Supplementary Material for "The Modified-Half-Normal Distribution: Properties and an Efficient Sampling Scheme"

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1. The Fox-Wright function and a few of its properties.

In this section, we present the general form of the Fox-Wright function (Fox, 1928; Wright, 1935) and identify a specific case that is required for studying the Modified-Half-Normal distribution. We include a brief discourse to familiarize the related notions and to investigate certain aspects of the function.

The general class of the Fox-Wright Psi function, introduced by Fox (1928) and Wright (1935), is defined as

$$ {}_p\Psi_q \left[\begin{matrix} (a_1, A_1) & (a_2, A_2) & \dots & (a_p, A_p) \\ (b_1, B_1) & (b_2, B_2) & \dots & (b_q, B_q) \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_q + B_q n)} \frac{z^n}{n!}, $$

where p, q are non-negative integers, $A_l \geq 0$ for $l = 1 \dots p$ and $B_m \geq 0$ for $m = 1, \dots q$. The constants a_l for $l = 1, \dots, p$ and b_m for $m = 1, \dots, q$ are allowed to be any nonzero complex numbers(Craven and Csordas, 2006; Mainardi and Pagnini, 2007) but for this manuscript we only need the case when they are positive real numbers. The above series converges absolutely for $z \in \mathbb{C}$ if $\sum_{l=1}^{q} B_l - \sum_{m=1}^{p} A_m + 1$. Though the function is defined for $z \in \mathbb{C}$, in this manuscript we will only be needing the case when z belongs to the real line. The usage of the function is seen in different branches of science. Many of its properties are documented for years while the related scientific investigation is still an active area of research (Wang, 2019; Wei, 2019). In the context of the current article, we streamline our focus on

$$ {}_1\Psi_1 \left[\begin{matrix} (\frac{\alpha}{2}, \frac{1}{2}) \\ (1, 0) \end{matrix} ; x \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\alpha}{2} + \frac{1}{2} n)}{n!} x^n, \tag{1} $$

a specific case of Fox-Wright function which is required not only to represent the normalizing constant of the MHN (α, β, γ) distribution (See Section 3) but also to apprise the efficiency of the rejection sampling algorithms developed in Section 3. Note that for the brevity of the expression we have used the notation $\Psi\left[\frac{\alpha}{2}, x\right] := {}_1\Psi_1 \left[\begin{matrix} (\frac{\alpha}{2}, \frac{1}{2}) \\ (1, 0) \end{matrix} ; x \right]$ for the rest of the document. The availability of computational algorithms for $\Psi\left[\frac{\alpha}{2}, x\right]$ and/or the ratio of the

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appropriate Fox-Wright functions are crucial for computing mean, variance, skewness and other moment based statistics (See Section 3 for details).

As a note in passing, we would like to mention that the Fox-Wright Functions are closely connected with ${}_pF_q$, the hypergeometric functions (Srivastava, 2007; Mehrez and Sitnik, 2019). In particular,

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,1)\ldots(a_p,1)\\(b_1,1)\ldots(b_q,1)\end{matrix};z\right]=\frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}{}_pF_q\left[\begin{matrix}a_1\ldots a_p\\b_1\ldots b_q\end{matrix};z\right] \tag{2} $$

when $A_1,...,A_p=1$ and $B_1,...,B_p=1$(Mehrez and Sitnik, 2019). As a result, many of the attributes including computation of the Fox-Wright function follows directly from that of the ${}_pF_q$ whenever the functional equality in Equation 2 is applicable. Unfortunately, it is not applicable to the specific case of the Fox-Wright function (as in Equation 1) that corresponds to the MHN (α,β,γ) distribution. Therefore, we invested a nontrivial effort design strategies for its computation which is require for the subsequent developments. Additionally, we develop a few generic properties of Fox-Wright Functions that we utilize in the later sections.

Lemma 9. *Let $\Psi\left[\frac{\alpha+n}{2},x\right]$ denotes the Fox-Wright Function. The following functional properties holds for all $x\in\mathbb{R}$.*

(a) If $\alpha+n>2$ then

$$\Psi\left[\frac{\alpha+n}{2},x\right]=\frac{\alpha+n-2}{2}\Psi\left[\frac{\alpha+n-2}{2},x\right]+\frac{x}{2}\Psi\left[\frac{\alpha+n-1}{2},x\right].$$

(b) If $\alpha>1$ then

$$\frac{\Psi\left[\frac{\alpha+1}{2},x\right]}{\Psi\left[\frac{\alpha}{2},x\right]}=\frac{x}{2}+\frac{\alpha-1}{2}\cfrac{1}{\frac{x}{2}+\cdots\cfrac{\vdots}{\frac{x}{2}+\frac{1+\delta_\alpha}{2}\cfrac{1}{\cfrac{\Psi\left[\frac{1+\delta_\alpha}{2},x\right]}{\Psi\left[\frac{\delta_\alpha}{2},x\right]}}}},$$

$\delta_\alpha=1$ if α is an integer otherwise $\delta_\alpha=\alpha-\lfloor\alpha\rfloor$ where $\lfloor\alpha\rfloor$ denotes the largest integer less than equal to α.

(c) If we denote the cumulative distribution function of the standard normal distribution as $\Phi(\cdot)$ then, $\frac{\Psi\left[\frac{2}{2},x\right]}{\Psi\left[\frac{1}{2},x\right]}=\frac{x}{2}+\frac{1}{2\sqrt{\pi}}\frac{\exp(-\frac{x^2}{4})}{1-\Phi(-\frac{x}{\sqrt{2}})}$, and $\Psi\left[\frac{1}{2},x\right]=2\sqrt{\pi}e^{\frac{x^2}{4}}\left(1-\Phi(-\frac{x}{\sqrt{2}})\right)$.

The proof of part(a) of the Lemma 9 essentially relies on the recurrence relation of the Gamma function while part(b) is a consequence of the part(a). The Part (c) of the proof utilizes the specific from of the normal probability density function restricted to the positive part of the real line. Though the statement of Lemma 9 does not show it explicitly, it can be employed for computation of the Fox-Wright function and their ratios when the α is a positive integer. Using part(c), we may evaluate the ratios of the FoxWright function without any further need of approximation. In the case when α is a positive integer, the continued fraction type recursive relation is an option for computing the moments, variance and skewness of the corresponding MHN (α, β, γ) distribution (see Lemma 2). The following Lemma is motivated by the series representation of the Fox-Wright function that is shown in the Equation 1.

Lemma 10. *Let $\alpha > 0, x \in \mathbb{R}$ then for arbitrary $\epsilon > 0$, let $A(k) = \frac{\Gamma(\frac{\alpha}{2}+k)x^{2k}}{(2k)!}$ and $B(k) = \frac{\Gamma(\frac{\alpha+1}{2}+k)x^{2k+1}}{(2k+1)!}$. Let $\lceil x \rceil$ denotes the smallest integer larger than x.*

(a) For a given constant $0 < q < 1$, $\frac{A(k+1)}{A(k)} < q$ if $k > C_1$, where the constant

$$C_1 = \max\left\{\left\lceil \frac{-(6q - x^2) + \sqrt{(6q - x^2)^2 - 8q(4q - \alpha x)}}{8q} \right\rceil, 1\right\}.$$

(b) For a given constant $0 < q < 1$, $\frac{B(k+1)}{B(k)} < q$ if $k > C_2$, where the constant

$$C_2 = \max\left\{\left\lceil \frac{-(10q - x^2) + \sqrt{(10q - x^2)^2 - 8q(12q - (\alpha+1)x)}}{8q} \right\rceil, 1\right\}.$$

(c) $A(k)$ is strictly decreasing function when $k \geq C_1$ and $B(k)$ is strictly decreasing function when $k \geq C_2$. Moreover $\lim_{k\to\infty} A(k) = 0$ and $\lim_{k\to\infty} B(k) = 0$.

(d) For a given $\epsilon > 0$, if $K = \max\{K_1, K_2\}$, $K_1 = \max\{\min\{k : A(K) \leq \frac{\epsilon}{4}$ for all $K \geq k\}, C_1\}$, $K_2 = \max\{\min\{k : |B(K)| \leq \frac{\epsilon}{4}$ for all $K \geq k\}, C_2\}$ then

$$\left| \Psi\left[\frac{\alpha}{2}, x\right] - \left(\sum_{k=0}^{K} A(k) + \sum_{k=0}^{K} B(k)\right)\right| \leq \epsilon.$$

The Lemma 10 provides a way to compute $\Psi\left[\frac{\alpha}{2}, x\right]$ by truncation of a infinite series in a systematic manner. Specifically from the part(d) of the Lemma identifies a truncation point K so that the finite summation $\sum_{k=0}^{K} A(k) + \sum_{k=0}^{K} B(k)$ can be used to approximate the intended function with a maximum predetermined error bound $\epsilon > 0$.

In the case when $\gamma < 0$, the series approximation procedure appears to be inefficient. From empirical experiments we observed that the accumulated errors contributed by the computation of the each terms of the $A(k)$ and $B(k)$ (Computation error that involving 'lgamma' function that is implemented in base R) appears to be significant compared to the functional value of Fox-Wright function. Hence we consider an alternative procedure premised on numerical integration. The strategy appears to perform efficiently for computing the function when $\gamma < 0$. In the following lemma, we truncate the infinite integral to that over a finite region in a systematic manner so that the error of the approximation can be controlled.

Lemma 11. *Let $\alpha > 0, \beta > 0$ and $\gamma < 0$. For any $m > 0$,*

$$\left| \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] - 2\beta^{\frac{\alpha}{2}} \int_0^{M_\epsilon} x^{\alpha-1} e^{-\beta x^2 - |\gamma| x} dx \right| < \epsilon.$$

when $M_\epsilon > \frac{1}{b}\gamma^{-1}\left(a, \frac{[\Gamma(a)]^2}{b^a} - \frac{\epsilon\Gamma(a)(2\beta m+|\gamma|)}{2\beta^{\frac{\alpha}{2}} m^\alpha(\beta m+|\gamma|)}\right)$. The constants $a = \frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}$, $b = \beta m^2 + |\gamma| m$, and the function $\gamma(a, bM_\epsilon)$ is a lower incomplete gamma function.

An implementation for the inverse of the lower incomplete gamma function is available in the R package "zipfR" (Evert and Baroni, 2007). So far, we have developed properties of the Fox-Wright function primarily aiming at its numerical computation for different cases. The following result provides functional approximation to the Fox-Wright function.

Lemma 12. *Let $\alpha > 1, \beta > 0$.*

(a) If $\gamma \leq 0$ then for any $m > 0$

$$\Psi\left[\frac{\alpha}{2}, \frac{-|\gamma|}{\sqrt{\beta}}\right] \geq \frac{\beta^{\frac{\alpha}{2}} \exp\left(-\frac{m|\gamma|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\left(\beta + \frac{|\gamma|}{2m}\right)^{\frac{\alpha}{2}}}.$$

(b) If $\gamma > 0$ then for any positive $\alpha_0 > 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\Psi\left[\frac{\alpha}{2}, \gamma\right] \geq \frac{2p^{\frac{\alpha+(2-\alpha_0)p-1}{2} - \frac{p}{q}} \left[\frac{\Gamma\left(\frac{\alpha+p\alpha_0-1}{2p}\right)}{2}\right]^p (q\gamma)^{p(\alpha_0-1)+\frac{p}{q}}}{\left[\Gamma\left(q\alpha_0 - q + 1\right)\right]^{\frac{p}{q}}}.$$

In particular if $p = \frac{\alpha}{\alpha-1}, q = \alpha, \alpha_0 = \frac{\alpha+1}{p}$ then

$$\Psi\left[\frac{\alpha}{2}, \gamma\right] \geq \frac{\left[\Gamma(\alpha-1)\right]^{\frac{\alpha}{\alpha-1}} (\alpha\gamma)^{\alpha}}{\left[2\Gamma(\alpha(\alpha-1))\right]^{\frac{1}{\alpha-1}}}.$$

The Lemma is utilized for establishing uniform efficiency of the accept reject algorithms that are designed later in this article.

2. Proof of the Lemma and Theorems

2.1. Proof of Lemma 9

2.1.1. Proof of Lemma 9 (a)

In the case when $\alpha > 1$, from the standard recurrence formula of the Gamma function we obtain that $\Gamma(\frac{\alpha+1+i}{2}) = \left(\frac{\alpha-1+i}{2}\right)\Gamma(\frac{\alpha-1+i}{2})$ for all $i \geq 0$. As a result

$$\begin{aligned}
\Psi\left[\frac{\alpha+1}{2}, z\right] &= \sum_{i=0}^{\infty} \frac{z^i \Gamma(\frac{\alpha+1+i}{2})}{i!} \\
&= \sum_{i=0}^{\infty} \frac{z^i \left(\frac{\alpha-1+i}{2}\right)\Gamma(\frac{\alpha-1+i}{2})}{i!} \\
&= \left(\frac{\alpha-1}{2}\right)\sum_{i=0}^{\infty} \frac{z^i \Gamma(\frac{\alpha-1+i}{2})}{i!} + \frac{z}{2}\sum_{i=1}^{\infty} \frac{z^{i-1}\Gamma(\frac{\alpha+i-1}{2})}{(i-1)!} \\
&= \left(\frac{\alpha-1}{2}\right)\sum_{i=0}^{\infty} \frac{z^i \Gamma(\frac{\alpha-1+i}{2})}{i!} + \frac{z}{2}\sum_{i=0}^{\infty} \frac{z^{i}\Gamma(\frac{\alpha+i}{2})}{i!} \\
&= \left(\frac{\alpha-1}{2}\right)\Psi\left[\frac{\alpha-1}{2}, z\right] + \frac{z}{2}\Psi\left[\frac{\alpha}{2}, z\right].
\end{aligned}$$

2.1.2. Proof of Lemma 9 (b)

The continued fraction representation can be obtained via repeated use of the recursive relation established in part(a) the Lemma.

2.1.3. Proof of Lemma 9 (c)

From Lemma 1, we get that $\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] = 2\beta^{\frac{\alpha}{2}} \int_0^\infty y^{\alpha-1} \exp\left(-\beta y^2 + \gamma y\right) dy$. Therefore,

$$\Psi\left[\frac{1}{2}, x\right] = 2e^{\frac{x^2}{4}} \int_0^\infty e^{-(y-\frac{x}{2})^2} dy = 2\sqrt{\pi} e^{\frac{x^2}{4}} \int_{-\frac{x}{\sqrt{2}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\sqrt{\pi} e^{\frac{x^2}{4}} [1 - \Phi(-\frac{x}{\sqrt{2}})] \text{ and}$$

$$\begin{aligned}
\Psi\left[\frac{2}{2}, x\right] = 2e^{\frac{x^2}{4}} \int_0^\infty y e^{-(y-\frac{x}{2})^2} dy &= 2\sqrt{\pi} e^{\frac{x^2}{4}} \int_{-\frac{x}{\sqrt{2}}}^\infty (\frac{t}{2} + \frac{x}{2\sqrt{2}}) e^{-\frac{t^2}{2}} dt \\
&= 2\sqrt{\pi} e^{\frac{x^2}{4}} \left[-\frac{1}{2} e^{-\frac{t^2}{2}} \Big|_{-\frac{x}{\sqrt{2}}}^\infty + \frac{x}{2} \sqrt{\pi} (1 - \Phi(-\frac{x}{\sqrt{2}})) \right] \\
&= 1 + \sqrt{\pi} x e^{\frac{x^2}{4}} [1 - \Phi(-\frac{x}{\sqrt{2}})].
\end{aligned}$$

2.2. Proof of Lemma 10

Let $\alpha > 0, \in \mathbb{R}$ then for arbitrary $\epsilon > 0$, let $A(k) = \frac{\Gamma(\frac{\alpha}{2}+k) ArgFW^{2k}}{(2k)!}$ and $B(k) = \frac{\Gamma(\frac{\alpha+1}{2}+k) ArgFW^{2k+1}}{(2k+1)!}$. Let $\lceil x \rceil$ denotes the smallest integer larger than x.

2.2.1. Proof of part(a) of Lemma 10

If $0 < q < 1$ then

$$\frac{A(k+1)}{A(k)} \le q \implies \frac{(\frac{\alpha}{2} + k) z^2}{(2k+2)(2k+1)} \le q \implies 4qk^2 + (6q - z^2)k + (2q - \frac{\alpha}{2} z^2) \ge 0 \implies k \ge C_1,$$

where $k \ge C_1$, where $C_1 = \max\left\{ \left\lceil \frac{-(6q-z^2) + \sqrt{(6q-z^2)^2 - 8q(4q - \alpha z^2)}}{8q} \right\rceil, 1 \right\}$.

2.2.2. Proof of part(b) of Lemma 10

If $0 < q < 1$ then

$$\frac{B(k+1)}{B(k)} \le q \implies \frac{(\frac{\alpha+1}{2} + k) z^2}{(2k+3)(2k+2)} \le q \implies 4qk^2 + (10q - z^2)k + (6q - \frac{\alpha+1}{2} z^2 \ge 0 \implies k \ge C_2,$$

where $C_2 = \max\left\{ \left\lceil \frac{-(10q-z^2) + \sqrt{(10q-z^2)^2 - 8q(12q - (\alpha+1) z^2)}}{8q} \right\rceil, 1 \right\}$.

2.2.3. Proof of part(c) of Lemma 10

By Lemma 10 (a) and Lemma 10 (b), we get that the sequences $\{A(k)\}_{k\geq C_1}$ and $\{B(k)\}_{k\geq C_2}$ are strictly decreasing. Also, if $k > C_1$ then $A(k) \leq q^{k-C_1}A(C_1)$. Hence

$$\lim_{k\to\infty} A(k) \leq A(C_1) \lim_{k\to\infty} q^{k-C_1} = 0$$

as $0 < q < 1$ and $|A(C_1)| < \infty$. In a similar fashion we can show that $\lim_{k\to\infty}|B(k)| = 0$.

2.2.4. Proof of part(d) of Lemma 10

Given any $\epsilon > 0$, let $K_1 = \min\{k : A(k) \leq (1-q)\frac{\epsilon}{2}$ for all $k \geq C_1\}$ where $0 < q < 1$ is a fraction of our choice and C_1 is the constant as it is define in the previosu part. A possible value for q is $\frac{1}{2}$. The integer K_1 is well defined because $\lim_{k\to\infty} A(k) = 0$.
As $K_1 \geq C_1$, we get from the previous part of the Lemma that $A(n) \leq A(K_1)q^{n-K_1} \leq (1-q)\frac{\epsilon}{2}q^{n-K_1}$ for all $n \geq K_1$. As a result,

$$\left|\sum_{k=0}^{\infty} A(k) - \sum_{k=0}^{K_1} A(k)\right| = \left|\sum_{k=K_1}^{\infty} A(k)\right| \leq \sum_{k=K_1}^{\infty} \left((1-q)\frac{\epsilon}{2}\right) q^{k-K_1} = \frac{\epsilon}{2}.$$

In a similar fashion, it can be shown that $\left|\sum_{k=0}^{\infty} B(k) - \sum_{k=0}^{K_2} B(k)\right| \leq \frac{\epsilon}{2}$, when $K_2 = \min\{k : |B(k)| \leq (1-q)\frac{\epsilon}{2}$ for all $k \geq C_2\}$. Define $K = \max\{K_1, K_2\}$.

$$\begin{aligned}\left|\Psi\left[\frac{\alpha}{2}, z\right] - \left(\sum_{k=0}^{K} A(k) + \sum_{k=0}^{K} B(k)\right)\right| &= \left|\sum_{k=0}^{\infty} A(k) - \sum_{k=0}^{K} A(k) + \left[\sum_{k=0}^{\infty} B(k) - \sum_{k=0}^{K} B(k)\right]\right| \\ &\leq \left|\sum_{k=0}^{\infty} A(k) - \sum_{k=0}^{K} A(k)\right| + \left|\sum_{k=0}^{\infty} B(k) - \sum_{k=0}^{K} B(k)\right| \\ &\leq \epsilon\end{aligned} \tag{3}$$

for a given value for the error of the approximation ϵ and $K = \max\{K_1, K_2\}$ is a truncated point for the infinite series for even as well as odd order terms.

2.3. Proof of Lemma 11

From Lemma 1, we get that

$$\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] = 2\beta^{\frac{\alpha}{2}} \int_0^{\infty} x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) dx$$

If we would like to find a positive constant T, depending on α, β, γ such that

$$\left|\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] - 2\beta^{\frac{\alpha}{2}} \int_0^T x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) dx\right| = 2\beta^{\frac{\alpha}{2}} \int_T^\infty x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) dx \leq \epsilon.$$

Using the inequality in part(a) of Theorem 3, we get

$$\int_T^\infty x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) dx \leq \int_T^\infty x^{\alpha-1} \exp\left(-(\beta m^2 + |\gamma| m)(\frac{x}{m})^{\frac{2\beta m+|\gamma|}{\beta m+|\gamma|}}\right) dx,$$

It follows from the change of variable $t = (\frac{x}{m})^{\frac{2\beta m+|\gamma|}{\beta m+|\gamma|}}$ that

$$\begin{aligned}
\int_T^\infty x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) dx &\leq m^\alpha \frac{\beta m + |\gamma|}{2\beta m + |\gamma|} \int_{T^\star}^\infty t^{\alpha-1} e^{-bt} dt \\
&= m^\alpha \frac{\beta m + |\gamma|}{2\beta m + |\gamma|} \left[\int_0^\infty t^{\alpha-1} e^{-bt} - \int_0^{T^\star} t^{\alpha-1} e^{-bt}\right] \\
&= m^\alpha \frac{\beta m + |\gamma|}{2\beta m + |\gamma|} \left[\frac{\Gamma(a)}{b^a} - \frac{\gamma(a, bT^\star)}{\Gamma(a)}\right],
\end{aligned}$$

where $T^\star = (\frac{T}{m})^{\frac{2\beta m+|\gamma|}{\beta m+|\gamma|}}$. Let $2\beta^{\frac{\alpha}{2}} m^\alpha \frac{\beta m+|\gamma|}{2\beta m+|\gamma|} \left[\frac{\Gamma(a)}{b^a} - \frac{\gamma(a,bT^\star)}{\Gamma(a)}\right] = \epsilon$, the truncated point of the numerical integral is

$$T = m(T^\star)^{\frac{\beta m+|\gamma|}{2\beta m+|\gamma|}}, \text{ where } T^\star = \frac{1}{b}\gamma^{-1}\left(a, \frac{[\Gamma(a)]^2}{b^a} - \frac{\epsilon\Gamma(a)}{2\beta^{\frac{\alpha}{2}} m^\alpha \frac{\beta m+|\gamma|}{2\beta m+|\gamma|}}\right),$$

where $a = \frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}$, $b = \beta m^2 + |\gamma| m)$, and $\gamma(a, bT)$ is a lower incomplete gamma function. The inverse of the lower incomplete gamma function can be calculated by the Igamma.inv() in the "zipfR" package in R.

2.4. Proof of Lemma 12

2.4.1. Proof of part(a) of Lemma 12

Consider the fact that

$$x = \frac{x}{\sqrt{m}}\sqrt{m} \leq \frac{1}{2}[(\frac{x}{\sqrt{m}})^2 + (\sqrt{m})^2] = \frac{x^2}{2m} + \frac{m}{2}.$$

Therefore,

$$\begin{aligned}
\Psi\left[\frac{\alpha}{2}, \frac{-|\gamma|}{\sqrt{\beta}}\right] &= 2\beta^{\frac{\alpha}{2}} \int_0^\infty x^{\alpha-1} \exp\left(-\beta x^2 - |\gamma| x\right) dx \\
&\geq 2\beta^{\frac{\alpha}{2}} \int_0^\infty x^{\alpha-1} \exp\left(-\beta x^2 - \frac{|\gamma| x^2}{2m} - \frac{|\gamma| m}{2}\right) dx \\
&= \beta^{\frac{\alpha}{2}} \exp\left(-\frac{|\gamma| m}{2}\right) \int_0^\infty y^{\frac{\alpha}{2}-1} \exp\left(-(\beta + \frac{|\gamma|}{2m})y\right) dy
\end{aligned}$$

Let $\alpha > 1, \beta > 0$ and $\gamma < 0$, then for any $m > 0$,

$$\Psi\left[\frac{\alpha}{2}, \frac{-|\gamma|}{\sqrt{\beta}}\right] \geq \frac{\beta^{\frac{\alpha}{2}} \exp\left(-\frac{m|\gamma|}{2}\right)\Gamma(\frac{\alpha}{2})}{\left(\beta + \frac{|\gamma|}{2m}\right)^{\frac{\alpha}{2}}}.$$

2.4.2. Proof of part(b) of Lemma 12

Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ then, using Holder's inequality we get that

$$\begin{aligned} & \int_0^\infty x^{\frac{\alpha+p\alpha_0-1}{p}-1} e^{-\frac{x^2}{p}} dx \\ = & \int_0^\infty \left[x^{\frac{\alpha-1}{p}} e^{-\frac{x^2}{p}+\frac{\gamma x}{p}}\right]\left[x^{\alpha_0-1}e^{-\frac{\gamma x}{p}}\right] dx \\ \leq & \left[\int_0^\infty x^{\alpha-1}e^{-x^2+\gamma x}dx\right]^{\frac{1}{p}} \left[\int_0^\infty x^{q(\alpha_0-1)}e^{-\frac{q\gamma x}{p}} dx\right]^{\frac{1}{q}}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \Psi\left[\frac{\alpha}{2}, \gamma\right] &= 2\int_0^\infty x^{\alpha-1}e^{-x^2+\gamma x}dx \\ &\geq \frac{2\left[\int_0^\infty x^{\frac{\alpha+p\alpha_0-1}{p}-1}e^{-\frac{x^2}{p}}dx\right]^p}{\left[\int_0^\infty x^{(q\alpha_0-q+1)-1}e^{-\frac{q\gamma x}{p}}dx\right]^{\frac{p}{q}}} \\ &= \frac{2\left[\frac{\Gamma\left(\frac{\alpha+p\alpha_0-1}{2p}\right)}{2}p^{\frac{\alpha+p\alpha_0-1}{2p}}\right]^p}{\left[\Gamma\left(q\alpha_0-q+1\right)\left(\frac{p}{q\gamma}\right)^{q\alpha_0-q+1}\right]^{\frac{p}{q}}} \\ &= \frac{2p^{\frac{\alpha+p\alpha_0-1}{2}-\frac{p(q\alpha_0-q+1)}{q}}\left[\frac{\Gamma\left(\frac{\alpha+p\alpha_0-1}{2p}\right)}{2}\right]^p}{\left[\Gamma\left(q\alpha_0-q+1\right)\right]^{\frac{p}{q}}(q\gamma)^{-\frac{p(q\alpha_0-q+1)}{q}}} \\ &= \frac{2p^{\frac{\alpha+(2-\alpha_0)p-1}{2}-\frac{p}{q}}\left[\frac{\Gamma\left(\frac{\alpha+p\alpha_0-1}{2p}\right)}{2}\right]^p(q\gamma)^{p(\alpha_0-1)+\frac{p}{q}}}{\left[\Gamma\left(q\alpha_0-q+1\right)\right]^{\frac{p}{q}}}. \end{aligned} \tag{4}$$

Our recommended values are $p = \frac{\alpha}{\alpha-1}, q = \alpha, \alpha_0 = \frac{\alpha+1}{p}$. Then the bound takes the following form

$$\Psi\left[\frac{\alpha}{2}, \gamma\right] \geq \frac{\left[\Gamma(\alpha-1)\right]^{\frac{\alpha}{\alpha-1}}(\alpha\gamma)^\alpha}{\left[2\Gamma(\alpha(\alpha-1))\right]^{\frac{1}{\alpha-1}}}.$$

In particular when $\gamma = \sqrt{\alpha}$ then

$$\Psi\left[\frac{\alpha}{2}, \sqrt{\alpha}\right] \geq \frac{[\Gamma(\alpha-1)]^{\frac{\alpha}{\alpha-1}}\,\alpha^{\frac{3\alpha}{2}}}{[2\Gamma(\alpha(\alpha-1))]^{\frac{1}{\alpha-1}}}.$$

2.5. Proof of Lemma 1

2.5.1. Proof of Lemma 1 (a)

The density function of the MHN (α, β, γ), $f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) \propto \cdot x^{\alpha-1}\exp(-\beta x^2+\gamma x)\mathbb{I}(x>0)$. The corresponding normalizing constant is

$$\int_0^\infty x^{\alpha-1}\exp(-\beta x^2+\gamma x)dx = \int_0^\infty x^{\alpha-1}\exp(-\beta x^2)\sum_{i=0}^\infty \frac{\gamma^i x^i}{i!}dx.$$

Using a change of variable $t = x^2$, the above integral turns out to be

$$\sum_{i=0}^\infty \frac{\gamma^i \Gamma(\frac{\alpha+i}{2})}{2i!\beta^{\frac{\alpha+i}{2}}}\int_0^\infty \frac{\beta^{\frac{\alpha+i}{2}}}{\Gamma(\frac{\alpha+i}{2})}t^{\frac{\alpha+i}{2}-1}\exp(-\beta t)dt = \sum_{i=0}^\infty \frac{\gamma^i\Gamma(\frac{\alpha+i}{2})}{2i!\beta^{\frac{\alpha+i}{2}}} = \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]},$$

where $\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] = \sum_{i=0}^\infty \frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{\Gamma(1)i!}$, which denotes the Fox-Wright function of the appropriate order. Consequently the corresponding density function is given as

$$f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) = \frac{2\beta^{\frac{\alpha}{2}}x^{\alpha-1}\exp(-\beta x^2+\gamma x)}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}\mathbb{I}(x>0).$$

2.5.2. Proof of Lemma 1 (b)

The cumulative distribution function $F(t)$ is given as

$$\begin{aligned}
F_{\text{MHN}}(t \mid \alpha, \beta, \gamma) &= \int_0^t \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}x^{\alpha-1}\exp(-\beta x^2+\gamma x)dx \\
&= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}\int_0^t x^{\alpha-1}\exp(-\beta x^2)\sum_{i=0}^\infty \frac{\gamma^i x^i}{i!}dx \\
&\quad \text{by considering th change of variable } y = \beta x^2 \\
&= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}\sum_{i=0}^\infty \frac{\gamma^i}{2i!}\beta^{-\frac{\alpha+i}{2}}\int_0^{\beta t^2} y^{\frac{\alpha+i}{2}-1}e^{-y}dy \\
&= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}\sum_{i=0}^\infty \frac{\gamma^i}{2i!}\beta^{-\frac{\alpha+i}{2}}\gamma(\frac{\alpha+i}{2}, \beta t^2),
\end{aligned}$$

where $\gamma(s,t) = \int_0^x t^{s-1}e^{-t}dt$ denotes the lower incomplete gamma function.

2.6. Proof of Lemma 2

2.6.1. Proof of part (a) of Lemma 2

A moment is a specific quantitative measure of the shape of a function. The k^{th} moment of the random variable following a MHN (α, β, γ) distribution can be expressed as:

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x) dx \\ &= \int_0^\infty x^k \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp(-\beta x^2 + \gamma x) dx \\ &= \frac{\Psi\left[\frac{\alpha+k}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}{\beta^{\frac{k}{2}} \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}. \end{aligned}$$

2.6.2. Proof of part (b) Lemma 2

Based on the property of FoxWright function

$$\Psi\left[\frac{\alpha+k}{2}, x\right] = \frac{\alpha+k-2}{2} \Psi\left[\frac{\alpha+k-2}{2}, x\right] + \frac{x}{2} \Psi\left[\frac{\alpha+k-1}{2}, x\right],$$

we could get the conclusion that

$$E(X^k) = \frac{\alpha+k-2}{2\beta} E(X^{k-2}) + \frac{\gamma}{2\beta} E(X^{k-1}).$$

2.6.3. Proof of part (c) of Lemma 2

The variance of a random variable X, which follows a MHN (α, β, γ) (α, β, γ) distribution is

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{\sum_{i=0}^\infty \frac{\gamma^i \Gamma(\frac{\alpha+i+2}{2})}{i! \beta^{\frac{i}{2}}}}{\beta \sum_{i=0}^\infty \frac{\gamma^i \Gamma(\frac{\alpha+i}{2})}{i! \beta^{\frac{i}{2}}}} - \left[\frac{\sum_{i=0}^\infty \frac{\gamma^i \Gamma(\frac{\alpha+i+1}{2})}{i! \beta^{\frac{i}{2}}}}{\beta^{\frac{1}{2}} \sum_{i=0}^\infty \frac{\gamma^i \Gamma(\frac{\alpha+i}{2})}{i! \beta^{\frac{i}{2}}}}\right]^2 \\ &= \frac{\Psi\left[\frac{\alpha+2}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}{\beta \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} - \left[\frac{\Psi\left[\frac{\alpha+1}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}{\sqrt{\beta} \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}\right]^2. \end{aligned} \tag{5}$$

Using the property of Fox-Wright function

$$\Psi\left[\frac{\alpha+k}{2}, x\right] = \frac{\alpha+k-2}{2} \Psi\left[\frac{\alpha+k-2}{2}, x\right] + \frac{x}{2} \Psi\left[\frac{\alpha+k-1}{2}, x\right],$$

$$Var(X) = \frac{\alpha}{2\beta} + E(X) \left(\frac{\gamma}{2\beta} - E(X)\right). \tag{6}$$

2.6.4. Proof of part (d) of Lemma 2

Let $X \sim \mathrm{MHN}\,(\alpha, \beta, \gamma)$. Then the moment-generating function of the random variable X is

$$\begin{aligned}
M_x(t) &= E[e^{tX}] \\
&= \int_0^\infty e^{tX} \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp(-\beta x^2 + \gamma x) dx \\
&= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} \frac{\Psi\left[\frac{\alpha}{2}, \frac{\gamma+t}{\sqrt{\beta}}\right]}{2\beta^{\frac{\alpha}{2}}} \int_0^\infty \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma+t}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp(-\beta x^2 + (\gamma+t)x) dx \\
&= \frac{\Psi\left[\frac{\alpha}{2}, \frac{\gamma+t}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}.
\end{aligned}$$

2.7. Proof of Lemma 3

(a) The density function for the $\mathrm{MHN}\,(\alpha, \beta, \gamma)$ distribution is

$$f_{\mathrm{MHN}}(x \mid \alpha, \beta, \gamma) = \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp(-\beta x^2 + \gamma x)$$

.
Therefore $\frac{d}{dx} \log\left(f_{\mathrm{MHN}}(x \mid \alpha, \beta, \gamma)\right) = \frac{\alpha-1}{x} - 2\beta x + \gamma$. If X_{mode} denotes the mode of the distribution then

$$\frac{\alpha-1}{X_{\text{mode}}} - 2\beta X_{\text{mode}} + \gamma = 0 \implies X_{\text{mode}} = \frac{\gamma + \sqrt{\gamma^2 + 8(\alpha-1)\beta}}{4\beta},$$

while we ignore the other solution $\frac{\gamma - \sqrt{\gamma^2+8(\alpha-1)\beta}}{4\beta}$ as it does not belong to the support of the distribution. Additionally,

$$\frac{d^2}{dx^2} \log\left(f_{\mathrm{MHN}}(x \mid \alpha, \beta, \gamma)\right)\Big|_{x=X_{\text{mode}}} = -\frac{\alpha-1}{X^2_{\text{mode}}} - 2\beta < 0 \text{ for } \alpha \geq 1.$$

(b) If $1-\frac{\gamma^2}{8\beta} \leq \alpha < 1$ and $\gamma > 0$ the equation $\frac{\alpha-1}{z} - 2\beta z + \gamma = 0$ has two real positive solutions, corresponding to the local minima and the point of local maxima. As a consequence, the density has a local maxima at $\frac{\gamma+\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}$ and a local minima at $\frac{\gamma-\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}$ because

$$\frac{d^2}{dx^2} \log\left(f_{\mathrm{MHN}}(x \mid \alpha, \beta, \gamma)\right)\Big|_{x=\frac{\gamma+\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}} < 0, \text{ and}$$

$$\frac{d^2}{dx^2} \log\left(f_{\mathrm{MHN}}(x \mid \alpha, \beta, \gamma)\right)\Big|_{x=\frac{\gamma-\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}} > 0.$$

(c) If $0 < \alpha < 1 - \frac{\gamma^2}{8\beta}$ and $\gamma > 0$ then

$$\frac{d}{dx} \log\left(f_{\text{MHN}}(x \mid \alpha, \beta, \gamma)\right) = -\frac{1}{x}\left[(\sqrt{2\beta x} - \frac{\gamma}{\sqrt{2\beta x}})^2 + \left(1 - \frac{\gamma^2}{8\beta} - \alpha\right)\right] < 0$$

for all $x > 0$. Therefore, in this case, the density function is gradually decreasing on $\mathbb{R}_+$ and mode of the distribution doesn't exist. The same is true for the case $\gamma \leq 0$ and $\alpha \leq 1$.

2.8. Proof of Lemma 4

2.8.1. Proof of part(a) of Lemma 4

If $\alpha > 1$, then from Lemma 2, we have that $E(X) = \frac{\alpha-1}{2\beta} E(\frac{1}{X}) + \frac{\gamma}{2\beta}$. As the function $x \mapsto \frac{1}{x}$ is convex on $\mathbb{R}_+$, using Jensen's inequality $E(\frac{1}{X}) > \frac{1}{E(X)}$ when $X \sim \text{MHN}(\alpha, \beta, \gamma)$. Hence,$E(X) \geq \frac{\alpha-1}{2\beta} \frac{1}{E(X)} + \frac{\gamma}{2\beta}$. Consequently,

$$E(X) \geq \frac{\gamma + \sqrt{\gamma^2 + 8\beta(\alpha - 1)}}{4\beta}. \tag{7}$$

On the other hand, using lemma 2 we get the recursive relation $E(X^2) = \frac{\alpha}{2\beta} E(X^0) + \frac{\gamma}{2\beta} E(X) = \frac{\alpha}{2\beta} + \frac{\gamma}{2\beta} E(X)$. As $E(X^2) \geq E(X)^2$,$E(X)^2 \leq \frac{\alpha}{2\beta} + \frac{\gamma}{2\beta} E(X)$. Therefore,

$$E(X) \leq \frac{\gamma + \sqrt{\gamma^2 + 8\beta\alpha}}{4\beta}. \tag{8}$$

From Lemma 3, we know that the mode of the distribution is $X_{\text{mode}} = \frac{\gamma+\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}$ when $\alpha > 1$. An implication of the inequality in 7 is that $E(X) > X_{\text{mode}}$. Thus $\text{MHN}(\alpha, \beta, \gamma)$ is a positively skewed distribution when $\alpha > 1$.

2.8.2. Proof of part(b) of Lemma 4

Let $X \sim \text{MHN}(\alpha, \beta, \gamma)$ for $\alpha \geq 4$, and $\gamma > 0$, then $E(\log(X)) \geq \log(X_{\text{mode}})$ where $X_{\text{mode}} = \frac{\gamma+\sqrt{\gamma^2+8\beta(\alpha-1)}}{4\beta}$. Without loss of generality we assume β to be 1 for this proof. Define the function

$$h(\gamma) = E(\log(X)) - \log(X_{\text{mode}}) = \int_{\mathbb{R}_+} \log(x) f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) dx - \log(X_{\text{mode}}).$$

Consider that

$$\frac{\partial X_{\text{mode}}}{\partial \gamma} = \frac{1}{4} + \frac{\gamma}{4\sqrt{\gamma^2 + 8(\alpha - 1)}} = \frac{X_{\text{mode}}}{\sqrt{\gamma^2 + 8(\alpha - 1)}}.$$

Therefor, we get

$$
\begin{aligned}
\frac{\partial h}{\partial \gamma} &= \int_{\mathbb{R}_+} x\log(x) f_{\text{MHN}}(x \mid \alpha, \beta, \gamma)dx - \frac{1}{X_{\text{mode}}}\frac{\partial X_{\text{mode}}}{\partial \gamma} \\
&= E(X\log(X)) - \frac{1}{X_{\text{mode}}}\frac{\partial X_{\text{mode}}}{\partial \gamma} \\
&\geq E(X)\log(E(X)) - \frac{1}{\sqrt{\gamma^2+8(\alpha-1)}},
\end{aligned} \tag{9}
$$

where the step us due to the Jensen's inequality because $x \mapsto x\log(x)$ is a convex function ($\frac{\partial^2}{\partial x^2}(x\log(x)) = \frac{1}{x} > 0$ for $x > 0$). Also, it follows from Lemma 2 that $E(X) \geq X_{\text{mode}} = \frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)}}{4}$. Therefore

$$
\begin{aligned}
\frac{\partial h}{\partial \gamma} &\geq \frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)}}{4}\log\left(\frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)}}{4}\right) - \frac{1}{\sqrt{\gamma^2+8(\alpha-1)}} \\
&\overset{(\dagger\dagger)}{\geq} \log\left(\frac{\sqrt{8(\alpha-1)}}{4}\right)\frac{\sqrt{8(\alpha-1)}}{4} - \frac{1}{\sqrt{8(\alpha-1)}} \\
&\geq 0.
\end{aligned} \tag{10}
$$

for $\alpha \geq 4$. The inequality in $(\dagger\dagger)$ is an implication of the fact that $\frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)}}{4}\log\left(\frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)}}{4}\right) - \frac{1}{\sqrt{\gamma^2+8(\alpha-1)}}$ is an increasing function in $\gamma > 0$. Altogether, the function $\gamma \mapsto h(\gamma)$ is an increasing function in γ. As a result, for $\alpha \geq 4$ and $\gamma \geq 0$,

$$
\begin{aligned}
h(\gamma) &\geq h(0) \\
&= \int_0^\infty \log(x)\frac{2x^{\alpha-1}e^{-x^2}}{\Gamma(\frac{\alpha}{2})}dx - \log(\frac{\sqrt{8(\alpha-1)}}{4}) \\
&= \frac{\frac{\partial}{\partial\alpha}\int_0^\infty 2x^{\alpha-1}e^{-x^2}dx}{\Gamma(\frac{\alpha}{2})} - \log(\frac{\sqrt{8(\alpha-1)}}{4}) \\
&= \frac{1}{2}\frac{\frac{\partial\Gamma(\frac{\alpha}{2})}{\partial\alpha}}{\Gamma(\frac{\alpha}{2})} - \log(\sqrt{\frac{(\alpha-1)}{2}}) \\
&\overset{(\dagger\star)}{>} \frac{1}{2}\log(\frac{\alpha}{\alpha-1}) - \frac{1}{2\alpha} - \frac{1}{6\alpha^2} \\
&\geq 0.
\end{aligned} \tag{11}
$$

The inequality in $(\dagger\star)$ is due to the fact that $\frac{\frac{d}{d\alpha}(\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \geq \log(\frac{\alpha}{2}) - \frac{1}{\alpha} - \frac{1}{3\alpha^2}$ (Batir, 2005). On the other hand,

$$
\frac{d}{d\alpha}\left[\frac{1}{2}\log(\frac{\alpha}{\alpha-1}) - \frac{1}{2\alpha} - \frac{1}{6\alpha^2}\right] = -\frac{1}{2\alpha(\alpha-1)} + \frac{1}{2\alpha^2} + \frac{1}{3\alpha^3} = -\frac{\alpha+2}{6\alpha^3(\alpha-1)} < 0 \text{ for } \alpha > 1.
$$

Therefore the function $\alpha \mapsto \frac{1}{2}\log(\frac{\alpha}{\alpha-1}) - \frac{1}{2\alpha} - \frac{1}{6\alpha^2}$ is decreasing in $\alpha > 1$ and

$$\frac{1}{2}\log(\frac{\alpha}{\alpha-1}) - \frac{1}{2\alpha} - \frac{1}{6\alpha^2} \geq \lim_{\alpha\to\infty}\frac{1}{2}\log(\frac{\alpha}{\alpha-1}) - \frac{1}{2\alpha} - \frac{1}{6\alpha^2} = 0.$$

Finally, if follows from the the definition of the function $\gamma \mapsto h(\gamma)$ and the Equation 11 that

$$E(\log(X)) \geq \log(X_{\text{mode}}),$$

for all $\gamma \geq 0$ when $X \sim \text{MHN}\,(\alpha, 1, \gamma)$, $\alpha \geq 4$. On the other hand, using Jensen's inequality the previous part of the lemma we get that

$$E(\log(X)) \leq \log(E(X)) \leq \log\left(\frac{\gamma + \sqrt{\gamma^2 + 8\alpha\beta}}{4\beta}\right). \tag{12}$$

2.8.3. **Proof of part (c) of Lemma 4**

If $\alpha \geq 1$, then from part(a) of Lemma 4, it appears that

$$E(X) \geq \frac{\gamma + \sqrt{\gamma^2 + 8\beta(\alpha-1)}}{4\beta} \implies \left(E(X) - \frac{\gamma}{4\beta}\right)^2 \geq \frac{\gamma^2 + 8\beta(\alpha-1)}{16\beta^2}. \tag{13}$$

Additionally, from Lemma 2 we get that

$$\begin{aligned}
\text{Var}(X) = E(X^2) - E^2(X) &= \frac{\alpha}{2\beta} + \frac{\gamma}{2\beta}E(X) - E^2(X) \\
&= \frac{\alpha}{2\beta} + \frac{\gamma^2}{16\beta^2} - \left(E(X) - \frac{\gamma}{4\beta}\right)^2 \\
&\overset{(\dagger\dagger\dagger)}{\leq} \frac{\alpha}{2\beta} + \frac{\gamma^2}{16\beta^2} - \frac{\gamma^2 + 8\beta(\alpha-1)}{16\beta^2} \\
&= \frac{1}{2\beta},
\end{aligned}$$

where the inequality in $(\dagger\dagger\dagger)$ follows from the Equation 13.

2.9. Proof of Lemma 5

2.9.1. Proof of part(a) of Lemma 5:

Let $\gamma > 0$ when $X \sim \text{MHN}\,(\alpha, \beta, \gamma)$. Consider a random variable V such that the conditional probability distribution of V given X is a Poisson distribution with parameter γX which has the probability mass function

$$f_{Poi}(V = v \mid X) = \frac{e^{-\gamma X}(\gamma X)^v}{v!}. \tag{14}$$

Consequently the conditional probability density of the random variable X given V

$$\begin{aligned} f_{X|V}(x \mid v) &= \frac{f_{X,V}(x,v)}{f_V(v)} \\ &= \frac{\frac{2\beta^{\frac{\alpha}{2}} x^{\alpha-1} e^{-\beta x^2+\gamma x}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} \left\{\frac{e^{-\gamma x}(\gamma x)^v}{v!}\right\} \mathbb{I}(x>0)}{\frac{\gamma^v \Gamma(\frac{\alpha+v}{2})}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right] v! \beta^{\frac{v}{2}}}} \\ &= \frac{2\beta^{\frac{\alpha+v}{2}}}{\Gamma(\frac{\alpha+v}{2})} x^{\alpha+v-1} e^{-\beta x^2} \mathbb{I}(x>0). \end{aligned} \tag{15}$$

As a result, the conditional distribution of the random variable X given V is the square root of a Gamma random variable with shape parameter $\frac{\alpha+v}{2}$ and rate parameter β. Obviously the conditional distribution of V given X is a Poisson random variable with parameter γX.

2.9.2. Proof of part(b) of Lemma 5 :

Let $\gamma < 0$ when $X \sim$ MHN (α, β, γ). Consider a random variable U such that the conditional probability distribution of U given X is a Generalized Inverse Gaussian distribution, i.e.

$$U \mid X \sim GIG(\frac{1}{2}, 1, \gamma^2 X^2)$$

with the probability density function

$$f_{U|X}(u \mid x) = \frac{1}{\sqrt{2\pi}} e^{|\gamma| x} u^{\frac{1}{2}-1} e^{-\frac{1}{2}\left(u+\frac{\gamma^2 x^2}{u}\right)}. \tag{16}$$

Hence the conditional probability density of the random variable X given U

$$\begin{aligned} f_{X|U}(x \mid u) &= \frac{f_{X,U}(x,u)}{f_U(u)} \\ &= \frac{\frac{2\beta^{\frac{\alpha}{2}} x^{\alpha-1} e^{-\beta x^2-|\gamma| x}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} \left\{\frac{1}{\sqrt{2\pi}} e^{|\gamma| x} u^{\frac{1}{2}-1} e^{-\frac{1}{2}\left(u+\frac{\gamma^2 x^2}{u}\right)}\right\} \mathbb{I}(x>0)}{\frac{\Gamma(\frac{\alpha}{2})}{2(\beta+\frac{\gamma^2}{u})^{\frac{\alpha}{2}}} \frac{2\beta^{\frac{\alpha}{2}}}{\sqrt{2\pi}\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} e^{-\frac{1}{2}u} u^{\frac{1}{2}-1}} \\ &= \frac{2(\beta+\frac{\gamma^2}{2})^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{u})} x^{\alpha-1} e^{-\left(\beta+\frac{\gamma^2}{u}\right)x^2} \mathbb{I}(x>0). \end{aligned} \tag{17}$$

As a result, the conditional distribution of the random variable X given U is actually the square root of a Gamma random variable with shape parameter $\frac{\alpha}{2}$ and rate parameter $\beta + \frac{\gamma^2}{u}$. The above result can be utilized to design hierarchical models by introducing additional variables U, V that can bypass the sampling step that involves sampling from the MHN (α, β, γ) distribution directly. But, this strategy of introducing additional variables is expected to lead to slower mixing Markov chains.

2.10. Proof of Lemma 6

The results in Lemma 6 are the direct implication of the corresponding restrictions that we assume.

2.11. Proof of Theorem 1

2.11.1. Proof of Theorem 1 (a)

Let $\gamma > 0$.

$$\begin{aligned} f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\{-\beta x^2 + \gamma x\} \\ &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\{-\beta x^2 + \gamma x\} \exp\{\beta(x-\mu)^2\} \exp\{-\beta(x-\mu)^2\} \\ &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\{-(2\beta\mu - \gamma)x + \beta\mu^2\} \exp\{-\beta(x-\mu)^2\} \end{aligned} \tag{18}$$

Assuming $\mu > \frac{\gamma}{2\beta}$, it can be shown that $x^{\alpha-1}\exp\{-(2\beta\mu - \gamma)x\} \le \left(\frac{\alpha-1}{2\beta\mu-\gamma}\right)^{\alpha-1} \exp\{-(\alpha-1)\}$ for all $x > 0$. Therefore,

$$\begin{aligned} f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) &\le \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} \left(\frac{\alpha-1}{2\beta\mu-\gamma}\right)^{\alpha-1} \exp\{-(\alpha-1) + \beta\mu^2\} \exp\{-\beta(x-\mu)^2\} \\ &= K_1 f_{\text{Normal}}(x \mid \mu, \frac{1}{2\beta}), \end{aligned} \tag{19}$$

where $K_1 = \frac{2\sqrt{\pi}\left(\frac{\sqrt{\beta}(\alpha-1)}{2\beta\mu-\gamma}\right)^{\alpha-1}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} e^{\{-(\alpha-1)+\beta\mu^2\}}$. On the other hand, if $0 < \delta < \beta$ be any constant then

$$\begin{aligned} f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\left(-\beta x^2 + \gamma x\right) \\ &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} \exp\left(-(\beta-\delta)x^2 + \gamma x\right) x^{\alpha-1} e^{-\delta x^2} \end{aligned} \tag{20}$$

As $\gamma x - (\beta - \delta)x^2 = \frac{\gamma^2}{4(\beta-\delta)} - \left\{\frac{\gamma}{2\sqrt{(\beta-\delta)}} - \sqrt{(\beta-\delta)}x\right\}^2 \le \frac{\gamma^2}{4(\beta-\delta)}$, from Equation 20 we get that

$$
\begin{aligned}
f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) &\leq \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} e^{\frac{\gamma^2}{4(\beta-\delta)}} x^{\alpha-1} e^{-\delta x^2} \\
&= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} e^{\frac{\gamma^2}{4(\beta-\delta)}} \frac{\Gamma(\alpha/2)}{2\delta^{\frac{\alpha}{2}}} f_{\sqrt{\text{Gam}}}(x|\alpha,\delta) \\
&= K_2 f_{\sqrt{\text{Gam}}}(x|\alpha,\delta), \qquad (21)
\end{aligned}
$$

where $K_2 = \frac{\Gamma(\frac{\alpha}{2})\beta^{\frac{\alpha}{2}} \exp\left(\frac{\gamma^2}{4(\beta-\delta)}\right)}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] \delta^{\frac{\alpha}{2}}}$.

2.11.2. Proof of part(b), Theorem 1

The inequalities in part(a) is efficient when the constant $K_1(\mu,\alpha,\beta,\gamma)$ and $K_2((\delta,\alpha,\beta,\gamma))$ are small. Therefore, we find the optimum values for μ and δ by minimizing $K_1(\mu,\alpha,\beta,\gamma)$ and $K_2(\delta,\alpha,\beta,\gamma)$ correspondingly.

The function $\mu \mapsto \frac{\exp(\beta\mu^2)}{(2\beta\mu-\gamma)^{\alpha-1}}$ from $(\frac{\gamma}{2\beta},\infty)$ to $\mathbb{R}_+$ is minimized when $\mu = \frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)\beta}}{4\beta}$. Thus, the best possible choice for μ is $\mu_{\text{opt}} = \frac{\gamma+\sqrt{\gamma^2+8(\alpha-1)\beta}}{4\beta}$ and $K_1(\mu_{\text{opt}}) = \frac{2\sqrt{\pi}\beta^{\frac{\alpha-1}{2}}\mu^{\alpha-1}\exp\{-(\alpha-1)+\beta\mu^2\}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}$.

On the other, the function $\delta \mapsto \frac{\exp\left(\frac{\gamma^2}{4(\beta-\delta)}\right)}{\delta^{\frac{\alpha}{2}}}$ from $(0,\beta)$ to $\mathbb{R}_+$ is minimized when $\delta = \beta + \frac{\gamma^2-\gamma\sqrt{\gamma^2+8\alpha\beta}}{4\alpha}$. Therefore, the optimum choice for δ is given as $\delta_{\text{opt}} = \beta + \frac{\gamma^2-\gamma\sqrt{\gamma^2+8\alpha\beta}}{4\alpha}$.

2.11.3. Proof of part(c), Theorem 1

Let $\Delta = \frac{\gamma}{\sqrt{\beta}}$. As $\sqrt{\beta}\mu_{\text{opt}} = \frac{\Delta+\sqrt{\Delta^2+8(\alpha-1)}}{4}$,

$$
\begin{aligned}
K_1(\mu_{\text{opt}},\alpha,\beta,\gamma) &= \frac{2\sqrt{\pi}\left(\frac{\sqrt{\beta}(\alpha-1)}{2\beta\mu_{\text{opt}}-\gamma}\right)^{\alpha-1} \exp\left(-(\alpha-1)+\beta\mu_{\text{opt}}^2\right)}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} \\
&= \frac{2\sqrt{\pi}\left(\sqrt{\beta}\mu_{\text{opt}}\right)^{\alpha-1} \exp\left(-(\alpha-1)+\beta\mu_{\text{opt}}^2\right)}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} \\
&= \frac{2\sqrt{\pi}\left(\frac{\Delta+\sqrt{\Delta^2+8(\alpha-1)}}{4}\right)^{\alpha-1} \exp\left(-\frac{(\alpha-1)}{2}+\frac{\Delta^2+\Delta\sqrt{\Delta^2+8(\alpha-1)}}{8}\right)}{\Psi\left[\frac{\alpha}{2},\Delta\right]}. \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
K_1(\delta_{\text{opt}},\alpha,\beta,\gamma) &= \frac{\left(\sqrt{\beta}\right)^{\alpha}\Gamma(\frac{\alpha}{2})\ \exp\left(\frac{\gamma^2}{4(\beta-\delta_{\text{opt}})}\right)}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]\left(\sqrt{\delta_{\text{opt}}}\right)^{\alpha}} \\
&= \frac{\left(\frac{\beta}{\delta_{\text{opt}}}\right)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\ \exp\left(\frac{4\alpha\gamma^2}{4(\gamma\sqrt{\gamma^2+8\alpha\beta}-\gamma^2)}\right)}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} \\
&= \frac{\Gamma(\frac{\alpha}{2})\ \exp\left(\frac{\alpha}{(\sqrt{1+8\frac{\alpha}{\Delta^2}}-1)}\right)}{\left(1+\frac{\Delta^2-\Delta\sqrt{\Delta^2+8\alpha}}{4\alpha}\right)^{\frac{\alpha}{2}}\ \Psi\left[\frac{\alpha}{2},\Delta\right]}.
\end{aligned}
\tag{23}
$$

Consequently, we will denote $K_1(\alpha,\Delta) := K_1(\mu_{\text{opt}},\alpha,\beta,\gamma)$ and $K_2(\alpha,\Delta) := K_1(\delta_{\text{opt}},\alpha,\beta,\gamma)$ where $\Delta = \frac{\gamma}{\sqrt{\beta}}$.

Result 1. *Let $\alpha > 0$ and $\Delta \in \mathbb{R}$.*

(a) Then $\frac{\partial}{\partial\Delta}\left\{\Psi\left[\frac{\alpha}{2},\Delta\right]\right\} = \Psi\left[\frac{\alpha+1}{2},\Delta\right]$

(b) $\frac{\Delta+\sqrt{\Delta^2+8(\alpha-1)}}{4} \le \frac{\Psi\left[\frac{\alpha+1}{2},\Delta\right]}{\Psi\left[\frac{\alpha}{2},\Delta\right]} \le \frac{\Delta+\sqrt{\Delta^2+8(\alpha)}}{4}$

Proof of the part(a) of Result 1

$$
\begin{aligned}
\frac{\partial}{\partial\Delta}\left\{\Psi\left[\frac{\alpha}{2},\Delta\right]\right\} &= \frac{\partial}{\partial\Delta}\left\{2\int_0^{\infty} y^{\alpha-1}\exp\left(-y^2+\Delta y\right)dy\right\} \\
&= 2\int_0^{\infty} y^{\alpha+1-1}\exp\left(-y^2+\Delta y\right)dy \\
&= \Psi\left[\frac{\alpha+1}{2},\Delta\right].
\end{aligned}
\tag{24}
$$

Proof of the part(b) of Result 1 Let $X \sim \text{MHN}(\alpha,1,\Delta)$ then from Lemma 2, we get that $E(X) = \frac{\Psi\left[\frac{\alpha+1}{2},\Delta\right]}{\Psi\left[\frac{\alpha}{2},\Delta\right]}$. From Lemma 4 we get that

$$
\frac{\Delta+\sqrt{\Delta^2+8(\alpha-1)}}{4} \le \frac{\Psi\left[\frac{\alpha+1}{2},\Delta\right]}{\Psi\left[\frac{\alpha}{2},\Delta\right]} \le \frac{\Delta+\sqrt{\Delta^2+8(\alpha)}}{4}.
$$

2.12. Proof of Theorem 2

2.12.1. Proof of part(a), Theorem 2

From Theorem 1, we get that

$$
f_{\text{MHN}}(x \mid \alpha,\beta,\gamma) \le I K_1(\alpha,\Delta)\left\{f_{\text{Normal}}(x \mid \mu,\frac{1}{2\beta})\right\} + (1-I)K_2(\alpha,\Delta)\left\{f_{\sqrt{\text{Gam}}}(x \mid \alpha,\delta)\right\},
$$

where $\Delta = \frac{\gamma}{\sqrt{\beta}}$ and $I = \mathbb{I}(K_1(\alpha,\Delta) \leq K_2(\alpha,\Delta))$. Therefore

$$\begin{aligned}
\mathcal{A}_{\text{pos}}(\alpha,\Delta) &= \frac{1}{\int_0^\infty \left\{ I K_1(\alpha,\Delta) f_{\text{Normal}}(x \mid \mu, \frac{1}{2\beta}) + (1-I) K_2(\alpha,\Delta) f_{\sqrt{\text{Gam}}}(x \mid \alpha,\delta) \right\} dx} \\
&= \frac{1}{I K_1(\alpha,\Delta) + (1-I) K_2(\alpha,\Delta)} \\
&= \frac{I}{K_1(\alpha,\Delta)} + \frac{1-I}{K_2(\alpha,\Delta)} \\
&= \max\left\{ \frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)} \right\}. \qquad (25)
\end{aligned}$$

2.12.2. Proof of part(b), Theorem 2

To prove that $\Delta \mapsto K_1(\alpha,\Delta)$ is decreasing in Δ.

$$\begin{aligned}
& K_1(\alpha,\Delta) = \frac{2\sqrt{\pi}\,(B_{\Delta})^{\alpha-1}\ \exp\left(-(\alpha-1) + B_{\Delta}^2\right)}{\Psi\left[\frac{\alpha}{2},\Delta\right]} \\
\Longrightarrow\quad & \log(K_1(\alpha,\Delta)) = (\alpha-1)\log(B_{\Delta}) + B_{\Delta}^2 - \log\left(\Psi\left[\frac{\alpha}{2},\Delta\right]\right) + \log(2\sqrt{\pi}\exp(-(\alpha-1))). \qquad (26)
\end{aligned}$$

where $B_{\Delta} = \frac{\Delta + \sqrt{\Delta^2 + 8(\alpha-1)}}{4}$. Note that

$$\begin{aligned}
\frac{\partial B_{\Delta}}{\partial \Delta} &= \frac{\partial}{\partial \Delta}\left(\frac{\Delta + \sqrt{\Delta^2 + 8(\alpha-1)}}{4}\right) \\
&= \frac{1}{4}\left(1 + \frac{\Delta}{\sqrt{\Delta^2 + 8(\alpha-1)}}\right) \\
&= \left(\frac{\Delta + \sqrt{\Delta^2 + 8(\alpha-1)}}{4}\right)\left(\frac{1}{\sqrt{\Delta^2 + 8(\alpha-1)}}\right) \\
&= \left(\frac{B_{\Delta}}{\sqrt{\Delta^2 + 8(\alpha-1)}}\right). \qquad (27)
\end{aligned}$$

Therefore, Result 1 and Equation 26, it follows that

$$\begin{aligned}
\frac{\partial \log(K_1(\alpha,\Delta))}{\partial \Delta} &= \left\{\frac{(\alpha-1)}{B_{\Delta}} + 2B_{\Delta}\right\}\frac{\partial B_{\Delta}}{\partial \Delta} - \frac{\frac{\partial}{\partial \Delta}\left\{\Psi\left[\frac{\alpha}{2},\Delta\right]\right\}}{\Psi\left[\frac{\alpha}{2},\Delta\right]} \\
&= \left\{\frac{(\alpha-1)}{B_{\Delta}} + 2B_{\Delta}\right\}\frac{\partial B_{\Delta}}{\partial \Delta} - \frac{\Psi\left[\frac{\alpha+1}{2},\Delta\right]}{\Psi\left[\frac{\alpha}{2},\Delta\right]} \\
&< \frac{(\alpha-1)+2B_{\Delta}^2}{\sqrt{\Delta^2+8(\alpha-1)}} - \frac{\Delta+\sqrt{\Delta^2+8(\alpha-1)}}{4} \\
&< \frac{(\alpha-1)+2B_{\Delta}^2}{\sqrt{\Delta^2+8(\alpha-1)}} - B_{\Delta} \\
&= \frac{2(\alpha-1)+\Delta B_{\Delta}}{\sqrt{\Delta^2+8(\alpha-1)}} - B_{\Delta},
\end{aligned}$$

because B_{Δ} satisfy the condition $2B_{\Delta}^2 - \Delta B_{\Delta} - (\alpha-1) = 0$. Consider the fact that

$$\begin{aligned}
\left(2(\alpha-1)+\Delta B_{\Delta}\right)^2 &= 4(\alpha-1)^2 + 4(\alpha-1)\Delta B_{\Delta} + \Delta^2 B_{\Delta}^2 \\
&= 4(\alpha-1)\left[(\alpha-1)+\Delta B_{\Delta}\right] + \Delta^2 B_{\Delta}^2 \\
&= 4(\alpha-1)\left[2B_{\Delta}^2\right] + \Delta^2 B_{\Delta}^2 \\
&= B_{\Delta}^2\left(\Delta^2+8(\alpha-1)\right). \qquad (28)
\end{aligned}$$

Therefore it follows from Equation 28 and Equation 28 it follows that,

$$\begin{aligned}
\frac{\partial \log(K_1(\alpha,\Delta))}{\partial \Delta} &< \frac{2(\alpha-1)+\Delta B_{\Delta}}{\sqrt{\Delta^2+8(\alpha-1)}} - B_{\Delta} \\
&= \frac{\sqrt{B_{\Delta}^2\left(\Delta^2+8(\alpha-1)\right)}}{\sqrt{\Delta^2+8(\alpha-1)}} - B_{\Delta} \\
&= 0.
\end{aligned}$$

To Prove that $\Delta \mapsto K_2(\alpha,\Delta)$ is increasing in Δ.
Consider that

$$K_2(\alpha,\Delta) = \frac{\left(\frac{\sqrt{\Delta^2+8\alpha}+\Delta}{2}\right)^{\alpha}\Gamma(\frac{\alpha}{2})\ \exp\left(\frac{\Delta(\sqrt{\Delta^2+8\alpha}+\Delta)}{8}\right)}{(2\alpha)^{\frac{\alpha}{2}}\ \Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} = \frac{\left(2D_{\Delta}\right)^{\alpha}\Gamma(\frac{\alpha}{2})\ \exp\left(\frac{\Delta D_{\Delta}}{2}\right)}{(2\alpha)^{\frac{\alpha}{2}}\ \Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]},$$

where $D_{\Delta} = \frac{\Delta+\sqrt{\Delta^2+8(\alpha)}}{4}$. Hence

$$\begin{aligned}
&\log(K_2(\alpha,\Delta)) \\
&= \alpha\log(D_{\Delta}) + \frac{\Delta D_{\Delta}}{2} - \log(\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]) + \log\left(\frac{(2)^{\alpha}\,\Gamma(\frac{\alpha}{2})}{(2\alpha)^{\frac{\alpha}{2}}}\right) \\
&= \alpha\log(D_{\Delta}) + D_{\Delta}^2 - \frac{\alpha}{2} - \log(\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]) + \log\left(\frac{(2)^{\alpha}\,\Gamma(\frac{\alpha}{2})}{(2\alpha)^{\frac{\alpha}{2}}}\right), \qquad (29)
\end{aligned}$$

because $D_{\vartriangle}$ satisfy the condition $2D_{\vartriangle}^2 - \Delta D_{\vartriangle} - \alpha = 0$. Note that

$$\begin{aligned}
\frac{\partial D_{\vartriangle}}{\partial \Delta} &= \frac{\partial}{\partial \Delta}\left(\frac{\Delta + \sqrt{\Delta^2 + 8\alpha}}{4}\right) \\
&= \frac{1}{4}\left(1 + \frac{\Delta}{\sqrt{\Delta^2 + 8\alpha}}\right) \\
&= \left(\frac{\Delta + \sqrt{\Delta^2 + 8\alpha}}{4}\right)\left(\frac{1}{\sqrt{\Delta^2 + 8\alpha}}\right) \\
&= \left(\frac{D_{\vartriangle}}{\sqrt{\Delta^2 + 8\alpha}}\right). \qquad (30)
\end{aligned}$$

Therefore, from Result 1 and Equation 29, we infer that

$$\begin{aligned}
\frac{\partial \log(K_2(\alpha, \Delta))}{\partial \Delta} &= \left\{\frac{\alpha}{D_{\vartriangle}} + 2D_{\vartriangle}\right\}\frac{\partial D_{\vartriangle}}{\partial \Delta} - \frac{\frac{\partial}{\partial \Delta}\left\{\Psi\left[\frac{\alpha}{2}, \Delta\right]\right\}}{\Psi\left[\frac{\alpha}{2}, \Delta\right]} \\
&= \left\{\frac{\alpha}{D_{\vartriangle}} + 2D_{\vartriangle}\right\}\frac{\partial D_{\vartriangle}}{\partial \Delta} - \frac{\Psi\left[\frac{\alpha+1}{2}, \Delta\right]}{\Psi\left[\frac{\alpha}{2}, \Delta\right]} \\
&> \frac{\alpha + 2D_{\vartriangle}^2}{\sqrt{\Delta^2 + 8\alpha}} - \frac{\Delta + \sqrt{\Delta^2 + 8\alpha}}{4} \\
&> \frac{\alpha + 2D_{\vartriangle}^2}{\sqrt{\Delta^2 + 8\alpha}} - D_{\vartriangle} \\
&= \frac{2\alpha + \Delta D_{\vartriangle}}{\sqrt{\Delta^2 + 8\alpha}} - D_{\vartriangle}, \qquad (31)
\end{aligned}$$

where the last equality follows as the constant $D_{\vartriangle}$ satisfy the condition $2D_{\vartriangle}^2 - \Delta D_{\vartriangle} = \alpha$. Consider the fact that

$$\begin{aligned}
\left(2\alpha + \Delta D_{\vartriangle}\right)^2 &= 4\alpha^2 + 4\alpha\Delta D_{\vartriangle} + \Delta^2 D_{\vartriangle}^2 \\
&= 4\alpha\left[\alpha + \Delta D_{\vartriangle}\right] + \Delta^2 D_{\vartriangle}^2 \\
&= 4\alpha\left[2D_{\vartriangle}^2\right] + \Delta^2 D_{\vartriangle}^2 \\
&= D_{\vartriangle}^2\left(\Delta^2 + 8\alpha\right). \qquad (32)
\end{aligned}$$

Consequently, it follows from Equation 31 and Equation 32 that

$$\frac{\partial \log(K_2(\alpha, \Delta))}{\partial \Delta} > \frac{2\alpha + \Delta D_{\vartriangle}}{\sqrt{\Delta^2 + 8\alpha}} - D_{\vartriangle} = \frac{\sqrt{D_{\vartriangle}^2\left(\Delta^2 + 8\alpha\right)}}{\sqrt{\Delta^2 + 8\alpha}} - D_{\vartriangle} = 0. \qquad (33)$$

2.12.3. Proof of part(c), Theorem 2

Let $B_\alpha = \frac{\sqrt{\alpha}+\sqrt{\alpha+8(\alpha-1)}}{4} = \frac{\sqrt{\alpha}+\sqrt{9\alpha-8}}{4}$ for $\alpha \geq 1$.

$$\begin{aligned}
\lim_{\alpha\to\infty} \frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})} &= \lim_{\alpha\to\infty} \frac{2\sqrt{\pi}\,(B_\alpha)^{\alpha-1}\ \exp\left(-(\alpha-1)+B_\alpha^2\right)}{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\exp(\frac{\alpha}{2})} \\
&= \lim_{\alpha\to\infty} \frac{2\sqrt{\pi}\,(B_\alpha)^{\alpha-1}\ \exp\left(-(\alpha-1)+B_\alpha^2\right)}{2^{\frac{\alpha}{2}}\left[2\sqrt{\pi}\alpha^{\frac{\alpha-1}{2}}\exp(-\frac{\alpha}{2})\,(2)^{-\frac{\alpha}{2}}\right]\exp(\frac{\alpha}{2})} \times \left\{\lim_{\alpha\to\infty}\frac{2\sqrt{\frac{\pi}{\alpha}}\left(\frac{\alpha}{2e}\right)^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}\right\} \\
&= \lim_{\alpha\to\infty} \frac{2\sqrt{\pi}\,(B_\alpha)^{\alpha-1}\ \exp\left(-(\alpha-1)+B_\alpha^2\right)}{2^{\frac{\alpha}{2}}\left[2\sqrt{\pi}\alpha^{\frac{\alpha-1}{2}}\exp(-\frac{\alpha}{2})\,(2)^{-\frac{\alpha}{2}}\right]\exp(\frac{\alpha}{2})},
\end{aligned} \tag{34}$$

because $\lim_{\alpha\to\infty} \frac{2\sqrt{\frac{\pi}{\alpha}}\left(\frac{\alpha}{2e}\right)^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})} = 1$ using the Starling's approximation for the Gamma function. Consequently,

$$\begin{aligned}
\lim_{\alpha\to\infty} \frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})} &= \lim_{\alpha\to\infty} \frac{(B_\alpha)^{\alpha-1}\ \exp\left(-(\alpha-1)+B_\alpha^2\right)}{\alpha^{\frac{\alpha-1}{2}}} \times \{1\} \\
&= \lim_{\alpha\to\infty} \left(\frac{B_\alpha}{\sqrt{\alpha}}\right)^{\alpha-1} \exp\left(-(\alpha-1)+B_\alpha^2\right) \\
&= \lim_{\alpha\to\infty} \left(\frac{B_\alpha}{\sqrt{\alpha}}\right)^{\alpha-1} \exp\left(-\frac{(\alpha-1)}{2}+\frac{\sqrt{\alpha}B_\alpha}{2}\right),
\end{aligned} \tag{35}$$

because B_α satisfies $2B_\alpha^2 - \sqrt{\alpha}B_\alpha = (\alpha-1)$. Now consider that

$$\begin{aligned}
\left(\frac{B_\alpha}{\sqrt{\alpha}}\right)^{\alpha-1} = \left(\frac{\sqrt{\alpha}+\sqrt{9\alpha-8}}{4\sqrt{\alpha}}\right)^{\alpha-1} &= \left(1-\frac{\sqrt{9\alpha}-\sqrt{9\alpha-8}}{4\sqrt{\alpha}}\right)^{\alpha-1} \\
&= \left(1-\frac{2}{\sqrt{\alpha}(\sqrt{9\alpha}+\sqrt{9\alpha-8})}\right)^{\alpha-1} \\
&= \left(1-\frac{2}{(3\alpha+\sqrt{\alpha(9\alpha-8)})}\right)^{\alpha-1}.
\end{aligned} \tag{36}$$

Therefore

$$\lim_{\alpha\to\infty}\left(\frac{B_\alpha}{\sqrt{\alpha}}\right)^{\alpha-1} = \exp\left(-\lim_{\alpha\to\infty}\frac{2(\alpha-1)}{(3\alpha+\sqrt{\alpha(9\alpha-8)})}\right) = \exp(-\frac{1}{3}). \tag{37}$$

On the other hand

$$\begin{aligned}
\lim_{\alpha\to\infty} -\frac{(\alpha-1)}{2}+\frac{\sqrt{\alpha}B_\alpha}{2} &= \lim_{\alpha\to\infty}\frac{1}{2}-\frac{\alpha}{2}+\frac{\alpha+\sqrt{\alpha(9\alpha-8)}}{8} \\
&= \lim_{\alpha\to\infty}\frac{1}{2}-\sqrt{\alpha}\left(\frac{3\sqrt{\alpha}-\sqrt{\alpha(9\alpha-8)}}{8}\right) \\
&= \lim_{\alpha\to\infty}\frac{1}{2}-\left(\frac{\sqrt{\alpha}}{(3\sqrt{\alpha}+\sqrt{\alpha(9\alpha-8)})}\right) \\
&= \frac{1}{2}-\frac{1}{6} \\
&= \frac{1}{3}.
\end{aligned} \tag{38}$$

Hence, it follows from Equations 35, 37, 38 that

$$\lim_{\alpha\to\infty}\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}=1. \tag{39}$$

From Equation 26, we get that

$$\log(K_1(\alpha,\sqrt{\alpha}))=(\alpha-1)\log(B_\alpha)+B_\alpha^2-(\alpha-1)-\log\left(\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right)+\log(2\sqrt{\pi}).$$

On the other hand,

$$\log(K_2(\alpha,\sqrt{\alpha}))=\log\left(\frac{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\exp(\frac{\alpha}{2})}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]}\right).$$

Therefore

$$\begin{aligned}
&\frac{d}{d\alpha}\log(\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}) \\
=\ &\frac{d}{d\alpha}\log(K_1(\alpha,\sqrt{\alpha}))-\frac{d}{d\alpha}\log(K_2(\alpha,\sqrt{\alpha})) \\
=\ &\log(B_\alpha)+\frac{B_\alpha}{2\sqrt{\alpha}}-\frac{d}{d\alpha}\left(\frac{\alpha}{2}\log(2)+\log(\Gamma(\frac{\alpha}{2})+\frac{\alpha}{2}\right) \\
=\ &\log(B_\alpha)+\frac{B_\alpha}{2\sqrt{\alpha}}-\left(\frac{1}{2}\log(2)+\frac{1}{2}\frac{\frac{d}{d\alpha}(\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})}+\frac{1}{2}\right) \\
\leq\ &\log(B_\alpha)+\frac{B_\alpha}{2\sqrt{\alpha}}-\left(\frac{1}{2}\log(2)+\frac{1}{2}\left(\log(\frac{\alpha}{2})-\frac{1}{\alpha}-\frac{1}{3\alpha^2}\right)+\frac{1}{2}\right)
\end{aligned} \tag{40}$$

because $\log(\frac{\alpha}{2})-\frac{1}{\alpha}-\frac{1}{3\alpha^2}\leq\frac{\frac{d}{d\alpha}(\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})}\leq\log(\frac{\alpha}{2})-\frac{1}{\alpha}$ that follows from an inequality in (Batir, 2005). Therefore,

$$\begin{aligned} \frac{d}{d\alpha}\log\left(\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}\right) &\leq \log\left(\frac{B_\alpha}{\sqrt{\alpha}}\right)+\frac{B_\alpha}{2\sqrt{\alpha}}+\frac{1}{2\alpha}+\frac{1}{6\alpha^2}-\frac{1}{2} \\ &\leq 0. \end{aligned} \tag{41}$$

In order to establish the last inequality, we define $p(\alpha)=\log\left(\frac{B_\alpha}{\sqrt{\alpha}}\right)+\frac{B_\alpha}{2\sqrt{\alpha}}+\frac{1}{2\alpha}+\frac{1}{6\alpha^2}-\frac{1}{2}$. Note that $p(\alpha)$ is an increasing function as its derivative $\frac{dp(\alpha)}{d\alpha}=\frac{1}{B_\alpha\alpha\sqrt{\alpha(9\alpha-8)}}+\frac{1}{2\alpha\sqrt{\alpha(9\alpha-8)}}-\frac{1}{2\alpha^2}-\frac{1}{3\alpha^3}>0$ and $\lim_{\alpha\to\infty}p(\alpha)=0$.
An implication of Equation 41 is that the function $\alpha\mapsto\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}$ is decreasing in $\alpha>1$. As a result, it follows from Equation 41 and Equation 39

$$\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}\geq\lim_{\alpha\to\infty}\frac{K_1(\alpha,\sqrt{\alpha})}{K_2(\alpha,\sqrt{\alpha})}=1 \text{ for all } \alpha>1.$$

Consequently, $K_1(\alpha,\sqrt{\alpha})\geq K_2(\alpha,\sqrt{\alpha})$ for all $\alpha>1$.

2.12.4. Proof of part(d), Theorem 2

To prove $K_1(\alpha,\sqrt{\alpha})$ is decreasing.
From the definition of $K_1(\alpha,\Delta)$, we get that

$$K_1(\alpha,\sqrt{\alpha})=\frac{2\sqrt{\pi}\,(B_\alpha)^{\alpha-1}\ \exp\left(-(\alpha-1)+B_\alpha^2\right)}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \tag{42}$$

where $B_\alpha=\frac{\sqrt{\alpha}+\sqrt{9\alpha-8}}{4}$. As

$$\begin{aligned} (\alpha-1)+2B_\alpha^2=(\alpha-1)+\left(\frac{5\alpha-4+\sqrt{\alpha(9\alpha-8)}}{4}\right) &= \left(\frac{9\alpha-8+\sqrt{\alpha(9\alpha-8)}}{4}\right) \\ &= (\sqrt{9\alpha-8})B_\alpha, \end{aligned} \tag{43}$$

it turns out that

$$\begin{aligned} \frac{\partial B_\alpha}{\partial\Delta}=\frac{1}{4}\left[\frac{1}{2\sqrt{\alpha}}+\frac{9}{2\sqrt{9\alpha-8}}\right] &= \frac{1}{8}\left[\frac{\sqrt{9\alpha-8}+9\sqrt{\alpha}}{\sqrt{\alpha(9\alpha-8)}}\right] \\ &= \frac{B_\alpha}{2\sqrt{\alpha(9\alpha-8)}}+\frac{1}{\sqrt{(9\alpha-8)}}. \end{aligned} \tag{44}$$

Observe that

$$\log(K_1(\alpha,\Delta))=(\alpha-1)\log(B_\alpha)+B_\alpha^2-(\alpha-1)-\log\left(\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right)+\log(2\sqrt{\pi}).$$

Therefore,

$$
\begin{aligned}
&\frac{\partial \log(K_1(\alpha,\Delta))}{\partial\alpha} \\
= \;& \log(B_\alpha) - 1 + \left\{\frac{(\alpha-1)}{B_\alpha} + 2B_\alpha\right\}\frac{\partial B_\alpha}{\partial\alpha} - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \\
= \;& \log(B_\alpha) - 1 + \left\{\frac{(\alpha-1)}{B_\alpha} + 2B_\alpha\right\}\frac{\partial B_\alpha}{\partial\alpha} - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \\
= \;& \log(B_\alpha) - 1 + \frac{(\alpha-1)+2B_\alpha^2}{B_\alpha}\left(\frac{B_\alpha}{2\sqrt{\alpha(9\alpha-8)}} + \frac{1}{\sqrt{(9\alpha-8)}}\right) - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \\
\overset{(\dagger)}{=} \;& \log(B_\alpha) - 1 + \frac{(\sqrt{9\alpha-8})B_\alpha}{B_\alpha}\left(\frac{B_\alpha}{2\sqrt{\alpha(9\alpha-8)}} + \frac{1}{\sqrt{(9\alpha-8)}}\right) - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \\
= \;& \log(B_\alpha) - 1 + \left(\frac{B_\alpha}{2\sqrt{\alpha}} + 1\right) - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} \\
= \;& \log(B_\alpha) + \frac{B_\alpha}{2\sqrt{\alpha}} - \frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]}
\end{aligned} \tag{45}
$$

where the equality in (†) is due to Equation 43. On the other hand,

$$
\frac{\frac{\partial}{\partial\alpha}\left\{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]\right\}}{\Psi\left[\frac{\alpha}{2},\sqrt{\alpha}\right]} = E(\log(X)) + \frac{E(X)}{2\sqrt{\alpha}} \tag{46}
$$

where $X \sim \text{MHN}\,(\alpha, 1, \sqrt{\alpha})$. Therefore, it follows from Equation 45 that

$$
\begin{aligned}
\frac{\partial \log(K_1(\alpha,\Delta))}{\partial\alpha} &= \log(B_\alpha) + \frac{B_\alpha}{2\sqrt{\alpha}} - E(\log(X)) - \frac{E(X)}{2\sqrt{\alpha}} \\
&= -\left[E(\log(X)) - \log(B_\alpha)\right] - \frac{[E(X)-B_\alpha]}{2\sqrt{\alpha}} \\
&= -E\left(\log\left(\frac{X}{B_\alpha}\right)\right) - \frac{[E(X)-B_\alpha]}{2\sqrt{\alpha}} \\
&< 0,
\end{aligned} \tag{47}
$$

when $\alpha \geq 4$. The last inequality in Equation 47 follows from part(a) and part(b) of the Lemma 4 as B_α is also the mode of the MHN $(\alpha, 1, \sqrt{\alpha})$ distribution.

2.12.5. Proof of part(e), Theorem 2

Result 2. *For arbitrary* $\alpha > 1$, $\min\limits_{\Delta>0}\max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\} \geq \max\limits_{\Delta>0}\min\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\}$.

Proof of Result 2: According to the part(b) of the Theorem 2, the functions $\Delta \mapsto K_1(\alpha,\Delta)$ is non-increasing and $\Delta \mapsto K_2(\alpha,\Delta)$ is non-decreasing for all $\alpha > 1$. Therefore,

$$\left(\frac{1}{K_1(\alpha,\Delta_1)} - \frac{1}{K_1(\alpha,\Delta_2)}\right)\left(\frac{1}{K_2(\alpha,\Delta_1)} - \frac{1}{K_2(\alpha,\Delta_2)}\right) \le 0 \text{ for arbitrary } \Delta_1,\Delta_2 > 0 \text{ and } \alpha > 1. \quad (48)$$

If the statement of the result were not true then

$$\min_{\Delta>0} \max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\} < \max_{\Delta>0} \min\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\}.$$

In that case, there would exist $\Delta_1,\Delta_2 > 0, \alpha > 1$ such that

$$\begin{aligned}
& \max\left\{\frac{1}{K_1(\alpha,\Delta_1)}, \frac{1}{K_2(\alpha,\Delta_1)}\right\} < \min\left\{\frac{1}{K_1(\alpha,\Delta_2)}, \frac{1}{K_2(\alpha,\Delta_2)}\right\} \\
\implies & \begin{cases} \frac{1}{K_1(\alpha,\Delta_1)} < \min\left\{\frac{1}{K_1(\alpha,\Delta_2)}, \frac{1}{K_2(\alpha,\Delta_2)}\right\} & \text{and ,} \\ \frac{1}{K_2(\alpha,\Delta_1)} < \min\left\{\frac{1}{K_1(\alpha,\Delta_2)}, \frac{1}{K_2(\alpha,\Delta_2)}\right\} \end{cases} \\
\implies & \begin{cases} \frac{1}{K_1(\alpha,\Delta_1)} < \frac{1}{K_1(\alpha,\Delta_2)} & \text{and ,} \\ \frac{1}{K_2(\alpha,\Delta_1)} < \frac{1}{K_2(\alpha,\Delta_2)} \end{cases} \\
\implies & \left(\frac{1}{K_1(\alpha,\Delta_1)} - \frac{1}{K_1(\alpha,\Delta_2)}\right)\left(\frac{1}{K_2(\alpha,\Delta_1)} - \frac{1}{K_2(\alpha,\Delta_2)}\right) > 0, \quad (49)
\end{aligned}$$

which would contradict the fact in Equation 48. Therefore the statement of the result holds true.

Proof of part(e), Theorem 2: From part(a) of the Theorem 2, we have $\mathcal{A}_{\text{pos}}(\alpha,\Delta) = \max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\}$. An implication of the Part(b) of the Theorem 2 is that, for any $\alpha > 1$ the functions $\Delta \mapsto \frac{1}{K_1(\alpha,\Delta)}$ and $\Delta \mapsto \frac{1}{K_2(\alpha,\Delta)}$ are increasing and decreasing in $\Delta > 0$ correspondingly. On the other hand, Result 2 implicates that

$$\max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\} \ge \min\left\{\frac{1}{K_1(\alpha,\Delta_\star)}, \frac{1}{K_2(\alpha,\Delta_\star)}\right\} \text{ for arbitrary } \Delta > 0 \text{ and } \Delta_\star > 0. \quad (50)$$

In particular, if we choose $\Delta_\star$ to be $\sqrt{\alpha}$ then for any $\Delta > 0$

$$\max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\} \ge \min\left\{\frac{1}{K_1(\alpha,\sqrt{\alpha})}, \frac{1}{K_2(\alpha,\sqrt{\alpha})}\right\} = \frac{1}{K_1(\alpha,\sqrt{\alpha})}, \quad (51)$$

where the last equality is due to the Part(c) of the Theorem 2 which conveys that $\frac{1}{K_1(\alpha,\sqrt{\alpha})} \le \frac{1}{K_2(\alpha,\sqrt{\alpha})}$ for all $\alpha \ge 4$. Additionally, from the part(d) of Theorem 2 we infer that $\frac{1}{K_1(4,\sqrt{4})} \le \frac{1}{K_1(\alpha,\sqrt{\alpha})}$ for all $\alpha \ge 4$. Altogether, we conclude that

$$\mathcal{A}_{\text{pos}}(\alpha,\Delta) = \max\left\{\frac{1}{K_1(\alpha,\Delta)}, \frac{1}{K_2(\alpha,\Delta)}\right\} \ge \frac{1}{K_1(\alpha,\sqrt{\alpha})} \ge \frac{1}{K_1(4,2)} \ge 0.8$$

for all $\Delta > 0$ and $\alpha \ge 4$. As the point $\Delta > 0$ is arbitrary, we can write

$$\mathcal{A}_{\text{pos}}(\alpha, \frac{\gamma}{\sqrt{\beta}}) \ge \frac{1}{K_1(\alpha,\sqrt{\alpha})} \ge \frac{1}{K_1(4,2)} \ge 0.8 \text{ for all } \alpha \ge 4, \beta > 0, \gamma > 0.$$

2.13. Proof of Lemma 7

The MHN (α, β, γ) density function could be expressed as the summing of a infinite series of square root gamma density function.

$$\begin{aligned} f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} e^{\gamma x} e^{-\beta x^2} \\ &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \sum_{i=0}^{\infty} \frac{(\gamma x)^i}{i!} e^{-\beta x^2} \\ &= \sum_{i=0}^{\infty} \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} \frac{\gamma^i}{i!} \frac{\Gamma(\frac{\alpha+i}{2})}{2\beta^{\frac{\alpha+i}{2}}} \frac{2\beta^{\frac{\alpha+i}{2}}}{\Gamma(\frac{\alpha+i}{2})} x^{\alpha+i-1} e^{-\beta x^2} \\ &= \sum_{i=0}^{\infty} p_i f_i(x|\alpha, \beta), \end{aligned} \tag{52}$$

where $p_i = \frac{\Gamma(\frac{\alpha+i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] i!}$ and $f_i(x|\alpha, \beta) = \frac{2\beta^{\frac{\alpha+i}{2}}}{\Gamma(\frac{\alpha+i}{2})} x^{\alpha+i-1} e^{-\beta x^2}$. $f_i(x|\alpha, \beta)$ is the square root gamma density function. The density function of square root gamma distribution is $f_{\sqrt{\text{Gam}}}(y|\alpha, \beta, \gamma) = \frac{2\beta^\alpha}{\Gamma(\alpha)} y^{2\alpha-1} e^{-\beta y^2}$, which is derived by the transformation $y = \sqrt{x}$ from the Gamma distribution density $f_{\text{Gam}}(x|\alpha, \beta, \gamma) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$.

2.14. Proof of Lemma 8

2.14.1. Proof of Lemma 8 (a)

$q_i = \frac{\Gamma(\frac{\alpha+i}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^i}{i!}$ for $i \geq 0$ where $\alpha \geq 1$, $\gamma > 0, \beta > 0$. For an arbitrary positive number $\epsilon_1 > 0$, suppose $M^\dagger = Max\left\{[\alpha], \left[\frac{\gamma^2}{\epsilon_1^2 \beta}\right]\right\}$ where $[\alpha]$ and $\left[\frac{\gamma^2}{\epsilon_1^2 \beta}\right]$ denotes the largest integer less than equal to α and $\frac{\gamma^2}{\epsilon_1^2 \beta}$ respectively.

$$\frac{q_{M+1}}{q_M} = \frac{\frac{\Gamma(\frac{\alpha+M+1}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^{M+1}}{(M+1)!}}{\frac{\Gamma(\frac{\alpha+M}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^{M}}{M!}} = \frac{\gamma \Gamma(\frac{\alpha+M+1}{2})}{\sqrt{\beta}(M+1)\Gamma(\frac{\alpha+M}{2})}, \tag{53}$$

Using the fact that $\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \leq \sqrt{x + \frac{1}{4} + \left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}$ for $x \geq -\frac{1}{2}$ (see (Kazarinoff, 1956; Watson, 1959; Qi et al., 2012)) we get that,

$$
\begin{aligned}
\frac{q_{M+1}}{q_M} \leq \frac{\gamma\sqrt{\frac{\alpha+M-1}{2}+\frac{1}{4}+\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}}{\sqrt{\beta}(M+1)} \quad &= \quad \frac{\gamma\sqrt{\frac{M}{2}+\left\{\frac{\alpha-1}{2}+\frac{1}{4}+\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2\right\}}}{\sqrt{\beta}(M+1)} \\
&\leq \quad \frac{\gamma\sqrt{\frac{M}{2}+\left\{\frac{\alpha-1}{2}+\frac{1}{2}\right\}}}{\sqrt{\beta}(M+1)} \\
&= \quad \frac{\gamma\sqrt{\frac{M}{2}+\frac{\alpha}{2}}}{\sqrt{\beta}(M+1)} \\
&\overset{(ix)}{\leq} \quad \frac{\gamma\sqrt{M+1}}{\sqrt{\beta}(M+1)} \\
&= \quad \frac{\gamma}{\sqrt{\beta(M+1)}}, \qquad (54)
\end{aligned}
$$

where the inequality in (ix) follows from the assumption that $M \geq [\alpha]$. From equation 54 and the assumption $M \geq \left[\frac{\gamma^2}{\epsilon_1^2\beta}\right]$, it follows that

$$
M+1 \geq \frac{\gamma^2}{\epsilon_1^2\beta} \quad \Longrightarrow \quad \frac{q_{M+1}}{q_M} \leq \epsilon_1.
$$

2.14.2. Proof of Lemma 8 (b)

$$
\begin{aligned}
\frac{q_{i+2}q_i}{q_{i+1}^2} \quad &= \quad \frac{\frac{\Gamma(\frac{\alpha+i+2}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^{i+2}}{(i+2)!}\frac{\Gamma(\frac{\alpha+i}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^{i}}{i!}}{\left[\frac{\Gamma(\frac{\alpha+i+1}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^{i+1}}{(i+1)!}\right]^2} \\
&= \quad \frac{\frac{\Gamma(\frac{\alpha+i+2}{2})}{(i+2)!}\frac{\Gamma(\frac{\alpha+i}{2})}{i!}}{\left[\frac{\Gamma(\frac{\alpha+i+1}{2})}{(i+1)!}\right]^2} \\
&= \quad \frac{[(i+1)!]^2}{(i+2)!(i)!}\frac{\Gamma(\frac{\alpha+i+2}{2})\Gamma(\frac{\alpha+i}{2})}{\left[\Gamma(\frac{\alpha+i+1}{2})\right]^2} \\
&= \quad \frac{(i+1)}{(i+2)}\frac{\Gamma(\frac{\alpha+i+2}{2})\Gamma(\frac{\alpha+i}{2})}{\left[\Gamma(\frac{\alpha+i+1}{2})\right]^2} \\
&= \quad \frac{(i+1)}{(i+2)}\frac{\Gamma(\frac{\alpha+i}{2}+1)}{\Gamma(\frac{\alpha+i}{2}+\frac{1}{2})}\frac{\Gamma(\frac{\alpha+i-1}{2}+\frac{1}{2})}{\Gamma(\frac{\alpha+i-1}{2}+1)}. \qquad (55)
\end{aligned}
$$

From Watson's monotonicity properties (Kazarinoff, 1956; Watson, 1959; Qi et al., 2012) of the ratio of Gamma functions, we have the double inequality

$$
\sqrt{x+\frac{1}{4}} \leq \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \leq \sqrt{x+\frac{1}{4}+\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2} \text{ for } x \geq -\frac{1}{2}. \qquad (56)
$$

Using equation 56 in Equation 55 we get that

$$\begin{aligned}
\frac{q_{i+2}q_i}{q_{i+1}^2} &\leq \frac{i+1}{i+2}\frac{\sqrt{\frac{\alpha+i}{2}+\frac{1}{4}+\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}}{\sqrt{\frac{\alpha+i-1}{2}+\frac{1}{4}}} \\
&\leq \frac{i+1}{i+2}\frac{\sqrt{\frac{1+i}{2}+\frac{1}{4}+\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}}{\sqrt{\frac{i}{2}+\frac{1}{4}}} \\
&\leq \frac{i+1}{i+2}\frac{\sqrt{1+i+\frac{1}{2}+2\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}}{\sqrt{i+\frac{1}{2}}} \\
&\leq \frac{i+1}{i+2}\frac{\sqrt{i+\frac{3}{2}+2\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2}}{\sqrt{i+\frac{1}{2}}}. \qquad (57)
\end{aligned}$$

As $\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^2 < \frac{1}{4}$ we get that

$$\frac{q_{i+2}q_i}{q_{i+1}^2} < \frac{i+1}{i+2}\frac{\sqrt{i+2}}{\sqrt{i+\frac{1}{2}}} = \frac{i+1}{\sqrt{(i+2)(i+\frac{1}{2})}} = \frac{i+1}{\sqrt{(i+1)^2+\frac{i}{2}}} \leq 1$$

for all $i \geq 0$.

2.14.3. Proof of Lemma 8 (c)

Let $M \geq M^{\dagger}$, then using part (a) we get that

$$\frac{q_{M+1}}{q_M} \leq \epsilon_1, \qquad (58)$$

while from part (b) it follows that

$$\frac{q_{M+1}}{q_M} > \frac{q_{M+2}}{q_{M+1}} > \dots$$

Consequently, $q_{M+1} < q_M\epsilon_1, \dots, q_{j+1} \leq q_M\epsilon_1^{j-M}$ for $j > M$. As a result

$$\sum_{j=M}^{\infty} q_j = \sum_{j=M}^{\infty} \epsilon_1^{j-M} q_M = q_M \sum_{j=0}^{\infty} \epsilon_1^j = \frac{q_M}{1-\epsilon_1}.$$

part(d): From the definition of the the fox wright function we have

$$\sum_{i=1}^{\infty} q_i = \sum_{i=0}^{\infty} \frac{\Gamma(\frac{\alpha+i}{2})\left(\frac{\gamma}{\sqrt{\beta}}\right)^i}{i!} \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] < \infty,$$

which is a (absolutely) convergent series. Hence $lim_{i\to\infty} q_i = 0$.

2.15. Proof of Theorem 3

The argument presented depends on the the following theorem.
Theorem: (AM-GM Inequality with non uniform Weights (Steele, 2004)) Suppose that $p_1, p_2, \cdots, p_n$ are nonnegative real numbers such that $\sum_{i=1}^{n} p_i = 1$. For any nonnegative real numbers $a_1, a_2. \cdots, a_n$,

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}.$$

$$\begin{aligned} & f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) \\ = & \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} e^{\gamma x} e^{-\beta x^2} \\ = & \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\left(-\beta m^2 (x/m)^2 - |\gamma| m (x/m)\right) \\ = & \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\left(-(\beta m^2 + |\gamma| m)(\frac{\beta m^2}{\beta m^2 + |\gamma| m}(x/m)^2 + \frac{|\gamma| m}{\beta m^2 + |\gamma| m}(x/m))\right). \end{aligned}$$

Applying the AM-GM Inequality with non uniform Weights (that we stated above) we get that

$$f_{\text{MHN}}(x \mid \alpha, \beta, \gamma) \leq g_{\text{kernel}}(x; \alpha, \beta, \gamma) \tag{59}$$

where $g_{\text{kernel}}(x|\alpha, \beta, \gamma) = K_0(m, \alpha, \beta, \gamma) x^{\alpha-1} \ \exp\left(-(\beta m^2 + |\gamma| m)(\frac{x}{m})^{\frac{2\beta m + |\gamma|}{\beta m + |\gamma|}}\right)$,

$$K_0(m, \alpha, \beta, \gamma) = \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}.$$

2.16. Proof of Theorem 4

2.16.1. Proof of part(a), Theorem 4:

The proposal kernel for the Accept-Reject algorithm is (see Theorem 3)

$$g_{\text{kernel}}(x \mid \alpha, \beta, \gamma) = \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} x^{\alpha-1} \exp\left(-(\beta m^2 + |\gamma| m)(\frac{x}{m})^{\frac{2\beta m + |\gamma|}{\beta m + |\gamma|}}\right).$$

Therefore,

$$\int_0^\infty g_{\text{kernel}}(x|\alpha,\beta,\gamma)dx = \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}\int_0^\infty x^{\alpha-1}\exp\left(-(\beta m^2+|\gamma|m)(\frac{x}{m})^{\frac{2\beta m+|\gamma|}{\beta m+|\gamma|}}\right)dx.$$

Applying the change of variable $y=(\frac{x}{m})^{\frac{2\beta m+|\gamma|}{\beta m+|\gamma|}}$, we get that

$$\begin{aligned}\int_0^\infty g(x|\alpha,\beta,\gamma)dx &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}m^\alpha\frac{\beta m+|\gamma|}{2\beta m+|\gamma|}\int_0^\infty y^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}\exp[-(\beta m^2+|\gamma|m)y]dy\\ &= \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}m^\alpha\frac{\beta m+|\gamma|}{2\beta m+|\gamma|}\frac{\Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})}{[m(\beta m+|\gamma|)]^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}}}.\end{aligned}$$

Then the acceptance rate

$$\begin{aligned}\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma) &= \frac{\int_0^\infty f_{\text{MHN}}(x\mid\alpha,\beta,\gamma)dx}{\int_0^\infty g_{\text{kernel}}(x|\alpha,\beta,\gamma)dx}\\ &= \frac{1}{\int_0^\infty g_{\text{kernel}}(x|\alpha,\beta,\gamma)dx}\\ &= \frac{\frac{1}{2(\beta m^2)^{\frac{\alpha}{2}}}\Psi\left[\frac{\alpha}{2},\frac{-|\gamma|}{\sqrt{\beta}}\right]}{\frac{\beta m+|\gamma|}{2\beta m+|\gamma|}\frac{\Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})}{[m(\beta m+|\gamma|)]^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}}}}.\end{aligned}$$

2.16.2. Proof of part(b), Theorem 4:

$$\begin{aligned}&\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma))\\ =\;& \frac{\alpha\beta|\gamma|}{(2\beta m+|\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}\right)-\log(\beta m^2+m|\gamma|)\right]+\frac{2\beta}{2\beta m+|\gamma|}-\frac{\beta}{\beta m+|\gamma|}\\ =\;& \frac{\alpha\beta|\gamma|}{(2\beta m+|\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}\right)-\log(\beta m^2+m|\gamma|)\right]+\frac{\alpha\beta|\gamma|}{(\beta m+|\gamma|)(2\beta m+|\gamma|)}\\ =\;& \frac{\alpha\beta|\gamma|}{(2\beta m+|\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}\right)+\frac{2\beta m+|\gamma|}{\alpha(\beta m+|\gamma|)}-\log(\beta m^2+m|\gamma|)\right],\end{aligned}\tag{60}$$

where $\psi(\cdot)$ denotes the digamma function. The function $x\mapsto\psi(x)+\frac{1}{x}$ is a strictly increasing function because

$$\frac{d}{dx}\left(\psi(x)+\frac{1}{x}\right)=\left(\frac{d\psi(x)}{dx}-\frac{1}{x^2}\right)\overset{(\star\star)}{>}\frac{e^{\frac{1}{x+1}}-e^{-\frac{1}{x}}}{2}>0,$$

where the inequality in $(\star\star)$ is due to Equation 1.12 in (Qi and Mortici, 2015; Yang, Chu and Tao, 2014).
The function $m \mapsto \frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}$ is strictly decreasing. Therefore their composition, $m \mapsto \psi(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}) + \frac{2\beta m+|\gamma|}{\alpha(\beta m+|\gamma|)}$ is a strictly decreasing function. As the functions $m \mapsto \frac{\alpha\beta|\gamma|}{(2\beta m+|\gamma|)^2}$ and $m \mapsto -\log(\beta m^2+|\gamma|m)$ are strictly decreasing as well, it follows from Equation 60 that $\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma))$ is strictly decreasing in m for $m>0$. Moreover,

$$\lim_{m\to 0}\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma)) = \infty,\ \lim_{m\to\infty}\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma)) = -\infty,$$

and $m \mapsto \frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma))$ is a continuous function. Consequently, we conclude that that a unique $m_{\text{opt}}>0$ such that $\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma))\big|_{m=m_{\text{opt}}} = 0$.

Now we will prove that $m_{\text{opt}} > X_{\text{mode}}$ the mode of the MHN (α,β,γ) distribution. X_{mode} satisfies the equation

$$\frac{\alpha-1}{X_{\text{mode}}} - 2\beta X_{\text{mode}} - |\gamma| = 0 \quad\Longrightarrow\quad 2\beta X_{\text{mode}}^2 + |\gamma|X_{\text{mode}} = \alpha-1. \tag{61}$$

As a result, it follows that

$$\begin{aligned}
&\left[\psi\left(\frac{\alpha(\beta X_{\text{mode}}+|\gamma|)}{2\beta X_{\text{mode}}+|\gamma|}\right) - \log(\beta X_{\text{mode}}^2 + X_{\text{mode}}|\gamma|)\right]\\
=\ &\left[\psi\left(\frac{\alpha(\beta X_{\text{mode}}^2+|\gamma|X_{\text{mode}})}{\alpha-1}\right) - \log(\beta X_{\text{mode}}^2 + X_{\text{mode}}|\gamma|)\right].
\end{aligned}\tag{62}$$

Consider the inequality $\log(x)-\psi(x) < \frac{1}{x}$ for all $x>0$ (Alzer, 1997; Anderson et al., 1995). Thus it follows from Equation 62 that

$$\begin{aligned}
&\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta X_{\text{mode}}+|\gamma|)}{2\beta X_{\text{mode}}+|\gamma|}\right) - \log(\beta X_{\text{mode}}^2 + X_{\text{mode}}|\gamma|)\right]\\
>\ &\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\left\{\log\left(\frac{\alpha(\beta X_{\text{mode}}^2+|\gamma|X_{\text{mode}})}{\alpha-1}\right)\right.\\
&\qquad\left.-\frac{\alpha-1}{\alpha(\beta X_{\text{mode}}^2+|\gamma|X_{\text{mode}})} - \log(\beta X_{\text{mode}}^2 + X_{\text{mode}}|\gamma|)\right\}\\
=\ &\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) - \frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\frac{\alpha-1}{\alpha(\beta X_{\text{mode}}^2+|\gamma|X_{\text{mode}})}\\
=\ &\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) - \frac{\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)(2\beta X_{\text{mode}}^2+|\gamma|X_{\text{mode}})}\frac{\alpha-1}{(\beta X_{\text{mode}}+|\gamma|)}\\
\overset{(*)}{=}\ &\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) - \frac{\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)(\alpha-1)}\frac{\alpha-1}{(\beta X_{\text{mode}}+|\gamma|)}\\
=\ &\frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) - \frac{\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)(\beta X_{\text{mode}}+|\gamma|)},
\end{aligned}$$

where the equality in $(*)$ is a consequence of the Equation 61. Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial m}\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma))\Big|_{m=X_{\text{mode}}} \\
= \;& \frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta X_{\text{mode}}+|\gamma|)}{2\beta X_{\text{mode}}+|\gamma|}\right)-\log(\beta X_{\text{mode}}^2+X_{\text{mode}}|\gamma|)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{2\beta}{2\beta X_{\text{mode}}+|\gamma|}-\frac{\beta}{\beta X_{\text{mode}}+|\gamma|} \\
> \;& \frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) \\
&\qquad -\frac{\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)(\beta X_{\text{mode}}+|\gamma|)}+\frac{\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)(\beta X_{\text{mode}}+|\gamma|)} \\
= \;& \frac{\alpha\beta|\gamma|}{(2\beta X_{\text{mode}}+|\gamma|)^2}\log\left(\frac{\alpha}{\alpha-1}\right) \\
> \;& 0.
\end{aligned} \tag{63}$$

The function $\log(\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma)))$ is increasing in the region $(0, m_{\text{opt}})$ whereas it decreases on (m_{opt}, ∞). As the slope of the function at X_{mode} is positive (see Equation 63) we conclude that $X_{\text{mode}} \in (0, m_{\text{opt}})$ and $m_{\text{opt}} > X_{\text{mode}}$.

2.16.3. **Proof of part(c), Theorem 4:**

The arguments used to prove this part of the Themorem utilizes the following Theorem on the Gamma function.

Theorem (Ramanujan's Double Inequality(Alzer, 2003)) For $z > 0$,

$$\sqrt{\pi}\left(\frac{z}{e}\right)^z\left(8z^3+4z^2+z+\frac{1}{100}\right)^{\frac{1}{6}} < \Gamma(1+z) < \sqrt{\pi}\left(\frac{z}{e}\right)^z\left(8z^3+4z^2+z+\frac{1}{30}\right)^{\frac{1}{6}}.$$

From Theorem 3 and the Lemma 12 we get that,

$$\begin{aligned}
\mathcal{A}_{\text{neg}}(m,\alpha,\beta,\gamma) &= \frac{(2\beta m+|\gamma|)(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}\Psi\left[\frac{\alpha}{2},\frac{-|\gamma|}{\sqrt{\beta}}\right]}{2\beta^{\frac{\alpha}{2}}\ m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}}\ \Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})} \\
&\geq \frac{(2\beta m+|\gamma|)(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}\left[\frac{\beta^{\frac{\alpha}{2}}\exp(-\frac{m|\gamma|}{2})\Gamma(\frac{\alpha}{2})}{\left(\beta+\frac{|\gamma|}{2m}\right)^{\frac{\alpha}{2}}}\right]}{2\beta^{\frac{\alpha}{2}}\ m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}}\ \Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})} \\
&\geq \frac{(2\beta m+|\gamma|)(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}\left[\exp(-\frac{m|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{2\ m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}}\ \Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})\left[\left(\beta+\frac{|\gamma|}{2m}\right)^{\frac{\alpha}{2}}\right]} \\
&\geq \frac{(2\beta m+|\gamma|)(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}}{2\ m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}}\ \left[\left(\beta+\frac{|\gamma|}{2m}\right)^{\frac{\alpha}{2}}\right]} \times \frac{\left[\exp(-\frac{m|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{\Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})} \\
&\geq \frac{(2\beta m+|\gamma|)(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}}{2^{-\frac{\alpha}{2}+1}\ m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}-\frac{\alpha}{2}}\ \left[(2\beta m+|\gamma|)^{\frac{\alpha}{2}}\right]} \times \frac{\left[\exp(-\frac{m|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{\Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})} \\
&\geq \frac{2^{\frac{\alpha}{2}-1}(\beta m+|\gamma|)^{\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|}-1}}{m^{\frac{\alpha\beta m}{2\beta m+|\gamma|}-\frac{\alpha}{2}}\ (2\beta m+|\gamma|)^{\frac{\alpha}{2}-1}} \times \frac{\left[\exp(-\frac{m|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{\Gamma(\frac{\alpha(\beta m+|\gamma|)}{2\beta m+|\gamma|})}. \qquad (64)
\end{aligned}$$

Consider a point $m_\star > 0$ such that $2\beta m_\star^2 + |\gamma| m_\star = \alpha$. Note that, X_{mode}, the mode of the MHN (α,β,γ), $\alpha > 1, \gamma < 0$ satisfies the equation $2\beta X_{\text{mode}}^2 + |\gamma| X_{\text{mode}} = \alpha - 1$. Therefore $m_\star$ is greater than X_{mode}. It follows from Equation 64 that

$$\begin{aligned}
\mathcal{A}_{\text{neg}}(m_\star,\alpha,\beta,\gamma) &\geq \frac{2^{\frac{\alpha}{2}-1}(\beta m_\star+|\gamma|)^{\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}-1}}{m_\star^{\frac{\alpha\beta m_\star}{2\beta m_\star+|\gamma|}-\frac{\alpha}{2}}\ (2\beta m_\star+|\gamma|)^{\frac{\alpha}{2}-1}} \times \frac{\left[\exp(-\frac{m_\star|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{\Gamma(\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|})} \\
&= \frac{2^{\frac{\alpha}{2}-1}(\beta m_\star^2+|\gamma| m_\star)^{\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}-1}}{(2\beta m_\star^2+|\gamma| m_\star)^{\frac{\alpha}{2}-1}} \times \frac{\left[\exp(-\frac{m_\star|\gamma|}{2})\Gamma(\frac{\alpha}{2})\right]}{\Gamma(\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|})}.
\end{aligned} \qquad (65)$$

Applying Ramanujan's Double Inequality for the Gamma function (stated above), we get that

$$\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|})} > \left(\frac{\alpha-2}{2e}\right)^{\frac{\alpha}{2}-1}\left(\frac{1}{e}\left\{\frac{\alpha(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}-1\right\}\right)^{-\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}+1}\Upsilon(\alpha,\beta,\gamma,m_\star), \qquad (66)$$

where

$$\Upsilon(\alpha,\beta,\gamma,m_\star)=\frac{\left((\alpha-2)^3+(\alpha-2)^2+\frac{\alpha-2}{2}+\frac{1}{100}\right)^{\frac{1}{6}}}{\left[(\alpha-2)^3(1+c_{\beta,\gamma,m_\star})^3+(\alpha-2)^2(1+c_{\beta,\gamma,m_\star})^2+\frac{\alpha-2}{2}(1+c_{\beta,\gamma,m_\star})+\frac{1}{30}\right]^{\frac{1}{6}}},$$

and

$$c_{\beta,\gamma,m_\star}=\frac{\alpha\gamma}{(\alpha-2)(2\beta m_\star+\gamma)}.$$

In the case when $\alpha\ge 3$

$$\begin{aligned} c_{\beta,\gamma,m_\star}=\frac{\alpha\gamma}{(\alpha-2)(2\beta m_\star+\gamma)} &= \frac{\alpha\gamma m_\star}{(\alpha-2)(2\beta m_\star^2+\gamma m_\star)} \\ &= \frac{\alpha\gamma m_\star}{(\alpha-2)(\alpha)} \\ &= \frac{2}{(\alpha-2)\left(\sqrt{(1+\frac{8\alpha\beta}{\gamma^2})}+1\right)} \\ &\le 1. \end{aligned} \tag{67}$$

Moreover, $\Upsilon(\alpha,\beta,\gamma,m_\star)\ge\frac{\left((\alpha-2)^3+(\alpha-2)^2+\frac{\alpha-2}{2}+\frac{1}{100}\right)^{\frac{1}{6}}}{\left[(\alpha-2)^3(2)^3+(\alpha-2)^2(2)^2+\frac{\alpha-2}{2}(2)+\frac{1}{30}\right]^{\frac{1}{6}}}\ge\frac{1}{\sqrt{2}}$ because $\alpha\mapsto\frac{\left(\alpha^3+\alpha^2+\frac{\alpha}{2}+\frac{1}{100}\right)}{\left(8\alpha^3+4\alpha^2+\alpha+\frac{1}{30}\right)}$ is strictly decreasing in α and $\lim_{\alpha\to\infty}\frac{\left(\alpha^3+\alpha^2+\frac{\alpha}{2}+\frac{1}{100}\right)}{\left(8\alpha^3+4\alpha^2+\alpha+\frac{1}{30}\right)}=\frac{1}{8}$. Therefore,

$$\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|})} > \left(\frac{\alpha-2}{2e}\right)^{\frac{\alpha}{2}-1}\left(\frac{1}{e}\left\{\frac{\alpha(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}-1\right\}\right)^{-\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}+1}\frac{1}{\sqrt{2}}. \tag{68}$$

Consider the fact that, for any $c>0, x>0$,

$$\begin{aligned} & (1+\frac{1}{x+c})^{x+c} > (1+\frac{1}{x})^x \\ \Longrightarrow\quad & x^x(x+c+1)^{x+c} > (x+1)^x(x+c)^{x+c} \\ \Longrightarrow\quad & x^x(x+c)^{-(x+c)} > (x+1)^x(x+c+1)^{-(x+c)}. \end{aligned} \tag{69}$$

Utilizing the inequality with $x=\frac{\alpha}{2}-1$ and $x+c=\frac{\alpha(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}-1$, it follows from Equation 68 that

$$\begin{aligned} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|})} &> \left(\frac{\alpha}{2e}\right)^{\frac{\alpha}{2}-1}\left(\frac{1}{e}\left\{\frac{\alpha(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}\right\}\right)^{-\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}+1}\frac{1}{\sqrt{2}} \\ &= \left(\frac{e}{\alpha}\right)^{\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}-\frac{\alpha}{2}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2}-1}\left(\frac{(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}\right)^{-\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}+1}\frac{1}{\sqrt{2}} \\ &= 2^{-\frac{\alpha-1}{2}}\left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star+|\gamma|)}}\left(\frac{(\beta m_\star+|\gamma|)}{(2\beta m_\star+|\gamma|)}\right)^{-\frac{\alpha(\beta m_\star+|\gamma|)}{2\beta m_\star+|\gamma|}+1}. \end{aligned} \tag{70}$$

It follows from Equations 65 and 70 that

$$
\begin{aligned}
& \mathcal{A}_{\text{neg}}(m_\star, \alpha, \beta, \gamma) \\
\geq \quad & \frac{2^{\frac{\alpha}{2}-1}(\beta m_\star^2 + |\gamma| m_\star)^{\frac{\alpha(\beta m_\star + |\gamma|)}{2\beta m_\star + |\gamma|}-1}}{(2\beta m_\star^2 + |\gamma| m_\star)^{\frac{\alpha}{2}-1}} \times \frac{\left[\exp\left(-\frac{m_\star|\gamma|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\right]}{\Gamma\left(\frac{\alpha(\beta m_\star + |\gamma|)}{2\beta m_\star + |\gamma|}\right)}. \\
\geq \quad & \frac{2^{\frac{\alpha}{2}-1}(\beta m_\star^2 + |\gamma| m_\star)^{\frac{\alpha(\beta m_\star + |\gamma|)}{2\beta m_\star + |\gamma|}-1} \exp\left(-\frac{m_\star|\gamma|}{2}\right)}{(2\beta m_\star^2 + |\gamma| m_\star)^{\frac{\alpha}{2}-1}} \\
& \times \left[2^{-\frac{\alpha-1}{2}} \left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}} \left(\frac{(\beta m_\star + |\gamma|)}{(2\beta m_\star + |\gamma|)}\right)^{-\frac{\alpha(\beta m_\star + |\gamma|)}{2\beta m_\star + |\gamma|}+1}\right] \\
\geq \quad & \frac{2^{\frac{\alpha}{2}-1}\exp\left(-\frac{m_\star|\gamma|}{2}\right)}{(2\beta m_\star^2 + |\gamma| m_\star)^{-\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}} \times \left[2^{-\frac{\alpha-1}{2}} \left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}\right] \\
= \quad & \frac{2^{-\frac{1}{2}}\exp\left(-\frac{m_\star|\gamma|}{2}\right)}{(2\beta m_\star^2 + |\gamma| m_\star)^{-\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}} \times \left[\left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}\right].
\end{aligned} \tag{71}
$$

As the point $m_\star$ is such that, $2\beta m_\star^2 + |\gamma| m_\star = \alpha$. It appears that

$$
\begin{aligned}
\mathcal{A}_{\text{neg}}(m_\star, \alpha, \beta, \gamma) \quad \geq \quad & \frac{2^{-\frac{1}{2}}\exp\left(-\frac{m_\star|\gamma|}{2}\right)}{(2\beta m_\star^2 + |\gamma| m_\star)^{-\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}} \times \left[\left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}\right]. \\
= \quad & \frac{2^{-\frac{1}{2}}\exp\left(-\frac{m_\star|\gamma|}{2}\right)}{(\alpha)^{-\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}} \times \left[\left(\frac{e}{\alpha}\right)^{\frac{\alpha|\gamma|}{2(2\beta m_\star + |\gamma|)}}\right]. \\
= \quad & 2^{-\frac{1}{2}}\exp\left(-\frac{m_\star|\gamma|}{2}\right) \times \left[(e)^{\frac{\alpha|\gamma| m_\star}{2(\alpha)}}\right]. \\
= \quad & \frac{1}{\sqrt{2}}.
\end{aligned} \tag{72}
$$

The point m_{opt} maximizes the function $m \mapsto \mathcal{A}_{\text{neg}}(m, \alpha, \beta, \gamma)$. Therefore, we conclude that

$$
\mathcal{A}_{\text{neg}}(m_{\text{opt}}, \alpha, \beta, \gamma) \geq \mathcal{A}_{\text{neg}}(m_\star, \alpha, \beta, \gamma) \geq \frac{1}{\sqrt{2}}.
$$

2.17. Iterative algorithm to find the optimal Matching Point

Our goal is to find the m making $\mathcal{A}_{\text{neg}}(m_{\text{opt}}, \alpha, \beta, \gamma)$ maximum, which is same as finding the m maximizing $\log(\mathcal{A}_{\text{neg}}(m_{\text{opt}}, \alpha, \beta, \gamma))$. Use Newton-Raphson Method to estimate the optimal m.

- Start with the proposed initial value m_{init},which is calculated based on the parameters α,β and γ.

- For iteration time $t+1$, $m_{t+1} = m_t - \frac{l'(m_t)}{l''(m_t)}$

- Repeat this process until $|m_{t+1} - m_t| < \epsilon$, where $\epsilon > 0$ is a predetermined small number to identify the convergence of the algorithm.

$$\begin{aligned}
l &= \log(\mathcal{A}_{\text{neg}}(m_{\text{opt}}, \alpha, \beta, \gamma)) \\
&= \log(\Psi\left[\frac{\alpha}{2}, \frac{-|\gamma|}{\sqrt{\beta}}\right]) + \frac{\alpha(\beta m + |\gamma|)}{2\beta m + |\gamma|}\log(\beta m^2 + m|\gamma|) + \log(2\beta m + |\gamma|) - \log 2 \\
&- \frac{\alpha}{2}\log(\beta m^2) - \log\Gamma(\frac{\alpha(\beta m + |\gamma|)}{2\beta m + |\gamma|}) - \log(\beta m + |\gamma|),
\end{aligned}$$

$$l' = \frac{\alpha\beta|\gamma|}{(2\beta m + |\gamma|)^2}\left[\psi\left(\frac{\alpha(\beta m + |\gamma|)}{2\beta m + |\gamma|}\right) - \log(\beta m^2 + m|\gamma|)\right] + \frac{2\beta}{2\beta m + |\gamma|} - \frac{\beta}{\beta m + |\gamma|}$$

$$\begin{aligned}
l'' = &\frac{4\alpha\beta^2|\gamma|}{(2\beta m + |\gamma|)^3}\left[\psi\left(\frac{\alpha(\beta m + |\gamma|)}{2\beta m + |\gamma|}\right) - \log(\beta m^2 + m|\gamma|)\right] \\
&+ \frac{\alpha\beta|\gamma|}{(2\beta m + |\gamma|)^2}\left[\frac{-\alpha\beta|\gamma|}{(2\beta m + |\gamma|)^2}\psi'\left(\frac{\alpha(\beta m + |\gamma|)}{2\beta m + |\gamma|}\right) - \frac{2\beta m + |\gamma|}{\beta m^2 + m|\gamma|}\right] - \frac{1}{(m + \frac{|\gamma|}{2\beta})^2} + \frac{1}{(m + \frac{|\gamma|}{\beta})^2},
\end{aligned}$$

where ψ and ψ' denote the digamma and trigamma functions respectively.

2.18. Method of Moment Estimators

First, based on lemma 2, we could express the expected value and variance of the random variable as functions of the parameters of interest. The number of such equations is the same as the number of parameters to be estimated.

$$\begin{aligned}
\text{Var}(X) &= \frac{\alpha}{2\beta} + E(X)\left(\frac{\gamma}{2\beta} - E(X)\right) \\
E(X^3) &= \frac{\alpha+1}{2\beta}E(X) + \frac{\gamma}{2\beta}E(X^2) \\
E(X^4) &= \frac{\alpha+2}{2\beta}E(X^2) + \frac{\gamma}{2\beta}E(X^3) \qquad (73)
\end{aligned}$$

The variance and moments are $\sigma^2 = \text{Var}(X)$ and $\mu_k = E(X^k)$. If $X_1, \cdot, X_N$ are i.i.d. random variables from the distribution, the k^{textth} sample moment can be expressed as $m_k = \sum_{i=0}^n x_i^k$ where k=1,2,3, and 4. From the law of large number, $m_k \to \mu_k$ in probability as $N \to \infty$. In addition, we could replace $\text{Var}(X) = E(X^2) - E(X)^2$ in equation 73. Then, all expected values acould be set equal to the sample moments. Finally, the solution of equation 73 is

$$\begin{aligned}
\hat{\alpha} &= \frac{m_2^3 - m_3^2 - m_1 m_2 m_3 + m_2 m_4}{G} - 1 \\
\hat{\beta} &= \frac{2m_2^2 - m_1 m_3 - m_1^2 m_2}{2G} \\
\hat{\gamma} &= \frac{2m_2 m_3 - m_1 m_2^2 - m_1 m_4}{G}
\end{aligned} \tag{74}$$

where $G = 2m_1 m_2 m_3 - m_1^2 m_4 + m_2 m_4 - m_2^3 - m_3^2$.

2.18.1. Asymptotic Distribution of Estimated Parameters

Consider a random vector $Y_i = (X_i, X_i^2, X_i^3, X_i^4)^T$ in $\mathbb{R}^4$, with the mean vector $\mu = E(Y_i)$ and the covariance matrix Σ. $(Y_1, \cdots, Y_n)$ are independent and identically distributed random variables, where $X_i \sim \text{MHN}(\alpha, \beta, \gamma)$. Using the multivariate central limit theorem,

$$\sqrt{n}(\overline{Y}_n = \mu) \xrightarrow{d} N_4(\mathbf{0}, \Sigma), \tag{75}$$

where $\overline{Y}_n = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}$, and $\Sigma = \begin{pmatrix} \mu_2 - \mu_1^2 & \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_1\mu_3 & \mu_5 - \mu_1\mu_4 \\ \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_2^2 & \mu_5 - \mu_2\mu_3 & \mu_6 - \mu_2\mu_4 \\ \mu_4 - \mu_1\mu_3 & \mu_5 - \mu_2\mu_3 & \mu_6 - \mu_3^2 & \mu_7 - \mu_3\mu_4 \\ \mu_5 - \mu_1\mu_4 & \mu_6 - \mu_2\mu_4 & \mu_7 - \mu_3\mu_4 & \mu_8 - \mu_4^2 \end{pmatrix}$.

Here, $m_k = \sum_{i=0}^{n} X_i^k$ and $\mu_k = E(X_i^k)$ for i in $(1, 2, \cdots, n)$.

Define a function $g : \mathbb{R}^4 \to \mathbb{R}^3$ such that g applied to the vector of moments in Equation 75 yields the vector $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ as defined in expression 74.

$$\nabla g \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{\alpha}}{\partial m_1} & \frac{\partial \hat{\beta}}{\partial m_1} & \frac{\partial \hat{\gamma}}{\partial m_1} \\ \frac{\partial \hat{\alpha}}{\partial m_2} & \frac{\partial \hat{\beta}}{\partial m_2} & \frac{\partial \hat{\gamma}}{\partial m_2} \\ \frac{\partial \hat{\alpha}}{\partial m_3} & \frac{\partial \hat{\beta}}{\partial m_3} & \frac{\partial \hat{\gamma}}{\partial m_3} \\ \frac{\partial \hat{\alpha}}{\partial m_4} & \frac{\partial \hat{\beta}}{\partial m_4} & \frac{\partial \hat{\gamma}}{\partial m_4} \end{bmatrix}$$

By the delta method,

$$\sqrt{n}\left(\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\right) \xrightarrow{d} N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma^* = \left(\nabla g \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}\right)^T \Sigma \left(\nabla g \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}\right)\right) \tag{76}$$

It is easy to find that all elements in Σ^* only depend on μ_k for $k = (1, 2, \cdots, 8)$.

2.19. Deribatives required for the finding MLE

The likelihood function for the

$$L(\theta) = L(\alpha,\beta,\gamma) = \prod_{j=1}^{n} \frac{2\beta^{\frac{\alpha}{2}}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} {x_j}^{\alpha-1} \exp(-\beta {x_j}^2 + \gamma x_j)$$

The log-likelihood function is

$$\begin{aligned} l(\theta) &= l(\alpha,\beta,\gamma) = n\ \log(2) + \frac{n\alpha}{2}\log(\beta) - n\ \log(\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]) \\ & \quad +(\alpha-1)\sum_{j=1}^{n}\log(x_j) - \beta\sum_{j=1}^{n}{x_j}^2 + \gamma\sum_{j=1}^{n}x_j. \end{aligned} \tag{77}$$

The deribatives required for the optimization is given as following.

$$\frac{\partial l}{\partial \alpha} = \frac{n}{2}\log(\beta) - n\frac{{}_1\Psi_1^{*}\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2}) \\ (1,0)\end{matrix}; \frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} + \sum_{j=1}^{n}\log(x_j).$$

Since ${}_1\Psi_1^{*}\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2}) \\ (1,0)\end{matrix}; \frac{\gamma}{\sqrt{\beta}}\right] \succ \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]$ as α increase, so $\lim_{\alpha\to\infty} \frac{{}_1\Psi_1^{*}\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2}) \\ (1,0)\end{matrix}; \frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} = \infty$

$$\frac{\partial l}{\partial \beta} = \frac{n\alpha}{2\beta} + \frac{n}{2\beta}\frac{\sum_{i=1}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{(i-1)!}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} - \sum_{j=1}^{n}{x_j}^2$$

$$\frac{\partial l}{\partial \gamma} = -\frac{n}{\gamma}\frac{\sum_{i=1}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{(i-1)!}}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} + \sum_{j=1}^{n}x_j$$

Since $\sum_{i=1}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{(i-1)!} \geq \Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]$ as α increase, so $\lim_{\alpha\to\infty} \frac{\Psi\left[\frac{\alpha+1}{2}, \frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]} = \infty,$

$$\frac{\partial^2 l}{\partial \alpha^2} = -n\frac{{}_1\Psi_1^{**}\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2}) \\ (1,0)\end{matrix}; \frac{\gamma}{\sqrt{\beta}}\right]\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right] - \left[{}_1\Psi_1^{*}\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2}) \\ (1,0)\end{matrix}; \frac{\gamma}{\sqrt{\beta}}\right]\right]^2}{\left[\Psi\left[\frac{\alpha}{2}, \frac{\gamma}{\sqrt{\beta}}\right]\right]^2}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta} = \frac{n}{2\beta} - n\frac{(-\frac{1}{2\beta})\sum_{i=1}^{\infty}\frac{\Gamma'(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{2(i-1)!}\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right] - {}_1\Psi_1^*\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2})\\(1,0)\end{matrix};\frac{\gamma}{\sqrt{\beta}}\right](-\frac{1}{2\beta})\sum_{i=1}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{(i-1)!}}{\left[\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]\right]^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\gamma} = -n\frac{(-\frac{1}{\gamma})\sum_{i=1}^{\infty}\frac{\Gamma'(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{2(i-1)!}\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right] - {}_1\Psi_1^*\left[\begin{matrix}(\frac{\alpha}{2},\frac{1}{2})\\(1,0)\end{matrix};\frac{\gamma}{\sqrt{\beta}}\right] - (\frac{1}{\gamma})\sum_{i=1}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+\frac{i}{2})(\frac{\gamma}{\sqrt{\beta}})^i}{(i-1)!}}{\left[\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]\right]^2},$$

$$\frac{\partial^2 l}{\partial\beta^2} = -\frac{4n\alpha\beta + n\alpha\gamma^2}{8\beta^3} - \frac{6n\beta\gamma + n\gamma^3}{8\beta^{\frac{7}{2}}}\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} + \frac{n\gamma^2}{4\beta^3}\left[\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}\right]^2,$$

$$\frac{\partial^2 l}{\partial\beta\partial\gamma} = \frac{n\alpha\gamma}{4\beta^2} + \frac{2n\beta + n\gamma^2}{4\beta^{\frac{5}{2}}}\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} - \frac{n\gamma}{2\beta^2}\left[\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}\right]^2,$$

$$\frac{\partial^2 l}{\partial\gamma^2} = -\frac{n\alpha}{2\beta} - \frac{n\gamma}{2\beta^{\frac{3}{2}}}\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]} + \frac{n}{\beta}\left[\frac{\Psi\left[\frac{\alpha+1}{2},\frac{\gamma}{\sqrt{\beta}}\right]}{\Psi\left[\frac{\alpha}{2},\frac{\gamma}{\sqrt{\beta}}\right]}\right]^2.$$

We apply the Newtown Raphson and Gradient based procedures to obtain the MLE.

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