Supplementary material to "Open-Ended Recursive Approach for the Calculation of Multiphoton Absorption Matrix Elements"

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1 Intermediate quantities for the cubic response function

In this section we list some definitions of intermediate quantities that are needed for the description of the cubic response function.

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}_{1,2}^{abcd} =& \boldsymbol{\mathcal{E}}^{2,a}(\mathbf{D}^{bc})\mathbf{D}^{d} + \boldsymbol{\mathcal{E}}^{2,a}(\mathbf{D}^{bd})\mathbf{D}^{c} + \boldsymbol{\mathcal{E}}^{2,a}(\mathbf{D}^{cd})\mathbf{D}^{b} + \boldsymbol{\mathcal{E}}^{3,a}(\mathbf{D}^{b})\mathbf{D}^{c}\mathbf{D}^{d} + \mathbf{F}_{2}^{bcd}\mathbf{D}^{a} && (1)\\
\mathbf{F}_{2}^{bcd} =& \boldsymbol{\mathcal{E}}^{2,b}(\mathbf{D}^{cd}) + \boldsymbol{\mathcal{E}}^{2,c}(\mathbf{D}^{bd}) + \boldsymbol{\mathcal{E}}^{2,d}(\mathbf{D}^{bc}) + && (2)\\
& \boldsymbol{\mathcal{E}}^{3,b}(\mathbf{D}^{c},\mathbf{D}^{d}) + \boldsymbol{\mathcal{E}}^{3,c}(\mathbf{D}^{b},\mathbf{D}^{d}) + \boldsymbol{\mathcal{E}}^{3,d}(\mathbf{D}^{b},\mathbf{D}^{c}) + \\
& \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{bc},\mathbf{D}^{d}) + \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{b},\mathbf{D}^{cd}) + \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{bd},\mathbf{D}^{c}) + \boldsymbol{\mathcal{E}}^{4}(\mathbf{D}^{b},\mathbf{D}^{c},\mathbf{D}^{d}),\\
\mathbf{Y}_{2'}^{bcd} =& [\mathbf{F}^{bc}\mathbf{D}^{d}\mathbf{S} + \mathbf{F}^{bd}\mathbf{D}^{c}\mathbf{S} + \mathbf{F}^{cd}\mathbf{D}^{b}\mathbf{S} + \mathbf{F}^{b}\mathbf{D}^{cd}\mathbf{S} + \mathbf{F}^{c}\mathbf{D}^{bd}\mathbf{S} + && (3)\\
& \mathbf{F}^{d}\mathbf{D}^{bc}\mathbf{S} + \mathbf{F}_{2}^{bcd}\mathbf{D}\mathbf{S}]^{\ominus},\\
\mathbf{Z}_{2'}^{bcd} =& \mathbf{D}^{bc}\mathbf{S}\mathbf{D}^{d} + \mathbf{D}^{bd}\mathbf{S}\mathbf{D}^{c} + \mathbf{D}^{cd}\mathbf{S}\mathbf{D}^{b} + \mathbf{D}^{b}\mathbf{S}\mathbf{D}^{cd} + \mathbf{D}^{c}\mathbf{S}\mathbf{D}^{bd} + \mathbf{D}^{d}\mathbf{S}\mathbf{D}^{bc}. && (4)
\end{aligned}
$$

2 Intermediate quantities for the residues of the cubic response function

In this section we list some definitions of intermediate quantities that are needed for the calculation of the residues of the cubic response function.

$$
\begin{aligned}
\mathbf{F}_{2}^{bc(d\to p)} =& \boldsymbol{\mathcal{E}}^{2,b}(\mathbf{D}^{c(d\to p)}) + \boldsymbol{\mathcal{E}}^{2,c}(\mathbf{D}^{b(d\to p)}) + \boldsymbol{\mathcal{E}}^{3,b}(\mathbf{D}^{c},\mathbf{D}^{d\to p}) + \boldsymbol{\mathcal{E}}^{3,c}(\mathbf{D}^{b},\mathbf{D}^{d\to p}) + && (5)\\
& \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{bc},\mathbf{D}^{d\to p}) + \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{b},\mathbf{D}^{c(d\to p)}) + \boldsymbol{\mathcal{E}}^{3}(\mathbf{D}^{b(d\to p)},\mathbf{D}^{c}) + \boldsymbol{\mathcal{E}}^{4}(\mathbf{D}^{b},\mathbf{D}^{c},\mathbf{D}^{d\to p}),\\
\mathbf{Y}_{2'}^{bc(d\to p)} =& [\mathbf{F}^{bc}\mathbf{D}^{d\to p}\mathbf{S} + \mathbf{F}^{b(d\to p)}\mathbf{D}^{c}\mathbf{S} + \mathbf{F}^{c(d\to p)}\mathbf{D}^{b}\mathbf{S} + \mathbf{F}^{b}\mathbf{D}^{c(d\to p)}\mathbf{S} + && (6)\\
& \mathbf{F}^{c}\mathbf{D}^{b(d\to p)}\mathbf{S} + \mathbf{F}^{d\to p}\mathbf{D}^{bc}\mathbf{S} + \mathbf{F}_{2}^{bc(d\to p)}\mathbf{D}\mathbf{S}]^{\ominus},\\
\mathbf{Z}_{2'}^{bc(d\to p)} =& \mathbf{D}^{bc}\mathbf{S}\mathbf{D}^{(d\to p)} + \mathbf{D}^{b(d\to p)}\mathbf{S}\mathbf{D}^{c} + \mathbf{D}^{c(d\to p)}\mathbf{S}\mathbf{D}^{b} + \mathbf{D}^{b}\mathbf{S}\mathbf{D}^{c(d\to p)} + && (7)\\
& \mathbf{D}^{c}\mathbf{S}\mathbf{D}^{b(d\to p)} + \mathbf{D}^{d\to p}\mathbf{S}\mathbf{D}^{bc}.
\end{aligned}
$$

3 Four-photon absorption

In this section we list some definitions of intermediate quantities that are needed for the description of four-photon absorption.

$$
\begin{aligned}
\mathbf{Y}_{2'}^{bcd(e\to p)} =& \mathbf{F}_2^{bcd}\mathbf{D}^{e\to p}\mathbf{S} + \mathbf{F}^{bc}\mathbf{D}^{d(e\to p)}\mathbf{S} + \mathbf{F}^{bd}\mathbf{D}^{c(e\to p)}\mathbf{S} + \mathbf{F}^{cd}\mathbf{D}^{b(e\to p)}\mathbf{S} + \\
& \mathbf{F}^{b(e\to p)}\mathbf{D}^{cd}\mathbf{S} + \mathbf{F}^{c(e\to p)}\mathbf{D}^{bd}\mathbf{S} + \mathbf{F}^{d(e\to p)}\mathbf{D}^{bc}\mathbf{S} + \mathbf{F}_2^{bc(e\to p)}\mathbf{D}^{d}\mathbf{S} + \\
& \mathbf{F}_2^{bd(e\to p)}\mathbf{D}^{c}\mathbf{S} + \mathbf{F}_2^{cd(e\to p)}\mathbf{D}^{b}\mathbf{S} + \mathbf{F}_2^{bcd(e\to p)}\mathbf{D}\mathbf{S},
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\mathbf{F}_{2}^{bcd(e\to p)} =& \boldsymbol{\mathcal{E}}^{3,b}(\mathbf{D}^{cd})\mathbf{D}^{e\to p} + \boldsymbol{\mathcal{E}}^{3,b}(\mathbf{D}^{c(e\to p)})\mathbf{D}^{d} + \boldsymbol{\mathcal{E}}^{3,b}(\mathbf{D}^{c})\mathbf{D}^{d(e\to p)} + \\
& \boldsymbol{\mathcal{E}}^{3,c}(\mathbf{D}^{bd})\mathbf{D}^{e\to p} + \boldsymbol{\mathcal{E}}^{3,c}(\mathbf{D}^{b(e\to p)})\mathbf{D}^{d} + \boldsymbol{\mathcal{E}}^{3,c}(\mathbf{D}^{b})\mathbf{D}^{d(e\to p)} + \\
& \boldsymbol{\mathcal{E}}^{3,d}(\mathbf{D}^{bc})\mathbf{D}^{e\to p} + \boldsymbol{\mathcal{E}}^{3,d}(\mathbf{D}^{b(e\to p)})\mathbf{D}^{c} + \boldsymbol{\mathcal{E}}^{3,d}(\mathbf{D}^{b})\mathbf{D}^{c(e\to p)} + \\
& \boldsymbol{\mathcal{E}}^{4,b}(\mathbf{D}^{c})\mathbf{D}^{d}\mathbf{D}^{e\to p} + \boldsymbol{\mathcal{E}}^{4,c}(\mathbf{D}^{b})\mathbf{D}^{d}\mathbf{D}^{e\to p} + \boldsymbol{\mathcal{E}}^{4,d}(\mathbf{D}^{b})\mathbf{D}^{c}\mathbf{D}^{e\to p} + \\
& \boldsymbol{\mathcal{E}}^{5}(\mathbf{D}^{b})\mathbf{D}^{c}\mathbf{D}^{d}\mathbf{D}^{e\to p}
\end{aligned}
\tag{9}
$$

$$
\mathbf{Z}_{2'}^{bcd(e\to p)} = \left[\mathbf{D}^{bc}\mathbf{S}\mathbf{D}^{d(e\to p)} + \mathbf{D}^{bd}\mathbf{S}\mathbf{D}^{c(e\to p)} + \mathbf{D}^{b(e\to p)}\mathbf{S}\mathbf{D}^{cd}\right]^{\oplus}, \tag{10}
$$

$$
\lambda_a^d = [\mathbf{D}^{ad}\mathbf{S}\mathbf{D}]^{\ominus} + [\mathbf{D}^{a}\mathbf{S}\mathbf{D}^{d}]^{\ominus}, \tag{11}
$$

$$
\lambda_a^{e\to p} = [\mathbf{D}^{a(e\to p)}\mathbf{S}\mathbf{D}]^{\ominus} + [\mathbf{D}^{a}\mathbf{S}\mathbf{D}^{e\to p}]^{\ominus}, \tag{12}
$$

$$
\zeta_a^d = [\mathbf{F}^{ad}(\mathbf{D}\mathbf{S} - \tfrac{1}{2})]^{\oplus} + [\mathbf{F}^{a}\mathbf{D}^{d}\mathbf{S}]^{\oplus}, \tag{13}
$$

$$
\zeta_a^{e\to p} = [\mathbf{F}^{a(e\to p)}(\mathbf{D}\mathbf{S} - \tfrac{1}{2})]^{\oplus} + [\mathbf{F}^{a}\mathbf{D}^{e\to p}\mathbf{S}]^{\oplus}. \tag{14}
$$