Supplementary Material to *Testing Nowcast Monotonicity with Estimated Factors*

May 2, 2018

S1 General Remarks

This supplementary material is organised into different sections. Section S2 provides some further details explicitly showing how the factor-augmented model set-up fits the moment inequality setting and introduces a representation of the factor estimation error, which is used in Sections S4 and S5. Section S3 on the other hand contains additional factor model assumptions required for various technical lemmas outlined in Section S4, as well as in Section S5. The latter section uses the results from Section S4 and deals with parameter estimation error in the test statistic. Finally, based on these results, the central result of the paper, Theorem 1, is proven in Section S6. Section S7 describes the technical Assumption SM2 mentioned in Section 3.2 of the paper. It also states Theorem S1, which shows that the results of Theorem 7.1 in Chernozhukov et al. (2014b)'s paper continue to hold for the case of unbounded random variables subject to Assumption SM2. Section S8 contains the proof of Theorem 2 in the paper, and Section S9 outlines an ad-hoc method to choose the small-large block combination in practice and presents a small Monte Carlo simulation to that effect. Finally, Sections S10 and S11 provide additional set-ups and tables of results for the Monte Carlo simulations, and then the set of variables used in the empirical application along with additional sets of empirical results.

S2 Factor Augmented Set-up & Representation

For the results derived in the remainder of this Supplementary Material, we require a representation of the factor estimation error along similar lines to the representation in (A.1) of Bai (2003) or the one adapted to out-of-sample estimation in Gonçalves et al. (2017) and Fosten (2016), but with the extension involving the forecast errors used to address the ragged edge problem. However, while explicitly accounting for the ragged edge in the factor estimation problem we will, for the rest of this supplement, abstract from the mixed frequency aspect of the data and assume that the predictor variables X_{jt} are released at the same (quarterly) frequency as the target variable y_t. The reason for doing so is that neglecting mixed frequency allows to highly reduce the complexity of the notation by setting the number of nowcast updates equal to the number of variables in the data (i.e., $S = N$) without altering the technical aspects of the problem. For instance, accounting for the monthly frequency in the proofs below would require a distinction between cases where the ragged edge $i = 1, \ldots, S$ occurs in the first ($i \leq N$), the second ($N \leq i \leq 2N$), or third month ($2N \leq i \leq 3N$) of the last quarter t. Also, as pointed out in Remark 1 in Section S4, skip sampling the monthly series to obtain quarterly series $\widehat{F}_t^{(i,t)}$, $\widehat{F}_{t-1/3}^{(i,t)}$, and $\widehat{F}_{t-2/3}^{(i,t)}$ does, by construction, lead to a sum of squared estimation error lower than the one for the entire series, and does therefore not alter the conclusions drawn from the uni-frequency set-up.

S2.1 Monotonicity Testing in the Factor-Augmented Set-up

Before we derive a representation of the factor estimation error for our set-up, we recall some notation from the main paper adapting it to the uni-frequency case as mentioned above. That is, the window of monthly observations in the matrix $X^{(t)}$ is now composed of $[X_1, X_2, ..., X_t]'$, where:

$$X_t = \Lambda F_t + u_t \tag{S-1}$$

and the quarterly regression for y_t is:

$$y_t = \beta' F_t + \varepsilon_t \tag{S-2}$$

which is the uni-frequency equivalent of unrestricted factor-MIDAS model in the main paper. This corresponds to the standard factor-augmented model of Stock and Watson (2002a,b) and Bai and Ng (2006), where we ignore the presence of 'must-have' regressors W_t for simplicity. Note that $X_t = \Lambda F_t + u_t$ can be re-written as:

$$F_t = (\Lambda'\Lambda)^{-1}\Lambda' X_t + (\Lambda'\Lambda)^{-1}\Lambda' u_t \tag{S-3}$$

However, in the out-of-sample nowcasting context, at each quarterly out of sample point $t = R+1, ..., T$ and nowcast point $i = 1, ..., S$, we do not observe the matrix $X^{(t)}$. Rather, we must use the estimated $\widehat{X}^{(i,t)}$ which solves the ragged-edge problem, as described in the main text. To apply PCA for factor estimation, we require the balanced recursive data matrix:

$$\underset{t\times N}{\widehat{X}^{(i,t)}} = \begin{bmatrix} X_{1,1} & \dots & \dots & \dots & \dots & X_{N,1} \\ \vdots & & & & & \vdots \\ X_{1,t-1} & \dots & \dots & \dots & \dots & X_{N,t-1} \\ X_{1t} & \dots & X_{it} & \widehat{X}_{i+1,t} & \dots & \widehat{X}_{Nt} \end{bmatrix} \quad \text{for } t = R+1, ..., T; i = 1, ..., S \tag{S-4}$$

A typical element of this matrix of predicted observations from the naïve forecasting model is given by $\{\widehat{X}_j^{(i,t)} : t = R+1, ..., T; j = 1, ..., t; i = 1, ..., S\}$. The relationship between the two matrices X_t and $\widehat{X}^{(i,t)}$ is given by:

$$\widehat{X}^{(i,t)} = X^{(t)} + \widehat{A}^{(i,t)}$$

where:

$$\widehat{A}_{t\times N}^{(i,t)} = \begin{bmatrix} 0_{(t-1)\times N} \\ \widehat{\mu}_t^{(i)} \end{bmatrix}.$$

That is, $\widehat{\mu}_t^{(i)}$ is a $1 \times N$ vector which contains 0 values for the variables which have already been released at point i:

$$\widehat{\mu}_t^{(i)} = [0, \dots, 0, \widehat{\mu}_{i+1,t}, \dots, \widehat{\mu}_{N,t}].$$

Since $\widehat{\mu}_t^{(i)}$ is usually the result of estimating the parameters of some naïve model for each variable to solve the ragged edge problem, we make the distinction between the predictions $\widehat{X}_t^{(i,t)}$ based on estimated parameters, and $\ddot{X}_t^{(i,t)}$, which is free of estimated parameters. The corresponding prediction errors $\ddot{\mu}_t^{(i)}$ is as follows:

$$\ddot{\mu}_t^{(i)} = [0, \ldots, 0, \ddot{\mu}_{i+1,t}, \ldots, \ddot{\mu}_{N,t}].$$

For example, if the $AR(1)$ interpolation is used, as in Kim and Swanson (2017) and the empirical section in the main text of this paper, then the distinction between $\widehat{X}_t^{(i,t)}$ and $\ddot{X}_t^{(i,t)}$ would be $\widehat{X}_t^{(i,t)} = \widehat{\rho}_{it} X_{t-1}$ whereas $\ddot{X}_t^{(i,t)} = \rho X_{t-1}$. The relationship between X_t and $\ddot{X}^{(i,t)}$ is therefore:

$$\ddot{X}^{(i,t)} = X^{(t)} + \ddot{A}^{(i,t)}$$

where:

$$\ddot{A}_{t\times N}^{(i,t)} = \begin{bmatrix} 0_{(t-1)\times N} \\ \ddot{\mu}_t^{(i)} \end{bmatrix}.$$

With the above in mind, we replace X_t in Equation (S-3) with $\ddot{X}^{(i,t)}$ for each i and t:

$$F_j^{(i,t)} = (\Lambda'\Lambda)^{-1}\Lambda'\ddot{X}_j^{(i,t)} + (\Lambda'\Lambda)^{-1}\Lambda' u_j \qquad t = R+1, ..., T; j = 1, ..., t; i = 1, ..., S \tag{S-5}$$

which gives us an expression of the true factors as a function of the known data at time t and nowcast point i. We call these factors $F_j^{(i,t)}$ the "pseudo true factors". Therefore the pseudo true factors relate to the true factors as follows:

$$F_j^{(i,t)} \equiv F_j + (\Lambda'\Lambda)^{-1}\Lambda'\ddot{\mu}_j^{(i,t)} \tag{S-6}$$

and they can be seen as a potentially biased version of the true factors, caused by the predictions used to solve the ragged-edge. Later we show results on the limiting properties of the estimated factors $\widehat{F}^{(i,t)}$ based on the pseudo true factors when Λ has been estimated using PCA.

Finally, we can re-write the quarterly regression model $y_t = \beta' F_t + \varepsilon_t$ for each i and t by expanding around the "pseudo" factors from Equation (S-6):

$$\begin{aligned} y_j &= \beta' F_j^{(i,t)} + \varepsilon_j + \beta'\left(F_j - F_j^{(i,t)}\right) \\ &= \beta' F_j^{(i,t)} + \ddot{\varepsilon}_j^{(i,t)} \qquad t = R+1, ..., T; j = 1, ..., t; i = 1, ..., S \end{aligned} \tag{S-7}$$

where $\ddot{\varepsilon}_j^{(i,t)} = \varepsilon_j + \beta'(F_j - F_j^{(i,t)})$. The moment inequalities for our test are based on the end-of-window "pseudo" error $\ddot{\varepsilon}_t^{(i,t)}$ and therefore correspondingly $\theta_i = [F_t^{(i,t)\prime}, \beta']'$. For example, for MSFE loss the moment inequalities can be written as:

$$L(y_t - y_{i+k,t}(\theta_{i+k})) - L(y_t - y_{i,t}(\theta_i)) = \left(\ddot{\varepsilon}_t^{(i+k,t)}\right)^2 - \left(\ddot{\varepsilon}_t^{(i,t)}\right)^2$$

In this factor model case, if we have correctly specified the model for the true data generating process, and in the absence of measurement errors, then we expect the null hypothesis of monotonicity to hold. This is because the pseudo-true factors approach the true factors as i grows towards S, meaning that $\left(F_j - F_j^{(i,t)}\right)$ declines to zero and $\ddot{\varepsilon}_j^{(i,t)}$ becomes comprised only of ε_j. Under correct model specification, we expect that MSFE declines towards the end of the nowcast period.

In order to derive the representation, which will be used in the technical lemmas of Sections S4 and S5, respectively, we define the estimated and 'infeasible' $t \times r$ matrices of bias terms:

$$\widehat{B}^{(i,t)} \equiv \frac{1}{tN}\left(X^{(t)}\widehat{A}^{(i,t)\prime} + \widehat{A}^{(i,t)}X^{(t)\prime} + \widehat{A}^{(i,t)}\widehat{A}^{(i,t)\prime}\right)\widehat{F}^{(i,t)}\widehat{V}^{(i,t)-1} \tag{S-8}$$

and

$$\ddot{B}_j^{(i,t)} = \frac{1}{tN}\left(X^{(t)}\ddot{A}^{(i,t)\prime} + \ddot{A}^{(i,t)}X^{(t)\prime} + \ddot{A}^{(i,t)}\ddot{A}^{(i,t)\prime}\right)F^{(t)}H_0^{(i,t)}V_0^{-1}. \tag{S-9}$$

Here, $\widehat{H}^{(i,t)} = \widehat{V}^{(i,t)-1}(\widehat{F}^{(i,t)\prime}F^{(t)}/t)(\Lambda'\Lambda/N)$ denotes the rotation matrix and $H_0^{(i,t)} = \text{diag}(\pm 1)$ is its probability limit. Likewise, $\widehat{V}^{(i,t)}$ is a $r \times r$ diagonal matrix containing the largest r eigenvalues of the covariance matrix $\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}/Nt$, and $V_0 = \Sigma_\Lambda^{\frac{1}{2}}\Sigma_F\Sigma_\Lambda^{\frac{1}{2}}$ is the probability limit of $\widehat{V}^{(i,t)}$.

S2.2 Representation

Expanding window PCA estimation proceeds to estimate $\widehat{F}^{(i,t)}$ as the $t \times r$ window corresponding to the r eigenvectors corresponding to the r largest eigenvalues of the covariance matrix of $\widehat{X}^{(i,t)}$. Therefore we have the following identity for each $t = R+1, ..., T$ and $i = 1, ..., S$:

$$\left(\frac{\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}}{tN}\right)\widehat{F}^{(i,t)} = \widehat{F}^{(i,t)}\widehat{V}^{(i,t)} \tag{S-10}$$

and so:

$$\widehat{F}^{(i,t)} = \left(\frac{\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}}{tN}\right)\widehat{F}^{(i,t)}\widehat{V}^{(i,t)-1} \tag{S-11}$$

In order to relate this back to the standard setting where $X^{(t)}$ is used for estimation, noting that the forecast errors only affect the last row of $\widehat{X}^{(i,t)}$:

$$\widehat{X}^{(i,t)} = X^{(t)} + \widehat{A}^{(i,t)}$$

with the identity in Equation (S-11) becomes:

$$\widehat{F}^{(i,t)} = \frac{1}{tN}\left(X^{(t)}X^{(t)\prime} + X^{(t)}\widehat{A}^{(i,t)\prime} + \widehat{A}^{(i,t)}X^{(t)\prime} + \widehat{A}^{(i,t)}\widehat{A}^{(i,t)\prime}\right)\widehat{F}^{(i,t)}\widehat{V}^{(i,t)-1}$$

Note that the sum of the second, third, and fourth term correspond to the estimated $t \times r$ matrix $\widehat{B}^{(i,t)}$ in Equation (S-8) which contains the forecast errors $\widehat{\mu}_t^{(i)}$. Finally, using the same argument as in Bai (2003), by substituting in the factor model in matrix notation, which is written:

$$\underset{t\times N}{X^{(t)}} = \underset{t\times r}{F^{(t)}}\ \underset{r\times N}{\Lambda'} + \underset{t\times N}{u^{(t)}} \tag{S-12}$$

we have that:

$$\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j - \widehat{B}_j^{(i,t)} = \widehat{V}^{(i,t)-1}\left(\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kj} + \frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kj}\right.$$

$$+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kj}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kj}\Bigg) \tag{S-13}$$

and, as defined in Bai and Ng (2002), we let:

$$\gamma_{kj}=\mathrm{E}\left(\frac{1}{N}\sum_{i=1}^{N}u_{ik}u_{ij}\right)$$
$$\zeta_{kj}=\left(\frac{1}{N}\sum_{i=1}^{N}u_{ij}u_{ik}\right)-\gamma_{kj}$$
$$\eta_{kj}=\frac{1}{N}\sum_{i=1}^{N}F_k'\lambda_i u_{ij}=F_k'\frac{\Lambda' u_j}{N}$$
$$\xi_{kj}=\frac{1}{N}\sum_{i=1}^{N}\lambda_i' F_j u_{ik}=\frac{u_k'\Lambda}{N}F_j$$

Now, by defining:

$$\widetilde{F}_j^{(i,t)}\equiv F_j-\widehat{H}^{(i,t)-1}\widehat{B}_j^{(i,t)} \tag{S-14}$$

we can rewrite Equation S-13 to get:

$$\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_j^{(i,t)}=\widehat{V}^{(i,t)-1}\Bigg(\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kj}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kj}$$
$$+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kj}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kj}\Bigg) \tag{S-15}$$

Equation (S-15) gives us an expression for the deviation of the estimated factors from a rotation of the true factors plus a contamination due to the presence of the forecast errors $\widehat{\mu}_t^{(i)}$. The right hand side (RHS) of Equation (S-15) will be treated in the same way as in Bai (2003) and more recently Fosten (2016) and Gonçalves et al. (2017).

Finally, in each window $t=R+1,...,T$ and for observations $j=1,...,t$ we define the 'pseudo-true factors' as:

$$F_j^{(i,t)}\equiv F_j-H_0^{(i,t)-1}\ddot{B}_j^{(i,t)} \tag{S-16}$$

which is equal to the true factors plus a contamination which depends on the true forecast errors $\ddot{\mu}_t^{(i)}$ rather than estimated forecast errors $\widehat{\mu}_t^{(i)}$. The matrix $\ddot{B}_j^{(i,t)}$ is defined in (S-9). In order to purge $\widetilde{F}_j^{(i,t)}$ in Equation (S-14) of the effect of the estimation error in $\widehat{\mu}_t^{(i)}$ we will therefore make use of the following representation:

$$F_j^{(i,t)}=\widetilde{F}_j^{(i,t)}-H_0^{(i,t)-1}\left(\ddot{B}_j^{(i,t)}-\widehat{B}_j^{(i,t)}\right)-\left(H_0^{(i,t)-1}-\widehat{H}^{(i,t)-1}\right)\widehat{B}_j^{(i,t)}$$
$$+\left(H_0^{(i,t)-1}-\widehat{H}^{(i,t)-1}\right)\left(\ddot{B}_j^{(i,t)}-\widehat{B}_j^{(i,t)}\right). \tag{S-17}$$

S3 Factor Assumptions

The following assumptions are used to prove Theorem 1 as well as various auxiliary lemmas listed below. Before outlining the conditions, let $\sup_t$ stand short for $\sup_{R\leq t\leq T}$ and $\sup_{i,t}$ stand short for $\sup_{1\leq i\leq N}\sup_{R\leq t\leq T}$ (we will refer to the latter as 'uniformly in i and t' in what follows). Also, recall that we imposed the identification assumptions $F^{(t)\prime}F^{(t)}/t = I_r$ and $\Lambda'\Lambda/N$ is diagonal.

Assumption SM1. *1. For every $i = 1,\ldots,N$, the data $\{F_t, \varepsilon_t, \mu_{it}, u_{it}\}_{t=1}^T$ is strictly stationary and β-mixing with size of the mixing coefficient being $b_{r_P} = o\left(r_P^{-2d_{r_P}/(d_{r_P}-1)}\right)$ and $d_r > 1$, where d_{r_P} was defined in Assumption 4 in the main text.*

2. *$E\left[\|F_t\|^{4d_r}\right] \leq C$, and $\frac{1}{T}\sum_{j=1}^t F_jF_j' \xrightarrow{p} \Sigma_F$ uniformly in $R \leq t \leq T$ ('uniformly in t' in the following), with the $r\times r$ positive definite matrix Σ_F; The loadings λ_i for $i = 1,...,N$ are either deterministic such that $\|\lambda_i\| \leq C$ or stochastic such that $E\left[\|\lambda_i\|^{4d_r}\right] \leq C$. In any case $\Lambda'\Lambda/N \xrightarrow{p} \Sigma_\Lambda$, with Σ_Λ an $r\times r$ positive definite matrix Σ_Λ; The eigenvalues of the $r\times r$ matrix $(\Sigma_\Lambda\cdot\Sigma_F)$ are unique.*

3. *$E[u_{it}] = 0$, $E\left[|u_{it}|^{8d_r}\right] \leq C$; $E\left[\frac{1}{N}\sum_{i=1}^N u_{is}u_{it}\right] = \gamma_{st}$, $|\gamma_{ss}| \leq C$ for all s, and $\frac{1}{T}\sum_{j=1}^t\sum_{k=1}^t\left|\gamma_{jk}\right| \leq C$ for all j,k, as well as $\frac{1}{P}\sum_{t=R+1}^T\sum_{k=1}^t\gamma_{tk}^2 \leq C$ uniformly in t; For all (t,s), $E\left[\left|N^{-1/2}\sum_{i=1}^N u_{it}u_{is} - E(u_{it}u_{is})\right|^8\right] \leq C$; $E[u_{it}u_{jt}] = \tau_{ij,t}$ with $|\tau_{ij,t}| \leq |\tau_{ij}|$ for some τ_{ij} and all t; in addition, $\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|\tau_{ij}| \leq C$; for some (k,h), let $E[u_{ik}u_{jh}] = \tau_{ij,kh}$, and assume for all t, $\frac{1}{NT}\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^t\sum_{h=1}^t|\tau_{ij,kh}| \leq C$.*

4. *For all s, $E\left[\left\|\sup_t\frac{1}{\sqrt{NT}}\sum_{k=1}^t\sum_{i=1}^N F_k\left(u_{is}u_{ik} - E(u_{is}u_{ik})\right)\right\|^2\right] \leq C$; For all s, and $h \geq 0$, $E\left[\left|\sup_t\frac{1}{\sqrt{NT}}\sum_{k=1}^{t-h}\sum_{i=1}^N\left(u_{is}u_{ik} - E(u_{is}u_{ik})\right)\varepsilon_k\right|^2\right] \leq C$; $E\left[\left\|\frac{1}{\sqrt{TN}}\sup_t\sum_{j=1}^t\Lambda'u_j\varepsilon_j\right\|^2\right] \leq C$,and $E[\lambda_iu_{it}\varepsilon_t] = 0$ for all (i,t); $E\left[\left(\frac{1}{N}\sum_{i=1}^N\left\|\sup_t\frac{1}{\sqrt{T}}\sum_{j=1}^t F_ju_{ij}\right\|\right)^2\right] \leq C$, and $E[F_tu_{it}] = 0$ for all (i,t); $E\left[\left\|\frac{1}{\sqrt{TN}}\sup_t\sum_{j=1}^t F_ju_j'\Lambda\right\|^2\right] \leq C$, and $E[\lambda_iu_{it}F_t] = 0$ for all (i,t); $E\left[\frac{1}{T}\sup_t\sum_{j=1}^t\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^N\lambda_iu_{ij}\right\|^2\right] \leq C$, and $E[\lambda_iu_{ij}] = 0$ for all (i,j)*

5. *$E[F_t\varepsilon_t] = 0$ and $E\left[|\varepsilon_t|^{4d_r}\right] \leq C$; $E\left[\sup_t\left(\frac{1}{T}\sum_{k=1}^t F_k\varepsilon_k\right)^2\right] \leq C$.*

6. *Uniformly in i and t, $\|\widehat{\mu}_t^{(i)} - \ddot{\mu}_t^{(i)}\| = O_p(1/\sqrt{R})$ and $E\left[\left\|\ddot{\mu}_t^{(i,t)}\right\|^{4d_r}\right] \leq C$.*

S4 Auxiliary Lemmas (Factors)

Lemma B.1. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\left\| \widehat{V}^{(i,t)-1} \right\| = O_p(1) \qquad \textit{and} \qquad \left\| \widehat{H}^{(i,t)} \right\|^2 = O_p(1)$$

uniformly in i and t.

Lemma B.2. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{T} \sum_{j=1}^{t} \left\| \widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)} \widetilde{F}_j^{(i,t)} \right\|^2 = O_p\left(\max\left\{ \frac{1}{N}, \frac{1}{R} \right\} \right)$$

uniformly in i and t.

Lemma B.3. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{T} \sum_{j=1}^{t} \left\| \ddot{B}_j^{(i,t)} \right\|^2 = O_p\left(\frac{1}{R} \right)$$

uniformly in i and t.

Lemma B.4. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{T} \sum_{j=1}^{t} \left\| \widehat{B}_j^{(i,t)} \right\|^2 = O_p\left(\frac{1}{R} \right)$$

uniformly in i and t.

Lemma B.5. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{T} \sum_{j=1}^{t} \left\| \widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)} F_j \right\|^2 = O_p\left(\max\left\{ \frac{1}{N}, \frac{1}{R} \right\} \right)$$

uniformly in i and t, where F_j are the the true factors.

Lemma B.6. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{T} \sum_{j=1}^{t} \left(\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)} F_j \right) F_j' = O_p\left(\max\left\{ \frac{1}{N}, \frac{1}{R} \right\} \right)$$

uniformly in i and t.

Lemma B.7. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\left\| \widehat{H}^{(i,t)} - H_0^{(i,t)} \right\| = O_p\left(\max\left\{ \frac{1}{R}, \frac{1}{N} \right\} \right)$$

uniformly in i and t, where $H_0^{(i,t)} = diag(\pm 1)$.

Lemma B.8. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\left\|\widehat{V}^{(i,t)-1} - V_0^{-1}\right\| = O_p\left(\max\left\{\frac{1}{R}, \frac{1}{N}\right\}\right)$$

uniformly in i and t, with $V_0 = \Sigma_\Lambda^{\frac{1}{2}}\Sigma_F\Sigma_\Lambda^{\frac{1}{2}}$.

Lemma B.9. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\sup_{i,t}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\| = O_p\left(\max\left\{\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{N}}\right\}\right).$$

Lemma B.10. *Under Assumption SM1 above and Assumption 4 of the paper, it holds that:*

$$\frac{1}{P}\sum_{t=R+1}^{T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\|^2 = O_p\left(\max\left\{\frac{1}{R}, \frac{1}{N}\right\}\right)$$

uniformly in i.

Remark 1. *Note that the technical results of Lemmas B.1 through B.10 carry directly over to the mixed frequency case with skip-sampled series. To see this, assume the monthly case from the paper with $j = 1/3, 2/3, 1, \ldots, t-2/3, t-1/3, t$. Then, taking Lemma B.5 as an illustrative example, the result of this lemma for the uni-frequency case and the rate conditions of Assumption 4 imply that:*

$$\frac{1}{3t}\sum_{j=1/3}^{t}\left\|\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j\right\|^2 = \left(\frac{T}{3t}\right)\frac{1}{T}\sum_{j=1/3}^{t}\left\|\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j\right\|^2 = O(1)O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$$

for each $R+1 \leq t \leq T$. Now, skip-sampling over say months 1 and 2 of the quarter (i.e., $k = 1, 2, 3, \ldots, t-2, t-1, t$) yields automatically the bound:

$$\frac{1}{t}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)}F_k\right\|^2 \leq \frac{1}{t}\sum_{j=1/3}^{t}\left\|\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j\right\|^2 = O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right).$$

for each skip-sampled series, since the additional scaling of the average by $1/3$ does not affect the asymptotic rate. A similar line of argument can be used for the other Lemmas.

Proof of Lemma B.1. We start with the first claim and note that Lemma A.5 of Gonçalves et al. (2017) shows uniformity over t in the standard set-up where the data matrix $X^{(t)}X^{(t)\prime}$ is used to estimate the factors and not $\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}$. In this proof it therefore suffices to show that:

$$\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}/tN = X^{(t)}X^{(t)\prime}/tN + o_p(1)$$

uniformly across i, which removes dependence on i. From Section S2, we know that:

$$\widehat{X}^{(i,t)}\widehat{X}^{(i,t)\prime}/tN = X^{(t)}X^{(t)\prime}/tN + \left(X^{(t)}\widehat{A}^{(i,t)\prime} + \widehat{A}^{(i,t)}X^{(t)\prime} + \widehat{A}^{(i,t)}\widehat{A}^{(i,t)\prime}\right)/tN. \quad \text{(S-18)}$$

For the term in brackets, we have that uniformly in i and t:

$$\|X^{(t)}\widehat{A}^{(i,t)\prime} + \widehat{A}^{(i,t)}X^{(t)\prime} + \widehat{A}^{(i,t)}\widehat{A}^{(i,t)\prime}\|/tN = O_p\left(\frac{1}{RN}\right),$$

which follows from a similar line of reasoning as in the proof of Lemma B.4 below using the fact that the first $(t-1)$ rows and/or columns of the first, second, and third term have zero entries (see proofs of Lemma B.3 and B.4 for details) and Assumption SM1, as well as the fact that $(t/R) = O(1)$ for every $R \leq t \leq T$ by Assumption 4. Moreover, using the matrix representation of $X^{(t)}$ in Equation (S-12), we obtain for the first term on the RHS of (S-18) that:

$$\mathrm{E}\left[X^{(t)}X^{(t)\prime}\right]/tN = \mathrm{E}\left[F^{(t)}\Lambda'\Lambda F^{(t)\prime}\right]/tN + \mathrm{E}\left[u^{(t)}u^{(t)\prime}\right]/tN.$$

given that $\mathrm{E}\left[F^{(t)}\Lambda' u^{(t)\prime}\right] = 0$ by 4. in Assumption SM1. Therefore, repeating the line of argument as in the proof of Lemma A.5 in Gonçalves et al. (2017), we can deduce that the smallest eigenvalue of $\widehat{V}^{(i,t)}$ is bounded from below by the smallest eigenvalue of $\Sigma_\Lambda^{\frac{1}{2}}\Sigma_F\Sigma_\Lambda^{\frac{1}{2}}$, the probability limit of $\Lambda' F^{(t)\prime}F^{(t)}\Lambda/tN$ (because the non-zero eigenvalues of $X^{(t)}X^{(t)\prime}$ are identical to those of $X^{(t)\prime}X^{(t)}$), if we can show that:

$$\left\|X^{(t)}X^{(t)\prime}/tN - \mathrm{E}\left[X^{(t)}X^{(t)\prime}\right]/tN\right\| = o_p(1)$$

uniformly in i and t. But this follows from SM1 and an argument similar to the proof of Lemma A.5 in Gonçalves et al. (2017), which in turn implies that $\|\widehat{V}^{(i,t)-1}\| = O_p(1)$ uniformly in i and t. Moreover, given this result and since $\widehat{F}^{(i,t)\prime}\widehat{F}^{(i,t)}/t = I_r$, one can again follow the proof of Lemma A.5 in Gonçalves et al. (2017) to verify the second claim in Lemma B.1, namely $\|\widehat{H}^{(i,t)}\|^2 = O_p(1)$ uniformly in i and t.

■

Proof of Lemma B.2. Since $\widehat{V}^{(i,t)-1}$ is of order $O_p(1)$ uniformly in i and t by Lemma B.1, its presence will be ignored in the following. First, observe that:

$$\begin{aligned}
&\frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}\widetilde{F}_j^{(i,t)}\right\|^2 \\
\leq\; &\left(\frac{t}{T}\right)\frac{4}{t}\left(\sum_{j=1}^{t}\left(\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kj}\right\|^2 + \left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kj}\right\|^2\right.\right. \\
&\left.\left. + \left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kj}\right\|^2 + \left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kj}\right\|^2\right)\right)
\end{aligned}$$

Since $(t/T) = O(1)$ by Assumption 4, the remaining arguments are as in the proof of Theorem 1 in Gonçalves et al. (2017) noting that they hold uniformly in i and t. Therefore, we have that the first term on the RHS is of order $O_p(R^{-1})$, the second term is $O_p(N^{-1})$, the third term is of order $O_p(N^{-1})$, and the fourth term is also of order $O_p(N^{-1})$ uniformly in i and t. This completes the proof of the Lemma. ■

Proof of Lemma B.3. We firstly obtain an expression for $\ddot{B}_j^{(i,t)}$ for a given t and i. Recall that $\ddot{B}^{(i,t)}$ is the $t \times r$ matrix:

$$\begin{aligned}\ddot{B}^{(i,t)} &\equiv \frac{1}{Nt}\left(X^{(t)}\ddot{A}^{(i,t)\prime} + \ddot{A}^{(i,t)}X^{(t)\prime} + \ddot{A}^{(i,t)}\ddot{A}^{(i,t)\prime}\right)F^{(t)}H_0^{(i,t)}V_0^{-1} \\ &= I + II + III\end{aligned}$$

Starting with I we note that:

$$\begin{aligned}X^{(t)}\ddot{A}^{(i,t)\prime}F^{(t)} &= X^{(t)}\left[0_{N\times(t-1)}, \ddot{\mu}_t^{(i)}\right]F^{(t)} \\ &= \left[0_{t\times(t-1)}, X^{(t)}\ddot{\mu}_t^{(i)}\right]F^{(t)} \\ &= \left[0_{t\times(t-1)}, \begin{matrix}\sum_{h=i+1}^N X_{h,1}\ddot{\mu}_{ht} \\ \vdots \\ \sum_{h=i+1}^N X_{h,t}\ddot{\mu}_{ht}\end{matrix}\right]\begin{bmatrix}F_1' \\ \vdots \\ F_t'\end{bmatrix} \\ &= \begin{bmatrix}\sum_{h=i+1}^N X_{h,1}\ddot{\mu}_{ht}F_t' \\ \vdots \\ \sum_{h=i+1}^N X_{h,t}\ddot{\mu}_{ht}F_t'\end{bmatrix} \\ &= \begin{bmatrix}\sum_{h=i+1}^N(\lambda_h' F_1 + u_{h,1})\ddot{\mu}_{ht}F_t' \\ \vdots \\ \sum_{h=i+1}^N(\lambda_h' F_t + u_{h,t})\ddot{\mu}_{ht}F_t'\end{bmatrix},\end{aligned}$$

For part II we have:

$$\begin{aligned}\ddot{A}^{(i,t)}X^{(t)\prime}F^{(t)} &= \begin{bmatrix}0_{(t-1)\times N} \\ \ddot{\mu}_t^{(i)\prime}\end{bmatrix}X^{(t)\prime}F^{(t)} \\ &= \begin{bmatrix}0_{(t-1)\times t} \\ \ddot{\mu}_t^{(i)\prime}X^{(t)\prime}\end{bmatrix}F^{(t)} \\ &= \begin{bmatrix}0_{(t-1)\times t} \\ \sum_{h=i+1}^N \ddot{\mu}_{ht}X_{h,1}, \ldots, \sum_{h=i+1}^N \ddot{\mu}_{ht}X_{h,t}\end{bmatrix}\begin{bmatrix}F_1' \\ \vdots \\ F_t'\end{bmatrix} \\ &= \begin{bmatrix}0_{(t-1)\times r} \\ \sum_{h=i+1}^N\sum_{k=1}^t \ddot{\mu}_{ht}X_{h,k}F_k'\end{bmatrix} \\ &= \begin{bmatrix}0_{(t-1)\times r} \\ \sum_{h=i+1}^N\sum_{k=1}^t \ddot{\mu}_{ht}X_{h,k}F_k'\end{bmatrix}\end{aligned}$$

Finally, for part III:

$$\ddot{A}^{(i,t)}\ddot{A}^{(i,t)\prime}F^{(t)} = \left[0_{N\times(t-1)}, \ddot{\mu}_t^{(i)}\right]\begin{bmatrix}0_{(t-1)\times N} \\ \ddot{\mu}_t^{(i)\prime}\end{bmatrix}F^{(t)}$$

$$= \begin{bmatrix} 0_{(t-1)\times(t-1)} & 0_{(t-1)\times 1} \\ 0_{1\times(t-1)} & \ddot{\mu}_t^{(i)\prime}\ddot{\mu}_t^{(i)} \end{bmatrix} F^{(t)}$$

$$= \begin{bmatrix} 0_{(t-1)\times(t-1)} & 0_{(t-1)\times 1} \\ 0_{1\times(t-1)} & \sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2 \end{bmatrix} \begin{bmatrix} F_1' \\ \vdots \\ F_t' \end{bmatrix}$$

$$= \begin{bmatrix} 0_{(t-1)\times r} \\ \sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2 F_t' \end{bmatrix}$$

Therefore we see that parts II and III only affect the last row of the $t \times r$ matrix $B^{(i,t)}$ whereas part I has a contribution to every row.

$$\ddot{B}^{(i,t)} = \frac{1}{tN} \begin{bmatrix} \sum_{h=i+1}^{N} X_{h,1}\ddot{\mu}_{ht}F_t' \\ \vdots \\ \sum_{h=i+1}^{N} X_{h,t-1}\ddot{\mu}_{ht}F_t' \\ \sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{h,k}F_k' + \sum_{h=i+1}^{N} X_{h,t}\ddot{\mu}_{ht}F_t' + \sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2 F_t' \end{bmatrix} H_0^{(i,t)} V_0^{-1}$$

Now, we note that for $j = 1, ..., t-1$:

$$\ddot{B}_j^{(i,t)} = \frac{1}{tN}\sum_{h=i+1}^{N} X_{h,j}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}$$

Whereas for the final observation of the window at $j = t$:

$$\ddot{B}_j^{(i,t)} = \frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{h,k}F_k' H_0^{(i,t)} V_0^{-1} + \frac{1}{tN}\sum_{h=i+1}^{N} X_{h,t}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1} + \frac{1}{tN}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2 F_t' H_0^{(i,t)} V_0^{-1}$$

Therefore we can write the statement in Lemma B.3 as:

$$\sum_{j=1}^{t}\left\|\ddot{B}_j^{(i,t)}\right\|^2 = \sum_{j=1}^{t}\left\|\frac{1}{tN}\sum_{h=i+1}^{N} X_{h,t}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right\|^2 + \left\|\frac{1}{tN}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2 F_t' H_0^{(i,t)} V_0^{-1}\right\|^2 + \left\|\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{h,k}F_k' H_0^{(i,t)} V_0^{-1}\right\|^2 \tag{S-19}$$

noting that the last two terms do not have $\sum_{j=1}^{t}$ because these terms only apply to the last row of the window.

We start with the first term on the RHS of Equation (S-19). Note that $H_0^{(i,t)}$ and of V_0^{-1} are the probability limits of $\widehat{H}^{(i,t)}$ and $\widehat{V}^{(i,t)-1}$, which are both bounded uniformly in i and t. Thus, we can bound this term as follows:

$$\begin{aligned}
&\sum_{j=1}^{t}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}X_{j,t}\ddot{\mu}_{ht}F_t'H_0^{(i,t)}V_0^{-1}\right\|^2\\
\leq&\frac{C}{t}\left\|F_t\right\|^2\left(\frac{1}{tN^2}\sum_{j=1}^{t}\sum_{h=i+1}^{N}\left\|\left(\lambda_h'F_j+u_{hj}\right)\ddot{\mu}_{ht}\right\|^2\right)\\
\leq&C\left(\frac{T}{t}\right)\frac{1}{N}\left(\frac{1}{t}\left\|F_t\right\|^2\right)\left(\frac{1}{N}\sum_{h=i+1}^{N}\left\|\ddot{\mu}_{ht}\right\|^4\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{h=i+1}^{N}\left(\frac{1}{T}\sum_{j=1}^{t}\left(\lambda_h'F_j+u_{hj}\right)^2\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

While $\left(\frac{T}{t}\right)=O(1)$ by the rate conditions of Assumption 4, the term $t^{-1}\left\|F_t\right\|^2$ is of order $O_p(1)$ uniformly in i and t since $t^{-1}\sum_{k=1}^{t}\left\|F_k\right\|^2=O_p(1)$ uniformly in i and t. Moreover, note that:

$$\frac{1}{N}\sum_{h=i+1}^{N}\left(\frac{1}{T}\sum_{j=1}^{t}\left(\lambda_h'F_jF_j'\lambda_h+u_{hj}^2+2\lambda_h'F_ju_{hj}\right)\right)^2=O_p(1)$$

uniformly in i and t by Assumption SM1 and an application of Markov's inequality. Also, by Assumption SM1, the second term on the RHS is:

$$\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}^4=O_p(1)$$

uniformly in i and t and so the whole term is therefore of order $O_p(1)$ uniformly in i and t.

Next, consider the second term on the RHS of Equation (S-19):

$$\left\|\frac{1}{tN}\sum_{h=i+1}^{N}X_{h,t}\ddot{\mu}_{ht}F_t'\right\|^2\leq C\frac{1}{t^2}\left\|F_t\right\|^2\left(\frac{1}{N^2}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^4\right).$$

By Assumption SM1 and application of Markov's inequality, it holds that:

$$\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^4=O_p(1)$$

uniformly in i and t. The whole term is therefore of order $o_p(R^{-1})$ for all i and t since $t^{-1}\leq R^{-1}$.

The final term from the RHS of Equation (S-19) to consider is:

$$\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}F_k'\left(\lambda_h'F_k+u_{hk}\right)\right\|^2,$$

which can be bounded by:

$$C\left(\frac{1}{t}\sum_{k=1}^{t}\|F_k\|^2\right)\left(\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2\right)\left(\frac{T}{t}\right)\left(\frac{1}{N}\sum_{h=i+1}^{N}\left(\frac{1}{T}\sum_{k=1}^{t}\left(\lambda_h'F_k+u_{hk}\right)^2\right)\right) \tag{S-20}$$

The term $t^{-1}\sum_{k=1}^{t}\|F_k\|^2$ is of order $O_p(1)$ uniformly in i and t by the arguments from before. Moreover, by Assumption SM1 and Markov's inequality:

$$\left(\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2\right)\left(\frac{T}{t}\right)\left(\frac{1}{N}\sum_{h=i+1}^{N}\left(\frac{1}{T}\sum_{k=1}^{t}\left(\lambda_h'F_k+u_{hk}\right)^2\right)\right)=O_p(1)$$

uniformly in i and t, and by Assumption SM1 for all i and t:

$$\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^2=O_p(1).$$

Thus, the bound for Equation (S-20) is of order $O_p(1)$, which holds uniformly in i and t. The claim follows noting that $(R/T)=O(1)$ by assumption.

■

Proof of Lemma B.4. A similar decomposition as in the proof of Lemma B.3 yields:

$$\begin{aligned}\sum_{j=1}^{t}\left\|\widehat{B}_j^{(i,t)}\right\|^2=&\sum_{j=1}^{t}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}X_{h,j}\widehat{\mu}_{ht}F_t'\widehat{H}^{(i,t)}\widehat{V}^{(i,t)-1}\right\|^2\\&+\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\widehat{\mu}_{ht}^2F_t'\widehat{H}^{(i,t)}\widehat{V}^{(i,t)-1}\right\|^2\\&+\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{h,k}F_k'\widehat{H}^{(i,t)}\widehat{V}^{(i,t)-1}\right\|^2.\end{aligned} \tag{S-21}$$

We will only sketch the steps for the first term on the RHS of S-21, the others follow by similar arguments. Since $\widehat{H}^{(i,t)}$ and of $\widehat{V}^{(i,t)-1}$ are both bounded in probability uniformly in i and t by Lemma B.1, we can use the same argument as in the previous lemma to derive the bound:

$$\begin{aligned}&\sum_{j=1}^{t}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}X_{j,t}\widehat{\mu}_{ht}F_t'\widehat{H}^{(i,t)}\widehat{V}^{(i,t)-1}\right\|^2\\\le&C\left(\frac{T}{t}\right)\frac{1}{N}\left(\frac{1}{t}\|F_t\|^2\right)\left(\frac{1}{N}\sum_{h=i+1}^{N}|\widehat{\mu}_{ht}|^4\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{h=i+1}^{N}\left(\frac{1}{T}\sum_{j=1}^{t}\left(\lambda_h'F_j+u_{hj}\right)^2\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

While the first, second, and fourth term are identical to the proof of Lemma B.3, the third term on the RHS is:

$$\frac{1}{N}\sum_{h=i+1}^{N}\widehat{\mu}_{ht}^4=\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^4+o_p(1)=O_p(1)$$

uniformly in i and t by Assumption SM1 and an application of Markov's inequality. The remaining terms can be analyzed using analogous arguments to above and Lemma B.3.

■

Proof of Lemma B.5. Note that:

$$\begin{aligned}\frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}F_j\right\|^2 &= \frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}\left(\widetilde{F}_j^{(i,t)}+\widehat{H}^{(i,t)-1}\widehat{B}_j^{(i,t)}\right)\right\|^2\\ &= \frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_j^{(i,t)}-\widehat{B}_j^{(i,t)}\right\|^2\\ &\leq \frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_j^{(i,t)}\right\|^2\\ &\quad+\frac{1}{T}\sum_{j=1}^{t}\left\|\widehat{B}_j^{(i,t)}\right\|^2.\end{aligned}$$

By Lemma B.2, the first term after the inequality is of order $O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)$, while the second is of order $O_p\left(\frac{1}{R}\right)$ by Lemma B.4, and so the entire term is of order $O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)$ uniformly in i and t. ■

Proof of Lemma B.6. The proof of this lemma has been omitted for brevity. It follows very closely Lemma B.2 of Bai (2003), replacing use of his Lemma A.1 by our Lemma B.5 above, and noting that results can be shown to hold uniformly in i and t. ■

Proof of Lemma B.7. The proof closely follows the argument of Bai and Ng (2013), Appendix B (p.27), extending the results to hold uniformly in i and t. Starting with the rotation matrix $\widehat{H}^{(i,t)}=\widehat{V}^{(i,t)-1}(\widehat{F}^{(i,t)\prime}F^{(t)}/t)(\Lambda'\Lambda/N)$ defined in the text, we decompose the term $\widehat{F}^{(i,t)\prime}F^{(t)}/t$ into:

$$\begin{aligned}\frac{\widehat{F}^{(i,t)\prime}F^{(t)}}{t} &= \frac{\left(\widehat{F}^{(i,t)}-F^{(t)}\widehat{H}^{(i,t)\prime}\right)'F^{(t)}}{t}+\widehat{H}^{(i,t)}\frac{F^{(t)\prime}F^{(t)}}{t}\\ &= \widehat{H}^{(i,t)}+O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)\end{aligned}\tag{S-22}$$

since we use the normalisation $\frac{F^{(t)\prime}F^{(t)}}{t}=I_r$ for all $t=R+1,...,T$, and since $\left(\widehat{F}^{(i,t)}-F^{(t)}\widehat{H}^{(i,t)\prime}\right)'F^{(t)}/t=(T/t)\left(\widehat{F}^{(i,t)}-F^{(t)}\widehat{H}^{(i,t)\prime}\right)'F^{(t)}/T=O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)$ uniformly in i and t by Lemma B.6.

Now post-multiplying Equation (S-22) further by $\widehat{H}^{(i,t)\prime}$ we have:

$$\frac{\widehat{F}^{(i,t)\prime}F^{(t)}}{t}\widehat{H}^{(i,t)\prime}=\widehat{H}^{(i,t)}\widehat{H}^{(i,t)\prime}+O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)$$

and expanding this further:

$$\frac{\widehat{F}^{(i,t)\prime}\left(F^{(t)}\widehat{H}^{(i,t)\prime}-\widehat{F}^{(i,t)}+\widehat{F}^{(i,t)}\right)}{t}=\widehat{H}^{(i,t)}\widehat{H}^{(i,t)}+O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)$$

which finally gives:

$$\widehat{H}^{(i,t)}\widehat{H}^{(i,t)} = I_r + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$$

since we normalise the estimates $\frac{\widehat{F}^{(i,t)\prime}\widehat{F}^{(i,t)}}{t} = I_r$ for all $t = R+1, ..., T$, $i = 1, ..., N$, and since it also holds from Lemma B.6 and a similar result to Bai and Ng (2006) Lemmas A1(ii) and A1(iii) that $\widehat{F}^{(i,t)\prime}\left(F^{(t)}\widehat{H}^{(i,t)\prime} - \widehat{F}^{(i,t)}\right)/t = O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$ uniformly in i and t.

The result that $\widehat{H}^{(i,t)}\widehat{H}^{(i,t)} = I_r + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$ uniformly in i and t means that $\widehat{H}^{(i,t)}$ is an orthogonal matrix with eigenvalues equal to ± 1 up to a term which is negligible uniformly in i and t. Finally, in order to show what is required, that $\widehat{H}^{(i,t)}$ is a diagonal matrix with ± 1 on the principal diagonal, up to a term which is of order $O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$, it suffices to show that $\widehat{H}^{(i,t)}$ is symmetric, which follows an identical proof to that of Bai and Ng (2013). ■

Proof of Lemma B.8. Since we know from the proof of Lemma B.7 that

$$\widehat{H}^{(i,t)}\widehat{H}^{(i,t)} = I_r + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right),$$

uniformly in i and t, and, as in Bai and Ng (2013), Appendix B (p.27),

$$\frac{\Lambda' F^{(t)\prime} F^{(t)}\Lambda}{Nt}\widehat{H}^{(i,t)} = \widehat{H}^{(i,t)}\widehat{V}^{(i,t)} + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right),$$

we can conclude that:

$$\widehat{V}^{(i,t)} = V_0 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{R}\right\}\right)$$

uniformly in i and t. Moreover, it holds that:

$$\left\|\widehat{V}^{(i,t)-1} - V_0^{-1}\right\| = \left\|V_0^{-1}\right\| \times \left\|V_0 - \widehat{V}^{(i,t)}\right\| \times \left\|\widehat{V}^{(i,t)-1}\right\|.$$

Since V_0 is positive definite by Assumption SM1 above and Lemma B.1 that $\left\|\widehat{V}^{(i,t)-1}\right\| = O_p(1)$ uniformly in i and t, it follows that for all i and t:

$$\left\|\widehat{V}^{(i,t)-1} - V_0^{-1}\right\| = O_p\left(\max\left\{\frac{1}{R}, \frac{1}{N}\right\}\right).$$

■

Proof of Lemma B.9. Using the representation derived in the proof of Lemma B.3, we have that:

$$\begin{aligned}
&\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\| \\
= &\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\left(X_{ht}\widehat{\mu}_{ht}\widehat{F}_t^{(i,t)\prime}\widehat{V}^{(i,t)-1} - X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)}V_0^{-1}\right)\right\| \\
&+ \left\|\frac{1}{tN}\sum_{h=i+1}^{N}\left(\widehat{\mu}_{ht}^2\widehat{F}_t^{(i,t)\prime}\widehat{V}^{(i,t)-1} - \ddot{\mu}_{ht}^2 F_t' H_0^{(i,t)}V_0^{-1}\right)\right\|
\end{aligned}$$

$$+\left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \left(\widehat{\mu}_{ht} X_{hk} \widehat{F}_k^{(i,t)\prime} \widehat{V}^{(i,t)-1} - \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right) \right\|.$$

For simplicity, we will only examine the leading term involving the double sum $\sum_{h=i+1}^{N}\sum_{k=1}^{t}$, the remaining terms follow by similar arguments and are of smaller order than the last term. First note, that this term can be further decomposed as:

$$\begin{aligned}
& \left\| \frac{1}{tN} \left(\sum_{h=i+1}^{N} \sum_{k=1}^{t} \widehat{\mu}_{ht} X_{hk} \widehat{F}_k^{(i,t)\prime} \widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right) \right\| \\
\leq & \left\| \frac{1}{tN} \left(\sum_{h=i+1}^{N} \sum_{k=1}^{t} \widehat{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} - \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right) \right\| \qquad \text{(S-23)} \\
& + \left\| \frac{1}{tN} \left(\sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} \widehat{F}_k^{(i,t)\prime} V_0^{-1} - \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right) \right\| \\
& + \left\| \frac{1}{tN} \left(\sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} \widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right) \right\| \\
& + o_p(1),
\end{aligned}$$

where the $o_p(1)$ term holds uniformly in i and t and contains the cross-products. We start with the last term on the RHS of the inequality, which can be written as:

$$\begin{aligned}
& \left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} \left(V_0^{-1} - \widehat{V}^{(i,t)-1} \right) \right\| \\
\leq & \sup_{i,t} \left\| V_0^{-1} - \widehat{V}^{(i,t)-1} \right\| \left(\left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' H_0^{(i,t)} \right\| \right) \\
\leq & \sup_{i,t} \left\| V_0^{-1} - \widehat{V}^{(i,t)-1} \right\| \left(\frac{T}{t} \right) \left(\left(\frac{1}{TN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| \ddot{\mu}_{ht} X_{hk} \|^2 \right)^{\frac{1}{2}} \left(\frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 \right)^{\frac{1}{2}} \right) \\
\leq & \sup_{i,t} \left\| V_0^{-1} - \widehat{V}^{(i,t)-1} \right\| \left(\frac{T}{t} \right) \left(\left(\frac{1}{N} \sum_{h=i+1}^{N} \ddot{\mu}_{ht}^4 \right)^{\frac{1}{4}} \left(\frac{1}{TN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} (\lambda_h' F_k + u_{hk})^4 \right)^{\frac{1}{4}} \right. \\
& \left. \times \left(\frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 \right)^{\frac{1}{2}} \right) \\
= & O_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right) O(1) O_p(1) O_p(1) O_p(1),
\end{aligned}$$

where the last equality follows from Lemma B.8, Assumption SM1, and the fact that:

$$\left(\frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 \right) = O_p(1)$$

for every i and t. For the second term on the RHS of Equation (S-23), we get that:

$$\begin{aligned}
&\left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} \left(\widehat{F}_k^{(i,t)\prime} - F_k' H_0^{(i,t)} \right) V_0^{-1} \right\| \\
\leq\ & C \left(\left(\frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| \ddot{\mu}_{ht} X_{hk} \|^2 \right)^{\frac{1}{2}} \left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)\prime} - H_0^{(i,t)} F_k' \|^2 \right)^{\frac{1}{2}} \right) \\
\leq\ & C \left(\frac{T}{t} \right)^{\frac{1}{2}} \left(\left(\frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| \ddot{\mu}_{ht} \|^4 \right)^{\frac{1}{4}} \left(\frac{1}{TN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| X_{hk} \|^4 \right)^{\frac{1}{4}} \right. \\
& \left. \times \left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)\prime} - H_0^{(i,t)} F_k' \|^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

where we used the fact that $\|V_0^{-1}\| = O(1)$. While the first term in brackets is again of order $O_p(1)$ uniformly in i and t by Assumption SM1 and the same argument as above, note that the second term can be decomposed as follows:

$$\begin{aligned}
& \frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} - H_0^{(i,t)} F_k \|^2 \\
\leq\ & \left(\frac{T}{t} \right) \frac{1}{T} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)} F_k \|^2 + \left(\| \widehat{H}^{(i,t)} - H_0^{(i,t)} \|^2 \right) \frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 = O_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right)
\end{aligned}$$

uniformly in i and t by Lemma B.5 and B.7, which implies this term is of order $O_p\left(\max\left\{\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{N}}\right\}\right)$ uniformly in i and t. For the first term on the RHS of Equation (S-23), we obtain instead:

$$\begin{aligned}
& \left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \left(\ddot{\mu}_{ht} - \widehat{\mu}_{ht} \right) X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right\| \\
\leq\ & C \sup_{i,t} |\ddot{\mu}_{ht} - \widehat{\mu}_{ht}|^2 \left(\frac{T}{t} \right)^{\frac{1}{2}} \left(\left(\frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 \right)^{\frac{1}{2}} \left(\frac{1}{NT} \sum_{h=i+1}^{N} \sum_{k=1}^{t} (\lambda_h' F_k + u_{hk})^2 \right)^{\frac{1}{2}} \right) \\
=\ & O_p \left(\frac{1}{\sqrt{R}} \right) O(1) O_p(1) O_p(1),
\end{aligned}$$

where we used again the fact that $\|V_0^{-1}\| = O(1)$ and $H_0^{(i,t)} H_0^{(i,t)} = I_r$ uniformly in i and t.

■

Proof of Lemma B.10. Unlike in Lemma B.3, we now look at $\frac{1}{P} \sum_{t=R+1}^{T} \left\| \widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)} \right\|^2$, which involves averages of the last row of the window, across all $t = R, ..., T$.

First, note that:

$$\ddot{B}_t^{(i,t)} = \frac{1}{tN} \left(\sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} F_k' + \sum_{h=i+1}^{N} X_{ht} \ddot{\mu}_{ht} F_t' + \sum_{h=i+1}^{N} \ddot{\mu}_{ht}^2 F_t' \right) H_0^{(i,t)} V_0^{-1} \quad \text{(S-24)}$$

whereas:

$$\widehat{B}_t^{(i,t)} = \frac{1}{tN}\left(\sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime} + \sum_{h=i+1}^{N}X_{ht}\widehat{\mu}_{ht}\widehat{F}_t^{(i,t)\prime} + \sum_{h=i+1}^{N}\widehat{\mu}_{ht}^2\widehat{F}_t^{(i,t)\prime}\right)\widehat{V}^{(i,t)-1}. \tag{S-25}$$

It is straightforward to see that the leading term of $\frac{1}{P}\sum_{t=R+1}^{T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\|^2$ is given by the expression involving the double sum $\sum_{h=i+1}^{N}\sum_{k=1}^{t}$, which can be written as:

$$\begin{aligned}
& \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{hk}F_k'H_0^{(i,t)}V_0^{-1} - \sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\widehat{V}^{(i,t)-1}\right)\right\|^2 \\
\leq\ & \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\widehat{V}^{(i,t)-1}\right)\right\|^2 \\
& +\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}F_k'H_0^{(i,t)}\widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\widehat{V}^{(i,t)-1}\right)\right\|^2 \\
& +\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}V_0^{-1} - \sum_{h=i+1}^{N}\sum_{k=1}^{t}\widehat{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\widehat{V}^{(i,t)-1}\right)\right\|^2 \\
& +o_p(1),
\end{aligned} \tag{S-26}$$

where the $o_p(1)$ term holds uniformly in i and t and contains the cross-products. We start with the last term on the RHS of the inequality. Since $|\ddot{\mu}_{ht} - \widehat{\mu}_{ht}| = o_p(1)$ uniformly in i and t by Assumption SM1, note that this term can be written as:

$$\begin{aligned}
& \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\left(V_0^{-1} - \widehat{V}^{(i,t)-1}\right)\right\|^2 \\
\leq\ & \sup_{i,t}\left\|V_0^{-1} - \widehat{V}^{(i,t)-1}\right\|^2\left(\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\ddot{\mu}_{ht}X_{hk}\widehat{F}_k^{(i,t)\prime}\right\|^2\right) \\
\leq\ & \sup_{i,t}\left\|V_0^{-1} - \widehat{V}^{(i,t)-1}\right\|^2\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{tN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}\|\ddot{\mu}_{ht}X_{hk}\|^2\right)\left(\frac{1}{t}\sum_{k=1}^{t}\|\widehat{F}_k^{(i,t)}\|^2\right)\right) \\
\leq\ & \sup_{i,t}\left\|V_0^{-1} - \widehat{V}^{(i,t)-1}\right\|^2\left(\frac{T}{t}\right)^{\frac{1}{2}}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^4\right)^{\frac{1}{2}}\left(\frac{1}{TN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}(\lambda_h'F_k + u_{hk})^4\right)^{\frac{1}{2}}\right. \\
& \left.\times\left(\frac{1}{t}\sum_{k=1}^{t}\|\widehat{F}_k^{(i,t)}\|^2\right)\right) \\
\leq\ & \sup_{i,t}\left\|V_0^{-1} - \widehat{V}^{(i,t)-1}\right\|^2\left(\frac{T}{t}\right)^{\frac{1}{2}}\left(\left(\frac{1}{N}\sum_{h=i+1}^{N}\ddot{\mu}_{ht}^4\right)^{\frac{1}{2}}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{TN}\sum_{h=i+1}^{N}\sum_{k=1}^{t}(\lambda_h'F_k + u_{hk})^4\right)\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$
\begin{aligned}
& \times \left(\frac{1}{P} \sum_{t=R}^{T} \left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} \|^2 \right)^2 \right)^{\frac{1}{2}} \\
= \;& o_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right) O(1) O_p(1) O_p(1) O_p(1),
\end{aligned}
$$

where the last equality follows from Lemma B.8, Assumption SM1, and the fact that:

$$
\left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} \|^2 \right) = O_p(1)
$$

for every i and t due to the normalisation. After replacing again $\widehat{\mu}_{ht}$ and $\widehat{V}^{(i,t)-1}$ by its counterparts $\ddot{\mu}_{ht}$ and V_0^{-1}, respectively, using Assumption SM1 and the uniformity over i and t, we have for the second term on the RHS of Equation (S-26) that:

$$
\begin{aligned}
& \frac{1}{P} \sum_{t=R+1}^{T} \left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \ddot{\mu}_{ht} X_{hk} \left(\widehat{F}_k^{(i,t)\prime} - F_k' H_0^{(i,t)} \right) V_0^{-1} \right\|^2 \\
\leq \;& C \frac{1}{P} \sum_{t=R+1}^{T} \left(\left(\frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| \ddot{\mu}_{ht} X_{hk} \|^2 \right) \left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)\prime} - H_0^{(i,t)} F_k' \|^2 \right) \right) \\
\leq \;& C \left(\frac{1}{P} \sum_{t=R+1}^{T} \left(\frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \| \ddot{\mu}_{ht} X_{hk} \|^2 \right)^2 \right)^{\frac{1}{2}} \left(\frac{1}{P} \sum_{t=R+1}^{T} \left(\frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)\prime} - H_0^{(i,t)} F_k' \|^2 \right)^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

where we used the fact that $\|V_0^{-1}\| = O(1)$. While the first term in brackets is again of order $O_p(1)$ uniformly in i and t by Assumption SM1 and the same argument as above, note that the second term can be decomposed as follows:

$$
\begin{aligned}
& \frac{1}{t} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} - H_0^{(i,t)} F_k \|^2 \\
\leq \;& \frac{T}{t} \frac{1}{T} \sum_{k=1}^{t} \| \widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)} F_k \|^2 + \left(\| \widehat{H}^{(i,t)} - H_0^{(i,t)} \| \right) \frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 = O_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right)
\end{aligned}
$$

uniformly in i and t by Lemma B.5 and B.7 and $(T/t) = O(1)$ by Assumption 4. For the first term on the RHS of Equation (S-26), after the same replacements as above, we obtain instead:

$$
\begin{aligned}
& \frac{1}{P} \sum_{t=R+1}^{T} \left\| \frac{1}{tN} \sum_{h=i+1}^{N} \sum_{k=1}^{t} \left(\ddot{\mu}_{ht} - \widehat{\mu}_{ht} \right) X_{hk} F_k' H_0^{(i,t)} V_0^{-1} \right\|^2 \\
\leq \;& C \sup_{i,t} | \ddot{\mu}_{ht} - \widehat{\mu}_{ht} |^2 \frac{1}{P} \sum_{t=R+1}^{T} \left(\left(\frac{1}{t} \sum_{k=1}^{t} \| F_k \|^2 \right) \left(\frac{T}{t} \right) \left(\frac{1}{NT} \sum_{h=i+1}^{N} \sum_{k=1}^{t} (\lambda_h' F_k + u_{hk})^2 \right) \right) \\
= \;& O_p \left(\frac{1}{R} \right) O_p(1),
\end{aligned}
$$

where we used again the fact that $\|V_0^{-1}\| = O(1)$ and $H_0^{(i,t)} H_0^{(i,t)} = I_r$ uniformly in i and t.

Next, we examine the second term of $\frac{1}{P}\sum_{t=R+1}^{T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\|^2$, given in the expressions of (S-24) and (S-25):

$$
\begin{aligned}
& \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N} X_{ht}\widehat{\mu}_{ht}\widehat{F}_t^{(i,t)\prime}\widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right)\right\|^2 \\
\leq\ & \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N} X_{ht}\widehat{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1} - \sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right)\right\|^2 \\
& +\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}\widehat{F}_t^{(i,t)\prime} V_0^{-1} - \sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right)\right\|^2 \qquad \text{(S-27)} \\
& +\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\left(\sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)}\widehat{V}^{(i,t)-1} - \sum_{h=i+1}^{N} X_{ht}\ddot{\mu}_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right)\right\|^2 \\
& +o_p(1),
\end{aligned}
$$

where the $o_p(1)$ term holds uniformly in i and t again and collects the cross-products. Starting with the first term on the right hand side of (S-27), it is straightforward to see that similar arguments to before yield:

$$
\begin{aligned}
& \frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{tN}\sum_{h=i+1}^{N}\left(\ddot{\mu}_{ht} - \widehat{\mu}_{ht}\right) X_{ht}F_t' H_0^{(i,t)} V_0^{-1}\right\|^2 \\
\leq\ & \frac{C}{R^2}\sup_{i,t}|\ddot{\mu}_{ht} - \widehat{\mu}_{ht}|^2\left(\frac{1}{P}\sum_{t=R+1}^{T}\|F_t\|^2\right)^{\frac{1}{2}}\left(\frac{1}{NP}\sum_{t=R+1}^{T}\sum_{h=i+1}^{N}(\lambda_h' F_k + u_{hk})^2\right)^{\frac{1}{2}} \\
=\ & o_p\left(\frac{1}{R}\right) O_p(1)O_p(1),
\end{aligned}
$$

where we used again the normalisation $H_0^{(i,t)} H_0^{(i,t)} = I_r$ and the fact that $(R/t) = O(1)$. Using similar arguments we can bound the second term on the RHS of (S-27) by:

$$
\frac{C}{R^2}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{N}\sum_{h=i+1}^{N}\|\ddot{\mu}_{ht}X_{ht}\|^2\right)^2\right)^{\frac{1}{2}}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\|\widehat{F}_t^{(i,t)\prime} - H_0^{(i,t)}F_t'\|^2\right)^2\right)^{\frac{1}{2}}. \qquad \text{(S-28)}
$$

Now while the first term in brackets is clearly $O_p(1)$ uniformly in i and t since

$$
\frac{1}{N}\sum_{h=i+1}^{N}\|\ddot{\mu}_{ht}X_{ht}\|^2 \leq \left(\frac{1}{N}\sum_{h=i+1}^{N}|\ddot{\mu}_{ht}|^4\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{h=i+1}^{N}\|X_{ht}\|^4\right)^{\frac{1}{2}} = O_p(1)
$$

by Assumption SM1, we note that for the second term the crude bound:

$$\|\widehat{F}_t^{(i,t)\prime} - H_0^{(i,t)} F_t'\|^2 \leq \sum_{k=1}^{t} \|\widehat{F}_k^{(i,t)\prime} - H_0^{(i,t)} F_k'\|^2.$$

Therefore, re-inserting this term into the bound in Equation (S-28) and using the same arguments as for the second term of Equation (S-26) term gives us that the second term is of order $O_p(1/R)$ uniformly in i and t. Finally, for the third term on the RHS of (S-27) we get the bound:

$$\frac{1}{R}\sup_{i,t} \|\widehat{V}^{(i,t)-1} - V_0^{-1}\|^2 \left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{N}\sum_{h=i+1}^{N}\|\ddot{\mu}_{ht}X_{ht}\|^2\right)^2\right)^{\frac{1}{2}} \left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{t}\|F_t\|^2\right)^2\right)^{\frac{1}{2}},$$

which is of order $O_p\left(\frac{1}{R}\right)$ uniformly in i and t. Since similar arguments to above, also yield that the third term of $\frac{1}{P}\sum_{t=R+1}^{T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\|^2$, given in the expressions of (S-24) and (S-25), is of order $O_p(1/R)$ uniformly in i and t. This establishes the claim. ■

S5 Auxiliary Lemmas (Test Statistic)

Recall that c, c', C, and C' denote generic positive constants whose values may vary from line to line and which are assumed to depend exclusively on $0 < c_1 \leq C_1 < 1$ and on $0 < c_2 < 1/4$. Also, let $\Pr^*(\cdot)$ denote the bootstrap probability measure, conditional on the data $\{\Delta L_t(.)\}_{t=1}^{T}$, defined on a given probability space $(\Omega, \mathcal{F}, \Pr)$. Also, recall that κ denotes the cardinality of the set $\mathcal{C}_S$.

Lemma C.1. *Grant the assumptions of Theorem 1. For some ζ_{P1} satisfying Assumption 4 ($\zeta_{P1} \leq C_1 P^{-c_2}$ in particular), where all constants have been defined above, it holds that:*

$$Pr\left(\max_{i,k\in\mathcal{C}_S}\sqrt{P}\left|\frac{1}{P}\sum_{t=R+1}^{T}\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{it}) - \Delta L_t(\theta_{i+k},\theta_i)\right| > \zeta_{P1}\right) \leq C_1 P^{-c_2}$$

Lemma C.2. *Grant the assumptions of Theorem 1. For some ζ_{P1} as in Lemma C.1 and all constants have been defined above, it holds that:*

$$Pr\left(\max_{i,k\in\mathcal{C}_S}\left(\frac{1}{\sqrt{m_P q_P}}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{it}) - \Delta L_t(\theta_{i+k},\theta_i)\right)\right)^2\right)^{\frac{1}{2}} > \zeta_{P1}\right) \leq C_1 P^{-c_2}.$$

Lemma C.3. *Grant the assumptions of Theorem 1. For some φ_P with $0 < \varphi_P \leq CP^{-c}$, it holds that:*

$$Pr\left(c^{BMB}(\alpha) \geq c_0(\alpha + \varphi_P)\right) \geq 1 - CP^{-c}$$

and

$$Pr\left(c^{BMB}(\alpha) \leq c_0(\alpha - \varphi_P)\right) \geq 1 - CP^{-c},$$

where $c_0(\gamma)$, $\gamma \in (0,1)$, denotes the $(1-\gamma)$ quantile of the distribution of $\max_{1\leq l\leq\kappa} Y_l$.

Proof of Lemma C.1. Applying Markov's inequality, the probability is bounded by:

$$\frac{\mathrm{E}\left[\max_{i,k\in\mathcal{C}_S}\left|\frac{\sqrt{P}}{P}\sum_{t=R+1}^{T}\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{it})-\Delta L_t(\theta_{i+k},\theta_i)\right|\right]}{\zeta_{P1}}. \tag{S-29}$$

For simplicity, we focus on $\widehat{\theta}_{it}=[\widehat{F}_t^{(i,t)\prime},\widehat{\beta}_{it}^{\prime}]^{\prime}$, ignoring any other set of estimated factors $\widehat{F}_t^{(i+k,t)\prime}$ as they will follow an identical proof. Thus, taking a mean value expansion around $\widehat{H}^{(i,t)}F_t^{(i,t)}$ and $\widehat{H}^{(i,t)\prime-1}\beta$ we obtain:

$$\begin{aligned}
&\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{it})-\Delta L_t(\theta_{i+k},\theta_i)\right)\\
=\;&\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}F_t^{(i,t)}\right)\\
&+\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{\beta}_{it}-\widehat{H}^{(i,t)\prime-1}\beta\right).
\end{aligned} \tag{S-30}$$

In the following we establish that:

$$\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}F_t^{(i,t)}\right)=O_p\left(\max\left\{\frac{P}{R},\frac{\sqrt{P}}{R}\right\}\right)$$

and

$$\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{\beta}_{it}-\widehat{H}^{(i,t)\prime-1}\beta\right)=O_p\left(\max\left\{\frac{P}{R},\frac{\sqrt{P}}{R}\right\}\right)$$

uniformly in i and t (note that since the same arguments apply to terms with k, uniformity holds also in k). The first term in Equation (S-30) can be split into the following two components:

$$\mathcal{A}_{1T}=\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_t^{(i,t)}\right)$$

and

$$\mathcal{A}_{2T}=\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\widehat{H}^{(i,t)}\left(\widetilde{F}_t^{(i,t)}-F_t^{(i,t)}\right).$$

Starting with $\mathcal{A}_{1T}$, we use the representation of Equation (S-15) to obtain:

$$\begin{aligned}
&\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_t^{(i,t)}\right)\\
=\;&\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Bigg(\widehat{V}^{(i,t)-1}\Bigg(\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kt}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kt}
\end{aligned}$$

$$+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kt}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kt}\Bigg)\Bigg) \tag{S-31}$$

$$= \frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{V}^{(i,t)-1}\left(I_{1t}+I_{2t}+I_t+I_{4t}\right)\right)$$

The RHS of Equation (S-31) can be treated using standard arguments relying on Lemma B.2. We will therefore only sketch the key steps.

Since $\|\widehat{V}^{(i,t)-1}\|=O_p(1)$ and $\|\widehat{H}^{(i,t)}\|^2=O_p(1)$ uniformly in i and t by Lemma B.1, we can ignore the presence of $\widehat{V}^{(i,t)-1}$ in what follows as it can just be replaced by some constant. Thus, note that the first term of Equation (S-31) can be decomposed as:

$$\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\frac{1}{t}\sum_{k=1}^{t}\widehat{H}^{(i,t)}F_k^{(i,t)}\gamma_{kt}\right)+\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\frac{1}{t}\sum_{k=1}^{t}\left(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right)\gamma_{kt}\right)$$

The second term can be bounded (uniformly in i and t) as follows:

$$\begin{aligned}
&\frac{1}{P}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\frac{1}{t}\sum_{k=1}^{t}\left(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right)\gamma_{kt}\right)\\
\leq\ &\frac{1}{tP}\left(\sum_{t=R+1}^{T}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\right)^{1/2}\left(\sum_{t=R+1}^{T}\left\|\sum_{k=1}^{t}\left(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right)\gamma_{kt}\right\|^2\right)^{1/2}\\
=\ &\left(\frac{1}{P}\sum_{t=R+1}^{T}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\right)^{1/2}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{t}\sum_{k=1}^{t}\left(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right)\gamma_{kt}\right\|^2\right)^{1/2}\\
=\ &O_p(1)\,O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{R}}\right\}\right)O_p\left(\frac{1}{\sqrt{R}}\right),
\end{aligned}$$

where the last line follows from an argument similar to Lemma A.2 and A.3 in Goncalves and Perron (2014) using Lemma B.2. That is:

$$\begin{aligned}
&\frac{1}{P}\sum_{t=R+1}^{T}\left\|\frac{1}{t}\sum_{k=1}^{t}\left(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right)\gamma_{kt}\right\|^2\\
&\left(\frac{1}{t}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right\|^2\right)\left(\frac{1}{tP}\sum_{t=R+1}^{T}\sum_{k=1}^{t}\gamma_{kt}^2\right)\\
&\leq\left(\frac{T}{t}\right)^2\left(\frac{1}{T}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k^{(i,t)}\right\|^2\right)\left(\frac{1}{PT}\sum_{t=R+1}^{T}\sum_{k=1}^{t}\gamma_{kt}^2\right)\\
&=O(1)O_p\left(\max\left\{\frac{1}{N},\frac{1}{R}\right\}\right)O_p\left(\frac{1}{R}\right)
\end{aligned}$$

uniformly in i and t using the fact that $t/T=O(1)$ by Assumption 4. Similarly, since $\sqrt{P}/\sqrt{R}\longrightarrow 0$, it follows that the second term is of order $o_p\left(\frac{\sqrt{P}}{R},\frac{\sqrt{P}}{N}\right)$ uniformly in i and t. By contrast, for the first term, we have that:

$$
\begin{aligned}
& \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \left(\frac{1}{t} \sum_{k=1}^{t} \widehat{H}^{(i,t)} F_k^{(i,t)} \gamma_{kt} \right) \\
= & \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \left(\frac{1}{t} \sum_{k=1}^{t} \widehat{H}^{(i,t)} F_k \gamma_{kt} \right) - \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \left(\frac{1}{t} \sum_{k=1}^{t} \widehat{B}_k^{(i,t)} \gamma_{kt} \right) \\
= & Q_{1T} + Q_{2T}.
\end{aligned}
$$

The first term Q_{1T} can be dealt with using standard arguments. That is:

$$
\begin{aligned}
& \mathrm{E}\left[\left\| \frac{1}{t\sqrt{P}} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \widehat{H}^{(i,t)} F_j \gamma_{jt} \right\| \right] \\
\leq & \frac{1}{\sqrt{P}R} \sum_{t=R+1}^{T} \sum_{j=1}^{t} |\gamma_{jt}| \left(\mathrm{E}\left[\left\| \widehat{H}^{(i,t)} F_j \right\|^2 \right] \right)^{1/2} \left(\mathrm{E}\left[\left\| \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \right\|^2 \right] \right)^{1/2} \\
= & \frac{\sqrt{P}}{R} \frac{1}{P} \sum_{t=R+1}^{T} \sum_{j=1}^{t} |\gamma_{jt}| \times O_p(1) \\
= & O_p\left(\frac{\sqrt{P}}{R} \right),
\end{aligned}
$$

where the last two lines hold uniformly in i and t, and follow from $\frac{1}{P} \sum_{t=R+1}^{T} \sum_{j=1}^{t} |\gamma_{jt}| = O_p(1)$ in Assumption SM1. For the second term, Q_{2T}, observe that:

$$
\begin{aligned}
& \frac{1}{t\sqrt{P}} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \widehat{B}_j^{(i,t)} \gamma_{jt} \\
= & \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \Big] \frac{1}{t\sqrt{P}} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \widehat{B}_j^{(i,t)} \gamma_{jt} \\
& + \frac{1}{t\sqrt{P}} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \Big(\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] - \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \Big] \Big) \widehat{B}_j^{(i,t)} \gamma_{jt}.
\end{aligned}
$$

The first term on the RHS of the above equation can be bounded by:

$$
C\, \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \Big] \left(\frac{1}{T} \sum_{j=1}^{t} \left\| \widehat{B}_j^{(i,t)} \right\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{PT} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \gamma_{jt}^2 \right)^{\frac{1}{2}},
$$

which is of order $O_p\left(\frac{1}{R}\right)$ for all i and t by Assumption SM1 and Lemma B.4. For the second term, note that for all i and t:

$$
\frac{1}{t\sqrt{P}} \sum_{t=R+1}^{T} \sum_{j=1}^{t} \Big(\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] - \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] \Big] \Big) \widehat{B}_j^{(i,t)} \gamma_{jt}
$$

$$
\begin{aligned}
&\leq \sup_t \left(\left\| \frac{1}{R} \sum_{j=1}^{t} \widehat{B}_j^{(i,t)} \gamma_{jt} \right\| \right) \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \left(\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] - \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \Big] \right) \\
&\leq \left(\frac{T}{R} \right) \sup_t \left(\frac{1}{T} \sum_{j=1}^{t} \gamma_{jt}^2 \right)^{\frac{1}{2}} \sup_t \left(\frac{1}{T} \sum_{j=1}^{t} \left\| \widehat{B}_j^{(i,t)} \right\|^2 \right)^{\frac{1}{2}} \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \left(\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] - \mathrm{E}\Big[\nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \Big] \right) \\
&= O(1) O_p\left(\frac{1}{\sqrt{R}} \right) O_p\left(\frac{1}{\sqrt{R}} \right) O_p(1) = O_p\left(\frac{1}{R} \right),
\end{aligned}
$$

where the first term on the last line follows from the rate conditions, the second term from Assumption SM1, while the third term follows from Lemma B.4. Finally, the last term is a result of an argument similar to the proof of Lemma A4 in West (1996). Q_{2T} is thus of order $O_p\left(\frac{1}{R}\right)$ uniformly in i and t.

The remaining terms involving I_{2t}, I_t, and I_{4t} can be treated in a similar manner to before using and combining again arguments from Bai and Ng (2006) with arguments from Lemma B.2,B.4, and B.9.

Turning to $\mathcal{A}_{2T}$ from in Equation (S-30), note that we can use expressions (S-14) and (S-16), along with the representation in Equation (S-17) to obtain:

$$
\begin{aligned}
& \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \widehat{H}^{(i,t)} \left(\widehat{H}^{(i,t)-1} \widehat{B}_t^{(i,t)} - H_0^{(i,t)-1} \ddot{B}_t^{(i,t)} \right) \\
= & \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \widehat{H}^{(i,t)} \left(\widehat{H}^{(i,t)-1} - H_0^{(i,t)-1} \right) \ddot{B}_t^{(i,t)} \\
& + \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \widehat{H}^{(i,t)} H_0^{(i,t)-1} \left(\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)} \right) + o_p(1) \\
= & \mathcal{A}_{21T} + \mathcal{A}_{22T} + o_p(1),
\end{aligned}
$$

where the $o_p(1)$ term contains the cross-product and holds uniformly in i and t, which follows once we have established the uniform convergence of $\mathcal{A}_{21T}$ and $\mathcal{A}_{22T}$.

We start with $\mathcal{A}_{21T}$:

$$
\begin{aligned}
& \left\| \frac{1}{P} \sum_{t=R+1}^{T} \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \widehat{H}^{(i,t)} \left(\widehat{H}^{(i,t)-1} - H_0^{(i,t)-1} \right) \ddot{B}_t^{(i,t)} \right\| \\
\leq & C \sup_{i,t} \left\| \widehat{H}^{(i,t)-1} - H_0^{(i,t)-1} \right\| \left(\frac{1}{P} \sum_{t=R}^{T} \left\| \ddot{B}_t^{(i,t)} \right\|^2 \right)^{1/2} \left(\frac{1}{P} \sum_{t=R+1}^{T} \left\| \nabla_F \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \right\|^2 \right)^{1/2},
\end{aligned}
$$

which uses the fact that $\|\widehat{H}^{(i,t)}\|^2 = O_p(1)$ uniformly in i and t. The last term on the RHS is $O_p(1)$ by Assumption SM1. For $\frac{1}{P}\sum_{t=R}^{T} \left\| \ddot{B}_t^{(i,t)} \right\|^2$ we have the following crude bound:

$$
\frac{1}{P} \sum_{t=R+1}^{T} \left\| \ddot{B}_t^{(i,t)} \right\|^2 \leq \sup_{1 \leq i \leq N} \sup_{R \leq t \leq T} \left\| \ddot{B}_t^{(i,t)} \right\|^2 \leq \sup_{1 \leq i \leq N} \sup_{R \leq t \leq T} \sum_{j=1}^{t} \left\| \ddot{B}_j^{(i,t)} \right\|^2
$$

which is $O_p(1)$ by Lemma B.3.

Finally, since $\widehat{H}^{(i,t)-1} - H_0^{(i,t)-1} = H_0^{(i,t)-1}\left(H_0^{(i,t)} - \widehat{H}^{(i,t)}\right)\widehat{H}^{(i,t)-1}$ and since Lemma B.7 shows that $\left\|\widehat{H}^{(i,t)} - H_0^{(i,t)}\right\| = O_p\left(\max\left\{\frac{1}{R}, \frac{1}{N}\right\}\right)$ uniformly in i and t, it therefore follows that:

$$\mathcal{A}_{21T} = \frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\widehat{H}^{(i,t)}\left(\widehat{H}^{(i,t)-1} - H_0^{(i,t)-1}\right)\ddot{B}_t^{(i,t)} = O_p\left(\max\left\{\frac{\sqrt{P}}{R}, \frac{\sqrt{P}}{N}\right\}\right)$$

Next for $\mathcal{A}_{22T}$ we have:

$$\begin{aligned}
& \frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\widehat{H}^{(i,t)}H_0^{(i,t)-1}\left(\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right) \\
= & \frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\mathrm{E}\left[\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\right]\widehat{H}^{(i,t)}H_0^{(i,t)-1}\left(\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right) \\
& + \frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] - \mathrm{E}\left[\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\right]\right)\widehat{H}^{(i,t)}H_0^{(i,t)-1}\left(\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right)
\end{aligned}$$

For the first part, since $\|\widehat{H}^{(i,t)}\| = O_p(1)$ uniformly in i and t, and $\mathrm{E}\left[\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\right] < C$ by Assumption 2, by Lemma B.10 it follows that this expression is of order $O_p\left(\max\left\{\frac{\sqrt{P}}{R}, \frac{\sqrt{P}}{N}\right\}\right)$. For the second term instead, we note that this expression can be bounded by:

$$C\sup_{1\leq i\leq N}\sup_{R\leq t\leq T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\|\left(\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right] - \mathrm{E}\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\right)\right).$$

Now, while $\sup_{1\leq i\leq N}\sup_{R\leq t\leq T}\left\|\widehat{B}_t^{(i,t)} - \ddot{B}_t^{(i,t)}\right\| = O_p\left(\max\left\{\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{N}}\right\}\right)$ by Lemma B.9 and the fact that all but the last row are zero, the second term satisfies again a CLT for mixing data (cf. West, 1996), and so we obtain that $\mathcal{A}_{22T} = O_p\left(\max\left\{\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{N}}\right\}\right)$ uniformly in i and t. We therefore have that $\mathcal{A}_{2T} = \mathcal{A}_{21T} + \mathcal{A}_{22T}$ is the sum of an $O_p\left(\max\left\{\frac{\sqrt{P}}{R}, \frac{\sqrt{P}}{N}\right\}\right)$ term and an $O_p\left(\max\left\{\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{N}}\right\}\right)$ term. Therefore, since $P/(R)^{\frac{1}{2}} \longrightarrow \infty$ and $P/N = O(1)$ by Assumption 4, it follows that $\mathcal{A}_{2T} = O_p\left(\max\left\{\frac{P}{R}, \frac{\sqrt{P}}{N}\right\}\right)$.

We therefore have shown that the first term on the RHS of Equation (S-30) is:

$$\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i)\right]\left(\widehat{F}_t^{(i,t)} - \widehat{H}^{(i,t)}F_t^{(i,t)}\right) = O_p\left(\max\left\{\frac{P}{R}, \frac{\sqrt{P}}{N}\right\}\right)$$

uniformly in i and t.

Now we can we turn to the second term on the RHS of Equation (S-30). For this we need to analyse the OLS estimator $\widehat{\beta}_{it}$, remarking that these OLS estimates use the whole window from $1, ..., t$, for which only the last row of factor estimates are contaminated by the forecast errors $\widehat{\mu}_t^{(i)}$.

$$\widehat{\beta}_{it} = (\widehat{F}^{(i,t)\prime}\widehat{F}^{(i,t)})^{-1}\widehat{F}^{(i,t)\prime}y^{(t)}$$

$$= \frac{1}{t}\widehat{F}^{(i,t)\prime}y^{(t)}$$

since we normalize $(\widehat{F}^{(i,t)\prime}\widehat{F}^{(i,t)})/t = I_r$. Now manipulating Equation $y_t = \beta' F_t + \varepsilon_t$, ignoring the presence of W_t, we get:

$$y_t = \beta' F_t + \varepsilon_t = \beta' \widehat{H}^{(i,t)-1}\widehat{F}_t^{(i,t)} + \varepsilon_t + \beta'\widehat{H}^{(i,t)-1}\left(\widehat{H}^{(i,t)}F_t - \widehat{F}_t^{(i,t)}\right)$$

and so:

$$\begin{aligned}
\widehat{\beta}_{it} - \widehat{H}^{(i,t)\prime-1}\beta &= \frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\varepsilon_k + \frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\left(\widehat{H}^{(i,t)}F_k - \widehat{F}_k^{(i,t)}\right)'\widehat{H}^{(i,t)\prime-1}\beta \\
&= \frac{1}{t}\sum_{k=1}^{t}\widehat{H}^{(i,t)}F_k\varepsilon_k + \frac{1}{t}\sum_{k=1}^{t}(\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)}F_k)\varepsilon_k \\
&\quad + \frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\left(\widehat{H}^{(i,t)}F_k - \widehat{F}_k^{(i,t)}\right)'\widehat{H}^{(i,t)\prime-1}\beta,
\end{aligned} \tag{S-32}$$

Inserting the representation from Equation (S-32) into the second term on the RHS of Equation (S-30) we get:

$$\begin{aligned}
&\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Bigg(\frac{1}{t}\sum_{k=1}^{t}\widehat{H}^{(i,t)}F_k\varepsilon_k + \frac{1}{t}\sum_{k=1}^{t}(\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)}F_k)\varepsilon_k \\
&+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\left(\widehat{H}^{(i,t)}F_k - \widehat{F}_k^{(i,t)}\right)'\widehat{H}^{(i,t)\prime-1}\beta\Bigg).
\end{aligned} \tag{S-33}$$

For the second term on the RHS of Equation (S-33), note that this expression can be decomposed as:

$$\begin{aligned}
&\mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\frac{1}{t\sqrt{P}}\sum_{t=R+1}^{T}\sum_{j=1}^{t}(\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j)\varepsilon_j \\
&+\frac{1}{t\sqrt{P}}\sum_{t=R+1}^{T}\sum_{j=1}^{t}\Big(\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right] - \mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\Big)(\widehat{F}_j^{(i,t)} - \widehat{H}^{(i,t)}F_j)\varepsilon_j.
\end{aligned}$$

The first term is bounded by:

$$\begin{aligned}
&C\,\mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\sqrt{P}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{T}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)}F_k\right\|^2\right)\right)^{\frac{1}{2}}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{T}\sum_{k=1}^{t}\varepsilon_k^2\right)\right)^{\frac{1}{2}} \\
&= O_p\left(\max\left\{\frac{\sqrt{P}}{R},\frac{\sqrt{P}}{N}\right\}\right)
\end{aligned}$$

uniformly in i and t by Assumption SM1 and Lemma B.5. The second term can be bounded in a similar manner to before:

$$\sup_t\left(\left\|\frac{1}{t}\sum_{k=1}^{t}(\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)}F_k)\varepsilon_k\right\|\right)\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right] - \mathrm{E}\Big[\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\right)$$

$$
\begin{aligned}
\leq & \left(\frac{T}{R}\right)\sup_t\left(\frac{1}{T}\sum_{k=1}^{t}\varepsilon_k^2\right)^{\frac{1}{2}}\sup_t\left(\frac{1}{T}\sum_{k=1}^{t}\left\|(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k)\right\|^2\right)^{\frac{1}{2}} \\
& \times\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]-\mathrm{E}\Big[\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\right) \\
= & \; O\left(1\right)O_p\left(\max\left\{\frac{1}{R},\frac{1}{N}\right\}\right)O_p(1)=O_p\left(\max\left\{\frac{1}{R},\frac{1}{N}\right\}\right),
\end{aligned}
$$

where the last line holds for all i and t. Using similar arguments to the above and noting that $\widehat{H}^{(i,t)\prime-1}\beta$ is uniformly bounded in i and t, we obtain:

$$
\frac{1}{t\sqrt{P}}\sum_{t=R+1}^{T}\sum_{j=1}^{t}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\widehat{F}_j^{(i,t)}\left(\widehat{H}^{(i,t)}F_j-\widehat{F}_j^{(i,t)}\right)'\widehat{H}^{(i,t)\prime-1}\beta=O_p\left(\max\left\{\frac{\sqrt{P}}{R},\frac{\sqrt{P}}{N}\right\}\right).
$$

Finally, we address the first term of Equation (S-33), namely:

$$
\frac{1}{t\sqrt{P}}\sum_{t=R+1}^{T}\sum_{j=1}^{t}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\widehat{H}^{(i,t)}F_j\varepsilon_j. \tag{S-34}
$$

But this expression can again be decomposed by:

$$
\begin{aligned}
& \mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\sqrt{P}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\frac{1}{t}\sum_{k=1}^{t}F_k\varepsilon_{ik}\right)\right) \\
& +\sqrt{P}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]-\mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\right)\left(\frac{1}{t}\sum_{k=1}^{t}F_k\varepsilon_{ik}\right)\right),
\end{aligned} \tag{S-35}
$$

where $\widehat{H}^{(i,t)}$ has been omitted since it is of order $O_p(1)$ uniformly in i and t. The second term will follow by the same arguments as the first term after noting that it can be bounded by:

$$
\left(\frac{T}{R}\right)\left(\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]-\mathrm{E}\Big[\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\Big]\right)\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\frac{1}{T}\sum_{k=1}^{t}F_k\varepsilon_{ik}\right)^{\frac{1}{2}},
$$

where the first term is of order $O(1)$, the second part of order $O_p(1)$ uniformly in i and t using again arguments from West (1996). Turning to the first term of Equation (S-35), we note that $F_k\varepsilon_{ik}$ is a mixingale for all i, $1\leq l\leq\kappa$, satisfying $\mathrm{E}\left[\sup_{1\leq t\leq T}|(F_1\varepsilon_{i,1}+\ldots+\Delta L_t\varepsilon_{i,t})^2|\right]\leq CT$ (see Hall and Heyde, 1980). Thus, for all i, note that:

$$
\begin{aligned}
& P\,\mathrm{E}\left[\sup_{(R+1)\leq t\leq T}\left|(T)^{-2}(F_1\varepsilon_{i,1}+\ldots+F_t\varepsilon_{i,t})^2\right|\right] \\
=& P(t)^{-2}\,\mathrm{E}\left[\sup_{(R+1)\leq t\leq T}\left|(F_1\varepsilon_{i,1}+\ldots+F_t\varepsilon_{i,t})^2\right|\right]
\end{aligned}
$$

$$\leq P(R)^{-2}\,\mathrm{E}\left[\sup_{1\leq t\leq T}\left|(F_1\varepsilon_{i,1}+\ldots+F_t\varepsilon_{i,t})^2\right|\right]$$
$$\leq P(R)^{-2}CT,$$

which converges to zero at rate $O(P(R)^{-1})$ uniformly in i. By Markov's inequality and Assumption 4, this implies that Equation (S-34) is of order $O_p\left(\frac{P}{R}\right)$ uniformly in i and t. This completes the proof. Turing again to Equation (S-30), it therefore follows that:

$$\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{\beta}_{it}-\widehat{H}^{(i,t)\prime-1}\beta\right)=O_p\left(\max\left\{\frac{P}{R},\frac{\sqrt{P}}{N}\right\}\right)$$

uniformly in i and t.

Thus, putting all pieces together, it follows from Assumption 4 that:

$$\Pr\left(\max_{1\leq l\leq\kappa}\sqrt{P}\left|\frac{1}{P}\sum_{t=R+1}^{T}\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right|>\zeta_{P1}\right)\leq\frac{C\max\left\{\frac{P}{R},\frac{\sqrt{P}}{R}\right\}}{\zeta_{P1}}\leq C_1P^{-c_2}$$

and hence the claim of Lemma C.1 follows.

■

Proof of Lemma C.2. By Markov's inequality, we can bound the probability by:

$$\Pr\left(\max_{i,k\in\mathcal{C}_S}\left(\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})\right)^2\right)^{\frac{1}{2}}>\zeta_{P1}\right)$$
$$\leq\frac{\mathrm{E}\left[\max_{i,k\in\mathcal{C}_S}\left(\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right)\right)^2\right)^{\frac{1}{2}}\right]}{\zeta_{P1}}. \tag{S-36}$$

As in the proof of Lemma C.1, we will start with a mean value expansion of the term inside the expectation around the 'pseudo-true' factors and the other population parameters, and then then establish convergence rates in probability uniformly over i,k and t, which in turn implies uniformity in $l=1,\ldots,\kappa$. Thus, a mean value expansion of the term inside the expectation around $F^{(i,t)}\widehat{H}^{(i,t)}$ and $\widehat{H}^{(i,t)\prime-1}\beta$ (ignoring again terms with k) yields:

$$\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right)^2$$
$$=\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}F_t^{(i,t)}\right\|^2 \tag{S-37}$$
$$+\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\widehat{\beta}_{it}-\widehat{H}^{(i,t)\prime-1}\beta\right\|^2$$

$$-\frac{2}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]'\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right)\left(\widehat{\beta}_{it}-\widehat{H}^{(i,t)\prime-1}\beta\right)'\left(\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}F_j^{(i,t)}\right),$$

where the rate of convergence of the cross-product will follow once the convergence rate of the first two terms has been established. Noting that as in the proof of Lemma C.1, $\|\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}F_t^{(i,t)}\| = \|\widehat{F}_t^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_t^{(i,t)}\|+\|\widehat{H}^{(i,t)}\widetilde{F}_t^{(i,t)}-\widehat{H}^{(i,t)}F_t^{(i,t)}\|$, we can insert the representation of Equation (S-15) into the first part of the RHS of Equation (S-37) to obtain:

$$\begin{aligned}
&\frac{1}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\left(\widehat{F}_j^{(i,t)}-\widehat{H}^{(i,t)}\widetilde{F}_j^{(i,t)}\right)\\
=\;&\frac{1}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\widehat{V}^{(i,t)-1}\right\|^2\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kt}\right.\\
&\left.+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kt}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kt}+\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kt}\right\|^2 \qquad \text{(S-38)}\\
\leq\;&\frac{4}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\widehat{V}^{(i,t)-1}\right\|^2\left(\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\gamma_{kt}\right\|^2\right.\\
&\left.+\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kt}\right\|^2+\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kt}\right\|^2+\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\xi_{kt}\right\|^2\right)\\
=\;&\frac{4}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\widehat{V}^{(i,t)-1}\right\|^2\Bigg(I_{1t}+I_{2t}+I_t+I_{4t}\Bigg),
\end{aligned}$$

where the inequality follows since $(w+x+y+z)^2\leq 4(w^2+x^2+y^2+z^2)$. Since $V^{(i,t)-1}=O_p(1)$ uniformly in i and t by Lemma B.1, which in turn implies that $\|V^{(i,t)-1}\|^2=O_p(1)$ uniformly in i and t. Similar to before, we will therefore ignore the presence of $V^{(i,t)-1}$ in the following as it can just be replaced by a constant C. Starting with the term involving I_{1t}, note that this term can be bounded by:

$$\sup_t\left(\frac{1}{t}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)}\right\|^2\right)\left(\frac{4}{tm_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\sum_{k=1}^{t}\gamma_{kt}^2\right).$$

The first term is of order $O_p(1)$ uniformly in i and t since for every i and t, we impose the normalization $\widehat{F}^{(i,t)\prime}\widehat{F}^{(i,t)}/t=I_r$, where I_r denotes the $r\times r$ identity matrix. For the second term note that:

$$\begin{aligned}
&\mathrm{E}\left[\frac{4}{tm_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\sum_{j=1}^{t}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\gamma_{jt}^2\right]\\
\leq\;&\frac{4}{m_P q_P R}\sum_{h=1}^{m_P}\sum_{t\in I_h}\sum_{j=1}^{t}\left(\mathrm{E}\left[\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\right]\right)\gamma_{jt}^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{T}{R}\right)\left(\sup_t \mathrm{E}\left[\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\right]\right)\left(\frac{4}{m_P q_P T}\sum_{h=1}^{m_P}\sum_{t\in I_h}\sum_{j=1}^{t}\gamma_{jt}^2\right)\\
&= O(1)O((R)^{-1})
\end{aligned}
$$

uniformly in i and t, where the last equality follows from Assumptions SM1 and the fact that $(T/R) = O(1)$. By Markov's inequality, the term involving I_{1t} is therefore of order $O_p(R^{-1})$ uniformly in i and t.

Next we turn to the term involving I_{2t}, which can be bounded by:

$$
\frac{4}{m_P q_P}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^4\right)^{\frac{1}{2}}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kt}\right\|^4\right)^{\frac{1}{2}}.
$$

The first term is of order $O_p(1)$ for all i and t by Assumption SM1, while the second term can be bounded as follows:

$$
\begin{aligned}
&\frac{1}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\zeta_{kt}\right\|^4\\
&= \frac{1}{m_P q_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\frac{1}{(t)^2}\sum_{k=1}^{t}\sum_{k^*=1}^{t}\widehat{F}_k^{(i,t)\prime}\widehat{F}_{k^*}^{(i,t)}\zeta_{kt}\zeta_{k^*t}\right)^2 \qquad \text{(S-39)}\\
&\leq \left(\frac{T}{R}\right)\frac{1}{m_P q_P}\left(\frac{1}{t}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)}\right\|^2\right)^2\left(\frac{1}{(T)^4}\sum_{k=1}^{t}\sum_{k^*=1}^{t}\sum_{j=1}^{t}\sum_{j^*=1}^{t}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\zeta_{kt}\zeta_{k^*t}\zeta_{jt}\zeta_{j^*t}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

As in Bai and Ng (2002), note that:

$$
\mathrm{E}\left[\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\zeta_{kt}\zeta_{k^*t}\zeta_{jt}\zeta_{j^*t}\right)^2\right] \leq \Big(m_P q_P\Big)^2 \max_{s,t}\mathrm{E}\Big[|\zeta_{st}|^8\Big]
$$

and, by Assumption SM1:

$$
\mathrm{E}\Big[|\zeta_{st}|^8\Big] = \mathrm{E}\left[\frac{1}{N^4}\left|N^{-\frac{1}{2}}\sum_{i=1}^{N}\Big(e_{is}e_{it}-\mathrm{E}\Big[e_{is}e_{it}\Big]\Big)\right|^8\right] \leq N^{-4}C.
$$

The last term of Equation (S-39) is therefore of order $O_p\left(N^{-2}\right)$ uniformly in i and t, and so the term involving I_{2t} is of order $O_p\left(N^{-2}\right)$ uniformly in i and t.

By the same arguments as above, the term involving I_t is bounded by:

$$
\begin{aligned}
&\frac{4}{m_P q_P}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^4\right)^{\frac{1}{2}}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\eta_{kt}\right\|^4\right)^{\frac{1}{2}}\\
&= \frac{4}{m_P q_P}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_F\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^4\right)^{\frac{1}{2}}\left(\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}F_k'\frac{\Lambda' u_t}{N}\right\|^4\right)^{\frac{1}{2}} \qquad \text{(S-40)}
\end{aligned}
$$

As in Bai and Ng (2002), we can bound the second term by:

$$\begin{aligned}
& \left(\frac{1}{t}\sum_{k=1}^{t}\left\|\widehat{F}_k^{(i,t)}\right\|^2\right)^2\left(\frac{1}{t}\sum_{k=1}^{t}\left\|F_k\right\|^2\right)^2\left(\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{\Lambda' u_t}{N}\right\|^4\right) \\
= \quad & O_p(1)O_p(1)\left(\frac{1}{m_Pq_PN^2}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\frac{\Lambda' u_t}{\sqrt{N}}\right\|^4\right) \\
= \quad & O_p\left(\frac{1}{N^2}\right)
\end{aligned}$$

uniformly in i and t by Assumption SM1. Since the first term of Equation(S-40) is of order $O_p(1)$ uniformly in i and t, the expression involving I_t is of order $O_p(N^{-2})$ uniformly in i and t. The last expression involving I_{4t} can be treated in an analogous manner and is therefore of order $O_p(N^{-2})$ for all i and t. Finally, the term involving $\|\widehat{H}^{(i,t)}\widetilde{F}_t^{(i,t)} - \widehat{H}^{(i,t)}F_t^{(i,t)}\|$ can be treated as $\mathcal{A}_{2T}$ in the proof of Lemma C.1 to obtain that the first part on the RHS of Equation (S-37) is $O_p(\max\{R^{-1}, N^{-1}\})$ uniformly in i and t.

Next, we address the second term on the RHS of Equation (S-37). Using the representation of Equation (S-32), we obtain:

$$\begin{aligned}
& \frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left(\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{H}^{(i,t)}F_k\varepsilon_k\right\|^2+\left\|\frac{1}{t}\sum_{k=1}^{t}(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k)\varepsilon_k\right\|^2\right. \\
& \left.+\left\|\frac{1}{t}\sum_{k=1}^{t}\widehat{F}_k^{(i,t)}\left(\widehat{H}^{(i,t)}F_k-\widehat{F}_k^{(i,t)}\right)'\widehat{H}^{(i,t)\prime-1}\beta\right\|^2\right), \qquad \text{(S-41)}
\end{aligned}$$

where the cross-products have again been neglected since they will follow once the convergence rates of the squared terms have been established. Starting with the first term and omitting again $\widehat{H}^{(i,t)}$, note that this expression is bounded by:

$$\frac{C}{R}\left(\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^4\right)^{\frac{1}{2}}\left(\frac{1}{m_Pq_P(t)^2}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left(\sum_{k=1}^{t}\|F_k\|^2\right)^2\left(\sum_{k=1}^{t}\varepsilon_k^2\right)^2\right)^{\frac{1}{2}}.$$

The first term is of order $O_p(1)$ uniformly in i and t by Assumption SM1 and Markov's inequality, and the second one is also of order $O_p(1)$ uniformly in i and t by SM1. The entire expression is therefore of order $O_p(R^{-1})$ uniformly in i and t.

The second term on the RHS of Equation (S-41) can be bounded as follows:

$$\begin{aligned}
& \frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left\|\frac{1}{t}\sum_{k=1}^{t}(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k)\varepsilon_k\right\|^2 \\
\leq \quad & \left(\frac{T}{R}\right)\frac{1}{m_Pq_P}\sum_{h=1}^{m_P}\sum_{t\in I_h}\left\|\nabla_\beta\left[\Delta L_t(\overline{\theta}_{i+k},\overline{\theta}_i)\right]\right\|^2\left(\frac{1}{T}\sum_{k=1}^{t}\left\|(\widehat{F}_k^{(i,t)}-\widehat{H}^{(i,t)}F_k)\right\|^2\right)\left(\frac{1}{T}\sum_{k=1}^{t}\varepsilon_k^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sup_t \left(\frac{1}{T} \sum_{k=1}^{t} \left\| (\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)} F_k) \right\|^2 \right) \sup_t \left(\frac{1}{T} \sum_{k=1}^{t} \varepsilon_k^2 \right) \frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{t \in I_h} \left\| \nabla_\beta \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \right\|^2 \\
&= O_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right) O_p(1) \, O_p(1),
\end{aligned}
$$

where the first part of the last equality follows from Lemma B.5, the second and the last part from Assumption SM1 (all terms hold uniformly in i and t).

For the last term on the RHS of Equation (S-41), noting that $\widehat{H}^{(i,t)\prime-1}\beta$ is uniformly bounded in i and t, we can derive the following bound:

$$
\begin{aligned}
&\frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{t \in I_h} \left\| \nabla_\beta \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \right\|^2 \left\| \frac{1}{t} \sum_{k=1}^{t} \widehat{F}_k^{(i,t)} \left(\widehat{H}^{(i,t)} F_k - \widehat{F}_k^{(i,t)} \right)' \widehat{H}^{(i,t)\prime-1} \beta \right\|^2 \\
\leq\; & \frac{C}{m_P q_P} \sum_{h=1}^{m_P} \sum_{t \in I_h} \left\| \nabla_\beta \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \right\|^2 \left(\frac{1}{T} \sum_{k=1}^{t} \left\| (\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)} F_k) \right\|^2 \right) \left(\frac{1}{t} \sum_{k=1}^{t} \left\| \widehat{F}_k^{(i,t)} \right\|^2 \right) \\
\leq\; & \sup_t \left(\frac{1}{T} \sum_{k=1}^{t} \left\| (\widehat{F}_k^{(i,t)} - \widehat{H}^{(i,t)} F_k) \right\|^2 \right) \sup_t \left(\frac{1}{t} \sum_{k=1}^{t} \left\| \widehat{F}_k^{(i,t)} \right\|^2 \right) \frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{t \in I_h} \left\| \nabla_\beta \left[\Delta L_t(\overline{\theta}_{i+k}, \overline{\theta}_i) \right] \right\|^2 \\
=\; & O_p \left(\max \left\{ \frac{1}{R}, \frac{1}{N} \right\} \right) O_p(1) \, O_p(1),
\end{aligned}
$$

Turning back to the inequality in Equation (S-36), the conclusion of the lemma follows by Assumption 4 and the same argument as in the proof of Lemma C.1. ■

Proof of Lemma C.3. The proof of this Lemma follows by similar arguments as the proof of Theorem 4.3 of Chernozhukov et al. (2014b). That is, to establish the first claim, note that for any $x \in \mathbb{R}$:

$$
\begin{aligned}
\Pr{}^*\left(W^{BMB} \leq x \right) \;\leq\;& \Pr{}^*\left(\overline{W}^{BMB} \leq x + \zeta''_{P1} \right) \\
&+ \Pr{}^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right),
\end{aligned} \tag{S-42}
$$

where ζ''_{P1} was defined as $\zeta''_{P1} = \zeta_{P1}^{\frac{1}{2}} \log^{\frac{1}{2}} \kappa$ and the second part of the RHS follows since:

$$
\begin{aligned}
&\Pr{}^*\left(\left\{ \overline{W}^{BMB} + \left(W^{BMB} - \overline{W}^{BMB} \right) \leq x \right\} \cap \left\{ \left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right\} \right) \\
\leq\; & \Pr{}^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right).
\end{aligned}
$$

The RHS of Equation (S-42) can be further bounded by:

$$
\Pr\left(\max_{1 \leq l \leq \kappa} Y_l \leq x + \zeta''_{P1} \right) + \rho_P + \Pr{}^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right),
$$

where ρ_P is defined in Equation (S-46) below. Moreover, by the same arguments as in the proof of Theorem 4.3, for any $\gamma \in (0, 1 - 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa)$ and sufficiently large P one can show that:

$$\Pr\left(\max_{1 \le l \le \kappa} Y_l \le c_0(\gamma + 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa) + \zeta''_{P1} \right) \le 1 - \gamma$$

and thus:

$$c_0(\gamma + 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa) + \zeta''_{P1} \le c_0(\gamma), \tag{S-43}$$

which follows since $\max_{1 \le l \le \kappa} Y_l$ has no point masses. Now, setting $x = c_0(\alpha + CP^{-c} + C'P^{-c'} + 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa)$ in Equation (S-42), it holds that:

$$\begin{aligned}
&\Pr^*\left(W^{BMB} \le c_0(\alpha + CP^{-c} + C'P^{-c'} + 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa) \right) \\
\le\ & 1 - \alpha - CP^{-c} - C'P^{-c'} + \rho_P + \Pr^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right) \\
\le\ & 1 - \alpha
\end{aligned}$$

on the events $\rho_P < CP^{-c}$ and $\Pr^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right) \le C'P^{-c'}$, which hold with probability $1 - CP^{-c}$ by Equations (S-46) and (S-48). Since $CP^{-c} + 8\zeta''_{P1} \log^{\frac{1}{2}} \kappa \le C'P^{-c'}$, this implies the first claim. The second claim follows by analogous arguments starting with:

$$\begin{aligned}
\Pr^*\left(W^{BMB} \le x \right) \ \ge\ & \Pr^*\left(\overline{W}^{BMB} \le x - \zeta''_{P1} \right) \\
& - \Pr^*\left(\left| W^{BMB} - \overline{W}^{BMB} \right| > \zeta''_{P1} \right),
\end{aligned}$$

and mirroring the steps from before. ■

S6 Proofs of Theorem 1

Before stating the proof of Theorem 1, we define various quantities:

$$\overline{U} \equiv \max_{i,k \in \mathcal{C}_S} \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \left(\Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \mathrm{E}\Big[\Delta L_t(\theta_{i+k}, \theta_i)\Big] \right)$$

and

$$U_0 \equiv \max_{i,k \in \mathcal{C}_S} \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \left(\Delta L_t(\theta_{i+k}, \theta_i) - \mathrm{E}\Big[\Delta L_t(\theta_{i+k}, \theta_i)\Big] \right).$$

Moreover, let:

$$W_{BMB} \equiv \max_{i,k \in \mathcal{C}_S} \left(\frac{1}{\sqrt{m_P q_P}} \sum_{h=1}^{m_P} \epsilon_h \sum_{j \in I_h} \left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \frac{1}{P} \sum_{t=R+1}^{T} \Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) \right) \right),$$

and

$$\overline{W}_{BMB} \equiv \max_{i,k\in\mathcal{C}_S}\left(\frac{1}{\sqrt{m_P q_P}}\sum_{h=1}^{m_P}\epsilon_h\sum_{j\in I_h}\left(\Delta L_j(\theta_{i+k},\theta_i)-\frac{1}{P}\sum_{t=R+1}^{T}\Delta L_t(\theta_{i+k},\theta_i)\right)\right),$$

and

$$W^0_{BMB} \equiv \max_{i,k\in\mathcal{C}_S}\left(\frac{1}{\sqrt{m_P q_P}}\sum_{h=1}^{m_P}\epsilon_h\sum_{j\in I_h}\left(\Delta L_j(\theta_{i+k},\theta_i)-\mathrm{E}\Big[\Delta L_t(\theta_{i+k},\theta_i)\Big]\right)\right),$$

where $\epsilon_1,\ldots,\epsilon_{m_P}$ are standard normal random variables independent of the quarterly data $\{\Delta L_t(\cdot)\}_{t=1}^{T}$. In Theorem S7 of Section S7 below, we show that under the assumptions of Theorem 1, the results of Theorem 7.1 of Chernozhukov et al. (2014b) continue to hold, namely:

$$\rho^U_P \equiv \sup_{x\in\mathbb{R}}\left|\Pr\Big(U_0\le x\Big)-\Pr\Big(\max_{1\le l\le\kappa}Y_l\le x\Big)\right|\le CP^{-c}, \tag{S-44}$$

and

$$\rho^W_P \equiv \sup_{x\in\mathbb{R}}\left|\Pr^*\Big(W^0_{BMB}\le x\Big)-\Pr\Big(\max_{1\le l\le\kappa}Y_l\le x\Big)\right|\le C'P^{-c'}, \tag{S-45}$$

where $Y=(Y_1,\ldots,Y_\kappa)'$ is a centered normal random vector with covariance matrix:

$$\mathrm{E}\Big[YY'\Big]=(1/m_Pq_P)\sum_{h=1}^{m_P}\mathrm{E}\Big[\Big(\sum_{t\in I_h}\boldsymbol{\Delta L}_t(\boldsymbol{\theta})-\mathrm{E}\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big)\Big(\sum_{t\in I_h}\boldsymbol{\Delta L}_t(\boldsymbol{\theta})-\mathrm{E}\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big)'\Big],$$

Here, $\boldsymbol{\Delta L}_t(\boldsymbol{\theta})$ denotes a stacked vector in $\mathbb{R}^\kappa$.

Proof of Theorem 1. First, note that under H_0, it holds that:

$$U^*=\max_{i,k\in\mathcal{C}_S}\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})\le\max_{i,k\in\mathcal{C}_S}\frac{1}{\sqrt{P}}\sum_{t=R+1}^{T}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\mathrm{E}\Big[\Delta L_t(\theta_{i+k},\theta_i)\Big]\right)=\overline{U},$$

where the equality holds when $\mathrm{E}\Big[\Delta L_t(\theta_{i+k},\theta_i)\Big]=0$ for all $i,k\in\mathcal{C}_S$. Moreover, invoking Assumption 4 and in particular $(r_P/q_P)\log^2\kappa\le C_1P^{-c_2}$ and $q_P\log^{\frac{5}{2}}(\kappa P)\le C_1P^{\frac{1}{2}-c_2}$ thereof, and using the same steps as in the proof of Theorem 7.1 in Chernozhukov et al. (2014b), the following result can be verified conditional on the data:

$$\Pr^*\left(\left|\overline{W}_{BMB}-W^0_{BMB}\right|>\zeta'_{P1}\right)\le CP^{-c'},$$

where $\zeta'_{P1}=C'P^{-\frac{1}{4}+\frac{3c_2}{4}}\log\kappa$. Thus, from ρ^W_P in Equation (S-45) it follows that:

$$\sup_{x\in\mathbb{R}}\left|\Pr^*\Big(\overline{W}_{BMB}\le x\Big)-\Pr\Big(\max_{1\le l\le\kappa}Y_l\le x\Big)\right|\le C'P^{-c'} \tag{S-46}$$

with probability larger than $1 - CP^{-c}$. Likewise, from Equation (S-44) we know that $\rho_P^U \leq CP^{-c}$. Now, to prove Theorem 1, we proceed in two steps: first, we establish that:

$$\Pr\left(\left|\overline{U} - U_0\right| > \zeta_{P1}\right) \leq CP^{-c} \tag{S-47}$$

and

$$\Pr\left(\Pr^*\left(\left|W^{BMB} - \overline{W}^{BMB}\right| > \zeta''_{P1}\right) > C'P^{-c'}\right) \leq C'P^{-c'}, \tag{S-48}$$

which deal with the estimation error in the test and the bootstrap statistic, respectively. Then, in a second step, we verify the first (only some moment inequalities are binding) and the second (all moment inequalities are binding) claim of Theorem 1.

We start with the expression in (S-47). Using Lemma C.1, it follows that:

$$\left|\overline{U} - U_0\right| \leq \max_{i,k \in \mathcal{C}_S} \left|\sqrt{P}\left(\frac{1}{P}\sum_{t=R+1}^{T} \Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \Delta L_t(\theta_{i+k}, \theta_i)\right)\right| \leq \zeta_{P1}$$

with probability $1 - CP^{-c}$. Next, we turn to Equation (S-48). Start again with:

$$\begin{aligned}
\left|W^{BMB} - \overline{W}^{BMB}\right| \quad &\leq \quad \max_{i,k \in \mathcal{C}_S} \left|\frac{1}{\sqrt{m_P q_P}} \sum_{h=1}^{m_P} \epsilon_h \sum_{j \in I_h} \left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \Delta L_j(\theta_{i+k}, \theta_i)\right)\right. \\
&\qquad \left. - \frac{1}{P}\sum_{t=R+1}^{T} \left(\Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \Delta L_t(\theta_{i+k}, \theta_i)\right)\right|.
\end{aligned}$$

Conditional on $\{\Delta L_t(\cdot)\}_{t=1}^T$, the vector:

$$\frac{1}{\sqrt{m_P q_P}} \sum_{h=1}^{m_P} \epsilon_h \sum_{j \in I_h} \left(\left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \Delta L_j(\theta_{i+k}, \theta_i)\right) - \frac{1}{P}\sum_{t=R+1}^{T} \left(\Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \Delta L_t(\theta_{i+k}, \theta_i)\right)\right),$$

$\max_{i,k \in \mathcal{C}_S}$ is normal with mean zero and all diagonal elements of the covariance matrix bounded by:

$$\max_{i,k \in \mathcal{C}_S} \frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{j \in I_h} \left(\left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \Delta L_j(\theta_{i+k}, \theta_i)\right) - \frac{1}{P}\sum_{t=R+1}^{T} \left(\Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \Delta L_t(\theta_{i+k}, \theta_i)\right)\right)^2, \tag{S-49}$$

For the last expression, observe that:

$$\begin{aligned}
&\left(\frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{j \in I_h} \left(\left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \Delta L_j(\theta_{i+k}, \theta_i)\right) - \frac{1}{P}\sum_{t=R+1}^{T} \left(\Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{i,t}) - \Delta L_t(\theta_{i+k}, \theta_i)\right)\right)^2\right)^{\frac{1}{2}} \\
\leq \quad &\left(\frac{1}{m_P q_P} \sum_{h=1}^{m_P} \sum_{j \in I_h} \left(\Delta L_j(\widehat{\theta}_{i+k,j}, \widehat{\theta}_{i,j}) - \Delta L_j(\theta_{i+k}, \theta_i)\right)^2\right)^{\frac{1}{2}}
\end{aligned} \tag{S-50}$$

$$+\left|\frac{1}{P}\sum_{t=R+1}^{T}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right)\right|$$

where the inequality follows by an application of Minkowski's inequality and the fact that:

$$\begin{aligned}&\frac{1}{m_P q_P}\sum_{h=1}^{m_P}\sum_{j\in I_h}\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right)\right)^2\\ =\;&\left(\frac{1}{P}\sum_{t=R+1}^{T}\left(\Delta L_t(\widehat{\theta}_{i+k,t},\widehat{\theta}_{i,t})-\Delta L_t(\theta_{i+k},\theta_i)\right)\right)^2.\end{aligned}$$

The first term on the RHS of (S-50) is bounded by ζ_{P1} with probability larger than $1-CP^{-c}$ by Lemma C.2. The second term on the RHS of (S-50) is also bounded by ζ_{P1} with probability larger than $1-CP^{-c}$ by Lemma C.1. Hence Equation (S-49) is bounded by $C\zeta_{P1}^2$ with probability larger than $1-CP^{-c}$. Then, by Markov's inequality (conditional on the data):

$$\Pr{}^*\left(\left|W^{BMB}-\overline{W}^{BMB}\right|>\zeta_{P1}^{\frac{1}{2}}\log^{\frac{1}{2}}\kappa\right)\leq\frac{C\zeta_{P1}\log^{\frac{1}{2}}\kappa}{\zeta_{P1}^{\frac{1}{2}}\log^{\frac{1}{2}}\kappa}\leq CP^{-c},$$

which follows from Proposition 1.1.3 in Talagrand (2003) and the rate condition in Assumption 4.[1] The claim from Equation (S-48) follows by setting $\zeta''_{P1}=\zeta_{P1}^{\frac{1}{2}}\log^{\frac{1}{2}}\kappa$.

We now prove the claims of Theorem 1 starting with the case where $\mathrm{E}\left[\Delta L_t(\theta_{i+k},\theta_i)\right]<0$ for at least some $i,k\in\mathcal{C}_S$. Under H_0 it holds that:

$$\begin{aligned}\Pr\left(U^*>c^{BMB}(\alpha)\right)&\leq\Pr\left(\overline{U}>c^{BMB}(\alpha)\right)\\ &\leq\Pr\left(U_0>c^{BMB}(\alpha)-\zeta_{P1}\right)+\Pr\left(\left|\overline{U}-U_0\right|>\zeta_{P1}\right)\\ &\leq\Pr\left(U_0>c_0(\alpha+CP^{-c}+8\zeta''_{P1}\log^{\frac{1}{2}}\kappa)-\zeta_{P1}\right)+C'P^{-c'}\\ &\leq\Pr\left(U_0>c_0(\alpha+CP^{-c}+16\zeta''_{P1}\log^{\frac{1}{2}}\kappa)\right)+C'P^{-c'}\\ &\leq\Pr\left(\max_{1\leq l\leq\kappa}Y_l>c_0(\alpha+CP^{-c}+16\zeta''_{P1}\log^{\frac{1}{2}}\kappa)\right)+\rho_P^U+C'P^{-c'}\\ &=\alpha+CP^{-c}+16\zeta''_{P1}\log^{\frac{1}{2}}\kappa+\rho_P^U+C'P^{-c'}\\ &\leq\alpha+CP^{-c},\end{aligned}$$

[1] Proposition 1.1.3 in Talagrand (2003): For centered normal random variables ξ_l, $l=1,\ldots,\kappa$ with $\max_{1\leq l\leq\kappa}\mathrm{E}[\xi_l^2]<\infty$, it holds that: $\mathrm{E}\left[\max_{1\leq l\leq\kappa}\xi_l\right]\leq\sqrt{2\max_{1\leq l\leq\kappa}\mathrm{E}[\xi_l^2]\log\kappa}$.

where the second inequality holds since $0 < \zeta_{P1} \leq CP^{-c}$ and $\Pr\Big(\Big\{U_0 + \Big(\overline{U} - U_0\Big) > c^{BMB}(\alpha)\Big\} \cap \Big\{\overline{U} - U_0 > \zeta_{P1}\Big\}\Big) \leq \Pr\Big(\Big\{\Big|\overline{U} - U_0\Big| > \zeta_{P1}\Big\}\Big)$, the third inequality follows from Equation (S-47) and the first claim of Lemma C.3, the fourth inequality from Equation (S-43) in the proof of Lemma C.3, the fifth inequality from Equation (S-44), while the equality follows from the fact that $\max_{1\leq l\leq\kappa}$ has no point masses. This establishes the first claim of Theorem 1. For the second claim, suppose that $\mathrm{E}\Big[\Delta L_t(\theta_{i+k}, \theta_i)\Big)\Big] = 0$ for all $i, k \in \mathcal{C}_S$. Then, under H_0:

$$\begin{aligned}
\Pr\Bigg(U^* > c^{BMB}(\alpha)\Bigg) &= \Pr\Bigg(\overline{U} > c^{BMB}(\alpha)\Bigg) \\
&\geq \Pr\Bigg(U_0 > c^{BMB}(\alpha) + \zeta_{P1}\Bigg) - \Pr\Bigg(\Big|\overline{U} - U_0\Big| > \zeta_{P1}\Bigg) \\
&\geq \Pr\Bigg(U_0 > c_0(\alpha - CP^{-c} - 8\zeta''_{P1}\log^{\frac{1}{2}}\kappa) + \zeta_{P1}\Bigg) - C'P^{-c'} \\
&\geq \Pr\Bigg(U_0 > c_0(\alpha - CP^{-c} - 16\zeta''_{P1}\log^{\frac{1}{2}}\kappa)\Bigg) - C'P^{-c'} \\
&\geq \Pr\Bigg(\max_{1\leq l\leq\kappa} Y_l > c_0(\alpha - CP^{-c} - 16\zeta''_{P1}\log^{\frac{1}{2}}\kappa)\Bigg) - \rho_P^U - C'P^{-c'} \\
&= \alpha - CP^{-c} - 16\zeta''_{P1}\log^{\frac{1}{2}}\kappa - \rho_P^U - C'P^{-c'} \\
&\geq \alpha - CP^{-c},
\end{aligned}$$

where the first inequality holds again since $0 < \zeta_{P1} \leq CP^{-c}$ and $\Pr\Big(\Big\{U_0 + \Big(\overline{U} - U_0\Big) > c^{BMB}(\alpha)\Big\} \cap \Big\{\overline{U} - U_0 > \zeta_{P1}\Big\}\Big) \leq \Pr\Big(\Big\{\Big|\overline{U} - U_0\Big| > \zeta_{P1}\Big\}\Big)$, the second inequality follows from Equation (S-47) and the second claim of Lemma C.3, the third inequality from the reverse of Equation (S-43) in the proof of Lemma C.3, and the fourth inequality from Equation (S-44). Together with the previous set of inequalities, this establishes the second claim of Theorem 1. ■

S7 Extension to Unbounded R.V.'s

In this section, we show formally that the results of Theorem 7.1 (more specifically, Theorems B.1 and B.2 therein) of Chernozhukov et al. (2014b) remain valid for unbounded random variables, which satisfy the assumptions stated in the main text and Assumption SM2 below. Before stating Assumption SM2 and Theorem S1 below, we introduce the following notation from here onwards: we will use the more compact shorthand $\boldsymbol{\theta}_l = (\theta'_{i+k}, \theta'_i)'$, where l corresponds uniquely to a specific pair i, k from $\mathcal{C}_S$, and thus $\Delta L_t(\boldsymbol{\theta}_{lt}) = \Delta L_t(\theta_{i+k}, \theta_i)$. Likewise, $\widehat{\boldsymbol{\theta}}_{lt} = (\widehat{\theta}'_{i+k,t}, \widehat{\theta}'_{it})'$ with $\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt}) = \Delta L_t(\widehat{\theta}_{i+k,t}, \widehat{\theta}_{it})$ for every $i, k \in \mathcal{C}_S$ and $l = 1, \ldots, \kappa$. Finally, let $\boldsymbol{\Delta L}_t(\boldsymbol{\theta})$ denote a stacked vector in $\mathbb{R}^\kappa$. We write:

$$S_v(\boldsymbol{\theta}) = \sum_{t\in I_v}\Big(\boldsymbol{\Delta L}_t(\boldsymbol{\theta}) - \mathrm{E}\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big) \qquad S'_v(\boldsymbol{\theta}) = \sum_{t\in J_v}\Big(\boldsymbol{\Delta L}_t(\boldsymbol{\theta}) - \mathrm{E}\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big),$$

where I_v and J_v denote sets of observations from large and small blocks, respectively. Next, let $\{\widetilde{S}_v(\boldsymbol{\theta})\}_{v=1}^m$ and $\{\widetilde{S}'_v(\boldsymbol{\theta})\}_{v=1}^m$ be two independent sequences of random vectors in $\mathbb{R}^\kappa$ such that $\widetilde{S}_v(\boldsymbol{\theta}) \stackrel{d}{=} S_v(\boldsymbol{\theta})$ and $\widetilde{S}'_v(\boldsymbol{\theta}) \stackrel{d}{=} S'_v(\boldsymbol{\theta})$, where $\stackrel{d}{=}$ denotes equality in distribution. The small and large block sizes are q_P and r_P. Also:

$$\overline{\sigma}^2(q) \equiv \max_{1\leq l\leq\kappa} \max_{I_q} \operatorname{Var}\left(q_P^{-\frac{1}{2}} \sum_{t\in I_q} \Delta L_t(\boldsymbol{\theta}_l)\right),$$

$$\overline{\sigma}^2(r) \equiv \max_{1\leq l\leq\kappa} \max_{I_r} \operatorname{Var}\left(r_P^{-\frac{1}{2}} \sum_{t\in I_r} \Delta L_t(\boldsymbol{\theta}_l)\right)$$

as well as:

$$\underline{\sigma}^2(q) \equiv \min_{1\leq l\leq\kappa} \min_{I_q} \operatorname{Var}\left(q_P^{-\frac{1}{2}} \sum_{t\in I_q} \Delta L_t(\boldsymbol{\theta}_l)\right),$$

Finally, recall the definitions of U_0, $\overline{W}_{BMB}$, and W^0_{BMB} from Section S6 above.

Assumption SM2. *Assume that the following expressions are bounded as follows:*

$$\begin{aligned}
E\left[\max_{1\leq l\leq\kappa}\left|\widetilde{S}'_v(\boldsymbol{\theta}_l)\right|^2\right] &< r_P\overline{\sigma}^2(r)\\
E\left[\max_{1\leq l,k\leq\kappa}\left|\widetilde{S}_v(\boldsymbol{\theta}_l)\widetilde{S}_v(\boldsymbol{\theta}_k)\right|^2\right] &< Cq_P^4\, E\left[\max_{1\leq l\leq\kappa}\left|\Big(\Delta L_t(\boldsymbol{\theta}_l) - E\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\right|^4\right]\\
E\left[\left|\widetilde{S}_v(\boldsymbol{\theta}_l)\widetilde{S}_v(\boldsymbol{\theta}_k)\right|^2\right] &< Cq_P^3\overline{\sigma}^2(q)\, E\left[\max_{1\leq l\leq\kappa}\left|\Big(\Delta L_t(\boldsymbol{\theta}_l) - E\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\right|^2\right]
\end{aligned}$$

as well as:

$$E\left[\exp\left(\left|\widetilde{S}_v(\boldsymbol{\theta}_l)(m_pP^{-1})^{\frac{1}{2}}\right| \Big/ Cq_P^{\frac{1}{2}}\, E\left[\max_{1\leq l\leq\kappa}\left|\Big(\Delta L_t(\boldsymbol{\theta}_l) - E\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\right|\right]\right)\right] < 3.$$

$$\begin{aligned}
E\left[\left|\widetilde{S}_v(\boldsymbol{\theta}_l)(m_pP^{-1})^{\frac{1}{2}}\right|^3\right] &\leq Cq_P^{\frac{1}{2}}\, E\left[\max_{1\leq l\leq\kappa}\left|\Big(\Delta L_t(\boldsymbol{\theta}_l) - E\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\right|^3\right]\\
E\left[\left|\widetilde{S}_v(\boldsymbol{\theta}_l)(m_pP^{-1})^{\frac{1}{2}}\right|^4\right] &\leq Cq_P\, E\left[\max_{1\leq l\leq\kappa}\left|\Big(\Delta L_t(\boldsymbol{\theta}_l) - E\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\right|^4\right].
\end{aligned}$$

Finally, assume that:

$$c_1 \leq \underline{\sigma}^2(q) \leq \max\{\overline{\sigma}^2(q), \overline{\sigma}^2(r)\} \leq C_1$$

As noted in the paper, Assumption SM2 allows to relax the bounded support assumption for the case of dependent data, imposed by Chernozhukov et al. (2014b) for illustrative purposes. The following Theorem S1 therefore establishes formally that the results of Theorem 7.1 in their paper continue hold under the regularity set out in Theorem S7.

Theorem S1. *Under Assumptions 1 through 5 and the definitions of Theorem 1 in the paper, and Assumption SM2 above, the results of Theorem 7.1 in Chernozhukov et al. (2014b, p. 30) continue to hold, namely:*

$$\sup_{x\in\mathbb{R}}\left|Pr\bigg(U_0 \leq x\bigg) - Pr\bigg(\max_{1\leq l\leq\kappa} Y_l \leq x\bigg)\right| \leq CP^{-c}, \tag{S-51}$$

$$\sup_{x\in\mathbb{R}}\left|Pr^*\bigg(W^0_{BMB} \leq x\bigg) - Pr\bigg(\max_{1\leq l\leq\kappa} Y_l \leq x\bigg)\right| \leq C'P^{-c'}, \tag{S-52}$$

where $Y = (Y_1,\ldots,Y_\kappa)'$ *is a centered normal random vector with covariance matrix* $E\Big[YY'\Big] = (1/m_Pq_P)\sum_{h=1}^{m_P} E\Big[\Big(\sum_{t\in I_h}\Big(\boldsymbol{\Delta L}_t(\boldsymbol{\theta}) - E\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big)\Big)\Big(\sum_{t\in I_h}\Big(\boldsymbol{\Delta L}_t(\boldsymbol{\theta}) - E\Big[\boldsymbol{\Delta L}_t(\boldsymbol{\theta})\Big]\Big)\Big)'\Big]$ *(*$\boldsymbol{\Delta L}_t(\boldsymbol{\theta})$ *denotes a stacked vector in* $\mathbb{R}^\kappa$*).*

Proof of Theorem S1. The proof of Theorem 7.1 in Chernozhukov et al. (2014b) starts off by establishing (S-51) through Theorem B.1 (pp. 53-55) and continues to verify (S-52) through Theorem B.2 (p. 56). In the following, we therefore outline the changes in the proofs of Theorems B.1 and B.2, which are required for the conclusions in (S-51) and (S-52) to continue to hold. Since most arguments are identical to the ones used in Chernozhukov et al. (2014b) and only involve replacing certain bounds in a suitable manner, we just sketch the key differences.

We start with (S-51) and the changes in Theorem B.1: as in Chernozhukov et al. (2014b), the first step involves a reduction of $\sum_{t=R+1}^{T}\Big(\Delta L_t(\boldsymbol{\theta}_l) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)$ in U_0 into a sum of independent blocks $\sum_{v=1}^{m}\widetilde{S}_v(\boldsymbol{\theta}_l)$ by showing that

$$\begin{aligned}
& \mathrm{Pr}\bigg(\max_{1\leq l\leq\kappa}\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}_v(\boldsymbol{\theta}_l) \leq x - CP^{-c}\log^{-\frac{1}{2}}\kappa\bigg) - P^{-c} - 2(m_P-1)b_r \\
\leq\ & \mathrm{Pr}\bigg(U_0 \leq x\bigg) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{S-53}) \\
\leq\ & \mathrm{Pr}\bigg(\max_{1\leq l\leq\kappa}\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}_v(\boldsymbol{\theta}_l) \leq x + CP^{-c}\log^{-\frac{1}{2}}\kappa\bigg) + P^{-c} + 2(m_P-1)b_r.
\end{aligned}$$

Focusing on the second inequality only, first note that:

$$\sum_{t=R+1}^{T}\Big(\Delta L_t(\boldsymbol{\theta}_l) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) = \sum_{v=1}^{m}S_v(\boldsymbol{\theta}_l) + \sum_{v=1}^{m}S'_v(\boldsymbol{\theta}_l) + S'_{m+1}(\boldsymbol{\theta}_l),$$

and thus:

$$\left|\max_{1\leq l\leq\kappa}\sum_{t=R+1}^{T}\Big(\Delta L_t(\boldsymbol{\theta}_l) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) - \max_{1\leq l\leq\kappa}\sum_{v=1}^{m}S_v(\boldsymbol{\theta}_l)\right| \leq \left|\max_{1\leq l\leq\kappa}\sum_{v=1}^{m}S'_v(\boldsymbol{\theta}_l) + \max_{1\leq l\leq\kappa}S'_{m+1}(\boldsymbol{\theta}_l)\right|.$$

By Corollary 2.7 in Yu (1994), it holds that:

$$\sup_{x\in\mathbb{R}}\left|\mathrm{Pr}\bigg(\max_{1\leq l\leq\kappa}\sum_{v=1}^{m}S_v(\boldsymbol{\theta}_l) \leq x\bigg) - \mathrm{Pr}\bigg(\max_{1\leq l\leq\kappa}\sum_{v=1}^{m}\widetilde{S}_v(\boldsymbol{\theta}_l) \leq x\bigg)\right| \leq (m_P-1)b_{r_P}$$

and

$$\sup_{x\in\mathbb{R}}\left|\Pr\left(\max_{1\le l\le\kappa}\sum_{v=1}^{m}S'_v(\boldsymbol{\theta}_l)\le x\right)-\Pr\left(\max_{1\le l\le\kappa}\sum_{v=1}^{m}\widetilde{S}'_v(\boldsymbol{\theta}_l)\le x\right)\right|\le(m_P-1)b_{q_P},$$

where b_{r_P} and b_{q_P} are the mixing coefficients. Hence for every δ_1, $\delta_2>0$:

$$\begin{aligned}\Pr\left(U_0\le x\right) &\le \Pr\left(\max_{1\le l\le\kappa}\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}_v(\boldsymbol{\theta}_l)\le x+\delta_1+\delta_2\right)+\Pr\left(\max_{1\le l\le\kappa}\left|\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}'_v(\boldsymbol{\theta}_l)\right|\le\delta_1\right)\\ &\quad+\Pr\left(\max_{1\le l\le\kappa}\left|S'_{m+1}(\boldsymbol{\theta}_l)\right|\le\sqrt{P}\delta_2\right)+2(m_P-1)b_r\\ &= I+II+III+IV.\end{aligned}$$

The second inequality of (S-53) follows once we have established that II and III are of order P^{-c} and $\delta_1+\delta_2\le CP^{-c}\log^{-\frac{1}{2}}\kappa$. We start with II. By Markov's inequality, for every $\epsilon>0$ setting

$$\delta_1=\epsilon^{-1}\,\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}'_v(\boldsymbol{\theta}_l)\Big|\Big]$$

yields $II\le\epsilon$. The expectation $\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}'_v(\boldsymbol{\theta}_l)\Big|\Big]$ on the other hand can be bounded as follows: noting that $\widetilde{S}'_v(\boldsymbol{\theta}_l)$, $1\le v\le m$, are independent, we can invoke Assumptions 2 and SM2 to obtain $\Big(\max_{1\le l\le\kappa}\sum_{v=1}^{m}\mathrm{E}\Big[\Big|\widetilde{S}'_v(\boldsymbol{\theta}_l)\Big|^2\Big]\Big)^{\frac{1}{2}}\le m_P^{\frac{1}{2}}r_P^{\frac{1}{2}}\Big(\overline{\sigma}^2(r)\Big)^{\frac{1}{2}}$. By Lemma A.3 in Chernozhukov et al. (2014b, p. 34):

$$\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|\frac{1}{\sqrt{P}}\sum_{v=1}^{m}\widetilde{S}'_v(\boldsymbol{\theta}_l)\Big|\Big]\le C\Big(\Big(\frac{m_P}{P}\Big)^{\frac{1}{2}}r_P\sqrt{\overline{\sigma}^2(r)}\log^{\frac{1}{2}}\kappa+P^{-\frac{1}{2}}r_P\overline{\sigma}^2(r)\Big)\le CP^{-2c}\log^{-\frac{1}{2}}\kappa,$$

where the last inequality follows by the rate conditions in Assumption 4. Finally, setting $\epsilon=P^{-c}$ leads to the required result with $\delta_1\le CP^{-c}\log^{-\frac{1}{2}}\kappa$. Next, consider III. Applying again Markov's inequality with $\delta_2=\epsilon^{-1}(\sqrt{P})^{-1}\,\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|S'_{m+1}(\boldsymbol{\theta}_l)\Big|\Big]$, we have that $III\le\epsilon$. Moreover, note that:

$$\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|S'_{m+1}(\boldsymbol{\theta}_l)\Big|\Big]\le(q_P+r_P-1)C\,\mathrm{E}\Big[\max_{1\le l\le\kappa}\Big|\Delta L_t(\boldsymbol{\theta}_l)\Big|\Big],$$

where $\mathrm{E}\Big[\max_{1\le i\le\kappa}\Big|\Delta L_t(\boldsymbol{\theta}_l)\Big|\Big]<C'$ by Assumption 2. Thus, setting $\epsilon=P^{-c'}$ for some c' sufficiently small, it holds that $\delta_2\le CP^{-c}\log^{-\frac{1}{2}}\kappa$ by the rate conditions in Assumption 4 and the fact that $P=(q_P+r_P)m_P$ as well as $m_P\ge C^{-1}P^c$.

The next step, which requires modification, is the normal approximation of the sum of independent blocks from the first step. Chernozhukov et al. (2014b) verify the conditions of Corollary 2.1(i) in Chernozhukov et al. (2013) for independent random variables $\widetilde{S}_v(\boldsymbol{\theta}_l)$, $v=1,\ldots,m$. Note that as in their paper, it holds that:

$$c_1/4<\underline{\sigma}^2(q)/4<\mathrm{var}\Big(\widetilde{S}_v(\boldsymbol{\theta}_l)(m_pP^{-1})^{\frac{1}{2}}\Big)\le\overline{\sigma}^2(q)<C_1,$$

which follows since:

$$\frac{1}{\sqrt{P}} \sum_{v=1}^{m} \widetilde{S}_v(\boldsymbol{\theta}_l) = \frac{1}{\sqrt{m_P}} \sum_{v=1}^{m} \widetilde{S}_v(\boldsymbol{\theta}_l) \sqrt{\frac{m_P}{P}}$$

and $m_P = P/(q_P + r_P)$ so that $\sqrt{q_p} \leq \sqrt{m_P/P} \leq 2\sqrt{q_P}$. Now, setting $B_n = Cq_P^{\frac{1}{2}} \mathrm{E}\Big[\max_{1\leq l\leq\kappa}\Big|\Big(\Delta L_t(\boldsymbol{\theta}_l) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big)\Big|\Big]$ with C sufficiently large and invoking Assumptions 2 and SM2, the conditions of Corollary 2.1(i) can be straightforwardly verified noting that $\Big(Cq_P^{\frac{1}{2}} \mathrm{E}\Big[\max_{1\leq l\leq\kappa}\Big|\Delta L_t(\boldsymbol{\theta}_l)\Big|\Big]\Big)^2 \log^7(\kappa P)P^{-1} \leq C'P^{-c}$ as $q_P \leq P/(4q_P) \leq CP^{1-c}$. The remaining parts of the proof of Theorem B.1 are as in Chernozhukov et al. (2014b).

The validity of the Block Multiplier Bootstrap, i.e. the verification of (S-52), is established through Theorem B.2, which in turn relies on Theorem 2 from Chernozhukov et al. (2015) and again Corollary 2.7 of Yu (1994). The only difference w.r.t. their paper consists again in appropriately bounding the expression:

$$\mathrm{E}\Big[\max_{1\leq l,j\leq\kappa}\Big|\frac{1}{q_P m_P}\sum_{v=1}^{m} S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j) - \mathrm{E}\Big[S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j)\Big]\Big|\Big].$$

By Assumptions 4 and SM2, we have that:

$$\Bigg(\max_{1\leq l,j\leq\kappa}\sum_{v=1}^{m}\mathrm{E}\Big[\Big|S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j)\Big|^2\Big]\Bigg)^{\frac{1}{2}} \leq Cm_P^{\frac{1}{2}}q_P^{\frac{3}{2}}\overline{\sigma}(q)$$

and

$$\mathrm{E}\Big[\Big(\max_{1\leq l,j\leq\kappa}\Big|S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j)\Big|\Big)^2\Big] \leq Cq_P^4.$$

Hence, by Lemma A.3 in Chernozhukov et al. (2014b), it follows that:

$$\begin{aligned}
\mathrm{E}\Big[\max_{1\leq l,j\leq\kappa}\Big|\frac{1}{q_P m_P}\sum_{v=1}^{m} S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j) - \mathrm{E}\Big[S_v(\boldsymbol{\theta}_l)S_v(\boldsymbol{\theta}_j)\Big]\Big|\Big] &\leq C'\Big(\frac{m_P^{\frac{1}{2}}q_P^{\frac{3}{2}}\overline{\sigma}(q)\log^{\frac{1}{2}}\kappa}{m_P q_P} + \frac{q_P^2\log\kappa}{m_P q_P}\Big) \\
&\leq C\Big(\frac{q_P\log^{\frac{1}{2}}\kappa}{P^{\frac{1}{2}}} + \frac{q_P^2\log\kappa}{P}\Big).
\end{aligned}$$

By Assumption 4, this is bounded by $C'P^{-c_2}\log^{-2}\kappa$.

■

S8 Proof of Theorem 2

Proof of Theorem 2. The proof follows the steps of the proof of Theorem 4.4 in Chernozhukov et al. (2014b) and adapts them to the case of dependent data. First note that from Lemma B.3 of the paper, we know that $c^{BMB}(\alpha) \geq c_0(\alpha + \varphi_P)$ and $c^{BMB}(\alpha) \leq c_0(\alpha - \varphi_P)$ with probability larger than $1 - CP^{-c}$ for some $0 < \varphi_P \leq CP^{-c}$. Moreover, recall the definition:

$$\Delta\overline{L}_l \equiv \frac{1}{P}\sum_{t=R+1}^{T}\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt}).$$

as well as:

$$J \equiv \Big\{ l \in \{1,\dots,\kappa\} : \sqrt{P}\,\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] > -c_0(\beta_P + \varphi_P) \Big\}$$

and the complement set $J^c = \{1,\dots,\kappa\} \setminus J$. In a first step, we want to show that with probability larger than $1 - CP^{-c} - \beta_P - 2(m_P - 1)b_{r_P}$, it holds that $\Delta \overline{L}_l \leq 0$ for all $l \in J^c$. First, observe that $\Delta \overline{L}_l > 0$ for some $l \in J^c$ implies that:

$$\max_{1 \leq l \leq \kappa} \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \Big(\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt}) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] \Big) > c_0(\alpha + \varphi_P).$$

Thus, it is sufficient to prove that:

$$\begin{aligned} \Pr\Big(\overline{U} > c_0(\beta_P + \varphi_P) \Big) &= \Pr\Big(\max_{1 \leq l \leq \kappa} \frac{1}{\sqrt{P}} \sum_{t=R+1}^{T} \Big(\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt}) - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] \Big) > c_0(\beta_P + \varphi_P) \Big) \\ &\leq CP^{-c} + \beta_P + (m_P - 1)b_{r_P}. \end{aligned}$$

First, for some φ_P' only dependent on φ_P, note that:

$$\begin{aligned} \Pr\Big(\overline{U} > c_0(\beta_P + \varphi_P') \Big) &\leq \Pr\Big(U_0 > c_0(\beta_P + \varphi_P') - \zeta_P \Big) + \Pr\Big(\Big|\overline{U} - U_0\Big| > \zeta_P \Big) \\ &\leq \Pr\Big(U_0 > c_0(\beta_P + \varphi_P' + 8\zeta_P \log^{\frac{1}{2}} \kappa) \Big) + CP^{-c}, \end{aligned}$$

where the first inequality follows by the fact that $\Pr\Big(\Big\{ U_0 + \Big(\overline{U} - U_0\Big) > c_0(\beta_P + \varphi_P') \Big\} \cap \Big\{ \Big(\overline{U} - U_0\Big) < \zeta_P \Big\} \Big) \leq \Pr\Big(U_0 > c_0(\beta_P + \varphi_P') - \zeta_P \Big)$, and the second inequality follows from Equation (S-47) and the proof of Theorem 1. Moreover, since by the argument in the proof of Theorem S7:

$$\sup_{x \in \mathbb{R}} \Big| \Pr\Big(U_0 \leq x \Big) - \Pr\Big(\max_{1 \leq l \leq \kappa} Y_l \leq x \Big) \Big| \leq CP^{-c} + 2(m_P - 1)b_{r_P} \leq C'P^{-c'},$$

and by Corollary B.1 in Chernozhukov et al. (2014b):

$$\sup_{t \in \mathbb{R}} \Big| \Pr\Big(\max_{1 \leq l \leq \kappa} \frac{1}{\sqrt{m_P q_P}} \sum_{l=1}^{m} S_l(\boldsymbol{\theta}_l) \leq x \Big) - \Pr\Big(\max_{1 \leq l \leq \kappa} Y_l \leq x \Big) \Big| \leq CP^{-c} + (m_P - 1)b_{r_P} \leq C'P^{-c'},$$

it holds that:

$$\begin{aligned} &\Pr\Big(U_0 > c_0(\beta_P + \varphi_P' + 8\zeta_P \log^{\frac{1}{2}} \kappa) \Big) + CP^{-c} \\ &\leq \Pr\Big(\max_{1 \leq i \leq \kappa} \frac{1}{\sqrt{m_P q_P}} \sum_{l=1}^{m} S_l(\boldsymbol{\theta}_l) > c_0(\beta_P + \varphi_P' + 8\zeta_P \log^{\frac{1}{2}} \kappa) \Big) \end{aligned}$$

$$+ C'P^{-c'} + 2(m_P-1)b_{r_P}$$

By Corollary 2.1 in Chernozhukov et al. (2014a) setting $\varphi_P = \varphi'_P + 8\zeta_P \log^{\frac{1}{2}}\kappa$ and using the fact that $\varphi_P = C'P^{-c'}$, it follows that:

$$\Pr\bigg(\overline{U} > c_0(\beta_P + \varphi_P)\bigg) \leq \beta_P + CP^{-c} + 2(m_P-1)b_{r_P}.$$

This establishes the first step.

For the second step, we wish to prove that with probability larger than $1 - CP^{-c} - \beta_P - 2(m_P-1)b_{r_P}$, $\widehat{J}_{BMB} \supset J$. Thus, start with:

$$\begin{aligned}
\Pr\bigg(\widehat{J}_{BMB} \not\supset J\bigg) &= \Pr\bigg(\Big\{\exists l \in \{1,\ldots,\kappa\} : \Big\{\sqrt{P}\,\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] > -c_0(\beta_P+\varphi_P)\Big\} \cap \Big\{\sqrt{P}\Delta\overline{L}_l > -2c^{BMB}(\beta_P)\Big\}\Big\}\bigg) \\
&\leq \Pr\bigg(\max_{1\leq i\leq\kappa}\sqrt{P}\Big(\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] - \Delta\overline{L}_l\Big) > 2c^{BMB}(\beta_P) - c_0(\beta_P+\varphi_P)\bigg).
\end{aligned}$$

On the event $c^{BMB}(\beta_P) > c_0(\beta_P+\varphi_P)$, it holds that $2c^{BMB}(\beta_P) - c_0(\beta_P+\varphi_P) \geq c_0(\beta_P+\varphi_P)$. Thus:

$$\begin{aligned}
\Pr\bigg(\widehat{J}_{BMB} \not\supset J\bigg) &\leq \Pr\bigg(\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] - \Delta\overline{L}_l\Big) > c_0(\beta_P+\varphi_P)\bigg) + \Pr\bigg(c^{BMB}(\beta_P) < c_0(\beta_P+\varphi_P)\bigg) \\
&\leq \Pr\bigg(\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] - \Delta\overline{L}_l\Big) > c_0(\beta_P+\varphi_P)\bigg) + CP^{-c}
\end{aligned}$$

By arguments similar to step one above, the first term can be bounded by:

$$\Pr\bigg(\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] - \Delta\overline{L}_l\Big) > c_0(\beta_P+\varphi_P)\bigg) \leq \beta_P + CP^{-c} + 2(m_P-1)b_{r_P},$$

which leads to the conclusion of step two.

The last step completes the proof. Assume that $J = \emptyset$. Then, with probability larger than $1 - \beta_P - CP^{-c} - 2(m_P-1)b_{r_P}$, it must hold that $U^* \leq 0$ by step one above and Lemma C.1. But as $c^{BMB,2S}(\alpha) \geq 0$, it holds that:

$$\Pr\bigg(U^* > c^{BMB,2S}(\alpha)\bigg) \leq \beta_P + CP^{-c} + 2(m_P-1)b_{r_P} \leq \alpha + C'P^{-c'}.$$

By contrast, consider the case where J is not the empty set and define $c^{BMB,2S}(\alpha, J)$ by the same bootstrap procedure as $c^{BMB,2S}(\alpha)$ with $\widehat{J}_{BMB}$ replaced by J. Then:

$$\begin{aligned}
\Pr\bigg(U^* > c^{BMB,2S}(\alpha)\bigg) &= \Pr\bigg(\Big\{U^* > c^{BMB,2S}(\alpha)\Big\} \cap \Big\{\max_{l\in J^c}\Delta\overline{L}_l \leq 0\Big\}\bigg) \\
&\quad + \Pr\bigg(\Big\{U^* > c^{BMB,2S}(\alpha)\Big\} \cap \Big\{\max_{l\in J^c}\Delta\overline{L}_l > 0\Big\}\bigg)
\end{aligned}$$

$$\leq \Pr\left(\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha)\right) + \beta_P + CP^{-c} + 2(m_P-1)b_{r_P},$$

where the inequality follows from step one above and the fact that $\Big\{U^* > c^{BMB,2S}(\alpha)\Big\}\cap\Big\{\max_{l\in J^c}\Delta\overline{L}_l \leq 0\Big\} \subset \Big\{\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha)\Big\}$. The probability of the last line can be further decomposed as:

$$\begin{aligned}
\Pr\left(\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha)\right) &= \Pr\left(\Big\{\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha)\Big\}\cap\Big\{\widehat{J}_{BMB}\supset J\Big\}\right)\\
&\quad + \Pr\left(\Big\{\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha)\Big\}\cap\Big\{\widehat{J}_{BMB}\not\supset J\Big\}\right)\\
&\leq \Pr\left(\max_{l\in J}\sqrt{P}\Delta\overline{L}_l > c^{BMB,2S}(\alpha, J)\right) + \Pr\left(\widehat{J}_{BMB}\not\supset J\right)\\
&\leq \Pr\left(\max_{l\in J}\sqrt{P}\Big(\Delta\overline{L}_l - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) > c^{BMB,2S}(\alpha, J)\right)\\
&\quad + \beta_P + CP^{-c} + 2(m_P-1)b_{r_P},
\end{aligned}$$

where the first inequality follows from the fact that $c^{BMB,2S}(\alpha) \geq c^{BMB,2S}(\alpha, J)$ on the event $\widehat{J}_{BMB}\supset J$, and the second inequality follows from step two above. Thus, by similar steps as for step one above:

$$\Pr\left(U^* > c^{BMB,2S}(\alpha)\right) \leq \alpha - 2\beta_P + 2\beta_P + 4(m_P-1)b_{r_P} + CP^{-c} = \alpha + 4(m_P-1)b_{r_P} + CP^{-c}.$$

If $\mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big] = 0$ for all $1\leq l\leq\kappa$, it holds that $J = \{1,\ldots,\kappa\}$. Hence, $c^{BMB,2S}(\alpha) = c^{BMB,2S}(\alpha, J)$ with probability larger than $1-\beta_P-2(m_P-1)b_{r_P}$ and so:

$$\begin{aligned}
\Pr\left(U^* > c^{BMB,2S}(\alpha)\right) &= \Pr\left(\overline{U} > c^{BMB,2S}(\alpha)\right)\\
&= \Pr\left(\Big\{\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\Delta\overline{L}_l - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) > c^{BMB,2S}(\alpha)\Big\}\cap\Big\{\widehat{J}_{BMB} = J\Big\}\right)\\
&\quad + \Pr\left(\Big\{\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\Delta\overline{L}_l - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) > c^{BMB,2S}(\alpha)\Big\}\cap\Big\{\widehat{J}_{BMB} \neq J\Big\}\right)\\
&\geq \Pr\left(\max_{1\leq l\leq\kappa}\sqrt{P}\Big(\Delta\overline{L}_l - \mathrm{E}\Big[\Delta L_t(\boldsymbol{\theta}_l)\Big]\Big) > c^{BMB,2S}(\alpha, J)\right) - \beta_P - 2(m_P-1)b_{r_P} - CP^{-c}\\
&\geq \alpha - 3\beta_P - 2(m_P-1)b_{r_P} - CP^{-c},
\end{aligned}$$

where the last inequality follows again by similar arguments as in step one.

■

S9 Optimal Block Length Procedure

S9.1 Description of Optimal Block Length Procedure

As standard selection methods of the block size for dependent data have not yet been extended to high-dimensional data, we follow and adapt an ad-hoc nested bootstrap procedure suggested by Zhang and Cheng (2014) for the BMB applied to a max statistic to estimate the optimal small-large block combination: firstly, define a grid (r_P, q_P) of possible small and large block combinations and pick an initial block size for both the small and the large block, say (r_{int}, q_{int}), such that $(r_{int} + q_{int})m_{int} = P$. Conditional on the sample $\{\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt})\}_{t=R+1}^{T}$, let $s_1, \ldots, s_{m_{int}}$ be i.i.d. uniform random variables on the set $\{0, \ldots, m_{int} - 1\}$ and define $\Delta L^*_{(j-1)(r_{int}+q_{int})+i}(\widehat{\boldsymbol{\theta}}_{lt}) = \Delta L_{s_j(r_{int}+q_{int})+i}(\widehat{\boldsymbol{\theta}}_{lt})$ with $1 \leq j \leq m_{int}$ and $1 \leq i \leq (q_{int} + r_{int})$. In other words, $\{\Delta L^*_t(\widehat{\boldsymbol{\theta}}_{lt})\}_{t=R+1}^{T}$ is a block bootstrap sample of the original sample with non-overlapping blocks and initiates the *outer bootstrap* procedure. Since, conditional on the sample, $\Delta\overline{L}_l$ is the true mean of this block bootstrap sample, compute the times that the sample mean $\Delta\overline{L}_l$ satisfies $P^{\frac{1}{2}} \max_{1\leq l\leq\kappa}(\Delta\overline{L}^*_l - \Delta\overline{L}_l) \leq c(\alpha)$ for each grid value of (r_P, q_P), where $c(\alpha)$ is computed on the basis of $\{\Delta L^*_t(\widehat{\boldsymbol{\theta}}_{lt})\}_{t=R+1}^{T}$ repeating steps 1 through 3 from the main text B_I times (the *inner bootstrap* procedure). Drawing B_O different block bootstrap samples allows to compute the times that $\Delta\overline{L}_l$ is contained in the set $\{\Delta\overline{L}_l : P^{\frac{1}{2}} \max_{1\leq l\leq\kappa}(\Delta\overline{L}^*_l - \Delta\overline{L}_l) \leq c(\alpha), l = 1, \ldots, \kappa | \{\Delta L_t(\widehat{\boldsymbol{\theta}}_{lt})\}_{t=R+1}^{T}\}$ (empirical coverage probability) for each grid choice (r_P, q_P). The pair (r_P, q_P) associated with the empirical coverage probability closest to $1 - \alpha$ can viewed as an estimate of the optimal block choice(s) that captures the dependence structure of the actual data best. This 'optimal choice' can then subsequently be employed to perform the moment inequality test of interest on the original sample.

S9.2 Monte Carlo Simulation

Table S1 presents a small set of power results for the optimal block length procedure outlined above, for the data generating process DGP-A1 described in Section S10.2 below with sample size $P = 100$ and dependence combination $\rho = 0.2$ and $\phi = 0.2$. We have set the number of inner and outer bootstrap replications described above to $B_I = 299$ and $B_O = 400$, respectively, while the number of Monte Carlo and of bootstrap replications to calculate the critical values are as in Section S10.2 below.

Table S1: Optimal Block Choice Procedure - Rejection Frequencies

$\kappa = 105$	$\kappa = 300$
0.962	0.951

The results suggest that the optimal block length procedure does a reasonable job in picking a suitable (r_P, q_P) combination from the grid as the rejection rates are around the upper end of the values seen in the equivalent Table S8, which presents results for fixed block length combinations. We leave it as a topic for future research whether additional iterations of the optimal block length procedure may improve results further.

S10 Additional Monte Carlo Simulations

In this section we present two additional sets of Monte Carlo set-ups. The first set-up in Section S10.1, as mentioned in the main text, expands on the simulations involving parameter estimation error by adding in a nowcast bias component in generating violations of monotonicity. The second set-up in Section S10.2 gives a set-up without parameter or factor estimation which was used in previous versions of the paper to assess the performance of the test under various cross-section and time series dependence scenarios.

S10.1 Additional Set-up 1: With Parameter Estimation and Nowcast Bias

Here the DGP for y_t is the same as in the main text, namely:

$$y_t = \beta_0' F_t + \beta_1' F_{t-1/3} + \beta_2' F_{t-2/3} + \varepsilon_t \tag{S-54}$$

where the number of factors is $r = 1$ and we use the scalar parameters $(\beta_0, \beta_1, \beta_2) = (0.25, 0.25, 0.25)$. The nowcaster makes predictions using the estimated equivalent of Equation (S-54), with factors estimated by PCA and coefficients by OLS. However, in addition to this they also adds a bias B_i to their prediction of y_t, in other words:

$$\widehat{y}_{it} = \widehat{\beta}_{0it} \widehat{F}_t^{(i,t)} + \widehat{\beta}_{1it} \widehat{F}_{t-1/3}^{(i,t)} + \widehat{\beta}_{2it} \widehat{F}_{t-2/3}^{(i,t)} + B_i \tag{S-55}$$

for each $t = R+1, ..., T$ and $i = 1, ..., S$ where it is clear that the bias only depends on the nowcast point i and does not vary across t. In other words at each nowcast point the prediction $\widehat{y}_{it}$ is formed of the factor-augmented model-based prediction plus the bias term. With the model correctly specified, we have an otherwise declining MSFE profile plus the square of the bias term. We can therefore use the bias to control the degree of non-monotonicity of MSFE.

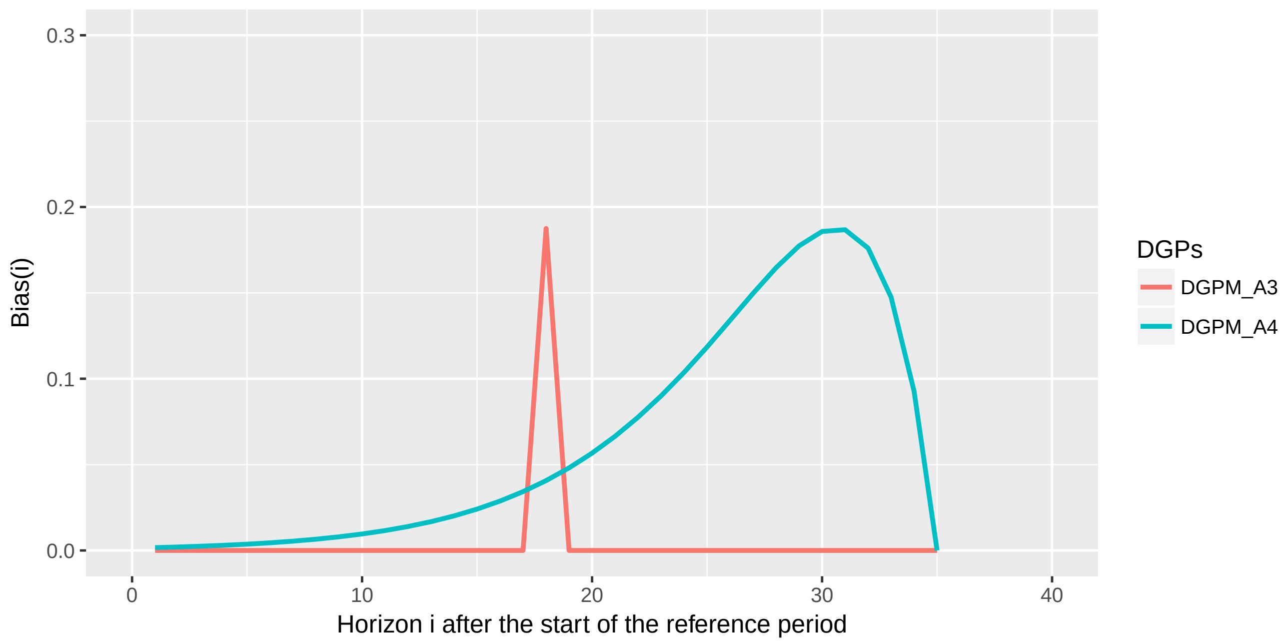
In order to check the power of the test, under the alternative there are many different ways in which monotonicity can be violated. From the perspective of nowcaster bias, the interpretation is that the systematic bias to the nowcast depends on which point the nowcaster is throughout the prediction period. We follow the literature on monotonicity testing in formulating two different types of violation of monotonicity, differing in the degree and persistence of violation (e.g., Ghosal et al., 2000). These bias terms are all depicted graphically in Figure 1.

DGPF-A3: For $i = \frac{S+1}{2}$, $B_i = K$ with $K > 0$, else for all $i \neq \frac{S+1}{2}$, $B_i = 0$.

DGPF-A4: For all i, $B_i = \frac{8K}{Se^{-1}}(S-i)e^{-\frac{8(S-i)}{S}}$, with $K > 0$.

In both of the above cases, we set $K = \sigma_\varepsilon$, the standard deviation of the factor-augmented model error. This means that the bias term causes a peak violation in the MSFE profile equal to $2\sigma_\varepsilon^2$, which is double that of the lowest point. Case A3 is a violation of monotonicity at a single point (spike) occurring at the point $i = S/2$, which should help to detect the test's sensitivity to small periods of violations of the null. The second case A4 is a standard non-monotonic function often used in simulation studies on tests for monotonicity where the MSFE follows a humped shape.

Figure 1: Bias Terms Corresponding to DGPF-A3 and DGPF-A4



Notes: This example has been created for $S = 35$ with parameter value $K = \sigma_\varepsilon$ where, as in the main text, $\sigma_\varepsilon^2 = \beta_0^2 + \beta_1^2 + \beta_2^2 = 3 \times 0.25^2$.

Table S2: Rejection Rates

		All Inequalities				Spacing $k_S^c = 5$			
		$T = 200, (R, P) = (133, 67)$				$T = 200, (R, P) = (133, 67)$			
		$N = 10, \kappa = 435$		$N = 20, \kappa = 1770$		$N = 10, \kappa = 300$		$N = 20, \kappa = 1485$	
		DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4
$r_P = 0$	$q_P = 4$	0.990	0.914	0.989	0.878	0.904	0.925	0.978	0.893
$r_P = 0$	$q_P = 5$	0.992	0.912	0.990	0.890	0.907	0.938	0.981	0.925
$r_P = 1$	$q_P = 4$	0.987	0.905	0.989	0.879	0.904	0.931	0.983	0.938
$r_P = 1$	$q_P = 5$	0.988	0.918	0.987	0.876	0.904	0.942	0.983	0.932
$r_P = 2$	$q_P = 4$	0.983	0.911	0.987	0.910	0.905	0.910	0.979	0.891
$r_P = 2$	$q_P = 5$	0.987	0.909	0.989	0.908	0.897	0.934	0.980	0.925
		$T = 200, (R, P) = (67, 133)$				$T = 200, (R, P) = (67, 133)$			
		$N = 10, \kappa = 435$		$N = 20, \kappa = 1770$		$N = 10, \kappa = 300$		$N = 20, \kappa = 1485$	
		DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4
$r_P = 0$	$q_P = 4$	1.000	1.000	1.000	0.998	0.993	0.999	1.000	0.999
$r_P = 0$	$q_P = 5$	1.000	1.000	1.000	0.998	0.993	0.999	1.000	1.000
$r_P = 1$	$q_P = 4$	1.000	1.000	1.000	0.998	0.990	0.999	1.000	0.999
$r_P = 1$	$q_P = 5$	1.000	1.000	1.000	0.998	0.993	0.999	1.000	0.999
$r_P = 2$	$q_P = 4$	1.000	1.000	1.000	0.999	0.990	0.998	1.000	0.997
$r_P = 2$	$q_P = 5$	1.000	1.000	1.000	0.998	0.994	0.999	1.000	0.999

		$T = 300, (R, P) = (200, 100)$				$T = 300, (R, P) = (200, 100)$			
		$N = 10, \kappa = 435$		$N = 20, \kappa = 1770$		$N = 10, \kappa = 300$		$N = 20, \kappa = 1485$	
		DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4
$r_P = 0$	$q_P = 4$	1.000	0.994	1.000	0.997	0.984	0.992	0.996	0.996
$r_P = 0$	$q_P = 5$	1.000	0.991	1.000	0.995	0.984	0.992	0.996	0.995
$r_P = 1$	$q_P = 4$	1.000	0.991	0.999	0.998	0.980	0.986	0.998	0.996
$r_P = 1$	$q_P = 5$	1.000	0.993	1.000	0.994	0.984	0.990	0.996	0.992
$r_P = 2$	$q_P = 4$	1.000	0.992	1.000	0.998	0.984	0.987	0.996	0.989
$r_P = 2$	$q_P = 5$	1.000	0.994	0.999	0.994	0.984	0.987	0.996	0.990
		$T = 300, (R, P) = (100, 200)$				$T = 300, (R, P) = (100, 200)$			
		$N = 10, \kappa = 435$		$N = 20, \kappa = 1770$		$N = 10, \kappa = 300$		$N = 20, \kappa = 1485$	
		DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4	DGPF-A3	DGPF-A4
$r_P = 0$	$q_P = 4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$r_P = 0$	$q_P = 5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$r_P = 1$	$q_P = 4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$r_P = 1$	$q_P = 5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$r_P = 2$	$q_P = 4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$r_P = 2$	$q_P = 5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

S10.2 Additional Set-up 2: Without Estimation Error

In this set-up we directly simulate the nowcast errors ε_{it} for $i = 1, ..., S$ and for each $t = R + 1, ..., T$. The general form of the data generating process (DGP) for each of the forecast errors ε_{it} was closely follows that of Zhang and Cheng (2014) and Jin et al. (2017) allowing for cross-sectional and time series dependence. We describe it in more detail here:

$$\varepsilon_{it} = \rho\varepsilon_{i,t-1} + (1-\rho)\left(\sqrt{1-\phi}v_{i,t} + \sqrt{\phi}v_{0,t}\right) \tag{S-56}$$

where ρ and ϕ are scalars such that $|\rho| < 1$ and $0 \leq \phi < 1$.

In the results below, we let the number of nowcasts be $S \in \{15, 25, 35\}$, which implies a maximum number of $\kappa = S(S-1)/2 \in \{105, 300, 595\}$ moment inequalities, while a small dimensional example with $S \in \{3, 5\}$ is also investigated. We consider ruling out adjacent and nearby moment inequalities with additional spacings than the previous section, namely $k_S^c \in \{1, 2, 5\}$. We will also consider the adjacent-only set of moment inequalities, as in the test $\mathbf{\Delta^e}$ of Patton and Timmermann (2012). The sample sizes are $P \in \{100, 200\}$ which give scenarios in which the number of moment inequalities is larger than the number of observations.

In the DGP in Equation (S-56) we let $v_{i,t} \sim i.i.d.\mathcal{N}(0, \sigma_i^2)$ for all $i = 0, ..., S$. The term $v_{0,t}$ is introduced only as a device to produce cross-sectional dependence. Note that in this set-up when $\sigma_i^2 = 1$ for all i, it follows that for some errors ε_{it} and ε_{iv}, we have $\text{Cor}(\varepsilon_{it}, \varepsilon_{iv}) = \rho^{|t-v|}\phi$. Therefore we can choose ρ and ϕ to vary the degree of time-series and cross-section dependence. We consider cases $(\rho, \phi) \in \{(0.2, 0.2), (0.5, 0.5)\}$. Note that in the main text, only the $(0.5, 0.5)$ case was presented and only for the null case.

The parameter σ_i^2 can be varied across i to generate scenarios of non-monotonicity to assess the power of the test. When the null hypothesis is true, we use a parameterisation of the DGP with constant forecast error variance across all i:

DGP-N: $\sigma_i^2 = 1$ for all $i = 0, ..., S$.

This 'least favourable case' is often chosen to assess the performance of the test when the null is satisfied as it is for this case that rejection rates should be close to the nominal level α.

Under the alternative there are many different ways in which monotonicity can be violated. In a similar way to the case described in Section S10.1 above, we will use standard shapes of monotonicity violation. The difference is that here we are seeing how these affect the results in a case with no parameter estimation error, and with different sized violations of the null. The null and alternative cases are depicted graphically in Figure 2.

DGP-A1: For $i = \frac{S+1}{2}$, $\sigma_i^2 = \frac{3}{2} - \frac{i}{S} + K$ with $K > 0$, else for all $i \neq \frac{S+1}{2}$, $\sigma_i^2 = \frac{3}{2} - \frac{i}{S}$.

DGP-A2: For all i, $\sigma_i^2 = 1 + \frac{8K}{Se^{-1}}(S-i)e^{-\frac{8(S-i)}{S}}$, with $K > 0$.

In both of the above cases, we can set $K = \widetilde{K}/(1-\phi)$ so that the largest violation of monotonicity is of magnitude $\widetilde{K}$. The rescaling by $(1-\phi)^{-1}$ is required because violations in monotonicity due to changes in σ_i^2 are dampened when ϕ is large according to the DGP in Equation (S-56). We can therefore use this parameter, which we set to $\widetilde{K} = 1.5$, to analyse the power properties of the test.

Figure 2: MSFE Profiles Under the Null (DGP-N) and Alternatives (DGP-A1, DGP-A2)

Notes: This example has been created for $S = 35$ with parameter value $\widetilde{K} = 1.5$.

Below we provide the results tables for this Monte Carlo set-up without parameter estimation error, starting with dependence case $(\rho, \phi) = (0.5, 0.5)$ for the five different versions of the test with different sets of moment inequalities, and then $(\rho, \phi) = (0.2, 0.2)$. Note that the results for $(\rho, \phi) = (0.5, 0.5)$ and DGP-N correspond to what was presented in the main text. After these results, for the case $(\rho, \phi) = (0.5, 0.5)$, we also provide results for the 'small dimensional' scenario where $S = \{3, 5\}$ using all and only adjacent moment comparisons.

Table S3: Rejection Frequencies - Without Estimation Error - All Inequalities

$P = 100$, $(\rho, \phi) = (0.5, 0.5)$										
		$\kappa = 105$			$\kappa = 300$			$\kappa = 595$		
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.059	0.818	0.943	0.042	0.844	0.907	0.044	0.871	0.891
$r_P = 0$	$q_P = 5$	0.059	0.827	0.919	0.034	0.827	0.953	0.037	0.773	0.898
$r_P = 0$	$q_P = 6$	0.051	0.836	0.907	0.033	0.806	0.899	0.024	0.785	0.866
$r_P = 0$	$q_P = 7$	0.037	0.809	0.790	0.021	0.761	0.869	0.019	0.799	0.849
$r_P = 0$	$q_P = 8$	0.051	0.827	0.929	0.021	0.785	0.828	0.024	0.795	0.855
$r_P = 1$	$q_P = 4$	0.074	0.848	0.917	0.048	0.837	0.912	0.024	0.824	0.900
$r_P = 1$	$q_P = 5$	0.057	0.833	0.932	0.023	0.768	0.927	0.034	0.817	0.867
$r_P = 1$	$q_P = 6$	0.039	0.855	0.900	0.033	0.815	0.909	0.024	0.820	0.880
$r_P = 1$	$q_P = 7$	0.032	0.868	0.895	0.019	0.785	0.875	0.020	0.787	0.830
$r_P = 1$	$q_P = 8$	0.034	0.830	0.859	0.031	0.754	0.823	0.014	0.772	0.820
$r_P = 2$	$q_P = 4$	0.038	0.841	0.898	0.046	0.817	0.923	0.023	0.825	0.955
$r_P = 2$	$q_P = 5$	0.037	0.828	0.879	0.022	0.785	0.853	0.027	0.785	0.862
$r_P = 2$	$q_P = 6$	0.029	0.799	0.816	0.026	0.812	0.909	0.010	0.799	0.910
$r_P = 2$	$q_P = 7$	0.033	0.799	0.888	0.025	0.742	0.830	0.011	0.754	0.809
$r_P = 2$	$q_P = 8$	0.028	0.775	0.853	0.033	0.791	0.917	0.008	0.770	0.845
$P = 200$, $(\rho, \phi) = (0.5, 0.5)$										
		$\kappa = 105$			$\kappa = 300$			$\kappa = 595$		
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.066	0.989	0.999	0.059	0.994	1.000	0.043	0.994	1.000
$r_P = 0$	$q_P = 5$	0.061	0.991	0.999	0.059	0.994	1.000	0.037	0.991	1.000
$r_P = 0$	$q_P = 6$	0.048	0.987	0.998	0.019	0.992	1.000	0.032	0.989	1.000
$r_P = 0$	$q_P = 7$	0.049	0.991	0.999	0.044	0.990	1.000	0.033	0.990	1.000
$r_P = 0$	$q_P = 8$	0.038	0.987	0.999	0.061	0.988	1.000	0.021	0.989	1.000
$r_P = 1$	$q_P = 4$	0.052	0.989	0.999	0.061	0.994	1.000	0.030	0.990	1.000
$r_P = 1$	$q_P = 5$	0.043	0.990	0.999	0.061	0.992	1.000	0.029	0.992	1.000
$r_P = 1$	$q_P = 6$	0.046	0.984	0.998	0.055	0.988	1.000	0.032	0.989	1.000
$r_P = 1$	$q_P = 7$	0.041	0.985	0.998	0.041	0.990	1.000	0.019	0.988	1.000
$r_P = 1$	$q_P = 8$	0.042	0.989	0.998	0.036	0.994	1.000	0.023	0.990	1.000
$r_P = 2$	$q_P = 4$	0.051	0.991	0.999	0.056	0.987	1.000	0.029	0.989	1.000
$r_P = 2$	$q_P = 5$	0.041	0.987	0.998	0.039	0.992	1.000	0.031	0.988	1.000
$r_P = 2$	$q_P = 6$	0.044	0.985	0.998	0.049	0.991	1.000	0.025	0.990	1.000
$r_P = 2$	$q_P = 7$	0.041	0.990	0.998	0.019	0.992	1.000	0.021	0.985	1.000
$r_P = 2$	$q_P = 8$	0.030	0.984	0.998	0.041	0.992	1.000	0.025	0.992	1.000

Table S4: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 1$

		$P = 100$, $(\rho, \phi) = (0.5, 0.5)$								
			$\kappa = 91$			$\kappa = 276$			$\kappa = 561$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.055	0.842	0.958	0.048	0.807	0.948	0.041	0.847	0.948
$r_P = 0$	$q_P = 5$	0.045	0.809	0.968	0.046	0.810	0.940	0.039	0.817	0.952
$r_P = 0$	$q_P = 6$	0.028	0.820	0.923	0.034	0.771	0.929	0.030	0.827	0.914
$r_P = 0$	$q_P = 7$	0.040	0.807	0.914	0.034	0.801	0.882	0.026	0.759	0.853
$r_P = 0$	$q_P = 8$	0.026	0.817	0.954	0.033	0.790	0.894	0.026	0.803	0.931
$r_P = 1$	$q_P = 4$	0.059	0.854	0.965	0.046	0.820	0.925	0.054	0.843	0.890
$r_P = 1$	$q_P = 5$	0.052	0.783	0.917	0.039	0.792	0.870	0.026	0.790	0.880
$r_P = 1$	$q_P = 6$	0.049	0.804	0.943	0.041	0.782	0.892	0.025	0.825	0.890
$r_P = 1$	$q_P = 7$	0.045	0.789	0.923	0.030	0.788	0.912	0.026	0.759	0.858
$r_P = 1$	$q_P = 8$	0.035	0.783	0.905	0.031	0.788	0.927	0.023	0.783	0.887
$r_P = 2$	$q_P = 4$	0.046	0.849	0.976	0.038	0.786	0.918	0.026	0.795	0.889
$r_P = 2$	$q_P = 5$	0.034	0.786	0.919	0.036	0.795	0.907	0.046	0.788	0.913
$r_P = 2$	$q_P = 6$	0.053	0.871	0.968	0.037	0.773	0.840	0.024	0.742	0.911
$r_P = 2$	$q_P = 7$	0.035	0.842	0.906	0.021	0.780	0.872	0.023	0.784	0.916
$r_P = 2$	$q_P = 8$	0.020	0.817	0.891	0.030	0.775	0.896	0.025	0.748	0.874
		$P = 200$, $(\rho, \phi) = (0.5, 0.5)$								
			$\kappa = 91$			$\kappa = 276$			$\kappa = 561$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.061	0.987	1.000	0.064	0.990	1.000	0.090	0.993	1.000
$r_P = 0$	$q_P = 5$	0.066	0.991	1.000	0.038	0.992	1.000	0.061	0.990	1.000
$r_P = 0$	$q_P = 6$	0.060	0.983	1.000	0.047	0.990	1.000	0.048	0.988	1.000
$r_P = 0$	$q_P = 7$	0.039	0.986	1.000	0.034	0.985	1.000	0.072	0.994	1.000
$r_P = 0$	$q_P = 8$	0.042	0.987	1.000	0.034	0.978	1.000	0.032	0.988	1.000
$r_P = 1$	$q_P = 4$	0.051	0.990	1.000	0.062	0.992	1.000	0.051	0.995	1.000
$r_P = 1$	$q_P = 5$	0.042	0.991	1.000	0.048	0.990	1.000	0.031	0.993	1.000
$r_P = 1$	$q_P = 6$	0.042	0.983	1.000	0.042	0.986	1.000	0.042	0.994	1.000
$r_P = 1$	$q_P = 7$	0.036	0.982	1.000	0.028	0.990	1.000	0.045	0.995	1.000
$r_P = 1$	$q_P = 8$	0.039	0.983	1.000	0.034	0.986	1.000	0.032	0.993	1.000
$r_P = 2$	$q_P = 4$	0.041	0.987	1.000	0.073	0.990	1.000	0.049	0.996	1.000
$r_P = 2$	$q_P = 5$	0.054	0.987	1.000	0.027	0.982	1.000	0.037	0.994	1.000
$r_P = 2$	$q_P = 6$	0.036	0.987	1.000	0.031	0.987	1.000	0.024	0.994	1.000
$r_P = 2$	$q_P = 7$	0.031	0.987	1.000	0.024	0.989	1.000	0.032	0.988	1.000
$r_P = 2$	$q_P = 8$	0.031	0.986	1.000	0.026	0.985	1.000	0.032	0.993	1.000

Table S5: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 2$

					$P = 100$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 78$			$\kappa = 253$			$\kappa = 528$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.050	0.833	0.981	0.026	0.770	0.953	0.059	0.795	0.975
$r_P = 0$	$q_P = 5$	0.052	0.792	0.967	0.026	0.778	0.961	0.035	0.799	0.897
$r_P = 0$	$q_P = 6$	0.032	0.809	0.968	0.023	0.793	0.933	0.042	0.811	0.930
$r_P = 0$	$q_P = 7$	0.033	0.808	0.967	0.023	0.803	0.912	0.026	0.789	0.941
$r_P = 0$	$q_P = 8$	0.038	0.803	0.949	0.023	0.844	0.961	0.020	0.782	0.917
$r_P = 1$	$q_P = 4$	0.033	0.800	0.969	0.046	0.809	0.935	0.026	0.835	0.939
$r_P = 1$	$q_P = 5$	0.040	0.822	0.966	0.026	0.803	0.965	0.026	0.825	0.900
$r_P = 1$	$q_P = 6$	0.034	0.797	0.962	0.019	0.765	0.936	0.020	0.795	0.950
$r_P = 1$	$q_P = 7$	0.032	0.809	0.958	0.024	0.803	0.955	0.024	0.796	0.897
$r_P = 1$	$q_P = 8$	0.033	0.737	0.939	0.022	0.781	0.929	0.019	0.733	0.934
$r_P = 2$	$q_P = 4$	0.034	0.835	0.989	0.017	0.797	0.955	0.040	0.788	0.945
$r_P = 2$	$q_P = 5$	0.029	0.822	0.967	0.023	0.748	0.961	0.017	0.775	0.830
$r_P = 2$	$q_P = 6$	0.040	0.777	0.972	0.023	0.786	0.955	0.020	0.796	0.919
$r_P = 2$	$q_P = 7$	0.029	0.749	0.925	0.017	0.742	0.886	0.020	0.780	0.905
$r_P = 2$	$q_P = 8$	0.047	0.820	0.967	0.023	0.804	0.922	0.024	0.736	0.911
					$P = 200$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 78$			$\kappa = 253$			$\kappa = 528$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.054	0.990	1.000	0.047	0.991	1.000	0.053	0.992	1.000
$r_P = 0$	$q_P = 5$	0.049	0.992	1.000	0.047	0.991	1.000	0.033	0.989	1.000
$r_P = 0$	$q_P = 6$	0.045	0.991	1.000	0.039	0.991	1.000	0.027	0.992	1.000
$r_P = 0$	$q_P = 7$	0.050	0.982	1.000	0.028	0.991	1.000	0.031	0.992	1.000
$r_P = 0$	$q_P = 8$	0.036	0.984	1.000	0.025	0.985	1.000	0.018	0.980	1.000
$r_P = 1$	$q_P = 4$	0.047	0.988	1.000	0.057	0.991	1.000	0.051	0.988	1.000
$r_P = 1$	$q_P = 5$	0.044	0.990	1.000	0.032	0.991	1.000	0.040	0.988	1.000
$r_P = 1$	$q_P = 6$	0.043	0.991	1.000	0.021	0.974	1.000	0.019	0.984	1.000
$r_P = 1$	$q_P = 7$	0.026	0.982	1.000	0.023	0.991	1.000	0.027	0.991	1.000
$r_P = 1$	$q_P = 8$	0.035	0.983	1.000	0.024	0.991	1.000	0.019	0.980	1.000
$r_P = 2$	$q_P = 4$	0.071	0.991	1.000	0.063	0.991	1.000	0.027	0.992	1.000
$r_P = 2$	$q_P = 5$	0.054	0.987	1.000	0.021	0.985	1.000	0.024	0.992	1.000
$r_P = 2$	$q_P = 6$	0.025	0.988	1.000	0.036	0.991	1.000	0.017	0.988	1.000
$r_P = 2$	$q_P = 7$	0.036	0.985	1.000	0.024	0.991	1.000	0.018	0.980	1.000
$r_P = 2$	$q_P = 8$	0.044	0.982	1.000	0.021	0.991	1.000	0.011	0.988	1.000

Table S6: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 5$

					$P = 100$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 45$			$\kappa = 190$			$\kappa = 435$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.061	0.785	0.989	0.057	0.797	0.992	0.061	0.816	0.988
$r_P = 0$	$q_P = 5$	0.031	0.745	0.993	0.057	0.824	0.985	0.032	0.803	0.987
$r_P = 0$	$q_P = 6$	0.055	0.730	0.979	0.047	0.794	0.993	0.032	0.792	0.985
$r_P = 0$	$q_P = 7$	0.029	0.749	0.987	0.038	0.755	0.987	0.018	0.803	0.970
$r_P = 0$	$q_P = 8$	0.029	0.741	0.987	0.039	0.766	0.970	0.038	0.753	0.986
$r_P = 1$	$q_P = 4$	0.045	0.751	0.981	0.056	0.800	0.997	0.061	0.796	0.993
$r_P = 1$	$q_P = 5$	0.039	0.741	0.987	0.056	0.770	0.998	0.031	0.760	0.992
$r_P = 1$	$q_P = 6$	0.034	0.741	0.982	0.023	0.751	0.965	0.008	0.800	0.989
$r_P = 1$	$q_P = 7$	0.032	0.724	0.981	0.035	0.785	0.990	0.012	0.790	0.940
$r_P = 1$	$q_P = 8$	0.029	0.741	0.986	0.045	0.787	0.988	0.014	0.715	0.950
$r_P = 2$	$q_P = 4$	0.065	0.741	0.994	0.025	0.775	0.980	0.032	0.760	0.989
$r_P = 2$	$q_P = 5$	0.034	0.664	0.984	0.032	0.784	0.986	0.031	0.736	0.965
$r_P = 2$	$q_P = 6$	0.022	0.724	0.991	0.032	0.749	0.962	0.032	0.759	0.972
$r_P = 2$	$q_P = 7$	0.029	0.740	0.989	0.035	0.783	0.982	0.016	0.741	0.965
$r_P = 2$	$q_P = 8$	0.032	0.741	0.986	0.026	0.766	0.986	0.013	0.714	0.969
					$P = 200$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 45$			$\kappa = 190$			$\kappa = 435$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.081	0.974	1.000	0.061	0.990	1.000	0.058	0.989	1.000
$r_P = 0$	$q_P = 5$	0.047	0.970	1.000	0.043	0.987	1.000	0.053	0.990	1.000
$r_P = 0$	$q_P = 6$	0.059	0.964	1.000	0.039	0.985	1.000	0.040	0.986	1.000
$r_P = 0$	$q_P = 7$	0.043	0.974	1.000	0.039	0.983	1.000	0.037	0.983	1.000
$r_P = 0$	$q_P = 8$	0.037	0.975	1.000	0.038	0.985	1.000	0.041	0.986	1.000
$r_P = 1$	$q_P = 4$	0.052	0.974	1.000	0.067	0.988	1.000	0.052	0.986	1.000
$r_P = 1$	$q_P = 5$	0.063	0.971	1.000	0.061	0.986	1.000	0.041	0.989	1.000
$r_P = 1$	$q_P = 6$	0.042	0.960	1.000	0.051	0.986	1.000	0.039	0.974	1.000
$r_P = 1$	$q_P = 7$	0.043	0.972	1.000	0.030	0.982	1.000	0.026	0.986	1.000
$r_P = 1$	$q_P = 8$	0.053	0.965	1.000	0.022	0.968	1.000	0.039	0.979	1.000
$r_P = 2$	$q_P = 4$	0.070	0.975	1.000	0.065	0.981	1.000	0.058	0.989	1.000
$r_P = 2$	$q_P = 5$	0.054	0.971	1.000	0.039	0.987	1.000	0.039	0.987	1.000
$r_P = 2$	$q_P = 6$	0.064	0.971	1.000	0.039	0.976	1.000	0.026	0.987	1.000
$r_P = 2$	$q_P = 7$	0.037	0.972	1.000	0.051	0.984	1.000	0.020	0.979	1.000
$r_P = 2$	$q_P = 8$	0.033	0.957	1.000	0.040	0.982	1.000	0.026	0.977	1.000

Table S7: Rejection Frequencies - Without Estimation Error - Adjacent-only Inequalities

					$P = 100$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 14$			$\kappa = 24$			$\kappa = 34$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.053	0.845	0.136	0.064	0.823	0.091	0.064	0.817	0.048
$r_P = 0$	$q_P = 5$	0.051	0.847	0.127	0.058	0.837	0.078	0.044	0.817	0.061
$r_P = 0$	$q_P = 6$	0.044	0.830	0.116	0.053	0.803	0.092	0.025	0.792	0.053
$r_P = 0$	$q_P = 7$	0.039	0.809	0.092	0.042	0.824	0.084	0.035	0.793	0.042
$r_P = 0$	$q_P = 8$	0.050	0.853	0.136	0.044	0.804	0.064	0.030	0.797	0.042
$r_P = 1$	$q_P = 4$	0.042	0.806	0.138	0.052	0.826	0.104	0.061	0.839	0.060
$r_P = 1$	$q_P = 5$	0.051	0.843	0.140	0.064	0.848	0.073	0.043	0.795	0.046
$r_P = 1$	$q_P = 6$	0.036	0.806	0.126	0.053	0.830	0.073	0.031	0.807	0.043
$r_P = 1$	$q_P = 7$	0.041	0.845	0.111	0.040	0.779	0.071	0.020	0.811	0.061
$r_P = 1$	$q_P = 8$	0.042	0.813	0.096	0.053	0.830	0.070	0.027	0.785	0.042
$r_P = 2$	$q_P = 4$	0.050	0.833	0.140	0.058	0.854	0.100	0.034	0.784	0.044
$r_P = 2$	$q_P = 5$	0.035	0.834	0.129	0.045	0.799	0.059	0.035	0.802	0.054
$r_P = 2$	$q_P = 6$	0.043	0.833	0.101	0.044	0.790	0.064	0.030	0.784	0.066
$r_P = 2$	$q_P = 7$	0.044	0.808	0.108	0.037	0.813	0.054	0.030	0.778	0.046
$r_P = 2$	$q_P = 8$	0.044	0.852	0.118	0.050	0.780	0.056	0.030	0.792	0.050
					$P = 200$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 14$			$\kappa = 24$			$\kappa = 34$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.069	0.989	0.195	0.071	0.985	0.102	0.065	0.993	0.084
$r_P = 0$	$q_P = 5$	0.061	0.987	0.135	0.066	0.988	0.081	0.035	0.991	0.078
$r_P = 0$	$q_P = 6$	0.052	0.989	0.135	0.046	0.988	0.103	0.045	0.990	0.077
$r_P = 0$	$q_P = 7$	0.038	0.989	0.178	0.048	0.989	0.095	0.045	0.991	0.067
$r_P = 0$	$q_P = 8$	0.027	0.986	0.139	0.046	0.993	0.093	0.033	0.990	0.066
$r_P = 1$	$q_P = 4$	0.057	0.992	0.173	0.075	0.988	0.126	0.065	0.993	0.074
$r_P = 1$	$q_P = 5$	0.060	0.990	0.199	0.051	0.987	0.088	0.069	0.991	0.050
$r_P = 1$	$q_P = 6$	0.039	0.989	0.135	0.065	0.982	0.098	0.047	0.991	0.074
$r_P = 1$	$q_P = 7$	0.027	0.992	0.195	0.063	0.984	0.103	0.060	0.991	0.069
$r_P = 1$	$q_P = 8$	0.053	0.989	0.164	0.056	0.985	0.099	0.050	0.990	0.071
$r_P = 2$	$q_P = 4$	0.067	0.992	0.172	0.057	0.988	0.104	0.054	0.993	0.071
$r_P = 2$	$q_P = 5$	0.045	0.992	0.183	0.063	0.987	0.098	0.033	0.989	0.078
$r_P = 2$	$q_P = 6$	0.038	0.989	0.215	0.050	0.986	0.102	0.039	0.990	0.048
$r_P = 2$	$q_P = 7$	0.045	0.992	0.195	0.060	0.988	0.104	0.053	0.989	0.050
$r_P = 2$	$q_P = 8$	0.040	0.989	0.159	0.036	0.984	0.092	0.030	0.991	0.050

Table S8: Rejection Frequencies - Without Estimation Error - All Inequalities

		$P = 100$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 105$			$\kappa = 300$			$\kappa = 595$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.036	0.929	0.988	0.024	0.925	0.993	0.038	0.934	0.998
$r_P = 0$	$q_P = 5$	0.028	0.908	0.995	0.018	0.910	0.997	0.027	0.913	0.998
$r_P = 0$	$q_P = 6$	0.026	0.931	0.992	0.018	0.925	0.997	0.024	0.926	0.996
$r_P = 0$	$q_P = 7$	0.017	0.910	0.985	0.011	0.930	0.994	0.015	0.928	0.995
$r_P = 0$	$q_P = 8$	0.028	0.941	0.990	0.014	0.912	0.985	0.018	0.934	0.996
$r_P = 1$	$q_P = 4$	0.027	0.920	0.995	0.027	0.935	0.993	0.015	0.924	0.995
$r_P = 1$	$q_P = 5$	0.031	0.928	0.992	0.011	0.907	0.997	0.016	0.932	0.997
$r_P = 1$	$q_P = 6$	0.035	0.937	0.990	0.019	0.938	0.994	0.024	0.913	0.995
$r_P = 1$	$q_P = 7$	0.026	0.952	0.989	0.011	0.917	0.984	0.013	0.924	0.988
$r_P = 1$	$q_P = 8$	0.030	0.937	0.987	0.018	0.922	0.989	0.022	0.914	0.989
$r_P = 2$	$q_P = 4$	0.017	0.929	0.965	0.010	0.939	0.991	0.013	0.924	0.995
$r_P = 2$	$q_P = 5$	0.027	0.936	0.980	0.013	0.920	0.982	0.031	0.903	0.991
$r_P = 2$	$q_P = 6$	0.031	0.926	0.980	0.015	0.926	0.995	0.012	0.932	0.994
$r_P = 2$	$q_P = 7$	0.022	0.916	0.988	0.011	0.907	0.994	0.016	0.915	0.988
$r_P = 2$	$q_P = 8$	0.024	0.921	0.987	0.026	0.909	0.993	0.009	0.910	0.995
		$P = 200$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 105$			$\kappa = 300$			$\kappa = 595$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.040	0.999	1.000	0.026	1.000	1.000	0.023	0.998	1.000
$r_P = 0$	$q_P = 5$	0.045	1.000	1.000	0.037	1.000	1.000	0.022	0.999	1.000
$r_P = 0$	$q_P = 6$	0.035	0.998	1.000	0.022	0.999	1.000	0.022	0.997	1.000
$r_P = 0$	$q_P = 7$	0.032	1.000	1.000	0.024	1.000	1.000	0.031	0.998	1.000
$r_P = 0$	$q_P = 8$	0.033	0.999	1.000	0.037	1.000	1.000	0.017	0.998	1.000
$r_P = 1$	$q_P = 4$	0.036	1.000	1.000	0.029	1.000	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 5$	0.033	1.000	1.000	0.024	0.999	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 6$	0.039	0.999	1.000	0.031	0.999	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 7$	0.032	0.999	1.000	0.042	0.999	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 8$	0.035	1.000	1.000	0.024	1.000	1.000	0.018	0.999	1.000
$r_P = 2$	$q_P = 4$	0.035	1.000	1.000	0.034	0.999	1.000	0.019	0.997	1.000
$r_P = 2$	$q_P = 5$	0.033	0.999	1.000	0.017	1.000	1.000	0.022	0.997	1.000
$r_P = 2$	$q_P = 6$	0.033	0.999	1.000	0.024	0.999	1.000	0.018	0.998	1.000
$r_P = 2$	$q_P = 7$	0.032	1.000	1.000	0.022	0.999	1.000	0.020	0.997	1.000
$r_P = 2$	$q_P = 8$	0.033	0.998	1.000	0.024	0.999	1.000	0.031	0.999	1.000

Table S9: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 1$

		$P = 100$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 91$			$\kappa = 276$			$\kappa = 561$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.035	0.919	0.997	0.017	0.898	0.993	0.012	0.928	0.996
$r_P = 0$	$q_P = 5$	0.035	0.923	0.998	0.017	0.916	0.995	0.018	0.927	1.000
$r_P = 0$	$q_P = 6$	0.031	0.929	0.992	0.019	0.883	0.992	0.010	0.911	0.979
$r_P = 0$	$q_P = 7$	0.027	0.933	0.982	0.015	0.927	0.992	0.011	0.906	0.990
$r_P = 0$	$q_P = 8$	0.016	0.946	0.994	0.010	0.898	0.991	0.010	0.906	0.994
$r_P = 1$	$q_P = 4$	0.035	0.923	0.993	0.014	0.907	0.991	0.014	0.929	0.989
$r_P = 1$	$q_P = 5$	0.035	0.892	0.982	0.016	0.900	0.986	0.012	0.922	0.989
$r_P = 1$	$q_P = 6$	0.027	0.920	0.992	0.009	0.890	0.992	0.014	0.922	0.989
$r_P = 1$	$q_P = 7$	0.027	0.909	0.992	0.008	0.899	0.991	0.009	0.913	0.989
$r_P = 1$	$q_P = 8$	0.030	0.933	0.995	0.015	0.893	0.992	0.009	0.911	0.987
$r_P = 2$	$q_P = 4$	0.035	0.934	0.987	0.016	0.895	0.991	0.005	0.905	0.986
$r_P = 2$	$q_P = 5$	0.024	0.908	0.985	0.013	0.900	0.991	0.012	0.889	0.989
$r_P = 2$	$q_P = 6$	0.034	0.935	0.997	0.010	0.890	0.984	0.009	0.892	0.990
$r_P = 2$	$q_P = 7$	0.024	0.923	0.992	0.006	0.886	0.991	0.008	0.905	0.986
$r_P = 2$	$q_P = 8$	0.026	0.926	0.990	0.007	0.898	0.991	0.005	0.911	0.989
		$P = 200$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 91$			$\kappa = 276$			$\kappa = 561$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.036	0.995	1.000	0.032	0.999	1.000	0.038	0.999	1.000
$r_P = 0$	$q_P = 5$	0.042	0.996	1.000	0.022	1.000	1.000	0.038	0.999	1.000
$r_P = 0$	$q_P = 6$	0.035	0.995	1.000	0.032	0.999	1.000	0.033	0.999	1.000
$r_P = 0$	$q_P = 7$	0.036	0.995	1.000	0.020	0.998	1.000	0.041	0.999	1.000
$r_P = 0$	$q_P = 8$	0.044	0.997	1.000	0.022	0.998	1.000	0.034	0.999	1.000
$r_P = 1$	$q_P = 4$	0.024	0.996	1.000	0.031	0.999	1.000	0.027	0.999	1.000
$r_P = 1$	$q_P = 5$	0.024	0.997	1.000	0.032	0.998	1.000	0.028	0.999	1.000
$r_P = 1$	$q_P = 6$	0.025	0.995	1.000	0.021	0.998	1.000	0.034	0.999	1.000
$r_P = 1$	$q_P = 7$	0.024	0.995	1.000	0.041	1.000	1.000	0.036	0.999	1.000
$r_P = 1$	$q_P = 8$	0.024	0.996	1.000	0.027	0.999	1.000	0.036	0.999	1.000
$r_P = 2$	$q_P = 4$	0.024	0.997	1.000	0.025	0.999	1.000	0.034	0.999	1.000
$r_P = 2$	$q_P = 5$	0.031	0.997	1.000	0.010	0.999	1.000	0.031	0.999	1.000
$r_P = 2$	$q_P = 6$	0.031	0.994	1.000	0.020	1.000	1.000	0.034	0.999	1.000
$r_P = 2$	$q_P = 7$	0.021	0.996	1.000	0.009	0.999	1.000	0.029	0.999	1.000
$r_P = 2$	$q_P = 8$	0.025	0.996	1.000	0.017	0.999	1.000	0.028	0.999	1.000

Table S10: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 2$

					$P = 100$, $(\rho, \phi) = (0.2, 0.2)$					
			$\kappa = 78$			$\kappa = 253$			$\kappa = 528$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.032	0.933	0.999	0.020	0.909	0.995	0.021	0.899	0.999
$r_P = 0$	$q_P = 5$	0.022	0.907	0.997	0.024	0.901	0.997	0.018	0.923	0.996
$r_P = 0$	$q_P = 6$	0.023	0.922	0.997	0.019	0.909	0.996	0.030	0.935	0.999
$r_P = 0$	$q_P = 7$	0.023	0.901	0.996	0.026	0.920	0.993	0.020	0.922	0.994
$r_P = 0$	$q_P = 8$	0.023	0.922	0.997	0.022	0.937	0.997	0.014	0.929	0.995
$r_P = 1$	$q_P = 4$	0.023	0.917	0.996	0.024	0.903	0.994	0.012	0.946	0.995
$r_P = 1$	$q_P = 5$	0.026	0.905	0.997	0.019	0.881	0.997	0.015	0.935	0.994
$r_P = 1$	$q_P = 6$	0.022	0.898	0.997	0.025	0.913	0.997	0.012	0.933	0.996
$r_P = 1$	$q_P = 7$	0.017	0.920	0.996	0.020	0.911	0.996	0.014	0.917	0.993
$r_P = 1$	$q_P = 8$	0.017	0.896	0.993	0.017	0.915	0.997	0.015	0.900	0.994
$r_P = 2$	$q_P = 4$	0.017	0.921	0.997	0.016	0.918	0.993	0.015	0.911	0.997
$r_P = 2$	$q_P = 5$	0.018	0.912	0.995	0.017	0.901	0.997	0.011	0.913	0.994
$r_P = 2$	$q_P = 6$	0.017	0.901	0.997	0.017	0.910	0.996	0.011	0.948	0.994
$r_P = 2$	$q_P = 7$	0.018	0.907	0.995	0.015	0.887	0.990	0.010	0.922	0.994
$r_P = 2$	$q_P = 8$	0.030	0.904	0.997	0.017	0.891	0.996	0.007	0.903	0.996
					$P = 200$, $(\rho, \phi) = (0.2, 0.2)$					
			$\kappa = 78$			$\kappa = 253$			$\kappa = 528$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.030	1.000	1.000	0.021	0.999	1.000	0.030	0.998	1.000
$r_P = 0$	$q_P = 5$	0.032	1.000	1.000	0.028	0.999	1.000	0.022	0.998	1.000
$r_P = 0$	$q_P = 6$	0.036	1.000	1.000	0.029	0.999	1.000	0.017	0.998	1.000
$r_P = 0$	$q_P = 7$	0.039	1.000	1.000	0.022	0.999	1.000	0.021	0.998	1.000
$r_P = 0$	$q_P = 8$	0.030	1.000	1.000	0.022	0.999	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 4$	0.023	1.000	1.000	0.028	0.999	1.000	0.018	0.998	1.000
$r_P = 1$	$q_P = 5$	0.029	1.000	1.000	0.028	0.999	1.000	0.019	0.998	1.000
$r_P = 1$	$q_P = 6$	0.031	1.000	1.000	0.020	0.998	1.000	0.017	0.998	1.000
$r_P = 1$	$q_P = 7$	0.025	1.000	1.000	0.020	0.999	1.000	0.017	0.998	1.000
$r_P = 1$	$q_P = 8$	0.025	1.000	1.000	0.022	0.999	1.000	0.015	0.998	1.000
$r_P = 2$	$q_P = 4$	0.034	1.000	1.000	0.024	0.999	1.000	0.017	0.998	1.000
$r_P = 2$	$q_P = 5$	0.020	1.000	1.000	0.010	0.999	1.000	0.017	0.999	1.000
$r_P = 2$	$q_P = 6$	0.025	1.000	1.000	0.018	0.999	1.000	0.019	0.998	1.000
$r_P = 2$	$q_P = 7$	0.031	1.000	1.000	0.013	0.999	1.000	0.019	0.998	1.000
$r_P = 2$	$q_P = 8$	0.018	1.000	1.000	0.018	0.999	1.000	0.017	0.998	1.000

Table S11: Rejection Frequencies - Without Estimation Error - Inequality Spacing $k_S^c = 5$

		$P = 100$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 45$			$\kappa = 190$			$\kappa = 435$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.036	0.841	1.000	0.030	0.867	1.000	0.031	0.890	1.000
$r_P = 0$	$q_P = 5$	0.023	0.814	1.000	0.034	0.909	1.000	0.021	0.893	1.000
$r_P = 0$	$q_P = 6$	0.031	0.783	1.000	0.021	0.894	1.000	0.019	0.878	1.000
$r_P = 0$	$q_P = 7$	0.028	0.802	1.000	0.030	0.881	1.000	0.022	0.901	1.000
$r_P = 0$	$q_P = 8$	0.016	0.805	1.000	0.027	0.894	1.000	0.022	0.872	1.000
$r_P = 1$	$q_P = 4$	0.030	0.789	1.000	0.027	0.870	1.000	0.021	0.893	1.000
$r_P = 1$	$q_P = 5$	0.022	0.823	1.000	0.025	0.859	1.000	0.021	0.870	1.000
$r_P = 1$	$q_P = 6$	0.028	0.813	1.000	0.026	0.848	1.000	0.010	0.891	1.000
$r_P = 1$	$q_P = 7$	0.030	0.783	1.000	0.025	0.885	1.000	0.009	0.879	1.000
$r_P = 1$	$q_P = 8$	0.030	0.819	1.000	0.033	0.901	1.000	0.010	0.878	0.998
$r_P = 2$	$q_P = 4$	0.030	0.796	1.000	0.011	0.869	1.000	0.012	0.878	1.000
$r_P = 2$	$q_P = 5$	0.022	0.744	1.000	0.017	0.901	1.000	0.012	0.874	0.998
$r_P = 2$	$q_P = 6$	0.022	0.786	1.000	0.026	0.870	1.000	0.014	0.879	1.000
$r_P = 2$	$q_P = 7$	0.014	0.808	1.000	0.024	0.873	1.000	0.007	0.886	0.998
$r_P = 2$	$q_P = 8$	0.022	0.795	1.000	0.017	0.891	1.000	0.010	0.869	1.000
		$P = 200$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 45$			$\kappa = 190$			$\kappa = 435$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.065	0.995	1.000	0.035	0.997	1.000	0.044	0.993	1.000
$r_P = 0$	$q_P = 5$	0.044	0.995	1.000	0.021	0.995	1.000	0.035	0.993	1.000
$r_P = 0$	$q_P = 6$	0.044	0.988	1.000	0.019	0.995	1.000	0.041	0.992	1.000
$r_P = 0$	$q_P = 7$	0.045	0.995	1.000	0.018	0.995	1.000	0.028	0.992	1.000
$r_P = 0$	$q_P = 8$	0.035	0.993	1.000	0.027	0.995	1.000	0.028	0.993	1.000
$r_P = 1$	$q_P = 4$	0.038	0.986	1.000	0.041	0.995	1.000	0.020	0.992	1.000
$r_P = 1$	$q_P = 5$	0.045	0.990	1.000	0.022	0.996	1.000	0.027	0.993	1.000
$r_P = 1$	$q_P = 6$	0.037	0.982	1.000	0.026	0.995	1.000	0.027	0.992	1.000
$r_P = 1$	$q_P = 7$	0.038	0.988	1.000	0.021	0.995	1.000	0.019	0.993	1.000
$r_P = 1$	$q_P = 8$	0.037	0.995	1.000	0.017	0.994	1.000	0.018	0.993	1.000
$r_P = 2$	$q_P = 4$	0.046	0.988	1.000	0.023	0.994	1.000	0.021	0.993	1.000
$r_P = 2$	$q_P = 5$	0.037	0.982	1.000	0.022	0.995	1.000	0.027	0.993	1.000
$r_P = 2$	$q_P = 6$	0.045	0.991	1.000	0.016	0.995	1.000	0.014	0.993	1.000
$r_P = 2$	$q_P = 7$	0.044	0.987	1.000	0.021	0.993	1.000	0.013	0.993	1.000
$r_P = 2$	$q_P = 8$	0.048	0.986	1.000	0.019	0.995	1.000	0.016	0.991	1.000

Table S12: Rejection Frequencies - Without Estimation Error - Adjacent-only Inequalities

		$P = 100$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 14$			$\kappa = 24$			$\kappa = 34$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.037	0.941	0.103	0.040	0.961	0.078	0.021	0.944	0.050
$r_P = 0$	$q_P = 5$	0.039	0.947	0.137	0.041	0.952	0.078	0.018	0.935	0.055
$r_P = 0$	$q_P = 6$	0.037	0.939	0.135	0.042	0.948	0.075	0.014	0.917	0.045
$r_P = 0$	$q_P = 7$	0.036	0.937	0.078	0.039	0.950	0.100	0.016	0.928	0.045
$r_P = 0$	$q_P = 8$	0.035	0.938	0.118	0.032	0.946	0.059	0.014	0.933	0.042
$r_P = 1$	$q_P = 4$	0.027	0.946	0.098	0.041	0.952	0.053	0.029	0.942	0.050
$r_P = 1$	$q_P = 5$	0.037	0.938	0.157	0.047	0.943	0.059	0.016	0.942	0.054
$r_P = 1$	$q_P = 6$	0.035	0.919	0.111	0.036	0.951	0.071	0.016	0.926	0.046
$r_P = 1$	$q_P = 7$	0.033	0.945	0.102	0.037	0.938	0.062	0.011	0.935	0.054
$r_P = 1$	$q_P = 8$	0.035	0.938	0.089	0.043	0.952	0.075	0.015	0.932	0.047
$r_P = 2$	$q_P = 4$	0.037	0.941	0.131	0.038	0.955	0.078	0.012	0.925	0.045
$r_P = 2$	$q_P = 5$	0.035	0.939	0.124	0.036	0.943	0.055	0.016	0.928	0.050
$r_P = 2$	$q_P = 6$	0.036	0.944	0.107	0.037	0.943	0.067	0.014	0.926	0.054
$r_P = 2$	$q_P = 7$	0.028	0.944	0.135	0.035	0.937	0.062	0.016	0.932	0.062
$r_P = 2$	$q_P = 8$	0.035	0.948	0.124	0.035	0.948	0.075	0.015	0.927	0.048
		$P = 200$, $(\rho, \phi) = (0.2, 0.2)$								
			$\kappa = 14$			$\kappa = 24$			$\kappa = 34$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.048	1.000	0.202	0.052	1.000	0.108	0.032	1.000	0.068
$r_P = 0$	$q_P = 5$	0.036	1.000	0.226	0.054	1.000	0.080	0.031	1.000	0.065
$r_P = 0$	$q_P = 6$	0.037	1.000	0.172	0.054	1.000	0.097	0.035	1.000	0.078
$r_P = 0$	$q_P = 7$	0.034	1.000	0.190	0.048	1.000	0.072	0.036	1.000	0.062
$r_P = 0$	$q_P = 8$	0.029	1.000	0.186	0.051	1.000	0.090	0.030	1.000	0.056
$r_P = 1$	$q_P = 4$	0.031	1.000	0.179	0.054	1.000	0.109	0.055	1.000	0.057
$r_P = 1$	$q_P = 5$	0.040	1.000	0.246	0.048	1.000	0.079	0.035	1.000	0.060
$r_P = 1$	$q_P = 6$	0.030	1.000	0.181	0.051	1.000	0.103	0.037	1.000	0.060
$r_P = 1$	$q_P = 7$	0.023	1.000	0.175	0.051	1.000	0.091	0.029	1.000	0.062
$r_P = 1$	$q_P = 8$	0.031	1.000	0.187	0.031	1.000	0.081	0.031	1.000	0.054
$r_P = 2$	$q_P = 4$	0.034	1.000	0.227	0.051	1.000	0.082	0.030	1.000	0.060
$r_P = 2$	$q_P = 5$	0.023	1.000	0.230	0.050	1.000	0.082	0.028	1.000	0.064
$r_P = 2$	$q_P = 6$	0.037	1.000	0.209	0.054	1.000	0.101	0.032	1.000	0.043
$r_P = 2$	$q_P = 7$	0.038	1.000	0.220	0.051	1.000	0.106	0.029	1.000	0.049
$r_P = 2$	$q_P = 8$	0.024	1.000	0.172	0.048	1.000	0.099	0.030	1.000	0.060

Table S13: Rejection Frequencies - Small $S \in \{3, 5\}$ - All Inequalities

		$P = 100$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 3$			$\kappa = 10$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.060	0.793	0.490	0.070	0.861	0.814
$r_P = 0$	$q_P = 5$	0.048	0.790	0.460	0.063	0.840	0.775
$r_P = 0$	$q_P = 6$	0.067	0.802	0.459	0.057	0.781	0.802
$r_P = 0$	$q_P = 7$	0.057	0.740	0.438	0.056	0.803	0.788
$r_P = 0$	$q_P = 8$	0.048	0.762	0.479	0.061	0.772	0.759
$r_P = 1$	$q_P = 4$	0.070	0.823	0.504	0.076	0.856	0.815
$r_P = 1$	$q_P = 5$	0.055	0.783	0.514	0.076	0.842	0.815
$r_P = 1$	$q_P = 6$	0.054	0.788	0.440	0.075	0.845	0.791
$r_P = 1$	$q_P = 7$	0.040	0.814	0.457	0.061	0.830	0.803
$r_P = 1$	$q_P = 8$	0.055	0.768	0.441	0.047	0.839	0.784
$r_P = 2$	$q_P = 4$	0.054	0.835	0.554	0.076	0.868	0.828
$r_P = 2$	$q_P = 5$	0.045	0.757	0.486	0.074	0.825	0.788
$r_P = 2$	$q_P = 6$	0.049	0.775	0.487	0.093	0.776	0.726
$r_P = 2$	$q_P = 7$	0.070	0.773	0.440	0.059	0.802	0.750
$r_P = 2$	$q_P = 8$	0.045	0.793	0.478	0.056	0.835	0.814
		$P = 200$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 3$			$\kappa = 10$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.070	0.983	0.810	0.062	0.993	0.987
$r_P = 0$	$q_P = 5$	0.070	0.983	0.779	0.052	0.990	0.994
$r_P = 0$	$q_P = 6$	0.065	0.983	0.729	0.049	0.990	0.991
$r_P = 0$	$q_P = 7$	0.057	0.980	0.782	0.053	0.982	0.988
$r_P = 0$	$q_P = 8$	0.057	0.985	0.810	0.045	0.984	0.994
$r_P = 1$	$q_P = 4$	0.081	0.986	0.773	0.075	0.994	0.991
$r_P = 1$	$q_P = 5$	0.051	0.983	0.781	0.053	0.990	0.988
$r_P = 1$	$q_P = 6$	0.044	0.985	0.778	0.045	0.990	0.992
$r_P = 1$	$q_P = 7$	0.058	0.980	0.795	0.049	0.990	0.991
$r_P = 1$	$q_P = 8$	0.065	0.987	0.800	0.036	0.982	0.987
$r_P = 2$	$q_P = 4$	0.065	0.983	0.778	0.062	0.984	0.990
$r_P = 2$	$q_P = 5$	0.059	0.983	0.794	0.053	0.990	0.991
$r_P = 2$	$q_P = 6$	0.076	0.981	0.750	0.045	0.981	0.987
$r_P = 2$	$q_P = 7$	0.058	0.980	0.760	0.049	0.981	0.987
$r_P = 2$	$q_P = 8$	0.058	0.982	0.794	0.045	0.993	0.994

Table S14: Rejection Frequencies - Small $S \in \{3, 5\}$ - Adjacent-only Inequalities

		$P = 100$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 2$			$\kappa = 4$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.083	0.795	0.511	0.055	0.799	0.416
$r_P = 0$	$q_P = 5$	0.091	0.819	0.509	0.084	0.848	0.437
$r_P = 0$	$q_P = 6$	0.070	0.832	0.439	0.069	0.792	0.428
$r_P = 0$	$q_P = 7$	0.069	0.796	0.507	0.063	0.800	0.477
$r_P = 0$	$q_P = 8$	0.083	0.773	0.435	0.073	0.790	0.481
$r_P = 1$	$q_P = 4$	0.074	0.804	0.562	0.077	0.789	0.449
$r_P = 1$	$q_P = 5$	0.085	0.809	0.496	0.068	0.790	0.437
$r_P = 1$	$q_P = 6$	0.071	0.823	0.460	0.041	0.796	0.415
$r_P = 1$	$q_P = 7$	0.083	0.814	0.492	0.062	0.834	0.396
$r_P = 1$	$q_P = 8$	0.072	0.823	0.468	0.059	0.824	0.425
$r_P = 2$	$q_P = 4$	0.087	0.807	0.562	0.088	0.840	0.477
$r_P = 2$	$q_P = 5$	0.084	0.768	0.457	0.062	0.792	0.414
$r_P = 2$	$q_P = 6$	0.085	0.823	0.492	0.058	0.830	0.448
$r_P = 2$	$q_P = 7$	0.075	0.805	0.512	0.058	0.780	0.403
$r_P = 2$	$q_P = 8$	0.069	0.806	0.492	0.050	0.811	0.403
		$P = 200$, $(\rho, \phi) = (0.5, 0.5)$					
			$\kappa = 2$			$\kappa = 4$	
		DGP-N	DGP-A1	DGP-A2	DGP-N	DGP-A1	DGP-A2
$r_P = 0$	$q_P = 4$	0.077	0.982	0.767	0.058	0.991	0.775
$r_P = 0$	$q_P = 5$	0.066	0.984	0.780	0.056	0.988	0.748
$r_P = 0$	$q_P = 6$	0.057	0.984	0.730	0.067	0.985	0.748
$r_P = 0$	$q_P = 7$	0.061	0.981	0.777	0.072	0.992	0.715
$r_P = 0$	$q_P = 8$	0.056	0.982	0.744	0.053	0.985	0.748
$r_P = 1$	$q_P = 4$	0.057	0.982	0.757	0.062	0.989	0.772
$r_P = 1$	$q_P = 5$	0.089	0.982	0.771	0.058	0.988	0.763
$r_P = 1$	$q_P = 6$	0.070	0.984	0.780	0.065	0.989	0.716
$r_P = 1$	$q_P = 7$	0.055	0.980	0.743	0.045	0.986	0.755
$r_P = 1$	$q_P = 8$	0.057	0.975	0.780	0.066	0.988	0.789
$r_P = 2$	$q_P = 4$	0.079	0.987	0.814	0.055	0.984	0.665
$r_P = 2$	$q_P = 5$	0.070	0.984	0.791	0.058	0.988	0.726
$r_P = 2$	$q_P = 6$	0.070	0.974	0.759	0.045	0.992	0.750
$r_P = 2$	$q_P = 7$	0.057	0.984	0.754	0.045	0.988	0.733
$r_P = 2$	$q_P = 8$	0.057	0.982	0.761	0.045	0.989	0.723

S11 Additional Tables: Empirical Application

S11.1 List of Variables

Table S15: Monthly Variables and Quarterly GDP Component Target Variables

FRED Code	Description	Publication Lag
UMCSENTx	Consumer Sentiment Index	-2
S&P 500	S&Ps Common Stock Price Index: Composite	0
TB3MS	3-Month Treasury Bill	0
GS10	10-Year Treasury Rate	0
TWEXMMTH	Trade Weighted U.S. Dollar Index: Major Currencies	0
EXSZUSx	Switzerland/U.S. Foreign Exchange Rate	0
EXJPUSx	Japan/U.S. Foreign Exchange Rate	0
EXUSUKx	U.S./U.K. Foreign Exchange Rate	0
EXCAUSx	Canada/U.S. Foreign Exchange Rate	0
CE16OV	Civilian Employment	1
UNRATE	Civilian Unemployment Rate	1
PAYEMS	All Employees: Total nonfarm	1
USGOOD	All Employees: Goods-Producing Industries	1
CES1021000001	All Employees: Mining and Logging: Mining	1
USCONS	All Employees: Construction	1
MANEMP	All Employees: Manufacturing	1
DMANEMP	All Employees: Durable goods	1
NDMANEMP	All Employees: Nondurable goods	1
SRVPRD	All Employees: Service-Providing Industries	1
USTPU	All Employees: Trade, Transportation & Utilities	1
USWTRADE	All Employees: Wholesale Trade	1
USTRADE	All Employees: Retail Trade	1
USFIRE	All Employees: Financial Activities	1
USGOVT	All Employees: Government	1
CES0600000007	Avg Weekly Hours: Goods-Producing	1
AWOTMAN	Avg Weekly Overtime Hours: Manufacturing	1
AWHMAN	Avg Weekly Hours: Manufacturing	1
M2SL	M2 Money Stock	2
BUSLOANS	Commercial and Industrial Loans	2
REALLN	Real Estate Loans at All Commercial Banks	2
CLAIMSx	Initial Claims	7
WPSFD49207	PPI: Finished Goods	13
RETAILx	Retail and Food Services Sales	15
INDPRO	IP Index	15
IPFINAL	IP: Final Products (Market Group)	15
IPCONGD	IP: Consumer Goods	15
IPDCONGD	IP: Durable Consumer Goods	15
IPNCONGD	IP: Nondurable Consumer Goods	15
IPBUSEQ	IP: Business Equipment	15
IPMAT	IP: Materials	15
IPDMAT	IP: Durable Materials	15
IPNMAT	IP: Nondurable Materials	15
IPMANSICS	IP: Manufacturing (SIC)	15
IPB51222S	IP: Residential Utilities	15
IPFUELS	IP: Fuels	15
CUMFNS	Capacity Utilization: Manufacturing	15
HOUST	Housing Starts: Total New Privately Owned	19
HOUSTNE	Housing Starts, Northeast	19
HOUSTMW	Housing Starts, Midwest	19
HOUSTS	Housing Starts, South	19

HOUSTW	Housing Starts, West	19
PERMIT	New Private Housing Permits (SAAR)	19
PERMITNE	New Private Housing Permits, Northeast (SAAR)	19
PERMITMW	New Private Housing Permits, Midwest (SAAR)	19
PERMITS	New Private Housing Permits, South (SAAR)	19
PERMITW	New Private Housing Permits, West (SAAR)	19
AMDMNOx	New Orders for Durable Goods	27
ANDENOx	New Orders for Nondefense Capital Goods	27
DPCERA3M086SBEA	Real personal consumption expenditures	29
CMRMTSPLx	Real Manu. and Trade Industries Sales	30
DTCOLNVHFNM	Consumer Motor Vehicle Loans Outstanding	48
DTCTHFNM	Total Consumer Loans and Leases Outstanding	48
GDP	Real Gross Domestic Product	28
CONS	Real Personal Consumption Expenditures	28
INV	Real Gross Private Domestic Investment	28
GOV	Real Gov Consumption Exp & Gross Inv	28
EXP	Real Exports of Goods & Services	28
IMP	Real Imports of Goods & Services	28

S11.2 Additional Results

This section provides several additional sets of results, as laid out in the main text these are as follows: (i) rolling estimation scheme, (ii) reduced dataset of $N = 9$, (iii) nowcasts evaluated at the end of each day (iv) nowcasts evaluated once every 10 days, (v) post-1984 sample, (vi) using the Linex loss function.

Table S16: Monotonicity Test Results by GDP Sub-Component - Rolling Estimation

		All Inequalities ($\kappa = 1378$)				Spacing k_S^c=5 ($\kappa = 1128$)				Adjacent-only ($\kappa = 52$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.212	1.871	2.162	0.764	0.150	1.555	1.964	0.840	0.150	0.682	0.856	0.687
	CONS	0.342	1.115	1.392	0.464	0.342	1.095	1.267	0.444	0.193	0.341	0.465	0.393
$R = 101$	INV	9.470	53.967	64.167	0.549	8.477	53.235	65.384	0.586	8.647	31.313	38.905	0.504
$P = 51$	GOV	2.220	1.724	1.979	0.038	2.220	1.766	2.232	0.050	0.608	0.638	0.773	0.118
	EXP	1.312	25.934	32.259	0.852	0.879	24.411	30.538	0.812	1.300	8.189	11.206	0.684
	IMP	1.446	29.942	39.007	0.852	1.446	33.422	44.297	0.820	0.920	8.687	10.886	0.860
	GDP	0.292	1.694	2.281	0.682	0.258	1.820	2.253	0.682	0.280	0.806	1.083	0.559
	CONS	0.243	0.924	1.175	0.456	0.243	1.211	1.521	0.534	0.083	0.261	0.351	0.609
$R = 87$	INV	11.262	38.042	46.693	0.566	10.306	41.313	51.586	0.561	10.915	30.748	40.586	0.496
$P = 65$	GOV	2.568	1.670	1.896	0.008	2.568	1.416	1.678	0.000	0.491	0.602	0.706	0.168
	EXP	1.574	19.583	25.329	0.744	1.145	19.645	25.364	0.827	1.503	7.289	8.752	0.657
	IMP	1.677	24.381	31.324	0.764	1.677	31.029	36.915	0.694	1.321	8.407	10.114	0.679
	GDP	0.303	1.728	2.238	0.679	0.280	1.334	1.670	0.574	0.209	0.689	0.822	0.596
	CONS	0.364	0.745	1.005	0.296	0.364	0.901	1.235	0.393	0.165	0.219	0.258	0.226
$R = 76$	INV	5.723	35.554	43.825	0.669	4.822	39.459	47.108	0.702	5.248	12.805	17.964	0.444
$P = 76$	GOV	2.401	1.279	1.552	0.000	2.401	1.286	1.510	0.003	0.607	0.547	0.674	0.075
	EXP	2.614	22.437	27.681	0.692	2.614	21.916	30.066	0.707	1.968	6.360	8.638	0.544
	IMP	1.915	22.963	29.666	0.807	1.915	14.617	16.941	0.742	1.042	11.376	13.943	0.865
	GDP	0.348	1.267	1.536	0.624	0.296	1.355	1.619	0.707	0.260	0.678	0.867	0.612
	CONS	0.246	0.896	1.197	0.451	0.246	0.939	1.242	0.501	0.098	0.262	0.294	0.639
$R = 65$	INV	4.869	44.934	54.034	0.832	4.869	41.239	50.405	0.840	3.244	17.375	26.570	0.789
$P = 87$	GOV	2.240	1.503	1.723	0.013	2.240	1.322	1.638	0.005	0.462	0.528	0.656	0.168
	EXP	3.578	21.273	26.691	0.759	3.578	21.324	26.357	0.752	3.401	12.424	15.140	0.607
	IMP	3.371	31.307	42.753	0.684	3.371	30.895	38.542	0.639	1.743	5.771	6.516	0.782
	GDP	0.398	1.367	1.690	0.561	0.365	1.253	1.668	0.649	0.288	0.469	0.562	0.313
	CONS	0.369	0.689	0.899	0.361	0.369	0.827	0.995	0.419	0.161	0.241	0.270	0.326
$R = 51$	INV	7.989	31.808	42.182	0.564	7.989	29.493	35.849	0.642	4.745	18.114	22.973	0.539
$P = 101$	GOV	2.345	1.589	1.816	0.013	2.345	1.859	2.211	0.033	0.570	0.603	0.692	0.125
	EXP	3.896	19.821	26.263	0.782	3.896	21.830	27.048	0.792	3.095	7.492	10.188	0.564
	IMP	2.297	24.640	30.984	0.732	2.297	24.122	29.306	0.722	1.515	12.348	15.133	0.810

Notes: The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

Table S17: Monotonicity Test Results by GDP Sub-Component - $N = 9$

		All Inequalities ($\kappa = 253$)				Spacing k_S^c=5 ($\kappa = 153$)				Adjacent-only ($\kappa = 22$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.396	2.423	2.837	0.599	0.065	1.938	2.441	0.905	0.298	1.141	1.507	0.476
	CONS	0.615	1.520	1.857	0.429	0.252	1.489	1.794	0.591	0.615	0.732	0.904	0.190
$R = 101$	INV	10.320	64.123	78.530	0.599	0.856	62.132	75.552	0.955	10.320	41.744	52.339	0.544
$P = 51$	GOV	1.710	1.562	1.826	0.075	1.710	1.606	1.944	0.070	0.504	0.770	0.932	0.398
	EXP	5.196	20.475	26.689	0.501	5.196	11.253	13.104	0.471	3.262	5.553	6.897	0.411
	IMP	5.276	35.738	50.652	0.612	-2.713	36.752	50.428	0.980	5.276	17.471	23.854	0.481
	GDP	0.499	1.676	2.223	0.449	0.112	1.639	1.989	0.860	0.432	0.853	1.071	0.356
	CONS	0.570	1.376	1.706	0.393	0.378	1.266	1.551	0.446	0.570	0.796	0.945	0.221
$R = 87$	INV	19.798	47.398	56.467	0.469	12.149	43.254	57.164	0.596	17.516	40.395	49.209	0.331
$P = 65$	GOV	2.541	1.414	1.627	0.000	2.541	1.401	1.802	0.008	0.516	0.765	0.920	0.404
	EXP	7.508	17.144	22.208	0.396	7.508	18.707	22.314	0.356	4.744	8.297	10.004	0.326
	IMP	6.300	30.234	38.494	0.534	2.239	30.564	38.372	0.712	6.300	11.498	15.411	0.348
	GDP	0.472	1.657	2.044	0.501	0.171	1.825	2.228	0.805	0.409	0.960	1.283	0.378
	CONS	0.558	1.304	1.688	0.336	0.359	1.209	1.430	0.461	0.558	0.621	0.697	0.160
$R = 76$	INV	19.176	34.161	38.575	0.333	10.460	48.689	65.887	0.622	16.552	21.419	26.353	0.173
$P = 76$	GOV	2.936	1.658	1.886	0.000	2.936	1.618	2.020	0.003	0.585	0.687	0.776	0.170
	EXP	6.932	20.457	22.560	0.386	6.932	19.597	25.605	0.358	3.591	7.050	8.606	0.373
	IMP	5.858	14.365	18.593	0.386	2.435	15.926	18.865	0.684	5.858	12.282	16.765	0.346
	GDP	0.417	1.777	2.451	0.654	0.255	1.851	2.275	0.744	0.371	1.100	1.325	0.554
	CONS	0.517	1.267	1.575	0.434	0.373	1.195	1.564	0.491	0.517	0.596	0.719	0.168
$R = 65$	INV	17.441	48.155	59.606	0.499	12.148	59.362	76.218	0.617	15.142	39.977	49.973	0.378
$P = 87$	GOV	2.591	1.790	2.047	0.005	2.591	1.572	1.920	0.003	0.587	0.791	0.920	0.278
	EXP	7.006	19.593	26.326	0.401	7.006	20.401	23.582	0.416	3.572	8.560	11.086	0.481
	IMP	5.500	30.463	37.627	0.632	2.036	30.473	39.209	0.734	5.500	14.303	18.276	0.456
	GDP	0.432	1.425	1.842	0.471	0.337	1.409	1.823	0.561	0.391	0.661	0.833	0.283
	CONS	0.469	1.140	1.400	0.411	0.412	0.700	0.843	0.346	0.469	0.648	0.802	0.263
$R = 51$	INV	16.006	46.778	56.228	0.516	9.215	46.743	56.185	0.672	12.816	26.670	33.750	0.363
$P = 101$	GOV	3.017	1.651	2.027	0.003	3.017	1.775	2.041	0.003	0.810	0.642	0.747	0.030
	EXP	12.137	16.219	19.558	0.195	12.137	16.662	21.083	0.198	3.457	5.753	7.115	0.388
	IMP	5.126	24.807	33.128	0.629	-0.541	27.378	35.031	0.932	5.126	10.636	13.376	0.436

Notes: The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

Table S18: Monotonicity Test Results by GDP Sub-Component - Nowcasts Evaluated Every 10 Days

		All Inequalities ($\kappa = 66$)				Spacing k_S^c=5 ($\kappa = 21$)				Adjacent-only ($\kappa = 11$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.069	1.669	1.905	0.967	-0.286	1.349	1.690	0.992	0.047	0.797	1.129	0.950
	CONS	0.100	1.247	1.592	0.564	-0.160	1.218	1.604	0.784	0.100	0.488	0.611	0.461
$R = 101$	INV	4.379	47.606	57.706	0.744	-5.014	43.866	57.129	0.952	1.929	23.764	29.727	0.835
$P = 51$	GOV	1.022	1.049	1.197	0.115	1.022	0.918	1.218	0.075	0.717	0.730	0.858	0.113
	EXP	0.882	16.203	19.903	0.794	-2.789	15.573	20.144	0.699	0.665	7.070	9.533	0.787
	IMP	1.693	28.765	37.859	0.717	-10.990	35.803	44.659	0.952	1.401	11.110	13.609	0.672
	GDP	0.090	1.294	1.636	0.877	-0.389	1.455	1.825	0.990	0.045	0.624	0.767	0.932
	CONS	0.130	1.377	1.679	0.516	-0.052	1.126	1.491	0.697	0.129	0.504	0.687	0.466
$R = 87$	INV	5.376	31.559	41.289	0.674	-4.783	36.982	45.612	0.950	2.323	22.914	29.771	0.764
$P = 65$	GOV	1.607	1.170	1.355	0.033	1.607	0.992	1.291	0.033	0.582	0.709	0.823	0.173
	EXP	0.995	11.515	14.684	0.815	-2.250	15.559	19.368	0.774	0.838	5.725	7.034	0.734
	IMP	1.529	27.694	34.859	0.694	-8.413	27.994	34.528	0.870	1.194	10.293	13.311	0.622
	GDP	0.083	1.312	1.634	0.867	-0.461	1.305	1.640	0.970	0.051	0.818	1.030	0.915
	CONS	0.170	1.254	1.674	0.499	-0.116	1.214	1.457	0.724	0.136	0.394	0.505	0.479
$R = 76$	INV	4.081	33.919	43.112	0.749	-3.551	35.470	45.846	0.920	2.360	19.696	24.247	0.764
$P = 76$	GOV	1.670	1.164	1.416	0.028	1.670	1.052	1.443	0.035	0.604	0.540	0.630	0.058
	EXP	0.835	18.798	22.067	0.885	-3.388	17.523	21.669	0.797	0.835	6.639	8.135	0.732
	IMP	1.627	12.359	16.659	0.684	-8.845	12.271	15.889	0.982	1.386	9.803	12.277	0.664
	GDP	0.096	1.419	1.829	0.890	-0.515	1.433	1.846	0.987	0.056	0.730	0.895	0.927
	CONS	0.187	0.988	1.279	0.466	-0.091	1.090	1.343	0.752	0.139	0.524	0.666	0.479
$R = 65$	INV	4.189	41.940	51.148	0.744	-4.585	44.864	57.769	0.927	2.500	21.682	29.294	0.772
$P = 87$	GOV	1.466	0.947	1.092	0.020	1.466	0.963	1.138	0.013	0.543	0.662	0.757	0.180
	EXP	0.820	17.008	21.316	0.862	-1.677	16.833	20.471	0.679	0.820	7.999	9.629	0.805
	IMP	2.929	28.044	40.418	0.642	-7.808	27.517	34.501	0.845	2.366	8.765	11.277	0.604
	GDP	0.107	1.161	1.395	0.817	-0.493	1.068	1.330	0.960	0.089	0.433	0.514	0.872
	CONS	0.157	1.089	1.364	0.614	-0.051	0.878	1.120	0.712	0.137	0.380	0.468	0.514
$R = 51$	INV	3.651	25.080	29.975	0.749	-4.983	24.070	31.561	0.900	2.461	15.602	18.589	0.729
$P = 101$	GOV	1.379	1.137	1.504	0.068	1.379	1.091	1.334	0.043	0.492	0.642	0.737	0.201
	EXP	0.713	9.627	11.852	0.855	-0.004	15.361	18.715	0.622	0.713	6.372	8.322	0.852
	IMP	3.527	27.127	33.402	0.546	-6.547	23.628	30.453	0.862	3.076	6.529	8.118	0.386

Notes: The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

Table S19: Monotonicity Test Results by GDP Sub-Component - Nowcasts Evaluated at end-of-day

		All Inequalities ($\kappa = 703$)				Spacing k_S^c=5 ($\kappa = 528$)				Adjacent-only ($\kappa = 37$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.265	1.949	2.213	0.772	0.181	1.535	1.962	0.835	0.190	0.688	0.920	0.604
	CONS	0.113	1.284	1.592	0.617	0.099	1.260	1.501	0.599	0.097	0.381	0.551	0.509
$R = 101$	INV	6.270	51.630	64.052	0.689	5.125	50.598	61.723	0.764	6.025	22.822	27.101	0.612
$P = 51$	GOV	1.040	1.230	1.428	0.140	1.040	1.248	1.431	0.165	0.729	0.769	0.930	0.118
	EXP	1.044	18.659	23.106	0.920	1.036	17.060	22.462	0.835	0.704	7.835	9.447	0.880
	IMP	1.826	34.240	43.300	0.724	1.826	36.066	48.349	0.707	0.849	11.200	14.166	0.810
	GDP	0.198	1.618	2.119	0.817	0.166	1.643	2.030	0.815	0.150	0.595	0.756	0.709
	CONS	0.213	1.159	1.332	0.466	0.213	1.257	1.588	0.506	0.119	0.381	0.480	0.496
$R = 87$	INV	5.376	42.746	50.830	0.774	5.376	44.075	52.992	0.722	4.870	21.654	29.515	0.647
$P = 65$	GOV	1.623	1.369	1.584	0.038	1.623	1.124	1.345	0.020	0.633	0.841	0.958	0.185
	EXP	1.342	11.680	15.872	0.850	1.167	13.317	17.089	0.812	0.788	6.160	7.409	0.794
	IMP	1.773	27.892	34.724	0.719	1.773	28.370	39.736	0.649	0.749	7.719	9.354	0.805
	GDP	0.234	1.486	1.946	0.717	0.166	1.564	1.979	0.742	0.168	0.609	0.836	0.632
	CONS	0.253	1.170	1.428	0.491	0.253	1.257	1.564	0.486	0.137	0.309	0.365	0.441
$R = 76$	INV	4.660	46.334	58.005	0.764	4.081	17.600	21.049	0.729	4.660	18.305	23.128	0.581
$P = 76$	GOV	1.670	1.217	1.574	0.040	1.670	1.274	1.504	0.033	0.621	0.574	0.670	0.073
	EXP	1.203	19.557	24.930	0.910	0.896	18.747	23.523	0.910	0.752	5.156	6.271	0.905
	IMP	1.733	13.616	17.207	0.749	1.733	14.514	17.953	0.727	0.768	9.926	12.734	0.787
	GDP	0.201	1.575	2.096	0.842	0.134	1.590	2.046	0.870	0.139	0.711	0.863	0.764
	CONS	0.263	1.073	1.354	0.449	0.263	1.055	1.320	0.439	0.136	0.426	0.546	0.424
$R = 65$	INV	4.271	45.484	58.874	0.872	4.271	44.499	55.974	0.877	4.217	26.155	32.888	0.767
$P = 87$	GOV	1.540	1.016	1.205	0.008	1.540	1.026	1.231	0.013	0.529	0.738	0.844	0.276
	EXP	1.432	15.506	20.947	0.832	1.432	16.583	20.441	0.762	0.887	6.058	7.730	0.845
	IMP	3.160	26.558	33.820	0.602	3.160	29.828	39.070	0.586	1.651	5.223	6.188	0.734
	GDP	0.194	1.127	1.492	0.752	0.130	1.278	1.495	0.895	0.150	0.353	0.421	0.607
	CONS	0.236	0.928	1.213	0.526	0.236	0.943	1.244	0.491	0.138	0.252	0.316	0.416
$R = 51$	INV	4.400	30.135	35.241	0.807	4.400	32.179	36.483	0.724	4.281	15.742	19.557	0.632
$P = 101$	GOV	1.466	1.250	1.372	0.043	1.466	1.093	1.255	0.028	0.497	0.615	0.779	0.180
	EXP	1.362	11.425	14.652	0.845	1.294	15.484	19.917	0.817	1.362	6.109	7.880	0.659
	IMP	3.802	22.650	27.311	0.514	3.802	22.606	28.326	0.489	1.914	6.465	8.015	0.566

Notes: The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

Table S20: Monotonicity Test Results by GDP Sub-Component - Post-1984

		All Inequalities ($\kappa = 1378$)				Spacing k_S^c=5 ($\kappa = 1128$)				Adjacent-only ($\kappa = 52$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.194	1.604	2.108	0.739	0.117	1.654	2.266	0.802	0.172	0.732	0.872	0.622
	CONS	0.564	1.493	1.844	0.316	0.564	1.277	1.785	0.326	0.317	0.403	0.486	0.195
$R = 86$	INV	11.837	51.257	68.996	0.471	10.146	59.163	70.372	0.559	10.999	34.008	46.145	0.401
$P = 43$	GOV	1.344	1.657	1.952	0.203	1.344	1.397	1.719	0.123	0.482	0.683	0.853	0.303
	EXP	1.367	29.219	36.187	0.777	0.865	29.693	34.219	0.842	1.364	11.216	13.941	0.714
	IMP	1.030	31.548	42.894	0.860	1.006	34.906	46.719	0.880	0.995	8.608	10.804	0.842
	GDP	0.175	1.481	1.812	0.739	0.115	1.686	2.032	0.832	0.156	0.697	0.924	0.637
	CONS	0.526	1.463	2.019	0.391	0.526	1.438	1.799	0.318	0.325	0.410	0.504	0.193
$R = 74$	INV	10.116	50.263	65.022	0.539	8.752	51.392	65.868	0.559	9.362	34.975	42.220	0.471
$P = 55$	GOV	2.332	1.612	1.873	0.020	2.332	1.685	1.871	0.020	0.686	0.855	1.006	0.193
	EXP	1.418	26.908	36.310	0.749	0.989	25.796	29.963	0.772	1.408	8.241	10.991	0.634
	IMP	1.057	30.956	40.764	0.875	1.057	29.220	35.255	0.845	0.812	7.617	9.554	0.877
	GDP	0.223	1.437	1.802	0.667	0.170	1.475	1.966	0.682	0.198	0.458	0.573	0.526
	CONS	0.599	1.185	1.482	0.321	0.599	1.167	1.452	0.273	0.352	0.432	0.506	0.185
$R = 65$	INV	8.840	34.860	42.610	0.591	7.447	38.120	46.462	0.692	8.374	24.402	32.094	0.471
$P = 64$	GOV	2.041	1.393	1.631	0.010	2.041	1.336	1.508	0.008	0.619	0.748	0.832	0.160
	EXP	1.542	19.101	26.378	0.709	1.077	20.367	26.096	0.774	1.454	6.745	9.197	0.619
	IMP	1.617	23.250	30.543	0.727	1.609	25.687	30.618	0.759	1.612	6.815	7.991	0.667
	GDP	0.206	1.605	1.975	0.627	0.152	1.256	1.558	0.622	0.184	0.539	0.647	0.591
	CONS	0.543	0.937	1.172	0.251	0.543	0.896	1.250	0.253	0.296	0.224	0.286	0.045
$R = 55$	INV	8.193	42.524	54.279	0.644	6.994	46.138	59.610	0.667	7.723	27.311	36.184	0.521
$P = 74$	GOV	1.895	1.256	1.461	0.025	1.895	1.365	1.589	0.018	0.650	0.653	0.849	0.105
	EXP	1.492	22.497	27.747	0.744	1.137	26.897	33.234	0.709	1.367	7.418	9.287	0.617
	IMP	1.618	26.007	31.394	0.707	1.618	28.942	36.195	0.777	1.549	7.631	9.502	0.699
	GDP	0.183	1.220	1.686	0.744	0.125	1.256	1.591	0.847	0.167	0.540	0.697	0.622
	CONS	0.476	0.907	1.148	0.276	0.476	0.891	1.119	0.298	0.252	0.314	0.370	0.188
$R = 43$	INV	8.002	48.192	60.732	0.669	6.834	47.334	62.343	0.669	7.521	23.796	27.842	0.476
$P = 86$	GOV	1.781	1.207	1.494	0.018	1.781	1.259	1.467	0.018	0.501	0.557	0.657	0.160
	EXP	1.379	24.693	29.812	0.722	1.009	21.606	27.291	0.742	1.270	7.394	9.757	0.627
	IMP	1.563	28.197	36.097	0.662	1.463	27.635	33.022	0.684	1.563	7.343	8.987	0.569

Notes: The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

Table S21: Monotonicity Test Results by GDP Sub-Component - Linex Loss

		All Inequalities ($\kappa = 1378$)				Spacing $k_S^c = 5$ ($\kappa = 1128$)				Adjacent-only ($\kappa = 52$)			
		U^*	90%	95%	p-val	U^*	90%	95%	p-val	U^*	90%	95%	p-val
	GDP	0.010	0.077	0.085	0.719	0.007	0.062	0.077	0.792	0.008	0.029	0.037	0.559
	CONS	0.006	0.045	0.057	0.549	0.006	0.046	0.055	0.546	0.002	0.014	0.020	0.619
$R = 101$	INV	0.622	4.938	6.049	0.556	0.561	4.869	5.862	0.634	0.600	1.511	1.793	0.529
$P = 51$	GOV	0.035	0.040	0.046	0.138	0.035	0.039	0.047	0.153	0.021	0.024	0.030	0.148
	EXP	0.104	1.524	1.998	0.737	0.063	1.639	2.083	0.712	0.102	0.601	0.745	0.669
	IMP	0.063	3.434	4.075	0.714	0.063	3.187	4.224	0.694	0.042	0.808	1.069	0.699
	GDP	0.008	0.057	0.071	0.744	0.005	0.065	0.084	0.820	0.007	0.023	0.028	0.554
	CONS	0.008	0.042	0.054	0.509	0.008	0.045	0.054	0.506	0.003	0.015	0.019	0.596
$R = 87$	INV	0.540	2.836	3.563	0.526	0.486	3.047	3.922	0.604	0.536	1.708	2.095	0.524
$P = 65$	GOV	0.058	0.047	0.061	0.060	0.058	0.049	0.062	0.058	0.019	0.023	0.030	0.155
	EXP	0.095	1.180	1.488	0.754	0.071	1.096	1.444	0.689	0.093	0.376	0.470	0.717
	IMP	0.064	2.336	3.059	0.647	0.064	2.337	3.164	0.637	0.039	0.593	0.759	0.727
	GDP	0.009	0.056	0.072	0.724	0.006	0.063	0.075	0.845	0.007	0.025	0.033	0.612
	CONS	0.008	0.034	0.044	0.451	0.008	0.034	0.043	0.446	0.003	0.016	0.021	0.586
$R = 76$	INV	0.500	3.410	4.557	0.526	0.452	3.696	5.289	0.591	0.497	1.324	1.660	0.429
$P = 76$	GOV	0.061	0.038	0.048	0.025	0.061	0.041	0.046	0.025	0.018	0.021	0.026	0.150
	EXP	0.089	1.272	1.715	0.807	0.068	1.416	1.682	0.812	0.087	0.472	0.578	0.712
	IMP	0.060	2.165	2.760	0.714	0.060	2.586	3.094	0.719	0.036	0.643	0.849	0.732
	GDP	0.008	0.062	0.078	0.787	0.005	0.061	0.077	0.862	0.006	0.026	0.032	0.692
	CONS	0.008	0.043	0.057	0.414	0.008	0.039	0.052	0.449	0.003	0.014	0.018	0.556
$R = 65$	INV	0.457	3.782	4.339	0.584	0.410	3.667	4.763	0.664	0.454	1.223	1.552	0.456
$P = 87$	GOV	0.053	0.044	0.058	0.065	0.053	0.049	0.060	0.083	0.016	0.023	0.027	0.253
	EXP	0.084	1.381	1.680	0.835	0.063	1.531	1.857	0.797	0.082	0.528	0.625	0.762
	IMP	0.089	2.160	3.032	0.604	0.089	2.290	3.208	0.594	0.046	0.616	0.798	0.687
	GDP	0.008	0.048	0.058	0.734	0.005	0.046	0.062	0.845	0.006	0.018	0.023	0.521
	CONS	0.007	0.035	0.045	0.531	0.007	0.034	0.044	0.501	0.003	0.011	0.014	0.581
$R = 51$	INV	0.438	2.521	3.170	0.619	0.394	2.332	3.066	0.612	0.435	0.967	1.181	0.436
$P = 101$	GOV	0.049	0.046	0.054	0.088	0.049	0.047	0.056	0.090	0.014	0.017	0.020	0.198
	EXP	0.083	0.974	1.170	0.817	0.058	0.793	1.130	0.810	0.082	0.315	0.368	0.717
	IMP	0.117	1.836	2.627	0.516	0.117	2.455	3.124	0.481	0.063	0.570	0.758	0.481

Notes: These results use the Linex loss function $L(x) = \exp(\alpha x) - \alpha x - 1$ and we set the parameter $\alpha = -1/4$ so that there is aversion towards negative values. The p-value is the one-sided rejection probability based on the empirical bootstrap distribution.

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