \searrow 0.$$

Definition (Physically Inspired). Running) Spectral Dimension $D(\tau)$ is defined by dropping $\tau \searrow 0$ condition such that

$$D(\tau) := -2 \frac{\mathrm{d} \log(P(\tau))}{\mathrm{d} \log(\tau)}.$$