Supplementary Material

1 MULTIPLE SCALES ANALYSIS

The solution of the leading order Eq. (29) is derived by making the following product ansatz

$$P^0(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{\eta}, T, t) = w(\boldsymbol{r}, \boldsymbol{\eta})\rho^0(\boldsymbol{R}, T, t)\,. \tag{S1}$$

Integrating by parts over the microscale variables $\boldsymbol{r}$ and $\boldsymbol{\eta}$, and using the solvability condition, the solution of Eq. (29) P^0 is constant over the period: $P^0(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{\eta}, T, t) = P^0(\boldsymbol{R}, \boldsymbol{\eta}, T, t)$. The leading order Eq. (29) becomes

$$\nabla_{\boldsymbol{\eta}} \cdot [\boldsymbol{\eta} w] + \frac{1}{2}\nabla^2_{\boldsymbol{\eta}} w = 0\,. \tag{S2}$$

Using the known result for a multi-dimensional Ornstein-Uhlenbeck process Ref. 138, the solution for w is given by

$$w(\eta_1, \eta_2, \eta_3) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}} e^{\dfrac{-\eta_i^2}{2}}\,. \tag{S3}$$

The solvability condition for $\mathcal{O}(\epsilon)$ equation is

$$\int d\boldsymbol{r} d\boldsymbol{\eta} \left(w\frac{\partial}{\partial T}\rho^0 + w\frac{\partial}{\partial R_3}\left[v\rho^0\right] + wP_a v_a \boldsymbol{\eta} \cdot \nabla_{\boldsymbol{R}}\rho^0 \right) = 0\,, \tag{S4}$$

which depends on the leading order result, P^0, from which we find

$$\frac{\partial}{\partial T}\rho^0 = -\frac{\partial}{\partial R_3}\left[v\rho^0\right]\,, \tag{S5}$$

and the $\mathcal{O}(\epsilon)$ equation becomes

$$\mathcal{L}P^1 = wP_a v_a \boldsymbol{\eta} \cdot \nabla_{\boldsymbol{R}}\rho^0\,. \tag{S6}$$

We assume that

$$P^1 = wP_a v_a \boldsymbol{\alpha} \cdot \nabla_{\boldsymbol{R}}\rho^0\,, \tag{S7}$$

after which we find that

$$\boldsymbol{\alpha} = -\frac{1}{P_A}\boldsymbol{\eta}\,. \tag{S8}$$

Substitution of P^1 into the $\mathcal{O}(\epsilon^2)$ equation and using the solvability condition, we obtain

$$\frac{\partial}{\partial t}\rho^0 = \frac{P_a^2 v_a^2}{2P_A}\nabla^2_{\boldsymbol{R}}\rho^0 - \beta_D\frac{c_h}{D_c}\nabla_{\boldsymbol{R}} \cdot \left[\rho^0 \nabla_{\boldsymbol{R}} c^0\right] + \nabla^2_{\boldsymbol{R}}\left[D\rho^0\right]\,, \tag{S9}$$

and in dimensional form, we have

$$\frac{\partial}{\partial \tilde{t}}\rho = -\beta_D \nabla_{\tilde{\boldsymbol{r}}} \cdot \left[\rho \nabla_{\tilde{\boldsymbol{r}}}\tilde{c}\right] - \frac{\partial}{\partial \tilde{z}}\left[\tilde{v}\rho\right] + \nabla^2_{\tilde{\boldsymbol{r}}}\left[(\tilde{D}_a + \tilde{D})\rho\right]\,. \tag{S10}$$

2 SOLUTION OF THE FOKKER-PLANCK EQUATION IN THE LARGE PÉCLET LIMIT

The analytic solution of Eq. (32) in the large Péclet number limit follows by expanding the probability density function, ρ perturbatively Ref. 139, as

$$\rho = \rho_0 + \rho_1 \,, \tag{S11}$$

with $\rho_1 = \mathcal{O}(\tilde{c})\rho_0$. The associated system of equations is

$$\frac{\partial}{\partial \tilde{t}}\rho_0 + \frac{\partial}{\partial \tilde{z}}\left[\tilde{v}\rho_0\right] - \nabla^2_{\tilde{r}}\left[(\tilde{D}_a + \tilde{D})\rho_0\right] = 0 \,, \tag{S12}$$

$$\frac{\partial}{\partial \tilde{t}}\rho_1 + \frac{\partial}{\partial \tilde{z}}\left[\tilde{v}\rho_1\right] - \nabla^2_{\tilde{r}}\left[(\tilde{D}_a + \tilde{D})\rho_1\right] = -\beta_D \nabla_{\tilde{\boldsymbol{r}}} \cdot \left[\rho_0 \nabla_{\tilde{\boldsymbol{r}}}\tilde{c}\right] \,. \tag{S13}$$

To derive the solutions for ρ_0 and ρ_1, we use the Green's function $G(\tilde{\boldsymbol{r}}, \tilde{t}|\tilde{\boldsymbol{r}}_0, \tilde{t}_0)$ Ref. 140 which satisfies

$$\frac{\partial}{\partial \tilde{t}}G + \frac{\partial}{\partial \tilde{z}}\left[\tilde{v}G\right] - \nabla^2_{\tilde{\boldsymbol{r}}}\left[(\tilde{D}_a + \tilde{D})G\right] = 0 \,. \tag{S14}$$

The solution of Eq.(32) was derived in the large Péclet number limit in Ref. 89, and is

$$\rho_0 = G(\tilde{\boldsymbol{r}}, \tilde{t}|\tilde{\boldsymbol{r}}_0, \tilde{t}_0) = \frac{\tilde{z}^3}{(\tilde{z}')^{3/4}} \exp\left(-\frac{\left[(\tilde{z}')^{1/4} - \tilde{z}_0\right]^2}{20 + 4\tilde{D}_a(\tilde{t} - \tilde{t}_0)}\right) \exp\left[-\frac{((\tilde{x} - \tilde{x}_0)^2 + (\tilde{y} - \tilde{y}_0)^2)}{\left(1 + 4\frac{\tilde{D}(\tilde{z})}{\tilde{v}(\tilde{z})}\left[(\tilde{z}')^{1/4} - \tilde{z}\right] + 4\tilde{D}_a(\tilde{t} - \tilde{t}_0)\right)}\right]$$
$$\times \frac{1}{2\sqrt{5\pi}\left(1 + 4\frac{\tilde{D}(\tilde{z})}{\tilde{v}(\tilde{z})}\left[(\tilde{z}')^{1/4} - \tilde{z}\right] + 4\tilde{D}_a(\tilde{t} - \tilde{t}_0)\right)} \,. \tag{S15}$$

Given the Green's function the formal solution of Eq. (32) is

$$\rho(\tilde{\boldsymbol{r}}, \tilde{t}) = G(\tilde{\boldsymbol{r}}, \tilde{t}|\tilde{\boldsymbol{r}}_0, \tilde{t}_0) - \beta_D \int_{\tilde{t}_0}^{\tilde{t}} d\tau \int d\tilde{\boldsymbol{r}}_1 G(\tilde{\boldsymbol{r}}, \tilde{t}, |\tilde{\boldsymbol{r}}_1, \tau)\left[\nabla_{\tilde{\boldsymbol{r}}_1}\left(G(\tilde{\boldsymbol{r}}_1, \tau|\tilde{\boldsymbol{r}}_0, \tilde{t}_0)\nabla_{\tilde{\boldsymbol{r}}_1}\tilde{c}(\tilde{\boldsymbol{r}}_1, \tau|\tilde{\boldsymbol{r}}_0, \tilde{t}_0)\right)\right] \,, \tag{S16}$$

with

$$\tilde{c}(\tilde{\boldsymbol{r}}, \tilde{t}|\tilde{\boldsymbol{r}}_0, \tilde{t}_0) = \frac{1}{\left(4\pi\tilde{D}_{ch}(\tilde{t} - \tilde{t}_0)\right)^{\frac{3}{2}}} \exp\left[-\frac{((\tilde{x} - \tilde{x}_0)^2 + (\tilde{y} - \tilde{y}_0)^2 + (\tilde{z} - \tilde{z}_0)^2)}{4\tilde{D}_{ch}(\tilde{t} - \tilde{t}_0)}\right] \,. \tag{S17}$$

At $\tilde{t}_0 = 0$ and $\tilde{x}_0 = \tilde{y}_0 = 0$, the initial distribution is, to leading order ρ_0, given by

$$\rho_0(\tilde{\boldsymbol{r}}, \tilde{t} = 0) = \frac{1}{2\sqrt{5\pi}} \exp\left[-\frac{(\tilde{z} - \tilde{z}_0)^2}{20} - (\tilde{x}^2 + \tilde{y}^2)\right] \,. \tag{S18}$$

Equations (S16) and (S17) give the analytic solution to Eqs. (32)-(33), with initial distribution given by Eq. (S18). When the nutrients are neglected we recover our previous result Ref. 89.