# The Reynolds stress model

# The Reynolds stress model

- The extra term appearing in the RANS/URANS equations is know as the Reynolds stress tensor,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \overline{p}) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

- Where $\tau^R$ is the Reynolds stress tensor, and it can be written as,

$$\tau^R = -\rho \left( \overline{\mathbf{u}'\mathbf{u}'} \right) = - \begin{pmatrix} \rho \overline{u'u'} & \rho \overline{u'v'} & \rho \overline{u'w'} \\ \rho \overline{v'u'} & \rho \overline{v'v'} & \rho \overline{v'w'} \\ \rho \overline{w'u'} & \rho \overline{w'v'} & \rho \overline{w'w'} \end{pmatrix}$$

- So far, we have modeled this term using the Boussinesq approximation.

# The Reynolds stress model

- The Reynolds stress tensor $\tau^R$, is the responsible for the increased mixing and larger wall shear stresses.
- Remember, increased mixing and larger wall shear stresses are properties of turbulent flows.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled, for example, by using eddy viscosity models (EVM).
- However, it is possible to derive its own governing equations (six new equations as the tensor is symmetric).
- This approach is known as Reynolds stress models (RSM).
- Probably, the RSM is the most physically sounded RANS/URANS approach as it avoids the use of hypothesis/assumptions to model the Reynolds stress tensor.
- However, it is computationally expensive, and less robust than EVM.
    - It can be unstable if proper boundary conditions and initial conditions are  not used.
    - And, as you may guess, it is heavily modeled.

# The Reynolds stress model

- The RSM models are more general than EVM models.
- They potentially have better accuracy than the EVM model.
- However, this does not mean that they are better than EVM models.
- RSM models perform better in situations where the EVM models have poor performance,
    - Flows with strong curvature or swirl (cyclone separators and flows with concentrated vortices).
    - Flows in corners with secondary motions.
    - Very complex 3D interacting flows.
    - Highly anisotropic flows.
- In general, RSM models can be considered in non-equilibrium conditions (production not equal to dissipation),

$$P \neq D$$

# The Reynolds stress model

- Let us recall the exact Reynolds stress transport equations,

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = \underbrace{-\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k}}_{} + \epsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right]$$

Production term – Notice that it does not require modeling, it is computed from the mean gradients

- Where the following terms require modeling,

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

Dissipation tensor

$$\Pi_{ij} = \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$$

Pressure-strain correlation tensor

$$\rho C_{ijk} = \rho \overline{u_i' u_j' u_k'} + \overline{p' u_i'}\delta_{jk} + \overline{p' u_j'}\delta_{ik}$$

Turbulent transport tensor

- The most critical term is the pressure-strain term.
- RSM models differ by how this term is modeled.

# The Reynolds stress model

- The dissipation tensor of the Reynolds stress equations is also a tensor and can be modeled as follows (assuming that it reaches isotropy for small scales),

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}$$

- Where $\epsilon$ denotes the dissipation rate of turbulence kinetic energy,

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}$$

- The use of this assumption avoids the need for employing a dissipation transport equation for each component of the Reynolds stress tensor.
- Which results in a reduction in the number of transport equation to be solved and thus the computational cost.
- It is clear that $\epsilon$ needs to be modeled.
- For this we use a similar approach to the one used in the two-equations models presented in the previous lectures.
- Most of the time, the turbulent dissipation rate transport equation $\epsilon$ is solved.

# The Reynolds stress model

- The turbulent transport tensor of the Reynolds stress equations is also a tensor and can be modeled as follows,

$$C_{ijk} = \frac{\partial}{\partial x_k} \left[ \frac{\mu_t}{\sigma} \frac{\partial}{\partial x_k} \left( \overline{u'_i u'_j} \right) \right]$$

- Using this approach [1], the turbulent transport tensor is modeled using a gradient-diffusion model (this is the easiest and most robust approach).
- And alternative approach is the one proposed by Daly and Harlow [2],

$$C_{ijk} = C_s \frac{2}{3} \frac{k^2}{\epsilon} \left[ \frac{\partial \tau_{jk}}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_k} \right]$$

- Have in mind that there are more complex forms to model the turbulent transport tensor, but they are not very robust for industrial applications.

[1] F. S. Lien, M. A. Leschziner. Assessment of Turbulent Transport Models Including Non-Linear RNG Eddy-Viscosity Formulation and Second-Moment Closure. 1994.
[2] B. J. Daly, F. H. Harlow. Transport Equations in Turbulence. 1970.

# The Reynolds stress model

- The modeling of the pressure-strain term is critical. It contains complex correlations that are difficult to measure.
- Major difference between RSM models is due to the approach taken to model this term.
- The pressure-strain tensor can be decomposed as follows,

$$\Pi_{ij} = \underbrace{\Pi_{ij,1}}_{\text{Slow pressure strain term}} + \underbrace{\Pi_{ij,2}}_{\text{Fast pressure strain term}}$$

- To most widely used approach to model this term is the LRR [1] method, which is given as follows,

$$\Pi_{ij,1} = -C_1 \frac{\epsilon}{k} \left( \tau_{ij} - \frac{2}{3}\delta_{ij} k \right) \qquad \Pi_{ij,2} = -C_2 \left( P_{ij} - \frac{2}{3}\delta_{ij} P \right)$$

$$P_{ij} = -\tau_{ki} \frac{\partial \overline{u}_j}{\partial x_k} - \tau_{kj} \frac{\partial \overline{u}_i}{\partial x_k} \qquad P = \frac{1}{2} P_{kk}$$

Capital P stand for production not pressure

[1] B. E. Launder, G. J. Reece, W. Rodi. Progress in the Development of a Reynolds-Stress Turbulence Closure. 1975.

# The Reynolds stress model

- The **solvable** RSM equations of the LRR model are given as follows,

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \epsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right]$$

- With the following auxiliary relationships,

$$C_{ijk} = \frac{\partial}{\partial x_k}\left[\frac{\mu_t}{\sigma}\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j'}\right)\right]$$

$$\Pi_{ij} = -C_1 \frac{\epsilon}{k}\left(\tau_{ij} - \frac{2}{3}\delta_{ij}k\right) - C_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P_k\right)$$

If you compare this term with the original formulation of the LRR method, you will notice that this term has been further simplified

$$P_{ij} = -\tau_{ki}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{kj}\frac{\partial \bar{u}_i}{\partial x_k}$$

$$P = \frac{1}{2}P_{kk}$$

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}$$ and $$\epsilon = \nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}$$

# The Reynolds stress model

- And the following closure coefficients,

$$\sigma = 0.82 \qquad C_1 = 1.8 \qquad C_2 = 0.6$$

- With the following relation for the kinematic eddy viscosity,

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$$

- As the turbulent eddy viscosity is based on the turbulent dissipation rate $\epsilon$, we also solve the transport equation for this variable,

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}}\epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[ \left( \nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

# The Reynolds stress model

- The Reynolds stress model (RSM) [1, 2, 3, 4] is the most elaborate type of RANS turbulence model.
- It abandons the isotropic eddy-viscosity hypothesis.
- But still it is computed as a scalar quantity.
- The RSM closes the RANS equations by solving transport equations for the Reynolds stresses, together with an equation for the turbulent dissipation rate or the specific dissipation rate.
- This means that five additional transport equations are required in 2D flows, and seven additional transport equations are solved in 3D.
- Then, the Reynolds stresses are inserted directly into the momentum equations.
- If additional scalars are present (temperature, passive scalars, and so on), three additional equations need to be added.
- If the turbulent kinetic energy equation is needed for specific terms, it is obtained by taking the trace of the Reynolds tress tensor.
- The most used versions of the RSM are the LRR [3] and the SSG [5].

[1] M. M. Gibson, B. E. Launder. Ground Effects on Pressure Fluctuations in the Atmospheric Boundary Layer. 1978.
[2] B. E. Launder. Second-Moment Closure: Present... and Future?. 1989.
[3] B. E. Launder, G. J. Reece, W. Rodi. Progress in the Development of a Reynolds-Stress Turbulence Closure. 1975.
[4] B. J. Daly, F. H. Harlow. Transport Equations in Turbulence. 1970.
[5] C. G. Speziale, S. Sarkar, T. B. Gatski. Modelling the Pressure-Strain Correlation of Turbulence: An Invariant Dynamical Systems Approach. 1991.

# The Reynolds stress model

- The RSM might not always yield results that are clearly superior to EVM models.
- However, the use of the RSM is a must when the flow features of interest are the result of anisotropy in the Reynolds stresses.
- Among the examples are cyclone flows, highly swirling flows in combustors, rotating flow passages, and the stress-induced secondary flows in ducts.
- Despite its apparent superiority over EVM models, the RSM is not widely used.
- Also, the RSM is not widely validated as other EVM models.
- There are also algebraic version of the RSM models that solve two equations.
  - Explicit Algebraic Reynolds Stress Model [1, 2].
  - They are usually an extension of the $k-\epsilon$ and $k-\omega$ family models.

[1] W. Rodi. A New Algebraic Relation for Calculating Reynolds Stress. 1976.
[2] S. Girimaji. Fully Explicit and Self-Consistent Algebraic Reynolds Stress Model. 1996.

# The Reynolds stress model

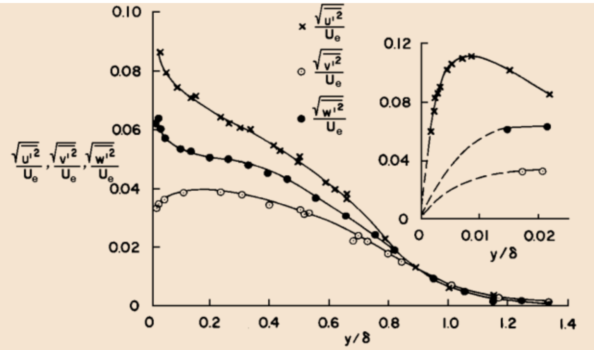
- The RSM model can be used with wall functions.
- The wall boundary conditions for the solution variables are all taken care of by the wall functions implementation.
- Therefore, when using commercial solvers you do not need to be concerned about the boundary conditions at the walls.
- If you are using a wall resolving approach, all Reynolds stresses must approach in an asymptotic way to zero at the wall.
- The freestream values can be computed as follows,

$$\overline{u_1'^2} = k$$

$$\overline{u_2'^2} = \overline{u_3'^2} = \frac{1}{2}k$$

$$\overline{u_i' u_j'} = 0 \qquad (i \neq j)$$

$$\overline{u'^2} : \overline{v'^2} : \overline{w'^2} \approx 4 : 2 : 3$$




Notice that the Reynolds stress are anisotropic

- The boundary condition for turbulent dissipation rate or specific dissipation rate are determined in the same manner as for the two-equations turbulence models.