On the closure coefficients

On the closure coefficients

- The coefficients used in the turbulence models do not come out of thin air.
- They have been calibrated using experiments, numerical simulations, or empirical correlations.
- During this calibration process, many assumptions are taken that sometimes might not be very realistic.
  - Such as, local equilibrium, local isotropy, two-dimensional flow, fully developed flow, and so on.
- Optimization methods, data driven simulations, and machine learning is also being used to calibrate these coefficients.
- Notice that we called then coefficients and not constants.
- They certainly can be adjusted.

On the closure coefficients

- Let us review the $k - \epsilon$ turbulence model,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- With the following closure coefficients,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- And auxiliary relationships,

$$\omega = \frac{\epsilon}{C_\mu k} \qquad\qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

On the closure coefficients

- Talking about canonical or simplified solutions, the RANS equations for a two-dimensional boundary layer (or pure shear flow) can be written as follows,

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial}{\partial y}\left(\overline{u'v'}\right)$$

$$\frac{\partial \overline{p}}{\partial y} \approx 0$$

- Where the following assumptions were taken,

$$v << u \qquad \frac{\partial}{\partial x} << \frac{\partial}{\partial y} \qquad \frac{\partial p}{\partial y} << \frac{\partial p}{\partial x}$$

- Under certain conditions, these equations provide high quality solutions of turbulent flows that can be used to validate models and calibrate coefficients.

On the closure coefficients

- Let us address how to calibrate the eddy viscosity coefficient C_μ which is used in the following eddy viscosity relation,

$$\nu_t = C_\mu \frac{k^2}{\epsilon}$$

- As we have seen, this equation is used in two-equation models.
- In the $k-\epsilon$ model, the coefficient C_μ is equal to 0.09.
- This coefficient also appears in the $k-\omega$ model, but it is called β^* instead.
- This coefficient can be calibrated using the approximation of the two-dimensional shear layer flow.
- That is, we are dealing with a simple pure shear flow, a big simplification.
- Additionally, let us assume local equilibrium and local isotropy,

$$P = \epsilon$$

- Which is not entirely true, as this quantities are not entirely in equilibrium, even in shear flows.

References:
S. Pope. Turbulent Flows, Cambridge University Press, 2000.
P. Bernard, J. Wallace. Turbulent Flow. Analysis, Measurement, and Prediction. Wiley. 2002.

On the closure coefficients

- If production is equal to dissipation, *i.e.*, $P = \epsilon$, we obtain,

$$\underbrace{\tau_{uv}\frac{\partial \overline{u}}{\partial y}}_{P} = \epsilon$$

- Using the Boussinesq hypothesis to derive τ_{uv},

$$\tau_{uv} = -\overline{u'v'} = \nu_t \frac{\partial \overline{u}}{\partial y}$$

- Combining the two previous equations, so we drop the derivative, we obtain,

$$\frac{\tau_{uv}^2}{\nu_t} = \epsilon$$

On the closure coefficients

- Finally, using the following relation of eddy viscosity,

$$\nu_t = C_\mu \frac{k^2}{\epsilon}$$

- Together with the following relationship (that we obtained in the previous slide),

$$\frac{\tau_{uv}^2}{\nu_t} = \epsilon$$

- We obtain the following relation to estimate the eddy viscosity coefficient C_μ,

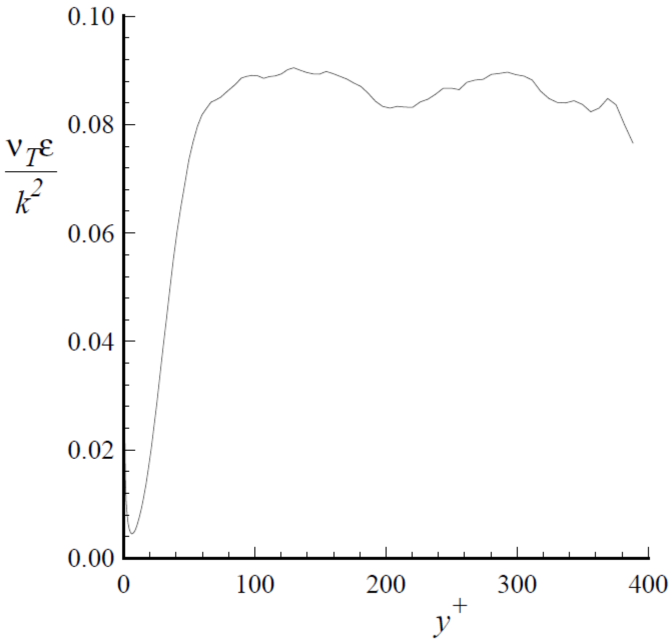
$$C_\mu = \frac{\tau_{uv}^2}{k^2}$$

This quantity can be measured using DNS or experiments
It represents the ratio of the shear Reynolds stress to turbulent kinetic energy

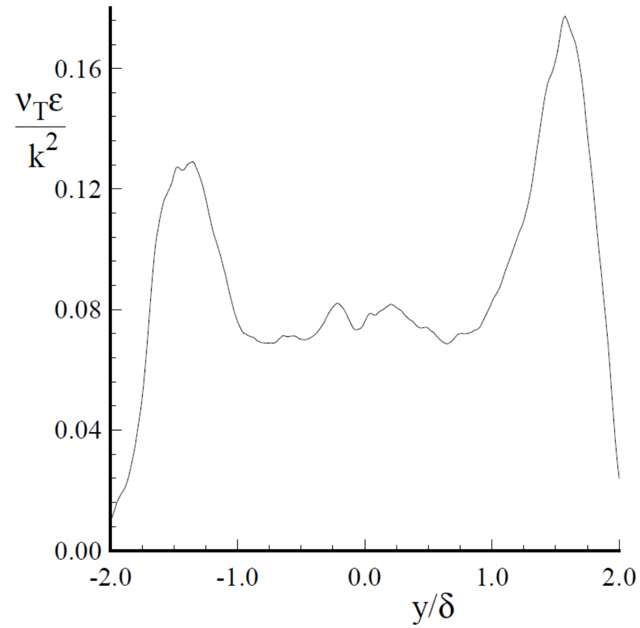
On the closure coefficients

- For simple shear flows the quantities k, ϵ, $\nu_t = -\overline{u'v'}/(\partial \overline{u}/\partial y)$, can be measured.

$$\nu_t \epsilon = \tau_{uv}^2$$



J. Kim, P. Moin, R. Moser. Turbulence statistics in fully developed channel flow at low Reynolds number. 1987.



M. Rogers, R. Moser. Direct simulation of a self-similar turbulent mixing layer. 1994.

- From these results, it can be seen that the eddy viscosity coefficient C_μ is approximately equal to 0.09
- This is where DNS and canonical flows come in handy.

On the closure coefficients

- The calibrated eddy viscosity coefficient C_μ can be expressed as,

$$C_\mu = \frac{\tau_{uv}^2}{k^2} = 0.09$$

- Sometimes in the literature you will find,

$$\sqrt{C_\mu} = \beta_r = \frac{\tau_{uv}}{k} = 0.3$$

- Where the ratio of the Reynolds stress to turbulent kinetic energy, *i.e.*, β_r, is often referred to as Bradshaw's constant [1], and sometimes as to Townsend's constant [2].

References:
[1] P. Bradshaw, D. Ferriss, N. Atwell. Calculation of Boundary Layer Development Using the Turbulent Energy Equation. 1967.
[2] A. Townsend. The Structure of Turbulent Shear Flow. 1976.

On the closure coefficients

- Calibration of the coefficient $C_{\epsilon 2}$.
- This constant can be calibrated using the hypothesis of decaying homogenous turbulence.
- In this type of flow the mean velocity and mean velocity gradients are equal to zero (huge simplification).
- Therefore, the governing equations can be expressed as follows,

$$\frac{\partial k}{\partial t} = -\epsilon \qquad \frac{\partial \epsilon}{\partial t} = -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

- Using experimental data and looking for power law solutions, an expression for $C_{\epsilon 2}$ is obtained as a function of a decay exponent.

$$k = \frac{k_0}{(t/t_0 + 1)^n} \qquad C_{\epsilon 2} = \frac{n+1}{n}$$

- Fitting experimental data, the decay coefficient n can be obtained.
- As can be seen, depending on the data, the value of the coefficient $C_{\epsilon 2}$ will vary.
- Values ranging from 1.5 to 2.2 are often found in the literature.

References:
S. Pope. Turbulent Flows, Cambridge University Press, 2000.
P. Durbin, B. Petterson-Reif. Statistical Theory and Modeling for Turbulent Flow. Wiley. 2011.

On the closure coefficients

- Let us address the coefficient $C_{\epsilon 1}$, its calibration is also related to experimental data.
- In this case we use the assumption of homogenous shear flow.
- Therefore, the governing equations can be expressed as follows,

$$\frac{\partial k}{\partial t} = P_k - \epsilon \qquad \qquad \frac{\partial \epsilon}{\partial t} = \frac{C_{\epsilon 1} P_k - C_{\epsilon 2} \epsilon}{k/\epsilon}$$

- Combining these equation we can find the following relationship,

$$C_{\epsilon 1} = \frac{C_{\epsilon 2} - 1}{P_k/\epsilon} + 1$$

- By using growth rates of homogenous sheared turbulence (for the spreading rate in a plane mixing layer), we can find the coefficient $C_{\epsilon 1}$.
- Values ranging from 1.2 to 2.0 are often found in the literature.

References:
S. Pope. Turbulent Flows, Cambridge University Press, 2000.
P. Durbin, B. Petterson-Reif. Statistical Theory and Modeling for Turbulent Flow. Wiley. 2011.

On the closure coefficients

- Let us address the coefficient σ_ϵ.
- This closure coefficient acts like an effective Prandtl number for dissipation diffusion and is specified to ensure the correct log-law slope of κ^{-1}.
- Using the previous coefficients, the value of σ_ϵ can be found using the following relationship,

$$\sigma_\epsilon = \frac{\kappa^2}{(C_{\epsilon 1} - C_{\epsilon 2})\sqrt{C_\mu}}$$

- This is the equation used by Jones and Launder (see references) to determine the value of the coefficient σ_ϵ.
- The original value obtained by Jones and Launder is equal to 1.3.
- Values ranging from 1.2 to 1.5 are often found in the literature.
- Notice that this relation can also be used to derive the value of the Karman constant κ by adjusting the value of the coefficients to produce a particular value of κ.
- This implies the use of data fitting or optimization methods.

References:
W. Jones, B. Launder. The prediction of laminarization with a two equation model of turbulence. Int. J. Heat Mass Transfer 15, 301–314. 1972.
P. Durbin, B. Petterson-Reif. Statistical Theory and Modeling for Turbulent Flow. Wiley. 2011.

On the closure coefficients

- Let us look back at the relationship for σ_ϵ presented in the previous slide.
- That relationship was obtained by manipulating the **solvable** dissipation rate equation and using some additional relations and assumptions.
- By rearranging the relationship, we obtain the following equation,

$$\frac{\sqrt{C_\mu}\sigma_\epsilon}{\kappa^2}\left(C_{\epsilon 1} - C_{\epsilon 2}\right) = 1$$

- Using the standard coefficients of the $k - \epsilon$ model,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- In the left-hand side of the previous condition, we obtain a value of 1.11, showing that this constraint is reasonably well satisfied.
- Which suggests that some kind of optimization method or data drive approach could be used to calibrate all these coefficients.
- In fact, machine learning methods are being used to calibrate turbulence models.

References:
S. Pope. Turbulent Flows, Cambridge University Press, 2000.
R. Bernard. Turbulent Fluid Flow. Wiley. 2019.

On the closure coefficients

- Finally, let us address the coefficient σ_k .
- This term affect the effective diffusivity in the TKE equation.
- Generally speaking, there is no consensus within the turbulence community about the value of this coefficient.
- Different values can be used depending on the set of experimental data available.
- In most cases this coefficient is assumed to be one.
- In fact, we have assumed that the value of this coefficient is one.
- Values ranging from 0.8 to 1.2 are often found in the literature.

On the closure coefficients

- Some observations of the effect of changing the closure coefficients in the standard $k-\epsilon$ turbulence model,

Coefficient	Value	Result of Increasing Value
C_μ	0.09	More mixing, more shear, greater change in pressure
$C_{\epsilon 1}$	1.44	Less mixing, lower shear, smaller change in pressure
$C_{\epsilon 2}$	1.92	More mixing, more shear, greater change in pressure
σ_k	1.0	Less mixing (Prandtl-Schmidt number)
σ_ϵ	1.3	Less mixing (Turbulent Prandtl number)

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- These observation can also be use with other variants of the $k-\epsilon$ turbulence models.
- Use with care as these are rough observations.

On the closure coefficients

- All the coefficients used in all turbulence models, undergo a similar calibration process.
- We briefly discussed how to calibrate the coefficients; the interest reader can refer to the original references of the turbulence models for a detailed description of the calibration of all coefficients.
- If you are interested in the standard $k - \epsilon$ turbulence model, these are the original references,

[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.

[2] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.

- If you are interested in the Wilcox 1988 $k - \omega$ turbulence model,

[1] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.

[2] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.

- Plus, many additional cross-references.

On the closure coefficients

At this point cynics might protest:

“This is pure empiricism! How can an entirely fictional equation, plucked out of thin air, and forced, through the judicious choice of some arbitrary coefficients, to reproduce one or two laboratory results, possibly hope to anticipate the evolution of a wide range of flows?”

The extraordinary thing, however, is that, by and large, it works reasonably well, at least much better than it ought to. So perhaps there is more to

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

than meets the eye. Perhaps there is some underlying rationale for this equation. It turns out that there is.

P. Davidson [1]

References:
[1] P. Davidson. Turbulence. An Introduction for Scientists and Engineers. Oxford University Press. 2015.