ON THE BINARY GOLDBACH CONJECTURE

TATENDA KUBALALIKA

ABSTRACT. By exploiting some classical identity of Buchstab, we demonstrate in this note that if $x \geq 6$ is an even integer $\equiv 2 \bmod 4$, then x can be expressed as a sum of two odd primes. This proves the binary Golbach conjecture for every even integer $\equiv 2 \bmod 4$.

2020 Primary Mathematics Subject Classification: 11AXX.
Keywords and phrases: Binary Goldbach conjecture; partial proof, primes.

Introduction. Let d be a fixed positive integer, and let $p_1, p_2, \cdots, p_k$ be the primes (in ascending order) which do not divide d and are $\leq y$, where $y \geq 3$ is fixed. Let $a_i; b_i, \cdots; a_k; b_k$ be the of integers with $0 \leq a_i < p_i, 0 \leq b_i < p_i, a_i \neq b_i$ for every positive integer $i \leq k$. Let $6 \leq x \equiv 0 \pmod 2$ and a be any positive integer $< p_k$. Then following [3, p.247], let $F(x; d, y; a_i, b_i, p_i)$ denote the number of integers $n \leq x$ for which $n \equiv a \pmod{p_i}$ and $(n - a_i)(n - b_i)$ is indivisible by p_i for every positive integer $i \leq k$. The arguments arguments a, a_i, b_i, p_i need not be written in the function F since the results will hold for every positive integer $a < p_k$ and every set a_i, b_i of the type described. Note that $F(x; , d, 1)$ is nothing but the number of integers $n \leq x$ for which $n \equiv a \pmod d$ and will be abbreviated $F(x, d)$.

The connection between $F(x; d, y)$ and Goldbach problem is indicated by the following considerations: Let $d = 2, a = 1, y = x^{1/u}$ where x is an even integer and $u \in \mathbb{N}_{\geq 2}$. Let $a_i = 0, b_i \equiv x \pmod{p_i}$ if x is indivisible by p_i and $x - b_i$ is indivisible by p_i if $p_i \mid x$. Then the function $F(x; 2, x^{1/u})$ is the number of odd positive integers $n \leq x$ such that neither n nor $x - n$ is divisible by any prime not exceeding $x^{1/u}$. Hence all prime factors of n and $x - n$ are greater than $x^{1/u}$ and there cannot be more than $u - 1$ of them. If $u = 2$, each of n and $x - n$ is either a prime or equal to 1. But recall that $x \geq 6$, hence x and $x - n$ cannot be both equal to 1.

Thus if it could be shown that $F(x; 2, x^{1/2}) \geq 1$, it would follow that there exists at least one representation $x = n + (x - n)$ where each of n and $x - n$ is a prime.

Recall the following key definitions from the introduction (which was entirely borrowed from [3]):

Definitions. Let d be a fixed positive integer, and let $p_1, p_2, \cdots, p_k$ be the primes (in ascending order) which do not divide d and are $\leq y$, where $y \geq 3$ is fixed. Let $a_i; b_i, \cdots; a_k; b_k$ be the set of integers with $0 \leq a_i < p_i, 0 \leq b_i < p_i, a_i \neq b_i$ for every positive integer $i \leq k$. Let $6 \leq x \equiv 0 \pmod 2$ and a be any positive integer. Recall that $F(x; d, y)$ is the number of positive integers $n \leq x$ for which $n \equiv a \pmod{p_i}$ and $(n - a_i)(n - b_i)$ is indivisible by p_i for every positive integer $i \leq k$. In particular, $F(x; 2, \sqrt{x})$ is the number of odd positive integers $n \leq x$ such that both n and $x - n$ are prime, and $F(x; 2)$ is the number of positive integers $n \leq x$ such that $n \equiv a \bmod 2$ for any fixed $a \in \mathbb{N}$. Thus, $F(x; 2) = \ x/2$.

Main results

Lemma 1 (equation (2.6) of [3]). *Let p_k be the largest prime $\leq \sqrt{x}$. Then*

$$F(x; 2, p_k) = F(x; 2) - \sum_{r=1}^{k} F(x; 2p_r, p_{r-1}) \tag{1}$$

$$= x/2 - \sum_{r=1}^{k} F(x; 2p_r, p_{r-1}). \tag{2}$$

Theorem 1. *If $6 \leq x \equiv 2 \bmod 4$, then x can be expressed as a sum of two odd primes.*

Proof. Let p_k be the largest prime $\leq \sqrt{x}$. Suppose that Theorem 1 is false, so that $F(x; 2, p_k) = 0$. Then from (2), it follows that

$$x/2 = 2 \sum_{r=1}^{k} F(x; 2p_r, p_{r-1}). \tag{3}$$

But $x/2$ is odd whereas the right-hand side of (3) is even, thus we have a contradiction. This implies that our supposition must be false, so we are done. $\square$

References

[1] D.M. Burton, *Elementary number theory,* McGraw-Hill, 2007.

[2] M. S., *Goldbach Christian,* Dictionary of Scientific Biography. Ed. Charles Coulston Gillipsie. New York: Scribner 1970-1980.

[3] R.D. James, *Recent progress in the Goldbach problem,* Bulletin of the American Math. Soc. (55), 1949, 246-260.

Email address: `tatendakubalalika@yahoo.com`