THE RIEMANN HYPOTHESIS IS FALSE

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ABSTRACT. Let Θ be the supremum of the real parts of the zeros of the Riemann zeta function. We demonstrate that $\Theta \geq \frac{3}{4}$. This disproves the Riemann Hypothesis, which asserts that $\Theta = \frac{1}{2}$.

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Introduction. The Riemann zeta function is a function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and in the whole complex plane by analytic continuation. Euler noticed that for $\Re(s) > 1$, $\zeta(s)$ can be expressed as a product $\prod_p (1-p^{-s})^{-1}$ over the entire set of primes, which entails that $\zeta(s) \neq 0$ for $\Re(s) > 1$. It can be shown that $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1. Define ρ to be a complex (non-real) zero of ζ. Let $\Lambda(n)$ denote the von Mangoldt function, which is equal to $\log p$ if $n = p^r$ for some prime p and $r \in \mathbb{N}$, and 0 otherwise. The importance of the $\rho's$ in the distribution of primes can be clearly seen from the Riemann explicit formula $\psi(x) := \sum_{n \leq x} \Lambda(n) = x - \sum_{|\rho| \leq x} \frac{x^\rho}{\rho} + O(\log^2 x)$. In the literature, ψ is sometimes referred to as the Chebyshev ψ function after P.L. Chebyshev, who pioneered its study. It can be shown that $\psi(x) - x \ll x^b (\log x)^2$ if $\zeta(s) \neq 0$ for $\Re(s) > b$. In particular, the Riemann Hypothesis (RH) is equivalent to the statement that $b = \frac{1}{2}$. For a far more thorough discussion of the RH, the interested reader is kindly referred to [3].

Main results

Lemma 1 (Plancherel's identity, [2, Theorem 5.4]). *Suppose that*

$$\nu(s) = \sum_{n=1}^{\infty} v_n n^{-s}$$

is a Dirichlet series whose abscissa of convergence is $c > 0$. Let $V(x) = \sum_{n \leq x} v_n$. Then for $\sigma = \Re(s) > c$, one has

$$2\pi \int_0^\infty |V(x)|^2 x^{-2\sigma-1} \mathrm{d}x = \int_{\mathbb{R}} \Big| \frac{\nu(\sigma+it)}{\sigma+it} \Big|^2 \mathrm{d}t.$$

Definitions. Let: Λ be the von Mangoldt function, μ be the Mobius function and p be a prime. Define $\theta(x) := \sum_{p \leq x} \log p, \psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{r=1}^{\infty} \theta(x^{1/r})$. Let $\gamma = 0.57721\cdots$ be the Euler-Mascheroni constant and $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$. Note that sometimes we shall take

$\sigma \in \mathbb{C}$, e.g. when considering complex-analytic continuations of some real-analytic functions of σ. *In all such cases, we shall make it clear that* $\sigma \in \mathbb{C}$. From now on, assume that $\sigma \in \mathbb{R}_{>1}$ unless specified otherwise. Let $k_1(s) = \frac{1}{s-1} - \gamma$

$$K_1(\sigma) = \int_{\mathbb{R}} \Big|\frac{k_1(\sigma+it)}{\sigma+it}\Big|^2 \mathrm{d}t = \frac{\pi((2\sigma^2-3\sigma+1)\gamma^2+(2-2\sigma)\gamma+1)}{\sigma(2\sigma-1)(\sigma-1)}, \tag{1}$$

$q = \sum_{n=2}^{\infty} \mu(n)\frac{\zeta'}{\zeta}(n), k_2(s) = k_1(s) - q$ and

$$K_2(\sigma) = \int_{\mathbb{R}} \Big|\frac{k_2(\sigma+it)}{\sigma+it}\Big|^2 \mathrm{d}t = \frac{\pi((2\sigma^2-3\sigma+1)(\gamma+q)^2+(2-2\sigma)(\gamma+q)+1)}{\sigma(2\sigma-1)(\sigma-1)}. \tag{2}$$

From (1) and (2), note that the function $K_2(\sigma) - K_1(\sigma)$ has a holomorphic (complex-analytic) continuation to $\Re(\sigma) > \frac{1}{2}$, though both K_1 and K_2 are $\mathbb{C} \mapsto \mathbb{R}$ functions. Define

$$\alpha(s) := -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} = s\int_1^{\infty} \psi(x) x^{-s-1}\mathrm{d}x \tag{3}$$

and

$$\beta(s) := -\sum_{n=1}^{\infty} \mu(n)\frac{\zeta'}{\zeta}(ns) = \sum_p \frac{\log p}{p^s} = s\int_1^{\infty} \theta(x) x^{-s-1}\mathrm{d}x \tag{4}$$

[2, p.28], hence $q = -\int_1^{\infty}(\psi(x)-\theta(x))x^{-2}\mathrm{d}x$. Let

$$f(\sigma) : = 2\pi \int_1^{\infty} \psi^2(x) x^{-2\sigma-1}\mathrm{d}x - K_1(\sigma), \tag{5}$$

$$g(\sigma) : = 2\pi \int_1^{\infty} \theta^2(x) x^{-2\sigma-1}\mathrm{d}x - K_2(\sigma) \tag{6}$$

and

$$h(\sigma) := f(\sigma) - g(\sigma) = 2\pi\int_1^{\infty} (\psi^2(x)-\theta^2(x))x^{-2\sigma-1}\mathrm{d}x + K_2(\sigma) - K_1(\sigma). \tag{7}$$

Theorem 1. *The function $h(\sigma)$ has a holomormphic continuation to $\Re(\sigma) > \frac{3}{4}$ and also a simple pole at $\sigma = \frac{3}{4}$.*

Proof. By the Prime Number Theorem, we know [2, p.179] that there exists some constant $d > 0$ such that $\psi(y) = y(1 + O(e^{-d\sqrt{\log y}}))$ uniformly for $y \geq 1$. Note that $\theta(y) = \psi(y) + O(\sqrt{y})$ [2, p.49]. Hence $\psi(x) - \theta(x) = \theta(\sqrt{x}) + O(x^{1/3}) = \sqrt{x}(1 + O(e^{-d\sqrt{\log x}}))$ uniformly for $x \geq 1$. Thus

$$\psi^2(x) - \theta^2(x) = (\psi(x)-\theta(x))(\psi(x)+\theta(x)) = 2x^{3/2}(1+E(x)) \tag{8}$$

uniformly for $x \geq 1$, where $E(x) \ll e^{-d\sqrt{\log x}}$. Inserting (8) into the integral on the extreme right-hand side of (7) yields

$$h(\sigma) = 2\pi\Big(\sigma - \frac{3}{4}\Big)^{-1} + (K_2(\sigma) - K_1(\sigma)) + A(\sigma), \tag{9}$$

where $A(\sigma) := 4\pi\int_1^{\infty} x^{\frac{3}{2}-2\sigma-1}E(x)\mathrm{d}x \ll 1$ uniformly for $\sigma \geq \frac{3}{4}$. Since the function $K_2(\sigma) - K_1(\sigma)$ has a holomorphic continuation to $\Re(\sigma) > \frac{1}{2}$, the claim follows from (9). $\square$

Theorem 2. *Let $\Theta \in [\frac{1}{2}, 1]$ be the supremum of the real parts of the zeros of ζ. If $\Theta < 1$, then $h(\sigma)$ has a holomorphic continuation to $\Re(\sigma) > \Theta$.*

Proof. Let ρ denote a complex (non-real) zero of ζ. By Lemma 12.1 of [2], we know that for $\sigma > \Theta$ and $s \neq 1$, one has

$$-\alpha(s) = \sum_{|\Im(s)-\Im(\rho)|\leq 1} \frac{1}{s-\rho} + O(\log|2s|) \tag{10}$$

as $\Im(s) \to \pm\infty$. Let $T \in \mathbb{R}_{>0}$. Define $N(T)$ to be the number of those ρ with $|\Im(\rho)| \leq T$. By Theorem 9.2 of [1], we know that $N(T+1) - N(T) \ll \log(T+1)$. Hence for fixed $\sigma > \Theta$, one has

$$\sum_{|\Im(s)-\Im(\rho)|\leq 1} \frac{1}{s-\rho} \ll \sum_{|\Im(s)-\Im(\rho)|\leq 1} 1 \ll \log|2s| \tag{11}$$

as $\Im(s) \to \pm\infty$. For fixed $\sigma > \frac{1}{2}$ and $n \geq 2$, note that

$$\left|\mu(n)\frac{\zeta'}{\zeta}(ns)\right| \leq \left|\frac{\zeta'}{\zeta}(ns)\right| \leq \sum_{m=1}^{\infty} \Lambda(m) m^{-n\sigma} \ll 2^{-n\sigma} \tag{12}$$

as $n \to \infty$. Combining (12) with (11) and (10) reveals that both $\alpha(s)$ and $\beta(s)$ are $\ll \log|2s|$ for fixed $\sigma \in \mathbb{R}_{>\Theta}$ and $s \neq 1$. Notice that this bound also holds for fixed $\sigma \in \mathbb{C}, \Re(\sigma) > \Theta$ and $s \neq 1$. Note that $\psi(y) = 0 = \theta(y)$ for every $y \in [0,1]$. Thus by Lemma 1, we have

$$f(\sigma) = 2\pi \int_1^{\infty} \psi^2(x) x^{-2\sigma-1} \mathrm{d}x - K_1(\sigma) = \int_{\mathbb{R}} \frac{|\alpha(\sigma+it)|^2 - |k_1(\sigma+it)|^2}{|\sigma+it|^2} \mathrm{d}t \tag{13}$$

and

$$g(\sigma) = 2\pi \int_1^{\infty} \theta^2(x) x^{-2\sigma-1} \mathrm{d}x - K_2(\sigma) = \int_{\mathbb{R}} \frac{|\beta(\sigma+it)|^2 - |k_2(\sigma+it)|^2}{|\sigma+it|^2} \mathrm{d}t \tag{14}$$

for $\sigma > 1$. Let

$$\alpha_0(s) := \frac{|\alpha(s)|^2 - |k_1(s)|^2}{|s|^2}$$

and

$$\beta_0(s) := \frac{|\beta(s)|^2 - |k_2(s)|^2}{|s|^2}.$$

Note that

$$\alpha(s) - \frac{s}{s-1} = s \int_1^{\infty} (\psi(x) - x) x^{-s-1} \mathrm{d}x. \tag{15}$$

Since $\psi(x) - x \ll x^{\Theta}(\log 2x)^2$ [2, p.430] and $\alpha(s) = (s-1)^{-1} - \gamma + O(|s-1|)$ around $s = 1$ [1, p.20], notice that both sides of (15) are holomorphic for $\sigma > \Theta$. Hence by the identity theorem for holomorphic functions, the domain of (15) extends to $\sigma > \Theta$ thus

$$\int_1^{\infty} (\psi(x) - x) x^{-2} \mathrm{d}x = -1 - \gamma. \tag{16}$$

Recall that $q = -\int_1^{\infty} (\psi(x) - \theta(x)) x^{-2} \mathrm{d}x$. Combining this with (16) gives

$$\int_1^{\infty} (\theta(x) - x) x^{-2} \mathrm{d}x = -1 - \gamma - q. \tag{17}$$

Since $k_1(s) = \frac{1}{s-1} - \gamma$ and $\alpha(s) = s\int_1^\infty \psi(x)x^{-s-1}\mathrm{d}x$, note that

$$\alpha_0(\sigma+it) = \frac{\left(\Re((\sigma+it)\int_1^\infty \psi(x)x^{-\sigma-1-it}\mathrm{d}x)\right)^2 + \left(\Im((\sigma+it)\int_1^\infty \psi(x)x^{-\sigma-1-it}\mathrm{d}x)\right)^2 - \frac{(1-\gamma\sigma+\gamma)^2+(\gamma t)^2}{(\sigma-1)^2+t^2}}{\sigma^2+t^2} \tag{18}$$

where

$$\int_1^\infty \psi(x)x^{-\sigma-1-it}\mathrm{d}x = \int_1^\infty \psi(x)x^{-\sigma-1}\cos(t\log x)\mathrm{d}x - i\int_1^\infty \psi(x)x^{-\sigma-1}\sin(t\log x)\mathrm{d}x.$$

We know [2, p.430] that $\psi(x) = x + O(x^\Theta(\log 2x)^2)$ uniformly for $x \geq 1$. Hence by writing $\psi(x) = x + (\psi(x) - x)$ in the integrals in (18), it follows from (16) and (18) that $\alpha_0(s)$ is regular at $s = 1$ and has a holomorphic continuation to $\Re(\sigma) > \Theta$. Since $\beta(s) = s\int_1^\infty \theta(x)x^{-s-1}\mathrm{d}x, k_2(s) = k_1(s) - q$ and $\theta(x) = x + O(x^\Theta(\log 2x)^2)$ uniformly for $x \geq 1$ [2, p.430], one deduces by a similar argument that $\beta_0(s)$ is also regular at $s = 1$ and has a holomorphic continuation to $\Re(\sigma) > \Theta$. Let $\alpha_1(s)$ be the holomorphic continuation of $\alpha_0(s)$ to $\Re(\sigma) > \Theta$, and let $\beta_1(s)$ be that of $\beta_0(s)$. For fixed $\sigma \in \mathbb{C}$, recall that both $\alpha_0(s)$ and $\beta_0(s)$ are $\ll \left(\frac{\log|2s|}{|s|}\right)^2$ for $\Re(\sigma) \geq \Theta + \varepsilon$ for any $\varepsilon > 0$. Since a uniformly absolutely convergent improper integral of a *holomorphic* function is also holomorphic [1], it follows that $f(\sigma)$ has a holomorphic continuation $\int_{\mathbb{R}} \alpha_1(\sigma+it)\mathrm{d}t$ to $\Re(\sigma) > \Theta$, and $g(\sigma)$ has a holomorphic continuation $\int_{\mathbb{R}} \beta_1(\sigma+it)\mathrm{d}t$ to $\Re(\sigma) > \Theta$. Hence $h(\sigma) = f(\sigma) - g(\sigma)$ also has a holomorphic continuation $\int_{\mathbb{R}} \alpha_1(\sigma+it)\mathrm{d}t - \int_{\mathbb{R}} \beta_1(\sigma+it)\mathrm{d}t$ to $\Re(\sigma) > \Theta$. This completes the proof □

Corollary 1. *Let Θ be as defined in Theorem 2. Then $\Theta \geq \frac{3}{4}$.*

Proof. Suppose that $\Theta < \frac{3}{4}$ and let $0 < \delta < \frac{3}{4} - \Theta$. Then by Theorem 2, $h(\sigma)$ must have a holomorphic continuation to $\Re(\sigma) > \frac{3}{4} - \delta$, contradicting Theorem 1. Thus our supposition must be false, so we are done. □

This disproves the Riemann hypothesis, which asserts that $\Theta = \frac{1}{2}$.

References

[1] E. C. Titchmarsh, *The theory of the Riemann zeta function,* second ed., The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.

[2] H. L. Montgomery, R. C. Vaughan, *Multiplicative Number Theory. I. Classical theory. Cambridge Studies in Advanced Mathematics*, **97**, Cambridge University Press, Cambridge, 2007.

[3] P.B. Borwein, S. Choi, B. Rooney and A. Weirathmueller, *The Riemann Hypothesis: A resource for the afficionado and virtuoso alike*, (2008), Springer-Berlin.

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[1] see e.g. https://mathstackexchange.com/questions/3495194/condition-for-an-integral-of-analytic-function-to-be-analytic.