Proceedings of

FEniCS 2021

22–26 March 2021

Editors

Igor Baratta

Jørgen S. Dokken

Chris Richardson

Matthew W. Scroggs

Hybridized discontinuous Galerkin methods for the Stokes and Navier–Stokes equations in FEniCSx: non-simplex cells and curved geometries

Joseph P. Dean (jpdean), University of Cambridge, United Kingdom
Sander Rhebergen, University of Waterloo, Canada
Chris N. Richardson (chrisrichardson), University of Cambridge, United Kingdom
Garth N. Wells, University of Cambridge, United Kingdom

26 March 2021

We investigate hybridized discontinuous Galerkin (HDG) methods for the Stokes and incompressible Navier–Stokes equations which yield approximate velocity fields that are pointwise divergence free in each cell and globally $H(\mathrm{div})$-conforming. The analysis of a recently developed method is restricted to simplex cells and affine geometries. Here, we explore the extension of the method to non-simplex cells and curved boundaries, both of which are important for engineering applications. Static condensation is used to reduce the size of the global system of equations. For the implementation, we make use of some new features of FEniCSx, which is composed of DOLFINx, FFCx, Basix, and UFL. We use UFL and FFCx to compile kernels for each block of the global matrix, which are then exposed to the Python interface using CFFI. These kernels are called from a custom kernel (compiled by Numba) to carry out the static condensation process. The smaller statically condensed system can then be solved using a block preconditioned iterative solver. We present analysis and numerical results demonstrating that the approximate velocity field is pointwise divergence free in each cell and globally $H(\mathrm{div})$-conforming on meshes with non-simplex cells and curved boundaries.

You can cite this talk as:

Joseph P. Dean, Sander Rhebergen, Chris N. Richardson, and Garth N. Wells. "Hybridized discontinuous Galerkin methods for the Stokes and Navier–Stokes equations in FEniCSx: non-simplex cells and curved geometries". In: *Proceedings of FEniCS 2021, online, 22–26 March* (eds Igor Baratta, Jørgen S. Dokken, Chris Richardson, Matthew W. Scroggs) (2021), 722–741. DOI: 10.6084/m9.figshare.14495634.

BibTeX for this citation can be found at https://mscroggs.github.io/fenics2021/talks/dean.html.

Hybridized discontinuous Galerkin methods for the Stokes and Navier-Stokes equations in FEniCSx: non-simplex cells and curved geometries

Joseph P. Dean[1] Sander Rhebergen[2] Chris N. Richardson[1] Garth N. Wells[1]

[1]*University of Cambridge* [2]*University of Waterloo*

Outline

1. The Stokes problem
2. Why not use conforming methods?
3. Hybridized discontinuous Galerkin
4. Non-simplex and curved cells
5. Implementation
6. Numerical results
7. The Navier-Stokes equations
8. Open questions

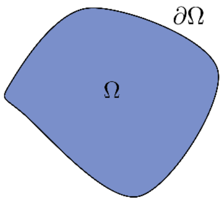
Problem statement

Stokes problem (weak form): *Given* $f \in [L^2(\Omega)]^d$, *find* $u \in V := [H_0^1(\Omega)]^d$ *and* $p \in Q := L_0^2(\Omega)$ *such that*

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \quad && \forall v \in V, \\ b(u,q) &= 0 && \forall q \in Q, \end{aligned}$$

where

$$a(u,v) := \int_\Omega \nu \nabla u : \nabla v \, \mathrm{d}x, \quad b(v,p) := -\int_\Omega p \nabla \cdot v \, \mathrm{d}x, \quad \text{and} \quad F(v) := \int_\Omega f \cdot v \, \mathrm{d}x.$$



Some observations

1. The problem is well-posed and $\exists \beta > 0$ such that

$$\inf_{q \in Q} \sup_{v \in V} \frac{\int_\Omega q \nabla \cdot v \, dx}{||v||_{1,\Omega} ||q||_{0,\Omega}} \geq \beta$$

2. The following invariance property[1] holds:

$$f \to f + \nabla \phi \implies (u, p) \to (u, p + \phi)$$

[1] Volker John et al. "On the Divergence Constraint in Mixed Finite Element Methods for Incompressible Flows". In: *SIAM Review* 59.3 (2017), pp. 492–544. DOI: `10.1137/15m1047696`.

Mass conservation?

Mass conservation (weak statement):

$$b(u, q) = 0 \quad \forall q \in Q$$

- The weak statement implies exact mass conservation, meaning $||\nabla \cdot u||_{0,\Omega} = 0$.

Mass conservation (discrete statement): *Let* $u_h \in V_h \subset V$, *then*

$$b(u_h, q_h) = 0 \quad \forall q_h \in Q_h \subset Q$$

- The discrete statement could imply global, local (cell), or exact mass conservation depending on V_h and Q_h. If $\nabla \cdot V_h \subseteq Q_h$, mass is conserved exactly.

With conforming methods, it is difficult to balance stability and incompressibility

Hybridized discontinuous Galerkin[2]

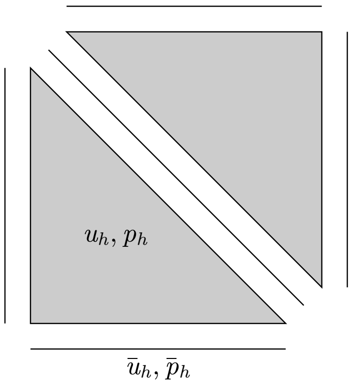
Let $\boldsymbol{u}_h := (u_h, \bar{u}_h) \in \boldsymbol{V}_h$ and $\boldsymbol{p}_h := (p_h, \bar{p}_h) \in \boldsymbol{Q}_h$, where $\boldsymbol{V}_h := V_h \times \bar{V}_h$, $\boldsymbol{Q}_h := Q_h \times \bar{Q}_h$, and

$$V_h := \left\{ v_h \in [L^2(\mathcal{T}_h)]^d;\ v_h|_K \in V_h(K)\ \forall K \in \mathcal{T}_h \right\},$$

$$\bar{V}_h := \left\{ \bar{v}_h \in [L^2(\mathcal{F}_h)]^d;\ \bar{v}_h|_F \in \bar{V}_h(F)\ \forall F \in \mathcal{F}_h,\ \bar{v}_h = 0 \text{ on } \partial\Omega \right\},$$

$$Q_h := \left\{ q_h \in L^2(\mathcal{T}_h);\ q_h|_K \in Q_h(K)\ \forall K \in \mathcal{T}_h \right\},$$

$$\bar{Q}_h := \left\{ \bar{q}_h \in L^2(\mathcal{F}_h);\ \bar{q}_h|_F \in \bar{Q}_h(F)\ \forall F \in \mathcal{F}_h \right\}.$$



[2] S. Rhebergen and G. N. Wells. “A hybridizable discontinuous Galerkin method for the Navier–Stokes equations with pointwise divergence-free velocity field”. In: *J. Sci. Comput.* 76.3 (2018), pp. 1484–1501. DOI: 10.1007/s10915-018-0671-4.

HDG formulation

Stokes problem (HDG formulation): *Find* $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{V}_h \times \boldsymbol{Q}_h$ *such that*

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(v_h, \boldsymbol{p}_h) &= F(v_h) \quad && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(u_h, \boldsymbol{q}_h) &= 0 && \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h, \end{aligned}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{K \in \mathcal{T}_h} \int_K \nu \nabla u_h : \nabla v_h \, \mathrm{d}x - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nu \left((u_h - \bar{u}_h) \cdot \partial_n v_h + \partial_n u_h \cdot (v_h - \bar{v}_h) \right) \mathrm{d}s \\ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nu \frac{\alpha}{h_K} (u_h - \bar{u}_h) \cdot (v_h - \bar{v}_h) \, \mathrm{d}s, \end{aligned}$$

and

$$b_h(v_h, \boldsymbol{p}_h) := - \sum_{K \in \mathcal{T}_h} \int_K p_h \nabla \cdot v_h \, \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} v_h \cdot n \bar{p}_h \, \mathrm{d}s.$$

Mapping functions

Let $\psi_K : V_h(K) \to V_h(\hat{K})$.

Lemma

If ψ_K is the pullback by the geometric mapping (as in the original method), and if $\nabla \cdot V_h(K) \subseteq Q_h(K)$ and $\bar{Q}_h(F) \supseteq \{v_h|_F \cdot n; v_h \in V_h(K)\}$, then the discrete velocity field is exactly divergence free.

Problem: what if the geometric mapping is not affine?

Lemma

If ψ_K is the contravariant Piola transform, then the above conditions can be relaxed; if $\nabla \cdot V_h(\hat{K}) \subseteq Q_h(\hat{K})$ and $\bar{Q}_h(\hat{F}) \supseteq \{\hat{v}_h|_{\hat{F}} \cdot \hat{n}; \hat{v}_h \in V_h(\hat{K})\}$ then the discrete velocity field is exactly divergence free.

A similar idea can be applied to Scott–Vogelius elements on curved domains.[3]

[3] Michael Neilan and M. Baris Otus. "Divergence–free Scott–Vogelius elements on curved domains". In: (2020), pp. 1–23. arXiv: 2008.06429. URL: http://arxiv.org/abs/2008.06429.

Suitable spaces

Simplex cells: *If $\hat{K}$ is the reference simplex and if ψ_K is the contravariant Piola transform, then the spaces*

$$V_h(\hat{K}) := [\mathbb{P}_k(\hat{K})]^d, \quad \bar{V}_h(\hat{F}) := [\mathbb{P}_k(\hat{F})]^d, \quad Q_h(\hat{K}) := \mathbb{P}_{k-1}(\hat{K}) \quad \textit{and} \quad \bar{Q}_h(\hat{F}) := \mathbb{P}_k(\hat{F})$$

give an exactly divergence free velocity field even if the geometric mapping is not affine.

Non-simplex cells: *If $\hat{K}$ is the reference quadrilateral or hexahedron and if ψ_K is the contravariant Piola transform, then the spaces*

$$V_h(\hat{K}) := \mathbb{RT}_k(\hat{K}), \quad \bar{V}_h(\hat{F}) := [\mathbb{Q}_k(\hat{F})]^d, \quad Q_h(\hat{K}) := \mathbb{Q}_k(\hat{K}), \quad \textit{and} \quad \bar{Q}_h(\hat{F}) := \mathbb{Q}_k(\hat{F})$$

give an exactly divergence free velocity field even if the geometric mapping is not affine.

More about the non-simplex case

- $H(\mathrm{div})$-conforming finite elements are introduced following the same ideas as divergence conforming DG[4] and HDG[5] methods.
- Other $H(\mathrm{div})$-conforming finite elements can be used, but care must be taken as some lose optimal order approximation in $[L^2(\Omega)]^d$ on general quadrilateral meshes.[6]

[4] Bernardo Cockburn, Guido Kanschat, and Dominik Schötzau. "A Note on Discontinuous Galerkin Divergence-free Solutions of the Navier-Stokes Equations". In: *Journal of Scientific Computing* 31.1-2 (2007), pp. 61–73. DOI: `10.1007/s10915-006-9107-7`.

[5] Christoph Lehrenfeld and Joachim Schöberl. "High order exactly divergence-free Hybrid Discontinuous Galerkin Methods for unsteady incompressible flows". In: *Computer Methods in Applied Mechanics and Engineering* 307 (2016), pp. 339–361. DOI: `10.1016/j.cma.2016.04.025`.

[6] Douglas N. Arnold, Daniele Boffi, and Richard S. Falk. "Quadrilateral H (div) Finite Elements". In: *SIAM Journal on Numerical Analysis* 42.6 (2005), pp. 2429–2451. DOI: `10.1137/S0036142903431924`.

Static condensation

The block structure of the element tensor is of the form

$$\begin{bmatrix} A_{uu} & B_{pu}^T & A_{\bar{u}u}^T & B_{\bar{p}u}^T \\ B_{pu} & 0 & 0 & 0 \\ A_{\bar{u}u} & 0 & A_{\bar{u}\bar{u}} & 0 \\ B_{\bar{p}u} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} U \\ P \\ \bar{U} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} F_u \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Eliminating the cell degrees of freedom gives the condensed element tensor

$$\begin{bmatrix} A_{\bar{u}\bar{u}} - BA^{-1}B^T & -BA^{-1}C^T \\ -CA^{-1}B^T & -CA^{-1}C^T \end{bmatrix} \begin{pmatrix} \bar{U} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} -BA^{-1}F \\ -CA^{-1}F \end{pmatrix},$$

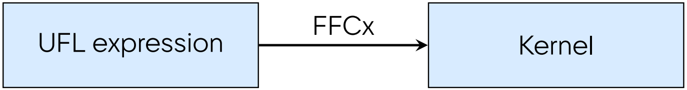
where

$$A = \begin{bmatrix} A_{uu} & B_{pu}^T \\ B_{pu} & 0 \end{bmatrix}, \; B = \begin{bmatrix} A_{\bar{u}u} & 0 \end{bmatrix}, \; C = \begin{bmatrix} B_{\bar{p}u} & 0 \end{bmatrix}, \text{ and } F = \begin{pmatrix} F_u \\ 0 \end{pmatrix}.$$

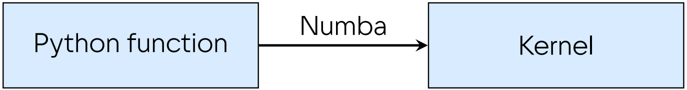
Implementation

Features of FEniCSx:

- Create kernels generated from UFL that are callable from python



- Create user defined kernels written in Python



- User defined kernels can call generated kernels

Implementation

Create kernels for each block of the element tensor (A_{uu}, ..., $A_{\bar{u}\bar{u}}$):

```
# UFL expressions for each block of the element tensor
A_uu_form = nu * inner(grad(u), grad(v)) * dx + nu * gamma * inner(u, v) * ds \
            - nu * (inner(u, dot(grad(v), n)) + inner(v, dot(grad(u), n))) * ds
...
A_ubar_ubar_form = nu * gamma * inner(ubar, vbar) * ds

# Compile forms with FFCx and expose to Python
forms = [A_uu_form, ..., A_ubar_ubar_form]
compiled_forms = ffcx.codegeneration.jit.compile_forms(forms)
A_uu_cell_kernel = compiled_forms[0][0].create_cell_integral().tabulate_tensor
A_uu_facet_kernel = \
    compiled_forms[0][0].create_exterior_facet_integral().tabulate_tensor
...

```

Implementation

Define a custom kernel to compute the top left block of the condensed element tensor ($K_{00} := A_{\bar{u}\bar{u}} - BA^{-1}B^T$):

```
@numba.cfunc(c_signature)
def tabulate_K00(K00_, w_, c_, coords_, entity_local_index, ...):
    K00 = numba.carray(K00_, (ubar_size, ubar_size))
    A_uu = np.zeros((u_size, u_size))
    ...
    # Compute cell integrals
    A_uu_cell_kernel(ffi.from_buffer(A_uu), w_, c_, coords_, entity_local_index, ...)
    ...
    for j in range(n_facets):
        # Compute facet integrals
        A_uu_facet_kernel(ffi.from_buffer(A_uu), w_, c_, coords_, fj, ...)
        ...
    # Static condensation
    K00 += A_ubar_ubar - B @ np.linalg.solve(A, B.T)

```

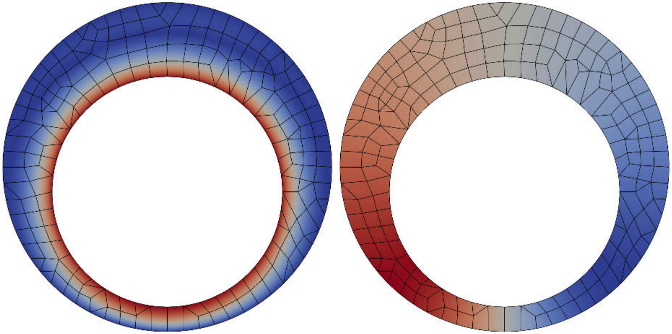
This kernel is passed to DOLFINx to assemble over the mesh.

Implementation: further work

- The above FEniCSx implementation has been tested on simplices.
- Until recently, FEniCSx did not have support for quadrilateral/hexahedral $H(\mathrm{div})$-conforming finite elements.
- Basix supports these elements, but some work is required to implement facet function spaces in a more general manner.
- To demonstrate the HDG scheme on meshes containing quadrilaterals, the method was also implemented in NGSolve.[7]

[7] Joachim Schöberl. "C++ 11 implementation of finite elements in NGSolve". In: *Technical Report ASC-2014-30, Institute for Analysis and Scientific Computing* (2014). URL: `https://www.asc.tuwien.ac.at/~schoeberl/wiki/publications/ngs-cpp11.pdf`.

Results: curved cells



(a) Velocity magnitude (b) Pressure

Figure: Computed solution

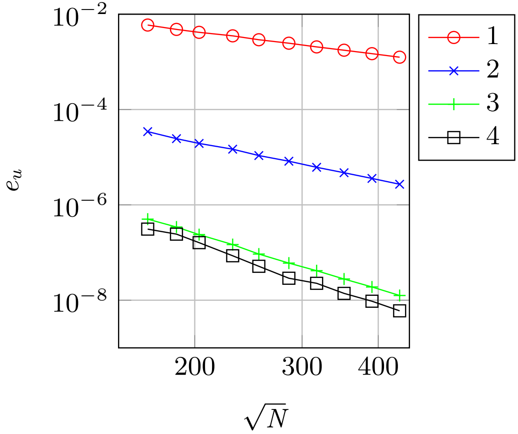


Figure: e_u against $\sqrt{N}$ for $k = 3$ with piecewise polynomial geometric mappings of degrees 1, 2, 3, and 4.

	N	e_u	$e_{\nabla\cdot u}$	$e_{[\![u]\!]}$
Present method	3870	6.17×10^{-4}	$\mathbf{5.45 \times 10^{-15}}$	4.68×10^{-14}
Original method	3870	6.71×10^{-4}	$\mathbf{3.02 \times 10^{-2}}$	8.51×10^{-13}

Extension to the Navier-Stokes equations

- Straightforward extension to the Navier-Stokes equations
- Divergence free velocity field on affine and non-affine simplex and non-simplex cells
- Local momentum conservation
- Arbitrarily high order

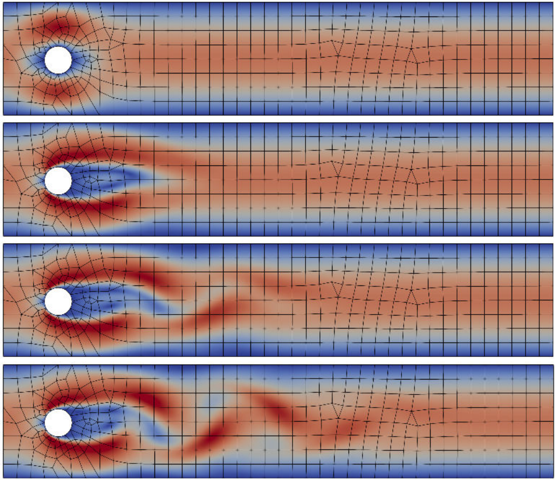


Figure: Velocity magnitude

Open questions

We are currently working on:

- Implementing a FEniCSx version of the method for meshes with quadrilateral and hexahedral cells.
- Rigorous proofs of the discrete inf-sup condition and error estimates on non-affine meshes.
- Optimal preconditioners and investigating the performance of the method at large scale.

Any suggestions/advice about these topics would be very much appreciated!

Thank you. Any questions?