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ORDINARY DIFFERENTIAL EQUATIONS

SCHOOL OF MATHEMATICS
UNIVERSITY OF BRISTOL

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Preface

This book consists of ten weeks of material given as a course on ordinary differential equations (ODEs) for second year mathematics majors at the University of Bristol. It is the first course devoted solely to differential equations that these students will take. An obvious question is "why does there need to be another textbook on ODEs"? From one point of view the answer is certainly that it is not needed. The classic textbooks of Coddington and Levinson, Hale, and Hartman[1] provide a thorough exposition of the topic and are essential references for mathematicians, scientists and engineers who encounter and must understand ODEs in the course of their research. However, these books are not ideal for use as a textbook for a student's first exposure to ODEs beyond the basic calculus course (more on that shortly). Their depth and mathematical thoroughness often leave students that are relatively new to the topic feeling overwhelmed and grasping for the essential ideas within the topics that are covered. Of course, (probably) no one would consider using these texts for a second year course in ODEs. That's not really an issue, and there is a large market for ODE texts for second year mathematics students (and new texts continue to appear each year). I spent some time examining some of these texts (many which sell for well over a hundred dollars) and concluded that none of them really would ''work" for the course that I wanted to deliver. So, I decided to write my own notes, which have turned into this small book. I have taught this course for three years now. There are typically about 160 students in the class, in their second year, and I have been somewhat surprised, and pleased, by how the course has been received by the students. So now I will explain a bit about my rationale, requirements and goals for the course.

[1] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Krieger, 1984; J. K. Hale. *Ordinary Differential Equations*. Dover, 2009; and P. Hartman. *Ordinary Differential Equations*. Society for industrial and Applied Mathematics, 2002

In the UK students come to University to study mathematics with a good background in calculus and linear algebra. Many have ''seen" some basic ODEs already. In their first year students have a year long course in calculus where they encounter the typical first order ODEs, second order linear constant coefficient ODEs, and two dimensional first order linear matrix ODEs. This material tends to form a substantial part of the traditional second year course in ODEs and since

I can consider the material as ''already seen, at least, once'', it allows me to develop the course in a way that makes contact with more contemporary concepts in ODEs and to touch on a variety of research issues. This is very good for our program since many students will do substantial projects that approach research level and require varying amounts of knowledge of ODEs.

This book consists of 10 chapters, and the course is 12 weeks long. Each chapter is covered in a week, and in the remaining two weeks I summarize the entire course, answer lots of questions, and prepare the students for the exam. I do not cover the material in the appendices in the lectures. Some of it is basic material that the students have already seen that I include for completeness and other topics are ''tasters'' for more advanced material that students will encounter in later courses or in their project work. Students are very curious about the notion of ''chaos'', and I have included some material in an appendix on that concept. The focus in that appendix is only to connect it with ideas that have been developed in this course related to ODEs and to prepare them for more advanced courses in dynamical systems and ergodic theory that are available in their third and fourth years.

There is a significant transition from first to second year mathematics at Bristol. For example, the first year course in calculus teaches a large number of techniques for performing analytical computations, e.g. the usual set of tools for computing derivatives and integrals of functions of one, and more variables. Armed with a large set of computational skills, the second year makes the transition to ''thinking about mathematics'' and ''creating mathematics''. The course in ODEs is ideal for making this transition. It is a course in ordinary differential ''equations'', and equations are what mathematicians learn how to solve. It follows then that students take the course with the expectation of learning how to solve ODEs. Therefore it is a bit disconcerting when I tell them that it is likely that almost all of the ODEs that they encounter throughout their career as a mathematician will *not* have analytical solutions. Moreover, even if they do have analytical solutions the complexity of the analytical solutions, even for ''simple'' ODEs, is not likely to yield much insight into the nature of the behavior of the solutions of ODEs. This last statement provides the entry into the nature of the course, which is based on the ''vision of Poincaré''–rather than seeking to find specific solutions of ODEs, we seek to understand how *all* possible solutions are related in their behavior in the geometrical setting of phase space. In other words, this course has been designed to be a beginning course in ODEs from the dynamical systems point of view.

I am grateful to all of the students who have taken this course over the past three years. Teaching the course was a very rewarding expe-

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rience for me and I very much enjoyed discussing this material with them in weekly office hours.

This book was typeset with the Tufte latex package. I am grateful to Edward R. Tufte for realizing his influential design ideas in this Latex book package.

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1
Getting Started: The Language of ODEs

This is a course about ordinary differential equations (ODEs). So we begin by defining what we mean by this term[1].

Definition 1 (Ordinary differential equation)**.** *An ordinary differential equation (ODE) is an equation for a function of one variable that involves (''ordinary") derivatives of the function (and, possibly, known functions of the same variable).*

We give several examples below.

1. $\frac{d^2x}{dt^2} + \omega^2 x = 0,$
2. $\frac{d^2x}{dt^2} - \alpha x \frac{dx}{dt} - x + x^3 = \sin \omega t,$
3. $\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0,$
4. $\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta\left(1 - \left(\frac{d^2f}{d\eta^2}\right)^2\right) = 0,$
5. $\frac{d^4y}{dx^4} + x^2\frac{d^2y}{dx^2} + x^5 = 0.$

ODEs can be succinctly written by adopting a more compact notation for the derivatives. We rewrite the examples above with this shorthand notation.

1′. $\ddot{x} + \omega^2 x = 0,$
2′. $\ddot{x} - \alpha x\dot{x} - x + x^3 = \sin \omega t,$
3′. $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0,$
4′. $f''' + ff'' + \beta\left(1 - (f'')^2\right) = 0,.$
5′. $y'''' + x^2y'' + x^5 = 0$

[1] The material in these lectures can be found in most textbooks on ODEs. It consists mainly of terminology and definitions. Generally such "defintion and terminology" introductions can be tedious and a bit boring. We probably have not entirely avoided that trap, but the material does introduce much of the essential language and some concepts that permeate the rest of the course. If you skip it, you will need to come back to it.

Now that we have defined the notion of an ODE, we will need to develop some additional concepts in order to more deeply describe the structure of ODEs. The notions of ''structure" are important since we will see that they play a key role in how we understand the nature of the behavior of solutions of ODEs.

Definition 2 (Dependent variable)**.** *The value of the function, e.g for example 1, $x(t)$.*

Definition 3 (Independent variable)**.** *The argument of the function, e.g for example 1, t.*

We summarize a list of the dependent and independent variables in the five examples of ODEs given above.

example	dependent variable	independent variable
1	x	t
2	x	t
3	x	t
4	f	η
5	y	x

Table 1.1: Identifying the independent and dependent variables for several examples.

The notion of ''order" is an important characteristic of ODEs.

Definition 4 (Order of an ODE)**.** *The number associated with the largest derivative of the dependent variable in the ODE.*

We give the order of each of the ODEs in the five examples above.

example	order
1	second order
2	second order
3	second order
4	third order
5	fourth order

Table 1.2: Identifying the order of the ODE for several examples.

Distinguishing between the independent and dependent variables enables us to define the notion of autonomous and nonautonomous ODEs.

Definition 5 (Autonomous, Nonautonomous)**.** *An ODE is said to be* autonomous *if none of the coefficients (i.e. functions) multiplying the dependent variable, or any of its derivatives, depend explicitly on the independent variable, and also if no terms not depending on the dependent variable or any of it derivatives depend explicitly on the independent variable. Otherwise, it is said to be* nonautonomous.

Or, more succinctly, an ODE is autonomous if the independent variable does not explicitly appear in the equation. Otherwise, it is nonautonomous.

We apply this definition to the five examples above, and summarize the results in the table below.

example	
1	autonomous
2	nonautonomous
3	autonomous
4	autonomous
5	nonautonomous

Table 1.3: Identifying autonomous and nonautonomous ODEs for several examples.

All scalar ODEs, i.e. the value of the dependent variable is a scalar, can be written as first order equations where the new dependent variable is a vector having the same dimension as the order of the ODE. This is done by constructing a vector whose components consist of the dependent variable and all of its derivatives *below* the highest order. This vector is the new dependent variable. We illustrate this for the five examples above.

1.

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\omega^2 x, \qquad (x, v) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

2.

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= \alpha x v + x - x^3 + \sin \omega t, \qquad (x, v) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

3.

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= \mu(1 - x^2)v - x, \qquad (x, v) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

4.

$$\begin{aligned} f' &= v, \\ f'' &= u, \\ f''' &= -ff'' - \beta(1 - (f'')^2) \end{aligned}$$

or

$$\begin{aligned} f' &= v, \\ v' &= f'' = u, \\ u' &= f''' = -fu - \beta(1-u^2) \end{aligned}$$

or

$$\begin{pmatrix} f' \\ v' \\ u' \end{pmatrix} = \begin{pmatrix} v \\ u \\ -fu - \beta(1-u^2) \end{pmatrix}, \qquad (f, v, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

5.

$$\begin{aligned} y' &= w, \\ y'' &= v, \\ y''' &= u, \\ y'''' &= -x^2 y'' - x^5 \end{aligned}$$

or

$$\begin{aligned} y' &= w, \\ w' &= y'' = v, \\ v' &= y''' = u, \\ u' &= y'''' = -x^2 v - x^5 \end{aligned}$$

or

$$\begin{pmatrix} y' \\ w' \\ v' \\ u' \end{pmatrix} = \begin{pmatrix} w \\ v \\ u \\ -x^2 v - x^5 \end{pmatrix}, \qquad (y, w, v, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Therefore without loss of generality, the general form of the ODE that we will study can be expressed as a *first order vector ODE*:

$$\dot{x} = f(x), \quad x(t_0) \equiv x_0, \qquad x \in \mathbb{R}^n, \quad \text{autonomous,} \tag{1.1}$$

$$\dot{x} = f(x,t), \quad x(t_0) \equiv x_0 \qquad x \in \mathbb{R}^n, \quad \text{nonautonomous,} \tag{1.2}$$

where $x(t_0) \equiv x_0$ is referred to as the *initial condition*.

This first order vector form of ODEs allows us to discuss many properties of ODEs in a way that is independent of the order of the ODE. It also lends itself to a natural geometrical description of the solutions of ODEs that we will see shortly.

A key characteristic of ODEs is whether or not they are linear or nonlinear.

Definition 6 (Linear and Nonlinear ODEs)**.** *An ODE is said to be* linear *if it is a linear function of the dependent variable. If it is not linear, it is said to be* nonlinear*.*

Note that the independent variable *does not* play a role in whether or not the ODE is linear or nonlinear.

example	
1	linear
2	nonlinear
3	nonlinear
4	nonlinear
5	linear

Table 1.4: Identifying linear and nonlinear ODEs for several examples.

When written as a first order vector equation the (vector) space of dependent variables is referred to as the phase space of the ODE. The ODE then has the geometric interpretation as a vector field on phase space. The structure of phase space, e.g. its dimension and geometry, can have a significant influence on the nature of solutions of ODEs. We will encounter ODEs defined on different types of phase space, and of different dimensions. Some examples are given in the following lists.

1-dimension

1. $\mathbb{R}$ –the real line,
2. $I \subset \mathbb{R}$ –an interval on the real line,
3. S^1–the circle.

''Solving" One dimensional Autonomous ODEs. Formally (we will explain what that means shortly) an expression for the solution of a one dimensional autonomous ODE can be obtained by integration. We explain how this is done, and what it means. Let $\mathcal{P}$ denote one of the one dimensional phase spaces described above. We consider the autonomous vector field defined on $\mathcal{P}$ as follows:

$$\dot{x} = \frac{dx}{dt} = f(x), \quad x(t_0) = x_0, \qquad x \in \mathcal{P}. \tag{1.3}$$

This is an example of a one dimensional separable ODE which can be written as follows:

$$\int_{x(t_0)}^{x(t)} \frac{dx'}{f(x')} = \int_{t_0}^{t} dt' = t - t_0. \tag{1.4}$$

If we can compute the integral on the left hand side of (1.4), then it may be possible to solve for $x(t)$. However, we know that not all functions $\frac{1}{f(x)}$ can be integrated. This is what we mean by we can ''formally" solve for the solution for this example. We may not be able to represent the solution in a form that is useful.

The higher dimensional phase spaces that we will consider will be constructed as Cartesian products of these three basic one dimensional phase spaces.

2-dimensions

1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$–the plane,
2. $\mathbb{T}^2 = \mathbb{S} \times \mathbb{S}$–the two torus,
3. $C = I \times \mathbb{S}$– the (finite) cylinder,
4. $C = \mathbb{R} \times \mathbb{S}$– the (infinite) cylinder.

In many applications of ODEs the independent variable has the interpretation of time, which is why the variable t is often used to denote the independent variable. Dynamics is the study of how systems change in time. When written as a first order system ODEs are often referred to as dynamical systems, and ODEs are said to generate *vector fields on phase space*. For this reason the phrases *ODE* and *vector field* tend to be used synonomously. Moreover, this geometrical view leads to an alternate, but synonomous, terminology for solutions of ODEs. In particular, solutions of ODEs may be referred to as *trajectories* or *orbits*.

Several natural questions arise when analysing an ODE . ''Does the ODE have a solution?" ''Are solutions unique?" (And what does ''unique" mean?) The standard way of treating this in an ODE course is to "prove a big theorem" about existence and uniqueness. Rather, than do that (you can find the proof in hundreds of books, as well as in many sites on the internet), we will consider some examples that illustrate the main issues concerning what these questions mean, and afterwards we will describe sufficent conditions for an ODE to have a unique solution (and then consider what ''uniqueness" means).

First, do ODEs have solutions? Not necessarily, as the following example shows.

Example 1 (An example of an ODE that has no solutions.)**.** *Consider the following ODE defined on* $\mathbb{R}$*:*

$$\dot{x}^2 + x^2 + t^2 = -1, \qquad x \in \mathbb{R}.$$

This ODE has no solutions since the left hand side is nonnegative and the right hand side is strictly negative.

Then you can ask the question–"if the ODE has solutions, are they unique?" Again, the answer is ''not necessarily", as the following example shows.

Example 2 (An example illustrating the meaning of uniqueness)**.**

$$\dot{x} = ax, \qquad x \in \mathbb{R}, \tag{1.5}$$

where a is an arbitrary constant. The solution is given by

$$x(t) = ce^{at}. \tag{1.6}$$

So we see that there are an infinite number of solutions, depending upon the choice of the constant c. So what could uniqueness of solutions mean? If we evaluate the solution (1.6) *at* $t = 0$ *we see that*

$$x(0) = c. \tag{1.7}$$

Substituting this into the solution (1.6)*, the solution has the form:*

$$x(t) = x(0)e^{at}. \tag{1.8}$$

From the form of (1.8) *we can see exactly what ''uniquess of solutions" means. For a given initial condition, there is* exactly one *solution of the ODE satisfying that initial condition.*

Example 3. *An example of an ODE with non-unique solutions.*

Consider the following ODE defined on $\mathbb{R}$*:*

$$\dot{x} = 3x^{\frac{2}{3}}, \quad x(0) = 0, \, x \in \mathbb{R}. \tag{1.9}$$

It is easy to see that a solution satisfying $x(0) = 0$ *is* $x = 0$*. However, one can verify directly by substituting into the equation that the following is also a solution satisfying* $x(0) = 0$*:*

$$x(t) = \begin{cases} 0, & t \leq a \\ (t-a)^3, & t > a \end{cases} \tag{1.10}$$

for any $a > 0$*. Hence, in this example, there are an infinite number of solutions satisfying* the same initial condition*. This example illustrates precisely what we mean by uniqueness. Given an initial condition, only one (''uniqueness") solution satisfies the initial condition at the chosen initial time.*

There is another question that comes up. If we have a unique solution does it exist *for all time*? Not necessarily, as the following example shows.

Example 4. *An example of an ODE with unique solutions that exists only for a finite time.*

Consider the following ODE on $\mathbb{R}$*:*

$$\dot{x} = x^2, \quad x(0) = x_0,\, x \in \mathbb{R}. \tag{1.11}$$

We can easily integrate this equation (it is separable) to obtain the following solution satisfying the initial condition:

$$x(t) = \frac{x_0}{1 - x_0 t} \tag{1.12}$$

The solution becomes infinite, or ''does not exist" or ''blows up" at $\frac{t}{x_0}$*. This is what ''does not exist" means. So the solution only exists for a finite time, and this ''time of existence" depends on the initial condition.*

These three examples contain the essence of the ''existence issues" for ODEs that will concern us. They are the ''standard examples" that can be found in many textbooks[2].[3]

Now we will state the standard ''existence and uniqueness" theorem for ODEs. The statement is an example of the power and flexibility of expressing a general ODE as a first order vector equation. The statement is valid for any (finite) dimension.

We consider the general vector field on $\mathbb{R}^n$

$$\dot{x} = f(x,t),\, x(t_0) = x_0, \quad x \in \mathbb{R}^n. \tag{1.13}$$

It is important to be aware that for the general result we are going to state it does not matter whether or not the ODE is autonomous or nonautonomous.

We define the domain of the vector field. Let $U \subset \mathbb{R}^n$ be an open set and let $I \subset \mathbb{R}$ be an interval. Then we express that the n-dimensional vector field is defined on this domain as follows:

$$\begin{aligned} f\,:\, U \times I &\to \mathbb{R}^n, \\ (x,t) &\to f(x,t) \end{aligned} \tag{1.14}$$

We need a definition to describe the ''regularity" of the vector field.

Definition 7 (C^r function)**.** *We say that* $f(x,t)$ *is* C^r *on* $U \times I \subset \mathbb{R}^n \times \mathbb{R}$ *if it is* r *times differentiable and each derivative is a continuous function (on the same domain). If* $r = 0$*,* $f(x,t)$ *is just said to be continuous.*

[2] J. K. Hale. *Ordinary Differential Equations*. Dover, 2009; P. Hartman. *Ordinary Differential Equations*. Society for industrial and Applied Mathematics, 2002; and E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Krieger, 1984

[3] The ''Existence and Uniqueness Theorem" is, traditionally, a standard part of ODE courses beyond the elementary level. This theorem, whose proof can be found in numerous texts (including those mentioned in the Preface), will not be given in this course. There are several reasons for this. One is that it requires considerable time to construct a detailed and careful proof, and I do not feel that this is the best use of time in a 12 week course. The other reason (not unrelated to the first) is that I do not feel that the understanding of the detailed ''Existence and Uniqueness" proof is particularly important at this stage of the students education. The subject has grown so much in the last forty years, especially with the merging of the ''dynamical systems point of view" with the subject of ODEs, that there is just not enough time to devote to all the topics that students ''should know". However, it is important to know what it means for an ODE to have a solution, what uniqueness of solutions means, and general conditions for when an ODE has unique solutions.

Now we can state *sufficient conditions* for (1.13) to have a unique solution. We suppose that $f(x,t)$ is C^r, $r \geq 1$. We choose any point $(x_0, t_0) \in U \times I$. Then there exists a unique solution of (1.13) satisfying this initial condition. We denote this solution by $x(t, t_0, x_0)$, and reflect in the notation that it satisfies the initial condition by $x(t_0, t_0, x_0) = x_0$. This unique solution exists for a time interval centered at the initial time t_0, denoted by $(t_0 - \epsilon, t_0 + \epsilon)$, for some $\epsilon > 0$. Moreover, this solution, $x(t, t_0, x_0)$, is a C^r function of t, t_0, x_0. Note that from Example 4 ϵ may depend on x_0. This also explains how a solution ''fails to exist"–it becomes unbounded (''blow up") in a finite time.

Finally, we remark that existence and uniqueness of ODEs is the mathematical manifestation of determinism. If the initial condition is specified (with 100% accuracy), then the past and the future is uniquely determined. The key phrase here is ''100% accuracy". Numbers cannot be specified with 100% accuracy. There will always be some imprecision in the specification of the initial condition. Chaotic dynamical systems are deterministic dynamical systems having the property that imprecisions in the initial conditions may be magnified by the dynamical evolution, leading to seemingly random behavior (even though the system is completely deterministic).

Problem Set 1

1. For each of the ODEs below, write it as a first order system, state the dependent and independent variables, state any parameters in the ODE (i.e. unspecified constants) and state whether it is linear or nonlinear, and autonomous or nonautonomous,

(a)
$$\ddot{\theta} + \delta\dot{\theta} + \sin\theta = F\cos\omega t, \quad \theta \in S^1.$$

(b)
$$\ddot{\theta} + \delta\dot{\theta} + \theta = F\cos\omega t, \quad \theta \in S^1.$$

(c)
$$\frac{d^3y}{dx^3} + x^2 y\frac{dy}{dx} + y = 0, \quad x \in \mathbb{R}^1.$$

(d)
$$\begin{aligned} \ddot{x} + \delta\dot{x} + x - x^3 &= \theta, \\ \ddot{\theta} + \sin\theta &= 0, \qquad (x,\theta) \in \mathbb{R}^1 \times S^1. \end{aligned}$$

(e)
$$\begin{aligned} \ddot{\theta} + \delta\dot{\theta} + \sin\theta &= x, \\ \ddot{x} - x + x^3 &= 0, \qquad (\theta, x) \in S^1 \times \mathbb{R}^1. \end{aligned}$$

2. Consider the vector field:

$$\dot{x} = 3x^{\frac{2}{3}}, \quad x(0) \neq 0, \quad x \in \mathbb{R}.$$

Does this vector field have unique solutions? [4]

3. Consider the vector field:

$$\dot{x} = -x + x^2, \quad x(0) = x_0, \quad x \in \mathbb{R}.$$

Determine the time interval of existence of all solutions as a function of the initial condition, x_0. [5]

4. Consider the vector field:

$$\dot{x} = a(t)x + b(t), \quad x \in \mathbb{R}.$$

Determine sufficient conditions on the coefficients $a(t)$ and $b(t)$ for which the solutions will exist for all time. Do the results depend on the initial condition? [6]

[4] Note the similarity of this exercise to Example 3. The point of this exercise is to think about how issues of existence and uniqueness depend on the initial condition.

[5] Here is the point of this exercise: $\dot{x} = -x$ has solutions that exist for all time (for any initial condition), $\dot{x} = x^2$ has solutions that ''blow up in finite time''. What happens when you ''put these two together''? In order to answer this you will need to solve for the solution, $x(t, x_0)$.

[6] The ''best'' existence and uniqueness theorems are when you can analytically solve for the ''exact'' solution of an ODE for arbitrary initial conditions. This is the type of ODE where that can be done. It is a first order, linear inhomogeneous ODE that can be solved using an ''integrating factor''. However, you will need to argue that the integrals obtained from this procedure ''make sense''.

2
Special Structure and Solutions of ODEs

A consistent theme throughout all of ODEs is that ''special structure" of the *equations* can reveal insight into the nature of the *solutions*. Here we look at a very basic and important property of autonomous equations:

> *''Time shifts of solutions of autonomous ODEs are also solutions of the ODE (but with a different initial condition)".*

Now we will show how to see this.

Throughout this course we will assume that existence and uniqueness of solutions holds on a domain and time interval sufficient for our arguments and calculations.

We start by establishing the setting. We consider an autonomous vector field defined on $\mathbb{R}^n$:

$$\dot{x} = f(x),\ x(0) = x_0, \quad x \in \mathbb{R}^n, \tag{2.1}$$

with solution denoted by:

$$x(t, 0, x_0), \quad x(0, 0, x_0) = x_0.$$

Here we are taking the initial time to be $t_0 = 0$. We will see, shortly, that for autonomous equations this can be done without loss of generality. Now we choose $s \in \mathbb{R}$ ($s \neq 0$, which is to be regarded as a fixed constant). We must show the following:

$$\dot{x}(t+s) = f(x(t+s)) \tag{2.2}$$

This is what we mean by the phrase time shifts of solutions are solutions. This relation follows immediately from the chain rule calculation:

$$\frac{d}{dt} = \frac{d}{d(t+s)}\frac{d(t+s)}{dt} = \frac{d}{d(t+s)}. \tag{2.3}$$

Finally, we need to determine the initial condition for the time shifted solution. For the original solution we have:

$$x(t, 0, x_0), \quad x(0, 0, x_0) = x_0, \tag{2.4}$$

and for the time shifted solution we have:

$$x(t + s, 0, x_0), \quad x(s, 0, x_0). \tag{2.5}$$

It is for this reason that, without loss of generality, for autonomous vector fields we can take the initial time to be $t_0 = 0$. This allows us to simplify the arguments in the notation for solutions of autonomous vector fields, i.e., $x(t, 0, x_0) \equiv x(t, x_0)$ with $x(0, 0, x_0) = x(0, x_0) = x_0$

Example 5 (An example illustrating the time-shift property of autonomous vector fields.)**.** *Consider the following one dimensional autonomous vector field:*

$$\dot{x} = \lambda x,\ x(0) = x_0, \quad x \in \mathbb{R},\ \lambda \in \mathbb{R}. \tag{2.6}$$

The solution is given by:

$$x(t, 0, x_0) = x(t, x_0) = e^{\lambda t} x_0. \tag{2.7}$$

The time shifted solution is given by:

$$x(t + s, x_0) = e^{\lambda(t+s)} x_0. \tag{2.8}$$

We see that it is a solution of the ODE with the following calculations:

$$\frac{d}{dt} x(t + s, x_0) = \lambda e^{\lambda(t+s)} x_0 = \lambda x(t + s, x_0), \tag{2.9}$$

with initial condition:

$$x(s, x_0) = e^{\lambda s} x_0. \tag{2.10}$$

In summary, we see that the solutions of autonomous vector fields satisfy the following three properties:

1. $x(0, x_0) = x_0$
2. $x(t, x_0)$ is C^r in x_0
3. $x(t + s, x_0) = x(t, x(s, x_0))$

Property one just reflects the notation we have adopted. Property 2 is a statement of the properties arising from existence and uniqueness of solutions. Property 3 uses two characteristics of solutions. One is the ''time shift" property for autonomous vector fields that we have proven. The other is ''uniquess of solutions" since the left hand side

and the right hand side of Property 3 satisfy the same initial condition at $t = 0$.

These three properties are the defining properties of a flow, i.e., a one-parameter group of transformations of the phase space. In other words, we view the solutions as defining a *map* of points in phase space. The group property arises from property 3, i.e., the time-shift property. In order to emphasise this ''map of phase space" property we introduce a general notation for the flow as follows

$$x(t, x_0) \equiv \phi_t(\cdot),$$

where the ''$\cdot$" in the argument of $\phi_t(\cdot)$ reflects the fact that the flow is a function on the phase space. With this notation the three properties of a flow are written as follows:

1. $\phi_0(\cdot)$ is the identity map.
2. $\phi_t(\cdot)$ is C^r for each t
3. $\phi_{t+s}(\cdot) = \phi_t \circ \phi_s(\cdot)$

We often use the phrase ''the flow generated by the (autonomous) vector field". Autonomous is in parentheses as it is understood that when we are considering flows then we are considering the solutions of autonomous vector fields. This is because nonautonomous vector fields do not necessarily satisfy the time-shift property, as we now show with an example.

Example 6 (An example of a nonautonomous vector field not having the time-shift property.)**.** *Consider the following one dimensional vector field on* $\mathbb{R}$*:*

$$\dot{x} = \lambda t x, \quad x(0) = x_0, \quad x \in \mathbb{R}, \lambda \in \mathbb{R}.$$

This vector field is separable and the solution is easily found to be:

$$x(t, 0, x_0) = x_0 e^{\frac{\lambda}{2} t^2}.$$

The time shifted ''solution" is given by:

$$x(t + s, 0, x_0) = x_0 e^{\frac{\lambda}{2}(t+s)^2}.$$

We show that this does not satisfy the vector field with the following calculation:

$$\begin{aligned} \frac{d}{dt} x(t + s, 0, x_0) &= x_0 e^{\frac{\lambda}{2}(t+s)^2} \lambda (t + s). \\ &\neq \lambda t x(t + s, 0, x_0). \end{aligned}$$

Perhaps a more simple example illustrating that nonautonomous vector fields do not satisfy the time-shift property is the following.

Example 7. *Consider the following one dimensional nonautonomous vector field:*

$$\dot{x} = e^t, \quad x \in \mathbb{R}.$$

The solution is given by:

$$x(t) = e^t.$$

It is easy to verify that the time-shifted function:

$$x(t+s) = e^{t+s},$$

does not satisfy the equation.

In the study of ODEs certain types of solutions have achieved a level of prominence largely based on their significance in applications. They are

- equilibrium solutions,
- periodic solutions,
- heteroclinic solutions,
- homoclinic solutions.

We define each of these.

Definition 8 (Equilibrium)**.** *A* point *in phase space $x = \bar{x} = \mathbb{R}^n$ that is a solution of the ODE , i.e.*

$$f(\bar{x}) = 0, \quad f(\bar{x}, t) = 0,$$

is called an equilibrium point. These may also be referred to as fixed points.

For example, $x = 0$ is an equilibrium point for the following autonomous and nonautonomous one dimensional vector fields, respectively,

$$\begin{aligned} \dot{x} &= x, \quad x \in \mathbb{R}, \\ \dot{x} &= tx, \quad x \in \mathbb{R}. \end{aligned}$$

A periodic solution is simply a solution that is periodic in time. Its definition is the same for both autonomous and nonautonomous vector fields.

Definition 9 (Periodic solutions)**.** *A solution $x(t, t_0, x_0)$ is periodic if there exists a $T > 0$ such that*

$$x(t, t_0, x_0) = x(t + T, t_0, x_0)$$

Homoclinic and heteroclinic solutions are important in a variety of applications. Their definition is not so simple as the definitions of equilibrium and periodic solutions since they can be defined and generalized to many different settings. We will only consider these special solutions for autonomous vector fields, and solutions homoclinic or heteroclinic to equilibrium solutions.

Definition 10 (Homoclinic and Heteroclinic Solutions)**.** *Suppose $\bar{x}_1$ and $\bar{x}_2$ are equilibrium points of an autonomous vector field, i.e.*

$$f(\bar{x}_1) = 0, \quad f(\bar{x}_2) = 0.$$

A trajectory $x(t, t_0, x_0)$ is said to be heteroclinic to $\bar{x}_1$ and $\bar{x}_2$ if

$$\begin{aligned} \lim_{t\to\infty} x(t, t_0, x_0) &= \bar{x}_2, \\ \lim_{t\to-\infty} x(t, t_0, x_0) &= \bar{x}_1 \end{aligned} \tag{2.11}$$

If $\bar{x}_1 = \bar{x}_2$ the trajectory is said to be homoclinic to $\bar{x}_1 = \bar{x}_2$.

Example 8. [1]

Here we give an example illustrating equilibrium points and heteroclinic orbits. Consider the following one dimensional autonomous vector field on $\mathbb{R}$:

$$\dot{x} = x - x^3 = x(1 - x^2), \quad x \in \mathbb{R}. \tag{2.12}$$

This vector field has three equilibrium points at $x = 0, \pm 1$.

In Fig. 2.1 we show the graph of the vector field (2.12) in panel a) and the phase line dynamics in panel b).

The solid black dots in panel b) correspond to the equilibrium points and these, in turn, correspond to the zeros of the vector field shown in panel a). Between its zeros, the vector field has a fixed sign (i.e. positive or negative), corresponding to $\dot{x}$ being either increasing or decreasing. This is indicated by the direction of the arrows in panel b).

Our discussion about trajectories, as well as this example, brings us to a point where it is natural to introduce the important notion of an invariant set. While this is a general idea that applies to both autonomous and nonautonomous systems, in this course we will only discuss this notion in the context of autonomous systems. Accordingly, let $\phi_t(\cdot)$ denote the flow generated by an autonomous vector field.

[1] There is a question that we will return to throughout this course. What does it mean to ''solve'' an ODE? We would argue that a more ''practical'' question might be, ''what does it mean to understand the nature of all possible solutions of an ODE?''. But don't you need to be able to answer the first question before you can answer the second? We would argue that Fig. 2.1 gives a complete ''qualitative'' understanding of (2.12) in a manner that is much simpler than one could directly obtain from its solutions. In fact, it would be an instructive exercise to first solve (2.12) and from the solutions sketch Fig. 2.1. This may seem a bit confusing, but it is even more instructive to think about, and understand, what it means.

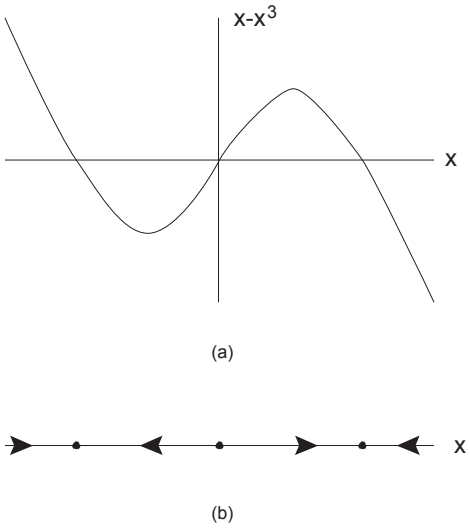


Figure 2.1: a) Graph of the vector field. b) The phase space.

Definition 11 (Invariant Set)**.** *A set $M \subset \mathbb{R}^n$ is said to be* invariant *if*

$$x \in M \Rightarrow \phi_t(x) \in M \quad \forall t.$$

In other words, a set is invariant (with respect to a flow) if you start in the set, and remain in the set, forever.

If you think about it, it should be clear that invariant sets are sets of trajectories. Any single trajectory is an invariant set. The entire phase space is an invariant set. The most interesting cases are those "in between". Also, it should be clear that the union of any two invariant sets is also an invariant set (just apply the definition of invariant set to the union of two, or more, invariant sets).

There are certain situations where we will be interested in sets that are invariant only for positive time–positive invariant sets.

Definition 12 (Positive Invariant Set)**.** *A set $M \subset \mathbb{R}^n$ is said to be* positive invariant *if*

$$x \in M \Rightarrow \phi_t(x) \in M \quad \forall t > 0.$$

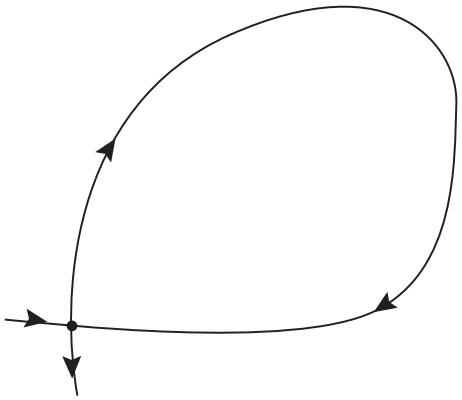
There is a similar notion of negative invariant sets, but the generalization of this from the definition of positive invariant sets should be obvious, so we will not write out the details.

Concerning example 8, the three equilibrium points are invariant sets, as well as the closed intervals $[-1, 0]$ and $[0, 1]$. Are there other invariant sets?

Problem Set 2

1. Consider an autonomous vector field on the plane having an equilibrium point with a homoclinic orbit connecting the equilibrium point, as illustrated in Fig. 1. We assume that existence and uniqueness of solutions holds. Can a trajectory starting at any point on the homoclinic orbit reach the equilibrium point in a finite time? (You must justify your answer.)[2]

[2] The main points to take into account for this problem are the fact that two trajectories cannot cross (in a finite time), and that an equilibrium point is a trajectory.



2. Can an autonomous vector field on $\mathbb{R}$ that has no equilibrium points have periodic orbits? We assume that existence and uniqueness of solutions holds.(You must justify your answer.)[3]

3. Can a nonautonomous vector field on $\mathbb{R}$ that has no equilibrium points have periodic orbits? We assume that existence and uniqueness of solutions holds.(You must justify your answer.)[4]

4. Can an autonomous vector field on the circle that has no equilibrium points have periodic orbits? We assume that existence and uniqueness of solutions holds. (You must justify your answer.)[5]

5. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= -\omega y, \\ \dot{y} &= \omega x, \qquad (x, y) \in \mathbb{R}^2, \end{aligned}$$

where $\omega > 0$.

- Show that the flow generated by this vector field is given by:[6]

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

[3] The main points to take into account in this problem is that the phase space is $\mathbb{R}$ and using this with the implication that trajectories of autonomous ODEs ''cannot cross".

[4] It is probably easiest to answer this problem by constructing a specific example.

[5] The main point to take into account here is that the phase space is ''periodic".

[6] Recall that the flow is obtained from the solution of the ODE for an arbitrary initial condition.

- Show that the flow obeys the time shift property.
- Give the initial condition for the time shifted flow.

6. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= \lambda y, \\ \dot{y} &= \lambda x, \qquad (x, y) \in \mathbb{R}^2, \end{aligned}$$

where $\lambda > 0$.

- Show that the flow generated by this vector field is given by:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh \lambda t & \sinh \lambda t \\ \sinh \lambda t & \cosh \lambda t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

- Show that the flow obeys the time shift property.
- Give the initial condition for the time shifted flow.

7. Show that the time shift property for autonomous vector fields implies that trajectories cannot ''cross each other", i.e. intersect, in phase space.

8. Show that the union of two invariant sets is an invariant set.

9. Show that the intersection of two invariant sets is an invariant set.

10. Show that the complement of a positive invariant set is a negative invariant set.

3
Behavior Near Trajectories and Invariant Sets: Stability

Consider the general nonautonomous vector field in n dimensions:

$$\dot{x} = f(x,t), \quad x \in \mathbb{R}^n, \tag{3.1}$$

and let $\bar{x}(t, t_0, x_0)$ be a solution of this vector field.

Many questions in ODEs concern understanding the behavior of neighboring solutions near a given, chosen solution. We will develop the general framework for considering such questions by transforming (3.1) to a form that allows us to explicitly consider these issues.

We consider the following (time dependent) transformation of variables:

$$x = y + \bar{x}(t, t_0, x_0). \tag{3.2}$$

We wish to express (3.1) in terms of the y variables. It is important to understand what this will mean in terms of (3.2). For y small it means that x is near the solution of interest, $\bar{x}(t, t_0, x_0)$. In other words, expressing the vector field in terms of y will provide us with an explicit form of the vector field for studying the behavior near $\bar{x}(t, t_0, x_0)$. Towards this end, we begin by transforming (3.1) using (3.2) as follows:

$$\dot{x} = \dot{y} + \dot{\bar{x}} = f(x,t) = f(y + \bar{x}, t), \tag{3.3}$$

or,

$$\begin{aligned} \dot{y} &= f(y + \bar{x}, t) - \dot{\bar{x}}, \\ &= f(y + \bar{x}, t) - f(\bar{x}, t) \equiv g(y,t), \qquad g(0,t) = 0. \end{aligned} \tag{3.4}$$

Hence, we have shown that solutions of (3.1) near $\bar{x}(t, t_0, x_0)$ are equivalent to solutions of (3.4) near $y = 0$.

The first question we want to ask related to the behavior near $\bar{x}(t, t_0, x_0)$ is whether or not this solution is *stable*? However, first we need to

mathematically define what is meant by this term ''stable". Now we should know that, without loss of generality, we can discuss this question in terms of the zero solution of (3.4).

We begin by defining the notion of ''Lyapunov stability" (or just ''stability").

Definition 13 (Lyapunov Stability)**.** *$y = 0$ is said to be Lyapunov stable at t_0 if given $\epsilon > 0$ there exists a $\delta = \delta(t_0, \epsilon)$ such that*

$$|y(t_0)| < \delta \Rightarrow |y(t)| < \epsilon, \quad \forall t > t_0 \tag{3.5}$$

If a solution is not Lyapunov stable, then it is said to be unstable.

Definition 14 (Unstable)**.** *If $y = 0$ is not Lyapunov stable, then it is said to be unstable.*

Then we have the notion of *asymptotic stability*.

Definition 15 (Asymptotic stability)**.** *$y = 0$ is said to be asymptotically stable at t_0 if:*

1. *it is Lyapunov stable at t_0,*
2. *there exists $\delta = \delta(t_0) > 0$ such that:*

$$|y(t_0)| < \delta \Rightarrow \lim_{t \to \infty} |y(t)| = 0 \tag{3.6}$$

We have several comments about these definitions.

- Roughly speaking, a Lyapunov stable solution means that if you start close to that solution, you stay close–forever. Asymptotic stability not only means that you start close and stay close forever, but that you actually get ''closer and closer" to the solution.
- Stability is an infinite time concept.
- If the ODE is autonomous, then the quantity $\delta = \delta(t_0, \epsilon)$ can be chosen to be independent of t_0.
- The *definitions* of stability do not tell us how to prove that a solution is stable (or unstable). We will learn two techniques for analyzing this question–linearization and Lyapunov's (second) method.
- Why is Lyapunov stability included in the definition of asymptotic stability? Because it is possible to construct examples where nearby solutions do get closer and closer to the given solution as $t \to \infty$, but in the process there are intermediate intervals of time where nearby solutions make ''large excursions" away from the given solution.

''Stability'' is a notion that applies to a ''neighborhood'' of a trajectory[1]. At this point we want to formalize various notions related to distance and neighborhoods in phase space. For simplicity in expressing these ideas we will take as our phase space $\mathbb{R}^n$. Points in this phase space are denoted $x \in \mathbb{R}^n$, $x \equiv (x_1, \ldots, x_n)$. The norm, or length, of x, denoted $|x|$ is defined as:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

[1] The notion that stability of a trajectory is a property of solutions in a *neighborhood* of a trajectory often causes confusion. To avoid confusion it is important to be clear about the notion of a ''neighborhood of a trajectory'', and then to realize that for solutions that are Lyapunov (or asymptotically) stable *all* solutions in the neighborhood have the same behavior as $t \to \infty$.

The distance between two points in $x, y \in \mathbb{R}^n$ is defined as:

$$\begin{aligned} d(x,y) &\equiv |x-y| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}, \\ &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \end{aligned} \tag{3.7}$$

Distance between points in $\mathbb{R}^n$ should be somewhat familiar, but now we introduce a new concept, the distance between a point and a set. Consider a set M, $M \subset \mathbb{R}^n$, let $p \in \mathbb{R}^n$. Then the distance from p to M is defined as follows:

$$\text{dist}(p, M) \equiv \inf_{x \in M} |p - x|. \tag{3.8}$$

We remark that it follows from the definition that if $p \in M$, then $\text{dist}(p, M) = 0$.

We have previously defined the notion of an invariant set. Roughly speaking, invariant sets are comprised of trajectories. We now have the background to discuss the notion of stability of invariant sets . Recall, that the notion of invariant set was only developed for autonomous vector fields. So we consider an autonomous vector field:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{3.9}$$

and denote the flow generated by this vector field by $\phi_t(\cdot)$. Let M be a closed invariant set (in many applications we may also require M to be bounded) and let $U \supset M$ denoted a neighborhood of M.

The definition of Lyapunov stability of an invariant set is as follows.

Definition 16 (Lyapunov Stability of M)**.** *M is said to be Lyapunov stable if for any neighborhood $U \supset M$, $x \in U \Rightarrow \phi_t(x) \in U$, $\forall t > 0$.*

SImilarly, we have the following definition of asymptotic stability of an invariant set.

Definition 17 (Asymptotic Stability of M)**.** *M is said to be asymptotically stable if*

1. *it is Lyapunov stable,*
2. *there exists a neighborhood $U \supset M$ such that $\forall x \in U$, $\text{dist}(\phi_t(x), M) \to 0$ as $t \to \infty$.*

In the dynamical systems approach to ordinary differential equations some alternative terminology is typically used.

Definition 18 (Attracting Set)**.** *If M is asymptotically stable it is said to be an attracting set.*

The significance of attracting sets is that they are the ''observable" regions in phase space since they are regions to which trajectories evolve in time. The set of points that evolve towards a specific attracting set is referred to as the basin of attraction for that invariant set.

Definition 19 (Basin of Attraction)**.** *Let $\mathcal{B} \subset \mathbb{R}^n$ denote the set of all points, $x \in B \subset \mathbb{R}^n$ such that*

$$\text{dist}(\phi_t(x), M) \to 0 \text{ as } t \to \infty$$

Then $\mathcal{B}$ is called the basin of attraction of M.

We now consider an example that allows us to explicitly explore these ideas[2].

Example 9. *Consider the following autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= y^2(1-y^2) \equiv f(y), \quad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{3.10}$$

First, it is useful to note that the x and y components of (3.10) are independent. Consequently, this may seem like a trivial example. However, we will see that such examples provide a great deal of insight, especially since they allow for simple computations of many of the mathematical ideas.

In Fig. 3.1 we illustrate the flow of the x and y components of (3.10) separately.
The two dimensional vector field (3.10) has equilibrium points at:

$$(x,y) = (0,0), \quad (0,1), \quad (0,-1).$$

In this example it is easy to identify three invariant horizontal lines *(examples of invariant sets). Since $y = 0$ implies that $\dot{y} = 0$, this implies that the x axis is invariant. Since $y = 1$ implies that $\dot{y} = 0$, this implies that the line $y = 1$ is invariant. Since $y = -1$ implies that $\dot{y} = 0$, which implies that the line $y = -1$ is invariant. This is illustrated in Fig. 3.2.*[3]*Below we provide some additional invariant sets for (3.10). It is instructive to understand why they are invariant, and whether or not there are other invariant sets.*

[2] Initially, this type of problem (two independent, one dimensional autonomous vector fields) might seem trivial and like a completely academic problem. However, we believe that there is quite a lot of insight that can be gained from such problems (that has been the case for the author). Generally, it is useful to think about breaking a problem up into smaller, understandable, pieces and then putting the pieces back together. Problems like this provide a controlled way of doing this. But also, these problems allow for exact computation by hand of concepts that do not lend themselves to such computations in the types of ODEs arising in typical applications. This gives some level of confidence that you understand the concept. Also, such examples could served as useful benchmarks for numerical computations, since checking numerical methods against equations where you have an analytical solution to the equation can be very helpful.

[3] Make sure you understand why these constraints on the coordinates imply the existence of invariant lines.

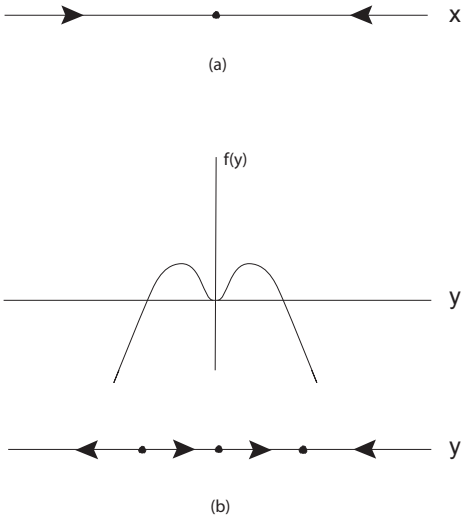


Figure 3.1: a) The phase '"line" of the x component of (3.10). b) The graph of $f(y)$ (top) and the phase "line" of the y component of (3.10) directly below.

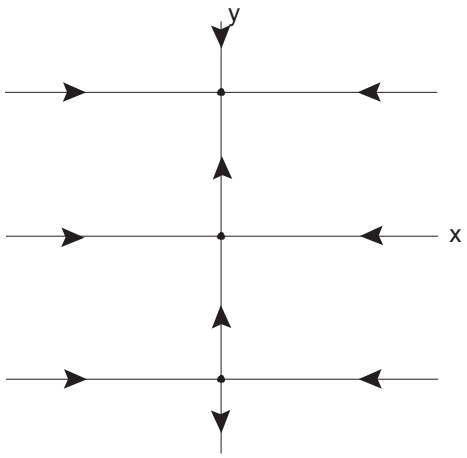


Figure 3.2: Phase plane of (3.10). The black dots indicate equilibrium points.

Additional invariant sets for (3.10).

$\{(x,y)|\ -\infty < x < 0,\ -\infty < y < -1\}$,
$\{(x,y)|\ 0 < x < \infty,\ -\infty < y < -1\}$,
$\{(x,y)|\ -\infty < x < 0,\ -1 < y < 0\}$,
$\{(x,y)|\ 0 < x < \infty,\ -1 < y < 0\}$,
$\{(x,y)|\ -\infty < x < 0,\ 0 < y < 1\}$,
$\{(x,y)|\ 0 < x < \infty,\ 0 < y < 1\}$,
$\{(x,y)|\ -\infty < x < 0,\ 1 < y < \infty\}$,
$\{(x,y)|\ 0 < x < \infty,\ 1 < y < \infty\}$,

Problem Set 3

1. Consider the following autonomous vector field on $\mathbb{R}$:

$$\dot{x} = x - x^3, \quad x \in \mathbb{R}. \tag{3.11}$$

   - Compute all equilibria and determine their stability, i.e., are they Lyapunov stable, asymptotically stable, or unstable?
   - Compute the flow generated by (3.11) and verify the stability results for the equilibria directly from the flow.

2. [4]Consider an autonomous vector field on $\mathbb{R}^n$:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{3.12}$$

   Suppose $M \subset \mathbb{R}^n$ is a bounded, invariant set for (3.12). Let $\phi_t(\cdot)$ denote the flow generated by (3.12).Suppose $p \in \mathbb{R}^n$, $p \notin M$. Is it possible for

$$\phi_t(p) \in M,$$

   for some finite t?

3. Consider the following vector field on the plane:

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y, \qquad , \qquad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{3.13}$$

   (a) Determine 0-dimensional, 1-dimensional, and 2-dimensional invariant sets.

   (b) Determine the attracting sets and their basins of attraction.

   (c) Describe the heteroclinic orbits and compute analytical expressions for the heteroclinic orbits.

   (d) [5]Does the vector field have periodic orbits?

   (e) Sketch the phase portrait.[6]

[4] This problem is ''essentially the same'' as Problem 1 from Problem Set 2.

[5] Keep in mind here that a trajectory is periodic if it is periodic, with the same period, *in each component*.

[6] There is a point to consider early on in this course. What exactly does ''sketch the phase portrait mean''? It means sketching trajectories through different initial conditions in the phase space in such a way that a ''complete picture'' of all *qualitatively distinct* trajectories is obtained. The phrase *qualitatively distinct* is the key here since you can only sketch trajectories for a finite number of initial conditions (unless you sketch an invariant set or manifold) and only a finite length of the trajectory, unless the trajectory is a fixed point, periodic, homoclinic, or heteroclinic.

4
Behavior Near Trajectories: Linearization

Now we are going to discuss a method for analyzing stability that utilizes linearization about the object whose stability is of interest. For now, the ''objects of interest" are specific solutions of a vector field.The structure of the solutions of linear, constant coefficient systems is covered in many ODE textbooks. My favorite is the book of Hirsch et al.[1]. It covers all of the linear algebra needed for analyzing linear ODEs that you probably did not cover in your linear algebra course. The book by Arnold[2] is also very good, but the presentation is more compact, with fewer examples.

[1] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012

[2] V. I. Arnold. *Ordinary differential equations*. M.I.T. press, Cambridge, 1973. ISBN 0262010372

We begin by considering a general nonautonomous vector field:

$$\dot{x} = f(x,t), \quad x \in \mathbb{R}^n, \tag{4.1}$$

and we suppose that

$$\bar{x}(t, t_0, x_0), \tag{4.2}$$

is the solution of (4.1) for which we wish to determine its stability properties. As when we introduced the definitions of stability, we proceed by localizing the vector field about the solution of interest. We do this by introducing the change of coordinates

$$x = y + \bar{x}$$

for (4.1) as follows:

$$\dot{x} = \dot{y} + \dot{\bar{x}} = f(y + \bar{x}, t),$$

or

$$\begin{aligned} \dot{y} &= f(y + \bar{x}, t) - \dot{\bar{x}}, \\ &= f(y + \bar{x}, t) - f(\bar{x}, t), \end{aligned} \tag{4.3}$$

where we omit the arguments of $\bar{x}(t, t_0, x_0)$ for the sake of a less cumbersome notation. Next we Taylor expand $f(y + \bar{x}, t)$ in y about the solution $\bar{x}$, but we will only require the leading order terms explicitly[3]:

[3] For the necessary background that you will need on Taylor expansions see Appendix A.

$$f(y + \bar{x}, t) = f(\bar{x}, t) + Df(\bar{x}, t)y + \mathcal{O}(|y|^2), \tag{4.4}$$

where Df denotes the derivative (i.e. Jacobian matrix) of the vector valued function f and $\mathcal{O}(|y|^2)$ denotes higher order terms in the Taylor expansion that we will not need in explicit form. Substituting this into (4.4) gives:

$$\begin{aligned} \dot{y} &= f(y + \bar{x}, t) - f(\bar{x}, t), \\ &= f(\bar{x}, t) + Df(\bar{x}, t)y + \mathcal{O}(|y|^2) - f(\bar{x}, t), \\ &= Df(\bar{x}, t)y + \mathcal{O}(|y|^2). \end{aligned} \tag{4.5}$$

Keep in mind that we are interested in the behaviour of solutions near $\bar{x}(t, t_0, x_0)$, i.e., for y small. Therefore, in that situation it seems reasonable that neglecting the $\mathcal{O}(|y|^2)$ in (4.5) would be an approximation that would provide us with the particular information that we seek. For example, would it provide sufficient information for us to determine stability? In particular,

$$\dot{y} = Df(\bar{x}, t)y, \tag{4.6}$$

is referred to as the linearization of the vector field $\dot{x} = f(x, t)$ about the solution $\bar{x}(t, t_0, x_0)$.

Before we answer the question as to whether or not (4.1) provides an adequate approximation to solutions of (4.5) for y ''small", we will first study linear vector fields on their own.

Linear vector fields can also be classified as nonautonomous or autonomous. Nonautonomous linear vector fields are obtained by linearizing a nonautonomous vector field about a solution (and retaining only the linear terms). They have the general form:

$$\dot{y} = A(t)y, \quad y(0) = y_0, \tag{4.7}$$

where

$$A(t) \equiv Df(\bar{x}(t, t_0, x_0), t) \tag{4.8}$$

is a $n \times n$ matrix. They can also be obtained by linearizing an autonomous vector field about a time-dependent solution.

An autonomous linear vector field is obtained by linearizing an autonomous vector field about an equilibrium point. More precisely, let $\dot{x} = f(x)$ denote an autonomous vector field and let $x = x_0$ denote an

equilibrium point, i.e. $f(x_0) = 0$. The linearized autonomous vector field about this equilibrium point has the form:

$$\dot{y} = Df(x_0)y, \quad y(0) = y_0, \tag{4.9}$$

or

$$\dot{y} = Ay, \quad y(0) = y_0, \tag{4.10}$$

where $A \equiv Df(x_0)$ is a $n \times n$ matrix of real numbers. This is significant because (4.10) can be solved using techniques of linear algebra, but (4.7), generally, cannot be solved in this manner. Hence, we will now describe the general solution of (4.10).

The general solution of (4.10) is given by:

$$y(t) = e^{At}y_0. \tag{4.11}$$

In order to verify that this is the solution, we merely need to substitute into the right hand side and the left hand side of (4.10) and show that equality holds. However, first we need to explain what e^{At} is, i.e. the exponential of the $n \times n$ matrix A (by examining (4.11) it should be clear that if (4.11) is to make sense mathematically, then e^{At} must be a $n \times n$ matrix).

Just like the exponential of a scalar, the exponential of a matrix is defined through the exponential series as follows:

$$\begin{aligned} e^{At} &\equiv \mathbb{I} + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots, \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}A^it^i, \end{aligned} \tag{4.12}$$

where $\mathbb{I}$ denotes the $n \times n$ identity matrix. But we still must answer the question, "does this exponential series involving products of matrices make mathematical sense"? Certainly we can compute products of matrices and multiply them by scalars. But we have to give meaning to an infinite sum of such mathematical objects. We do this by defining the norm of a matrix and then considering the convergence of the series in norm. When this is done the ''convergence problem" is exactly the same as that of the exponential of a scalar. Therefore the exponential series for a matrix converges absolutely for all t, and therefore it can be differentiated with respect to t term-by-term, and the resulted series of derivatives also converges absolutely.

Next we need to argue that (4.11) is a solution of (4.10). If we differentiate the series (4.12) term by term, we obtain that:

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A, \tag{4.13}$$

where we have used the fact that the matrices A and e^{At} commute (this is easy to deduce from the fact that A commutes with any power of A)[4]. It then follows from this calculation that:

$$\dot{y} = \frac{d}{dt}e^{At}y_0 = Ae^{At}y_0 = Ay. \tag{4.14}$$

Therefore the general problem of solving (4.10) is equivalent to computing e^{At}, and we now turn our attention to this task.

First, suppose that A is a diagonal matrix, say

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{4.15}$$

Then it is easy to see by substituting A into the exponential series (4.12) that:

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \tag{4.16}$$

Therefore our strategy will be to transform coordinates so that in the new coordinates A becomes diagonal (or as ''close as possible" to diagonal, which we will explain shortly). Then e^{At} will be easily computable in these coordinates. Once that is accomplished, then we use the inverse of the transformation to transform the solution back to the original coordinate system.

Now we make these ideas precise. We let

$$y = Tu, \qquad u \in \mathbb{R}^n,\ y \in \mathbb{R}^n, \tag{4.17}$$

where T is a $n \times n$ matrix whose precise properties will be developed in the following.

THIS IS A TYPICAL APPROACH IN ODES. WE PROPOSE A GENERAL COORDINATE TRANSFORMATION OF THE ODE, AND THEN WE CONSTRUCT IT IN A WAY THAT GIVES THE PROPERTIES OF THE ODE THAT WE DESIRE. Substituting (4.17) into (4.10) gives:

$$\dot{y} = T\dot{u} = Ay = ATu, \tag{4.18}$$

T will be constructed in a way that makes it invertible, so that we have:

$$\dot{u} = T^{-1}ATu, \quad u(0) = T^{-1}y(0). \tag{4.19}$$

[4] See Appendix B for another derivation of the solution of (4.10).

To simplify the notation we let:

$$\Lambda = T^{-1}AT, \tag{4.20}$$

or

$$A = T\Lambda T^{-1}. \tag{4.21}$$

Substituting (4.21) into the series for the matrix exponential (4.12) gives:

$$\begin{aligned} e^{At} &= e^{T\Lambda T^{-1}t}, \\ &= \mathbb{1} + T\Lambda T^{-1}t + \frac{1}{2!}\left(T\Lambda T^{-1}\right)^2 t^2 + \cdots + \frac{1}{n!}\left(T\Lambda T^{-1}\right)^n t^n + \cdots . \end{aligned} \tag{4.22}$$

Now note that for any positive integer n we have:

$$\begin{aligned} \left(T\Lambda T^{-1}\right)^n &= \underbrace{\left(T\Lambda T^{-1}\right)\left(T\Lambda T^{-1}\right)\cdots\left(T\Lambda T^{-1}\right)\left(T\Lambda T^{-1}\right)}_{\text{n factors}}, \\ &= T\lambda^n T^{-1}. \end{aligned} \tag{4.23}$$

Substituting this into (4.22) gives:

$$\begin{aligned} e^{At} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(T\Lambda T^{-1}\right)^n t^n, \\ &= T\left(\sum_{n=0}^{\infty} \frac{1}{n!}\Lambda^n t^n\right) T^{-1}, \\ &= Te^{\Lambda t}T^{-1}, \end{aligned} \tag{4.24}$$

or

$$e^{At} = Te^{\Lambda t}T^{-1}. \tag{4.25}$$

Now we arrive at our main result. If T is constructed so that

$$\Lambda = T^{-1}AT, \tag{4.26}$$

is diagonal, then it follows from (4.16) and (4.25) that e^{At} can always be computed. So the ODE problem of solving (4.10) becomes a problem in linear algebra. But can a general $n \times n$ matrix A always be diagonalized? If you have had a course in linear algebra, you know that the answer to this question is "no". There is a theory of the (real) that will apply here. However, that would take us into too great a diversion for this course. Instead, we will consider the three standard cases

for 2×2 matrices. That will suffice for introducing the the main ideas without getting bogged down in linear algebra. Nevertheless, it cannot be avoided entirely. You will need to be able to compute eigenvalues and eigenvectors of 2×2 matrices, and understand their meaning.

The three cases of 2×2 matrices that we will consider are characterized by their eigenvalues:

- two real eigenvalues, diagonalizable A,
- two identical eigenvalues, nondiagonalizable A,
- a complex conjugate pair of eigenvalues.

In the table below we summarize the form that these matrices can be transformed in to (referred to as the of A) and the resulting exponential of this canonical form.

eigenvalues of A	canonical form, Λ	e^{Λ}
λ, μ real, diagonalizable	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	$\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$
$\lambda = \mu$ real, nondiagonalizable	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\left(\mathbb{I} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix}$
complex conjugate pair, $\alpha \pm i\beta$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	$e^{\alpha}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

Once the transformation to Λ has been carried out, we will use these results to deduce e^{Λ}.

Problem Set 4

1. Suppose Λ is a $n \times n$ matrix and T is a $n \times n$ invertible matrix. Use mathematical induction to show that:

$$\left(T^{-1}\Lambda T\right)^k = T^{-1}\Lambda^k T,$$

for all natural numbers k, i.e., $k = 1, 2, 3, \ldots$.

2. Suppose A is a $n \times n$ matrix. Use the exponential series to give an argument that:

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

(You are allowed to use $e^{A(t+h)} = e^{At}e^{Ah}$ without proof, as well as the fact that A and e^{At} commute, without proof.)

3. Consider the following linear autonomous vector field:

$$\dot{x} = Ax, \quad x(0) = x_0, \qquad x \in \mathbb{R}^n,$$

where A is a $n \times n$ matrix of real numbers.

- Show that the solutions of this vector field exist for all time.
- Show that the solutions are infinitely differentiable with respect to the initial condition, x_0.

[5]

4. Consider the following linear autonomous vector field on the plane:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (x_1(0), x_2(0)) = (x_{10}, x_{20}). \tag{4.27}$$

(a) Describe the invariant sets.

(b) Sketch the phase portrait.

(c) Is the origin stable or unstable? Why?

5. Consider the following linear autonomous vector field on the plane:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (x_1(0), x_2(0)) = (x_{10}, x_{20}). \tag{4.28}$$

(a) Describe the invariant sets.

(b) Sketch the phase portrait.

(c) Is the origin stable or unstable? Why?

[5] The next two problems often give students difficulties. There are no hard calculations involved. Just a bit of thinking. The solutions for each ODE can be obtained easily. Once these are obtained you just need to think about what they mean in terms of the concepts involved in the questions that you are asked, e.g. Lyapunov stability means that if you ''start close, you stay close–forever''.

5
Behavior Near Equilibria: Linearization

Now we will consider several examples for solving, and understanding, the nature of the solutions, of

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2. \tag{5.1}$$

For all of the examples, the method for solving the system is the same.

Step 1. Compute the eigenvalues of A.

Step 2. Compute the eigenvectors of A.

Step 3. Use the eigenvectors of A to form the transformation matrix T.

Step 4. Compute $\Lambda = T^{-1}AT$.

Step 5. Compute $e^{At} = Te^{\Lambda t}T^{-1}$.

Once we have computed e^{At} we have the solution of (5.1) through any initial condiion y_0 since $y(t)$, $y(0) = y_0$, is given by $y(t) = e^{At}y_0$[1].

[1] Most of the linear algebra techniques necessary for this material are covered in Appendix A.

Example 10. *We consider the following linear, autonomous ODE:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.2}$$

where

$$A \equiv \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{5.3}$$

Step 1. Compute the eigenvalues of A.

The eigenvalues of A, denote by λ, are given by the solutions of the characteristic polynomial:

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} &= (2-\lambda)^2 - 1 = 0, \\ &= \lambda^2 - 4\lambda + 3 = 0, \end{aligned} \tag{5.4}$$

or

$$\lambda_{1,2} = 2 \pm \frac{1}{2}\sqrt{16-12} = 3,\, 1.$$

Step 2. Compute the eigenvectors of A.

For each eigenvalue, we compute the corresponding eigenvector. The eigenvector correponding to the eigenvalue 3 *is found by solving:*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.5}$$

or,

$$2x_1 + x_2 = 3x_1, \tag{5.6}$$
$$x_1 + 2x_2 = 3x_2. \tag{5.7}$$

Both of these equations yield the same equation since the two equations are dependent:

$$x_2 = x_1. \tag{5.8}$$

Therefore we take as the eigenvector corresponding to the eigenvalue 3*:*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.9}$$

Next we compute the eigenvector corresponding to the eigenvalue 1*. This is given by a solution to the following equations:*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.10}$$

or

$$2x_1 + x_2 = x_1, \tag{5.11}$$
$$x_1 + 2x_2 = x_2. \tag{5.12}$$

Both of these equations yield the same equation:

$$x_2 = -x_1. \tag{5.13}$$

Therefore we take as the eigenvector corresponding to the eigenvalue 1*:*

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.14}$$

Step 3. Use the eigenvectors of A to form the transformation matrix T.

For the columns of T we take the eigenvectors corresponding the the eigenvalues 1 *and* 3*:*

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{5.15}$$

with the inverse given by:

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5.16}$$

Step 4. Compute $\Lambda = T^{-1}AT$.

We have:

$$\begin{aligned} T^{-1}AT &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \equiv \Lambda. \end{aligned} \tag{5.17}$$

Therefore, in the $u_1 - u_2$ coordinates (5.2) *becomes:*

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{5.18}$$

In the $u_1 - u_2$ coordinates it is easy to see that the origin is an unstable equilibrium point.

Step 5. Compute $e^{At} = Te^{\Lambda t}T^{-1}$.

We have:

$$\begin{aligned} e^{At} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & -e^t \\ e^{3t} & e^{3t} \end{pmatrix}, \\ &= \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & -e^t + e^{3t} \\ -e^t + e^{3t} & e^t + e^{3t} \end{pmatrix}. \end{aligned} \tag{5.19}$$

We see that the origin is also unstable *in the original $x_1 - x_2$ coordinates. It is referred to as a source, and this is characterized by the fact that all of the eigenvalues of A have positive real part. The phase portrait is illustrated in Fig. 5.1.*

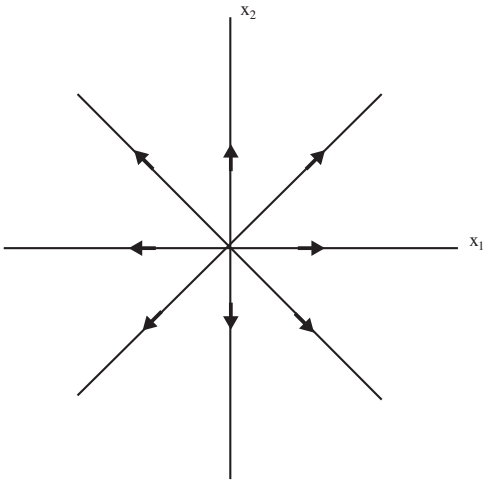


Figure 5.1: Phase plane of (5.19). The origin is unstable–a source.

We remark this it is possible to infer the behavior of e^{At} as $t \to \infty$ from the behavior of $e^{\Lambda t}$ as $t \to \infty$ since T does not depend on t.

Example 11. *We consider the following linear, autonomous ODE:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.20}$$

where

$$A \equiv \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix}. \tag{5.21}$$

Step 1. Compute the eigenvalues of A.

The eigenvalues of A, denote by λ, are given by the solutions of the characteristic polynomial:

$$\begin{aligned} \det \begin{pmatrix} -1-\lambda & -1 \\ 9 & -1-\lambda \end{pmatrix} &= (-1-\lambda)^2 + 9 = 0, \\ &= \lambda^2 + 2\lambda + 10 = 0. \end{aligned} \tag{5.22}$$

or,

$$\lambda_{1,2} = -1 \pm \frac{1}{2}\sqrt{4 - 40} = -1 \pm 3i.$$

The eigenvectors are complex, so we know it is not diagonalizable over the real numbers. What this means is that we cannot find real eigenvectors so that it can be transformed to a form where there are real numbers on the diagonal, and zeros in the off diagonal entries. The best we can do is to transform it to a form where the real parts of the eigenvalue are on the diagonal, and the imaginary parts are on the off diagonal locations, but the off diagonal elements differ by a minus sign.

Step 2. Compute the eigenvectors of A.

The eigenvector of A corresponding to the eigenvector $-1-3i$ is the solution of the following equations:

$$\begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1-3i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.23}$$

or,

$$\begin{aligned} -x_1 - x_2 &= -x_1 - 3ix_1, & (5.24)\\ 9x_1 - x_2 &= -x_2 - 3ix_2. & (5.25) \end{aligned}$$

A solution to these equations is given by:

$$\begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Step 3. Use the eigenvectors of A to form the transformation matrix T.

For the first column of T we take the real part of the eigenvector corresponding to the eigenvalue $-1-3i$, and for the second column we take the complex part of the eigenvector:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \tag{5.26}$$

with inverse

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}. \tag{5.27}$$

Step 4. Compute $\Lambda = T^{-1}AT$.

We have:

$$
\begin{aligned}
T^{-1}AT &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 9 & -3 \end{pmatrix}, \\
&= \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix} \equiv \Lambda. \qquad (5.28)
\end{aligned}
$$

With Λ in this form, we know from the previous chapter that:

$$
e^{\Lambda t} = e^{-t} \begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix}. \qquad (5.29)
$$

Then we have:

$$
e^{At} = Te^{\Lambda t}T^{-1}.
$$

From this expression we can conclude that $e^{At} \to 0$ as $t \to \infty$. Hence the origin is asymptotically stable. It is referred to as a sink and it is characterized by the real parts of the eigenvalues of A being negative. The phase plane is sketched in Fig. 5.2.

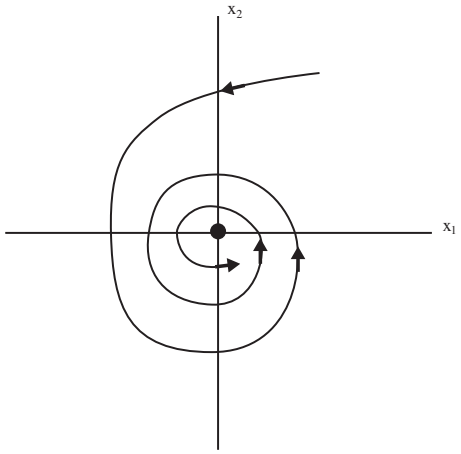


Figure 5.2: Phase plane of (5.20). The origin is a sink.

Example 12. *We consider the following linear autonomous ODE:*

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (5.30)
$$

where

$$
A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \qquad (5.31)
$$

Step 1. Compute the eigenvalues of A.

The eigenvalues are given by the solution of the characteristic equation:

$$\begin{aligned} \det \begin{pmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} &= (-1-\lambda)(1-\lambda) - 1 = 0, \\ &= \lambda^2 - 2 = 0, \end{aligned} \tag{5.32}$$

which are:

$$\lambda_{1,2} = \pm\sqrt{2}$$

Step 2. Compute the eigenvectors of A.

The eigenvector corresponding to the eigenvalue $\sqrt{2}$ is given by the solution of the following equations:

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.33}$$

or

$$-x_1 + x_2 = \sqrt{2}x_1, \tag{5.34}$$
$$x_1 + x_2 = \sqrt{2}x_2. \tag{5.35}$$

A solution is given by:

$$x_2 = (1+\sqrt{2})x_1,$$

corresponding to the eigenvector:

$$\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

The eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is given by the solution to the following equations:

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\sqrt{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.36}$$

or

$$-x_1 + x_2 = -\sqrt{2}x_1, \tag{5.37}$$
$$x_1 + x_2 = -\sqrt{2}x_2. \tag{5.38}$$

A solution is given by:

$$x_2 = (1-\sqrt{2})x_1,$$

corresponding to the eigenvector:

$$\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}.$$

Step 3. Use the eigenvectors of A to form the transformation matrix T.

For the columns of T we take the eigenvectors corresponding the the eigenvalues $\sqrt{2}$ and $-\sqrt{2}$:

$$T = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix}, \tag{5.39}$$

with T^{-1} given by:

$$T^{-1} = -\frac{1}{2\sqrt{2}} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1-\sqrt{2} & 1 \end{pmatrix}. \tag{5.40}$$

Step 4. Compute $\Lambda = T^{-1}AT$.

We have:

$$\begin{aligned} T^{-1}AT &= -\frac{1}{2\sqrt{2}} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1-\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix}, \\ &= -\frac{1}{2\sqrt{2}} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1-\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 2+\sqrt{2} & 2-\sqrt{2} \end{pmatrix}, \\ &= -\frac{1}{2\sqrt{2}} \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \equiv \Lambda. \end{aligned} \tag{5.41}$$

Therefore in the $u_1 - u_2$ coordinates (5.30) takes the form:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{5.42}$$

The phase portrait of (5.42) is shown in 5.3.

It is easy to see that the origin is unstable for (5.42). In fig. 5.3 we see that the origin has the structure of a saddle point, and we want to explore this idea further.

In the $u_1 - u_2$ coordinates the span of the eigenvector corresponding to the eigenvalue $\sqrt{2}$ is given by $u_2 = 0$, i.e. the u_1 axis. The span of the eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is given by $u_1 = 0$, i.e. the u_2 axis. Moreover, we can see from the form of (5.42) that these coordinate axes are invariant. The u_1 axis is referred to as the unstable subspace, denoted by E^u, and the u_2 axis is referred to as the stable subspace, denoted by E^s. In other words, the unstable subspace is the span of the eigenvector corresponding to the eigenvalue with positive real part and the stable subspace is the span of

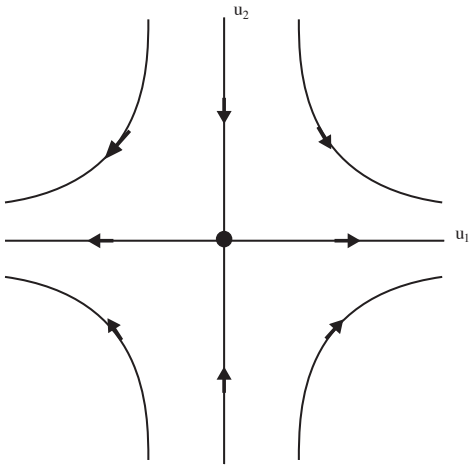


Figure 5.3: Phase portrait of (5.42).

the eigenvector corresponding to the eigenvalue having negative real part. The stable and unstable subspaces are invariant subspaces with respect to the flow generated by (5.42).

The stable and unstable subspaces correspond to the coordinate axes in the coordinate system given by the eigenvectors. Next we want to understand how they would appear in the original $x_1 - x_2$ coordinates. This is accomplished by transforming them to the original coordinates using the transformation matrix (5.39).

We first transform the unstable subspace from the $u_1 - u_2$ coordinates to the $x_1 - x_2$ coordinates. In the $u_1 - u_2$ coordinates points on the unstable subspace have coordinates $(u_1, 0)$. Acting on these points with T gives:

$$T \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.43}$$

which gives the following relation between points on the unstable subspace in the $u_1 - u_2$ coordinates to points in the $x_1 - x_2$ coordinates:

$$u_1 = x_1 \tag{5.44}$$
$$(1+\sqrt{2})u_1 = x_2, \tag{5.45}$$

or

$$(1+\sqrt{2})x_1 = x_2. \tag{5.46}$$

This is the equation for the unstable subspace in the $x_1 - x_2$ coordinates, which we illustrate in Fig. 5.4.

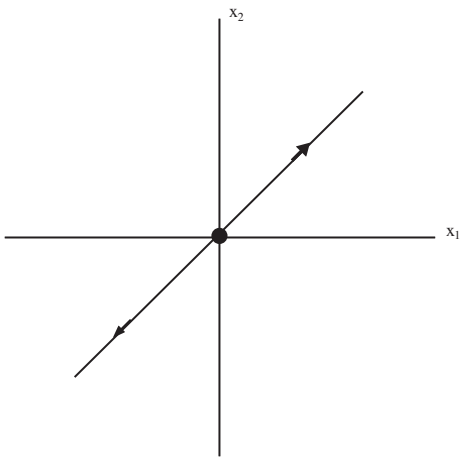


Figure 5.4: The unstable subspace in the original coordinates.

Next we transform the stable subspace from the $u_1 - u_2$ coordinates to the $x_1 - x_2$ coordinates. In the $u_1 - u_2$ coordinates points on the stable subspace have coordinates $(0, u_2)$. Acting on these points with T gives:

$$T \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (5.47)$$

which gives the following relation between points on the stable subspace in the $u_1 - u_2$ coordinates to points in the $x_1 - x_2$ coordinates:

$$u_2 = x_1 \qquad (5.48)$$

$$(1-\sqrt{2})u_2 = x_2 \qquad (5.49)$$

or

$$(1-\sqrt{2})x_1 = x_2. \qquad (5.50)$$

This is the equation for the stable subspace in the $x_1 - x_2$ coordinates, which we illustrate in Fig. 5.5.
In Fig. 5.6 we illustrate both the stable and the unstable subspaces in the original coordinates.

Now we want to discuss some general results from these three examples.

For all three examples, the real parts of the eigenvalues of A were nonzero, and stability of the origin was determined by the sign of the real parts of the eigenvalues, e.g., for example 10 the origin was unstable (the real parts of the eigenvalues of A were positive), for example 11 the origin was stable (the real parts of the eigenvalues of A were

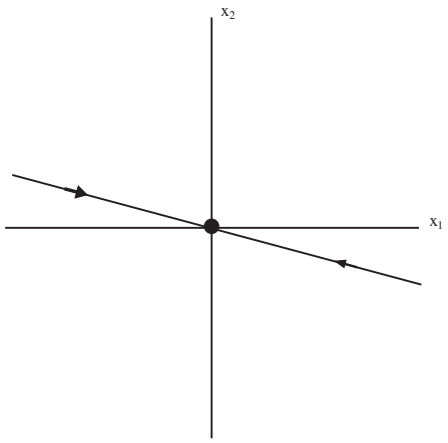


Figure 5.5: The stable subspace in the original coordinates.

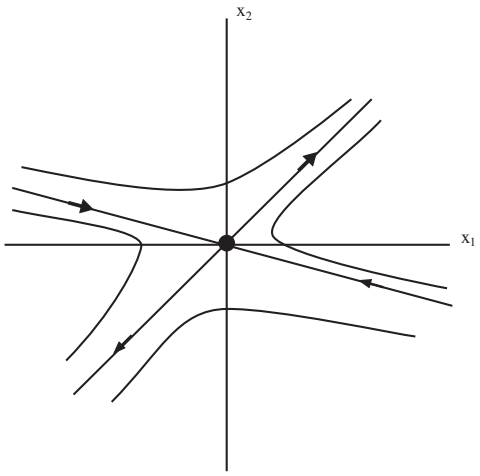


Figure 5.6: The stable and unstable subspaces in the original coordinates.

negative), and for example 12 the origin was unstable (A had one positive eigenvalue and one negative eigenvalue). This is generally true for all linear, autonomous vector fields. We state this more formally.

Consider a linear, autonomous vector field on $\mathbb{R}^n$:

$$\dot{y} = Ay, \quad y(0) = y_0, \quad y \in \mathbb{R}^n. \tag{5.51}$$

Then if A has no eigenvalues having zero real parts, the stability of the origin is determined by the real parts of the eigenvalues of A. If all of the real parts of the eigenvalues are strictly less than zero, then the origin is asymptotically stable. If at least one of the eigenvalues of A has real part strictly larger than zero, then the origin is unstable.

There is a term applied to this terminology that permeates all of dynamical systems theory.

Definition 20 (Hyperbolic Equilibrium Point)**.** *The origin of* (5.51) *is said to be hyperbolic if none of the real parts of the eigenvalues of A have zero real parts.*

It follows that hyperbolic equilibria of linear, autonomous vector fields on $\mathbb{R}^n$ can be either sinks, sources, or saddles. The key point is that the eigenvalues of A *all* have nonzero real parts.

If we restrict ourselves to two dimensions, it is possible to make a (short) list of all of the distinct canonical forms for A. These are given by the following six 2×2 matrices.

The first is a diagonal matrix with real, nonzero eigenvalues λ, $\mu \neq 0$, i.e. the origin is a hyperbolic fixed point:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \tag{5.52}$$

In this case the orgin can be a sink if both eigenvalues are negative, a source if both eigenvalues are positive, and a saddle if the eigenvalues have opposite sign.

The next situation corresponds to complex eigenvalues, with the real part, α, and imaginary part, β, both being nonzero. In this case the equilibrium point is hyperbolic, and a sink for $\alpha < 0$, and a source for $\alpha > 0$. The sign of β does not influence stability:

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{5.53}$$

Next we consider the case when the eigenvalues are real, identical, and nonzero, but the matrix is nondiagonalizable, i.e. two eigenvectors cannot be found. In this case the origin is hyperbolic for $\lambda \neq 0$, and is a sink for $\lambda < 0$ and a source for $\lambda > 0$:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{5.54}$$

Next we consider some cases corresponding to the origin being non-hyperbolic that would have been possible to include in the discussion of earlier cases, but it is more instructive to explicitly point out these cases separately.

We first consider the case where A is diagonalizable with one nonzero real eigenvalue and one zero eigenvalue:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.55}$$

We consider the case where the two eigenvalues are purely imagi-

nary, $\pm i\sqrt{\beta}$. In this case the origin is referred to as a center.

$$\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \tag{5.56}$$

For completeness, we consider the case where both eigenvalues are zero and A is diagonal.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.57}$$

Finally, we want to expand on the discussion related to the geometrical aspects of Example 12. Recall that for that example the span of the eigenvector corresponding to the eigenvalue with negative real part was an invariant subspace, referred to as the stable subspace. Trajectories with initial conditions in the stable subspace decayed to zero at an exponential rate as $t \to +\infty$. The stable invariant subspace was denoted by E^s. Similarly, the span of the eigenvector corresponding to the eigenvalue with positive real part was an invariant subspace, referred to as the unstable subspace. Trajectories with initial conditions in the unstable subspace decayed to zero at an exponential rate as $t \to -\infty$. The unstable invariant subspace was denoted by E^u.

We can easily see that (5.52) has this behavior when λ and μ have opposite signs. If λ and μ are both negative, then the span of the eigenvectors corresponding to these two eigenvalues is $\mathbb{R}^2$, and the entire phase space is the stable subspace. Similarly, if λ and μ are both positive, then the span of the eigenvectors corresponding to these two eigenvalues is $\mathbb{R}^2$, and the entire phase space is the unstable subspace.

The case (5.53) is similar. For that case there is not a pair of real eigenvectors corresponding to each of the complex eigenvalues. The vectors that transform the original matrix to this canonical form are referred to as *generalized eigenvectors*. If $\alpha < 0$ the span of the generalized eigenvectors is $\mathbb{R}^2$, and the entire phase space is the stable subspace. Similarly, if $\alpha > 0$ the span of the generalized eigenvectors is $\mathbb{R}^2$, and the entire phase space is the unstable subspace. The situation is similar for (5.54). For $\lambda < 0$ the entire phase space is the stable subspace, for $\lambda > 0$ the entire phase space is the unstable subspace.

The case (5.55) is different. The span of the eigenvector corresponding to λ is the stable subspace for $\lambda < 0$, and the unstable subspace for $\lambda > 0$ The space of the eigenvector corresponding to the zero eigenvalue is referred to as the center subspace.

For the case (5.56) there are not two real eigenvectors leading to the resulting canonical form. Rather, there are two generalized eigenvectors associated with this pair of complex eigenvalues having zero real part. The span of these two eigenvectors is a two dimensional center subspace corresponding to $\mathbb{R}^2$. An equilibrium point with purely

imaginary eigenvalues is referred to as a center.

Finally, the case (5.57) is included for completeness. It is the zero vector field where $\mathbb{R}^2$ is the center subspace.

We can characterize stability of the origin in terms of the stable, unstable, and center subspaces. The origin is asymptotically stable if $E^u = \emptyset$ and $E^c = \emptyset$. The origin is unstable if $E^u \neq \emptyset$.

Problem Set 5

1. Suppose A is a $n \times n$ matrix of real numbers. Show that if λ is an eigenvalue of A with eigenvector e, then $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{e}$.

2. Consider the matrices:

$$A_1 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad \omega > 0. \tag{5.58}$$

   Sketch the trajectories of the associated linear autonomous ordinary differential equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad i = 1, 2. \tag{5.59}$$

3. Consider the matrix

$$A = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \tag{5.60}$$

   (a) Show that the eigenvalues and eigenvectors are given by:

$$\begin{aligned} -1 - 3i: \quad & \begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ -1 + 3i: \quad & \begin{pmatrix} 1 \\ -3i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \end{aligned} \tag{5.61}$$

   (b) Consider the four matrices:

$$\begin{aligned} T_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \\ T_3 &= \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}. \end{aligned} \tag{5.62}$$

   Compute $\Lambda_i = T_i^{-1} A T_i$, $i = 1, \ldots, 4$.

(c) Discuss the form of T in terms of the eigenvectors of A. [2]

[2] The point ot this problem is to show that when you have complex eigenvalues (in the 2×2 case) there is a great deal of freedom in how you compute the real canonical form.

4. Consider the following two dimensional linear autonomous vector field:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (x_1(0), x_2(0)) = (x_{10}, x_{20}). \tag{5.63}$$

Show that the origin is Lyapunov stable. Compute and sketch the trajectories.

5. Consider the following two dimensional linear autonomous vector field:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (x_1(0), x_2(0)) = (x_{10}, x_{20}). \tag{5.64}$$

Show that the origin is a saddle. Compute the stable and unstable subspaces of the origin in the original coordinates, i.e. the $x_1 - x_2$ coordinates. Sketch the trajectories in the phase plane.

6. Compute e^A, where

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Hint. Write

$$A = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{\equiv S} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\equiv N}.$$

Then

$$A \equiv S + N, \quad \text{and} \quad NS = SN.$$

Use the binomial expansion fo compute $(S+N)^n$, $n \geq 1$,

$$(S+N)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} S^k N^{n-k},$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) \equiv \frac{n!}{k!(n-k)!}$$

and substitute the results into the exponential series.

6
Stable and Unstable Manifolds of Equilibria

For hyperbolic equilibria of autonomous vector fields, the linearization captures the local behavior near the equilibria for the nonlinear vector field. We describe the results justifying this statement in the context of two dimensional autonomous systems.[1]

We consider a C^r, $r \geq 1$ two dimensional autonomous vector field of the following form:

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{6.1}$$

Let $\phi_t(\cdot)$ denote the flow generated by (6.1). Suppose (x_0, y_0) is a hyperbolic equilibrium point of this vector field, i.e. the two eigenvalues of the Jacobian matrix:

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix},$$

have nonzero real parts. There are three cases to consider:

- (x_0, y_0) is a source for the linearized vector field,
- (x_0, y_0) is a sink for the linearized vector field,
- (x_0, y_0) is a saddle for the linearized vector field.

We consider each case individually. [2]

(x_0, y_0) is a source.

In this case (x_0, y_0) is a source for (6.1). More precisely, there exists a neighborhood $\mathcal{U}$ of (x_0, y_0) such that for any $p \in \mathcal{U}$, $\phi_t(p)$ leaves $\mathcal{U}$ as t increases.

[1] An extremely complete and thorough exposition of ''hyperbolic theory'' is given in .

Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge university press, 1997

[2] Details of the stability results for the hyperbolic source and sink can be found in .

Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012

(x_0, y_0) is a sink.

In this case (x_0, y_0) is a sink for (6.1). More precisely, there exists a neighborhood $\mathcal{S}$ of (x_0, y_0) such that for any $p \in \mathcal{S}$, $\phi_t(p)$ approaches (x_0, y_0) at an exponential rate as t increases. In this case (x_0, y_0) is an example of an attracting set and its basin of attraction is given by:

$$\mathcal{B} \equiv \bigcup_{t \leq 0} \phi_t(\mathcal{S}).$$

(x_0, y_0) is a saddle.

For the case of hyperbolic saddle points, the saddle point structure is still retained near the equilibrium point for nonlinear systems. We now explain precisely what this means. In order to do this we will need to examine (6.1) more closely. In particular, we will need to transform (6.1) to a coordinate system that ''localizes" the behavior near the equilibrium point and specifically displays the structure of the linear part. We have already done this several times in examining the behavior near specific solutions, so we will not repeat those details.

Transforming locally near (x_0, y_0) in this manner, we can express (6.1) in the following form:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} u(\xi, \eta) \\ v(\xi, \eta) \end{pmatrix}, \quad \alpha, \beta > 0, \qquad (\xi, \eta) \in \mathbb{R}^2, \tag{6.2}$$

where the Jacobian at the origin,

$$\begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{6.3}$$

reflects the hyperbolic nature of the equilibrium point. The linearization of (6.1) about the origin is given by:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{6.4}$$

It is easy to see for the linearized system that

$$E^s = \{(\xi, \eta) \,|\, \eta = 0\}, \tag{6.5}$$

is the invariant stable subspace and

$$E^u = \{(\xi, \eta) \,|\, \xi = 0\}, \tag{6.6}$$

is the invariant unstable subspace.

We now state how this saddle point structure is inherited by the nonlinear system by stating the results of the *stable and unstable manifold*

theorem for hyperbolic equilibria for two dimensional autonomous vector fields[3].

First, we consider two intervals of the coordinate axes containing the origin as follows:

$$I_\xi \equiv \{-\epsilon < \xi < \epsilon\}, \tag{6.7}$$

and

$$I_\eta \equiv \{-\epsilon < \eta < \epsilon\}, \tag{6.8}$$

for some small $\epsilon > 0$. A neighborhood of the origin is constructed by taking the cartesian product of these two intervals:

$$B_\epsilon \equiv \left\{(\xi, \eta) \in \mathbb{R}^2 \,|\, (\xi, \eta) \in I_\xi \times I_\eta\right\}, \tag{6.9}$$

and it is illustrated in Fig. 6.1. The stable and unstable manifold theorem for hyperbolic equilibrium points of autonomous vector fields states the following.

There exists a C^r curve, given by the graph of a function of the ξ variables:

$$\eta = S(\xi), \quad \xi \in I_\xi, \tag{6.10}$$

This curve has three important properties.

It passes through the origin, i.e. $S(0) = 0$.

It is tangent to E^s at the origin, i.e., $\frac{dS}{d\xi}(0) = 0$.

It is locally invariant in the sense that any trajectory starting on the curve approaches the origin at an exponential rate as $t \to \infty$, and it leaves B_ϵ as $t \to -\infty$.

Moreover, the curve satisfying these three properties is *unique*. For these reasons, this curve is referred to as the local stable manifold of the origin, and it is denoted by:

$$W^s_{\text{loc}}\left((0,0)\right) = \{(\xi, \eta) \in B_\epsilon \,|\, \eta = S(\xi)\}. \tag{6.11}$$

Similarly, there exists another C^r curve, given by the graph of a function of the η variables:

$$\xi = U(\eta), \quad \eta \in I_\eta, \tag{6.12}$$

This curve has three important properties.

It passes through the origin, i.e. $U(0) = 0$.

It is tangent to E^u at the origin, i.e., $\frac{dU}{d\eta}(0) = 0$.

[3] The stable and unstable manifold theorem for hyperbolic equilibria of autonomous vector fields is a fundamental result. The proof requires some preliminary work to develop the appropriate mathematical setting in order for the proof to proceed. See, for example, . Of course, before one proves a result one must have a through understanding of the result that one hopes to prove. For this reason we focus on explicit examples here.

E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Krieger, 1984; P. Hartman. *Ordinary Differential Equations*. Society for industrial and Applied Mathematics, 2002; J. K. Hale. *Ordinary Differential Equations*. Dover, 2009; and Chicone Carmen. *Ordinary differential equations with applications*. Springer, 2000

It is locally invariant in the sense that any trajectory starting on the curve approaches the origin at an exponential rate as $t \to -\infty$, and it leaves B_ϵ as $t \to \infty$.

For these reasons, this curve is referred to as the local unstable manifold of the origin, and it is denoted by:

$$W^u_{\text{loc}}\left((0,0)\right) = \left\{(\xi,\eta) \in B_\epsilon \,|\, \xi = U(\eta)\right\}. \tag{6.13}$$

The curve satisfying these three properties is *unique*.

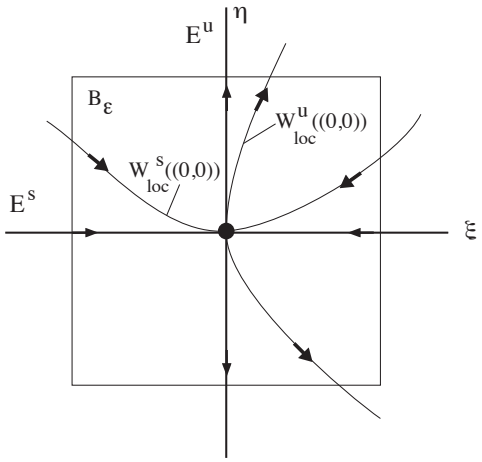


Figure 6.1: The neighborhood of the origin, B_ϵ, showing the local stable and unstable manifolds

These local stable and unstable manifolds are the ''seeds'' for the global stable and unstable manifolds that are defined as follows:

$$W^s\left((0,0)\right) \equiv \bigcup_{t\leq 0} \phi_t\left(W^s_{\text{loc}}\left((0,0)\right)\right), \tag{6.14}$$

and

$$W^u\left((0,0)\right) \equiv \bigcup_{t\geq 0} \phi_t\left(W^u_{\text{loc}}\left((0,0)\right)\right). \tag{6.15}$$

Now we will consider a series of examples showing how these ideas are used.

Example 13. *We consider the following autonomous, nonlinear vector field on the plane:*

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y + x^2, \qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{6.16}$$

This vector field has an equilibrium point at the origin, $(x, y) = (0, 0)$. The Jacobian of the vector field evaluated at the origin is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.17}$$

From this calculation we can conclude that the origin is a hyperbolic saddle point. Moreover, the x-axis is the unstable subspace for the linearized vector field and the y axis is the stable subspace for the linearized vector field.

Next we consider the nonlinear vector field (6.16). By inspection, we see that the y axis (i.e. $x = 0$) is the global stable manifold for the origin. We next consider the unstable manifold. Dividing the second equation by the first equation in (6.16) gives:

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -\frac{y}{x} + x. \tag{6.18}$$

This is a linear nonautonomous equation. A solution of this equation passing through the origin is given by:

$$y = \frac{x^2}{3}. \tag{6.19}$$

4

It is also tangent to the unstable subspace at the origin. It is the global unstable manifold[5].

We examine this statement further. It is easy to compute the flow generated by (6.16). The x-component can be solved and substituted into the y component to yield a first order linear nonautonomous equation. Hence, the flow generated by (6.16) is given by[6]*:*

$$\begin{aligned} x(t, x_0) &= x_0 e^t, \\ y(t, y_0) &= \left(y_0 - \frac{x_0^2}{3} \right) e^{-t} + \frac{x_0^2}{3} e^{2t}. \end{aligned} \tag{6.20}$$

The global unstable manifold of the origin is the set of initial conditions having the property that the trajectories through these initial conditions approach the origin at an exponential rate as $t \to -\infty$. On examining the two components of (6.20), we see that the x component approaches zero as $t \to -\infty$ for any choice of x_0. However, the y component will only approach zero as $t \to -\infty$ if y_0 and x_0 are chosen such that

$$y_0 = \frac{x_0^2}{3}. \tag{6.21}$$

Hence (6.21) is the global unstable manifold of the origin[7].

Example 14. *Consider the following nonlinear autonomous vector field on the plane:*

[4] This linear first order equation can be solved in the usual way with an integrating factor. See Appendix B for details of this procedure.

[5] You should verify that this curve is invariant. In terms of the vector field, invariance means that the vector field is tangent to the curve. Why?

[6] Again, the equation is solved in the usual way by using an integrating factor, see Appendix B for details.

[7] (6.19) characterizes the unstable manifold as an invariant curve passing through the origin and tangent to the unstable subspace at the origin. (6.21) characterizes the unstable manifold in terms of the asymptotic behavior of trajectories (as $t \to -\infty$) whose initial conditions satisfy a particular constraint, and that constraint is that they are on the unstable manifold of the origin.

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y, \qquad\qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{6.22}$$

Note that the x and y components evolve independently.

The equilibrium points and the Jacobians associated with their linearizations are given as follows:

$$(x,y) = (0,0); \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \textit{saddle} \tag{6.23}$$

$$(x,y) = (\pm 1,0); \quad \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}; \quad \textit{sinks} \tag{6.24}$$

We now compute the global stable and unstable manifolds of these equilibria. We begin with the saddle point at the origin.

$$(0,0): \quad \begin{array}{l} W^s((0,0)) = \{(x,y) | x = 0\} \\ W^u((0,0)) = \{(x,y) | -1 < x < 1,\ y = 0\} \end{array} \tag{6.25}$$

For the sinks the global stable manifold is synonomous with the basin of attraction for the sink.

$$(1,0): \quad W^s((1,0)) = \{(x,y) | x > 0\} \tag{6.26}$$

$$(-1,0): \quad W^s((-1,0)) = \{(x,y) | x < 0\} \tag{6.27}$$

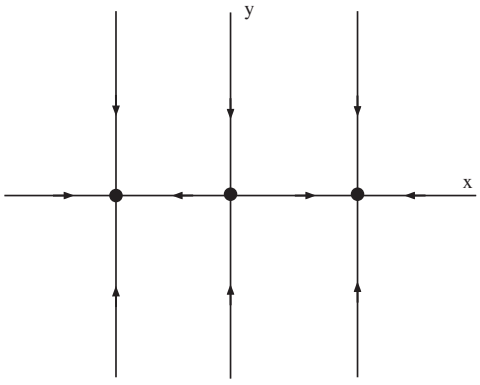


Figure 6.2: Invariant manifold structure of (6.22). The black dots indicate equilibrium points.

Example 15. *In this example we consider the following nonlinear autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= y^2(1-y^2), \qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{6.28}$$

Note that the x and y components evolve independently.

The equilibrium points and the Jacobians associated with their linearizations are given as follows:

$$(x,y) = (0,0),\ (0,\pm 1) \tag{6.29}$$

$$(x,y) = (0,0); \quad \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \textit{not hyperbolic} \tag{6.30}$$

$$(x,y) = (0,1); \quad \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}; \quad \textit{sink} \tag{6.31}$$

$$(x,y) = (0,-1); \quad \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}; \quad \textit{saddle} \tag{6.32}$$

We now compute the global invariant manifold structure for each of the equilibria, beginning with $(0,0)$.

$$\begin{aligned} (0,0); \quad & W^s\left((0,0)\right) = \{(x,y) | y = 0\} \\ & W^c\left((0,0)\right) = \{(x,y) | x = 0,\ -1 < y < 1\} \end{aligned} \tag{6.33}$$

The x-axis is clearly the global stable manifold for this equilibrium point. The segment on the y-axis between -1 *and* 1 *is invariant, but it does not correspond to a hyperbolic direction. It is referred to as the* center manifold of the origin*, and we will learn much more about invariant manifolds associated with nonhyperbolic directions later.*

The equilibrium point $(0,1)$ *is a sink. Its global stable manifold (basin of attraction) is given by:*

$$(0,1); \quad W^s\left((0,1)\right) = \{(x,y) | y > 0\}. \tag{6.34}$$

The equilibrium point $(0,-1)$ *is a saddle point with global stable and unstable manifolds given by:*

$$\begin{aligned} (0,-1); \quad & W^s\left((0,-1)\right) = \{(x,y) | y = -1\} \\ & W^u\left((0,-1)\right) = \{(x,y) | x = 0,\ -\infty < y < 0\} \end{aligned} \tag{6.35}$$

Example 16. *In this example we consider the following nonlinear autonomous vector field on the plane:*

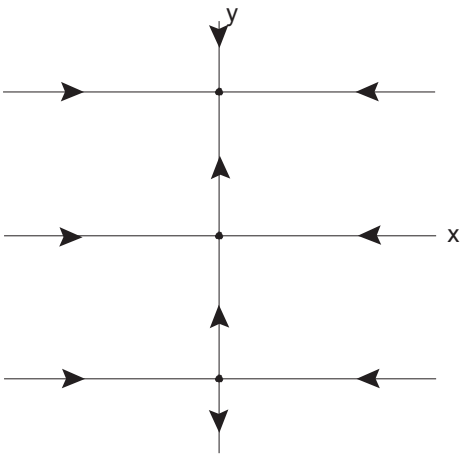


Figure 6.3: Invariant manifold structure of (6.28). The black dots indicate equilibrium points.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \delta y, \qquad (x,y) \in \mathbb{R}^2, \quad \delta \geq 0, \end{aligned} \tag{6.36}$$

where $\delta > 0$ is to be viewed as a parameter. The equilibrium points are given by:

$$(x,y) = (0,0),\ (\pm 1, 0). \tag{6.37}$$

We want to classify the linearized stability of the equilibria. The Jacobian of the vector field is given by:

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -\delta \end{pmatrix}, \tag{6.38}$$

and the eigenvalues of the Jacobian are:

$$\lambda_\pm = -\frac{\delta}{2} \pm \frac{1}{2}\sqrt{\delta^2 + 4 - 12x^2}. \tag{6.39}$$

We evaluate this expression for the eigenvalues at each of the equilibria to determine their linearized stability.

$$(0,0); \quad \lambda_\pm = -\tfrac{\delta}{2} \pm \tfrac{1}{2}\sqrt{\delta^2 + 4} \tag{6.40}$$

Note that

$$\delta^2 + 4 > \delta^2,$$

therefore the eigenvalues are always real and of opposite sign. This implies that $(0,0)$ is a saddle.

$$(\pm 1, 0) \quad \lambda_\pm = -\tfrac{\delta}{2} \pm \tfrac{1}{2}\sqrt{\delta^2 - 8} \tag{6.41}$$

First, note that

$$\delta^2 - 8 < \delta^2.$$

This implies that these two fixed points are always sinks. However, there are two subcases.

$\delta^2 - 8 < 0$: *The eigenvalues have a nonzero imaginary part.*

$\delta^2 - 8 \geq 0$: *The eigenvalues are purely real.*

In fig. 6.4 we sketch the local invariant manifold structure for these two cases.

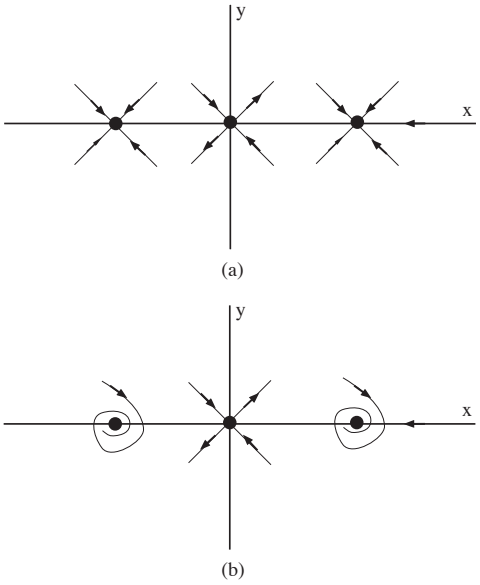


Figure 6.4: Local invariant manifold structure of (6.36). The black dots indicate equilibrium points. (a) $\delta^2 - 8 \geq 0$, (b) $\delta^2 - 8 < 0$

In fig. 6.5 we sketch the global invariant manifold structure for the two cases. In the coming lectures we will learn how we can justify this figure. However, note the role that the stable *manifold of the saddle plays in defining the basins of attractions of the two sinks.*

Problem Set 6

1. Consider the C^r, $r \geq 1$, autonomous vector field on $\mathbb{R}^2$:

$$\dot{x} = f(x),$$

with flow

$$\phi_t(\cdot),$$

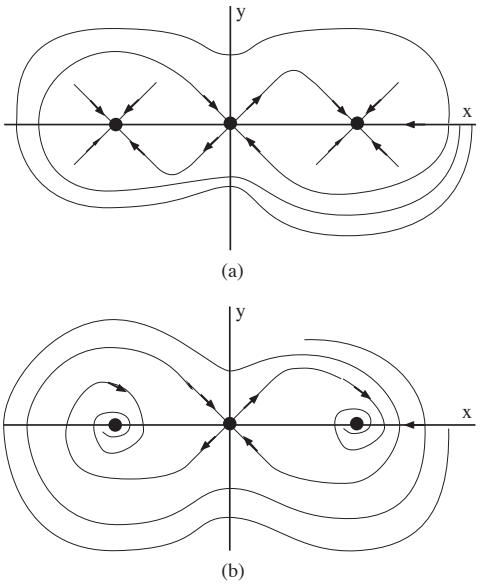


Figure 6.5: A sketch of the global invariant manifold structure of (6.36). The black dots indicate equilibrium points. (a) $\delta^2 - 8 \geq 0$, (b) $\delta^2 - 8 < 0$

and let $x = \bar{x}$ denote a hyperbolic saddle type equilibrium point for this vector field. We denote the local stable and unstable manifolds of this equilibrium point by:

$$W^s_{\text{loc}}(\bar{x}),\ W^u_{\text{loc}}(\bar{x}),$$

respectively. The global stable and unstable manifolds of $\bar{x}$ are defined by:

$$\begin{aligned} W^s(\bar{x}) &\equiv \bigcup_{t \leq 0} \phi_t \left(W^s_{\text{loc}}(\bar{x}) \right), \\ W^u(\bar{x}) &\equiv \bigcup_{t \geq 0} \phi_t \left(W^u_{\text{loc}}(\bar{x}) \right) \end{aligned}$$

(a) Show that $W^s(\bar{x})$ and $W^u(\bar{x})$ are invariant sets.

(b) Suppose that $p \in W^s(\bar{x})$, show that $\phi_t(p) \to \bar{x}$ at an exponential rate as $t \to \infty$.

(c) Suppose that $p \in W^u(\bar{x})$, show that $\phi_t(p) \to \bar{x}$ at an exponential rate as $t \to -\infty$.

2. Consider the C^r, $r \geq 1$, autonomous vector field on $\mathbb{R}^2$ having a hyperbolic saddle point. Can its stable and unstable manifolds intersect at an *isolated* point (which is *not* a fixed point of the vector field) as shown in figure 6.6?

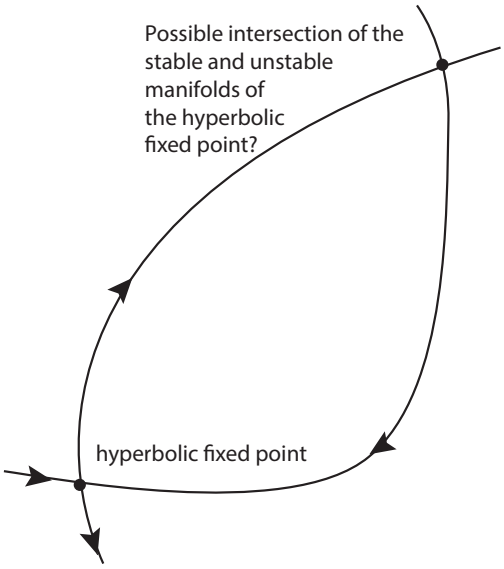


Figure 6.6: Possible intersection of the stable and unstable manifold of a hyperbolic fixed point?

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= \alpha x, \\ \dot{y} &= \beta y + \gamma x^{n+1}, \end{aligned} \tag{6.42}$$

where $\alpha < 0$, $\beta > 0$, γ is a real number, and n is a positive integer.

(a) Show that the origin is a hyperbolic saddle point.

(b) Compute and sketch the stable and unstable subspaces of the origin.

(c) Show that the stable and unstable subspaces are invariant under the linearized dynamics.

(d) Show the the flow generated by this vector field is given by:

$$\begin{aligned} x(t, x_0) &= x_0 e^{\alpha t}, \\ y(t, x_0, y_0) &= e^{\beta t}\left(y_0 - \frac{\gamma x_0^{n+1}}{\alpha(n+1) - \beta}\right) + \left(\frac{\gamma x_0^{n+1}}{\alpha(n+1) - \beta}\right) e^{\alpha(n+1)t} \end{aligned}$$

(e) Compute the global stable and unstable manifolds of the origin from the flow.

(f) Show that the global stable and unstable manifolds that you have computed are invariant.

(g) Sketch the global stable and unstable manifolds and discuss how they depend on γ and n.

4. Suppose $\dot{x} = f(x), \quad x \in \mathbb{R}^n$ is a C^r vector field having a hyperbolic fixed point, $x = x_0$, with a homoclinic orbit. Describe the homoclinic orbit in terms of the stable and unstable manifolds of x_0.

5. Suppose $\dot{x} = f(x), \quad x \in \mathbb{R}^n$ is a C^r vector field having hyperbolic fixed points, $x = x_0$ and x_1, with a heteroclinic orbit connecting x_0 and x_1. Describe the heteroclinic orbit in terms of the stable and unstable manifolds of x_0 and x_1.

7
Lyapunov's Method and the LaSalle Invariance Principle

We will next learn a method for determining stability of equilibria which may be applied when stability information obtained from the linearization of the ODE is not sufficient for determining stability information for the nonlinear ODE. The book by LaSalle[1] is an excellent supplement to this lecture. This is Lyapunov's method (or Lyapunov's second method, or the method of Lyapunov functions). [2]We begin by describing the framework for the method in the setting that we will use.

We consider a general C^r, $r \geq 1$ autonomous ODE

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{7.1}$$

having an equilibrium point at $x = \bar{x}$, i.e.,

$$f(\bar{x}) = 0. \tag{7.2}$$

For a scalar valued function defined on $\mathbb{R}^n$

$$\begin{aligned} V : \mathbb{R}^n &\to \mathbb{R}, \\ x &\mapsto V(x), \end{aligned} \tag{7.3}$$

we define the derivative of (7.3) along trajectories of (7.1) by:

$$\begin{aligned} \frac{d}{dt}V(x) = \dot{V}(x) &= \nabla V(x) \cdot \dot{x}, \\ &= \nabla V(x) \cdot f(x). \end{aligned} \tag{7.4}$$

We can now state Lyapunov's theorem on stability of the equilibrium point $x = \bar{x}$.

Theorem 1. *Consider the following C^r ($r \geq 1$) autonomous vector field on $\mathbb{R}^n$:*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{7.5}$$

[1] Joseph P LaSalle. *The stability of dynamical systems*, volume 25. SIAM, 1976

[2] The original work of Lyapunov is reprinted (Lyapunov). An excellent perspective of Lyapunov's work is given in Parks .

A.M. Lyapunov. *General Problem of the Stability Of Motion*. Control Theory and Applications Series. Taylor & Francis, 1992. ISBN 9780748400621. URL https://books.google.ie/books?id=4tmAvU3_SCoC; and P. C. Parks. A. M. Lyapunov's stability theory—100 years on. *IMA Journal of Mathematical Control and Information*, 9(4):275–303, 1992. DOI: 10.1093/imamci/9.4.275. URL http://imamci.oxfordjournals.org/content/9/4/275.abstract

Let $x = \bar{x}$ be an equilibrium point of (7.5) and let $V : U \to \mathbb{R}$ be a C^1 function defined in some neighborhood U of $\bar{x}$ such that:

1. *$V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$.*
2. *$\dot{V}(x) \leq 0$ in $U - \{\bar{x}\}$*

Then $\bar{x}$ is Lyapunov stable. Moreover, if

$$\dot{V}(x) < 0 \quad in \quad U - \{\bar{x}\}$$

then $\bar{x}$ is asymptotically stable.

The function $V(x)$ is referred to as a Lyapunov function.

We now consider an example.

Example 17.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \epsilon x^2 y, \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{7.6}$$

where ϵ is a parameter. It is clear that $(x, y) = (0, 0)$ is an equilibrium point of (7.6) and we want to determine the nature of its stability.

We begin by linearizing (7.6) about this equilibrium point. The matrix associated with this linearization is given by:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7.7}$$

and its eigenvalues are $\pm i$. Hence, the origin is not hyperbolic and therefore the information provided by the linearization of (7.6) about $(x, y) = (0, 0)$ does not provide information about stability of $(x, y) = (0, 0)$ for the nonlinear system (7.6).

Therefore we will attempt to apply Lyapunov's method to determine stability of the origin.

We take as a Lyapunov function:

$$V(x, y) = \frac{1}{2}(x^2 + y^2). \tag{7.8}$$

Note that $V(0, 0) = 0$ and $V(x, y) > 0$ in any neighborhood of the origin. Moreover, we have:

$$\begin{aligned} \dot{V}(x, y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}, \\ &= xy + y(-x - \epsilon x^2 y), \\ &= -\epsilon x^2 y^2, \\ &\leq 0, \qquad for \quad \epsilon \geq 0. \end{aligned} \tag{7.9}$$

Hence, it follows from Theorem 1 that the origin is Lyapunov stable[3].

[3] The obvious question that comes up at this point is, *how do I find the Lyapunov function?* We discuss this issue in Appendix C.

Next we will introduce the LaSalle invariance principle[4]. Rather than focus on the particular question of stability of an equilibrium solution as in Lyapunov's method, the LaSalle invariance principle gives conditions that describe the behavior as $t \to \infty$ of all solutions of an autonomous ODE.

We begin with an autonomous ODE defined on $\mathbb{R}^n$:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{7.10}$$

where $f(x)$ is C^r, $(r \geq 1)$. Let $\phi_t(\cdot)$ denote the flow generated by (7.10) and let $\mathcal{M} \subset \mathbb{R}^n$ denote a positive invariant set that is compact (i.e. closed and bounded in this setting). Suppose we have a scalar valued function

$$V : \mathbb{R}^n \to \mathbb{R}, \tag{7.11}$$

such that

$$\dot{V}(x) \leq 0 \quad \text{in} \quad \mathcal{M} \tag{7.12}$$

(Note the ''less than or equal to" in this inequality.)

Let

$$E = \left\{ x \in \mathcal{M} \mid \dot{V}(x) = 0 \right\}, \tag{7.13}$$

and

$$M = \left\{ \begin{array}{l} \text{the union of all trajectories that start} \\ \text{in E and remain in E for all } t \geq 0 \end{array} \right\} \tag{7.14}$$

Now we can state LaSalle's invariance principle.

Theorem 2. *For all $x \in \mathcal{M}$, $\phi_t(x) \to M$ as $t \to \infty$.*

We will now consider an example.

Example 18. *Consider the following vector field on $\mathbb{R}^2$:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \delta y, \quad (x, y) \in \mathbb{R}^2, \quad \delta > 0. \end{aligned} \tag{7.15}$$

This vector field has three equilibrium points–a saddle point at $(x, y) = (0, 0)$ and two sinks at $(x, y) = (\pm, 1, 0)$.

Consider the function

$$V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}, \tag{7.16}$$

and its level sets:

[4] The original paper is a technical report that can be found on the internet ( ). See also).

Joseph P LaSalle. An invariance principle in the theory of stability. Technical Report 66-1, Brown University, 1966; Joseph P LaSalle. *The stability of dynamical systems*, volume 25. SIAM, 1976; and Itzhak Barkana. Defending the beauty of the invariance principle. *International Journal of Control*, 87(1):186–206, 2014

$$V(x,y) = C.$$

We compute the derivative of V along trajectories of (7.15):

$$\begin{aligned} \dot{V}(x,y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}, \\ &= (-x + x^3)y + y(x - x^3 - \delta y). \\ &= -\delta y^2, \end{aligned} \tag{7.17}$$

from which it follows that

$$\dot{V}(x,y) \leq 0 \quad on \quad V(x,y) = C.$$

Therefore, for C sufficiently large, the corresponding level set of V bounds a compact positive invariant set, $\mathcal{M}$, containing the three equilibrium points of (7.15).

Next we determine the nature of the set

$$E = \{(x,y) \in \mathcal{M} \mid \dot{V}(x,y) = 0\}. \tag{7.18}$$

Using (7.17) we see that:

$$E = \{(x,y) \in \mathcal{M} \mid y = 0 \cap \mathcal{M}\}. \tag{7.19}$$

The only points in E that remain in E for all time are:

$$M = \{(\pm 1, 0), \quad (0,0)\}. \tag{7.20}$$

Therefore it follows from Theorem 2 that given any initial condition in $\mathcal{M}$, the trajectory starting at that initial condition approaches one of the three equilibrium points as $t \to \infty$.

Autonomous Vector Fields on the Plane; Bendixson's Criterion and the Index Theorem

Now we will consider some useful results that apply to vector fields on the plane.

First we will consider a simple, and easy to apply, criterion that rules out the existence of periodic orbits for autonomous vector fields *on the plane* (e.g., it is not valid for vector fields on the two torus).

We consider a C^r, $r \geq 1$ vector field on the plane of the following form:

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \qquad (x,y) \in \mathbb{R}^2 \end{aligned} \tag{7.21}$$

The following criterion due to Bendixson provides a simple, computable condition that rules out the existence of periodic orbits in certain regions of $\mathbb{R}^2$.

Theorem 3 (Bendixson's Criterion)**.** *If on a simply connected region $D \subset \mathbb{R}^2$ the expression*

$$\frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y). \tag{7.22}$$

is not identically zero and does not change sign then (7.21) has no periodic orbits lying entirely in D.

Example 19. *We consider the following nonlinear autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= y \equiv f(x,y), \\ \dot{y} &= x - x^3 - \delta y \equiv g(x,y), \qquad (x,y) \in \mathbb{R}^2, \quad \delta > 0. \end{aligned} \tag{7.23}$$

Computing (7.22) gives:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\delta. \tag{7.24}$$

Therefore this vector field has no periodic orbits for $\delta \neq 0$.

Example 20. *We consider the following linear autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= ax + by \equiv f(x,y), \\ \dot{y} &= cx + dy \equiv g(x,y), \qquad (x,y) \in \mathbb{R}^2, \quad a, b, c, d \in \mathbb{R} \end{aligned} \tag{7.25}$$

Computing (7.22) gives:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = a + d. \tag{7.26}$$

Therefore for $a + d \neq 0$ this vector field has no periodic orbits.

Next we will consider the index theorem. If periodic orbits exist, it provides conditions on the number of fixed points, and their stability, that are contained in the region bounded by the periodic orbit.

Theorem 4. *Inside any periodic orbit there must be* at least *one fixed point. If there is only one, then it must be a sink, source, or center. If all the fixed points inside the periodic orbit are hyperbolic, then there must be an odd number, $2n + 1$, of which n are saddles, and $n + 1$ are either sinks or sources.*

Example 21. *We consider the following nonlinear autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= y \equiv f(x,y), \\ \dot{y} &= x - x^3 - \delta y + x^2 y \equiv g(x,y), \qquad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{7.27}$$

where $\delta > 0$. The equilibrium points are given by:

$$(x,y) = (0,0),\ (\pm 1, 0).$$

The Jacobian of the vector field, denoted by A, is given by:

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 + 2xy & -\delta + x^2 \end{pmatrix} \tag{7.28}$$

Using the general expression for the eigenvalues for a 2×2 matrix A:

$$\lambda_{1,2} = \frac{tr\, A}{2} \pm \frac{1}{2}\sqrt{(tr\, A)^2 - 4\, det\, A},$$

we obtain the following expression for the eigenvalues of the Jacobian:

$$\lambda_{1,2} = \frac{-\delta + x^2}{2} \pm \frac{1}{2}\sqrt{(-\delta + x^2)^2 + 4(1 - 3x^2 + 2xy)} \tag{7.29}$$

If we substitute the locations of the equilibria into this expression we obtain the following values for the eigenvalues of the Jacobian of the vector field evaluated at the respective equilibria:

$$(0,0) \quad \lambda_{1,2} = -\tfrac{\delta}{2} \pm \tfrac{1}{2}\sqrt{\delta^2 + 4} \tag{7.30}$$

$$(\pm 1, 0) \quad \lambda_{1,2} = \tfrac{-\delta+1}{2} \pm \tfrac{1}{2}\sqrt{(-\delta+1)^2 - 8} \tag{7.31}$$

From these expressions we conclude that $(0,0)$ is a saddle for all values of δ and $(\pm 1, 0)$ are

$$\begin{array}{ll} \textit{sinks for} & \delta > 1 \\ \textit{centers for} & \delta = 1 \\ \textit{sources for} & 0 < \delta < 1 \end{array} \tag{7.32}$$

Now we will use Bendixson's criterion and the index theorem to determine regions in the phase plane where periodic orbits may *exist. For this example* (7.22) *is given by:*

$$-\delta + x^2. \tag{7.33}$$

Hence the two vertical lines $x = -\sqrt{\delta}$ and $x = \sqrt{\delta}$ divide the phase plane into three regions where periodic orbits cannot *exist entirely in one of these*

regions (or else Bendixson's criterion would be violated). There are two cases to be considered for the location of these vertical lines with respect to the equilibria: $\delta > 1$ *and* $0 < \delta < 1$.

In Fig. 7.1 we show three possibilities (they do not *exhaust all possible cases) for the existence of periodic orbits that would satisfy Bendixson's criterion in the case* $\delta > 1$. *However, (b) is not possible because it violates the index theorem.*

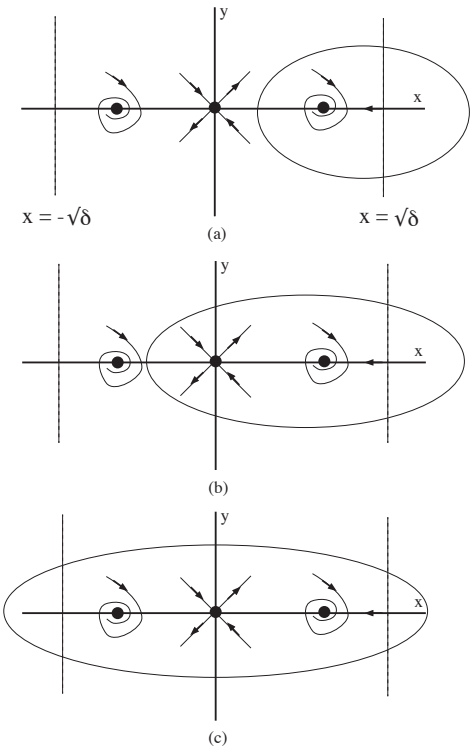


Figure 7.1: The case $\delta > 1$. Possibilities for periodic orbits that satisfy Bendixson's criterion. However, (b) is not possible because it violates the index theorem.

In Fig. 7.2 we show three possibilities (they do not *exhaust all possible cases) for the existence of periodic orbits that would satisfy Bendixson's criterion in the case* $0 < \delta < 1$. *However, (e) is not possible because it violates the index theorem.*

Problem Set 7

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \delta y, \qquad \delta \geq 0, \quad (x, y) \in \mathbb{R}^2. \end{aligned}$$

Use Lyapunov's method to show that the equilibria $(x, y) = (\pm 1, 0)$ are Lyapunov stable for $\delta = 0$ and asymptotically stable for $\delta > 0$.

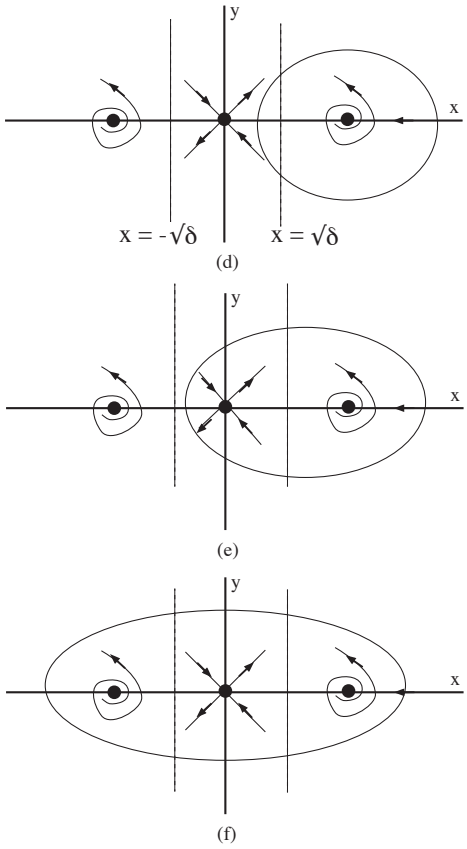


Figure 7.2: The case $0 < \delta < 1$. Possibilities for periodic orbits that satisfy Bendixson's criterion. However, (e) is not possible because it violates the index theorem.

2. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon x^2 y, \quad \varepsilon > 0, \, (x, y) \in \mathbb{R}^2. \end{aligned}$$

Use the LaSalle invariance principle to show that

$$(x, y) = (0, 0),$$

is asymptotically stable.

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \alpha x^2 y, \qquad \alpha > 0, \quad (x, y) \in \mathbb{R}^2, \end{aligned}$$

use the LaSalle invariance principle to describe the fate of all trajectories as $t \to \infty$.

4. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \alpha xy, \qquad (x,y) \in \mathbb{R}^2, \end{aligned}$$

where α is a real parameter. Determine the equilibria and discuss their linearized stability as a function of α.

5. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= cx + dy, \qquad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{7.34}$$

where $a, b, c, d \in \mathbb{R}$. In the questions below you are asked to give conditions on the constants a, b, c, and d so that particular dynamical phenomena are satisfied. You do *not* have to give all possible conditions on the constants in order for the dynamical condition to be satisfied. One condition will be sufficient, but you must justify your answer.

- Give conditions on a, b, c, d for which the vector field has no periodic orbits.
- Give conditions on a, b, c, d for which all of the orbits are periodic.
- Using

$$V(x,y) = \frac{1}{2}(x^2 + y^2)$$

as a Lyapunov function, give conditions on a, b, c, d for which $(x,y) = (0,0)$ is asymptotically stable.

- Give conditions on a, b, c, d for which $x = 0$ is the stable manifold of $(x,y) = (0,0)$ and $y = 0$ is the unstable manifold of $(x,y) = (0,0)$.

6. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \frac{x^2 y}{2}, \qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{7.35}$$

- Determine the linearized stability of $(x,y) = (0,0)$.

- Describe the invariant manifold structure for the linearization of (7.35) about $(x, y) = (0, 0)$.
- Using $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as a Lyapunov function, what can you conclude about the stability of the origin? Does this agree with the linearized stability result obtained above? Why or why not?
- Using the LaSalle invariance principle, determine the fate of a trajectory starting at an arbitrary initial condition as $t \to \infty$? What does this result allow you to conclude about stability of $(x, y) = (0, 0)$?

8
Bifurcation of Equilibria, I

We will now study the topic of bifurcation of equilibria of autonomous vector fields, or ''what happens as an equilibrium point loses hyperbolicity as a parameter is varied?'' We will study this question through a series of examples, and then consider what the examples teach us about the ''general situation'' (and what this might be).

Example 22 (The Saddle-Node Bifurcation)**.** *Consider the following nonlinear, autonomous vector field on* $\mathbb{R}^2$*:*

$$\begin{aligned} \dot{x} &= \mu - x^2, \\ \dot{y} &= -y, \qquad (x, y) \in \mathbb{R}^2 \end{aligned} \tag{8.1}$$

where μ *is a (real) parameter. The equilibrium points of* (8.1) *are given by:*

$$(x, y) = (\sqrt{\mu}, 0),\ (-\sqrt{\mu}, 0). \tag{8.2}$$

It is easy to see that there are no equilibrium points for $\mu < 0$*, one equilibrium point for* $\mu = 0$*, and two equilibrium points for* $\mu > 0$*.*

The Jacobian of the vector field evaluated at each equilibrium point is given by:

$$(\sqrt{\mu}, 0): \qquad \begin{pmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}, \tag{8.3}$$

from which it follows that the equilibria are hyperbolic and asymptotically stable for $\mu > 0$*, and nonhyperbolic for* $\mu = 0$*.*

$$(-\sqrt{\mu}, 0): \qquad \begin{pmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix} \tag{8.4}$$

from which it follows that the equilibria are hyperbolic saddle points for $\mu > 0$*, and nonhyperbolic for* $\mu = 0$*. We emphasize again that there are no equilibrium points for* $\mu < 0$*.*

As a result of the ''structure" of (8.1) *we can easily represent the behavior of the equilibria as a function of* μ *in a bifurcation diagram. That is, since the* x *and* y *components of* (8.1) *are ''decoupled", and the change in the number and stability of equilibria us completely captured by the* x *coordinates, we can plot the* x *component of the vector field as a function of* μ*, as we show in Fig. 8.1.*

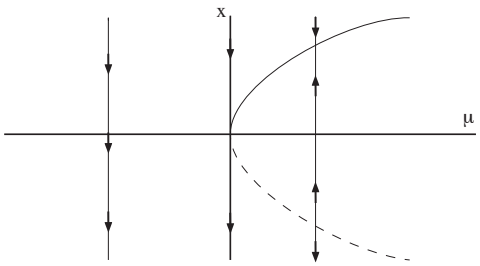


Figure 8.1: Bifurcation diagram for (8.1) in the $\mu - x$ plane. The curve of equilibria is given by $\mu = x^2$. The dashed line denotes the part of the curve corresponding to unstable equilibria, and the solid line denotes the part of the curve corresponding to stable equilibria.

In Fig. 8.2 we illustrate the bifurcation of equilibria for (8.1) *in the* $x - y$ *plane.*

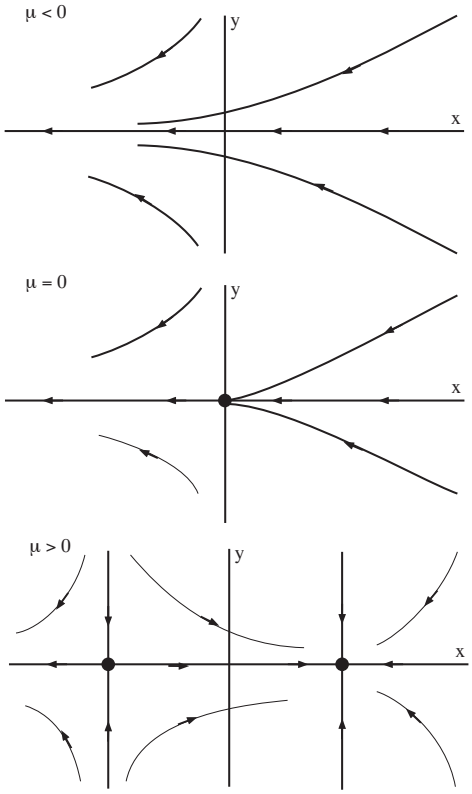


Figure 8.2: Bifurcation diagram for (8.1) in the $x - y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Compare with Fig. 8.1.

This type of bifurcation is referred to as a saddle-node bifurcation (occa-

sionally it may also be referred to as a fold bifurcation or tangent bifurcation, but these terms are used less frequently).

The key characteristic of the saddle-node bifurcation is the following. As a parameter (μ) is varied, the number of equilibria change from zero to two, and the change occurs at a parameter value corresponding to the two equilibria coalescing into one nonhyperbolic equilibrium.

μ is called the bifurcation parameter and $\mu = 0$ is called the bifurcation point.

Example 23 (The Transcritical Bifurcation)**.** *Consider the following nonlinear, autonomous vector field on $\mathbb{R}^2$:*

$$\begin{aligned} \dot{x} &= \mu x - x^2, \\ \dot{y} &= -y, \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{8.5}$$

where μ is a (real) parameter. The equilibrium points of (8.5) are given by:

$$(x, y) = (0, 0), (\mu, 0). \tag{8.6}$$

The Jacobian of the vector field evaluated at each equilibrium point is given by:

$$(0, 0) \qquad \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.7}$$

$$(\mu, 0) \qquad \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.8}$$

from which it follows that $(0, 0)$ is asymptotically stable for $\mu < 0$, and a hyperbolic saddle for $\mu > 0$, and $(\mu, 0)$ is a hyperbolic saddle for $\mu < 0$ and asymptotically stable for $\mu > 0$. These two lines of fixed points cross at $\mu = 0$, at which there is only one, nonhyperbolic fixed point.

In Fig. 8.3 we show the bifurcation diagram for (8.5) in the $\mu - x$ plane.

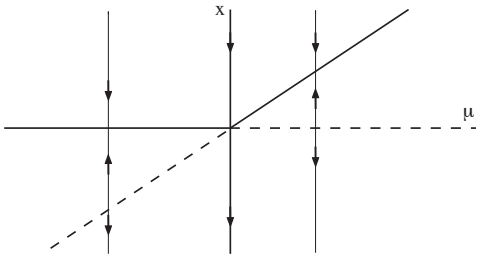


Figure 8.3: Bifurcation diagram for (8.5) in the $\mu - x$ plane. The curves of equilibria are given by $\mu = x$ and $x = 0$. The dashed line denotes unstable equilibria, and the solid line denotes stable equilibria.

In Fig. 8.4 we illustrate the bifurcation of equilibria for (8.5) in the $x - y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$.

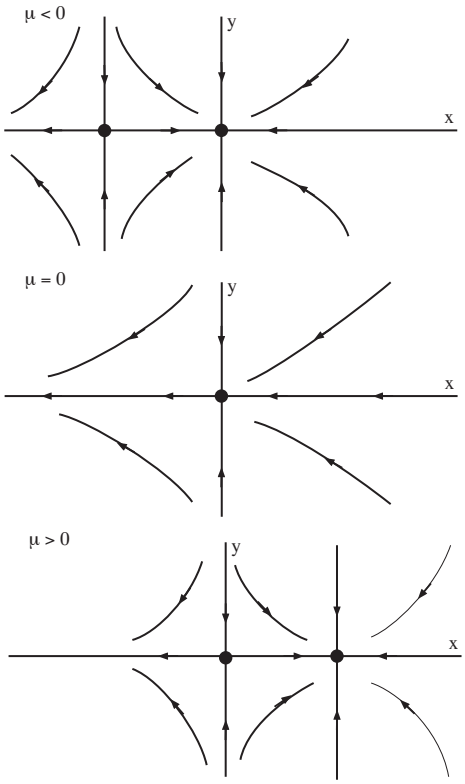


Figure 8.4: Bifurcation diagram for (8.1) in the $x-y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Compare with Fig. 8.3.

This type of bifurcation is referred to as a transcritical bifurcation.

The key characteristic of the transcritical bifurcation is the following. As a parameter (μ) is varied, the number of equilibria change from two to one, and back to two, and the change in number of equilibria occurs at a parameter value corresponding to the two equilibria coalescing into one nonhyperbolic equilibrium.

Example 24 (The (Supercritical) Pitchfork Bifurcation)**.** *Consider the following nonlinear, autonomous vector field on* $\mathbb{R}^2$*:*

$$\begin{aligned} \dot{x} &= \mu x - x^3, \\ \dot{y} &= -y, \qquad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{8.9}$$

where μ is a (real) parameter. The equilibrium points of (8.9) *are given by:*

$$(x,y) = (0,0),\ (\sqrt{\mu},0),\ (-\sqrt{\mu},0) \tag{8.10}$$

The Jacobian of the vector field evaluated at each equilibrium point is given

by:

$$(0,0) \qquad \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.11}$$

$$(\pm\sqrt{\mu},0) \qquad \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.12}$$

from which it follows that $(0,0)$ *is asymptotically stable for* $\mu < 0$*, and a hyperbolic saddle for* $\mu > 0$*, and* $(\pm\sqrt{\mu},0)$ *are asymptotically stable for* $\mu > 0$*, and do not exist for* $\mu < 0$*. These two curves of fixed points pass through zero at* $\mu = 0$*, at which there is only one, nonhyperbolic fixed point.*

In Fig. 8.5 we show the bifurcation diagram for (8.9) in the $\mu - x$ *plane.*

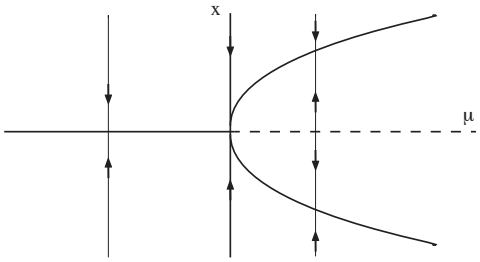


Figure 8.5: Bifurcation diagram for (8.9) in the $\mu - x$ plane. The curves of equilibria are given by $\mu = x^2$, and $x = 0$. The dashed line denotes unstable equilibria, and the solid line denotes stable equilibria.

In Fig. 8.6 we illustrate the bifurcation of equilibria for (8.9) in the $x - y$ *plane for* $\mu < 0$*,* $\mu = 0$*, and* $\mu > 0$*.*

Example 25 (The (Subcritical) Pitchfork Bifurcation)**.** *Consider the following nonlinear, autonomous vector field on* $\mathbb{R}^2$*:*

$$\begin{aligned} \dot{x} &= \mu x + x^3, \\ \dot{y} &= -y, \qquad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{8.13}$$

where μ *is a (real) parameter. The equilibrium points of (8.9) are given by:*

$$(x,y) = (0,0),\ (\sqrt{-\mu},0),\ (-\sqrt{-\mu},0). \tag{8.14}$$

The Jacobian of the vector field evaluated at each equilibrium point is given by:

$$(0,0) \qquad \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.15}$$

$$(\pm\sqrt{-\mu},0) \qquad \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix} \tag{8.16}$$

from which it follows that $(0,0)$ *is asymptotically stable for* $\mu < 0$*, and a hyperbolic saddle for* $\mu > 0$*, and* $(\pm\sqrt{-\mu},0)$ *are hyperbolic saddles for*

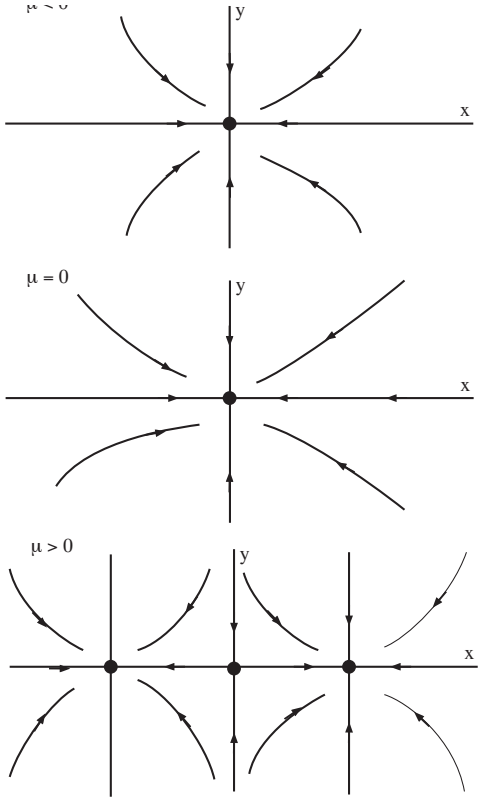


Figure 8.6: Bifurcation diagram for (8.9) in the $x - y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Compare with Fig. 8.5.

$\mu < 0$, and do not exist for $\mu > 0$. These two curves of fixed points pass through zero at $\mu = 0$, at which there is only one, nonhyperbolic fixed point.

In Fig. 8.7 we show the bifurcation diagram for (8.13) in the $\mu - x$ plane.

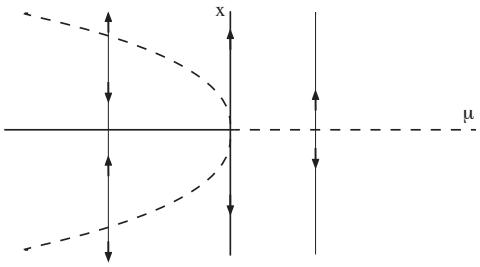


Figure 8.7: Bifurcation diagram for (8.13) in the $\mu - x$ plane. The curves of equilibria are given by $\mu = -x^2$, and $x = 0$. The dashed curves denotes unstable equilibria, and the solid line denotes stable equilibria.

In Fig. 8.8 we illustrate the bifurcation of equilibria for (8.9) in the $x - y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$.

We note that the phrase supercritical pitchfork bifurcation is also referred to as a soft loss of stability and the phrase subcritical pitchfork bifurcation is referred to as a hard loss of stability. What this means is the following. In the supercritical pitchfork bifurcation as μ goes

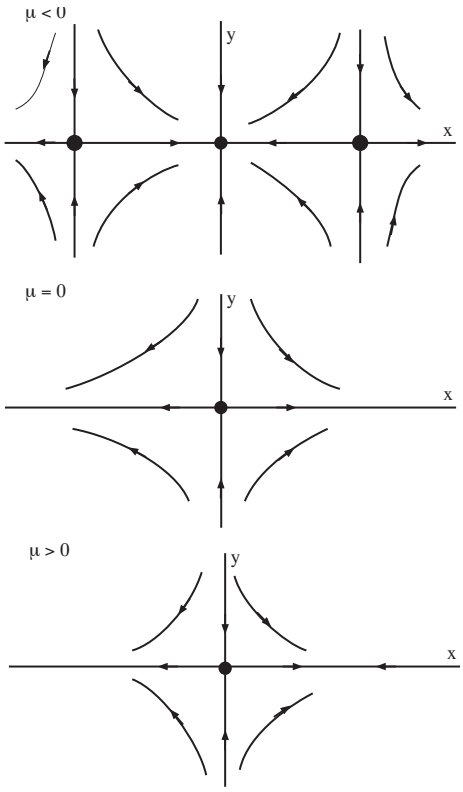


Figure 8.8: Bifurcation diagram for (8.13) in the $x - y$ plane for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Compare with Fig. 8.7.

from negative to positive the equilibrium point loses stability, but as μ increases past zero the trajectories near the origin are bounded in how far away from the origin they can move. In the subcritical pitchfork bifurcation the origin loses stability as μ increases from negative to positive, but trajectories near the unstable equilibrium can become unbounded.

It is natural to ask the question,

''WHAT IS COMMON ABOUT THESE THREE EXAMPLES OF BIFURCATIONS OF FIXED POINTS OF ONE DIMENSIONAL AUTONOMOUS VECTOR FIELDS?'' We note the following.

- A necessary (but *not* sufficient) for bifurcation of a fixed point is nonhyperbolicity of the fixed point.
- The ''nature'' of the bifurcation (e.g. numbers and stability of fixed points that are created or destroyed) is determined by the form of the nonlinearity.

But we could go further and ask what is in common about these examples that could lead to a definition of the bifurcation of a fixed

point for autonomous vector fields? From the common features we give the following definition.

Definition 21 (Bifurcation of a fixed point of a one dimensional autonomous vector field)**.** *We consider a one dimensional autonomous vector field depending on a parameter, μ. We assume that at a certain parameter value it has a fixed point that is not hyperbolic. We say that a bifurcation occurs at that ''nonhyperbolic parameter value" if for μ in a neighborhood of that parameter value the number of fixed points and their stability changes*[1].

[1] Consider this definition in the context of our examples (and pay particular attention to the transcritical bifurcation).

Finally, we finish the discussion of bifurcations of a fixed point of one dimensional autonomous vector fields with an example showing that a nonhyperbolic fixed point may not bifurcate as a parameter is varied, i.e. non-hyperbolicity is a necessary, but not sufficient, condition for bifurcation.

Example 26. *We consider the one dimensional autonomous vector field:*

$$\dot{x} = \mu - x^3, \quad x \in \mathbb{R}, \tag{8.17}$$

where μ is a parameter. This vector field has a nonhyperbolic fixed point at $x = 0$ for $\mu = 0$. The curve of fixed points in the $\mu - x$ plane is given by $\mu = x^3$, and the Jacobian of the vector field is $-3x^2$, which is strictly negative at all fixed points, except the nonhyperbolic fixed point at the origin.

In fig. 8.9 we plot the fixed points as a function of μ.

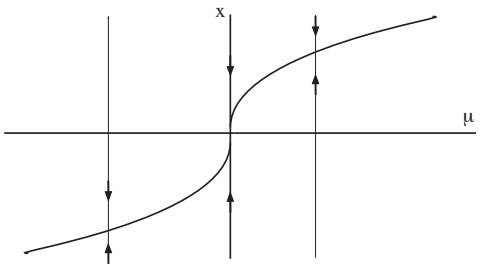


Figure 8.9: Bifurcation diagram for (8.17).

We see that there is no change in the number or stability of the fixed points for $\mu > 0$ and $\mu < 0$. Hence, no bifurcation.

Problem Set 8

1. Consider the following autonomous vector fields on the plane depending on a scalar parameter μ. Verify that each vector field has a fixed point at $(x, y) = (0, 0)$ for $\mu = 0$. Determine the linearized stability of this fixed point. Determine the nature (i.e. stability and number) of the fixed points for μ in a neighborhood of zero. (In other words, carry out a bifurcation analysis.) Sketch the flow in a

neighborhood of each fixed point for values of μ corresponding to changes in stability and/or numbers of fixed points.

(a)

$$\begin{aligned} \dot{x} &= \mu + 10x^2, \\ \dot{y} &= x - 5y. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= \mu x + 10x^2, \\ \dot{y} &= x - 2y. \end{aligned}$$

(c)

$$\begin{aligned} \dot{x} &= \mu x + x^5, \\ \dot{y} &= -y. \end{aligned}$$

9
Bifurcation of Equilibria, II

We have examined fixed points of one dimensional autonomous vector fields where the matrix associated with the linearization of the vector field about the fixed point has a zero eigenvalue.

It is natural to ask ''are there more complicated bifurcations, and what makes them complicated?''. If one thinks about the examples considered so far, there are two possibilities that could complicate the situation. One is that there could be more than one eigenvalue of the linearization about the fixed point with zero real part (which would necessarily require a consideration of higher dimensional vector fields), and the other would be more complicated nonlinearity (or a combination of these two). Understanding how dimensionality and nonlinearity contribute to the ''complexity'' of a bifurcation (and what that might mean) is a very interesting topic, but beyond the scope of this course. This is generally a topic explored in graduate level courses on dynamical systems theory that emphasize bifurcation theory. Here we are mainly just introducing the basic ideas and issues with examples that one might encounter in applications. Towards that end we will consider an example of a bifurcation that is very important in applications–the Poincaré-Andronov-Hopf bifurcation (or just Hopf bifurcation as it is more commonly referred to). This is a bifurcation of a fixed point of an autonomous vector field where the fixed point is nonhyperbolic as a result of the Jacobian having a pair of purely imaginary eigenvalues, $\pm i\,\omega$, $\omega \neq 0$. Therefore this type of bifurcation requires (at least two dimensions), and it is not characterize by a change in the number of fixed points, but by the creation of time dependent periodic solutions. We will analyze this situation by considering a specific example. References solely devoted to the Hopf bifurcation are the books of Marsden and McCracken[1] and Hassard, Kazarinoff, and Wan[2].

We consider the following nonlinear autonomous vector field on the plane:

[1] J.E. Marsden and M. McCracken. *The Hopf bifurcation and its applications*. Applied mathematical sciences. Springer-Verlag, 1976. ISBN 9780387902005. URL https://books.google.cl/books?id=KUHvAAAAMAAJ

[2] Brian D Hassard, Nicholas D Kazarinoff, and Y-H Wan. *Theory and applications of Hopf bifurcation*, volume 41. CUP Archive, 1981

$$\begin{aligned} \dot{x} &= \mu x - \omega y + (ax - by)(x^2 + y^2), \\ \dot{y} &= \omega x + \mu y + (bx + ay)(x^2 + y^2), \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{9.1}$$

where we consider a, b, ω as fixed constants and μ as a variable parameter. The origin, $(x, y) = (0, 0)$ is a fixed point, and we want to consider its stability. The matrix associated with the linearization about the origin is given by:

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}, \tag{9.2}$$

and its eigenvalues are given by:

$$\lambda_{1,2} = \mu \pm i\omega. \tag{9.3}$$

Hence, as a function of μ the origin has the following stability properties:

$$\begin{cases} \mu < 0 & \text{sink} \\ \mu = 0 & \text{center} \\ \mu > 0 & \text{source} \end{cases} \tag{9.4}$$

The origin is not hyperbolic at $\mu = 0$, and there is a change in stability as μ changes sign. We want to analyze the behaviour near the origin, both in phase space and in parameter space, in more detail.

Towards this end we transform (9.1) to polar coordinates using the standard relationship between cartesian and polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{9.5}$$

Differentiating these two expressions with respect to t, and substituting into (9.1) gives:

$$\begin{aligned} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta &= \mu r\cos\theta - \omega r\sin\theta \\ &+ (ar\cos\theta - br\sin\theta)r^2, \end{aligned} \tag{9.6}$$

$$\begin{aligned} \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta &= \omega r\cos\theta + \mu r\sin\theta \\ &+ (br\cos\theta + ar\sin\theta)r^2, \end{aligned} \tag{9.7}$$

from which we obtain the following equations for $\dot{r}$ and $\dot{\theta}$:

$$\dot{r} = \mu r + ar^3, \tag{9.8}$$

$$r\dot{\theta} = \omega r + br^3, \tag{9.9}$$

where (9.8) is obtained by multiplying (9.6) by $\cos\theta$ and (9.7) by $\sin\theta$ and adding the two results, and (9.9) is obtained by multiplying (9.6)

by $-\sin\theta$ and (9.7) by $\cos\theta$ and adding the two results. Dividing both equations by r gives the two equations that we will analyze:

$$\begin{aligned} \dot{r} &= \mu r + a r^3, && (9.10)\\ \dot{\theta} &= \omega + b r^2. && (9.11) \end{aligned}$$

Note that (9.10) has the form of the pitchfork bifurcation that we studied earlier. However, it is very important to realize that we are dealing with the equations in polar coordinates and to understand what they reveal to us about the dynamics in the original cartesian coordinates. To begin with, we must keep in mind that $r \geq 0$.

Note that (9.10) is independent of θ, i.e. it is a one dimensional, autonomous ODE which we rewrite below:

$$\dot{r} = \mu r + a r^3 = r(\mu + a r^2). \tag{9.12}$$

The fixed points of this equation are:

$$r = 0, \qquad r = \sqrt{-\frac{\mu}{a}} \equiv r^+. \tag{9.13}$$

(Keep in mind that $r \geq 0$.) Substituting r^+ into (9.10) and (9.11) gives:

$$\begin{aligned} \dot{r^+} &= \mu r^+ + a(r)^{+3} = 0, \\ \dot{\theta} &= \omega + b\left(-\frac{\mu}{a}\right) && (9.14) \end{aligned}$$

The θ component can easily be solved (using $r^+ = \sqrt{-\frac{\mu}{a}}$), after which we obtain:

$$\theta(t) = \left(\omega - \frac{\mu b}{a}\right) t + \theta(0). \tag{9.15}$$

Therefore r does not change in time at $r = r^+$ and θ evolves linearly in time. But θ is an angular coordinate. This implies that $r = r^+$ corresponds to a periodic orbit[3].

[3] At this point it is very useful to "pause" and think about the reasoning that led to the conclusion that $r = r^+$ is a periodic orbit.

Using this information, we analyze the behavior of (9.12) by constructing the bifurcation diagram. There are two cases to consider: $a > 0$ and $a < 0$.

In figure 9.1 we sketch the zeros of (9.12) as a function of μ for $a > 0$. We see that a periodic orbit bifurcates from the nonhyperbolic fixed point at $\mu = 0$. The periodic orbit is unstable and exists for $\mu < 0$. In Fig. 9.2 we illustrate the dynamics in the $x - y$ phase plane.

In figure 9.3 we sketch the zeros of (9.12) as a function of μ for $a < 0$. We see that a periodic orbit bifurcates from the nonhyperbolic fixed point at $\mu = 0$. The periodic orbit is stable in this case and exists for $\mu > 0$. In Fig. 9.4 we illustrate the dynamics in the $x - y$ phase plane.

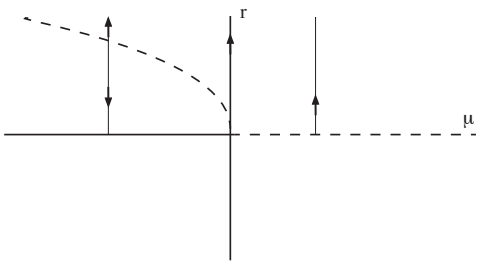


Figure 9.1: The zeros of (9.12) as a function of μ for $a > 0$.

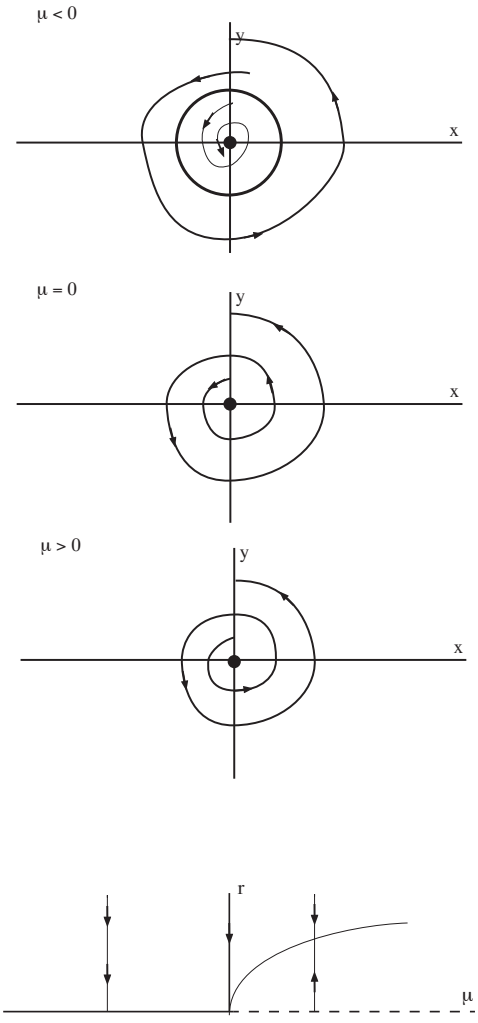


Figure 9.2: Phase plane as a function of μ in the $x - y$ plane for $a > 0$.

Figure 9.3: The zeros of (9.12) as a function of μ for $a < 0$.

In this example we have seen that a nonhyperbolic fixed point of a two dimensional autonomous vector field, where the nonhyperbolicity arises from the fact that the linearization at the fixed point has a pair of pure imaginary eigenvalues, $\pm i\omega$, can lead to the creation of periodic

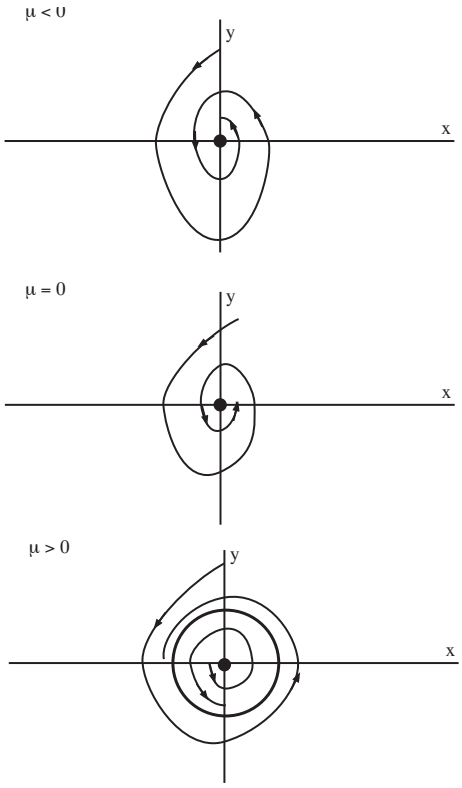


Figure 9.4: Phase plane as a function of μ in the $x - y$ plane for $a < 0$.

orbits as a parameter is varied. This is an example of what is generally called the Hopf bifurcation. It is the first example we have seen of the bifurcation of an equilibrium solution resulting in time-dependent solutions.

At this point it is useful to summarize the nature of the conditions resulting in a Hopf bifurcation for a two dimensional, autonomous vector field. To begin with, we need a fixed point where the Jacobian associated with the linearization about the fixed point has a pair of pure imaginary eigenvalues. This is a necessary condition for the Hopf bifurcation. Now just as for bifurcations of fixed points of one dimensional autonomous vector fields (e.g., the saddle-node, transcritical, and pitchfork bifurcations that we have studied in the previous chapter) the nature of the bifurcation for parameter values in a neighbourhood of the bifurcation parameter is determined by the form of the nonlinearity of the vector field[4]. We developed the idea of the Hopf bifurcation in the context of (9.1), and in that example the stability of the bifurcating periodic orbit was given by the sign of the coefficient a (stable for $a < 0$, unstable for $a > 0$). In the example the stability coefficient was evident from the simple structure of the nonlinearity. In more complicated examples, i.e. more complicated

[4] It would be very insightful to think about this statement in the context of the saddle-node, transcritical, and pitchfork bifurcations that we studied in the previous chapter.

nonlinear terms, the determination of the stability coefficient is more algebraically intensive. Explicit expressions for the stability coefficient are given in many dynamical systems texts. For example, it is given in Guckenheimer and Holmes[5] and Wiggins[6]. The complete details of the calculation of the stability coefficient are carried out in [7]. Problem 2 at the end of this chapter explores the nature of the Hopf bifurcation, e.g. the number and stability of bifurcating periodic orbits, for different forms of nonlinearity.

[5] John Guckenheimer and Philip J Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42. Springer Science & Business Media, 2013

[6] Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media, 2003

[7] Brian D Hassard, Nicholas D Kazarinoff, and Y-H Wan. *Theory and applications of Hopf bifurcation*, volume 41. CUP Archive, 1981

Next, we return to the examples of bifurcations of fixed points in one dimensional vector fields and give two examples of one dimensional vector fields where more than one of the bifurcations we discussed earlier can occur.

Example 27. *Consider the following one dimensional autonomous vector field depending on a parameter μ:*

$$\dot{x} = \mu - \frac{x^2}{2} + \frac{x^3}{3}, \qquad x \in \mathbb{R}. \tag{9.16}$$

The fixed points of this vector field are given by:

$$\mu = \frac{x^2}{2} - \frac{x^3}{3}, \tag{9.17}$$

and are plotted in Fig. 9.5.

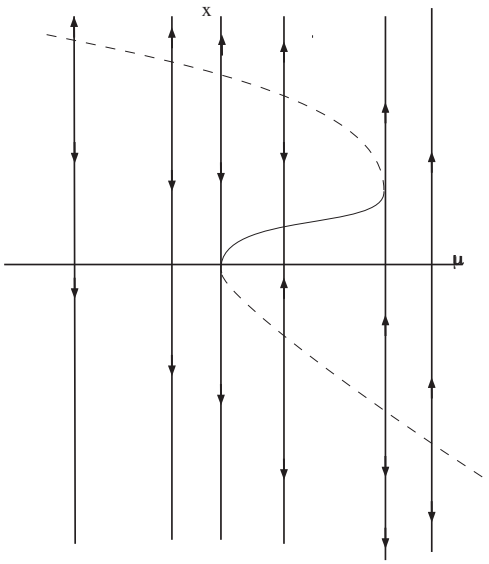


Figure 9.5: Fixed points of (9.16) plotted in the $\mu - x$ plane.

The curve plotted is where the vector field is zero. Hence, it is positive to the right of the curve and negative to the left of the curve. From this we conclude the stability of the fixed points as shown in the figure.

There are two saddle node bifurcations that occur where $\frac{d\mu}{dx}(x) = 0$. *These are located at*

$$(x, \mu) = (0, 0), \left(1, \frac{1}{6}\right). \tag{9.18}$$

Example 28. *Consider the following one dimensional autonomous vector field depending on a parameter* μ*:*

$$\begin{aligned} \dot{x} &= \mu x - \frac{x^3}{2} + \frac{x^4}{3}, \\ &= x\left(\mu - \frac{x^2}{2} + \frac{x^3}{3}\right) \end{aligned} \tag{9.19}$$

The fixed points of this vector field are given by:

$$\mu = \frac{x^2}{2} - \frac{x^3}{3}, \tag{9.20}$$

and

$$x = 0, \tag{9.21}$$

and are plotted in the $\mu - x$ *plane in Fig. 9.6.*

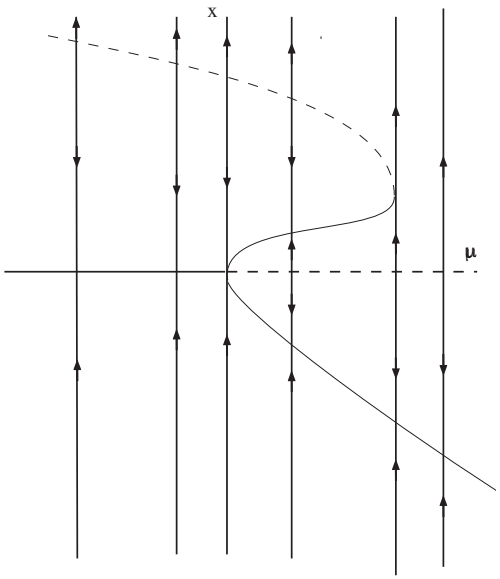


Figure 9.6: Fixed points of (9.19) plotted in the $\mu - x$ plane.

In this example we see that there is a pitchfork bifurcation and a saddle-node bifurcation.

Problem Set 9

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= \mu x - 3y - x(x^2+y^2)^3, \\ \dot{y} &= 3x + \mu y - y(x^2+y^2)^3, \end{aligned}$$

where μ is a parameter. Analyze possible bifurcations at $(x, y) = (0, 0)$ for μ in a neighborhood of zero. (Hint: use polar coordinates.)

2. These exercises are from the book of Marsden and McCracken[8]. Consider the following vector fields expressed in polar coordinates, i.e. $(r, \theta) \in \mathbb{R}^+ \times S^1$, depending on a parameter μ. Analyze the stability of the origin and the stability of all bifurcating periodic orbits as a function of μ.

[8] J.E. Marsden and M. McCracken. *The Hopf bifurcation and its applications*. Applied mathematical sciences. Springer-Verlag, 1976. ISBN 9780387902005. URL https://books.google.cl/books?id=KUHvAAAAMAAJ

(a)

$$\begin{aligned} \dot{r} &= -r(r-\mu)^2, \\ \dot{\theta} &= 1. \end{aligned}$$

(b)

$$\begin{aligned} \dot{r} &= r(\mu - r^2)(2\mu - r^2)^2, \\ \dot{\theta} &= 1. \end{aligned}$$

(c)

$$\begin{aligned} \dot{r} &= r(r+\mu)(r-\mu), \\ \dot{\theta} &= 1. \end{aligned}$$

(d)

$$\begin{aligned} \dot{r} &= \mu r(r^2 - \mu), \\ \dot{\theta} &= 1. \end{aligned}$$

(e)

$$\begin{aligned} \dot{r} &= -\mu^2 r(r+\mu)^2(r-\mu)^2, \\ \dot{\theta} &= 1. \end{aligned}$$

3. Consider the following vector field:

$$\dot{x} = \mu x - \frac{x^3}{2} + \frac{x^5}{4}, \quad x \in \mathbb{R},$$

where μ is a parameter. Classify all bifurcations of equilibria and, in the process of doing this, determine all equilibria and their stability type.

10
Center Manifold Theory

This chapter is about center manifolds, dimensional reduction, and stability of fixed points of autonomous vector fields.[1] We begin with a motivational example.

[1] Expositions of center manifold theory are mostly found in advanced dynamical systems textbooks. It is likely true that all such expositions have their roots in the monographs of Henry and Carr . The monograph of Henry is a bit obscure, but it is a seminal work in the field. The monograph of Carr is a real jewel. All of the theorems in this chapter are taken from Carr, where the proofs can also be found.

D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1993. ISBN 9783540105572. URL https://books.google.cl/books?id=ID3vAAAAMAAJ; and Jack Carr. *Applications of centre manifold theory*, volume 35. Springer Science & Business Media, 2012

Example 29. *Consider the following linear, autonomous vector field on* $\mathbb{R}^c \times \mathbb{R}^s$*:*

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{y} &= By, \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \tag{10.1}$$

where A *is a* $c \times c$ *matrix of real numbers having eigenvalues with zero real part and* B *is a* $s \times s$ *matrix of real numbers having eigenvalues with negative real part. Suppose we are interested in stability of the nonhyperbolic fixed point* $(x,y) = (0,0)$*. Then that question is determined by the nature of stability of* $x = 0$ *in the lower dimensional vector field:*

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^c. \tag{10.2}$$

This follows from the nature of the eigenvalues of B*, and the properties that* x *and* y *are decoupled in* (10) *and that it is linear. More precisely, the solution of* (10) *is given by:*

$$\begin{pmatrix} x(t,x_0) \\ y(t,y_0) \end{pmatrix} = \begin{pmatrix} e^{At}x_0 \\ e^{Bt}y_0 \end{pmatrix}. \tag{10.3}$$

From the assumption of the real parts of the eigenvalues of B *having negative real parts, it follows that:*

$$\lim_{t\to\infty} e^{Bt}y_0 = 0.$$

In fact, 0 *is approached at an exponential rate in time. Therefore it follows that stability, or asymptotic stability, or instability of* $x = 0$ *for* (10.2) *implies stability, or asymptotic stability, or instability of* $(x,y) = (0,0)$ *for* (10).

It is natural to ask if such a dimensional reduction procedure holds for nonlinear systems. This might seem unlikely since, in general, nonlinear systems are coupled and the superposition principle of linear systems does not hold. However, we will see that this is not the case.

INVARIANT MANIFOLDS LEAD TO A FORM OF DECOUPLING THAT RESULTS IN A DIMENSIONAL REDUCTION PROCEDURE *that gives, essentially, the same result as is obtained for this motivational linear example*[2]*.This is the topic of center manifold theory that we now develop.*

[2] In fact, this is one of the main uses of invariant manifolds. They can play an essential role in developing dimensional reduction schemes. The ''manifold" part is important because it is desirable for the reduced dimensional system to have properties where the usual techniques of calculus can be applied.

We begin by describing the set-up. It is important to realize that when applying these results to a vector field, it must be in the following form.

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \\ \dot{y} &= By + g(x,y), \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \tag{10.4}$$

where the matrices A and B are have the following properties:

$A-$ $c \times c$ matrix of real numbers
having eigenvalues with zero real parts,
$B-$ $s \times s$ matrix of real numbers
having eigenvalues with negative real parts,

and f and g are nonlinear functions. That is, they are of order two or higher in x and y, which is expressed in the following properties:

$$\begin{aligned} f(0,0) &= 0, \qquad Df(0,0) = 0, \\ g(0,0) &= 0, \qquad Dg(0,0) = 0, \end{aligned} \tag{10.5}$$

and they are C^r, r as large as required (we will explain what this means when we explicitly use this property later on).

With this set-up $(x,y) = (0,0)$ is a fixed point for (10.4) and we are interested in its stability properties.

The linearization of (10.4) about the fixed point is given by:

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{y} &= By, \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s. \end{aligned} \tag{10.6}$$

The fixed point is nonhyperbolic. It has a c dimensional invariant center subspace and a s dimensional invariant stable subspace given by:

$$E^c = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \,|\, y = 0\}, \tag{10.7}$$

$$E^s = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \,|\, x = 0\}, \tag{10.8}$$

respectively.

For the nonlinear system (10.4) there is a s dimensional, C^r passing through the origin and tangent to E^s at the origin. Moreover, trajectories in the local stable manifold inherit their behavior from trajectories in E^s under the linearized dynamics in the sense that they approach the origin at an exponential rate in time.

Similarly, there is a c dimensional C^r local center manifold that passes through the origin and is tangent to E^c are the origin. Hence, the center manifold has the form:

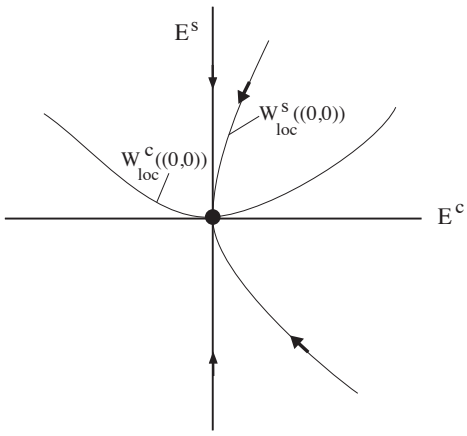
$$W^c(0) = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \,|\, y = h(x),\, h(0) = 0,\, Dh(0) = 0\}\,, \quad (10.9)$$

which is valid in a neighborhood of the origin, i.e. for $|x|$ sufficiently small[3].

We illustrate the geometry in Fig. 10.1.

[3] At this point it would be insightful to consider (10.9) and make sure you understand how it embodies the geometrical properties of the local center manifold of the origin that we have described.

Figure 10.1: The geometry of the stable and center manifolds near the origin.



The application of the center manifold theory for analyzing the behavior of trajectories near the origin is based on three theorems:

- existence of the center manifold and the vector field restricted to the center manifold,
- stability of the origin restricted to the center manifold and its relation to the stability of the origin in the full dimensional phase space,
- obtaining an approximation to the center manifold.

Theorem 5 (Existence and Restricted Dynamics)**.** *There exists a C^r center manifold of $(x,y) = (0,0)$ for (10.4). The dynamics of (10.4) restricted to the center manifold is given by:*

$$\dot{u} = Au + f(u, h(u)), \quad u \in \mathbb{R}^c, \quad (10.10)$$

for $|u|$ sufficiently small.

A natural question that arises from the statement of this theorem is ''why did we use the variable 'u' when it would seem that 'x' would be the more natural variable to use in this situation"? Understanding the answer to this question will provide some insight and understanding to the nature of solutions near the origin and the geometry of the invariant manifolds near the origin. The answer is that '' x and y are already used as variables for describing the coordinate axes in (10.4). We do not want to confuse a point in the center manifold with a point on the coordinate axis. A point on the center manifold is denoted by $(x, h(x))$. The coordinate u denotes a parametric representation of points along the center manifold. Moreover, we will want to compare trajectories of (10.10) with trajectories in (10.4). This will be confusing if x is used to denote a point on the center manifold. However, when computing the center manifold and when considering the (i.e. (10.10)) it is traditional to use the coordinate 'x', i.e. the coordinate describing the points in the center *subspace*. This does not cause ambiguities since we can name a coordinate anything we want. However, it would cause ambiguities when comparing trajectores in (10.4) with trajectories in (10.10), as we do in the next theorem[4].

[4] If this long paragraph is confusing it would be fruitful to spend some time considering each point. There is some useful insight to be gained.

Theorem 6. *i) Suppose that the zero solution of* (10.10) *is stable (asymptotically stable) (unstable), then the zero solution of* (10.4) *is also stable (asymptotically stable) (unstable). ii) Suppose that the zero solution of* (10.10) *is stable. Then if $(x(t), y(t))$ is a solution of* (10.4) *with $(x(0), y(0))$ sufficiently small, then there is a solution $u(t)$ of* (10.10) *such that as $t \to \infty$*

$$\begin{aligned} x(t) &= u(t) + \mathcal{O}(e^{-\gamma t}), \\ y(t) &= h(u(t)) + \mathcal{O}(e^{-\gamma t}), \end{aligned} \tag{10.11}$$

where $\gamma > 0$ is a constant.

Part i) of this theorem says that stability properties of the origin in the center manifold imply *the same* stability properties of the origin in the full dimensional equations. Part ii) gives much more precise results for the case that the origin is stable. It says that trajectories starting at initial conditions sufficiently close to the origin asymptotically approach a trajectory in the center manifold.

Now we would like to compute the center manifold so that we can use these theorems in specific examples. In general, it is not possible to compute the center manifold. However, it is possible to approximate it to ''sufficiently high accuracy" so that we can verify the stability results of Theorem 6 can be confirmed. We will show how this can be done. The idea is to derive an equation that the center manifold must satisfy, and then develop an approximate solution to that equation.

We develop this approach step-by-step.

The center manifold is realized as the graph of a function,

$$y = h(x), \quad x \in \mathbb{R}^c, \, y \in \mathbb{R}^s, \tag{10.12}$$

i.e. any point (x_c, y_c), sufficiently close to the origin, that is in the center manifold satisfies $y_c = h(x_c)$. In addition, the center manfold passes through the origin ($h(0) = 0$) and is tangent to the center subspace at the origin ($Dh(0) = 0$).

Invariance of the center manifold implies that the graph of the function $h(x)$ must also be invariant with respect to the dynamics generated by (10.4). Differentiating (10.12) with respect to time shows that $(\dot{x}, \dot{y})$ at any point on the center manifold satisfies:

$$\dot{y} = Dh(x)\dot{x}. \tag{10.13}$$

This is just the analytical manifestation of the fact that invariance of a surface with respect to a vector field implies that the vector field must be tangent to the surface.[5]

[5] It would be very insightful to think about this statement in the context of the examples from earlier chapters that involved determining invariant sets and invariant manifolds.

We will now use these properties to derive an equation that must be satisfied by the local center manifold.

The starting point is to recall that any point on the local center manifold obeys the dynamics generated by . Substituting $y = h(x)$ into gives:

$$\dot{x} = Ax + f(x, h(x)), \tag{10.14}$$

$$\dot{y} = Bh(x) + g(x, h(x)), \qquad (x, y) \in \mathbb{R}^c \times \mathbb{R}^s. \tag{10.15}$$

Substituting and into the invariance condition $\dot{y} = Dh(x)\dot{x}$ gives:

$$Bh(x) + g(x, h(x)) = Dh(x)\left(Ax + f(x, h(x))\right), \tag{10.16}$$

or

$$Dh(x)\left(Ax + f(x, h(x))\right) - Bh(x) - g(x, h(x)) \equiv \mathcal{N}(h(x) = 0. \tag{10.17}$$

This is an equation for $h(x)$. By construction, the solution implies invariance of the graph of $h(x)$, and we seek a solution satisfying the additional conditions $h(0) = 0$ and $Dh(0) = 0$. The basic result on approximation of the center manifold is given by the following theorem.

Theorem 7 (Approximation). *Let $\phi : \mathbb{R}^c \to \mathbb{R}^s$ be a C^1 mapping with*

$$\phi(0) = 0, \quad D\phi(0) = 0,$$

such that

$$\mathcal{N}(\phi(x)) = \mathcal{O}(|x|^q) \quad as \quad x \to 0,$$

for some $q > 1$. *Then*

$$|h(x) - \phi(x)| = \mathcal{O}(|x|^q) \quad as \quad x \to 0.$$

The theorem states that if we can find an approximate solution of (10.17) to a specified degree of accuracy, then that approximate solution is actually an approximation to the local center manifold, to the same degree of accuracy[6].

We now consider some examples showing how these results are applied.

[6] This answers the question of why we said that the original vector field, (10.4), was C^r, "r as large as required". r will need to be as large as needed in order to obtain a sufficiently accurate approximation to the local center manifold. "Sufficiently accurate" is determined by our ability to deduce stability properties of the zero solution of the vector field restricted to the center manifold.

Example 30. *We consider the following autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= x^2 y - x^5, \\ \dot{y} &= -y + x^2, \quad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{10.18}$$

or, in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 y - x^5 \\ x^2 \end{pmatrix}. \tag{10.19}$$

We are interested in determining the nature of the stability of $(x, y) = (0, 0)$. *The Jacobian associated with the linearization about this fixed point is:*

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which is nonhyperbolic, and therefore the linearization does not suffice to determine stability.

The vector field is in the form of (10.4)

$$\begin{aligned} \dot{x} &= Ax + f(x, y), \\ \dot{y} &= By + g(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{10.20}$$

where

$$A = 0,\ B = -1,\ f(x, y) = x^2 y - x^5,\ g(x, y) = x^2. \tag{10.21}$$

We assume a center manifold of the form:

$$y = h(x) = ax^2 + bx^3 + \mathcal{O}(x^4), \tag{10.22}$$

which satisfies $h(0) = 0$ *("passes through the origin") and* $Dh(0) = 0$ *(tangent to* E^c *at the origin). A center manifold of this type will require the*

vector field to be at least C^3 (hence, the meaning of the phrase C^r, r as large as necessary).

Substituting this expression into the equation for the center manifold (10.17) (using (10.21)) gives:

$$(2ax+3b^2+\mathcal{O}(x^3))(ax^4+bx^5+\mathcal{O}(x^6)-x^5)+ax^2+bx^3+\mathcal{O}(x^4)-x^2=0. \tag{10.23}$$

In order for this equation to be satisfied the coefficients on each power of x must be zero. Through third order this gives:

$$\begin{aligned} x^2 &: a-1=0 \Rightarrow a=1, \\ x^3 &: b=0. \end{aligned} \tag{10.24}$$

Substituting these values into (10.22) gives the following expression for the center manifold through third order:

$$y=x^2+\mathcal{O}(x^4). \tag{10.25}$$

Therefore the vector field restricted to the center manifold is given by:

$$\dot{x}=x^4+\mathcal{O}(x^5). \tag{10.26}$$

Hence, for x sufficiently small, $\dot{x}$ is positive for $x\neq 0$, and therefore the origin is unstable. We illustrate the flow near the origin in Fig. 10.2.

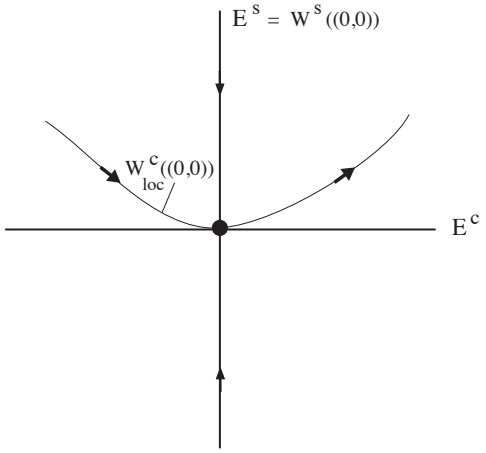


Figure 10.2: The flow near the origin for (10.19).

Example 31. *We consider the following autonomous vector field on the plane:*

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -y+x^3, \quad (x,y)\in\mathbb{R}^2, \end{aligned} \tag{10.27}$$

or, in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy \\ x^3 \end{pmatrix} \tag{10.28}$$

We are interested in determining the nature of the stability of $(x, y) = (0, 0)$. *The Jacobian associated with the linearization about this fixed point is:*

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

which is nonhyperbolic, and therefore the linearization does not suffice to determine stability.

The vector field is in the form of (10)

$$\begin{aligned} \dot{x} &= Ax + f(x, y), \\ \dot{y} &= By + g(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{10.29}$$

where

$$A = 0,\ B = -1,\ f(x, y) = xy,\ g(x, y) = x^3. \tag{10.30}$$

We assume a center manifold of the form:

$$y = h(x) = ax^2 + bx^3 + \mathcal{O}(x^4) \tag{10.31}$$

which satisfies $h(0) = 0$ *("passes through the origin") and* $Dh(0) = 0$ *(tangent to* E^c *at the origin). A center manifold of this type will require the vector field to be at least* C^3 *(hence, the meaning of the phrase* C^r, r *as large as necessary).*

Substituting this expression into the equation for the center manifold (10.17) *(using* (10.30)*) gives:*

$$(2ax + 3b^2 + \mathcal{O}(x^3))(ax^3 + bx^4 + \mathcal{O}(x^5)) + ax^2 + bx^3 + \mathcal{O}(x^4) - x^3 = 0. \tag{10.32}$$

In order for this equation to be satisfied the coefficients on each power of x *must be zero. Through third order this gives:*

$$\begin{aligned} x^2 &: \quad a = 0, \\ x^3 &: \quad b - 1 = 0 \Rightarrow b = 1. \end{aligned} \tag{10.33}$$

Substituting these values into (10.31) *gives the following expression for the center manifold through third order:*

$$y = x^3 + \mathcal{O}(x^4). \tag{10.34}$$

Therefore the vector field restricted to the center manifold is given by:

$$\dot{x} = x^4 + \mathcal{O}(x^5). \tag{10.35}$$

Since $\dot{x}$ is positive for x sufficiently small. the origin is unstable. We illustrate the flow near the origin in Fig. 10.3.

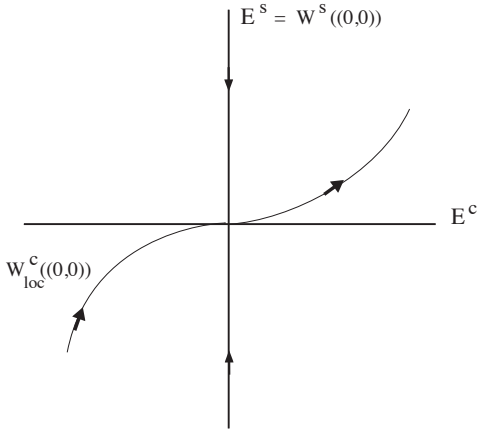


Figure 10.3: The flow near the origin for (10.27).

Problem Set 10

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= x^2 y - x^3, \\ \dot{y} &= -y + x^3, \qquad (x, y) \in \mathbb{R}^2. \end{aligned}$$

   Determine the stability of $(x, y) = (0, 0)$ using center manifold theory[7].

2. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y + x^2, \qquad (x, y) \in \mathbb{R}^2. \end{aligned}$$

   Determine the stability of $(x, y) = (0, 0)$ using center manifold theory. Does the fact that solutions of $\dot{x} = x^2$ "blow up in finite time" influence your answer (why or why not)?

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= -x + y^2, \\ \dot{y} &= -2x^2 + 2xy^2, \qquad (x, y) \in \mathbb{R}^2. \end{aligned}$$

[7] When the equation restricted to the center manifold is one dimensional, then stability can be deduced from the sign of the one dimensional vector field near the origin.

Show that $y = x^2$ is an invariant manifold. Show that there is a trajectory connecting $(0,0)$ to $(1,1)$, i.e. a heteroclinic trajectory.

4. Consider the following autonomous vector field on $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - x^2 y, \\ \dot{z} &= -z + xz^2, \qquad (x,y,z) \in \mathbb{R}^3. \end{aligned}$$

Determine the stability of $(x,y,z) = (0,0,0)$ using center manifold theory[8].

[8] In this problem the vector field restricted to the center manifold is two dimensional. The only techniques we learned for determining stability of a nonhypebolic fixed points for vector fields with more than one dimension was Lyapunov's method and the LaSalle invariance principle.

5. Consider the following autonomous vector field on $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - x^2 y + zxy, \\ \dot{z} &= -z + xz^2, \qquad (x,y,z) \in \mathbb{R}^3. \end{aligned}$$

Determine the stability of $(x,y,z) = (0,0,0)$ using center manifold theory.

6. Consider the following autonomous vector field on $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + zy^2, \\ \dot{z} &= -z + xz^2, \qquad (x,y,z) \in \mathbb{R}^3. \end{aligned}$$

Determine the stability of $(x,y,z) = (0,0,0)$ using center manifold theory.

A
Jacobians, Inverses of Matrices, and Eigenvalues

In this appendix we collect together some results on Jacobians and inverses and eigenvalues of 2×2 matrices that are used repeatedly in the material.

First, we consider the Taylor expansion of a vector valued function of two variables, denoted as follows:

$$H(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad (x,y) \in \mathbb{R}^2. \tag{A.1}$$

More precisely, we will need to Taylor expand such functions through second order:

$$H(x_0+h, y_0+k) = H(x_0,y_0) + DH(x_0,y_0) \begin{pmatrix} h \\ k \end{pmatrix} + \mathcal{O}(2). \tag{A.2}$$

The Taylor expansion of a scalar valued function of one variable should be familiar to most students at this level. Possibly there is less familiarity with the Taylor expansion of a vector valued function of a vector variable. However, to compute this we just Taylor expand each component of the function (which is a scalar valued function of a vector variable) in each variable, holding the other variable fixed for the expansion in that particular variable, and then we gather the results for each component into matrix form.

Carrying this procedure out for the $f(x,y)$ component of (A.1) gives:

$$\begin{aligned} f(x_0+h, y_0+k) &= f(x_0, y_0+k) + \frac{\partial f}{\partial x}(x_0, y_0+k)h + \mathcal{O}(h^2), \\ &= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)k + \mathcal{O}(k^2) + \frac{\partial f}{\partial x}(x_o,y_0)h \\ &+ \mathcal{O}(hk) + \mathcal{O}(h^2). \end{aligned} \tag{A.3}$$

The same procedure can be applied to $g(x,y)$. Recombining the terms back into the vector expresson for (A.1) gives:

$$\begin{aligned} H(x_0+h, y_0+k) &= \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + \mathcal{O}(2). \end{aligned} \tag{A.4}$$

Hence, the Jacobian of (A.1) at (x_0, y_0) is:

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}, \tag{A.5}$$

which is a 2×2 matrix of real numbers.

We will need to compute the inverse of such matrices, as well as its eigenvalues.

We denote a general 2×2 matrix of real numbers:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}. \tag{A.6}$$

It is easy to verify that the inverse of A is given by:

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{A.7}$$

Let $\mathbb{I}$ denote the 2×2 identity matrix. Then the eigenvalues of A are the solutions of the characteristic equation:

$$\det\,(A - \lambda \mathbb{I}) = 0. \tag{A.8}$$

where ''det" is notation for the determinant of the matrix. This is a quadratic equation in λ which has two solutions:

$$\lambda_{1,2} = \frac{\operatorname{tr} A}{2} \pm \frac{1}{2}\sqrt{(\operatorname{tr} A)^2 - 4 \det A}, \tag{A.9}$$

where we have used the notation:

$$\operatorname{tr} A \equiv \text{trace}\, A = a + d, \qquad \det A \equiv \text{determinant}\, A = ad - bc.$$

B
Integration of Some Basic Linear ODEs

In this appendix we collect together a few common ideas related to solving, explicitly, linear inhomogeneous differential equations. Our discussion is organized around a series of examples.

Example 32. *Consider the one dimensional, autonomous linear vector field:*

$$\dot{x} = ax, \quad x, a \in \mathbb{R}. \tag{B.1}$$

We often solve problems in mathematics by transforming them into simpler problems that we already know how to solve. Towards this end, we introduce the following (time-dependent) transformation of variables:

$$x = ue^{at}. \tag{B.2}$$

Differentiating this expression with respect to t, and using (B.1), gives the following ODE for u:

$$\dot{u} = 0, \tag{B.3}$$

which is trivial to integrate, and gives:

$$u(t) = u(0), \tag{B.4}$$

and it is easy to see from (36) that:

$$u(0) = x(0). \tag{B.5}$$

Using (36), as well as (B.4) and (B.5), it follows that:

$$x(t)e^{-at} = u(t) = u(0) = x(0), \tag{B.6}$$

or

$$x(t) = x(0)e^{at}. \tag{B.7}$$

Example 33. *Consider the following linear inhomogeneous nonautonomous ODE (due to the presence of the term $b(t)$):*

$$\dot{x} = ax + b(t), \quad a, x \in \mathbb{R}, \tag{B.8}$$

where $b(t)$ is a scalar valued function of t, whose precise properties we will consider a bit later. We will use exactly the same strategy and change of coordinates as in the previous example:

$$x = ue^{at}. \tag{B.9}$$

Differentiating this expression with respect to t, and using (B.8), gives:

$$\dot{u} = e^{-at}b(t). \tag{B.10}$$

(Compare with (B.3).) Integrating (B.10) gives:

$$u(t) = u(0) + \int_0^t e^{-at'}b(t')dt'. \tag{B.11}$$

Now using (B.9) (with the consequence $u(0) = x(0)$) with (B.11) gives:

$$x(t) = x(0)e^{at} + e^{at}\int_0^t e^{-at'}b(t')dt'. \tag{B.12}$$

Finally, we return to the necessary properties of $b(t)$ in order for this unique solution of (B.8) to ''make sense''. Upon inspection of (B.12) it is clear that all that is required is for the integrals involving $b(t)$ to be well-defined. Continuity is a sufficient condition.

Example 34. *Consider the one dimensional, nonautonomous linear vector field:*

$$\dot{x} = a(t)x, \quad x \in \mathbb{R}, \tag{B.13}$$

where $a(t)$ is a scalar valued function of t whose precise properties will be considered later. The similarity between (B.1) and (B.13) should be evident. We introduce the following (time-dependent) transformation of variables (compare with (36)):

$$x = ue^{\int_0^t a(t')dt'}. \tag{B.14}$$

Differentiating this expression with respect to t, and substituting (B.13) into the result gives:

$$\begin{aligned} \dot{x} &= \dot{u}e^{\int_0^t a(t')dt'} + ua(t)e^{\int_0^t a(t')dt'}, \\ &= \dot{u}e^{\int_0^t a(t')dt'} + a(t)x, \end{aligned} \tag{B.15}$$

which reduces to:

$$\dot{u} = 0. \tag{B.16}$$

Integrating this expression, and using (B.14), gives:

$$u(t) = u(0) = x(0) = x(t)e^{-\int_0^t a(t')dt'}, \tag{B.17}$$

or

$$x(t) = x(0)e^{\int_0^t a(t')dt'}. \tag{B.18}$$

As in the previous example, all that is required for the solution to be well-defined is for the integrals involving $a(t)$ to exist. Continuity is a sufficient condition.

Example 35. *Consider the one dimensional inhomogeneous nonautonomous linear vector field:*

$$\dot{x} = a(t)x + b(t), \quad x \in \mathbb{R}, \tag{B.19}$$

where $a(t)$, $b(t)$ are scalar valued functions whose required properties will be considered at the end of this example. We make the same transformation as (B.14):

$$x = ue^{\int_0^t a(t')dt'}, \tag{B.20}$$

from which we obtain:

$$\dot{u} = b(t)e^{-\int_0^t a(t')dt'}. \tag{B.21}$$

Integrating this expression gives:

$$u(t) = u(0) + \int_0^t b(t')e^{-\int_0^{t'} a(t'')dt''}dt'. \tag{B.22}$$

Using gives:

$$x(t)e^{-\int_0^t a(t')dt'} = x(0) + \int_0^t b(t')e^{-\int_0^{t'} a(t'')dt''}dt', \tag{B.23}$$

or

$$x(t) = x(0)e^{\int_0^t a(t')dt'} + e^{\int_0^t a(t')dt'}\int_0^t b(t')e^{-\int_0^{t'} a(t'')dt''}dt'. \tag{B.24}$$

As in the previous examples, that all that is required is for the integrals involving $a(t)$ and $b(t)$ to be well-defined. Continuity is a sufficient condition.

The previous examples were all one dimensional. Now we will consider two n dimensional examples.

Example 36. *Consider the n dimensional autonomous linear vector field:*

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n, \tag{B.25}$$

where A is a $n \times n$ matrix of real numbers. We make the following transformation of variables (compare with):

$$x = e^{At}u. \tag{B.26}$$

Differentiating this expression with respect to t, and using (B.25), gives:

$$\dot{u} = 0. \tag{B.27}$$

Integrating this expression gives:

$$u(t) = u(0). \tag{B.28}$$

Using (B.26) with (B.28) gives:

$$u(t) = e^{-At}x(t) = u(0) = x(0), \tag{B.29}$$

from which it follows that:

$$x(t) = e^{At}x(0). \tag{B.30}$$

Example 37. *Consider the n dimensional inhomogeneous nonautonomous linear vector field:*

$$\dot{x} = Ax + g(t), \quad x \in \mathbb{R}^n, \tag{B.31}$$

where $g(t)$ is a vector valued function of t whose required properties will be considered later on. We use the same transformation as in the previous example:

$$x = e^{At}u. \tag{B.32}$$

Differentiating this expression with respect to t, and using (B.31), gives:

$$\dot{u} = e^{-At}g(t), \tag{B.33}$$

from which it follows that:

$$u(t) = u(0) + \int_0^t e^{-At'}g(t')dt', \tag{B.34}$$

or, using (B.32):

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-At'}g(t')dt'. \tag{B.35}$$

C
Finding Lyapunov Functions

Lyapunov's method and the LaSalle invariance principle are very poweful techniques, but the obvious question always arises, ''how do I find *the* Lyapunov function? The unfortunate answer is that given an arbitrary ODE there is no general method to find a Lyapunov function appropriate for a given ODE for the application of these methods.

In general, to determine a Lyapunov function appropriate for a given ODE the ODE must have a structure that lends itself to the construction of the Lyapunov function. Therefore the next question is ''what is this structure?'' If the ODE arises from physical modelling there may be an ''energy function'' that is ''almost conserved''. What this means is that when certain terms of the ODE are neglected the resulting ODE has a conserved quantity, i.e. a scalar valued function whose time derivative along trajectories is zero, and this conserved quantity may be a candidate to for a Lyapunov function. If that sounds vague it is because the construction of Lyapunov functions often requires a bit of ''mathematical artistry''. We will consider this procedure with some examples. Energy methods are important techniques for understanding stability issues in science and engineering; see, for example see the book by Langhaar[1] and the article by Maschke[2].

To begin, we consider Newton's equations for the motion of a particle of mass m under a conservative force in one dimension:

$$m\ddot{x} = -\frac{d\Phi}{dx}(x), \quad x \in \mathbb{R}. \tag{C.1}$$

Writing this as a first order system gives:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{m}\frac{d\Phi}{dx}(x). \end{aligned} \tag{C.2}$$

It is easy to see that the time derivative of the following function is

[1] Henry Louis Langhaar. *Energy methods in applied mechanics*. John Wiley & Sons Inc, 1962

[2] Bernhard Maschke, Romeo Ortega, and Arjan J Van Der Schaft. Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation. *IEEE Transactions on automatic control*, 45(8): 1498–1502, 2000

zero

$$E = \frac{my^2}{2} + \Phi(x), \tag{C.3}$$

since

$$\begin{aligned} \dot{E} &= my\dot{y} + \frac{d\Phi}{dx}(x)\dot{x}, \\ &= -y\frac{d\Phi}{dx}(x) + y\frac{d\Phi}{dx}(x) = 0. \end{aligned} \tag{C.4}$$

In terms of dynamics, the function (C.3) has the interpretation as the conserved kinetic energy associated with (C.1).

Now we will consider several examples. In all cases we will simplify matters by taking $m = 1$.

Example 38. *Consider the following autonomous vector field on* $\mathbb{R}^2$*:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \delta y, \quad \delta \geq 0, \quad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{C.5}$$

For $\delta = 0$ (C.5) *has the form of* (C.1)*:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x, \quad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{C.6}$$

with

$$E = \frac{y^2}{2} + \frac{x^2}{2}. \tag{C.7}$$

It is easy to verify that $\frac{dE}{dt} = 0$ *along trajectories of* (C.6)*.*

Now we differentiate E along trajectories of (C.5) *and obtain:*

$$\frac{dE}{dt} = -\delta y^2. \tag{C.8}$$

(C.6) *has only one equilibrium point located at the origin. E is clearly positive everywhere, except for the origin, where it is zero. Using E as a Lyapunov function we can conclude that the origin is Lyapunov stable. If we use E to apply the LaSalle invariance principle, we can conclude that the origin is asymptotically stable. Of course, in this case we can linearize and conclude that the origin is a hyperbolic sink for* $\delta > 0$*.*

Example 39. *Consider the following autonomous vector field on* $\mathbb{R}^2$*:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \delta y, \quad \delta \geq 0, \quad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{C.9}$$

For $\delta = 0$ (C.9) has the form of (C.1):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3, \quad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{C.10}$$

with

$$E = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}. \tag{C.11}$$

It is easy to verify that $\frac{dE}{dt} = 0$ along trajectories of (C.10).

The question now is how we will use E to apply Lyapunov's method or the LaSalle invariance principle? (C.9) has three equilibrium points, a hyperbolic saddle at the origin for $\delta \geq 0$ and hyperbolic sinks at $(x, y) = (\pm 1, 0)$ for $\delta > 0$ and centers for $\delta = 0$. So linearization gives us complete information for $\delta > 0$. For $\delta = 0$ linearization is sufficient to allow is to conclude that the origin is a saddle. The equilibria $(x, y) = (\pm 1, 0)$ are Lyapunov stable for $\delta = 0$, but an argument involving the function E would be necessary in order to conclude this. Linearization allows us to conclude that the equilibria $(x, y) = (\pm 1, 0)$ are asymptotically stable for $\delta > 0$.

The function E can be used to apply the LaSalle invariance principle to conclude that for $\delta > 0$ all trajectories approach one of the three equilibria as $t \to \infty$.

D
Center Manifolds Depending on Parameters

In this appendix we describe the situation of center manifolds that depend on a parameter. The theoretical framework plays an important role in bifurcation theory.

As when we developed the theory earlier, we begin by describing the set-up. As before, it is important to realize that when applying these results to a vector field, it must be in the following form.

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \mu), \\ \dot{y} &= By + g(x, y, \mu), \qquad (x, y, \mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p, \end{aligned} \tag{D.1}$$

where $\mu \in \mathbb{R}^p$ is a vector of parameters and the matrices A and B are have the following properties:

$A-$ $c \times c$ matrix of real numbers
having eigenvalues with zero real parts,
$B-$ $s \times s$ matrix of real numbers
having eigenvalues with negative real parts,

and f and g are nonlinear functions. That is, they are of order two or higher in x, y and μ, which is expressed in the following properties:

$$\begin{aligned} f(0,0,0) &= 0, \qquad Df(0,0,0) = 0, \\ g(0,0,0) &= 0, \qquad Dg(0,0,0) = 0, \end{aligned} \tag{D.2}$$

and they are C^r, r as large as is required to compute an adequate approximation the center manifold. With this set-up $(x, y, \mu) = (0, 0, 0)$ is a fixed point for (D.1) and we are interested in its stability properties.

The conceptual ''trick" that reveals the nature of the parameter dependence of center manifolds is to include the parameter μ as a new *dependent* variable:[1]

[1] At this point it may be useful to go back to the first chapter and recall how nonlinearity of an ODE is defined in terms of the dependent variable.

$$\begin{aligned} \dot{x} &= Ax + f(x,y,\mu), \\ \dot{\mu} &= 0, \\ \dot{y} &= By + g(x,y,\mu), \qquad (x,y,\mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p, \end{aligned} \tag{D.3}$$

The linearization of (D.3) about the fixed point is given by:

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{\mu} &= 0, \\ \dot{y} &= By, \qquad (x,y,\mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p. \end{aligned} \tag{D.4}$$

Even after increasing the dimension of the phase space by p dimensions by including the parameters as new dependent variables, the fixed point $(x,y,\mu) = (0,0,0)$ remains a nonhyperbolic fixed point. It has a $c+p$ dimensional invariant center subspace and a s dimensional invariant stable subspace given by:

$$E^c = \{(x,y,\mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p \,|\, y = 0\}, \tag{D.5}$$

$$E^s = \{(x,y,\mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p \,|\, x = 0,\ \mu = 0\}, \tag{D.6}$$

respectively.

It should be clear that center manifold theory, as we have already developed, applies to (D.3). Including the parameters, μ as additional dependent variables has the effect of increasing the dimension of the ''center variables", but there is also an important consequence. Since μ are now *dependent* variables they enter in to the determination of the nonlinear terms in the equations. In particular, terms of the form

$$x_i^\ell \mu_j^m y_k^n,$$

now are interpreted as *nonlinear terms*, when $\ell + m + n > 1$, for nonnegative integers ℓ, m, n. We will see this in the example below.

Now we consider the information that center manifold theory provides us near the origin of (D.3).

1. In a neighborhood of the origin there exists a C^r center manifold that is represented as the graph of a function over the center variables, $h(x,\mu)$, it passes through the origin ($h(0,0) = 0$) and is tangent to the center subspace at the origin ($Dh(0,0) = 0$)

2. All solutions sufficiently close to the origin are attracted to a trajectory in the center manifold at an exponential rate.

3. The center manifold can be approximated by a power series expansion.

It is significant that the center manifold is defined in a neighborhood of the origin in both the x *and* μ coordinates since $\mu = 0$ is a *bifurcation value*. This means that *all bifurcating solutions* are contained in the center manifold. This is why, for example, that without loss of generality bifurcations from a single zero eigenvalue can be described by a parametrized family of *one* dimensional vector fields[2].

[2] This is a very significant statement and it explains why "bifurcation problems" are amenable to dimensional reduction. In particular, and understanding of the nature of bifurcation of equilibria for autonomous vector fieldscan be reduced to a lower dimensional problem, where the dimension of the problem is equal to the number of eigenvalues with zero real part.

Example 40. *We now consider an example which was exercise 1b from Problem Set 8.*

$$\begin{aligned} \dot{x} &= \mu x + 10x^2, \\ \dot{\mu} &= 0, \\ \dot{y} &= x - 2y, \qquad (x, y) \in \mathbb{R}^2,\ \mu \in \mathbb{R}. \end{aligned} \tag{D.7}$$

The Jacobian associated with the linearization about $(x, \mu, y) = (0, 0, 0)$ *is given by:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}. \tag{D.8}$$

It is easy to check that the eigenvalues of this matrix are 0, 0, *and* -2 *(as we would have expected). Each of these eigenvalues has an eigenvector. It is easily checked that two eigenvectors corresponding to the eigenvalue* 0 *are given by:*

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \tag{D.9}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{D.10}$$

and an eigenvector corresponding to the eigenvalue -2 *is given by:*

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{D.11}$$

From these eigenvectors we form the transformation matrix

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{D.12}$$

with inverse

$$T^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}. \tag{D.13}$$

The transformation matrix, T, defines the following transformation of the dependent variables of (D.7)*:*

$$\begin{pmatrix} x \\ \mu \\ y \end{pmatrix} = T \begin{pmatrix} u \\ \mu \\ v \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \mu \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ \mu \\ u+v \end{pmatrix}. \tag{D.14}$$

It then follows that the transformed vector field has the form:

$$\begin{pmatrix} \dot{u} \\ \dot{\mu} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ \mu \\ v \end{pmatrix} + T^{-1} \begin{pmatrix} \mu(2u) + 10(2u)^2 \\ 0 \\ 0 \end{pmatrix}, \tag{D.15}$$

or

$$\begin{pmatrix} \dot{u} \\ \dot{\mu} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ \mu \\ v \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu u + 40u^2 \\ 0 \\ 0 \end{pmatrix}, \tag{D.16}$$

or

$$\begin{aligned} \dot{u} &= \mu u + 20u^2, \\ \dot{\mu} &= 0, \\ \dot{v} &= -2v - \mu u - 20u^2 \end{aligned} \tag{D.17}$$

E
Dynamics of Hamilton's Equations

In this appendix we give a brief introduction to some of the characteristics and results associated with Hamiltonian differential equations (or, Hamilton's equations or Hamiltonian vector fields). The Hamiltonian formulation of Newton's equations reveals a great deal of structure about dynamics and it also gives rise to a large amount of deep mathematics that is the focus of much contemporary research. .

Our purpose here is not to derive Hamilton's equations from Newton's equations. Discussions of that can be found in many textbooks on mechanics (although it is often considered ''advanced mechanics''). For example, a classical exposition of this topic can be found in the classic book of Landau[1], and more modern expositions can be found in Abraham and Marsden[2] and Arnold[3]. Rather, our approach is to start with Hamilton's equations and to understand some simple aspects and consequences of the special structure associated with Hamilton's equations. Towards this end, our starting point will be Hamilton's equations. Keeping with the simple approach throughout these lectures, our discussion of Hamilton's equations will be for two dimensional systems.

[1] LD Landau and EM Lifshitz. Classical mechanics, 1960

[2] Ralph Abraham and Jerrold E Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978

[3] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013

We begin with a scalar valued function defined on $\mathbb{R}^2$

$$H = H(q,p), \quad (q,p) \in \mathbb{R}^2. \tag{E.1}$$

This function is referred to as the *Hamiltonian*. From the Hamiltonian, Hamilton's equations take the following form:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p), \quad (q,p) \in \mathbb{R}^2. \end{aligned} \tag{E.2}$$

The form of Hamilton's equations implies that the Hamiltonian is constant on trajectories. This can be seen from the following calculation:

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}, \\ &= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0. \end{aligned} \tag{E.3}$$

Furthermore, this calculation implies that the level sets of the Hamiltonian are invariant manifolds. We denote the level set of the Hamiltonian as:

$$H_E = \left\{ (q,p) \in \mathbb{R}^2 \,|\, H(q,p) = E \right\} \tag{E.4}$$

In general, the level set is a curve (or possibly an equilibrium point). Hence, in the two dimensional case, the trajectories of Hamilton's equations are given by the level sets of the Hamiltonian.

The Jacobian of the Hamiltonian vector field (E.2), denoted J, is given by:

$$J(q,p) \equiv \begin{pmatrix} \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial q^2} & -\frac{\partial^2 H}{\partial p \partial q} \end{pmatrix}, \tag{E.5}$$

at an arbitrary point $(q,p) \in \mathbb{R}^2$. Note that the trace of $J(q,p)$, denoted $\mathrm{tr}J(q,p)$, is zero. This implies that the eigenvalues of $J(q,p)$, denoted by $\lambda_{1,2}$, are given by:

$$\lambda_{1,2} = \pm\sqrt{-\det J(q,p)}, \tag{E.6}$$

where $\det J(q,p)$ denotes the determinant of $J(q,p)$. Therefore, if (q_0,p_0) is an equilibrium point of (E.1) and $\det J(q_0,p_0) \neq 0$, then the equilibrium point is a center for $\det J(q_0,p_0) > 0$ and a saddle for $\det J(q_0,p_0) < 0$. [4]

[4] Constraints on the eigenvalues of the matrix associated with the linearization of a Hamiltonian vector field at a fixed point in higher dimensions are described in Abraham and Marsden or Wiggins .

Ralph Abraham and Jerrold E Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978; and Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media, 2003

Next we describe some examples of two dimensional, linear autonomous Hamiltonian vector fields.

Example 41. *The Hamiltonian Saddle*

We consider the Hamiltonian:

$$H(q,p) = \frac{\lambda}{2}\left(p^2 - q^2\right) = \frac{\lambda}{2}(p-q)(p+q), \quad (q,p) \in \mathbb{R}^2 \tag{E.7}$$

with $\lambda > 0$. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) = \lambda p, \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) = \lambda q, \end{aligned} \tag{E.8}$$

or in matrix form:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{E.9}$$

The origin is a fixed point, and the eigenvalues associated with the linearization are given by $\pm\lambda$. Hence, the origin is a saddle point.The value of the Hamiltonian at the origin is zero. We also see from (E.7) that the Hamiltonian is zero on the lines $p - q = 0$ and $p + q = 0$. These are the unstable and stable manifolds of the origin, respectively. The phase portrait is illustrated in Fig. E.1.

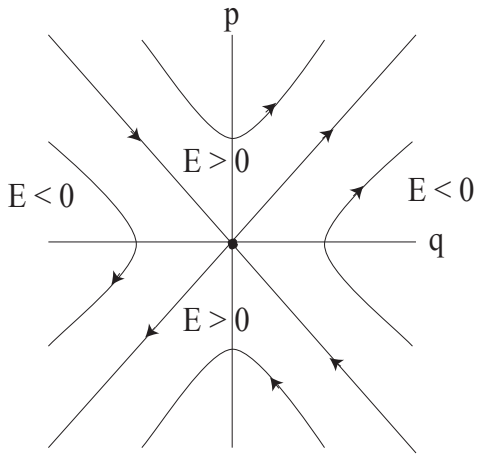


Figure E.1: The phase portrait of the linear Hamiltonian saddle. The stable manifold of the origin is given by $p - q = 0$ and the unstable manifold of the origin is given by $p + q = 0$.

The flow generated by this vector field is given in Chapter 2, Problem Set 2, problem 6.

Example 42 (The Hamiltonian Center)**.** *We consider the Hamiltonian:*

$$H(q,p) = \frac{\omega}{2}\left(p^2 + q^2\right), \quad (q,p) \in \mathbb{R}^2 \tag{E.10}$$

with $\omega > 0$. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) = \omega p, \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) = -\omega q, \end{aligned} \tag{E.11}$$

or, in matrix form:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{E.12}$$

The level sets of the Hamiltonian are circles, and are illustrated in Fig. E.2.

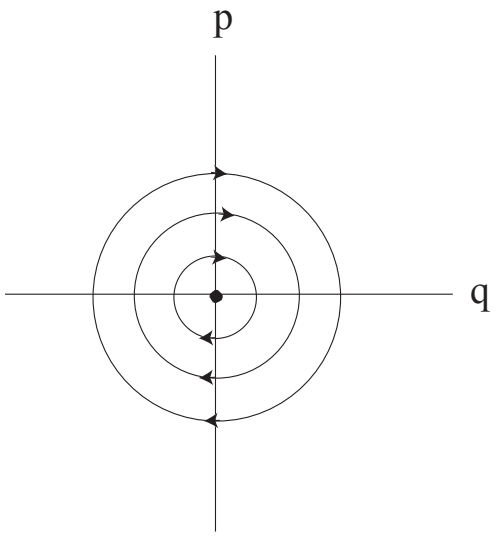


Figure E.2: The phase portrait for the linear Hamiltonian center.

The flow generated by this vector field is given in Chapter 2, Problem Set 2, problem 5.

We will now consider two examples of bifurcation of equilibria in two dimensional Hamiltonian systems. Bifurcation associated with one zero eigenvalue (as we studied in Chapter 8) is not possible since, following (E.6), if there is one zero eigenvalue the other eigenvalue must also be zero. We will consider examples of the Hamiltonian saddle-node and Hamiltonian pitchfork bifurcations. Discussions of the Hamiltonian versions of these bifurcations can also be found in Golubitsky et al.[5].

[5] Martin Golubitsky, Ian Stewart, and Jerrold Marsden. Generic bifurcation of hamiltonian systems with symmetry. *Physica D: Nonlinear Phenomena*, 24(1-3): 391–405, 1987

Example 43 (Hamiltonian saddle-node bifurcation)**.** *We consider the Hamiltonian:*

$$H(q,p) = \frac{p^2}{2} - \lambda q + \frac{q^3}{3}, \quad (q,p) \in \mathbb{R}^2. \tag{E.13}$$

where λ is considered to be a parameter that can be varied. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial q} = \lambda - q^2. \end{aligned} \tag{E.14}$$

The fixed points for (E.14) are:

$$(q,p) = (\pm\sqrt{\lambda}, 0), \tag{E.15}$$

from which it follows that there are no fixed points for $\lambda < 0$, one fixed point for $\lambda = 0$, and two fixed points for $\lambda > 0$. This is the scenario for a saddle-node bifurcation.

Next we examine stability of the fixed points. The Jacobian of (E.14) *is given by:*

$$\begin{pmatrix} 0 & 1 \\ -2q & 0 \end{pmatrix}. \tag{E.16}$$

The eigenvalues of this matrix are:

$$\lambda_{1,2} = \pm\sqrt{-2q}.$$

Hence $(q,p) = (-\sqrt{\lambda}, 0)$ is a saddle, $(q,p) = (\sqrt{\lambda}, 0)$ is a center, and $(q,p) = (0,0)$ has two zero eigenvalues. The phase portraits are shown in Fig. E.3.

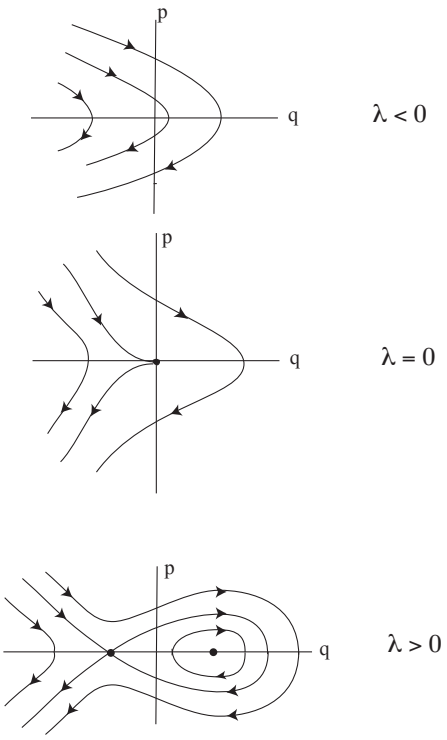


Figure E.3: The phase portraits for the Hamiltonian saddle-node bifurcation.

Example 44 (Hamiltonian pitchfork bifurcation)**.** *We consider the Hamiltonian:*

$$H(q,p) = \frac{p^2}{2} - \lambda\frac{q^2}{2} + \frac{q^4}{4}, \tag{E.17}$$

where λ is considered to be a parameter that can be varied. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial q} = \lambda q - q^3. \end{aligned} \tag{E.18}$$

The fixed points for (E.18) *are:*

$$(q, p) = (0, 0),\ (\pm\sqrt{\lambda}, 0), \tag{E.19}$$

from which it follows that there is one fixed point for $\lambda < 0$*, one fixed point for* $\lambda = 0$*, and three fixed points for* $\lambda > 0$*. This is the scenario for a pitchfork bifurcation.*

Next we examine stability of the fixed points. The Jacobian of (E.18) *is given by:*

$$\begin{pmatrix} 0 & 1 \\ \lambda - 3q^2 & 0 \end{pmatrix}. \tag{E.20}$$

The eigenvalues of this matrix are:

$$\lambda_{1,2} = \pm\sqrt{\lambda - 3q^2}.$$

Hence $(q, p) = (0, 0)$ *is a center for* $\lambda < 0$*, a saddle for* $\lambda > 0$ *and has two zero eigenvalues for* $\lambda = 0$*. The fixed points* $(q, p) = (\sqrt{\lambda}, 0)$ *are centers for* $\lambda > 0$*. The phase portraits are shown in Fig. E.4.*

We remark that, with a bit of thought, it should be clear that in *two dimensions* there is no analog of the Hopf bifurcation for Hamiltonian vector fields similar to to the situation we analyzed earlier in the non-Hamiltonian context. There is a situation that is referred to as the Hamiltonian Hopf bifurcation, but this notion requires at least four dimensions, see Van Der Meer[6].

In Hamiltonian systems a natural bifurcation parameter is the value of the level set of the Hamiltonian, or the ''energy''. From this point of view perhaps a more natural candidate for a Hopf bifurcation in a Hamiltonian system is described by the Lyapunov subcenter theorem, see Kelley[7]. The setting for this theorem also requires at least four dimensions, but the associated phenomena occur quite often in applications.

[6] Jan-Cees Van Der Meer. *The Hamiltonian Hopf bifurcation*. Springer, 1985

[7] Al Kelley. On the liapounov subcenter manifold. *Journal of mathematical analysis and applications*, 18(3):472–478, 1967

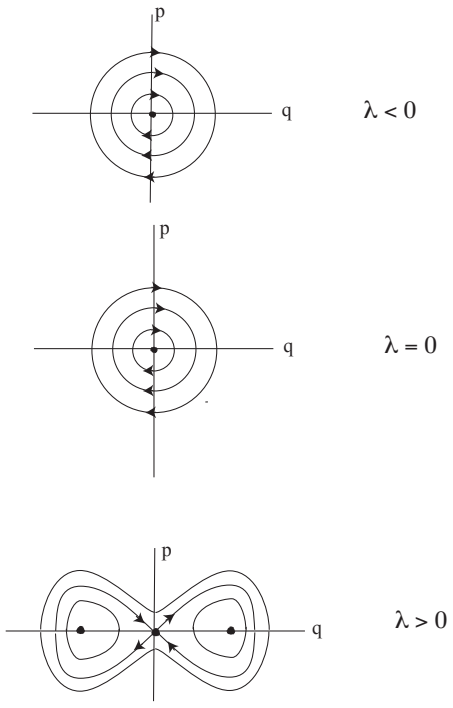


Figure E.4: The phase portraits for the Hamiltonian pitchfork bifurcation.

F
A Brief Introduction to the Characteristics of Chaos

In this appendix we will describe some aspects of the phenomenon of chaos as it arises in ODEs. Chaos is one of those notable topics that crosses disciplinary boundaries in mathematics, science, and engineering and captures the intrigue and curiousity of the general public. Numerous popularizations and histories of the topic, from different points of view, have been written; see, for example the books by Lorenz[1], Diacu and Holmes[2], Stewart[3], and Gleick[4].

Our goal here is to introduce some of the key characteristics of chaos based on notions that we have already developed so as to frame possible future directions of studies that the student might wish to pursue. Our discussion will be in the setting of a flow generated by an autonomous vector field.

The phrase ''chaotic behavior" calls to mind a form of randomness and unpredictability. But keep in mind, we are working in the setting of , i.e., our ODE satisfies the criteria for existence and uniqueness of solutions. Therefore specifying the initial condition *exactly* implies that the future evolution is uniquely determined, i.e. there is no ''randomness or unpredictability". The key here is the word ''exactly". Chaotic systems have an intrinsic property in their dynamics that can result in slight perturbations of the initial conditions leading to behavior, over time, that is unlike the behavior of the trajectory though the original initial condition. Often it is said that a chaotic system exhibits sensitive dependence on initial conditions. Now this is a lot of words for a mathematics course. Just like when we studied stability, we will give a mathematical definition of sensitive dependence on initial conditions, and then consider the meaning of the definition in the context of specific examples.

As mentioned above, we consider an autonomous, C^r, $r \geq 1$ vector field on $\mathbb{R}^n$:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{F.1}$$

[1] Edward N. Lorenz. *The essence of chaos*. University of Washington Press, Seattle, 1993

[2] Florin Diacu and Philip Holmes. *Celestial encounters: the origins of chaos and stability*. Princeton University Press, 1996

[3] Ian Stewart. *Does God play dice?: The new mathematics of chaos*. Penguin UK, 1997

[4] James Gleick. *Chaos: Making a New Science (Enhanced Edition)*. Open Road Media, 2011

and we denote the flow generated by the vector field by $\phi_t(\cdot)$, and we assume that it exists for all time. We let $\Lambda \subset \mathbb{R}^n$ denote an invariant set for the flow. Then we have the following definition.

Definition 22 (Sensitive dependence on initial conditions)**.** *The flow $\phi_t(\cdot)$ is said to have sensitive dependence on initial conditions on Λ if there exists $\epsilon > 0$ such that, for any $x \in \Lambda$ and any neighborhood $U \subset \Lambda$ of x there exists $y \in U$ and $t > 0$ such that $|\phi_t(x) - \phi_t(y)| > \epsilon$.*

Now we consider an example and analyze whether or not sensitive dependence on initial conditions is present in the example.

Example 45. *Consider the autonomous linear vector field on $\mathbb{R}^2$:*

$$\begin{aligned} \dot{x} &= \lambda x, \\ \dot{y} &= -\mu y, \qquad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{F.2}$$

with λ, $\mu > 0$. This is just a standard ''saddle point''. The origin is a fixed point of saddle type with its stable manifold given by the y axis (i.e. $x = 0$) and its unstable manifold given by the x axis (i.e. $y = 0$). The flow generated by this vector field is given by:

$$\phi_t(x_0, y_0) = \left(x_0 e^{\lambda t},\, y_0 e^{-\mu t} \right). \tag{F.3}$$

Following the definition, sensitive dependence on initial conditions is defined with respect to invariant sets. Therefore we must identify the invariant sets for which we want to determine whether or not they possess the property of sensitive dependence on initial condition.

The simplest invariant set is the fixed point at the origin. However, that invariant set clearly does not exhibit sensitive dependence on initial conditions.

Then we have the one dimensional stable (y axis) and unstable manifolds (x axis). We can consider the issue of sensitive dependence on initial conditions on these invariant sets. The stable and unstable manifolds divide the plane into four quadrants. Each of these is an invariant set (with a segment of the stable and unstable manifold forming part of their boundary), and the entire plane (i.e. the entire phase space) is also an invariant set.

We consider the unstable manifold, $y = 0$. The flow restricted to the unstable manifold is given by

$$\phi_t(x_0, 0) = \left(x_0 e^{\lambda t},\, 0 \right). \tag{F.4}$$

It should be clear the the unstable manifold is an invariant set that exhibits sensitive dependence on initial conditions. Choose an arbitrary point on the unstable manifold, $\bar{x}_1$. Consider another point arbitrarily close to $\bar{x}_1$, $\bar{x}_2$. Now consider any $\epsilon > 0$. We have

$$|\phi_t(\bar{x}_1, 0) - \phi_t(\bar{x}_2, 0)| = |\bar{x}_1 - \bar{x}_2| e^{\lambda t}. \tag{F.5}$$

Now since $|\bar{x}_1 - \bar{x}_2|$ is a fixed constant, we can clearly find a $t > 0$ such that

$$|\bar{x}_1 - \bar{x}_2| e^{\lambda t} > \epsilon. \tag{F.6}$$

Therefore the invariant unstable manifold exhibits sensitive dependence on initial conditions. Of course, this is not surprising because of the $e^{\lambda t}$ term in the expression for the flow since this term implies exponential growth in time of the x component of the flow.

The stable manifold,$x = 0$, does not exhibit sensitive dependence on initial conditions since the restriction to the stable manifold is given by:

$$\phi_t(0, y_0) = \left(0,\, y_0 e^{-\mu t}\right), \tag{F.7}$$

which implies that neighboring points actually get closer together as t increases.

Moreover, the term $e^{\lambda t}$ implies that the four quadrants separated by the stable and unstable manifolds of the origin also exhibit sensitive dependence on initial conditions.

Of course, we would not consider a linear autonomous ODE on the plane having a hyperbolic saddle point to be a chaotic dynamical system, even though it exhibits sensitive dependence on initial conditions. Therefore there must be something more to "chaos", and we will explore this through more examples.

Before we consider the next example we point out two features of this example that we will consider in the context of other examples.

1. *The invariant sets that we considered (with the exception of the fixed point at the origin) were unbounded. This was a feature of the linear nature of the vector field.*

2. *The ''separation of trajectories" occurred at an exponential rate. There was no requirement on the ''rate of separation" in the definition of sensitive dependence on initial conditions.*

3. *Related to these two points is the fact that trajectories continue to separate for all time, i.e. they never again ''get closer" to each other.*

Example 46. *Consider the autonomous vector field on the cylinder:*

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= r, \quad (r, \theta) \in \mathbb{R}^+ \times S^1. \end{aligned} \tag{F.8}$$

The flow generated by this vector field is given by:

$$\phi_t(r_0, \theta_0) = (r_0,\, r_0\, t + \theta_0). \tag{F.9}$$

Note that r is constant in time. This implies that any annulus is an invariant set. In particular, choose any $r_1 < r_2$. Then the annulus

$$\mathcal{A} \equiv \left\{ (r,\theta) \in \mathbb{R}^+ \times S^1 \,|\, r_1 \leq r \leq r_2,\, \theta \in S^1 \right\}, \tag{F.10}$$

is a bounded invariant set.

Now choose initial conditions in $\mathcal{A}$, (r_1', θ_1), (r_2', θ_2), with $r_1 \leq r_1' < r_2' \leq r_2$. Then we have that:

$$\begin{aligned} |\phi_t(r_1', \theta_1) - \phi_t(r_2', \theta_2)| &= |\left(r_1',\, r_1' \, t + \theta_1\right) - \left(r_2',\, r_2' \, t + \theta_2\right)|, \\ &= \left(r_1' - r_2',\, (r_1' - r_2')t + (\theta_1 - \theta_2)\right). \end{aligned}$$

Hence we see that the distance between trajectories will grow linearly in time, and therefore trajectories exhibit sensitive dependence on initial conditions. However, the distance between trajectories will not grow unboundedly (as in the previous example). This is because θ is on the circle. Trajectories will move apart (in θ, but their r values will remain constant) and then come close, then move apart, etc. Nevertheless, this is not an example of a chaotic dynamical system.

Example 47. *Consider the following autonomous vector field defined on the two dimensional torus (i.e. each variable is an angular variable):*

$$\begin{aligned} \dot{\theta}_1 &= \omega_1, \\ \dot{\theta}_2 &= \omega_2, \quad (\theta_1, \theta_2) \in S^1 \times S^1. \end{aligned} \tag{F.11}$$

This vector field is an example that is considered in many dynamical systems courses where it is shown that if $\frac{\omega_1}{\omega_2}$ is an irrational number, then the trajectory through any *initial condition ''densely fills out the torus". This means that given any point on the torus any trajectory will get arbitrarily close to that point at some time of its evolution, and this ''close approach" will happen infinitely often. This is the classic example of an ergodic system, and this fact is proven in many textbooks, e.g. Arnold[5] or Wiggins[6]. This behavior is very different from the previous examples. For the case $\frac{\omega_1}{\omega_2}$ an irrational number, the natural invariant set to consider is the entire phase space (which is bounded).*

[5] V. I. Arnold. *Ordinary differential equations*. M.I.T. press, Cambridge, 1973. ISBN 0262010372

[6] Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media, 2003

Next we consider the issue of sensitive dependence on initial conditions. The flow generated by this vector field is given by:

$$\phi_t(\theta_1, \theta_2) = \left(\omega_1 \, t + \theta_1,\, \omega_2 \, t + \theta_2\right). \tag{F.12}$$

We choose two initial conditions, (θ_1, θ_2), (θ_1', θ_2'). Then we have

$$\begin{aligned} |\phi_t(\theta_1, \theta_2) - \phi_t(\theta_1', \theta_2')| &= |\left(\omega_1 \, t + \theta_1,\, \omega_2 \, t + \theta_2\right) - \left(\omega_1 \, t + \theta_1',\, \omega_2 \, t + \theta_2'\right)|, \\ &= |(\theta_1 - \theta_1', \theta_2 - \theta_2')|, \end{aligned}$$

and therefore trajectories always maintain the same distance from each other during the course of their evolution.

Sometimes it is said that chaotic systems contain an infinite number of unstable periodic orbits. We consider an example.

Example 48. *Consider the following two dimensional autonomous vector field on the cylinder:*

$$\begin{aligned} \dot{r} &= \sin\frac{\pi}{r}, \\ \dot{\theta} &= r, \qquad\qquad (r,\theta) \in \mathbb{R}^+ \times S^1. \end{aligned}$$

Equilibrium points of the $\dot{r}$ component of this vector field correspond to periodic orbits. These equilibrium points are given by

$$r = \frac{1}{n}, \quad n = 0, 1, 2, 3, \ldots. \tag{F.13}$$

Stability of the periodic orbits can be determined by computing the Jacobian of the $\dot{r}$ component of the equation and evaluating it on the periodic orbit. This is given by:

$$-\frac{\pi}{r^2}\cos\frac{\pi}{r},$$

and evaluating this on the periodic orbits gives;

$$-\frac{\pi}{n^2}(-1)^n.$$

Therefore all of these periodic orbits are hyperbolic and stable for n even and unstable for n odd. This is an example of a two dimensional autonomous vector field that contains an infinite number of unstable hyperbolic periodic orbits in a bounded region, yet it is not chaotic.

Now we consider what we have learned from these four examples. In example 45 we identified invariant sets on which the trajectories exhibited sensitive dependence on initial conditions (i.e. trajectories separated at an exponential rate), but those invariant sets were unbounded, and the trajectories also became unbounded. This illustrates why boundedness is part of the definition of invariant set in the context of chaotic systems.

In example 46 we identified an invariant set, $\mathcal{A}$, on which all trajectories were bounded and they exhibited sensitive dependence on initial conditions, although they only separated linearly in time. However, the r coordinates of all trajectories remained constant, indicating that trajectories were constrained to lie on circles (''invariant circles'') within $\mathcal{A}$.

In example 47, for $\frac{\omega_1}{\omega_2}$ an irrational number, *every* trajectory densely fills out the entire phase space, the torus (which is bounded). However, the trajectories did not exhibit sensitive dependence on initial conditions.

Finally, in example 48 we gave an example having an infinite number of unstable hyperbolic orbits in a bounded region of the phase space. We did not explicitly examine the issue of sensitive dependence on initial conditions for this example.

So what characteristics would we require of a chaotic invariant set? A combination of examples 45 and 47 would capture many manifestations of ''chaotic invariant sets'':

1. the invariant set is bounded,
2. every trajectory comes arbitrarily close to every point in the invariant set during the course of its evolution in time, and
3. every trajectory has sensitive dependence on initial condition.

While simple to state, developing a unique mathematical framework that makes these three criteria mathematically rigorous, and provides a way to verify them in particular examples, is not so straightforward.

Property 1 is fairly straightforward, once we have identified a candidate invariant set (which can be very difficult in explicit ODEs). If the phase space is equipped with a norm, then we have a way of verifying whether or not the invariant set is bounded.

Property 2 is very difficult to verify, as well as to develop a universally accepted definition amongst mathematicians as to what it means for ''every trajectory to come arbitrarily close to every point in phase space during the course of its evolution''. Its definition is understood within the context of recurrence properties of trajectories. Those can be studied from either the topological point of view (see Akin[7]). or from the point of view of ergodic theory (see Katok and Hasselblatt[8] or Brin and Stuck[9]). The settings for both of these points of view utilize different mathematical structures (topology in the former case, measure theory in the latter case). A book that describes how both of these points of view are used in the application of mixing is Sturman et al.[10].

Verifying that Property 3 holds for *all* trajectories is also not straightforward. What ''all'' means is different in the topological setting (''open set'', Baire category) and the ergodic theoretic setting (sets of ''full measure''). What ''sensitive dependence on initial conditions'' means is also different in each setting. The definition we gave above was more in the spirit of the topological point of view (no specific ''rate of separation'' was given) and the ergodic theoretic framework focuses

[7] Ethan Akin. *The general topology of dynamical systems*, volume 1. American Mathematical Soc., 2010

[8] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge university press, 1997

[9] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, 2002

[10] Rob Sturman, Julio M Ottino, and Stephen Wiggins. *The mathematical foundations of mixing: the linked twist map as a paradigm in applications: micro to macro, fluids to solids*, volume 22. Cambridge University Press, 2006

on Lyapunov exponents (''Lyapunov's first method'') and exponential rate of separation of trajectories.

Therefore we have not succeeded in giving a specific example of an ODE whose trajectories behave in a chaotic fashion. We have been able to describe some of the issues, but the details will be left for other courses (which could be either courses in dynamical systems theory or ergodic theory or, ideally, a bit of both). But we have illustrated just how difficult it can be to formulate mathematically precise definitions that can be verified in specific examples.

All of our examples above were two dimensional, autonomous vector fields. The type of dynamics that can be exhibited by such systems is very limited, according to the Poincaré-Bendixson theorem (see Hirsch et al.[11] or Wiggins[12]). There are a number of variations of this theorem, so we will leave the exploration of this theorem to the interested student.

[11] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012

[12] Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media, 2003

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