Supplement to

"Testing Alphas in Conditional Time-Varying Factor Models with High Dimensional Assets"

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This supplement is composed of four parts. Section A contains the proofs of the main results in the paper. Section B provides the proofs of technical lemmas that are used in the proofs of the main results. Section C presents four additional simulation studies: (i) the simulation studies for mimicking the Chinese stock market; (ii) the simulation results of the PY test (Pesaran and Yamagata, 2012); (iii) the generation of the error terms $\mathbf{E}_t$ that is borrowed from Fan et al. (2015); (iv) the simulation for the case where the summation of the GARCH coefficients is smaller than 0.5. Section D presents the results of the real data analysis with a relatively short window of length $h = 60$ and a long window of length $h = 200$ for both the US and Chinese stock market data.

A Proofs of the Main Results in the Paper

This section contains four subsections A.1-A.4. Section A.1 contains two lemmas about the properties of B-splines that are used in the proofs of Theorems 1 and 2; A.2 shows Theorem 1 by employing Lemmas A.3-A.7; A.3 presents the proof of Theorem 2 by using Lemmas A.8-A.14 and also verifies Lemma A.8, which is the first result of Theorem 2; and A.4 provides the proof of Proposition 1.

A.1 Two Technical Lemmas

We first present two lemmas regarding some properties of the spline approximation and B-spline basis functions, and their proofs are given in Section B.

Lemma A.1. *Define* $\rho_{NT,i0t} \equiv \delta_i(t/T) - \boldsymbol{\lambda}_{i0}^{0\top}\widetilde{B}(t/T)$ *and* $\rho_{NT,ijt} \equiv \beta_{ij}(t/T) - \boldsymbol{\lambda}_{ij}^{0\top}B(t/T)$ *for* $1 \leq j \leq d$ *and* $1 \leq i \leq N$. *Then, under Assumption (A1), there exist* $\boldsymbol{\lambda}_{i0}^{0} \in \mathbb{R}^L$ *and* $\boldsymbol{\lambda}_{ij}^{0} \in \mathbb{R}^L$ *such that*

$$\sup_{1\leq t\leq T} |\rho_{NT,i0t}| = O(L^{-r}) \text{ and } \sup_{1\leq t\leq T} |\rho_{NT,ijt}| = O(L^{-r}) \text{ as } T \to \infty.$$

Lemma A.2. *Under Assumption (A2), there exist constants $0 < c_1 \le C_1 < \infty$ and $0 < C_2 < \infty$, with probability* 1,

$$c_1 L^{-1} \le \lambda_{\min}\{(\mathbf{Z}^\top \mathbf{Z})/T\} \le \lambda_{\max}\{(\mathbf{Z}^\top \mathbf{Z})/T\} \le C_1 L^{-1}, \tag{A.1}$$

$$C_1^{-1} L \le \lambda_{\min}[\{(\mathbf{Z}^\top \mathbf{Z})/T\}^{-1}] \le \lambda_{\max}[\{(\mathbf{Z}^\top \mathbf{Z})/T\}^{-1}] \le c_1^{-1} L, \tag{A.2}$$

as $T \to \infty$, and for any nonzero vector $\mathbf{a} \in \mathbb{R}^T$ with $||\mathbf{a}|| = 1$, $\mathbf{a}^\top\{(\mathbf{Z}\mathbf{Z}^\top)/T\}\mathbf{a} \le C_2 L^{-1}$.

A.2 Proof of Theorem 1

Using the fact that

$$\widehat{\delta}_i(t/T) + \widehat{\boldsymbol{\beta}}_i(t/T)^\top \mathbf{f}_t = \mathbf{Z}_t^\top (\mathbf{Z}^\top \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{R}_i,$$

the $\widehat{e}_{it}$ given in (4) can be re-expressed as

$$\widehat{e}_{it} = R_{it} - \mathbf{Z}_t^\top (\mathbf{Z}^\top \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{R}_i. \tag{A.3}$$

Then the R_{it} in (2) can be re-written as

$$R_{it} = \delta_i^0 + \mathbf{Z}_t^\top \boldsymbol{\lambda}_i^0 + e_{it} + \rho_{NT,it},$$

where $\boldsymbol{\lambda}_i^0 = (\boldsymbol{\lambda}_{ij}^{0\top}, 0 \le j \le d)^\top$ and $\rho_{NT,it} \equiv \rho_{NT,i0t} + \sum_{j=1}^d \rho_{NT,ijt} f_{jt}$. Note that, by Lemma A.1 and Assumption (A2), we have

$$\sup_{1 \le t \le T} |\rho_{NT,it}| = O(L^{-r}) \tag{A.4}$$

for each $1 \le i \le N$.

Using the expressions of R_{it} and $\widehat{e}_{it}$ given above, we further have

$$\mathbf{R}_i = \delta_i^0 \mathbf{1}_T + \mathbf{Z}\boldsymbol{\lambda}_i^0 + \mathbf{e}_i + \boldsymbol{\rho}_{NT,i} \tag{A.5}$$

and

$$\begin{aligned} \widehat{\mathbf{e}}_i &= M_{\mathbf{Z}} \mathbf{R}_i = M_{\mathbf{Z}}(\mathbf{Z}\boldsymbol{\lambda}_i^0 + \mathbf{e}_i + \boldsymbol{\rho}_{NT,i} + \delta_i^0 \mathbf{1}_T) \\ &= M_{\mathbf{Z}} \mathbf{e}_i + M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} + M_{\mathbf{Z}} \delta_i^0 \mathbf{1}_T, \end{aligned}$$

where $\mathbf{e}_i = (e_{i1}, \cdots, e_{iT})^\top \in \mathbb{R}^T$ and $\boldsymbol{\rho}_{NT,i} = (\rho_{NT,i1}, \cdots, \rho_{NT,iT})^\top$. Subsequently, the statistic J_{NT}^* defined in (7) can be written as

$$\begin{aligned} J_{NT}^* &= N^{-1}T^{-1} \sum\nolimits_{i=1}^N \widehat{\mathbf{e}}_i^\top \mathbf{1}_T \mathbf{1}_T^\top \widehat{\mathbf{e}}_i - N^{-1}T^{-1} \sum\nolimits_{i=1}^N \sum\nolimits_{t=1}^T e_{it}^2 \eta_t^2 \\ &= N^{-1}T^{-1} \sum\nolimits_{i=1}^N (M_{\mathbf{Z}} \mathbf{e}_i + M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} + M_{\mathbf{Z}} \delta_i^0 \mathbf{1}_T)^\top \mathbf{1}_T \mathbf{1}_T^\top (M_{\mathbf{Z}} \mathbf{e}_i + M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} + M_{\mathbf{Z}} \delta_i^0 \mathbf{1}_T) \\ &\quad - N^{-1}T^{-1} \sum\nolimits_{i=1}^N e_{it}^2 \eta_t^2 \\ &= (N^{-1}T^{-1} \sum\nolimits_{i=1}^N \mathbf{e}_i^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{e}_i - N^{-1}T^{-1} \sum\nolimits_{i=1}^N \sum\nolimits_{t=1}^T e_{it}^2 \eta_t^2) \\ &\quad + N^{-1}T^{-1} \sum\nolimits_{i=1}^N \left(\delta_i^0\right)^2 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \\ &\quad + N^{-1}T^{-1} \sum\nolimits_{i=1}^N \boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} + 2N^{-1}T^{-1} \sum\nolimits_{i=1}^N \boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{e}_i \\ &\quad + 2N^{-1}T^{-1} \sum\nolimits_{i=1}^N \delta_i^0 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{e}_i + 2N^{-1}T^{-1} \sum\nolimits_{i=1}^N \delta_i^0 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i}. \end{aligned}$$

Accordingly,

$$\begin{aligned}
& J_{NT}^{*} - N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\left(\delta_i^0\right)^2 \mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T \\
= \ & (N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i - N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t=1}^{T}e_{it}^2\eta_t^2) \\
& + N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} + 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i \\
& + 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i + 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} \\
\equiv \ & \varphi_{NT} + \zeta_{NT,1} + \zeta_{NT,2} + \zeta_{NT,3} + \zeta_{NT,4}. \qquad \text{(A.6)}
\end{aligned}$$

It suffices to show that $\sigma_{NT}^{-1}\varphi_{NT} \xrightarrow{d} N(0,1)$ and $\sigma_{NT}^{-1}\zeta_{NT,k} = o_p(1)$ for $k = 1, \cdots, 4$.

We begin to show $\sigma_{NT}^{-1}\varphi_{NT} \xrightarrow{d} N(0,1)$. The quantity φ_{NT} can be re-written as

$$\begin{aligned}
\varphi_{NT} &= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i - N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t=1}^{T}e_{it}^2\eta_t^2 \\
&= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t,s=1}^{T}e_{it}e_{is}\eta_t\eta_s - N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}e_{it}^2\eta_t^2 \\
&= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t\neq s}e_{it}e_{is}\eta_t\eta_s = N^{-1}T^{-1}\sum\nolimits_{t\neq s}\mathbf{E}_t^\top\mathbf{E}_s\eta_t\eta_s \\
&= \sum\nolimits_{t=2}^{T}2N^{-1}T^{-1}\mathbf{E}_t^\top\eta_t(\sum\nolimits_{s=1}^{t-1}\mathbf{E}_s\eta_s) = \sum\nolimits_{t=2}^{T}\varphi_{NT,t}. \qquad \text{(A.7)}
\end{aligned}$$

By Assumption (A3),

$$\mathbb{E}\left(\varphi_{NT,t}\left|\mathcal{F}_{NT,t-1}\right.\right) = 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\mathbb{E}\left(e_{it}\left|\mathcal{F}_{NT,t-1}\right.\right)\eta_t(\sum\nolimits_{s=1}^{t-1}e_{is}\eta_s) = 0.$$

In addition,

$$\begin{aligned}
\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^2\left|\mathcal{F}_{NT,t-1}\right.\right) &= \sum\nolimits_{t=2}^{T}4N^{-2}T^{-2}\mathbb{E}\left\{\left(\sum\nolimits_{s=1}^{t-1}\mathbf{E}_s^\top\mathbf{E}_t\eta_s\eta_t\right)^2\left|\mathcal{F}_{NT,t-1}\right.\right\} \\
&= \sum\nolimits_{t=2}^{T}4N^{-2}T^{-2}\sum\nolimits_{s_1,s_2=1}^{t-1}\mathbf{E}_{s_1}^\top\mathbb{E}\left(\mathbf{E}_t\mathbf{E}_t^\top\left|\mathcal{F}_{NT,t-1}\right.\right)\mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2} \\
&= \sum\nolimits_{t=2}^{T}4N^{-2}T^{-2}\sum\nolimits_{s_1,s_2=1}^{t-1}\mathbf{E}_{s_1}^\top\boldsymbol{\Sigma}\mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^2\right) &= \mathbb{E}\left\{\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^2\left|\mathcal{F}_{NT,t-1}\right.\right)\right\} \\
&= \sum\nolimits_{t=2}^{T}4N^{-2}T^{-2}\sum\nolimits_{s_1,s_2=1}^{t-1}\mathbb{E}\left\{\mathbf{E}_{s_1}^\top\boldsymbol{\Sigma}\mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2}\right\} \\
&= 4N^{-2}T^{-2}\sum\nolimits_{t=2}^{T}\sum\nolimits_{s_1,s_2=1}^{t-1}tr\left\{\mathbb{E}\left\{\mathbf{E}_{s_2}\mathbf{E}_{s_1}^\top\boldsymbol{\Sigma}\eta_t^2\eta_{s_1}\eta_{s_2}\right\}\right\}.
\end{aligned}$$

Since $\mathbb{E}\left(\mathbf{E}_s\mathbf{E}_s^\top\left|\mathcal{F}_{NT,s-1}\right.\right) = \boldsymbol{\Sigma}$ and $\mathbb{E}\left(\mathbf{E}_{s'}\mathbf{E}_s^\top\left|\mathcal{F}_{NT,s-1}\right.\right) = \mathbf{0}$ for $s' < s$, we have that

$$\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^2\right) = 4N^{-2}T^{-2}\sum\nolimits_{t=2}^{T}\sum\nolimits_{s=1}^{t-1}\mathbb{E}(\eta_t^2\eta_s^2)tr\left(\boldsymbol{\Sigma}^2\right) = \sigma_{NT}^2. \qquad \text{(A.8)}$$

Next we will show that $\sigma_{NT}^{-2}\sum_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^{2}\,|\mathcal{F}_{NT,t-1}\right)\xrightarrow{p}1$ and

$$\sum_{t=2}^{T}\sigma_{NT}^{-2}\mathbb{E}\left(\varphi_{NT,t}^{2}I(|\varphi_{NT,t}|>\epsilon\sigma_{NT})\,|\mathcal{F}_{NT,t-1}\right)\xrightarrow{p}0$$

via verifying Lemmas A.3-A.6 below. Then, by the Martingale Central Limit Theorem (e.g., Corollary 3.1 of Hall and Heyde, 1980), we have that $\sigma_{NT}^{-1}\varphi_{NT}\xrightarrow{d}N(0,1)$. By Lemma A.7 below, we have $\sigma_{NT}^{-1}\zeta_{NT,k}=o_p(1)$ for $k=1,\cdots,4$, where $\zeta_{NT,k}$ are given in (A.6). Then the proof of Theorem 1 is complete.

Lemma A.3. *Under Assumptions (A2)-(A3) and $L^3T^{-1}=o(1)$, we have $\sigma_{NT}^{-2}\sum_{t=2}^{T}\mathbb{E}(\varphi_{NT,t}^{2}|\mathcal{F}_{NT,t-1})\xrightarrow{p}1$, as $(N,T)\to\infty$.*

Lemma A.4. *Under Condition (C2), Assumptions (A2)-(A3), and $L^3T^{-1}=o(1)$, we have $\Psi_{NT,1}=\sigma_{NT}^{4}\{1+o(1)\}$, as $(N,T)\to\infty$.*

Lemma A.5. *Under Conditions (C1)-(C2), Assumptions (A2)-(A3), and $L^3T^{-1}=o(1)$, we have $\sum_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^{4}\right)=o(\sigma_{NT}^{4})$, as $(N,T)\to\infty$.*

Lemma A.6. *Under Conditions (C1)-(C2), Assumptions (A2)-(A3), and $L^3T^{-1}=o(1)$, we have for any $\epsilon>0$,*

$$\sum_{t=2}^{T}\sigma_{NT}^{-2}\mathbb{E}\left(\varphi_{NT,t}^{2}I(|\varphi_{NT,t}|>\epsilon\sigma_{NT})\,|\mathcal{F}_{NT,t-1}\right)\xrightarrow{p}0,$$

as $(N,T)\to\infty$.

Lemma A.7. *Under Condition (C3) (i) and (ii), Assumptions (A1)-(A3), $L^3T^{-1}=o(1)$, and the local alternative given in (8), we have $\sigma_{NT}^{-1}\zeta_{NT,k}=o_p(1)$ for $k=1,\cdots,4$, as $(N,T)\to\infty$.*

The proofs of Lemmas A.3-A.7 are presented in Section B.

A.3 Proof of Theorem 2

We present the detailed proofs of the first two results of Theorem 2, namely $\sigma_{NT}^{-1}(\widehat{J}_{NT}^{*}-J_{NT}^{*})=o_p(1)$ and $\widehat{\sigma}_{NT}^{2}=\sigma_{NT}^{2}\{1+o_p(1)\}$. The third result, asymptotic normality, follows directly from the first two results, Theorem 1, and Slutsky's theorem. The following lemma shows the first result.

Lemma A.8. *Under Condition (C3), Assumptions (A1)-(A3), $L^3T^{-1}=o(1)$, and the local alternative given in (8), we have $\sigma_{NT}^{-1}(\widehat{J}_{NT}^{*}-J_{NT}^{*})=o_p(1)$, as $(N,T)\to\infty$.*

Proof. By the fact that

$$a^2-b^2=(a-b)^2+2(a-b)b, \tag{A.9}$$

we can write

$$\widehat{J}_{NT}^{*}-J_{NT}^{*}\equiv\widehat{D}_{NT,1}+\widehat{D}_{NT,2},$$

where

$$\begin{aligned}
\widehat{D}_{NT,1} &= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t=1}^{T}(\widehat{e}_{it}-e_{it})^2\eta_t^2 \quad \text{and} \\
\widehat{D}_{NT,2} &= 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t=1}^{T}2(\widehat{e}_{it}-e_{it})e_{it}\eta_t^2.
\end{aligned}$$

To prove Lemma A.8, we will show that $\sigma_{NT}^{-1}\widehat{D}_{NT,1}=o_p(1)$ and $\sigma_{NT}^{-1}\widehat{D}_{NT,2}=o_p(1)$ given below.

Denote Ξ =diag$(\eta_1^2,\cdots,\eta_T^2)$. Since $\widehat{\mathbf{e}}_i=M_{\mathbf{Z}}\mathbf{e}_i+M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}+M_{\mathbf{Z}}\delta_i^0\mathbf{1}_T$, we have

$$\begin{aligned}
\widehat{D}_{NT,1} &= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}(-P_{\mathbf{Z}}\mathbf{e}_i+M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}+M_{\mathbf{Z}}\delta_i^0\mathbf{1}_T)^\top\Xi\times \\
&\quad (-P_{\mathbf{Z}}\mathbf{e}_i+M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}+M_{\mathbf{Z}}\delta_i^0\mathbf{1}_T) \\
&\le 3(\widehat{D}_{NT,11}+\widehat{D}_{NT,12}+\widehat{D}_{NT,13}),
\end{aligned}$$

where $\widehat{D}_{NT,11}=N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{e}_i^\top P_{\mathbf{Z}}\Xi P_{\mathbf{Z}}\mathbf{e}_i$, $\widehat{D}_{NT,12}=N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\Xi M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}$, and $\widehat{D}_{NT,13}=N^{-1}T^{-1}\sum_{i=1}^{N}(\delta_i^0)^2\mathbf{1}_T^\top M_{\mathbf{Z}}\Xi M_{\mathbf{Z}}\mathbf{1}_T$. Let

$$\sigma_{\max}=\max\nolimits_{1\le i\le n}(\sigma_{ii}). \tag{A.10}$$

By (A.1) and (B.8) that $\mathbb{E}(\eta_t^2)\le(dC'')^2\mathbb{E}(1+||\mathbf{f}_t||)^2\le C^{**}$ for some constants $0<C'',C^{**}<\infty$, we have $\{\lambda_{\min}(\mathbf{Z}^\top\mathbf{Z}/T)\}^{-2}=O_p(L^2)$ and $\lambda_{\max}(\mathbf{Z}^\top\Xi\mathbf{Z}/T)=O_p(L^{-1})$. In addition,

$$\begin{aligned}
N^{-1}T^{-2}\sum\nolimits_{i=1}^{N}\mathbb{E}\{(\mathbf{Z}^\top\mathbf{e}_i)^\top(\mathbf{Z}^\top\mathbf{e}_i)\} &= N^{-1}T^{-2}\sum\nolimits_{i=1}^{N}\mathbb{E}\{(\sum\nolimits_{t=1}^{T}e_{it}Z_{tk})^2\} \\
&= N^{-1}T^{-2}\sum\nolimits_{i=1}^{N}\sum\nolimits_{t=1}^{T}\mathbb{E}(e_{it}^2Z_{tk}^2) \\
&\le N^{-1}T^{-2}N\sigma_{\max}\sum\nolimits_{t=1}^{T}\mathbb{E}(Z_{tk}^2).
\end{aligned}$$

By B-spline properties, there exists a constant $0<\widetilde{C}<\infty$ such that $\mathbb{E}(Z_{tk}^2)\le\widetilde{C}L^{-1}$. Hence,

$$N^{-1}T^{-2}\sum\nolimits_{i=1}^{N}\mathbb{E}\{(\mathbf{Z}^\top\mathbf{e}_i)^\top(\mathbf{Z}^\top\mathbf{e}_i)\}\le\sigma_{\max}\widetilde{C}L^{-1}T^{-1},$$

and thus $N^{-1}T^{-2}\sum_{i=1}^{N}(\mathbf{Z}^\top\mathbf{e}_i)^\top(\mathbf{Z}^\top\mathbf{e}_i)=O_p(L^{-1}T^{-1})$. Using the above results, we have

$$\begin{aligned}
\widehat{D}_{NT,11} &\le \lambda_{\max}((\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}^\top\Xi\mathbf{Z}(\mathbf{Z}^\top\mathbf{Z})^{-1})\{N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}(\mathbf{Z}^\top\mathbf{e}_i)^\top(\mathbf{Z}^\top\mathbf{e}_i)\} \\
&\le \{\lambda_{\min}(\mathbf{Z}^\top\mathbf{Z}/T)\}^{-2}\lambda_{\max}(\mathbf{Z}^\top\Xi\mathbf{Z}/T)\{N^{-1}T^{-2}\sum\nolimits_{i=1}^{N}(\mathbf{Z}^\top\mathbf{e}_i)^\top(\mathbf{Z}^\top\mathbf{e}_i)\} \\
&= O_p(L^2)O_p(L^{-1})O_p(L^{-1}T^{-1})=O_p(T^{-1}).
\end{aligned}$$

By (B.14) that

$$2c_M^2N^{-2}tr\left(\boldsymbol{\Sigma}^2\right)\{1+o(1)\}\le\sigma_{NT}^2\le 2C_M^2N^{-2}tr\left(\boldsymbol{\Sigma}^2\right)\{1+o(1)\},$$

and Condition (C3)(iii), we have

$$\sigma_{NT}^{-1}\widehat{D}_{NT,11}=O_p[T^{-1}N\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}]=o_p(1). \tag{A.11}$$

It is worth noting that

$$M_{\mathbf{Z}}\Xi M_{\mathbf{Z}} = \Xi - 2P_{\mathbf{Z}}\Xi + P_{\mathbf{Z}}\Xi P_{\mathbf{Z}} \leq 2\Xi + 2P_{\mathbf{Z}}\Xi P_{\mathbf{Z}},$$

and by the result in Lemma A.2, we have $\lambda_{\max}(P_{\mathbf{Z}}\Xi P_{\mathbf{Z}}) = O_p(L^{-2})$. These results, together with (B.14), imply

$$\begin{aligned}
\sigma_{NT}^{-1}\widehat{D}_{NT,12} &\leq 2\sigma_{NT}^{-1}N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}\Xi\boldsymbol{\rho}_{NT,i} \\
&\quad +2\lambda_{\max}(P_{\mathbf{Z}}\Xi P_{\mathbf{Z}})\sigma_{NT}^{-1}(\ N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}\boldsymbol{\rho}_{NT,i}) \\
&= \sigma_{NT}^{-1}N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}\mathbb{E}(\Xi)\boldsymbol{\rho}_{NT,i}\{1+o_p(1)\} \\
&\quad +O_p(L^{-2})\sigma_{NT}^{-1}(\ N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}\boldsymbol{\rho}_{NT,i}) \\
&= O_p[N\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}N^{-1}T^{-1}NTL^{-2r}] \\
&= O_p[L^{-2r}N\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}] = o_p(1),
\end{aligned} \tag{A.12}$$

where the last equality follows from Condition (C3)(i). By Condition (C3)(iii) and (8), we have

$$\begin{aligned}
&\{tr(\boldsymbol{\Sigma}^2)\}^{-1}N^{1+\varrho}\sum_{i=1}^{N}\left(\delta_i^0\right)^2 \\
= \ &\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}N^{1+\varrho}T^{-1}N^{-1}\sum_{i=1}^{N}(c_i^0)^2 = o(1),
\end{aligned} \tag{A.13}$$

for any $\varrho > 0$. Also

$$\begin{aligned}
\mathbf{1}_T^{\top}M_{\mathbf{Z}}\Xi M_{\mathbf{Z}}\mathbf{1}_T &\leq 2\Big\{\sum_{t=1}^{T}\eta_t^2 + \lambda_{\max}(P_{\mathbf{Z}}\Xi P_{\mathbf{Z}})T\Big\} \\
&= O_p(T + TL^{-2}) = O_p(T).
\end{aligned}$$

The above results imply that, with probability approaching 1,

$$\begin{aligned}
&\sigma_{NT}^{-1}\widehat{D}_{NT,13} \\
\leq \ &(2c_M^2 + o(1))^{-1/2}N\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}N^{-1}T^{-1}O(T)\sum_{i=1}^{N}(\delta_i^0)^2 \\
= \ &(2c_M^2 + o(1))^{-1/2}N\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}N^{-1}O(1)\sum_{i=1}^{N}(\delta_i^0)^2 \\
= \ &O\Big[\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}\sum_{i=1}^{N}(\delta_i^0)^2\Big] = O\Big[\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1}N\sum_{i=1}^{N}(\delta_i^0)^2\Big] = o(1).
\end{aligned} \tag{A.14}$$

Consequently, (A.11), (A.12) and (A.14) conclude that $\sigma_{NT}^{-1}\widehat{D}_{NT,1} = o_p(1)$.

We next show the order of $\sigma_{NT}^{-1}\widehat{D}_{NT,2}$ by expressing

$$\begin{aligned}
\widehat{D}_{NT,2} &= 2N^{-1}T^{-1}\sum_{i=1}^{N}(-P_{\mathbf{Z}}\mathbf{e}_i + M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} + M_{\mathbf{Z}}\delta_i^0\mathbf{1}_T)^{\top}\Xi\mathbf{e}_i \\
&\equiv \widehat{D}_{NT,21} + \widehat{D}_{NT,22} + \widehat{D}_{NT,23},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{D}_{NT,21} &= -2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top P_{\mathbf{Z}}\Xi\mathbf{e}_i, \\
\widehat{D}_{NT,22} &= 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\Xi\mathbf{e}_i \quad \text{and} \\
\widehat{D}_{NT,23} &= 2N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\Xi\mathbf{e}_i.
\end{aligned}$$

It is worth noting that

$$\begin{aligned}
&|N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top P_{\mathbf{Z}}\Xi\mathbf{e}_i| \\
\leq\ &||(\mathbf{Z}^\top\mathbf{Z}/T)^{-1}||\Big\{N^{-1}\sum\nolimits_{i=1}^{N}||\mathbf{e}_i^\top\mathbf{Z}/T||||\mathbf{e}_i^\top\Xi\mathbf{Z}/T||\Big\} = O_p(LT^{-1}).
\end{aligned}$$

This, together with (B.14) and Condition (C3)(iii), implies that

$$\sigma_{NT}^{-1}\widehat{D}_{NT,21} = O_p\Big[LT^{-1}N\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}\Big] = o_p(1).$$

In addition, employing the same techniques as those used in the proof of Lemma A.7, we can show that $\sigma_{NT}^{-1}\widehat{D}_{NT,22} = o_p(1)$ and $\sigma_{NT}^{-1}\widehat{D}_{NT,23} = o_p(1)$. As a result, $\sigma_{NT}^{-1}\widehat{D}_{NT,2} = o_p(1)$, which completes the proof. □

Next, we show the second result that $\widehat{\sigma}_{NT}^2 = \sigma_{NT}^2\{1+o_p(1)\}$. Prior presenting Theorem 2, we have introduced the scaled estimator,

$$\widehat{\sigma}_{NT}^2 = 2N^{-2}T^{-2}\sum\nolimits_{t\neq s}\eta_t^2\eta_s^2\widehat{tr(\mathbf{\Sigma}^2)},$$

where $\widehat{tr(\mathbf{\Sigma}^2)} = T^2(T+(1+d)L-1)^{-1}(T-(1+d)L)^{-1}\Big\{tr(\widehat{\mathbf{\Sigma}}^2) - tr^2(\widehat{\mathbf{\Sigma}})/(T-(1+d)L)\Big\}$. By Lemmas A.9-A.10 below, we will demonstrate that

$$T^{-1/2}tr(\widehat{\mathbf{\Sigma}}) = T^{-1/2}tr(\mathbf{\Sigma}) + o_p\big\{tr^{1/2}(\mathbf{\Sigma}^2)\big\}. \tag{A.15}$$

By Condition (C3)(ii) that $T^{-1/2}N\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2} = O(1)$, we have

$$T^{-1/2}tr(\mathbf{\Sigma}) \leq \sigma_{\max}T^{-1/2}N = O\big\{tr(\mathbf{\Sigma}^2)\big\}. \tag{A.16}$$

Hence,

$$T^{-1}\{tr(\widehat{\mathbf{\Sigma}})\}^2 = T^{-1}\{tr(\mathbf{\Sigma})\}^2 + o_p\big\{tr(\mathbf{\Sigma}^2)\big\}. \tag{A.17}$$

By Lemmas A.11-A.14 below, we will show that

$$tr(\widehat{\mathbf{\Sigma}}^2) = tr(\mathbf{\Sigma}^2) + T^{-1}\{tr(\mathbf{\Sigma})\}^2 + o_p\big\{tr(\mathbf{\Sigma}^2)\big\}. \tag{A.18}$$

This, together with (A.17), implies that

$$tr(\widehat{\mathbf{\Sigma}}^2) - T^{-1}\{tr(\widehat{\mathbf{\Sigma}})\}^2 - tr(\mathbf{\Sigma}^2) = o_p\big\{tr(\mathbf{\Sigma}^2)\big\}.$$

Using the fact that $T^2(T+(1+d)L-1)^{-1}(T-(1+d)L)^{-1}=1+o(1)$ and $T(T-(1+d)L)^{-1}=1+o(1)$, we then have $\widehat{tr(\mathbf{\Sigma}^2)}=tr(\mathbf{\Sigma}^2)+o_p(tr(\mathbf{\Sigma}^2))$. Therefore, we have $\widehat{\sigma}^2_{NT}=\sigma^2_{NT}\{1+o_p(1)\}$.

To verify (A.15), we follow the definition of $T^{-1/2}tr(\widehat{\mathbf{\Sigma}})$ with $\widehat{\mathbf{e}}_i=M_{\mathbf{Z}}\mathbf{e}_i+M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}+M_{\mathbf{Z}}\delta_i^0\mathbf{1}_T$ and obtain

$$\begin{aligned}
& T^{-1/2}tr(\widehat{\mathbf{\Sigma}})-T^{-1/2}tr(\mathbf{\Sigma}) \\
= \ & T^{-1/2}\sum\nolimits_{i=1}^N\{T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_i-(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)\}-T^{-1/2}\sum\nolimits_{i=1}^N\mathbb{E}(e_{it}^2) \\
= \ & T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_i-T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\widehat{\mathbf{e}}_i^\top\mathbf{1}_T\mathbf{1}_T^\top\widehat{\mathbf{e}}_i-T^{-1/2}\sum\nolimits_{i=1}^N\mathbb{E}(e_{it}^2) \\
= \ & T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N(\mathbf{e}_i+\boldsymbol{\rho}_{NT,i}+\delta_i^0\mathbf{1}_T)^\top M_{\mathbf{Z}}(\mathbf{e}_i+\boldsymbol{\rho}_{NT,i}+\delta_i^0\mathbf{1}_T) \\
& -T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N(\mathbf{e}_i+\boldsymbol{\rho}_{NT,i}+\delta_i^0\mathbf{1}_T)^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}(\mathbf{e}_i+\boldsymbol{\rho}_{NT,i}+\delta_i^0\mathbf{1}_T) \\
& -T^{-1/2}\sum\nolimits_{i=1}^N\mathbb{E}(e_{it}^2) \\
= \ & \{T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_i-T^{-1/2}\sum\nolimits_{i=1}^N\mathbb{E}(e_{it}^2)\}+T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} \\
& +T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N(\delta_i^0)^2\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T+2T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} \\
& +2T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\delta_i^0\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T+2T^{-1/2}T^{-1}\sum\nolimits_{i=1}^N\delta_i^0\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T \\
& -T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i-T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\left(\delta_i^0\right)^2\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T \\
& -T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}-2T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i \\
& -2T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i-2T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} \\
\equiv \ & \sum\nolimits_{j=1}^6\xi_{NT,j}-\sum\nolimits_{j=1}^6\vartheta_{NT,j}. \qquad \text{(A.19)}
\end{aligned}$$

Lemma A.9. *Under Condition (C3), Assumptions (A1)-(A3), $L^3T^{-1}=o(1)$, and the local alternative given in (8), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,j}=o_p(1)$, for $j=1,3,\cdots,6$, and $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2}N^{\varrho}\vartheta_{NT,2}=o_p(1)$, as $(N,T)\to\infty$, where ϱ is given in Condition (C3)(iii).*

Lemma A.10. *Under Condition (C3), Assumptions (A1)-(A3), $L^3T^{-1}=o(1)$, and the local alternative given in (8), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,j}=o_p(1)$, as $(N,T)\to\infty$, for $j=1,\cdots,6$.*

By Lemmas A.9-A.10 and (A.19), we have verified (A.15). We next demonstrate (A.18). Let $\widehat{\sigma}_{ij}$ be the ij-th element of $\widehat{\mathbf{\Sigma}}$, which is

$$\widehat{\sigma}_{ij}=T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j-(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top\mathbf{1}_T). \qquad \text{(A.20)}$$

Then,

$$\begin{aligned}
tr(\widehat{\mathbf{\Sigma}}^2) &= \sum\nolimits_{i,j=1}^N\{T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j-(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top\mathbf{1}_T)\}^2 \\
&= \sum\nolimits_{i,j=1}^N(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)^2-2\sum\nolimits_{i,j=1}^N(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top\mathbf{1}_T) \\
&\quad +\{\sum\nolimits_{i=1}^N(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)^2\}^2.
\end{aligned}$$

By Lemmas A.11-A.14 below, we have

$$\{tr(\mathbf{\Sigma}^2)\}^{-1}\Big\{\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)^2 - tr(\mathbf{\Sigma}^2) - T^{-1}\{tr(\mathbf{\Sigma})\}^2\Big\} = o_p(1),$$

$\{tr(\mathbf{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top\mathbf{1}_T) = o_p(1)$, and $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)^2 = o_p(1)$. Thus, the result of (A.18) follows.

Lemma A.11**.** *Under Conditions (C1) and (C3), Assumptions (A1)-(A3), $L^rT^{-3/2} = o(1)$, $L^3T^{-1} = o(1)$, and the local alternative given in (8), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1}\{\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)^2 - tr(\mathbf{\Sigma}^2) - T^{-1}\{tr(\mathbf{\Sigma})\}^2\} = o_p(1)$, as $(N,T)\to\infty$.*

Lemma A.12**.** *Under Condition (C3), Assumptions (A2)-(A3), $L^rT^{-3/2} = o(1)$, $L^3T^{-1} = o(1)$, and the local alternative given in (8), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Delta_{NT,1} = o_p(1)$, as $(N,T)\to\infty$.*

Lemma A.13**.** *Under Condition (C3), Assumptions (A2)-(A3), $L^rT^{-3/2} = o(1)$, $L^3T^{-1} = o(1)$, and the local alternative given in (8), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Delta_{NT,2} = o_p(1)$, as $(N,T)\to\infty$.*

Lemma A.14**.** *Under Condition (C3), Assumptions (A1)-(A3), $L^rT^{-3/2} = o(1)$, $L^3T^{-1} = o(1)$, and the local alternative given in (8), we have*

$$\begin{aligned}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)^2 &= o_p(1) \text{ and}\\ \{tr(\mathbf{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)(T^{-1}\widehat{\mathbf{e}}_i^\top\mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top\mathbf{1}_T) &= o_p(1),\end{aligned}$$

as $(N,T)\to\infty$.

The proofs of Lemmas A.9-A.14 are given in Section B.

A.4 Proof of Proposition 1

Proof. Let $a_{ts} = \mathbf{Z}_t^\top(\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}_s - \mathbf{F}_t^\top(\mathbf{F}^\top\mathbf{F})^{-1}\mathbf{F}_s$. Then

$$\begin{aligned}(T-d-1)C_T^{(i)} &= \mathbf{e}_i^\top(P_{\mathbf{Z}} - P_{\mathbf{F}})\mathbf{e}_i/\widehat{\sigma}_i^2\\ &= \sum_{t=1}^{T}\sum_{s=1}^{T} e_{it}\left(\mathbf{Z}_t^\top(\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}_s - \mathbf{F}_t^\top(\mathbf{F}^\top\mathbf{F})^{-1}\mathbf{F}_s\right)e_{is}/\widehat{\sigma}_i^2\\ &= \sum_{t=1}^{T} e_{it}^2 a_{tt}/\widehat{\sigma}_i^2 + 2\sum_{t=2}^{T}(\sum_{s=1}^{t-1} e_{it}e_{is}a_{ts})/\widehat{\sigma}_i^2, \qquad (A.21)\end{aligned}$$

where $\widehat{\sigma}_i^2 = (T-d-1)^{-1}\mathbf{e}_i^\top M_{\mathbf{F}}\mathbf{e}_i$. We can readily show that

$$\sum_{t=1}^{T}\mathbb{E}(e_{it}^2 a_{tt}|\mathbf{F}) = \sigma_i^2\sum_{t=1}^{T} a_{tt} = \sigma_i^2\mathrm{tr}\,(P_{\mathbf{Z}} - P_{\mathbf{F}}) = \sigma_i^2(d+1)(L-1).$$

This result, together with $\widehat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2$, implies

$$\sum_{t=1}^{T} e_{it}^2 a_{tt}/\widehat{\sigma}_i^2 = (d+1)(L-1) + o_p(1). \qquad (A.22)$$

Using the fact that $P_{\mathbf{Z}}P_{\mathbf{F}} = P_{\mathbf{F}}P_{\mathbf{Z}} = P_F$, we have

$$
\begin{aligned}
& \sum_{t=2}^{T} \mathbb{E}[(2\sum_{s=1}^{t-1} e_{it}e_{is}a_{ts})^2 | \mathcal{F}_{t-1}] \\
= \;& 4\sigma_i^2 \sum_{t=2}^{T} \sum_{s=1}^{t-1} \sum_{r=1}^{t-1} e_{is}e_{ir}a_{ts}a_{tr} = 4\sigma_i^2 \sum_{t=2}^{T} \sum_{s=1}^{t-1} \mathbb{E}(e_{is}^2)a_{ts}^2 + o_p(1) \\
= \;& 2\sigma_i^4 \sum_{1\le t\ne s\le T} a_{ts}^2 + o_p(1) = 2\sigma_i^4 \mathrm{tr}[(P_{\mathbf{Z}} - P_{\mathbf{F}})^2] + o_p(1) \\
= \;& 2\sigma_i^4 \mathrm{tr}(P_{\mathbf{Z}} - P_{\mathbf{F}}) + o_p(1) = 2\sigma_i^4 (d+1)(L-1) + o_p(1).
\end{aligned}
$$

With the above result, and applying the Martingale Central Limit Theorem and subsequently verifying the Lindeberg's condition of $2\sum_{t=2}^{T}(\sum_{s=1}^{t-1} e_{it}e_{is}a_{ts})$, we have

$$
\frac{1}{\sqrt{2(d+1)(L-1)}} \frac{2}{\sigma_i^2} \sum_{t=2}^{T} (\sum_{s=1}^{t-1} e_{it}e_{is}a_{ts}) \xrightarrow{d} N(0,1).
$$

Since $\widehat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2$, we further have

$$
\frac{1}{\sqrt{2(d+1)(L-1)}} \frac{2}{\widehat{\sigma}_i^2} \sum_{t=2}^{T} (\sum_{s=1}^{t-1} e_{it}e_{is}a_{ts}) \xrightarrow{d} N(0,1). \tag{A.23}
$$

By (A.21), (A.22) and (A.23), we finally obtain that

$$
\frac{(T-d-1)C_T^{(i)} - (L-1)(d+1)}{\sqrt{2(d+1)(L-1)}} \xrightarrow{d} N(0,1) \quad \text{(Wilks phenomenon)}.
$$

□

B Proofs of the Technical Lemmas

In this section we provide proofs of the technical lemmas to which we referred in Section A.

B.1 Proof of Lemmas A.1-A.2 Used in the Proofs of Theorems 1-2

Proof of Lemma A.1. By Corollary 6.21 in Schumaker (1981), we have $\sup_{1\le t\le T}|\rho_{NT,ijt}| = O(L^{-r})$ for every $1 \le j \le d$. By the same corollary, there exists $\boldsymbol{\lambda}_{i0}^0 \in \mathbb{R}^L$ such that $\sup_{1\le t\le T}$ $|\alpha_i(t/T) - \boldsymbol{\lambda}_{i0}^{0\top} B(t/T)| = O(L^{-r})$. As a result,

$$
\begin{aligned}
& \sup_{1\le t\le T}|\rho_{NT,i0t}| \\
\le \;& \sup_{1\le t\le T}|\alpha_i(t/T) - \boldsymbol{\lambda}_{i0}^{0\top} B(t/T)| + \left| T^{-1}\sum_{t=1}^{T} \boldsymbol{\lambda}_{i0}^{0\top} B(t/T) - T^{-1}\sum_{t=1}^{T} \alpha_i(t/T)\right| \\
\le \;& \sup_{1\le t\le T}|\alpha_i(t/T) - \boldsymbol{\lambda}_{i0}^{0\top} B(t/T)| + T^{-1}\sum_{t=1}^{T} \sup_{1\le t\le T}|\alpha_i(t/T) - \boldsymbol{\lambda}_{i0}^{0\top} B(t/T)| = O(L^{-r}).
\end{aligned}
$$

□

Proof of Lemma A.2. Following a similar procedure to that in the proof for Lemma A.7 of Ma and Yang (2011), we have that, for any nonzero vector $\mathbf{a_2} \in \mathbb{R}^{L(d+1)}$ with $||\mathbf{a_2}|| = 1$, for sufficiently large T,

$$c'L^{-1} \leq \mathbf{a_2}^\top \{\mathbb{E}(\mathbf{Z}^\top \mathbf{Z})/T\} \mathbf{a_2} \leq C'L^{-1} \tag{B.1}$$

for some constants $0 < c' \leq C' < \infty$, and, with probability approaching 1,

$$\{c' + o(1)\}L^{-1} \leq \lambda_{\min}\{(\mathbf{Z}^\top \mathbf{Z})/T\} \leq \lambda_{\max}\{(\mathbf{Z}^\top \mathbf{Z})/T\} \leq \{C' + o(1)\}L^{-1},$$

as $T \to \infty$. Accordingly, this completes the proof of (A.1) by letting $c_1 = c' + o(1)$ and $C_1 = C' + o(1)$, and the result of (A.2) follows immediately from (A.1). Note that Ma and Yang (2011) used the B-spline basis function multiplied by $L^{1/2}$, hence, L^{-1} would disappear in both sides of the above inequalities.

For any $1 \leq t \leq T$ and $1 \leq \ell \leq L$, there exists some constant $0 < M_1 < \infty$ such that $0 \leq B_\ell(t/T) \leq M_1$. Also by Condition (A2), we have $\mathbb{E}||\mathbf{f}_t||^2 \leq M'$ for some constant $0 < M' < \infty$. Thus, for any nonzero vector $\mathbf{a} = (a_1, \cdots, a_T)^\top$ with $||\mathbf{a}|| = 1$, we have

$$\begin{aligned}
\mathbf{a}^\top \mathbb{E}(\mathbf{Z}\mathbf{Z}^\top)\mathbf{a} &\leq \sum_{j=1}^{d} \sum_{\ell=1}^{L} \mathbb{E}\{\sum_{t=1}^{T} B_\ell(t/T) f_{jt} a_t\}^2 \\
&\leq 2M'^2 \sum_{\ell=1}^{L} \sum_{t\in\{t:|\ell(t)-\ell|\leq q-1\}} \sum_{s\in\{s:|\ell(s)-\ell|\leq q-1\}} B_\ell(t/T)B_\ell(s/T)|a_t||a_s| \\
&\leq 2M'^2 M_1^2 \sum_{\ell=1}^{L} (\sum_{t\in\{t:|\ell(t)-\ell|\leq q-1\}} |a_t|)^2 \\
&\leq 2M'^2 M_1^2 C^* T L^{-1} \sum_{\ell=1}^{L} \sum_{t\in\{t:|\ell(t)-\ell|\leq q-1\}} |a_t|^2 \\
&\leq 2M'^2 M_1^2 C^* T L^{-1} q ||\mathbf{a}||^2 = C_2 ||\mathbf{a}||^2 T L^{-1} = C_2 T L^{-1}
\end{aligned}$$

for some constant $0 < C^* < \infty$, where $C_2 = 2M'^2 M_1^2 C^* q$. By Bernstein's inequality given in Bosq (1998), we obtain $|\mathbf{a}^\top(\mathbf{Z}\mathbf{Z}^\top)\mathbf{a} - \mathbf{a}^\top \mathbb{E}(\mathbf{Z}\mathbf{Z}^\top)\mathbf{a}| = o_{a.s.}(TL^{-1})$. This completes the proof of Lemma A.2. $\square$

B.2 Proofs of Lemmas A.3-A.7 Used in the Proof of Theorem 1

Proof of Lemma A.3. Clearly, $\mathbb{E}\left\{\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 \,|\mathcal{F}_{NT,t-1}\right)\right\} = \sigma_{NT}^2$. Hence, to prove this lemma, we only need to show that

$$\mathrm{Var}\left\{\sigma_{NT}^{-2} \sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 \,|\mathcal{F}_{NT,t-1}\right)\right\} = o(1). \tag{B.2}$$

Let

$$\mathbb{E}\left\{\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 \,|\mathcal{F}_{NT,t-1}\right)\right\}^2 = \Psi_{NT,1} + \Psi_{NT,2},$$

where

$$\begin{aligned}
\Psi_{NT,1} &= \mathbb{E}\left\{2\sum_{2\leq t_1<t_2}^{T} \mathbb{E}\left(\varphi_{NT,t_1}^2 \,|\mathcal{F}_{NT,t_1-1}\right) \mathbb{E}\left(\varphi_{NT,t_2}^2 \,|\mathcal{F}_{NT,t_2-1}\right)\right\} \text{ and} \\
\Psi_{NT,2} &= \sum_{t=2}^{T} \mathbb{E}\left\{\mathbb{E}\left(\varphi_{NT,t}^2 \,|\mathcal{F}_{NT,t-1}\right)\right\}^2 \leq \sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^4\right).
\end{aligned}$$

By Lemmas A.4 and A.5 below, we have that $\Psi_{NT,1} = \sigma_{NT}^4 \{1 + o(1)\}$ and $\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^4\right) = o(\sigma_{NT}^4)$. Hence, $\mathbb{E}\left\{\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 | \mathcal{F}_{NT,t-1}\right)\right\}^2 = \sigma_{NT}^4 + o(\sigma_{NT}^4)$ and

$$\begin{aligned}
& \mathrm{Var}\left\{\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 | \mathcal{F}_{NT,t-1}\right)\right\} \\
= \;& \mathbb{E}\left\{\sum_{t=2}^{T} \mathbb{E}\left(\varphi_{NT,t}^2 | \mathcal{F}_{NT,t-1}\right)\right\}^2 - \sigma_{NT}^4 = o(\sigma_{NT}^4).
\end{aligned}$$

This completes the proof of (B.2). □

Proof of Lemma A.4. We write

$$\begin{aligned}
\Psi_{NT,1} \;=\;& 2\sum_{2\le t_1<t_2}^{T} \mathbb{E}\left\{\left(4N^{-2}T^{-2}\sum_{s_1,s_2=1}^{t_1-1} \mathbf{E}_{s_1}^\top \boldsymbol{\Sigma} \mathbf{E}_{s_2} \eta_{t_1}^2 \eta_{s_1}\eta_{s_2}\right) \times \right. \\
& \left. \left(4N^{-2}T^{-2}\sum_{s_3,s_4=1}^{t_2-1} \mathbf{E}_{s_3}^\top \boldsymbol{\Sigma} \mathbf{E}_{s_4} \eta_{t_2}^2 \eta_{s_3}\eta_{s_4}\right)\right\}.
\end{aligned}$$

By Condition (A3), $\mathbb{E}\left(\mathbf{E}_{s_1}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_2}\mathbf{E}_{s_3}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_4} | f\right) \neq 0$, only holds under the following four scenarios: (1) $\mathbf{E}_{s_1} = \mathbf{E}_{s_2} \neq \mathbf{E}_{s_3} = \mathbf{E}_{s_4}$; (2) $\mathbf{E}_{s_1} = \mathbf{E}_{s_3} \neq \mathbf{E}_{s_2} = \mathbf{E}_{s_4}$; (3) $\mathbf{E}_{s_1} = \mathbf{E}_{s_4} \neq \mathbf{E}_{s_2} = \mathbf{E}_{s_3}$; (4) $\mathbf{E}_{s_1} = \mathbf{E}_{s_2} = \mathbf{E}_{s_3} = \mathbf{E}_{s_4}$. Thus,

$$\Psi_{NT,1} = \Psi_{NT,11} + \Psi_{NT,12} + \Psi_{NT,13},$$

where

$$\Psi_{NT,11} = 2\sum_{2\le t_1<t_2}^{T} \mathbb{E}\Big\{\left(4N^{-2}T^{-2}\right)^2 \sum_{s_1=1}^{t_1-1}\sum_{1\le s_3\neq s_1}^{t_2-1} \mathbf{E}_{s_1}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_1}\mathbf{E}_{s_3}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_3} \eta_{t_1}^2\eta_{s_1}^2\eta_{t_2}^2\eta_{s_3}^2\Big\},$$

$$\Psi_{NT,12} = 2\sum_{2\le t_1<t_2}^{T} \mathbb{E}\{\left(4N^{-2}T^{-2}\right)^2 \sum_{1\le s_1\neq s_2}^{t_1-1} \mathbf{E}_{s_1}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_2}\mathbf{E}_{s_2}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_1} \eta_{t_1}^2\eta_{s_1}^2\eta_{t_2}^2\eta_{s_2}^2\} \text{ and}$$

$$\Psi_{NT,13} = 2\sum_{2\le t_1<t_2}^{T} \mathbb{E}\left\{\left(4N^{-2}T^{-2}\right)^2 \sum_{s_1=1}^{t_1-1}\left(\mathbf{E}_{s_1}^\top \boldsymbol{\Sigma}\mathbf{E}_{s_1}\eta_{t_1}^2\eta_{s_1}^2\right)^2\right\}.$$

We next demonstrate that $\Psi_{NT,11} = \sigma_{NT}^4\{1 + o(1)\}$, $\Psi_{NT,12} = o(\sigma_{NT}^4)$ and $\Psi_{NT,13} = o(\sigma_{NT}^4)$, via items (i), (ii) and (iii), respectively.

(i) After simple calculation, we have

$$\Psi_{NT,11} = 2\sum_{2\le t_1<t_2}^{T}\sum_{s_1=1}^{t_1-1}\sum_{1\le s_3\neq s_1}^{t_2-1}\left(4N^{-2}T^{-2}\right)^2 tr^2(\boldsymbol{\Sigma}^2)\mathbb{E}(\eta_{t_1}^2\eta_{s_1}^2\eta_{t_2}^2\eta_{s_3}^2).$$

By Bernstein's inequality given in Bosq (1998) and the same proof for Lemma A.8 of Ma and Yang (2011), under $L^3T^{-1} = o(1)$, we obtain

$$||\mathbf{Z}^\top \mathbf{1}_T/T - \mathbb{E}(\mathbf{Z}^\top \mathbf{1}_T)/T||_\infty = O_{a.s.}(\log(T)/\sqrt{TL}) \tag{B.3}$$

and $\left\|\mathbf{Z}^\top\mathbf{Z}/T - \mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\|_\infty = O_{a.s.}(\log(T)/\sqrt{TL})$. Furthermore, using Assumption (A2) (ii), we have

$$
\begin{aligned}
||\mathbb{E}(\mathbf{Z}^\top\mathbf{1}_T)/T||_\infty &\leq \max_\ell T^{-1}\sum\nolimits_{t=1}^{T}|B_\ell(t/T)|(1+\sum\nolimits_{j=1}^{d}\mathbb{E}|f_{jt}|) \\
&\leq M^*T^{-1}\max_\ell \sum\nolimits_{t\in\{t:|\ell(t)-\ell|\leq q-1\}}|B_\ell(t/T)| \\
&\leq M^*M^{**}T^{-1}TL^{-1} = M^*M^{**}L^{-1}, \qquad \text{(B.4)}
\end{aligned}
$$

for some constants $0 < M^*, M^{**} < \infty$, which leads to $||\mathbf{Z}^\top\mathbf{1}_T/T||_\infty = O_{a.s.}(L^{-1})$.

By (A.2), (B.1) and the result in Demko (1986), we have, with probability 1,

$$
||\left\{(\mathbf{Z}^\top\mathbf{Z}/T)\right\}^{-1}||_\infty \leq C_3L \text{ and } ||\left\{\mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\}^{-1}||_\infty \leq C_3L, \qquad \text{(B.5)}
$$

as $T\to\infty$, where $0 < C_3 < \infty$. The above results imply that

$$
\begin{aligned}
\left\|(\mathbf{Z}^\top\mathbf{Z}/T)^{-1} - \left\{\mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\}^{-1}\right\|_\infty &= O_{a.s.}(L^2)\left\|\mathbf{Z}^\top\mathbf{Z}/T - \mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\|_\infty \\
&= O_{a.s.}(L^2\log(T)\sqrt{TL}). \qquad \text{(B.6)}
\end{aligned}
$$

Define $\widetilde{\eta}_t = 1 - \mathbf{Z}_t\left\{\mathbb{E}(\mathbf{Z}^\top\mathbf{Z})\right\}^{-1}\mathbb{E}(\mathbf{Z}^\top\mathbf{1}_T)$. Then by (B.4), (B.5) and the fact that $|\sum_{\ell=1}^{L}B_\ell(t/T)|$ is bounded, we have

$$
\begin{aligned}
|\widetilde{\eta}_t| &\leq 1+\sum\nolimits_{k=1}^{(1+d)L}|Z_{tk}|||\left\{\mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\}^{-1}||_\infty||\mathbb{E}(\mathbf{Z}^\top\mathbf{1}_T)/T||_\infty \\
&\leq 1+C'(1+\sum\nolimits_{j=1}^{d}|f_{jt}|)LL^{-1} \leq dC''(1+||\mathbf{f}_t||) \qquad \text{(B.7)}
\end{aligned}
$$

for some constants $0 < C', C'' < \infty$. Analogously, we have, with probability 1,

$$
|\eta_t| \leq 1+\sum\nolimits_{k=1}^{(1+d)L}|Z_{tk}|||(\mathbf{Z}^\top\mathbf{Z}/T)^{-1}||_\infty||(\mathbf{Z}^\top\mathbf{1}_T)/T||_\infty \leq dC''(1+||\mathbf{f}_t||). \qquad \text{(B.8)}
$$

The above two results, together with (B.3), (B.6) and the Lemma's assumption $L^3T^{-1} = o(1)$, imply that, with probability 1,

$$
\begin{aligned}
|\widetilde{\eta}_t - \eta_t| \leq & \Big(\sum_{\ell,j}|B_\ell(t/T)f_{jt}| + \sum_\ell\left|\widetilde{B}_\ell(t/T)\right|\Big)\left\|(\mathbf{Z}^\top\mathbf{Z}/T)^{-1} - \left\{\mathbb{E}(\mathbf{Z}^\top\mathbf{Z}/T)\right\}^{-1}\right\|_\infty \\
&\times ||\mathbf{Z}^\top\mathbf{1}_T/T - \mathbb{E}(\mathbf{Z}^\top\mathbf{1}_T)/T||_\infty \\
&\leq C'''L^2\sqrt{\log(T)/(TL)}\sqrt{\log(T)/(TL)}(1+||\mathbf{f}_t||) = (1+||\mathbf{f}_t||)o(1),
\end{aligned}
$$

for some constant $0 < C''' < \infty$. Hence, with probability 1,

$$
\left|\widetilde{\eta}_t^2 - \eta_t^2\right| \leq (|\widetilde{\eta}_t| + |\eta_t|)\,|\widetilde{\eta}_t - \eta_t| \leq 2dC''(1+||\mathbf{f}_t||)(1+||\mathbf{f}_t||)o(1).
$$

By the above results and Assumption (A2)(ii), we have

$$
\begin{aligned}
& |\mathbb{E}\{\widetilde{\eta}_{t_1}^2(\eta_{s_1}^2 - \widetilde{\eta}_{s_1}^2)\eta_{t_2}^2\eta_{s_3}^2\}| \\
= \;& \mathbb{E}[(1+||\mathbf{f}_{t_1}||)^2(1+||\mathbf{f}_{s_1}||)^2(1+||\mathbf{f}_{t_2}||)^2(1+||\mathbf{f}_{s_3}||)^2]o(1) \\
\leq \;& \{\mathbb{E}(1+||\mathbf{f}_{t_1}||)^8\mathbb{E}(1+||\mathbf{f}_{s_1}||)^8\mathbb{E}(1+||\mathbf{f}_{t_2}||)^8\mathbb{E}(1+||\mathbf{f}_{s_3}||)^8\}^{1/4}o(1) = o(1).
\end{aligned}
$$

Similarly, we have $|\mathbb{E}\{(\eta_{t_1}^2-\widetilde{\eta}_{t_1}^2)\eta_{s_1}^2\eta_{t_2}^2\eta_{s_3}^2\}|=o(1)$, $|\mathbb{E}\{\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2(\eta_{t_2}^2-\widetilde{\eta}_{t_2}^2)\eta_{s_3}^2\}|=o(1)$, and $|\mathbb{E}\{\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2(\eta_{s_3}^2-\widetilde{\eta}_{s_3}^2)\}|=o(1)$. Accordingly,

$$\begin{aligned}
& |\mathbb{E}(\eta_{t_1}^2\eta_{s_1}^2\eta_{t_2}^2\eta_{s_3}^2)-\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)| \\
\le\ & |\mathbb{E}\{(\eta_{t_1}^2-\widetilde{\eta}_{t_1}^2)\eta_{s_1}^2\eta_{t_2}^2\eta_{s_3}^2\}|+|\mathbb{E}\{\widetilde{\eta}_{t_1}^2(\eta_{s_1}^2-\widetilde{\eta}_{s_1}^2)\eta_{t_2}^2\eta_{s_3}^2\}| \\
& +|\mathbb{E}\{\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2(\eta_{t_2}^2-\widetilde{\eta}_{t_2}^2)\eta_{s_3}^2\}|+|\mathbb{E}\{\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2(\eta_{s_3}^2-\widetilde{\eta}_{s_3}^2)\}| \\
=\ & o(1). \qquad \text{(B.9)}
\end{aligned}$$

This leads to

$$\begin{aligned}
\Psi_{NT,11} &= 2\sum\nolimits_{2\le t_1<t_2}^{T}\sum\nolimits_{s_1=1}^{t_1-1}\sum\nolimits_{1\le s_3\ne s_1}^{t_2-1}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)+o(1)\} \\
&= T^4\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)o(1)+\Psi_{NT,11}^{(1)}+\Psi_{NT,11}^{(2)}+\Psi_{NT,11}^{(3)},
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{NT,11}^{(1)} &= 2\sum\nolimits_{s_1<t_1\le s_3<t_2}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2), \\
\Psi_{NT,11}^{(2)} &= 2\sum\nolimits_{s_1<s_3\le t_1<t_2}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2), \\
\Psi_{NT,11}^{(3)} &= 2\sum\nolimits_{s_3<s_1\le t_1<t_2}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2).
\end{aligned}$$

When $s_1<t_1\le s_3<t_2$, we have $\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)=\mathrm{cov}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2,\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)+\mathbb{E}\ (\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2)\mathbb{E}(\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)$. This yields

$$\Psi_{NT,11}^{(1)}=\Psi_{NT,111}^{(1)}+\Psi_{NT,112}^{(1)},$$

where

$$\begin{aligned}
\Psi_{NT,111}^{(1)} &= 2\sum\nolimits_{s_1<t_1\le s_3<t_2}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathrm{cov}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2,\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2) \text{ and} \\
\Psi_{NT,112}^{(1)} &= 2\sum\nolimits_{s_1<t_1\le s_3<t_2}\left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2)\mathbb{E}(\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2).
\end{aligned}$$

Applying Davydov's inequality given in Corollary 1.1 of Bosq (1998), we have for $\varkappa>0$,

$$\begin{aligned}
|\mathrm{cov}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2,\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)| &\le 2(2\varkappa^{-1}+1)\{\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2)^{2+\varkappa}\}^{1/(2+\varkappa)}\{\mathbb{E}(\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)^{2+\varkappa}\}^{1/(2+\varkappa)} \\
&\quad \times\{2\alpha(|t_1-s_3|)\}^{\varkappa/(2+\varkappa)}.
\end{aligned}$$

In addition, using (B.7) and Assumption (A2)(ii), we obtain

$$\begin{aligned}
\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2)^{2+\varkappa} &\le \{\mathbb{E}(\widetilde{\eta}_{t_1}^{4(2+\varkappa)})\mathbb{E}(\widetilde{\eta}_{s_1}^{4(2+\varkappa)})\}^{1/2} \\
&\le (dC'')^{4(2+\varkappa)}\{\mathbb{E}(1+||\mathbf{f}_{t_1}||)^{4(2+\varkappa)}\mathbb{E}(1+||\mathbf{f}_{s_1}||)^{4(2+\varkappa)}\}^{1/2}\le\widetilde{C}
\end{aligned}$$

for some constant $0<\widetilde{C}<\infty$. Analogously, we can show that $\mathbb{E}(\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)^{2+\varkappa}\le\widetilde{C}'$ for some constant $0<\widetilde{C}'<\infty$. Hence,

$$|\mathrm{cov}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2,\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)|\le 2(2\varkappa^{-1}+1)(\widetilde{C}\widetilde{C}')^{1/(2+\varkappa)}\{2\alpha(|t_1-s_3|)\}^{\varkappa/(2+\varkappa)}.$$

This leads to

$$\begin{aligned}\sum\nolimits_{s_1<t_1\le s_3<t_2} \operatorname{cov}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2, \widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2) &\le \widetilde{C}'' \sum\nolimits_{s_1<t_1\le s_3<t_2} \alpha(|t_1-s_3|)^{\varkappa/(2+\varkappa)} \\ &\le \widetilde{C}'' T^3 \sum\nolimits_{k=0}^{T} \alpha(k)^{\varkappa/(2+\varkappa)},\end{aligned}$$

where $\widetilde{C}'' = 2(2\varkappa^{-1}+1)(\widetilde{C}\widetilde{C}')^{1/(2+\varkappa)}2^{\varkappa/(2+\varkappa)}$. Accordingly,

$$|\Psi_{NT,111}^{(1)}| \le 2\widetilde{C}'' \left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2) T^3 \sum\nolimits_{k=0}^{T} \alpha(k)^{\varkappa/(2+\varkappa)},$$

which implies that $\Psi_{NT,111}^{(1)} = T^{-1}\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\sum_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\}O(1)$.

Following similar techniques to those used in the proof in (B.9), we have $|\mathbb{E}(\widetilde{\eta}_t^2\widetilde{\eta}_s^2) - \mathbb{E}(\eta_t^2\eta_s^2)| = o(1)$ and $|\mathbb{E}(\widetilde{\eta}_t^2) - \mathbb{E}(\eta_t^2)| = o(1)$. As a result,

$$\Psi_{NT,112}^{(1)} = 2\sum\nolimits_{s_1<t_1\le s_3<t_2} \left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathbb{E}(\eta_{t_1}^2\eta_{s_1}^2)\mathbb{E}(\eta_{t_2}^2\eta_{s_3}^2).$$

This, in conjunction with the above results, yields

$$\begin{aligned}\Psi_{NT,11}^{(1)} &= T^{-1}\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\sum\nolimits_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\}O(1) \\ &\quad +2\sum\nolimits_{s_1<t_1\le s_3<t_2} \left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathbb{E}(\eta_{t_1}^2\eta_{s_1}^2)\mathbb{E}(\eta_{t_2}^2\eta_{s_3}^2).\end{aligned}$$

In the scenarios of $s_1 < s_3 \le t_1 < t_2$ and $s_3 < s_1 \le t_1 < t_2$, we have $\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{t_2}^2\widetilde{\eta}_{s_3}^2)$ $=\operatorname{cov}(\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{s_3}^2, \widetilde{\eta}_{t_1}^2\widetilde{\eta}_{t_2}^2)$ $+\mathbb{E}(\widetilde{\eta}_{s_1}^2\widetilde{\eta}_{s_3}^2)\mathbb{E}(\widetilde{\eta}_{t_1}^2\widetilde{\eta}_{t_2}^2)$. Then, following the same procedure as above, we also have

$$\begin{aligned}\Psi_{NT,11}^{(2)} &= T^{-1}\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\sum\nolimits_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\}O(1) \\ &\quad +2\sum\nolimits_{s_1<s_3\le t_1<t_2} \left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathbb{E}(\eta_{s_1}^2\eta_{s_3}^2)\mathbb{E}(\eta_{t_1}^2\eta_{t_2}^2) \quad \text{and}\end{aligned}$$

$$\begin{aligned}\Psi_{NT,11}^{(3)} &= T^{-1}\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\{\sum\nolimits_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\}O(1) \\ &\quad +2\sum\nolimits_{s_3<s_1\le t_1<t_2} \left(4N^{-2}T^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\mathbb{E}(\eta_{s_1}^2\eta_{s_3}^2)\mathbb{E}(\eta_{t_1}^2\eta_{t_2}^2).\end{aligned}$$

Hence, employing Assumption (A2)(iii) that $\sum_{k=0}^{\infty}\alpha(k)^{\varkappa/(2+\varkappa)} < \infty$, we further obtain

$$\begin{aligned}\Psi_{NT,11} &= \left\{4N^{-2}T^{-2}\sum\nolimits_{t=2}^{T}\sum\nolimits_{s=1}^{t-1}\mathbb{E}(\eta_t^2\eta_s^2)tr\left(\mathbf{\Sigma}^2\right)\right\}^2\{1+o(1)\} \\ &\quad +T^{-1}\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)\Big\{\sum\nolimits_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\Big\}O(1) \\ &\quad +\left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)o(1) \\ &= \sigma_{NT}^4 + \sigma_{NT}^4 o(1) + \left(4N^{-2}\right)^2 tr^2(\mathbf{\Sigma}^2)o(1). \qquad \text{(B.10)}\end{aligned}$$

It is worth noting that, by (B.9), we have

$$\begin{aligned}
\sigma_{NT}^2 &= 4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\{\mathbb{E}(\widetilde{\eta}_t^2\widetilde{\eta}_s^2)+o(1)\} \\
&= 4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\mathbb{E}(\widetilde{\eta}_t^2)\mathbb{E}(\widetilde{\eta}_s^2)+4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\mathrm{cov}(\widetilde{\eta}_t^2,\widetilde{\eta}_s^2) \\
&\quad +4N^{-2}tr\left(\mathbf{\Sigma}^2\right)o(1). \qquad \text{(B.11)}
\end{aligned}$$

Also,

$$\begin{aligned}
&4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\mathbb{E}(\widetilde{\eta}_t^2)\mathbb{E}(\widetilde{\eta}_s^2) \\
=\;& 2N^{-2}T^{-2}tr\left(\mathbf{\Sigma}^2\right)(\sum_{t=1}^{T}\mathbb{E}(\eta_t^2))^2-2N^{-2}T^{-2}tr\left(\mathbf{\Sigma}^2\right)\sum_{t=1}^{T}\mathbb{E}^2(\eta_t^2) \\
&+4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)o(1) \\
=\;& 2N^{-2}T^{-2}tr\left(\mathbf{\Sigma}^2\right)\left(\mathbb{E}\left(\mathbf{1}_T^T M_{\mathbf{Z}}\mathbf{1}_T\right)\right)^2-2N^{-2}T^{-1}tr\left(\mathbf{\Sigma}^2\right)O(1)+4N^{-2}tr\left(\mathbf{\Sigma}^2\right)o(1).
\end{aligned}$$

Since $M_{\mathbf{Z}}$ is idempotent, its eigenvalues are either 0 or 1 and $\mathbf{a}^\top M_{\mathbf{Z}}\mathbf{a}\geq 0$ for any $\mathbf{a}\in\mathbb{R}^T$ with $||\mathbf{a}||=1$. In addition, $\mathbf{a}^\top M_{\mathbf{Z}}\mathbf{a}=\lambda_{\min}(M_{\mathbf{Z}})=0$ when $\mathbf{a}$ is an eigenvector corresponding to $\lambda_{\min}(M_{\mathbf{Z}})$. Since $\mathbf{1}_T/\sqrt{T}$ is not an eigenvector, we have $(\mathbf{1}_T/\sqrt{T})^\top M_{\mathbf{Z}}(\mathbf{1}_T/\sqrt{T})\geq c_M$ for some constant $0<c_M<\infty$. Thus, $\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\geq c_M T$. By the result in Lemma A.2, we further have, with probability 1, $\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\leq C_M T$ for some constant $0<C_M<\infty$. As a result,

$$c_M T\leq \mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T\leq C_M T,$$

which leads to

$$2c_M^2N^{-2}tr\left(\mathbf{\Sigma}^2\right)\leq 2N^{-2}T^{-2}tr\left(\mathbf{\Sigma}^2\right)\left(\mathbb{E}\left(\mathbf{1}_T^T M_{\mathbf{Z}}\mathbf{1}_T\right)\right)^2\leq 2C_M^2N^{-2}tr\left(\mathbf{\Sigma}^2\right).$$

The above results imply that

$$\begin{aligned}
2c_M^2N^{-2}tr\left(\mathbf{\Sigma}^2\right)\{1+o(1)\} &\leq 4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\mathbb{E}(\widetilde{\eta}_t^2)\mathbb{E}(\widetilde{\eta}_s^2) \\
&\leq 2C_M^2N^{-2}tr\left(\mathbf{\Sigma}^2\right)\{1+o(1)\}. \qquad \text{(B.12)}
\end{aligned}$$

By (B.7), $\mathbb{E}(\widetilde{\eta}_t^2)^{2+\varkappa}\leq (dC'')^{4+2\varkappa}\mathbb{E}(1+||\mathbf{f}_t||)^{2(2+\varkappa)}\leq c$ for some constant $0<c<\infty$. Hence, using Davydov's inequality given in Corollary 1.1 of Bosq (1998), we have

$$\begin{aligned}
|\mathrm{cov}(\widetilde{\eta}_t^2,\widetilde{\eta}_s^2)| &\leq 2(2\varkappa^{-1}+1)\{\mathbb{E}(\widetilde{\eta}_t^2)^{2+\varkappa}\}^{1/(2+\varkappa)}\{\mathbb{E}(\widetilde{\eta}_s^2)^{2+\varkappa}\}^{1/(2+\varkappa)}\{2\alpha(|t-s|)\}^{\varkappa/(2+\varkappa)} \\
&\leq c'\alpha(|t-s|)^{\varkappa/(2+\varkappa)}
\end{aligned}$$

for some constant $0<c'<\infty$, and thus,

$$\begin{aligned}
|4N^{-2}T^{-2}\sum_{t=2}^{T}\sum_{s=1}^{t-1} tr\left(\mathbf{\Sigma}^2\right)\mathrm{cov}(\widetilde{\eta}_t^2,\widetilde{\eta}_s^2)| &\leq 4c'N^{-2}T^{-2}tr\left(\mathbf{\Sigma}^2\right)\sum_{t=2}^{T}\sum_{s=1}^{t-1}\alpha(|t-s|)^{\varkappa/(2+\varkappa)} \\
&\leq 4c'N^{-2}T^{-1}tr\left(\mathbf{\Sigma}^2\right)\{\sum_{k=0}^{T}\alpha(k)^{\varkappa/(2+\varkappa)}\} \\
&= N^{-2}tr\left(\mathbf{\Sigma}^2\right)o(1). \qquad \text{(B.13)}
\end{aligned}$$

This, together with (B.11) and (B.12), leads to

$$2c_M^2 N^{-2} tr\left(\mathbf{\Sigma}^2\right)\{1+o(1)\} \leq \sigma_{NT}^2 \leq 2C_M^2 N^{-2} tr\left(\mathbf{\Sigma}^2\right)\{1+o(1)\}. \tag{B.14}$$

By (B.10) and (B.14), we have $\Psi_{NT,11} = \sigma_{NT}^4\{1+o(1)\}$, which completes the proof of (i).

(ii) By (B.7) and Assumption (A2)(ii), we have, for any $1 \leq t_1, s_1, t_2, s_2 \leq T$,

$$\begin{aligned}
& \mathbb{E}\{\widetilde{\eta}_{t_1}^2 \widetilde{\eta}_{s_1}^2 \widetilde{\eta}_{t_2}^2 \widetilde{\eta}_{s_2}^2\} \\
\leq \ & (dC'')^8 \mathbb{E}[(1+||\mathbf{f}_{t_1}||)^2(1+||\mathbf{f}_{s_1}||)^2(1+||\mathbf{f}_{t_2}||)^2(1+||\mathbf{f}_{s_2}||)^2] \\
\leq \ & (dC'')^8\{\mathbb{E}(1+||\mathbf{f}_{t_1}||)^8 \mathbb{E}(1+||\mathbf{f}_{s_1}||)^8 \mathbb{E}(1+||\mathbf{f}_{t_2}||)^8 \mathbb{E}(1+||\mathbf{f}_{s_2}||)^8\}^{1/4} \leq M',
\end{aligned}$$

where $0 < M' < \infty$. This, in conjunction with (B.9), implies that, for any $1 \leq t_1, s_1, t_2, s_2 \leq T$,

$$\mathbb{E}(\eta_{t_1}^2 \eta_{s_1}^2 \eta_{t_2}^2 \eta_{s_2}^2) \leq M'. \tag{B.15}$$

Thus,

$$\begin{aligned}
\Psi_{NT,12} &= 2\sum\nolimits_{2\leq t_1<t_2}^{T} \left(4N^{-2}T^{-2}\right)^2 \sum\nolimits_{1\leq s_1\neq s_2}^{t_1-1} tr(\mathbf{\Sigma}^4)\mathbb{E}(\eta_{t_1}^2\eta_{s_1}^2\eta_{t_2}^2\eta_{s_2}^2) \\
&\leq 2M'\left(4N^{-2}T^{-2}\right)^2 \sum\nolimits_{2\leq t_1<t_2}^{T} \sum\nolimits_{1\leq s_1\neq s_2}^{t_1-1} tr(\mathbf{\Sigma}^4) \\
&\leq 2C''''M'\left(4N^{-2}T^{-2}\right)^2 T^4 tr(\mathbf{\Sigma}^4) = 2C''''M'\left(4N^{-2}\right)^2 tr(\mathbf{\Sigma}^4)
\end{aligned}$$

for some constant $0 < C'''' < \infty$. By the condition that $tr\left(\mathbf{\Sigma}^4\right) = o\left\{tr^2\left(\mathbf{\Sigma}^2\right)\right\}$ in Condition (C2)(i) and (B.14), we obtain $\Psi_{NT,12} = o(\sigma_{NT}^4)$, which completes the proof of (ii).

(iii) By (B.15), we have

$$\begin{aligned}
\Psi_{NT,13} &\leq 2M'\left(4N^{-2}T^{-2}\right)^2 \sum\nolimits_{2\leq t_1<t_2}^{T} \sum\nolimits_{s_1=1}^{t_1-1} \mathbb{E}\left(\mathbf{E}_{s_1}^\top \mathbf{\Sigma}\mathbf{E}_{s_1}\mathbf{E}_{s_1}^\top \mathbf{\Sigma}\mathbf{E}_{s_1}\right) \\
&\leq M'\left(4N^{-2}\right)^2 T^{-2} \sum\nolimits_{s=1}^{T} \mathbb{E}\left(\mathbf{E}_s^\top \mathbf{\Sigma}\mathbf{E}_s\mathbf{E}_s^\top \mathbf{\Sigma}\mathbf{E}_s\right) \\
&= M'\left(4N^{-2}\right)^2 o(tr^2\left(\mathbf{\Sigma}^2\right)),
\end{aligned}$$

where the last step follows from Condition (C2)(ii). Hence, $\Psi_{NT,13} = o(\sigma_{NT}^4)$ and (iii) follows. □

Proof of Lemma A.5. By (A.7)

$$\begin{aligned}
& \varphi_{NT,t}^2 \\
= \ & \left(\sum\nolimits_{s=1}^{t-1} 2N^{-1}T^{-1}\mathbf{E}_t^\top \mathbf{E}_s \eta_t \eta_s\right)^2 = \left(2N^{-1}T^{-1}\right)^2 \sum\nolimits_{s_1=1}^{t-1}\sum\nolimits_{s_2=1}^{t-1} \mathbf{E}_t^\top \mathbf{E}_{s_1}\mathbf{E}_t^\top \mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2} \\
= \ & \left(2N^{-1}T^{-1}\right)^2 \sum\nolimits_{s=1}^{t-1} \mathbf{E}_t^\top \mathbf{E}_s \mathbf{E}_s^\top \mathbf{E}_t \eta_t^2\eta_s^2 + \left(2N^{-1}T^{-1}\right)^2 2\sum\nolimits_{s_2=2}^{t-1}\sum\nolimits_{s_1=1}^{s_2-1} \mathbf{E}_t^\top \mathbf{E}_{s_1}\mathbf{E}_t^\top \mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2} \\
\equiv \ & \psi_{NT,t1} + \psi_{NT,t2}.
\end{aligned}$$

Then,

$$\sum_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^{4}\right)\leq 2\left\{\sum_{t=2}^{T}\mathbb{E}\left(\psi_{NT,t1}^{2}\right)+\sum_{t=2}^{T}\mathbb{E}\left(\psi_{NT,t2}^{2}\right)\right\}. \tag{B.16}$$

In addition,

$$\begin{aligned}
\mathbb{E}\left(\psi_{NT,t1}^{2}\right) &= \left(2N^{-1}T^{-1}\right)^{4}\mathbb{E}\left(\sum_{s=1}^{t-1}\mathbf{E}_t^{\top}\mathbf{E}_s\mathbf{E}_s^{\top}\mathbf{E}_t\eta_t^2\eta_s^2\right)^2\\
&= \left(2N^{-1}T^{-1}\right)^{4}\sum_{s_1=1}^{t-1}\sum_{s_2=1}^{t-1}\mathbb{E}\left(\mathbf{E}_t^{\top}\mathbf{E}_{s_1}\mathbf{E}_{s_1}^{\top}\mathbf{E}_t\mathbf{E}_t^{\top}\mathbf{E}_{s_2}\mathbf{E}_{s_2}^{\top}\mathbf{E}_t\eta_t^4\eta_{s_1}^2\eta_{s_2}^2\right)\\
&\equiv \omega_{NT,t1}+\omega_{NT,t2},
\end{aligned} \tag{B.17}$$

where

$$\begin{aligned}
\omega_{NT,t1} &= \left(2N^{-1}T^{-1}\right)^{4}2\sum_{s_2=2}^{t-1}\sum_{s_1=1}^{s_2-1}\mathbb{E}\left(\mathbf{E}_t^{\top}\mathbf{E}_{s_1}\mathbf{E}_{s_1}^{\top}\mathbf{E}_t\mathbf{E}_t^{\top}\mathbf{E}_{s_2}\mathbf{E}_{s_2}^{\top}\mathbf{E}_t\eta_t^4\eta_{s_1}^2\eta_{s_2}^2\right) \text{ and}\\
\omega_{NT,t2} &= \left(2N^{-1}T^{-1}\right)^{4}\sum_{s=1}^{t-1}\mathbb{E}\left((\mathbf{E}_t^{\top}\mathbf{E}_s)^4\eta_t^4\eta_s^4\right).
\end{aligned}$$

Employing Assumptions (A3) (ii) and (iii) and (B.15), we obtain,

$$\begin{aligned}
\sum_{t=2}^{T}\omega_{NT,t1} &= \left(2N^{-1}T^{-1}\right)^{4}\sum_{t=2}^{T}\sum_{1=s_1\neq s_2}^{t-1}\mathbb{E}\left(\mathbf{E}_t^{\top}\mathbf{E}_{s_1}\mathbf{E}_{s_1}^{\top}\mathbf{E}_t\mathbf{E}_t^{\top}\mathbf{E}_{s_2}\mathbf{E}_{s_2}^{\top}\mathbf{E}_t\eta_t^4\eta_{s_1}^2\eta_{s_2}^2\right)\\
&= \left(2N^{-1}T^{-1}\right)^{4}\sum_{t=2}^{T}\sum_{1=s_1\neq s_2}^{t-1}\mathbb{E}\left(\mathbf{E}_t^{\top}\mathbf{\Sigma}\mathbf{E}_t\mathbf{E}_t^{\top}\mathbf{\Sigma}\mathbf{E}_t\eta_t^4\eta_{s_1}^2\eta_{s_2}^2\right)\\
&\leq \left(2N^{-1}T^{-1}\right)^{4}2M'T^2\sum_{t=2}^{T}\mathbb{E}\left(\mathbf{E}_t^{\top}\mathbf{\Sigma}\mathbf{E}_t\mathbf{E}_t^{\top}\mathbf{\Sigma}\mathbf{E}_t\right)\\
&= \left(2N^{-1}T^{-1}\right)^{4}2M'T^4 o(tr^2\left(\mathbf{\Sigma}^2\right)),
\end{aligned}$$

where the last step follows from (C2)(ii). This, in conjunction with (B.14), implies that

$$\sum_{t=2}^{T}\omega_{NT,t1}=2M'\left(2N^{-1}\right)^{4}o(tr^2\left(\mathbf{\Sigma}^2\right))=o(\sigma_{NT}^4). \tag{B.18}$$

By (B.15), we have

$$\sum_{t=2}^{T}\omega_{NT,t2}\leq\left(2N^{-1}T^{-1}\right)^{4}M'\sum_{t=2}^{T}\sum_{s=1}^{t-1}\mathbb{E}(\mathbf{E}_t^{\top}\mathbf{E}_s)^4.$$

Under Assumption (A3)(ii) and Condition (C1), we adopt the same procedure as given on page 24 of Chen and Qin (2010) and obtain $\mathbb{E}(\mathbf{E}_t^{\top}\mathbf{E}_s)^4=O(tr^2\left(\mathbf{\Sigma}^2\right))+O\left(tr\left(\mathbf{\Sigma}^4\right)\right)$. This, together with Condition (C2)(i) and (B.14), leads to

$$\begin{aligned}
\sum_{t=2}^{T}\omega_{NT,t2} &= M'\left(2N^{-1}T^{-1}\right)^{4}T^2\left\{O(tr^2\left(\mathbf{\Sigma}^2\right))+O\left(tr\left(\mathbf{\Sigma}^4\right)\right)\right\}\\
&= M'\left(2N^{-1}\right)^{4}T^{-2}O(tr^2\left(\mathbf{\Sigma}^2\right))=o(\sigma_{NT}^4).
\end{aligned} \tag{B.19}$$

Accordingly, (B.17), (B.18) and (B.19) yield

$$\sum_{t=2}^{T}\mathbb{E}\left(\psi_{NT,t1}^{2}\right)=o(\sigma_{NT}^4). \tag{B.20}$$

Using the result of (B.15), we have that

$$\begin{aligned}
\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\psi_{NT,t2}^2\right) &= \left(2N^{-1}T^{-1}\right)^4\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\sum\nolimits_{1\le s_1\neq s_2}^{t-1}\mathbf{E}_t^\top\mathbf{E}_{s_1}\mathbf{E}_t^\top\mathbf{E}_{s_2}\eta_t^2\eta_{s_1}\eta_{s_2}\right)^2\\
&\le \left(2N^{-1}T^{-1}\right)^4 M'\sum\nolimits_{t=2}^{T}\mathbb{E}\left(2\sum\nolimits_{1\le s_1<s_2}^{t-1}\mathbf{E}_t^\top\mathbf{E}_{s_1}\mathbf{E}_t^\top\mathbf{E}_{s_2}\right)^2\\
&= \left(2N^{-1}T^{-1}\right)^4 M'4\sum\nolimits_{t=2}^{T}\sum\nolimits_{1\le s_1<s_2}^{t-1}\sum\nolimits_{1\le s_3<s_4}^{t-1}\mathbb{E}(\mathbf{E}_t^\top\mathbf{E}_{s_1}\mathbf{E}_{s_3}^\top\mathbf{E}_t\mathbf{E}_t^\top\mathbf{E}_{s_2}\mathbf{E}_{s_4}^\top\mathbf{E}_t).
\end{aligned}$$

By Assumption (A3)(i), for $s_1\neq s_2$ and $s_3\neq s_4$, we have $\mathbb{E}(\mathbf{E}_t^\top\mathbf{E}_{s_1}\mathbf{E}_{s_2}^\top\mathbf{E}_t\mathbf{E}_t^\top\mathbf{E}_{s_3}\mathbf{E}_{s_4}^\top\mathbf{E}_t)\neq 0$ only when $s_1=s_3$ and $s_2=s_4$. The above results, together with Condition (C2)(ii) and (B.14), imply that

$$\begin{aligned}
\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\psi_{NT,t2}^2\right) &\le \left(2N^{-1}T^{-1}\right)^4 M'4\sum\nolimits_{t=2}^{T}\sum\nolimits_{1\le s_1<s_2}^{t-1}\mathbb{E}(\mathbf{E}_t^\top\mathbf{E}_{s_1}\mathbf{E}_{s_1}^\top\mathbf{E}_t\mathbf{E}_t^\top\mathbf{E}_{s_2}\mathbf{E}_{s_2}^\top\mathbf{E}_t)\\
&= \left(2N^{-1}T^{-1}\right)^4 M'4\sum\nolimits_{t=2}^{T}\sum\nolimits_{1\le s_1<s_2}^{t-1}\mathbb{E}(\mathbf{E}_t^\top\boldsymbol{\Sigma}\mathbf{E}_t\mathbf{E}_t^\top\boldsymbol{\Sigma}\mathbf{E}_t)\\
&\le 4M'\left(2N^{-1}\right)^4T^{-2}\sum\nolimits_{t=1}^{T}\mathbb{E}(\mathbf{E}_t^\top\boldsymbol{\Sigma}\mathbf{E}_t\mathbf{E}_t^\top\boldsymbol{\Sigma}\mathbf{E}_t)\\
&= 4M'\left(2N^{-1}\right)^4 o(tr^2\left(\boldsymbol{\Sigma}^2\right))=o(\sigma_{NT}^4). \qquad \text{(B.21)}
\end{aligned}$$

Consequently, by (B.16), (B.20) and (B.21), we have shown $\sum_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^4\right)=o(\sigma_{NT}^4)$. □

Proof of Lemma A.6. By Cauchy–Schwarz inequality and Chebyshev's inequality, we have

$$\begin{aligned}
&\sum\nolimits_{t=2}^{T}\sigma_{NT}^{-2}\mathbb{E}\left(\varphi_{NT,t}^2 I(|\varphi_{NT,t}|>\epsilon\sigma_{NT})\,|\mathcal{F}_{NT,t-1}\right)\\
\le\ &\sigma_{NT}^{-4}\epsilon^{-2}\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^4\,|\mathcal{F}_{NT,t-1}\right).
\end{aligned}$$

From Lemma A.5, we obtain

$$\mathbb{E}\Big\{\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^4\,|\mathcal{F}_{NT,t-1}\right)\Big\}=\sum\nolimits_{t=2}^{T}\mathbb{E}\left(\varphi_{NT,t}^4\right)=o(\sigma_{NT}^4).$$

The above two results lead to Lemma A.6, which completes the proof. □

Proof of Lemma A.7. Using the fact that $M_{\mathbf{Z}}$ is idempotent, we have $\lambda_{\max}(M_{\mathbf{Z}})=1$. Thus, $\lambda_{\max}(M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}})=\lambda_{\max}(\mathbf{1}_T^\top M_{\mathbf{Z}}M_{\mathbf{Z}}\mathbf{1}_T)\le\lambda_{\max}(M_{\mathbf{Z}})(\mathbf{1}_T^\top\mathbf{1}_T)=T$. This, in conjunction with (A.4) that $\sup_{1\le t\le T}|\rho_{NT,it}|=O(L^{-r})$ for each $1\le i\le N$, results in

$$\begin{aligned}
\zeta_{NT,1} &= N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i}\\
&\le \lambda_{\max}(M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}})N^{-1}T^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top\boldsymbol{\rho}_{NT,i}\\
&\le N^{-1}\sum\nolimits_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^\top\boldsymbol{\rho}_{NT,i}=O(L^{-2r}T).
\end{aligned}$$

By (B.14) and Condition (C3)(i) that $TL^{-2r}N\left\{tr\left(\boldsymbol{\Sigma}^2\right)\right\}^{-1/2} = o(1)$, we have $\sigma_{NT}^{-1}\zeta_{NT,1} = o_p(1)$.

Define $\varpi_{NT,i} = \sum_{s=1}^{T}\eta_s\rho_{NT,is}$. Then $\zeta_{NT,2} = 2N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\varpi_{NT,i}\eta_t e_{it}$. In addition, using the result under (B.8), we obtain

$$\mathbb{E}(\eta_t^2) \leq (dC'')^2\mathbb{E}(1+||\mathbf{f}_t||)^2 \leq C^{**} \tag{B.22}$$

for some constant $0 < C^{**} < \infty$. Furthermore, by Lemma A.1 and (B.22), we have $|\varpi_{NT,i}| = O(TL^{-r})$. This, together with Assumption (A3)(i) and (B.22), implies that

$$\begin{aligned}
\mathrm{Var}(\zeta_{NT,2}) &= \left(2N^{-1}T^{-1}\right)^2\sum_{t=1}^{T}\mathbb{E}\left(\sum_{i=1}^{N}\varpi_{NT,i}\eta_t e_{it}\right)^2 \\
&= \left(2N^{-1}T^{-1}\right)^2\sum_{t=1}^{T}\sum_{i,j=1}^{N}\mathbb{E}(\varpi_{NT,i}\varpi_{NT,j}\eta_t^2)\sigma_{ij} \\
&\leq \left(2N^{-1}T^{-1}\right)^2 N\sum_{t=1}^{T}\left(\sum_{i,j=1}^{N}\sigma_{ij}^2\right)^{1/2} O\{(TL^{-r})^2\} \\
&= O(N^{-1}TL^{-2r}tr^{1/2}(\boldsymbol{\Sigma}^2)),
\end{aligned}$$

where σ_{ij} is the ij-th element of $\boldsymbol{\Sigma}$. Hence, $|\zeta_{NT,2}| = O_p\left[N^{-1/2}T^{1/2}L^{-r}\left\{tr(\boldsymbol{\Sigma}^2)\right\}^{1/4}\right]$. Using this result, Condition (C3)(i), and (B.14), we have

$$\sigma_{NT}^{-1}\zeta_{NT,2} = O_p\left[N^{1/2}T^{1/2}L^{-r}\left\{tr(\boldsymbol{\Sigma}^2)\right\}^{-1/4}\right] = o_p(1).$$

Define $\varpi_{NT,i}^* = \delta_i^0\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{1}_T$. Then, we obtain, $\varpi_{NT,i}^* \leq T|\delta_i^0|$. This, in conjunction with (B.22), implies that

$$\begin{aligned}
\mathrm{Var}(\zeta_{NT,3}) &= \left(2N^{-1}T^{-1}\right)^2\sum_{t=1}^{T}\sum_{i,j=1}^{N}\mathbb{E}(\varpi_{NT,i}^*\varpi_{NT,j}^*\eta_t^2)\sigma_{ij} \\
&\leq \left(2N^{-1}T^{-1}\right)^2 T^2C^{**}T\sum_{i,j=1}^{N}|\delta_i^0||\delta_j^0||\sigma_{ij}| \\
&\leq \left(2N^{-1}T^{-1}\right)^2 T^2C^{**}T2^{-1}\sum_{i,j=1}^{N}(|\delta_i^0|^2+|\delta_j^0|^2)|\sigma_{ij}| \\
&\leq \left(2N^{-1}T^{-1}\right)^2 T^2C^{**}T\sum_{i=1}^{N}|\delta_i^0|^2\sum_{j=1}^{N}|\sigma_{ij}|.
\end{aligned}$$

Therefore,

$$|\zeta_{NT,3}| = O_p\left[N^{-1}T^{1/2}\{\sum_{i=1}^{N}|\delta_i^0|^2\}^{1/2}[\max_i\{\sum_{j=1}^{N}|\sigma_{ij}|\}]^{1/2}\right].$$

This, in conjunction with (B.14), leads to

$$\begin{aligned}
\sigma_{NT}^{-1}\zeta_{NT,3} &= O_p\left[\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}NN^{-1}T^{1/2}\{\sum_{i=1}^{N}|\delta_i^0|^2\}^{1/2}[\max_i\{\sum_{j=1}^{N}|\sigma_{ij}|\}]^{1/2}\right] \\
&= O_p\left[\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}T^{1/2}\{\sum_{i=1}^{N}|\delta_i^0|^2\}^{1/2}[\max_i\{\sum_{j=1}^{N}|\sigma_{ij}|\}]^{1/2}\right].
\end{aligned}$$

By the assumption that $\delta_i^0 = N^{-1/2}T^{-1/2}\{tr(\mathbf{\Sigma}^2)\}^{1/4}c_i^0$, we have

$$\begin{aligned}
\sigma_{NT}^{-1}\zeta_{NT,3} &= O_p\left[\{tr(\mathbf{\Sigma}^2)\}^{-1/4}\{N^{-1}\sum\nolimits_{i=1}^N (c_i^0)^2\}^{1/2}[\max_i\{\sum\nolimits_{j=1}^N |\sigma_{ij}|\}]^{1/2}\right] \\
&= O_p\left[\{tr(\mathbf{\Sigma}^2)\}^{-1/4}[\max_i\{\sum\nolimits_{j=1}^N |\sigma_{ij}|\}]^{1/2}\right] = o_p(1),
\end{aligned}$$

where the last equality follows from Condition (C3)(ii). By Lemma A.1 and (B.22), we have with probability approaching 1,

$$|\zeta_{NT,4}| \le 2N^{-1}\sum\nolimits_{i=1}^N |\delta_i^0| \sum_{t=1}^T |\eta_t\rho_{NT,it}| = O(N^{-1}TL^{-r}\sum\nolimits_{i=1}^N |\delta_i^0|).$$

This, together with (B.14) and Condition (C3)(i), we have

$$\begin{aligned}
\sigma_{NT}^{-1}|\zeta_{NT,4}| &= O_p[\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}TL^{-r}\sum\nolimits_{i=1}^N |\delta_i^0|] \\
&= O_p\Big[\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/4}T^{1/2}N^{1/2}L^{-r}N^{-1}\sum\nolimits_{i=1}^N |c_i^0|\Big] \\
&= O_p\Big[\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/4}T^{1/2}N^{1/2}L^{-r}\{N^{-1}\sum\nolimits_{i=1}^N |c_i^0|^2\}^{1/2}\Big] \\
&= O_p\Big[\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/4}T^{1/2}N^{1/2}L^{-r}\Big] = o_p(1).
\end{aligned}$$

The proof is complete. □

B.3 Proofs of Lemmas A.9-A.14 used in the Proof of the Second Result of Theorem 2

Proof of Lemma A.9. We decompose $\vartheta_{NT,1} = \vartheta_{NT,11} + \vartheta_{NT,12}$, where

$$\begin{aligned}
\vartheta_{NT,11} &= T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T\mathbf{1}_T^\top M_{\mathbf{Z}}\mathbf{e}_i - T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\sum\nolimits_{t=1}^T e_{it}^2\eta_t^2 \text{ and} \\
\vartheta_{NT,12} &= T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\sum\nolimits_{t=1}^T e_{it}^2\eta_t^2.
\end{aligned}$$

In addition, $\vartheta_{NT,11}$ can be expressed as

$$\begin{aligned}
\vartheta_{NT,11} &= T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\sum\nolimits_{t,s=1}^T e_{it}e_{is}\eta_t\eta_s - T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N e_{it}^2\eta_t^2 \\
&= T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N\sum\nolimits_{t\neq s} e_{it}e_{is}\eta_t\eta_s = T^{-1/2}T^{-2}\sum\nolimits_{t\neq s}\mathbf{E}_t^\top\mathbf{E}_s\eta_t\eta_s \\
&= \sum\nolimits_{t=2}^T 2T^{-1/2}T^{-2}\mathbf{E}_t^\top\eta_t(\sum\nolimits_{s=1}^{t-1}\mathbf{E}_s\eta_s) = \sum\nolimits_{t=2}^T \varkappa_{NT,t}.
\end{aligned}$$

By Assumption (A3),

$$\mathbb{E}\left(\varkappa_{NT,t}\,|\mathcal{F}_{NT,t-1}\right) = 2T^{-1/2}T^{-2}\sum\nolimits_{i=1}^N \mathbb{E}\left(e_{it}\,|\mathcal{F}_{NT,t-1}\right)\eta_t(\sum\nolimits_{s=1}^{t-1} e_{is}\eta_s) = 0.$$

Moreover, we have

$$\begin{aligned}\mathbb{E}(\eta_t^2\eta_s^2) &\le (dC'')^4\mathbb{E}(1+||\mathbf{f}_t||)^2(1+||\mathbf{f}_s||)^2 \\ &\le (dC'')^4\{\mathbb{E}(1+||\mathbf{f}_t||)^4\mathbb{E}(1+||\mathbf{f}_s||)^4\}^{1/2} \le M'' \end{aligned} \tag{B.23}$$

for some constant $0 < M'' < \infty$. Then applying the same techniques as those used in the proof of (A.8), we obtain

$$\mathbb{E}\Big(\sum_{t=2}^{T}\varkappa_{NT,t}\Big)^2 = \sum_{t=2}^{T}\mathbb{E}\left(\varkappa_{NT,t}^2\right) = 4T^{-5}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\mathbb{E}(\eta_t^2\eta_s^2)tr\left(\mathbf{\Sigma}^2\right) \le 2M''T^{-3}tr\left(\mathbf{\Sigma}^2\right).$$

Accordingly, $\vartheta_{NT,11} = o_p[T^{-3/2}\{tr\left(\mathbf{\Sigma}^2\right)\}^{1/2}]$, which implies that $\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,11} = o_p(1)$. By (B.22) and Condition (C3)(iii), we have

$$\begin{aligned}\mathbb{E}[\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,12}] &\le \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2+\varrho}C^{**}T^{-1/2}T^{-2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}(e_{it}^2) \\ &= \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2+\varrho}C^{**}T^{-1/2}T^{-1}\sum_{i=1}^{N}\sigma_{ii} \\ &\le \{tr(\mathbf{\Sigma}^2)\}^{-1/2}C^{**}T^{-1+\varrho}N\sigma_{\max} = o(1).\end{aligned}$$

Hence, $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,12} = o_p(1)$. Consequently,

$$\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,1} = o_p(1).$$

Moreover, $\vartheta_{NT,2} \le T^{-1/2}\sum_{i=1}^{N}\left(\delta_i^0\right)^2$. Hence,

$$\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2}N^{\varrho}\vartheta_{NT,2} \le \{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2}T^{-1/2}N^{\varrho}\sum_{i=1}^{N}\left(\delta_i^0\right)^2.$$

Using the result $\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2} = O[\{tr(\mathbf{\Sigma}^2)\}^{-1}N]$ and applying (A.13), we obtain

$$\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2}N^{\varrho}\vartheta_{NT,2} = o_p(1).$$

By (A.4), we have

$$\begin{aligned}\vartheta_{NT,3} &\le T^{-1/2}T^{-2}T\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} \le T^{-1/2}T^{-2}T\sum_{i=1}^{N}\boldsymbol{\rho}_{NT,i}^{\top}\boldsymbol{\rho}_{NT,i} \\ &= T^{-1/2}NL^{-2r}.\end{aligned}$$

Employing Condition (C3)(i), we then obtain that

$$\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2+\varrho}\vartheta_{NT,3} \le \{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{\varrho}NL^{-2r} = o(1).$$

Recall that $\varpi_{NT,i} = \sum_{s=1}^{T}\eta_t\rho_{NT,is}$ defined in Lemma A.7. Then

$$\vartheta_{NT,4} = 2T^{-1/2}T^{-2}\sum_{i=1}^{N}\sum_{t=1}^{T}\varpi_{NT,i}\eta_t e_{it}.$$

By Lemma A.1 and (B.22), we have $|\varpi_{NT,i}| = O(TL^{-r})$. This, together with Assumption (A3), implies

$$\begin{aligned}
\mathrm{Var}(\vartheta_{NT,4}) &= \left(2T^{-1/2}T^{-2}\right)^2 \sum_{t=1}^{T} \mathbb{E}\left(\sum_{i=1}^{N} \varpi_{NT,i}\eta_t e_{it}\right)^2 \\
&= \left(2T^{-1/2}T^{-2}\right)^2 \sum_{t=1}^{T} \sum_{i,j=1}^{N} \mathbb{E}(\varpi_{NT,i}\varpi_{NT,j}\eta_t^2)\sigma_{ij} \\
&\leq \left(2T^{-1/2}T^{-2}\right)^2 N \sum_{t=1}^{T} \left(\sum_{i,j=1}^{N} \sigma_{ij}^2\right)^{1/2} O\{(TL^{-r})^2\} \\
&= O(T^{-2}NL^{-2r}tr^{1/2}(\mathbf{\Sigma}^2)).
\end{aligned}$$

Under Condition (C3)(i), we thus have

$$\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/2}T^{1/2+\varrho}|\vartheta_{NT,4}| = O_p(\{tr\left(\mathbf{\Sigma}^2\right)\}^{-1/4}T^{-1/2+\varrho}N^{1/2}L^{-r}) = o_p(1).$$

Recall that $\varpi^*_{NT,i} = \delta_i^0 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T$ defined in Lemma A.7. Then, we obtain, $\varpi^*_{NT,i} \leq T|\delta_i^0|$. As a result,

$$\begin{aligned}
\mathrm{Var}(\vartheta_{NT,5}) &= \left(2T^{-1/2}T^{-2}\right)^2 \sum_{t=1}^{T} \sum_{i,j=1}^{N} \mathbb{E}(\varpi^*_{NT,i}\varpi^*_{NT,j}\eta_t^2)\sigma_{ij} \\
&\leq \left(2T^{-1/2}T^{-2}\right)^2 T^2C^{**}T \sum_{i,j=1}^{N} |\delta_i^0||\delta_j^0|\sigma_{ij} \\
&\leq \left(2T^{-1/2}T^{-2}\right)^2 T^3C^{**}tr(\mathbf{\Sigma}^2) + \left(2T^{-1/2}T^{-2}\right)^2 \tilde{C}^{*2}T^3C^{**} \left\{\sum_{i=1}^{N} \left(\delta_i^0\right)^2\right\}^2 \\
&= 4C^{**}T^{-2}tr(\mathbf{\Sigma}^2) + 4C^{**}T^{-2} \left\{\sum_{i=1}^{N} \left(\delta_i^0\right)^2\right\}^2
\end{aligned}$$

This, in conjunction with (A.13), leads to

$$\begin{aligned}
\left\{tr(\mathbf{\Sigma}^2)\right\}^{-1/2} T^{1/2+\varrho}|\vartheta_{NT,5}| &= O_p\left[T^{-1/2+\varrho} + \left\{tr(\mathbf{\Sigma}^2)\right\}^{-1/2} T^{-1/2+\varrho} \sum_{i=1}^{N} \left(\delta_i^0\right)^2\right] \\
&= O_p\left[T^{-1/2+\varrho} + \left\{tr(\mathbf{\Sigma}^2)\right\}^{-1} NT^{-1/2+\varrho} \sum_{i=1}^{N} \left(\delta_i^0\right)^2\right] = o_p(1).
\end{aligned}$$

By Lemma A.1, (B.22), we have

$$\begin{aligned}
\mathbb{E}\,|\vartheta_{NT,6}| &= \mathbb{E}|2T^{-1/2}T^{-2} \sum_{i=1}^{N} \delta_i^0 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i}| \\
&\leq \mathbb{E}\left\{2T^{-1/2}T^{-1} \sum_{i=1}^{N} \left|\delta_i^0\right| \sum_{t=1}^{T} |\eta_t \rho_{NT,it}|\right\} \\
&\leq (C^{**})^{1/2}\tilde{C}_\rho 2T^{-1/2}L^{-r} \sum_{i=1}^{N} \left|\delta_i^0\right|,
\end{aligned}$$

for some constant $0 < \tilde{C}_\rho < \infty$. By Condition (C3)(i), we thus have, with probability approaching 1,

$$\begin{aligned}
\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}T^{1/2+\varrho}\left|\vartheta_{NT,6}\right| &\leq (C^{**})^{1/2}\tilde{C}_\rho 2\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/2}T^{\varrho}L^{-r}\sum\nolimits_{i=1}^{N}\left|\delta_i^0\right| \\
&\leq C'\{tr\left(\boldsymbol{\Sigma}^2\right)\}^{-1/4}T^{\varrho-1/2}N^{1/2}L^{-r}\{N^{-1}\sum\nolimits_{i=1}^{N}(c_i^0)^2\}^{1/2} \\
&= o(1), \qquad\qquad (B.24)
\end{aligned}$$

where $C' = 2(C^{**})^{1/2}\tilde{C}_\rho$. This completes the proof. □

Proof of Lemma A.10. Write $\xi_{NT,1} = \xi_{NT,11} + \xi_{NT,12}$, where

$$\begin{aligned}
\xi_{NT,11} &= T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_i - T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top\mathbf{e}_i \quad \text{and} \\
\xi_{NT,12} &= T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top\mathbf{e}_i - T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbb{E}(\mathbf{e}_i^\top\mathbf{e}_i).
\end{aligned}$$

After simple calculation, we have

$$\xi_{NT,11} = T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top(M_{\mathbf{Z}} - \mathbf{I}_T)\mathbf{e}_i = -T^{-1/2}T^{-1}\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top\mathbf{Z}(\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}^\top\mathbf{e}_i.$$

By (A.2), we obtain that, with probability approaching 1,

$$|\xi_{NT,11}| \leq c_1^{-1}T^{-1/2}T^{-2}L\sum\nolimits_{i=1}^{N}\mathbf{e}_i^\top\mathbf{Z}\mathbf{Z}^\top\mathbf{e}_i,$$

as $T \to \infty$. In addition, by Assumption (A3),

$$\begin{aligned}
\mathbb{E}(\mathbf{e}_i^\top\mathbf{Z}\mathbf{Z}^\top\mathbf{e}_i) &= \sum\nolimits_{k=1}^{(1+d)L}\mathbb{E}(\sum\nolimits_{t=1}^{T}Z_{tk}e_{it})^2 = \sum\nolimits_{k=1}^{(1+d)L}\sum\nolimits_{t=1}^{T}\mathbb{E}(Z_{tk}e_{it})^2 \\
&= \sigma_{ii}\sum\nolimits_{k=1}^{(1+d)L}\sum\nolimits_{t=1}^{T}\mathbb{E}(Z_{tk})^2 \leq \tilde{C}_1\sigma_{ii}T \qquad\qquad (B.25)
\end{aligned}$$

for some constant $0 < \tilde{C}_1 < \infty$. This, together with Chebyshev's inequality, implies that

$$\begin{aligned}
|\xi_{NT,11}| &= O_p\left(T^{-1/2}T^{-2}L\sum\nolimits_{i=1}^{N}\sigma_{ii}T\right) = O_p\left(T^{-1/2}T^{-1}L\sum\nolimits_{i=1}^{N}\sigma_{ii}\right) \\
&= O_p\left(T^{-3/2}LN\sigma_{\max}\right).
\end{aligned}$$

Accordingly, by Condition (C3)(iii), we have

$$\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}|\xi_{NT,11}| = O_p[\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}T^{-3/2}LN\sigma_{\max}] = o(1).$$

By (B.31) demonstrated later, we have

$$\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{t=1}^{T}\mathbb{E}(e_{it}^2e_{jt}^2) = T\{tr(\boldsymbol{\Sigma})\}^2 + 2Ttr(\boldsymbol{\Sigma}^2)\{1 + o(1)\}.$$

Then

$$\begin{aligned}
\mathbb{E}(\sum_{i=1}^N \mathbf{e}_i^\top \mathbf{e}_i)^2 &= \sum_{i,j}\sum_{t,s}\mathbb{E}(e_{it}^2 e_{js}^2) = \sum_{i,j}\sum_t \mathbb{E}(e_{it}^2 e_{jt}^2) + \sum_{i,j}\sum_{t\neq s}\mathbb{E}(e_{it}^2)\mathbb{E}(e_{js}^2) \\
&= \sum_{i,j}\sum_t \mathbb{E}(e_{it}^2 e_{jt}^2) + \sum_{i,j}\sum_{t\neq s}\sigma_{ii}\sigma_{jj} \\
&= T\{tr(\mathbf{\Sigma})\}^2 + 2Ttr(\mathbf{\Sigma}^2)\{1+o(1)\} + T(T-1)\{tr(\mathbf{\Sigma})\}^2.
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathrm{var}(\xi_{NT,12}) &= T^{-3}[\mathbb{E}(\sum_{i=1}^N \mathbf{e}_i^\top \mathbf{e}_i)^2 - \{\sum_{i=1}^N \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_i)\}^2] \\
&= T^{-3}[T\{tr(\mathbf{\Sigma})\}^2 + 2Ttr(\mathbf{\Sigma}^2)\{1+o(1)\} + T(T-1)\{tr(\mathbf{\Sigma})\}^2 - \{Ttr(\mathbf{\Sigma})\}^2] \\
&= T^{-3}[2Ttr(\mathbf{\Sigma}^2)\{1+o(1)\}] = 2T^{-2}tr(\mathbf{\Sigma}^2)\{1+o(1)\}.
\end{aligned}$$

Therefore,

$$\mathrm{var}(\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,12}) = 2T^{-2}\{1+o(1)\} = o(1),$$

which implies that $\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,12} = o_p(1)$. Consequently, we have

$$\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,1} = o_p(1).$$

By (A.4) and Condition (C3)(i), we have

$$\begin{aligned}
\{tr(\mathbf{\Sigma}^2)\}^{-1/2}|\xi_{NT,2}| &\leq \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-3/2}\sum_{i=1}^N \boldsymbol{\rho}_{NT,i}^\top \boldsymbol{\rho}_{NT,i} \\
&\leq \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-3/2}TNL^{-2r} = o(1).
\end{aligned}$$

By (A.4) and Condition (C3)(iii), we also obtain that

$$\begin{aligned}
\{tr(\mathbf{\Sigma}^2)\}^{-1/2}|\xi_{NT,3}| &\leq \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-1/2}T^{-1}\sum_{i=1}^N(\delta_i^0)^2 T \\
&= \{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-1/2}\sum_{i=1}^N(\delta_i^0)^2 \\
&= O[\{tr(\mathbf{\Sigma}^2)\}^{-1}NT^{-1/2}\sum_{i=1}^N(\delta_i^0)^2] = o(1).
\end{aligned}$$

Denote $M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i} = \boldsymbol{\gamma}_{NT,i} = (\gamma_{NT,i1},\cdots,\gamma_{NT,iT})^\top$. Then

$$(\mathbb{E}|\gamma_{NT,it}|^2)^{1/2} \leq \{(T\max_{t'}|\rho_{NT,it'}|)^2\mathbb{E}|\eta_t|^2\}^{1/2} \leq \tilde{C}_2 TL^{-r} \tag{B.26}$$

for some constant $0<\tilde{C}_2<\infty$. By Assumption (A3), we obtain that

$$\begin{aligned}
&\mathbb{E}(T^{-1/2}T^{-1}\sum_{i=1}^N \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\rho}_{NT,i})^2 = T^{-3}\sum_{i,i'=1}^N\sum_{t=1}^T \mathbb{E}(e_{it}e_{i't}\gamma_{NT,it}\gamma_{NT,i't}) \\
&= T^{-3}\sum_{i,i'=1}^N\sum_{t=1}^T \sigma_{ii'}\mathbb{E}(\gamma_{NT,it}\gamma_{NT,i't}) \leq T^{-3}T(\tilde{C}_2)^2T^2L^{-2r}\sum_{i,i'=1}^N|\sigma_{ii'}| \\
&\leq (\tilde{C}_2)^2L^{-2r}N\{tr(\mathbf{\Sigma}^2)\}^{1/2}.
\end{aligned}$$

Applying Condition (C3)(i), we have $\mathbb{E}[\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,4}]^2 \leq 4(\tilde{C}_2)^2 L^{-2r} N\{tr(\mathbf{\Sigma}^2)\}^{-1/2} = o(1)$. Accordingly, we obtain

$$\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,4} = o_p(1).$$

After algebraic simplification and using (B.22), we have

$$\mathbb{E}(T^{-1/2}T^{-1}\sum_{i=1}^N \delta_i^0 \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{1}_T)^2 = T^{-3}\sum_{i,i'=1}^N \sum_{t=1}^T \delta_i^0\delta_{i'}^0 \mathbb{E}(e_{it}e_{i't}\eta_t^2)$$
$$\leq T^{-3}TC^{**}\sum_{i,i'=1}^N |\delta_i^0||\delta_{i'}^0||\sigma_{ii'}| \leq C^{**}T^{-2}(\sum_{i=1}^N |\delta_i^0|^2)^2 + C^{**}T^{-2}tr(\mathbf{\Sigma}^2).$$

This, together with (A.13), leads to

$$\begin{aligned}\mathbb{E}[\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,5}]^2 &\leq C^{**}\{tr(\mathbf{\Sigma}^2)\}^{-1}T^{-2}(\sum_{i=1}^N |\delta_i^0|^2)^2 + C^{**}T^{-2}\\ &= O[\{\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-1}(\sum_{i=1}^N |\delta_i^0|^2)\}^2 + T^{-2}]\\ &= O[\{\{tr(\mathbf{\Sigma}^2)\}^{-1}NT^{-1}(\sum_{i=1}^N |\delta_i^0|^2)\}^2 + T^{-2}] = o(1).\end{aligned}$$

As a result, we have

$$\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\xi_{NT,5} = o_p(1).$$

By (A.4) that $\sup_{1\leq t\leq T}|\rho_{NT,it}| = O(L^{-r})$, (B.22), and Condition (C3)(i), by following the same reasoning as the proof for (B.24), we have that

$$\begin{aligned}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}|\xi_{NT,6}| &\leq \{tr(\mathbf{\Sigma}^2)\}^{-1/2}\mathbb{E}\{2T^{-1/2}T^{-1}\sum_{i=1}^N |\delta_i^0|\sum_{t=1}^T |\rho_{NT,it}||\eta_t|\}\\ &\leq \tilde{C}_3(C^{**})^{1/2}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}T^{-1/2}L^{-r}\sum_{i=1}^N |\delta_i^0| = o(1),\end{aligned}$$

for some constant $0 < \tilde{C}_3 < \infty$. This completes the proof. □

Proof of Lemma A.11. By Assumption (A3), we have

$$T^{-2}\sum_{i,j=1}^N \mathbb{E}(\sum_{t=1}^T e_{it}e_{jt})^2 = T^{-2}\sum_{i,j=1}^N \sum_{t,t'=1}^T \mathbb{E}(e_{it}e_{jt}e_{it'}e_{jt'})$$
$$= T^{-2}\sum_{i,j=1}^N \sum_{t\neq t'}^T \mathbb{E}(e_{it}e_{jt})\mathbb{E}(e_{it'}e_{jt'}) + T^{-2}\sum_{i,j=1}^N \sum_{t=1}^T \mathbb{E}(e_{it}e_{jt}e_{it}e_{jt}). \quad \text{(B.27)}$$

Also,

$$T^{-2}\sum_{i,j=1}^N \sum_{t\neq t'}^T \mathbb{E}(e_{it}e_{jt})\mathbb{E}(e_{it'}e_{jt'}) = T^{-2}T(T-1)\sum_{i,j=1}^N \sigma_{ij}^2 = T^{-2}T(T-1)tr(\mathbf{\Sigma}^2). \quad \text{(B.28)}$$

Let Γ_{ih} be the (i,h) component of Γ. By Condition (C1), we have

$$\begin{aligned}
\mathbb{E}(e_{it}e_{jt}e_{it}e_{jt}) &= \sum\nolimits_{h_1,h_2,h_3,h_4} \mathbb{E}(w_{th_1}w_{th_2}w_{th_3}w_{th_4})\Gamma_{ih_1}\Gamma_{ih_2}\Gamma_{jh_3}\Gamma_{jh_4} \\
&= \sum\nolimits_{h} \mathbb{E}(w_{th}^4)\Gamma_{ih}\Gamma_{ih}\Gamma_{jh}\Gamma_{jh} + \sum\nolimits_{h_1,h_2} \mathbb{E}(w_{th_1}^2)\mathbb{E}(w_{th_2}^2)\Gamma_{ih_1}\Gamma_{ih_1}\Gamma_{jh_2}\Gamma_{jh_2} \\
&\quad + \sum\nolimits_{h_1,h_2} \mathbb{E}(w_{th_1}^2)\mathbb{E}(w_{th_2}^2)\Gamma_{ih_1}\Gamma_{ih_2}\Gamma_{jh_1}\Gamma_{jh_2} \\
&\quad + \sum\nolimits_{h_1,h_2} \mathbb{E}(w_{th_1}^2)\mathbb{E}(w_{th_2}^2)\Gamma_{ih_1}\Gamma_{ih_2}\Gamma_{jh_2}\Gamma_{jh_1}.
\end{aligned}$$

Since $\mathbb{E}(w_{th}^4) = 3 + \Delta$, $\mathbb{E}(w_{th_1}^2) = 1$ and $\sum_{h_1} \Gamma_{ih_1}\Gamma_{i'h_1} = \sigma_{ii'}$, we have

$$\begin{aligned}
& T^{-2}\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{t=1}^{T} \mathbb{E}(e_{it}e_{jt}e_{it}e_{jt}) = T^{-1}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(e_{it}e_{jt}e_{it}e_{jt}) \\
=\;& T^{-1}(3+\Delta)\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{h} \Gamma_{ih}\Gamma_{ih}\Gamma_{jh}\Gamma_{jh} + T^{-1}\sum\nolimits_{i,j=1}^{N} \sigma_{ii}\sigma_{jj} + 2T^{-1}\sum\nolimits_{i,j=1}^{N} \sigma_{ij}^2 \\
=\;& T^{-1}(3+\Delta)\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{h} \Gamma_{ih}^2\Gamma_{jh}^2 + T^{-1}\{tr(\mathbf{\Sigma})\}^2 + 2T^{-1}tr(\mathbf{\Sigma}^2). \qquad \text{(B.29)}
\end{aligned}$$

Using the fact that

$$(3+\Delta)\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{h} \Gamma_{ih}^2\Gamma_{jh}^2 = o\{\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{h}\Gamma_{ih}^2\sum\nolimits_{h}\Gamma_{jh}^2\} = o\{(\sum\nolimits_{i=1}^{N}\sigma_{ii})^2\} = o[\{tr(\mathbf{\Sigma})\}^2] \qquad \text{(B.30)}$$

we further obtain that

$$T^{-2}\sum\nolimits_{i,j=1}^{N}\sum\nolimits_{t=1}^{T} \mathbb{E}(e_{it}e_{jt}e_{it}e_{jt}) = T^{-1}\{tr(\mathbf{\Sigma})\}^2 + 2T^{-1}tr(\mathbf{\Sigma}^2)\{1+o(1)\}. \qquad \text{(B.31)}$$

This, together with (B.27), (B.28), and (B.31), leads to

$$\begin{aligned}
T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_j \mathbf{e}_i^\top \mathbf{e}_j) &= T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\sum\nolimits_{t=1}^{T} e_{it}e_{jt})^2 \\
&= tr(\mathbf{\Sigma}^2)\{1+o(1)\} + T^{-1}\{tr(\mathbf{\Sigma})\}^2\{1+o(1)\}.
\end{aligned}$$

By (A.16) that $T^{-1/2}tr(\mathbf{\Sigma}) = O(tr^{1/2}(\mathbf{\Sigma}^2))$, we have

$$T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_j \mathbf{e}_i^\top \mathbf{e}_j) = tr(\mathbf{\Sigma}^2) + T^{-1}\{tr(\mathbf{\Sigma})\}^2 + o\{tr(\mathbf{\Sigma}^2)\}. \qquad \text{(B.32)}$$

We next show that

$$T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j) - T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_j \mathbf{e}_i^\top \mathbf{e}_j) = o\{tr(\mathbf{\Sigma}^2)\}. \qquad \text{(B.33)}$$

Since $M_{\mathbf{Z}} = \mathbf{I}_T - P_{\mathbf{Z}}$, we have

$$\begin{aligned}
& T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j) - T^{-2}\sum\nolimits_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_j \mathbf{e}_i^\top \mathbf{e}_j) \\
=\;& -T^{-2}\sum_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top \mathbf{e}_j) - T^{-2}\sum_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top \mathbf{e}_j \mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j) + T^{-2}\sum_{i,j=1}^{N} \mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j).
\end{aligned}$$

Let $P_{tt'}$ be the (t,t') component of $P_{\mathbf{Z}}$. Then

$$\begin{aligned}
& T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_j)\\
= & T^{-2}\sum_{i,j=1}^{N}\sum_{t_1,t_2,t_3}\mathbb{E}(P_{t_1t_2}e_{it_1}e_{jt_2}e_{it_3}e_{jt_3})\\
= & T^{-2}\sum_{i,j=1}^{N}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1})\mathbb{E}(e_{it_1}e_{jt_1})\mathbb{E}(e_{it_3}e_{jt_3})+T^{-2}\sum_{i,j=1}^{N}\sum_{t_1}\mathbb{E}(P_{t_1t_1})\mathbb{E}(e_{it_1}e_{jt_1}e_{it_1}e_{jt_1})\\
= & T^{-2}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1})\sum_{i,j=1}^{N}\sigma_{ij}^2+T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1})\sum_{i,j=1}^{N}\mathbb{E}(e_{it_1}e_{jt_1}e_{it_1}e_{jt_1}).
\end{aligned}$$

In addition,

$$\begin{aligned}
&|T^{-2}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1})\sum_{i,j=1}^{N}\sigma_{ij}^2|\le T^{-1}\sum_{t_1}\mathbb{E}(P_{t_1t_1})tr(\mathbf{\Sigma}^2)\\
&=T^{-1}\mathbb{E}\{tr(P_{\mathbf{Z}})\}tr(\mathbf{\Sigma}^2)=T^{-1}\mathrm{rank}(P_{\mathbf{Z}})tr(\mathbf{\Sigma}^2)=T^{-1}(1+d)Ltr(\mathbf{\Sigma}^2)=o\{tr(\mathbf{\Sigma}^2)\}.
\end{aligned}$$

This, injunction with (A.16), (B.29), and (B.30), implies that

$$\begin{aligned}
& T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1})\sum_{i,j=1}^{N}\mathbb{E}(e_{it_1}e_{jt_1}e_{it_1}e_{jt_1})\\
= & T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1})\{(3+\Delta)\sum_{i,j=1}^{N}\sum_{h}\Gamma_{ih}^2\Gamma_{jh}^2+\{tr(\mathbf{\Sigma})\}^2+tr(\mathbf{\Sigma}^2)\}\\
= & T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1})[tr(\mathbf{\Sigma}^2)\{1+o(1)\}+\{tr(\mathbf{\Sigma})\}^2]\\
= & T^{-2}\mathbb{E}\{tr(P_{\mathbf{Z}})\}[tr(\mathbf{\Sigma}^2)\{1+o(1)\}+\{tr(\mathbf{\Sigma})\}^2]\\
= & T^{-2}(1+d)L[tr(\mathbf{\Sigma}^2)\{1+o(1)\}+\{tr(\mathbf{\Sigma})\}^2]=o\{tr(\mathbf{\Sigma}^2)\}.
\end{aligned}$$

Hence,

$$T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_j)=o\{tr(\mathbf{\Sigma}^2)\}. \tag{B.34}$$

Analogously, we can show that

$$T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)=o\{tr(\mathbf{\Sigma}^2)\}. \tag{B.35}$$

Moreover,

$$\begin{aligned}
&T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)=T^{-2}\sum_{i,j=1}^{N}\sum_{t_1,t_2,t_3,t_4}\mathbb{E}(P_{t_1t_2}P_{t_3t_4})\mathbb{E}(e_{it_1}e_{jt_2}e_{it_3}e_{jt_4})\\
&=T^{-2}\sum_{i,j=1}^{N}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1}P_{t_3t_3})\mathbb{E}(e_{it_1}e_{jt_1})\mathbb{E}(e_{it_3}e_{jt_3})\\
&+T^{-2}\sum_{i,j=1}^{N}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_1t_2})\mathbb{E}(e_{it_1}e_{it_1})\mathbb{E}(e_{jt_2}e_{jt_2})\\
&+T^{-2}\sum_{i,j=1}^{N}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_2t_1})\mathbb{E}(e_{it_1}e_{jt_1})\mathbb{E}(e_{jt_2}e_{it_2})\\
&+T^{-2}\sum_{i,j=1}^{N}\sum_{t_1}\mathbb{E}(P_{t_1t_1}P_{t_1t_1})\mathbb{E}(e_{it_1}e_{jt_1}e_{it_1}e_{jt_1})\\
&=T^{-2}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1}P_{t_3t_3})tr(\mathbf{\Sigma}^2)+T^{-2}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_1t_2})\{tr(\mathbf{\Sigma})\}^2\\
&+T^{-2}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_2t_1})tr(\mathbf{\Sigma}^2)+T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1}P_{t_1t_1})[tr(\mathbf{\Sigma}^2)\{1+o(1)\}+\{tr(\mathbf{\Sigma})\}^2].
\end{aligned}$$

It is worth noting that $\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1}P_{t_3t_3}) \le \mathbb{E}[\{tr(P_{\mathbf{Z}})\}^2] = \{\mathrm{rank}(P_{\mathbf{Z}})\}^2 = \{(1+d)L\}^2$. Thus,

$$T^{-2}\sum_{t_1\neq t_3}\mathbb{E}(P_{t_1t_1}P_{t_3t_3})tr(\mathbf{\Sigma}^2) \le T^{-2}\{(1+d)L\}^2 tr(\mathbf{\Sigma}^2) = o\{tr(\mathbf{\Sigma}^2)\}.$$

Since $\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_1t_2}) \le \mathbb{E}\{tr(P_{\mathbf{Z}}P_{\mathbf{Z}})\} = \mathbb{E}\{tr(P_{\mathbf{Z}})\} = (1+d)L$, we obtain

$$\begin{aligned} T^{-2}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_1t_2})\{tr(\mathbf{\Sigma})\}^2 &\le T^{-2}(1+d)L\{tr(\mathbf{\Sigma})\}^2 \\ &= o[T^{-1}\{tr(\mathbf{\Sigma})\}^2] = o\{tr(\mathbf{\Sigma}^2)\} \end{aligned}$$

and

$$T^{-2}\sum_{t_1\neq t_2}\mathbb{E}(P_{t_1t_2}P_{t_2t_1})tr(\mathbf{\Sigma}^2) \le T^{-2}(1+d)Ltr(\mathbf{\Sigma}^2) = o\{tr(\mathbf{\Sigma}^2)\}.$$

Using the fact that $\sum_{t_1}\mathbb{E}(P_{t_1t_1}P_{t_1t_1}) = O\{\mathbb{E}(\sum_{t_1}P_{t_1t_1})\} = O[\mathbb{E}\{tr(P_{\mathbf{Z}})\}] = O\{(1+d)L\}$, we have

$$\begin{aligned} & T^{-2}\sum_{t_1}\mathbb{E}(P_{t_1t_1}P_{t_1t_1})[tr(\mathbf{\Sigma}^2)\{1+o(1)\} + \{tr(\mathbf{\Sigma})\}^2] \\ = \;& O[T^{-2}(1+d)L\{tr(\mathbf{\Sigma}^2) + \{tr(\mathbf{\Sigma})\}^2\}] = o[T^{-1}\{tr(\mathbf{\Sigma}^2) + \{tr(\mathbf{\Sigma})\}^2\}] = o\{tr(\mathbf{\Sigma}^2)\}. \end{aligned}$$

Accordingly,

$$T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j) = o\{tr(\mathbf{\Sigma}^2)\}, \tag{B.36}$$

and the result of (B.33) follows directly from (B.34), (B.35) and (B.36). By (B.32) and (B.33), we have

$$T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j) = tr(\mathbf{\Sigma}^2) + T^{-1}\{tr(\mathbf{\Sigma})\}^2 + o\{tr(\mathbf{\Sigma}^2)\}. \tag{B.37}$$

This allows us to express

$$\begin{aligned} & \sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j)^2 - tr(\mathbf{\Sigma}^2) - T^{-1}\{tr(\mathbf{\Sigma})\}^2 \\ = \;& T^{-2}\sum_{i,j=1}^{N}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j - T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j) + o\{tr(\mathbf{\Sigma}^2)\} \\ = \;& \Big\{T^{-2}\sum_{i,j=1}^{N}\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j\widehat{\mathbf{e}}_i^\top\widehat{\mathbf{e}}_j - T^{-2}\sum_{i,j=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\Big\} \\ & + \Big\{T^{-2}\sum_{i,j=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j - T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)\Big\} + o\{tr(\mathbf{\Sigma}^2)\} \\ = \;& \Delta_{NT,1} + \Delta_{NT,2} + o\{tr(\mathbf{\Sigma}^2)\}. \end{aligned}$$

In the following two lemmas, we will show $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Delta_{NT,1} = o_p(1)$ and $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Delta_{NT,2} = o_p(1)$ to complete the proof of Lemma A.11. □

Proof of Lemma A.12. Let $\boldsymbol{\nu}_{NT,i} = (\nu_{NT,it}, 1 \le t \le T)^\top = \boldsymbol{\rho}_{NT,i} + \delta_i^0 \mathbf{1}_T$. Then

$$\begin{aligned}
& |T^{-2}\sum_{i,j=1}^{N} \widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j \widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j - T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j| \\
= \;& 2|2T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} + T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \boldsymbol{\nu}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} \\
& + T^{-2}\sum_{i,j=1}^{N} (\mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} + \mathbf{e}_j^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,i} + \boldsymbol{\nu}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j})^2| \\
\le \;& 2|2T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j}| + |T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j \boldsymbol{\nu}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j}| \\
& + 3T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} \\
& + 3T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_j^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,i} \mathbf{e}_j^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,i} + 3T^{-2}\sum_{i,j=1}^{N} \boldsymbol{\nu}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} \boldsymbol{\nu}_{NT,i}^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j} \\
\equiv \;& 2\sum_{k=1}^{5} \Upsilon_{NT,k}.
\end{aligned}$$

In the following, we will show that $\{tr(\boldsymbol{\Sigma}^2)\}^{-1}\Upsilon_{NT,k} = o_p(1)$ for $k = 1, \cdots, 5$. Moreover,

$$\begin{aligned}
\Upsilon_{NT,1} &= |2T^{-2}\sum_{i,j=1}^{N} \mathbf{e}_j^\top M_{\mathbf{Z}}\mathbf{e}_i \mathbf{e}_i^\top M_{\mathbf{Z}}\boldsymbol{\nu}_{NT,j}| \le 2T^{-2}\sum_{i,j=1}^{N} ||\mathbf{e}_i\mathbf{e}_i^\top|| \times ||M_{\mathbf{Z}}||^2 ||\mathbf{e}_j|| ||\boldsymbol{\nu}_{NT,j}|| \\
&\le 2T^{-2}\sum_{i,j=1}^{N} ||\mathbf{e}_i\mathbf{e}_i^\top|| ||\mathbf{e}_j|| ||\boldsymbol{\nu}_{NT,j}|| = 2T^{-2}\sum_{i=1}^{N} ||\mathbf{e}_i\mathbf{e}_i^\top|| \sum_{j=1}^{N} ||\mathbf{e}_j|| ||\boldsymbol{\nu}_{NT,j}||.
\end{aligned}$$

For any vector $\mathbf{a} \in \mathbb{R}^T$ with $||\mathbf{a}|| = 1$, $\mathbb{E}(\mathbf{a}^\top \mathbf{e}_i \mathbf{e}_i^\top \mathbf{a}) = \mathbb{E}(\sum_{t=1}^{T} a_t e_{it})^2 = \sigma_{ii}$. Hence, $\mathbb{E}||\mathbf{e}_i\mathbf{e}_i^\top|| = \sigma_{ii}$, which leads to

$$\sum_{i=1}^{N} ||\mathbf{e}_i\mathbf{e}_i^\top|| = O_p(\sum_{i=1}^{N} \sigma_{ii}) = O_p\{tr(\Sigma)\}.$$

In addition, $\mathbb{E}||\mathbf{e}_j|| \le (\mathbb{E}||\mathbf{e}_j||^2)^{1/2} = T^{1/2}\sigma_{jj}^{1/2}$. Thus, $\sum_{j=1}^{N} ||\mathbf{e}_j|| ||\boldsymbol{\nu}_{NT,j}|| = O_p\{T\sum_{j=1}^{N} \sigma_{jj}^{1/2}(L^{-r} + \left|\delta_j^0\right|)\}$. Since $\sigma_{\max}^{1/2} < \infty$, we further have

$$\sum_{j=1}^{N} ||\mathbf{e}_j|| ||\boldsymbol{\nu}_{NT,j}|| = O_p\{T\sum_{j=1}^{N} (L^{-r} + \left|\delta_j^0\right|)\} = O_p\{T(NL^{-r} + \sum_{j=1}^{N} \left|\delta_j^0\right|)\}.$$

By the above results, we have

$$\Upsilon_{NT,1} = O_p\{T^{-1}tr(\boldsymbol{\Sigma})(NL^{-r} + \sum_{j=1}^{N} \left|\delta_j^0\right|)\}.$$

Using the fact that $T^{-1/2}tr(\boldsymbol{\Sigma}) = O(tr^{1/2}(\boldsymbol{\Sigma}^2))$, we obtain

$$\begin{aligned}
\Upsilon_{NT,1} &= O_p\{T^{-1}tr(\boldsymbol{\Sigma})(NL^{-r} + \sum_{j=1}^{N} \left|\delta_j^0\right|)\} \\
&= O_p\{T^{-1/2}NL^{-r}tr^{1/2}(\boldsymbol{\Sigma}^2)\} + O_p\{T^{-1/2}tr^{1/2}(\boldsymbol{\Sigma}^2)\sum_{j=1}^{N} \left|\delta_j^0\right|\}.
\end{aligned}$$

Under the local alternative given in (8) and Condition (C3)(iii), we have

$$\begin{aligned}
T^{-1/2}\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}\sum_{j=1}^{N} \left|\delta_j^0\right| &\le T^{-1/2}\{tr(\boldsymbol{\Sigma}^2)\}^{-1/4} N N^{-1/2} T^{-1/2}\{N^{-1}\sum_{j=1}^{N} \left|c_j^0\right|^2\}^{1/2} \\
&= O[T^{-1}N^{1/2}\{tr(\boldsymbol{\Sigma}^2)\}^{-1/4}] = o(1).
\end{aligned}$$

This, together with Conditions (C3)(i) and the Lemma's assumption, $L^r T^{-3/2} = O(1)$, implies that

$$\{tr(\mathbf{\Sigma}^2)\}^{-1}\Upsilon_{NT,1} = O_p[T^{-1/2}NL^{-r}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}] + O_p[T^{-1/2}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}\sum_{j=1}^{N}|\delta_j^0|] = o_p(1).$$

It is worth noting that

$$\begin{aligned}
\Upsilon_{NT,2} &\leq T^{-2}\sum_{i,j=1}^{N}|\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j| \times ||M_{\mathbf{Z}}|| \times ||\boldsymbol{\nu}_{NT,i}|| \times ||\boldsymbol{\nu}_{NT,j}|| \\
&\leq T^{-2}\sum_{i,j=1}^{N}|\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j|T(\tilde{C}_4 L^{-r} + |\delta_j^0|)^2 \\
&\leq 2T^{-1}\sum_{i,j=1}^{N}|\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j|(\tilde{C}_4^2 L^{-2r} + |\delta_j^0|^2) \\
&\leq 2T^{-1}\sum_{i,j=1}^{N}||\mathbf{e}_i||||\mathbf{e}_j||(\tilde{C}_4^2 L^{-2r} + |\delta_j^0|^2),
\end{aligned}$$

for some positive constant $\tilde{C}_4$. Since $\mathbb{E}||\mathbf{e}_i|| \leq \{\mathbb{E}||\mathbf{e}_i||^2\}^{1/2} \leq T^{1/2}\sigma_{ii}^{1/2}$, we have

$$\begin{aligned}
&\mathbb{E}\{2T^{-1}\sum_{i,j=1}^{N}||\mathbf{e}_i||||\mathbf{e}_j||(\tilde{C}_4^2 L^{-2r} + |\delta_j^0|^2)\} \\
\leq\ & 2\tilde{C}_4^2(\sum_{i=1}^{N}\sigma_{ii}^{1/2})^2 L^{-2r} + 2\sigma_{\max}^{1/2}(\sum_{i=1}^{N}\sigma_{ii}^{1/2})\sum_{j=1}^{N}|\delta_j^0|^2 \\
\leq\ & 2\tilde{C}_4^2 N L^{-2r} tr(\mathbf{\Sigma}) + 2\sigma_{\max} N\sum_{j=1}^{N}|\delta_j^0|^2.
\end{aligned}$$

This, in conjunction with Condition (C3)(i) and (A.13), leads to

$$\begin{aligned}
&\{tr(\mathbf{\Sigma}^2)\}^{-1}\mathbb{E}\Big\{2C_M^2 T^{-1}\sum_{i,j=1}^{N}||\mathbf{e}_i||||\mathbf{e}_j||(\tilde{C}_4^2 L^{-2r} + |\delta_j^0|^2)\Big\} \\
=\ & O[T^{1/2}NL^{-2r}\{tr(\mathbf{\Sigma}^2)\}^{-1/2}] + O[\{tr(\mathbf{\Sigma}^2)\}^{-1}N\sum_{j=1}^{N}|\delta_j^0|^2] = o(1). \qquad \text{(B.38)}
\end{aligned}$$

Accordingly, $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Upsilon_{NT,2} = o_p(1)$.

After algebraic simplification, we obtain that

$$\begin{aligned}
\mathbb{E}(\Upsilon_{NT,3}) &\leq 3T^{-2}\sum_{i,j=1}^{N}\mathbb{E}(||\mathbf{e}_i||^2||M_{\mathbf{Z}}||^2||\boldsymbol{\nu}_{NT,j}||^2) \leq 3C_M^2 T^{-2}\sum_{i,j=1}^{N}\mathbb{E}||\mathbf{e}_i||^2||\boldsymbol{\nu}_{NT,j}||^2 \\
&\leq 3C_M^2(\sum_{i=1}^{N}\sigma_{ii})\sum_{j=1}^{N}(\tilde{C}_4 L^{-r} + |\delta_j^0|)^2 \leq 6C_M^2(\sum_{i=1}^{N}\sigma_{ii})(N\tilde{C}_4^2 L^{-2r} + \sum_{j=1}^{N}|\delta_j^0|^2) \\
&\leq 6C_M^2\tilde{C}_4^2 N L^{-2r} tr(\mathbf{\Sigma}) + 6C_M^2\sigma_{\max}N\sum_{j=1}^{N}|\delta_j^0|^2.
\end{aligned}$$

Employing the same techniques as those used in the proof of (B.38), we have $\{tr(\mathbf{\Sigma}^2)\}^{-1}\mathbb{E}(\Upsilon_{NT,3}) = o(1)$. Since $\Upsilon_{NT,3}$ is nonnegative, we have $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Upsilon_{NT,3} = o_p(1)$. Analogously, we can demonstrate that $\{tr(\mathbf{\Sigma}^2)\}^{-1}\Upsilon_{NT,4} = o_p(1)$.

Lastly,

$$\begin{aligned}\Upsilon_{NT,5} &\leq 3T^{-2}\sum_{i,j=1}^{N}||M_{\mathbf{Z}}||^2||\boldsymbol{\nu}_{NT,i}||^2||\boldsymbol{\nu}_{NT,j}||^2 \leq 3C_M^2(N\tilde{C}_4^2L^{-2r}+\sum_{j=1}^{N}\left|\delta_j^0\right|^2)^2\\ &\leq 6C_M^2\tilde{C}_4^4(NL^{-2r})^2+6C_M^2\{\sum_{j=1}^{N}\left|\delta_j^0\right|^2\}^2.\end{aligned}$$

Hence, by Condition (C3)(i) and (A.13), we have

$$\begin{aligned}\{tr(\boldsymbol{\Sigma}^2)\}^{-1}\Upsilon_{NT,5} &\leq 6C_M^2\tilde{C}_4^4[NL^{-2r}\left\{tr\left(\boldsymbol{\Sigma}^2\right)\right\}^{-1/2}]^2+6C_M^2[\left\{tr\left(\boldsymbol{\Sigma}^2\right)\right\}^{-1/2}\sum_{j=1}^{N}\left|\delta_j^0\right|^2]^2\\ &= O[[NL^{-2r}\left\{tr\left(\boldsymbol{\Sigma}^2\right)\right\}^{-1/2}]^2+\left\{tr\left(\boldsymbol{\Sigma}^2\right)\right\}^{-1}N\sum_{j=1}^{N}\left|\delta_j^0\right|^2]=o(1),\end{aligned}$$

which completes the whole proof. □

***Proof of Lemma* A.13.** We make the following decomposition

$$\begin{aligned}&\mathbb{E}(T^{-2}\sum_{i,j=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)^2\\ =\;& T^{-4}\sum_{i,j,i'j'}\mathbb{E}(\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)^2(\mathbf{e}_{i'}^\top M_{\mathbf{Z}}\mathbf{e}_{j'})^2\\ =\;& T^{-4}\sum_{i,j,i'j'}\mathbb{E}(\mathbf{e}_i^\top\mathbf{e}_j)^2(\mathbf{e}_{i'}^\top\mathbf{e}_{j'})^2+T^{-4}\sum_{i,j,i'j'}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)^2(\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'})^2\\ &-T^{-4}\sum_{i,j,i'j'}\mathbb{E}\{2\mathbf{e}_i^\top\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j(\mathbf{e}_{i'}^\top\mathbf{e}_{j'})^2\}-T^{-4}\sum_{i,j,i'j'}\mathbb{E}\{2\mathbf{e}_{i'}^\top\mathbf{e}_{j'}\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'}(\mathbf{e}_i^\top\mathbf{e}_j)^2\}\\ &-T^{-4}\sum_{i,j,i'j'}\mathbb{E}\{2(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)^2\mathbf{e}_{i'}^\top\mathbf{e}_{j'}\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'}\}-T^{-4}\sum_{i,j,i'j'}\mathbb{E}\{2(\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'})^2\mathbf{e}_i^\top\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\}\\ &+T^{-4}\sum_{i,j,i'j'}\mathbb{E}(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)^2(\mathbf{e}_{i'}^\top\mathbf{e}_{j'})^2+T^{-4}\sum_{i,j,i'j'}\mathbb{E}(\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'})^2(\mathbf{e}_i^\top\mathbf{e}_j)^2\\ &+T^{-4}\sum_{i,j,i'j'}\mathbb{E}(4\mathbf{e}_i^\top\mathbf{e}_j\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_{i'}^\top\mathbf{e}_{j'}\mathbf{e}_{i'}^\top P_{\mathbf{Z}}\mathbf{e}_{j'})\\ \equiv\;& \sum_{k=1}^{9}\psi_{NT,k}.\end{aligned}$$

In the following, we will show that

$$\{tr(\boldsymbol{\Sigma}^2)\}^{-2}[\psi_{NT,1}-\{\mathbb{E}(T^{-2}\sum_{i,j=1}^{N}\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)\}^2]=o(1) \tag{B.39}$$

and $\{tr(\boldsymbol{\Sigma}^2)\}^{-2}\psi_{NT,k}=o(1)$ for $k=2,\cdots,9$. Then Lemma A.13 follows immediately.

It is worth noting that

$$
\begin{aligned}
& \psi_{NT,1} \\
= \;& T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1,t_2,t_3,t_4}\mathbb{E}(e_{it_1}e_{it_2}e_{jt_1}e_{jt_2}e_{i't_3}e_{i't_4}e_{j't_3}e_{j't_4}) \\
= \;& T^{-4}\sum\nolimits_{t_1,t_2,t_3,t_4}\mathbb{E}\{(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2})^2(\mathbf{E}_{t_3}^\top\mathbf{E}_{t_4})^2\} \\
= \;& T^{-4}\sum_{t_1\neq t_2\neq t_3\neq t_4}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_3}^\top\mathbf{E}_{t_4}\mathbf{E}_{t_3}^\top\mathbf{E}_{t_4}) + T^{-4}\sum_{t_1\neq t_2}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_2}) \\
& +2T^{-4}\sum\nolimits_{t_1\neq t_2\neq t_3}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3}) + T^{-4}\sum\nolimits_t\mathbb{E}(\mathbf{E}_t^\top\mathbf{E}_t)^4 \\
& +4T^{-4}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}) + 2T^{-4}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}) \\
& +4T^{-4}\sum\nolimits_{t_1\neq t_2\neq t_3}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_3}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_3}) \\
\equiv \;& \sum\nolimits_{s=1}^7\psi_{NT,1s}.
\end{aligned}
$$

By (A.16), (B.28), and (B.31), we have

$$
\begin{aligned}
\psi_{NT,11} \;&\sim\; \{T^{-2}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_2})\}^2 \\
&=\; \{T^{-2}\sum\nolimits_{i,j=1}^N\sum\nolimits_{t_1\neq t_2}\mathbb{E}(e_{it_1}e_{jt_1})\mathbb{E}(e_{it_2}e_{jt_2})\}^2 \sim \{tr(\mathbf{\Sigma}^2)\}^2,
\end{aligned}
$$

$$
\begin{aligned}
\psi_{NT,12} \;&=\; T^{-4}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1})\mathbb{E}(\mathbf{E}_{t_2}^\top\mathbf{E}_{t_2}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_2}) \\
&\sim\; \{T^{-2}\sum\nolimits_{t_1}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1})\}^2 = \{T^{-2}\sum\nolimits_{i,j=1}^N\sum\nolimits_{t=1}^T\mathbb{E}(e_{it_1}e_{jt_1}e_{it_1}e_{jt_1})\}^2 \\
&=\; \{T^{-1}\{tr(\mathbf{\Sigma})\}^2 + 2T^{-1}tr(\mathbf{\Sigma}^2)\{1+o(1)\}\}^2 = \{T^{-1}\{tr(\mathbf{\Sigma})\}^2\}^2 + o[\{tr(\mathbf{\Sigma}^2)\}^2],
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_{NT,13} \;&=\; 2T^{-4}\sum\nolimits_{t_1\neq t_2\neq t_3}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1})\mathbb{E}(\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3})\} \\
&\sim\; 2\{T^{-2}\sum\nolimits_{t_1}\mathbb{E}(\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1}\mathbf{E}_{t_1}^\top\mathbf{E}_{t_1})\}\{T^{-2}\sum\nolimits_{t_2\neq t_3}\mathbb{E}(\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3}\mathbf{E}_{t_2}^\top\mathbf{E}_{t_3})\} \\
&\sim\; 2tr(\mathbf{\Sigma}^2)\{T^{-1}\{tr(\mathbf{\Sigma})\}^2 + T^{-1}tr(\mathbf{\Sigma}^2)\{1+o(1)\}\} = 2tr(\mathbf{\Sigma}^2)\{T^{-1}\{tr(\mathbf{\Sigma})\}^2\} + o[\{tr(\mathbf{\Sigma}^2)\}^2].
\end{aligned}
$$

Hence, $\psi_{NT,11}+\psi_{NT,12}+\psi_{NT,13} = [tr(\mathbf{\Sigma}^2)+T^{-1}\{tr(\mathbf{\Sigma})\}^2]^2 + o[\{tr(\mathbf{\Sigma}^2)\}^2]$. In addition, (B.37) leads to $\{\mathbb{E}(T^{-2}\sum_{i,j=1}^N\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)\}^2 = [tr(\mathbf{\Sigma}^2)+T^{-1}\{tr(\mathbf{\Sigma})\}^2]^2 + o[\{tr(\mathbf{\Sigma}^2)\}^2]$. Accordingly, we have

$$
\psi_{NT,11}+\psi_{NT,12}+\psi_{NT,13} - \{\mathbb{E}(T^{-2}\sum\nolimits_{i,j=1}^N\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j\mathbf{e}_i^\top M_{\mathbf{Z}}\mathbf{e}_j)\}^2 = o[\{tr(\mathbf{\Sigma}^2)\}^2].
$$

To complete the proof of (B.39), we will show that $\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,1s} = o(1)$, for $s = 4,\cdots,7$, given below. By Condition (C3)(iii), we have $\{tr(\mathbf{\Sigma}^2)\}\psi_{NT,14} = O[\{tr(\mathbf{\Sigma}^2)\}^{-2}T^{-4}TN^4] = o(1)$.

Also,

$$\begin{aligned}
& \{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,15} \\
= \;& \{tr(\mathbf{\Sigma}^2)\}^{-2}4T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(e_{it_1}^2 e_{jt_1}^2 e_{i't_1}e_{j't_1})\mathbb{E}(e_{i't_2}e_{j't_2}) \\
\leq \;& 4\tilde{C}_5\{tr(\mathbf{\Sigma}^2)\}^{-2}T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1\neq t_2}|\sigma_{i'j'}| \leq 4\tilde{C}_5\{tr(\mathbf{\Sigma}^2)\}^{-2}T^{-2}N^2N(\sum\nolimits_{i'j'}\sigma_{i'j'}^2)^{1/2} \\
\leq \;& 4\tilde{C}_5\{tr(\mathbf{\Sigma}^2)\}^{-3/2}T^{-2}N^3 = o(1),
\end{aligned}$$

for some constant $0 < \tilde{C}_5 < \infty$, Applying the same techniques as those used in the proof of (B.29), we have

$$\begin{aligned}
\psi_{NT,16} &= 2T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1\neq t_2}\mathbb{E}(e_{it_1}e_{i't_1}e_{jt_1}e_{j't_1})\mathbb{E}(e_{it_2}e_{i't_2}e_{jt_2}e_{j't_2}) \\
&= 2T^{-4}T(T-1)\sum\nolimits_{i,j,i'j'}(\sigma_{ii'}\sigma_{jj'}+\sigma_{ij'}\sigma_{i'j})^2(1+o(1)) \\
&\leq 4T^{-4}T(T-1)\Big(\sum\nolimits_{i,j,i'j'}\sigma_{ii'}^2\sigma_{jj'}^2+\sum\nolimits_{i,j,i'j'}\sigma_{ij'}^2\sigma_{i'j}^2\Big)(1+o(1)) \\
&= 8T^{-4}T(T-1)\{tr(\mathbf{\Sigma}^2)\}^2(1+o(1)).
\end{aligned}$$

As a result, $\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,16} \leq 8T^{-4}T(T-1)(1+o(1)) = o(1)$. Subsequently,

$$\begin{aligned}
\psi_{NT,17} &= 4T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1\neq t_2\neq t_3}\mathbb{E}(e_{it_1}e_{it_2}e_{jt_1}e_{jt_2}e_{i't_1}e_{i't_3}e_{j't_1}e_{j't_3}) \\
&= 4T^{-4}\sum\nolimits_{i,j,i'j'}\sum\nolimits_{t_1\neq t_2\neq t_3}\mathbb{E}(e_{it_1}e_{jt_1}e_{i't_1}e_{j't_1})\mathbb{E}(e_{it_2}e_{jt_2})\mathbb{E}(e_{i't_3}e_{j't_3}) \\
&= 4T^{-1}\sum\nolimits_{i,j,i'j'}(\sigma_{ii'}\sigma_{jj'}+\sigma_{ij'}\sigma_{i'j})\sigma_{ij}\sigma_{i'j'}(1+o(1)) \\
&= 4T^{-1}\sum\nolimits_{i,j,i'j'}(\sigma_{ii'}\sigma_{jj'}\sigma_{ij}\sigma_{i'j'}+\sigma_{ij'}\sigma_{i'j}\sigma_{ij}\sigma_{i'j'})(1+o(1)) \\
&\leq 2T^{-1}\sum\nolimits_{i,j,i'j'}(\sigma_{ii'}^2\sigma_{jj'}^2+\sigma_{ij}^2\sigma_{i'j'}^2+\sigma_{ij'}^2\sigma_{i'j}^2+\sigma_{ij}^2\sigma_{i'j'}^2)(1+o(1)) = 2T^{-1}\{tr(\mathbf{\Sigma}^2)\}^2(1+o(1)).
\end{aligned}$$

Thus, $\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,17} \leq 2T^{-1}(1+o(1)) = o(1)$, which completes the proof of (B.39).

We next only demonstrate $\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,2} = o(1)$, since the proofs of $\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,k} = o(1)$ for $k = 3,\cdots,9$ are quite similar and hence we omit them. By (B.36), we have $T^{-2}\sum_{i,j=1}^N(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)^2 = o_p\{tr(\mathbf{\Sigma}^2)\}$. Thus, $T^{-4}\{\sum_{i,j=1}^N(\mathbf{e}_i^\top P_{\mathbf{Z}}\mathbf{e}_j)^2\}^2 = o_p[\{tr(\mathbf{\Sigma}^2)\}^2]$. Consequently, we obtain

$$\{tr(\mathbf{\Sigma}^2)\}^{-2}\psi_{NT,2} = o[\{tr(\mathbf{\Sigma}^2)\}^{-2}\{tr(\mathbf{\Sigma}^2)\}^2] = o(1),$$

which completes the proof of the lemma. □

Proof of Lemma A.14. We decompose

$$
\begin{aligned}
& \sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T) \\
= & \sum_{i=1}^{N} T^{-2}(\mathbf{e}_i + \boldsymbol{\rho}_{NT,i} + \delta_i^0 \mathbf{1}_T)^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} (\mathbf{e}_i + \boldsymbol{\rho}_{NT,i} + \delta_i^0 \mathbf{1}_T) \\
= & T^{-2}\sum_{i=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{e}_i + T^{-2}\sum_{i=1}^{N} \boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} \\
& + T^{-2}\sum_{i=1}^{N} (\delta_i^0)^2 \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T + 2T^{-2}\sum_{i=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \boldsymbol{\rho}_{NT,i} \\
& + 2T^{-2}\sum_{i=1}^{N} \mathbf{e}_i^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \delta_i^0 \mathbf{1}_T + 2T^{-2}\sum_{i=1}^{N} \delta_i^0 \boldsymbol{\rho}_{NT,i}^\top M_{\mathbf{Z}} \mathbf{1}_T \mathbf{1}_T^\top M_{\mathbf{Z}} \mathbf{1}_T \\
\equiv & \omega_{NT,1} + \omega_{NT,2} + \omega_{NT,3} + \omega_{NT,4} + \omega_{NT,5} + \omega_{NT,6}.
\end{aligned}
$$

It can be seen that $\omega_{NT,j} = T^{1/2}\vartheta_{NT,j}$ for $j = 1, \cdots, 6$, and $\vartheta_{NT,j}$s are defined in (A.19). Then, by Lemma (A.9), we have $\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}\omega_{NT,j} = o_p(\chi_{NT})$ for $j = 1, \cdots, 6$, where $\chi_{NT} = N^{-\varrho} + T^{-\varrho}$. Accordingly,

$$
\{tr(\boldsymbol{\Sigma}^2)\}^{-1/2}\sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)^2 = o_p(\chi_{NT}) = o_p(1). \tag{B.40}
$$

In addition,

$$
\begin{aligned}
& \{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top \mathbf{1}_T) \\
\leq & \{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)^2 \chi_{NT}^{1/2} + \{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)^2(T^{-1}\widehat{\mathbf{e}}_j^\top \mathbf{1}_T)^2 \chi_{NT}^{-1/2} \\
= & \{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)^2 \chi_{NT}^{1/2} + \{tr(\boldsymbol{\Sigma}^2)\}^{-1}\{\sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)^2\}^2 \chi_{NT}^{-1/2}.
\end{aligned}
$$

By (B.40), we have $\{tr(\boldsymbol{\Sigma}^2)\}^{-1}\{\sum_{i=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)^2\}^2 \chi_{NT}^{-1/2} = o_p(\chi_{NT}^2 \chi_{NT}^{-1/2}) = o_p(1)$. Moreover, Lemma A.11 and (A.16) lead to $\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)^2 = O_p\{tr(\boldsymbol{\Sigma}^2)\}$. As a result,

$$
\{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)^2 \chi_{NT}^{1/2} = O_p\{\{tr(\boldsymbol{\Sigma}^2)\}^{-1} tr(\boldsymbol{\Sigma}^2)\chi_{NT}^{1/2}\} = O_p\{\chi_{NT}^{1/2}\} = o_p(1).
$$

Consequently, we have $\{tr(\boldsymbol{\Sigma}^2)\}^{-1}\sum_{i,j=1}^{N}(T^{-1}\widehat{\mathbf{e}}_i^\top \widehat{\mathbf{e}}_j)(T^{-1}\widehat{\mathbf{e}}_i^\top \mathbf{1}_T)(T^{-1}\widehat{\mathbf{e}}_j^\top \mathbf{1}_T) = o_p(1)$, which completes the proof. □

C Four Additional Simulation Results

In this section we present four additional simulation results: (i) the simulation studies for mimicking Chinese stock market; (ii) the simulation results of the PY test; (iii) the generation of the error terms $\mathbf{E}_t$ that is borrowed from Fan et al. (2015); (iv) the simulation for the case where the summation of the GARCH coefficients is smaller than 0.5.

C.1 Three Examples to Mimic the Chinese Stock Market Data

Example S1. The setting is similar to Example 1 except that f_t is generated from the coefficients in the Chinese stock market example in Section 5. Specifically, we assume that f_t follows an AR(1)-GARCH(1,1) process,

$$f_t - 0.24 = 0.07(f_{t-1} - 0.24) + h_t^{1/2}\zeta_t,$$

where ζ_t follows a standard normal distribution, h_t is generated from the process

$$h_t = 0.61 + 0.56h_{t-1} + 0.14h_{t-1}\zeta_{t-1}^2,$$

and the above coefficients are obtained by fitting the model to the Chinese stock data given in Section 5. The generations of the factor loadings, alphas and error terms are the same as those in Example 1.

The above process is simulated over the periods $t = -24, \cdots, 0, 1, \cdots, T$ with the initial values $R_{i,-25} = 0$, $h_{-25} = 1$, $z_{-25} = 0$ and $\sigma_{-25}^2 = 1$. To offset the start-up effects, we drop the first 25 simulated observations and use $t = 1, \cdots, T$ in our studies.

Example S2. The setting is similar to Example 2 except that f_t is generated from the coefficients in the Chinese stock market example in Section 5. Specifically, we assume that the three factors are correspondingly simulated from the following AR(1)-GARCH(1,1) processes,

$$\text{Market factor: } f_{1t} - 0.24 = 0.07(f_{1t-1} - 0.24) + h_{1t}^{1/2}\zeta_{1t},$$

$$\text{SMB factor: } f_{2t} - 0.14 = 0.03(f_{2t-1} - 0.14) + h_{2t}^{1/2}\zeta_{2t},$$

$$\text{HML factor: } f_{3t} - 0.09 = 0.04(f_{3t-1} - 0.09) + h_{3t}^{1/2}\zeta_{3t},$$

where ζ_{jt} ($j = 1, 2$ and 3) are simulated from a standard normal distribution, h_{jt} ($j = 1, 2$ and 3) are, respectively, generated through the following processes,

$$\text{Market factor: } h_{1t} = 0.61 + 0.56h_{1t-1} + 0.14h_{1t-1}\zeta_{1t-1}^2,$$

$$\text{SMB: } h_{2t} = 0.45 + 0.56h_{2t-1} + 0.14h_{2t-1}\zeta_{2t-1}^2,$$

$$\text{HML: } h_{3t} = 0.40 + 0.72h_{3t-1} + 0.01h_{3t-1}\zeta_{3t-1}^2,$$

and the above coefficients are obtained by fitting the model to the Chinese stock data given in Section 5. The generations of the factor loadings, alphas and error terms are the same as those in Example 2.

Example S3. The setting is similar to Example 3 except that we follow Example S2 to generate the three factors of the model.

The simulation results for the above three examples for three different sample sizes ($T = 100$, 200, 500) and four different numbers of stocks ($N = 3$, 200, 500, $1,000$) are summarized in Table S1 and they are similar to those in Table 1 of the manuscript.

Table S1:The empirical sizes of the HDA and LY tests from Examples S1–S3 for testing conditional alphas with a nominal level of 5%, where Normal Distribution, Exponential Distribution, and Mixture Distribution refer to the distribution from which the error term $\mathbf{E}_t$ is generated.

			Normal Distribution		Exponential Distribution		Mixture Distribution	
Example	N	T	HDA	LY-test	HDA	LY-test	HDA	LY-test
S1	3	100	0.052	0.058	0.062	0.057	0.041	0.047
		200	0.050	0.066	0.039	0.051	0.048	0.052
		500	0.059	0.037	0.048	0.050	0.039	0.042
S1	200	100	0.040	1	0.046	1	0.062	1
		200	0.055	1	0.062	1	0.039	1
		500	0.036	1	0.038	1	0.057	1
S1	500	100	0.055	1	0.047	1	0.071	1
		200	0.052	1	0.053	1	0.056	1
		500	0.043	1	0.051	1	0.065	1
S1	1000	100	0.045	1	0.035	1	0.044	1
		200	0.049	1	0.048	1	0.033	1
		500	0.039	1	0.042	1	0.058	1
S2	3	100	0.051	0.046	0.048	0.039	0.032	0.037
		200	0.052	0.068	0.057	0.049	0.043	0.041
		500	0.049	0.066	0.063	0.052	0.055	0.054
S2	200	100	0.043	1	0.062	1	0.058	1
		200	0.038	1	0.045	1	0.065	1
		500	0.047	1	0.039	1	0.038	1
S2	500	100	0.069	1	0.055	1	0.057	1
		200	0.060	1	0.061	1	0.047	1
		500	0.052	1	0.033	1	0.042	1
S2	1000	100	0.054	1	0.042	1	0.056	1
		200	0.052	1	0.051	1	0.054	1
		500	0.033	1	0.040	1	0.049	1
S3	3	100	0.054	0.037	0.043	0.061	0.058	0.052
		200	0.042	0.055	0.060	0.057	0.042	0.053
		500	0.059	0.038	0.046	0.039	0.054	0.044
S3	200	100	0.049	1	0.032	1	0.037	1
		200	0.044	1	0.042	1	0.054	1
		500	0.063	1	0.035	1	0.045	1
S3	500	100	0.040	1	0.062	1	0.052	1
		200	0.042	1	0.062	1	0.060	1
		500	0.046	1	0.055	1	0.043	1
S3	1000	100	0.033	1	0.064	1	0.047	1
		200	0.036	1	0.048	1	0.042	1
		500	0.046	1	0.060	1	0.057	1

C.2 Simulation Results of the PY Test

Since the PY test (Pesaran and Yamagata, 2012) can be used for testing alpha coefficients in high dimensional assets, we follow an anonymous referee's suggestion to conduct simulation studies for Example 1 across three different sample sizes, three different numbers of stocks, and three different error distributions. The results are summarized in Table S2, and indicate that the PY test exhibits serious size distortions. This finding is not surprising since the PY test is designed for time-invariant factor loadings, while the setting in Example 1 is based on time-varying factor loadings.

Table S2: The empirical sizes of the PY test for testing conditional alphas with a nominal level of 5%, where Normal, Exponential and Mixture refer to the distribution from which the error term $\mathbf{E}_t$ is generated.

N	T	Normal	Exponential	Mixture
200	100	0.481	0.493	0.512
	200	0.472	0.448	0.491
	500	0.491	0.469	0.472
500	100	0.672	0.631	0.645
	200	0.691	0.644	0.678
	500	0.701	0.722	0.682
1000	100	0.805	0.766	0.782
	200	0.817	0.789	0.794
	500	0.803	0.815	0.786

C.3 Simulation Results for an Alternative Error Structure

The simulation setting is similar to Example 1 except that the error terms $\mathbf{E}_t$ are generated as in Fan et al. (2015). Specifically, we set $\mathbf{\Sigma} = \text{diag}(A_1, \cdots, A_{N/4})$ to be a block-diagonal correlation matrix, and each diagonal block A_j for $j = 1, \cdots, N/4$ is a 4×4 positive definite matrix whose correlation matrix has equi-off-diagonal entry ρ_j, generated from the Uniform[0,0.5]. The simulation results are summarized in Table S2 and they are similar to those in Table 1 of the manuscript. Hence, our HDA test is robust to different error specifications.

C.4 Simulation Results for Different GARCH Coefficients

The simulation setting is similar to Example 1 except that $\{f_t\}$ is generated differently. Specifically, we assume that $\{f_t\}$ follows an AR(1)-GARCH(1,1) process, where

$$f_t - 0.42 = 0.06(f_{t-1} - 0.42) + h_t^{1/2}\zeta_t,$$

ζ_t follows a standard normal distribution, and h_t is generated from the process

$$h_t = 0.39 + 0.38h_{t-1} + 0.06h_{t-1}\zeta_{t-1}^2.$$

Note that the summation of the GARCH coefficients is smaller than 0.5. The results are summarized in Table S3 and they are similar to those in Table 1 of the manuscript.

Table S3: The empirical sizes of the HDA test for testing conditional alphas with a nominal level of 5%, where Normal, Exponential and Mixture refer to the distribution from which the error term $\mathbf{E}_t$ is generated.

N	T	Normal	Exponential	Mixture
200	100	0.054	0.062	0.061
	200	0.052	0.048	0.042
	500	0.038	0.051	0.049
500	100	0.033	0.049	0.040
	200	0.057	0.052	0.048
	500	0.046	0.044	0.034
1000	100	0.069	0.064	0.051
	200	0.034	0.043	0.042
	500	0.044	0.039	0.060

Table S4: The empirical sizes of the HDA test, where Normal, Exponential and Mixture refer to the distribution from which the error term $\mathbf{E}_t$ is generated.

N	T	Normal	Exponential	Mixture
200	100	0.056	0.067	0.055
	200	0.064	0.054	0.068
	500	0.051	0.057	0.052
500	100	0.054	0.049	0.048
	200	0.044	0.052	0.057
	500	0.050	0.059	0.062
1000	100	0.061	0.053	0.054
	200	0.045	0.040	0.047
	500	0.061	0.050	0.063

D Testing Market Efficiency with Different Window Length

For the purpose of robustness check, we present the results with window of length $h = 60$ as suggested by Pesaran and Yamagata (2012) for the US and Chinese stock market data in Figure S1. They exhibit a similar pattern to that of Figures 2 and 3 in the manuscript: FF is more efficient than CAPM, and the US stock market is more efficient than the Chinese stock market, both in terms of mean-variance efficiency.

To further assess the effect of window length on the estimation, we also consider a relative long window of length $h = 200$ for US stock market data and present the results in Figure S2. They exhibit a similar pattern to that of Figures 3 in the manuscript that the FF is more efficient than CAPM. Nevertheless, the p-values obtained for $h = 200$ tend to be smaller than those for $h = 100$.

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Figure S1: The dynamic movement of market efficiency in the US and Chinese stock market based on the p-values by testing the conditional CAPM and the conditional Fama-French three-factors model with window length $h = 60$.

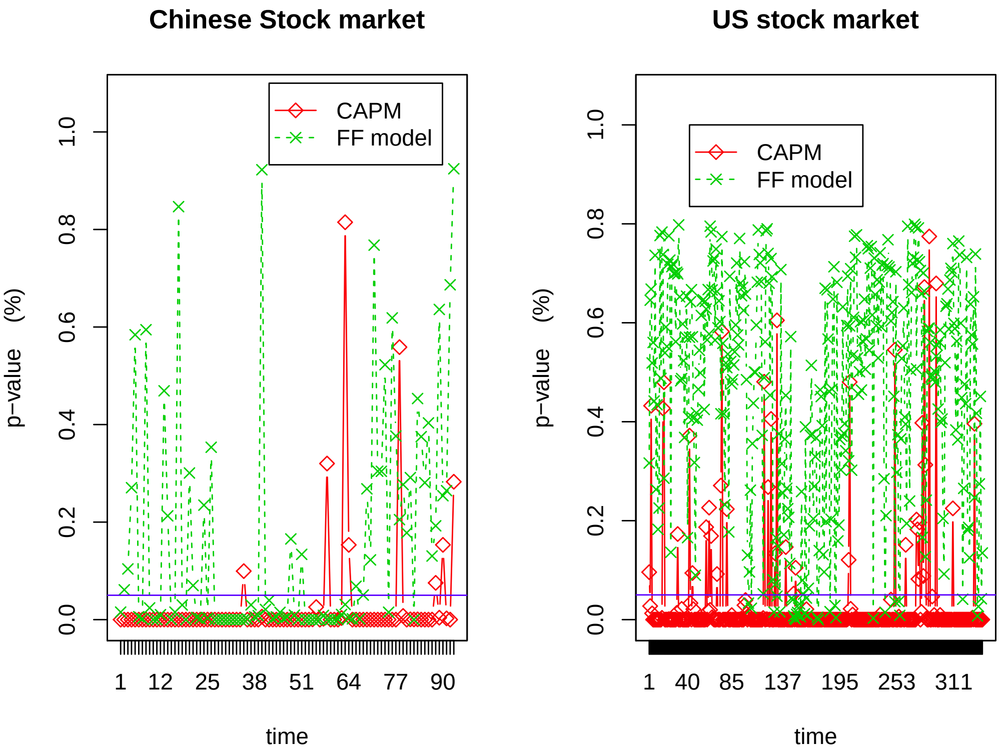

Figure S2: The dynamic movement of market efficiency in the US stock market based on the p-values by testing the conditional CAPM and the conditional Fama-French three-factors model with window length $h = 200$.