ARTICLE TEMPLATE

SUPPLEMENTARY MATERIAL: Birnbaum-Saunders sample selection model

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ARTICLE HISTORY
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Appendix A: Gradient and Hessian Matrix

Let $\boldsymbol{\theta} = (\boldsymbol{\gamma}^\top, \boldsymbol{\beta}^\top, \phi, \rho)$. The log-likelihood function for the Heckman-BS model based on a pair of observations (y, u) is

$$\begin{aligned}\mathcal{L}(\boldsymbol{\theta}) = u\Bigg\{&-\frac{\phi}{4}\left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu_1}}}-\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y(\phi+1)}}\right)^2\\&+\log\Phi\left[\frac{[\boldsymbol{\mu_2}-2]}{2\sqrt{\boldsymbol{\mu_2}(1-\rho^2)}}+\frac{\sqrt{\phi}\rho}{\sqrt{2(1-\rho^2)}}\left(\sqrt{\frac{(\phi+1)y}{\phi\boldsymbol{\mu_1}}}-\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y(\phi+1)}}\right)\right]\\&+\log\left(\frac{\sqrt{\phi+1}}{\sqrt{\phi\boldsymbol{\mu_1}y}}+\frac{\sqrt{\phi\boldsymbol{\mu_1}}}{\sqrt{y^3(\phi+1)}}\right)+\frac{1}{4}\sqrt{\frac{\phi}{\pi}}\Bigg\}\\&+(1-u)\log\Phi\left\{\sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{\boldsymbol{\mu_2}}}-\sqrt{\frac{\boldsymbol{\mu_2}}{2}}\right]\right\},\end{aligned}\tag{1}$$

where $\boldsymbol{\mu_1} = \exp(\boldsymbol{x}^\top\boldsymbol{\beta}) = \exp(\eta_1)$ and $\boldsymbol{\mu_2} = \exp(\boldsymbol{w}^\top\boldsymbol{\gamma}) = \exp(\eta_2)$. If y is observed, then $u = 1$, otherwise $u = 0$. To simplify the calculation of the Hessian matrix, we

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consider

$$t_0(y,\boldsymbol{\mu_1},\phi)=\left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu_1}}}-\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y(\phi+1)}}\right),\quad t_1(y,\boldsymbol{\mu_1},\phi)=-\frac{\phi}{4}[t_0(y,\boldsymbol{\mu_1},\phi)]^2,$$

$$t_2(y,\boldsymbol{\mu_1},\phi)=\left(\frac{\sqrt{\phi+1}}{\sqrt{\phi\boldsymbol{\mu_1}y}}+\frac{\sqrt{\phi\boldsymbol{\mu_1}}}{\sqrt{y^3(\phi+1)}}\right),\qquad t_3(\phi)=\frac{1}{4}\sqrt{\frac{\phi}{\pi}},$$

$$t_4(\boldsymbol{\mu_2},\rho)=\frac{\boldsymbol{\mu_2}-2}{2\sqrt{\boldsymbol{\mu_2}(1-\rho^2)}},\quad t_5(\phi,\rho)=\frac{\rho\sqrt{\phi}}{\sqrt{2(1-\rho^2)}},\quad t_6(\boldsymbol{\mu_2})=\sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{\boldsymbol{\mu_2}}}-\sqrt{\frac{\boldsymbol{\mu_2}}{2}}\right],$$

and for $k=1,\cdots,p$, and $i=1,\cdots,q$,

$$\frac{\partial t_0}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_1}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_2}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_3}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_5}{\partial\boldsymbol{\gamma}_i}=0,\frac{\partial t_4}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_4}{\partial\boldsymbol{\mu_2}}\frac{\partial\boldsymbol{\mu_2}}{\partial\eta_2}\frac{\partial\eta_2}{\partial\boldsymbol{\gamma}_i}=\frac{\boldsymbol{\mu_2}+2}{4\sqrt{\boldsymbol{\mu_2}(1-\rho^2)}}\mathbf{w}_i,$$

$$\frac{\partial t_6}{\partial\boldsymbol{\gamma}_i}=\frac{\partial t_6}{\partial\boldsymbol{\mu_2}}\frac{\partial\boldsymbol{\mu_2}}{\partial\eta_2}\frac{\partial\eta_2}{\partial\boldsymbol{\gamma}_i}=-\frac{1}{2\sqrt{2}}\left(\sqrt{\frac{2}{\boldsymbol{\mu_2}}}+\sqrt{\frac{\boldsymbol{\mu_2}}{2}}\right)\mathbf{w}_i,\frac{\partial t_3}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_4}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_5}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_6}{\partial\boldsymbol{\beta}_k}=0,$$

$$\frac{\partial t_0}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_0}{\partial\boldsymbol{\mu_1}}\frac{\partial\boldsymbol{\mu_1}}{\partial\eta_1}\frac{\partial\eta_1}{\partial\boldsymbol{\beta}_k}=-\frac{1}{2}\left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu_1}}}+\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y(\phi+1)}}\right)\mathbf{x}_{ki},\qquad \frac{\partial t_4}{\partial\phi}=\frac{\partial t_6}{\partial\phi}=0,$$

$$\frac{\partial t_1}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_1}{\partial\boldsymbol{\mu_1}}\frac{\partial\boldsymbol{\mu_1}}{\partial\eta_1}\frac{\partial\eta_1}{\partial\boldsymbol{\beta}_k}=-\frac{\phi}{2}t_0\frac{\partial t_0}{\partial\boldsymbol{\beta}_k},\qquad \frac{\partial t_0}{\partial\rho}=\frac{\partial t_1}{\partial\rho}=\frac{\partial t_2}{\partial\rho}=\frac{\partial t_3}{\partial\rho}=\frac{\partial t_6}{\partial\rho}=0,$$

$$\frac{\partial t_2}{\partial\boldsymbol{\beta}_k}=\frac{\partial t_2}{\partial\boldsymbol{\mu_1}}\frac{\partial\boldsymbol{\mu_1}}{\partial\eta_1}\frac{\partial\eta_1}{\partial\boldsymbol{\beta}_k}=\frac{1}{2}\left(\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y^3(\phi+1)}}-\sqrt{\frac{\phi+1}{\phi\boldsymbol{\mu_1}y}}\right)\mathbf{x}_k,\qquad \frac{\partial t_3}{\partial\phi}=\frac{1}{8\sqrt{\pi\phi}},$$

$$\frac{\partial t_0}{\partial\phi}=-\frac{1}{2}\left(\sqrt{\frac{y}{(\phi+1)\phi^3\boldsymbol{\mu_1}}}+\sqrt{\frac{\boldsymbol{\mu_1}}{y(\phi+1)^3\phi}}\right),\qquad \frac{\partial t_5}{\partial\phi}=\frac{\rho}{2\sqrt{2\phi(1-\rho^2)}},$$

$$\frac{\partial t_1}{\partial\phi}=-\frac{t_0^2}{4}+\frac{t_0}{4}\left(\sqrt{\frac{y}{(\phi+1)\phi\boldsymbol{\mu_1}}}+\sqrt{\frac{\phi\boldsymbol{\mu_1}}{y(\phi+1)^3}}\right),\qquad \frac{\partial t_4}{\partial\rho}=\frac{\rho t_4}{(1-\rho^2)},$$

$$\frac{\partial t_2}{\partial\phi}=\frac{1}{2}\left(\sqrt{\frac{\boldsymbol{\mu_1}}{y^3(\phi+1)^3\phi}}-\frac{1}{\sqrt{y\boldsymbol{\mu_1}(\phi+1)\phi^3}}\right),\qquad \frac{\partial t_5}{\partial\rho}=\frac{t_5}{\rho(1-\rho^2)}.$$

Define

$$f_1(y,\boldsymbol{\mu_1},\phi)=t_1(y,\boldsymbol{\mu_1},\phi),\qquad f_2(y,\boldsymbol{\mu_1},\phi)=\log\left[t_2(y,\boldsymbol{\mu_1},\phi)\right],\quad f_3(\phi)=\log\left[t_3(\phi)\right],$$
$$f_4(y,\boldsymbol{\mu_1},\boldsymbol{\mu_2},\phi,\rho)=\log\Phi(t_4+t_5t_0),\qquad f_5(\boldsymbol{\mu_2})=\log\Phi[t_6(\boldsymbol{\mu_2})].$$

The first-order derivatives of the above functions are

$$\frac{\partial f_1}{\partial\boldsymbol{\gamma}_i}=\frac{\partial f_2}{\partial\boldsymbol{\gamma}_i}=\frac{\partial f_3}{\partial\boldsymbol{\gamma}_i}=0,\qquad \frac{\partial f_4}{\partial\boldsymbol{\gamma}_i}=\frac{\phi(t_4+t_5t_0)}{\Phi(t_4+t_5t_0)}\frac{\partial t_4}{\partial\boldsymbol{\gamma}_i},\qquad \frac{\partial f_5}{\partial\boldsymbol{\gamma}_i}=\frac{\phi(t_6)}{\Phi(t_6)}\frac{\partial t_6}{\partial\boldsymbol{\gamma}_i},$$

$$\frac{\partial f_1}{\partial\boldsymbol{\beta}_k}=-\frac{\phi}{2}t_0\frac{\partial t_0}{\partial\boldsymbol{\beta}_k},\qquad \frac{\partial f_2}{\partial\boldsymbol{\beta}_k}=\frac{1}{t_2}\frac{\partial t_2}{\partial\boldsymbol{\beta}_k},\qquad \frac{\partial f_3}{\partial\boldsymbol{\beta}_k}=\frac{\partial f_5}{\partial\boldsymbol{\beta}_k}=0,$$

$$\frac{\partial f_4}{\partial\boldsymbol{\beta}_k}=\frac{\phi(t_4+t_5t_0)}{\Phi(t_4+t_5t_0)}t_5\frac{\partial t_0}{\partial\boldsymbol{\beta}_k},\qquad \frac{\partial f_1}{\partial\phi}=\frac{\partial t_1}{\partial\phi},\qquad \frac{\partial f_2}{\partial\phi}=\frac{1}{t_2}\frac{\partial t_2}{\partial\phi},$$

$$\frac{\partial f_3}{\partial\phi}=\frac{1}{2\phi},\qquad \frac{\partial f_4}{\partial\phi}=\frac{\phi(t_4+t_5t_0)}{\Phi(t_4+t_5t_0)}\left(\frac{\partial t_5}{\partial\phi}t_0+t_5\frac{\partial t_0}{\partial\phi}\right),\quad \frac{\partial f_5}{\partial\phi}=0,$$

$$\frac{\partial f_1}{\partial\rho}=\frac{\partial f_2}{\partial\rho}=\frac{\partial f_3}{\partial\rho}=\frac{\partial f_5}{\partial\rho}=0,\quad \frac{\partial f_4}{\partial\rho}=\frac{\phi(t_4+t_5t_0)}{\Phi(t_4+t_5t_0)}\left(\frac{\partial t_4}{\partial\rho}+\frac{\partial t_5}{\partial\rho}t_0\right).$$

Thus, we have that the maximum likelihood estimators are obtained by the solution of the system of equations

$$\begin{cases}
\dfrac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\gamma}_j} = \displaystyle\sum_{i=1}^n u_i\left(\frac{\partial f_{4i}}{\partial\boldsymbol{\gamma}_j}\right)+\sum_{i=1}^n(1-u_i)\left(\frac{\partial f_{5i}}{\partial\boldsymbol{\gamma}_j}\right) = 0,\ j=1,\cdots,q,\\[2ex]
\dfrac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}_k} = \displaystyle\sum_{i=1}^n u_i\left(\frac{\partial f_{1i}}{\partial\boldsymbol{\beta}_k}+\frac{\partial f_{2i}}{\partial\boldsymbol{\beta}_k}+\frac{\partial f_3}{\partial\boldsymbol{\beta}_k}+\frac{\partial f_{4i}}{\partial\boldsymbol{\beta}_k}\right) = 0,\ k=1,\cdots,p,\\[2ex]
\dfrac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\phi} = \displaystyle\sum_{i=1}^n u_i\left(\frac{\partial f_{1i}}{\partial\phi}+\frac{\partial f_{2i}}{\partial\phi}+\frac{\partial f_3}{\partial\phi}+\frac{\partial f_{4i}}{\partial\phi}\right) = 0,\\[2ex]
\dfrac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\rho} = \displaystyle\sum_{i=1}^n u_i\frac{\partial f_{4i}}{\partial\rho} = 0.
\end{cases}$$

Let $M(x)=\dfrac{\phi(x)}{\Phi(x)}, x\in\mathbb{R}$, follow that $\dfrac{\partial M(x)}{\partial a}=-\dfrac{\partial x}{\partial a}M(x)\left(x+M(x)\right), a\in(\boldsymbol{\gamma}^\top,\boldsymbol{\beta}^\top,\phi,\rho)$. The second-order derivatives of the above functions are

$$\frac{\partial^2 f_4}{\partial\gamma_i\partial\gamma_j}=\frac{\partial}{\partial\gamma_j}M(t_4+t_5t_0)\frac{\partial t_4}{\partial\gamma_i}+M(t_4+t_5t_0)\frac{\partial^2 t_4}{\partial\gamma_i\partial\gamma_j}$$

$$\frac{\partial^2 t_4}{\partial\gamma_i\partial\gamma_j} = \frac{t_4}{4}w_i w_j, \qquad \frac{\partial^2 f_5}{\partial\gamma_i\partial\gamma_j} = \frac{\partial}{\partial\gamma_j}M(t_6)\frac{\partial t_6}{\partial\gamma_i} + M(t_6)\frac{\partial^2 t_6}{\partial\gamma_i\partial\gamma_j}, \quad \frac{\partial^2 t_6}{\partial\gamma_i\partial\gamma_j} = \frac{1}{4}t_6 w_i w_j,$$

$$\frac{\partial^2 f_5}{\partial\gamma_i\partial\phi} = \frac{\partial}{\partial\phi}M(t_6)\frac{\partial t_6}{\partial\gamma_i}, \qquad \frac{\partial^2 f_4}{\partial\gamma_i\partial\beta_l} = \frac{\partial}{\partial\beta_l}M(t_4+t_5t_0)\frac{\partial t_4}{\partial\gamma_i}, \qquad \frac{\partial^2 f_5}{\partial\gamma_i\partial\beta_l} = \frac{\partial^2 f_5}{\partial\gamma_i\partial\rho} = 0,$$

$$\frac{\partial^2 f_4}{\partial\gamma_i\partial\phi} = \frac{\partial}{\partial\phi}M(t_4+t_5t_0)\frac{\partial t_4}{\partial\gamma_i}, \quad \frac{\partial^2 f_4}{\partial\beta_k\partial\gamma_j} = \frac{\partial}{\partial\gamma_j}M(t_4+t_5t_0)t_5\frac{\partial t_0}{\partial\beta_k}, \qquad \frac{\partial^2 t_4}{\partial\gamma_i\partial\rho} = \frac{\rho}{(1-\rho^2)}\frac{\partial t_4}{\partial\gamma_i},$$

$$\frac{\partial^2 t_0}{\partial\beta_k\partial\beta_l} = \frac{1}{4}t_0 x_k x_l, \qquad \frac{\partial^2 f_2}{\partial\beta_k\partial\beta_l} = \frac{\partial t_2^{-1}}{\partial\beta_l}\frac{\partial t_2}{\partial\beta_k} + \frac{1}{t_2}\frac{\partial^2 t_2}{\partial\beta_k\partial\beta_l}, \qquad \frac{\partial t_2^{-1}}{\partial\beta_l} = -\frac{1}{t_2^2}\frac{\partial t_2}{\partial\beta_l},$$

$$\frac{\partial^2 t_2}{\partial\beta_k\partial\beta_l} = \frac{1}{4}t_2 x_k x_l, \quad \frac{\partial^2 f_3}{\partial\beta_k\partial\beta_l} = 0, \qquad \frac{\partial^2 t_0}{\partial\phi\partial\beta_l} = \frac{1}{4}\left(\sqrt{\frac{\boldsymbol{\mu_1}}{y(\phi+1)^3\phi}} - \sqrt{\frac{y}{(\phi+1)\phi^3\boldsymbol{\mu_1}}}\right)x_l,$$

$$\frac{\partial^2 f_3}{\partial\beta_k\partial\phi} = 0, \qquad \frac{\partial^2 f_1}{\partial\beta_k\partial\rho} = 0, \qquad \frac{\partial^2 f_1}{\partial\phi\partial\gamma_j} = \frac{\partial^2 f_2}{\partial\phi\partial\gamma_j} = \frac{\partial^2 f_3}{\partial\phi\partial\gamma_j} = 0,$$

$$\frac{\partial^2 f_3}{\partial\beta_k\partial\rho} = 0, \qquad \frac{\partial t_2^{-1}}{\partial\phi} = -\frac{1}{t_2^2}\frac{\partial t_2}{\partial\phi}, \qquad \frac{\partial^2 f_4}{\partial\phi\partial\gamma_j} = \frac{\partial}{\partial\gamma_j}M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}t_0 + t_5\frac{\partial t_0}{\partial\phi}\right),$$

$$\frac{\partial^2 f_2}{\partial\beta_k\partial\rho} = 0, \qquad \frac{\partial^2 t_5}{\partial\phi\partial\rho} = \frac{1}{2\sqrt{2\phi(1-\rho^2)^3}}, \quad \frac{\partial^2 f_2}{\partial\phi^2} = -\frac{1}{t_2^2}\left(\frac{\partial t_2}{\partial\phi}\right)^2 + \frac{1}{t_2}\frac{\partial^2 t_2}{\partial\phi^2},$$

$$\frac{\partial^2 f_3}{\partial\phi^2} = -\frac{1}{2\phi^2}, \quad \frac{\partial^2 t_5}{\partial\phi^2} = -\frac{\rho}{4\sqrt{2\phi^3(1-\rho^2)}}, \quad \frac{\partial^2 f_2}{\partial\phi\partial\beta_l} = -\frac{1}{t_2^2}\frac{\partial t_2}{\beta_l} + \frac{1}{t_2}\frac{\partial^2 t_2}{\partial\phi\partial\beta_l},$$

$$\frac{\partial^2 f_4}{\partial\gamma_i\partial\rho} = \frac{\partial}{\partial\rho}M(t_4+t_5t_0)\frac{\partial t_4}{\partial\gamma_i} + M(t_4+t_5t_0)\frac{\partial^2 t_4}{\partial\gamma_i\partial\rho}, \quad \frac{\partial^2 f_1}{\partial\beta_k\partial\gamma_j} = \frac{\partial^2 f_2}{\partial\beta_k\partial\gamma_j} = \frac{\partial^2 f_3}{\partial\beta_k\partial\gamma_j} = 0,$$

$$\frac{\partial^2 f_1}{\partial\beta_k\partial\beta_l} = -\frac{\phi}{2}\left(\frac{\partial t_0}{\partial\beta_l}\frac{\partial t_0}{\partial\beta_k} + t_0\frac{\partial^2 t_0}{\partial\beta_k\partial\beta_l}\right), \qquad \frac{\partial^2 f_2}{\partial\beta_k\partial\phi} = \frac{\partial t_2^{-1}}{\partial\phi}\frac{\partial t_2}{\partial\beta_k} + \frac{1}{t_2}\frac{\partial^2 t_2}{\partial\beta_k\partial\phi},$$

$$\frac{\partial^2 f_1}{\partial\beta_k\partial\phi} = -\frac{1}{2}t_0\frac{\partial t_0}{\partial\beta_k} - \frac{\phi}{2}\left(\frac{\partial t_0}{\partial\phi}\frac{\partial t_0}{\partial\beta_k} + t_0\frac{\partial^2 t_0}{\partial\beta_k\partial\phi}\right), \qquad \frac{\partial^2 f_1}{\partial\phi\partial\rho} = \frac{\partial^2 f_2}{\partial\phi\partial\rho} = \frac{\partial^2 f_3}{\partial\phi\partial\rho} = 0,$$

$$\frac{\partial^2 f_4}{\partial\beta_k\partial\beta_l} = \frac{\partial}{\partial\beta_l}M(t_4+t_5t_0)t_5\frac{\partial t_0}{\partial\beta_k} + M(t_4+t_5t_0)\left(t_5\frac{\partial^2 t_0}{\partial\beta_k\partial\beta_l}\right),$$

$$\frac{\partial^2 t_0}{\partial\beta_k\partial\phi} = \frac{1}{4}\left(\sqrt{\frac{y}{(\phi+1)\phi^3\boldsymbol{\mu_1}}} + \sqrt{\frac{\boldsymbol{\mu_1}}{y(\phi+1)^3\phi}}\right)x_k,$$

$$\frac{\partial^2 t_2}{\partial\beta_k\partial\phi} = \frac{1}{4}\left(\sqrt{\frac{\boldsymbol{\mu_1}}{y^3\phi(\phi+1)^3}} + \frac{1}{\sqrt{\boldsymbol{\mu_1}y\phi^3(\phi+1)}}\right)x_k,$$

$$\frac{\partial^2 f_4}{\partial\beta_k\partial\rho} = \frac{\partial}{\partial\rho}M(t_4+t_5t_0)t_5\frac{\partial t_0}{\partial\beta_k} + M(t_4+t_5t_0)\frac{\partial t_5}{\partial\rho}\frac{\partial t_0}{\partial\beta_k},$$

$$\frac{\partial^2 t_2}{\partial\phi\partial\beta_l} = \frac{1}{4}\left(\sqrt{\frac{\boldsymbol{\mu_1}}{y^3\phi(\phi+1)^3}} + \frac{1}{\sqrt{\boldsymbol{\mu_1}y\phi^3(\phi+1)}}\right)x_l,$$

$$\frac{\partial^2 f_4}{\partial\phi\partial\beta_l} = \frac{\partial}{\partial\beta_l}M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}t_0 + t_5\frac{\partial t_0}{\partial\phi}\right) + M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}\frac{\partial t_0}{\partial\beta_l} + t_5\frac{\partial^2 t_0}{\partial\phi\partial\beta_l}\right),$$

$$\frac{\partial^2 t_2}{\partial\phi^2} = \frac{1}{4}\left(\frac{4\phi+3}{\phi^2\sqrt{\boldsymbol{\mu_1}y\phi(\phi+1)^3}} - \frac{4\phi+1}{(\phi+1)^2}\sqrt{\frac{\boldsymbol{\mu_1}}{y^3\phi^3(\phi+1)}}\right),$$

$$\frac{\partial^2 t_0}{\partial\phi^2} = \frac{1}{4}\left[\frac{4\phi+3}{\phi^4}\sqrt{\frac{y}{\boldsymbol{\mu_1}\phi(\phi+1)^3}} - \frac{4\phi+1}{(\phi+1)^4}\sqrt{\frac{\boldsymbol{\mu_1}}{y\phi^3(\phi+1)}}\right],$$

$$\frac{\partial^2 f_4}{\partial\beta_k\partial\phi} = \frac{\partial}{\partial\phi}M(t_4+t_5t_0)t_5\frac{\partial t_0}{\partial\beta_k} + M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}\frac{\partial t_0}{\partial\beta_k} + t_5\frac{\partial^2 t_0}{\partial\beta_k\partial\phi}\right),$$

$$\frac{\partial^2 f_4}{\partial\phi^2} = \frac{\partial}{\partial\phi}M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}t_0 + t_5\frac{\partial t_0}{\partial\phi}\right) + M(t_4+t_5t_0)\left(\frac{\partial^2 t_5}{\partial\phi^2}t_0 + 2\frac{\partial t_5}{\partial\phi}\frac{\partial t_0}{\partial\phi} + t_5\frac{\partial^2 t_0}{\partial\phi^2}\right),$$

$$\frac{\partial^2 f_4}{\partial\phi\partial\rho} = \frac{\partial}{\partial\rho}M(t_4+t_5t_0)\left(\frac{\partial t_5}{\partial\phi}t_0 + t_5\frac{\partial t_0}{\partial\phi}\right) + M(t_4+t_5t_0)\left(\frac{\partial^2 t_5}{\partial\phi\partial\rho}t_0 + \frac{\partial t_5}{\partial\rho}\frac{\partial t_0}{\partial\phi}\right),$$

$$\frac{\partial^2 f_4}{\partial\rho\partial\gamma_j} = \frac{\partial}{\partial\gamma_j}M(t_4+t_5t_0)\left(\frac{\partial t_4}{\partial\rho} + \frac{\partial t_5}{\partial\rho}t_0\right) + M(t_4+t_5t_0)\frac{\partial^2 t_4}{\partial\rho\partial\gamma_j},$$

$$\frac{\partial^2 f_4}{\partial\rho\partial\phi} = \frac{\partial}{\partial\phi}M(t_4+t_5t_0)\left(\frac{\partial t_4}{\partial\rho} + \frac{\partial t_5}{\partial\rho}t_0\right) + M(t_4+t_5t_0)\left(\frac{\partial^2 t_5}{\partial\rho\partial\phi}t_0 + \frac{\partial t_5}{\partial\rho}\frac{\partial t_0}{\partial\phi}\right),$$

$$\frac{\partial^2 f_4}{\partial \rho^2} = \frac{\partial}{\partial \rho} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right) + M(t_4 + t_5 t_0) \left(\frac{\partial^2 t_4}{\partial \rho^2} + \frac{\partial^2 t_5}{\partial \rho^2} t_0 + \frac{\partial t_5}{\partial \rho} \frac{\partial t_0}{\partial \rho} \right),$$

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial \phi \partial \beta_l} &= \frac{\partial^2 t_1}{\partial \phi \partial \beta_l}, \\
&= -\frac{1}{2} t_0 \frac{\partial t_0}{\partial \beta_l} + \frac{1}{4} \frac{\partial t_0}{\partial \beta_l} \left(\sqrt{\frac{y}{(\phi+1)\phi \boldsymbol{\mu_1}}} + \sqrt{\frac{\phi \boldsymbol{\mu_1}}{y(\phi+1)^3}} \right), \\
&\quad + \frac{t_0}{8} \left(\sqrt{\frac{\phi \boldsymbol{\mu_1}}{y(\phi+1)^3}} - \sqrt{\frac{y}{(\phi+1)\phi \boldsymbol{\mu_1}}} \right) x_l,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial \phi^2} &= \frac{\partial^2 t_1}{\partial \phi^2}, \\
&= -\frac{t_0}{2} \frac{\partial t_0}{\partial \phi} + \frac{1}{4} \frac{\partial t_0}{\partial \phi} \left(\sqrt{\frac{y}{(\phi+1)\phi \boldsymbol{\mu_1}}} + \sqrt{\frac{\phi \boldsymbol{\mu_1}}{y(\phi+1)^3}} \right), \\
&\quad - \frac{t_0}{8} \left((2\phi - 1) \sqrt{\frac{\boldsymbol{\mu_1}}{y(\phi+1)^5 \phi}} + (2\phi+1) \sqrt{\frac{y}{(\phi+1)^3 \phi \boldsymbol{\mu_1}}} \right),
\end{aligned}$$

$$\frac{\partial^2 t_4}{\partial \rho \partial \gamma_j} = \frac{\rho}{(1-\rho^2)} \frac{\partial t_4}{\partial \gamma_j}, \qquad \frac{\partial^2 f_4}{\partial \rho \partial \beta_l} = \frac{\partial}{\partial \beta_l} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right), \quad \frac{\partial^2 t_5}{\partial \rho^2} = \frac{3 t_5}{(1-\rho^2)^2},$$

$$\frac{\partial^2 t_5}{\partial \rho \partial \phi} = \frac{1}{2\sqrt{2\phi(1-\rho^2)^3}}, \quad \frac{\partial^2 t_4}{\partial \rho^2} = \frac{1+\rho^2}{(1-\rho^2)^2} t_4 + \frac{\rho}{(1-\rho^2)} \frac{\partial t_4}{\partial \rho}.$$

Let $S_{ab} = \dfrac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial a \partial b}$, with $a, b \in (\boldsymbol{\gamma}^\top, \boldsymbol{\beta}^\top, \phi, \rho)$. Therefore, for $i, j = 1, \cdots, q$ and

$k, l = 1, \cdots, p$, we have that the elements of the Hessian matrix are

$$
\begin{aligned}
S_{\gamma_i\gamma_j} &= u\frac{\partial^2 f_4}{\partial\gamma_i\partial\gamma_j} + (1-u)\frac{\partial^2 f_5}{\partial\gamma_i\partial\gamma_j},\\
S_{\gamma_i\beta_l} &= u\frac{\partial^2 f_4}{\partial\gamma_i\partial\beta_l} + (1-u)\frac{\partial^2 f_5}{\partial\gamma_i\partial\beta_l},\\
S_{\gamma_i\phi} &= u\frac{\partial^2 f_4}{\partial\gamma_i\partial\phi} + (1-u)\frac{\partial^2 f_5}{\partial\gamma_i\partial\phi},\\
S_{\gamma_i\rho} &= u\frac{\partial^2 f_4}{\partial\gamma_i\partial\rho} + (1-u)\frac{\partial^2 f_5}{\partial\gamma_i\partial\rho},\\
S_{\beta_k\gamma_j} &= u\left(\frac{\partial^2 f_1}{\partial\beta_k\partial\gamma_j} + \frac{\partial^2 f_2}{\partial\beta_k\partial\gamma_j} + \frac{\partial^2 f_3}{\partial\beta_k\partial\gamma_j} + \frac{\partial^2 f_4}{\partial\beta_k\partial\gamma_j}\right),\\
S_{\beta_k\beta_l} &= u\left(\frac{\partial^2 f_1}{\partial\beta_k\partial\beta_l} + \frac{\partial^2 f_2}{\partial\beta_k\partial\beta_l} + \frac{\partial^2 f_3}{\partial\beta_k\partial\beta_l} + \frac{\partial^2 f_4}{\partial\beta_k\partial\beta_l}\right),\\
S_{\beta_k\phi} &= u\left(\frac{\partial^2 f_1}{\partial\beta_k\partial\phi} + \frac{\partial^2 f_2}{\partial\beta_k\partial\phi} + \frac{\partial^2 f_3}{\partial\beta_k\partial\phi} + \frac{\partial^2 f_4}{\partial\beta_k\partial\phi}\right),\\
S_{\beta_k\rho} &= u\left(\frac{\partial^2 f_1}{\partial\beta_k\partial\rho} + \frac{\partial^2 f_2}{\partial\beta_k\partial\rho} + \frac{\partial^2 f_3}{\partial\beta_k\partial\rho} + \frac{\partial^2 f_4}{\partial\beta_k\partial\rho}\right),\\
S_{\phi\gamma_j} &= u\left(\frac{\partial^2 f_1}{\partial\phi\partial\gamma_j} + \frac{\partial^2 f_2}{\partial\phi\partial\gamma_j} + \frac{\partial^2 f_3}{\partial\phi\partial\gamma_j} + \frac{\partial^2 f_4}{\partial\phi\partial\gamma_j}\right),\\
S_{\phi\beta_l} &= u\left(\frac{\partial^2 f_1}{\partial\phi\partial\beta_l} + \frac{\partial^2 f_2}{\partial\phi\partial\beta_l} + \frac{\partial^2 f_3}{\partial\phi\partial\beta_l} + \frac{\partial^2 f_4}{\partial\phi\partial\beta_l}\right),
\end{aligned}
$$

$$
\begin{aligned}
S_{\phi\phi} &= u\left(\frac{\partial^2 f_1}{\partial\phi^2} + \frac{\partial^2 f_2}{\partial\phi^2} + \frac{\partial^2 f_3}{\partial\phi^2} + \frac{\partial^2 f_4}{\partial\phi^2}\right),\\
S_{\phi\rho} &= u\left(\frac{\partial^2 f_1}{\partial\phi\partial\rho} + \frac{\partial^2 f_2}{\partial\phi\partial\rho} + \frac{\partial^2 f_3}{\partial\phi\partial\rho} + \frac{\partial^2 f_4}{\partial\phi\partial\rho}\right),\\
S_{\rho\gamma_j} &= u\frac{\partial^2 f_4}{\partial\rho\partial\gamma_j},\\
S_{\rho\beta_l} &= u\frac{\partial^2 f_4}{\partial\rho\partial\beta_l},\\
S_{\rho\phi} &= u\frac{\partial^2 f_4}{\partial\rho\partial\phi},\\
S_{\rho\rho} &= u\frac{\partial^2 f_4}{\partial\rho^2}.
\end{aligned}
$$

Appendix B: Additional Simulations

We present additional simulations of the fit of the Heckman-BS selection model. These simulations aim to evaluate the behavior of the maximum likelihood estimators in finite samples under the Heckman-BS model with different settings for the covariates and different values for the correlation parameter. The models have been defined according to Scenarios 1 and 2 of the paper. In Scenario 1, the model is fitted in the presence of exclusion restriction. In Scenario 2, the model considers the absence of exclusion restriction. In both scenarios, we maintained average censorship of approximately 30%.

- **Scenario 1**

Table 1. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.7$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

			Configuration 1		Configuration 2		Configuration 3	
Parameters		n	Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.609	0.086	1.611	0.212	1.610	0.090
$\gamma_1 =$	1.600	1000	1.604	0.060	1.603	0.151	1.605	0.060
		2000	1.601	0.042	1.604	0.105	1.601	0.042
		500	0.816	0.096	0.822	0.200	0.811	0.120
$\gamma_2 =$	0.800	1000	0.803	0.067	0.808	0.144	0.808	0.079
		2000	0.803	0.042	0.801	0.097	0.805	0.062
		500	0.205	0.079	0.212	0.337	0.209	0.118
$\gamma_3 =$	0.200	1000	0.202	0.061	0.203	0.240	0.201	0.075
		2000	0.199	0.042	0.203	0.162	0.197	0.058
		500	0.711	0.093	0.716	0.058	0.715	0.091
$\gamma_4 =$	0.700	1000	0.705	0.061	0.708	0.040	0.705	0.063
		2000	0.706	0.044	0.704	0.027	0.704	0.040
		500	0.993	0.140	0.994	0.136	0.989	0.131
$\beta_1 =$	1.000	1000	0.996	0.091	0.994	0.099	0.995	0.090
		2000	0.998	0.059	0.998	0.067	0.998	0.061
		500	0.701	0.071	0.694	0.116	0.709	0.091
$\beta_2 =$	0.700	1000	0.703	0.047	0.703	0.079	0.699	0.058
		2000	0.700	0.031	0.701	0.054	0.701	0.039
		500	1.100	0.055	1.108	0.192	1.096	0.079
$\beta_3 =$	1.100	1000	1.100	0.041	1.100	0.139	1.101	0.053
		2000	1.099	0.029	1.099	0.092	1.099	0.038
		500	1.221	0.141	1.219	0.102	1.225	0.135
$\phi =$	1.200	1000	1.208	0.094	1.214	0.072	1.210	0.097
		2000	1.205	0.064	1.206	0.049	1.205	0.064
		500	-0.684	0.133	-0.707	0.085	-0.684	0.126
$\rho =$	-0.700	1000	-0.696	0.083	-0.697	0.064	-0.696	0.078
		2000	-0.698	0.053	-0.700	0.045	-0.698	0.052

Table 2. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.610	0.086	1.608	0.220	1.610	0.089
$\gamma_1 =$	1.600	1000	1.604	0.061	1.604	0.152	1.605	0.061
		2000	1.601	0.042	1.605	0.109	1.601	0.042
		500	0.817	0.097	0.826	0.209	0.812	0.126
$\gamma_2 =$	0.800	1000	0.803	0.067	0.806	0.151	0.810	0.082
		2000	0.802	0.043	0.801	0.102	0.806	0.064
		500	0.204	0.081	0.221	0.346	0.209	0.119
$\gamma_3 =$	0.200	1000	0.203	0.062	0.200	0.244	0.201	0.076
		2000	0.199	0.043	0.202	0.168	0.196	0.060
		500	0.713	0.094	0.718	0.060	0.715	0.095
$\gamma_4 =$	0.700	1000	0.705	0.064	0.707	0.041	0.706	0.068
		2000	0.706	0.045	0.704	0.027	0.704	0.042
		500	0.999	0.141	0.995	0.136	0.994	0.134
$\beta_1 =$	1.000	1000	0.997	0.096	0.993	0.100	0.996	0.094
		2000	1.001	0.064	0.997	0.068	0.999	0.066
		500	0.699	0.072	0.695	0.116	0.710	0.095
$\beta_2 =$	0.700	1000	0.703	0.051	0.704	0.081	0.700	0.061
		2000	0.700	0.034	0.701	0.055	0.701	0.042
		500	1.100	0.057	1.108	0.194	1.095	0.080
$\beta_3 =$	1.100	1000	1.100	0.041	1.099	0.141	1.101	0.054
		2000	1.099	0.030	1.099	0.094	1.099	0.039
		500	1.211	0.124	1.220	0.102	1.214	0.119
$\phi =$	1.200	1000	1.204	0.088	1.214	0.070	1.206	0.088
		2000	1.202	0.060	1.206	0.048	1.203	0.060
		500	-0.486	0.165	-0.509	0.113	-0.482	0.163
$\rho =$	-0.500	1000	-0.494	0.122	-0.493	0.087	-0.495	0.114
		2000	-0.500	0.080	-0.498	0.060	-0.497	0.079

Table 3. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.611	0.086	1.607	0.223	1.608	0.090
$\gamma_1 =$	1.600	1000	1.605	0.059	1.607	0.156	1.607	0.061
		2000	1.600	0.042	1.604	0.106	1.601	0.043
		500	0.818	0.098	0.821	0.206	0.810	0.127
$\gamma_2 =$	0.800	1000	0.803	0.067	0.805	0.154	0.811	0.087
		2000	0.803	0.044	0.802	0.103	0.806	0.064
		500	0.203	0.082	0.229	0.362	0.210	0.121
$\gamma_3 =$	0.200	1000	0.203	0.061	0.191	0.246	0.199	0.079
		2000	0.200	0.044	0.202	0.167	0.198	0.059
		500	0.716	0.094	0.718	0.061	0.714	0.097
$\gamma_4 =$	0.700	1000	0.704	0.063	0.706	0.041	0.705	0.070
		2000	0.705	0.045	0.704	0.028	0.704	0.042
		500	1.005	0.128	0.996	0.135	1.001	0.131
$\beta_1 =$	1.000	1000	0.999	0.090	0.995	0.099	0.999	0.090
		2000	1.002	0.062	0.998	0.068	1.000	0.063
		500	0.697	0.075	0.697	0.118	0.709	0.098
$\beta_2 =$	0.700	1000	0.703	0.054	0.704	0.082	0.700	0.066
		2000	0.699	0.036	0.701	0.056	0.701	0.044
		500	1.099	0.057	1.105	0.192	1.096	0.080
$\beta_3 =$	1.100	1000	1.101	0.043	1.098	0.141	1.100	0.055
		2000	1.099	0.030	1.099	0.096	1.100	0.040
		500	1.200	0.103	1.221	0.101	1.202	0.101
$\phi =$	1.200	1000	1.199	0.069	1.211	0.068	1.201	0.071
		2000	1.200	0.048	1.205	0.046	1.201	0.047
		500	-0.196	0.196	-0.211	0.140	-0.192	0.205
$\rho =$	-0.200	1000	-0.197	0.151	-0.195	0.105	-0.200	0.147
		2000	-0.202	0.103	-0.198	0.076	-0.198	0.100

Table 4. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.612	0.085	1.608	0.219	1.608	0.089
$\gamma_1 =$	1.600	1000	1.605	0.059	1.606	0.154	1.607	0.062
		2000	1.600	0.042	1.602	0.109	1.601	0.043
		500	0.815	0.097	0.824	0.209	0.809	0.127
$\gamma_2 =$	0.800	1000	0.804	0.066	0.804	0.152	0.810	0.087
		2000	0.802	0.044	0.803	0.105	0.805	0.063
		500	0.205	0.081	0.223	0.360	0.209	0.121
$\gamma_3 =$	0.200	1000	0.203	0.060	0.188	0.243	0.199	0.079
		2000	0.200	0.044	0.201	0.172	0.199	0.058
		500	0.714	0.094	0.717	0.060	0.713	0.098
$\gamma_4 =$	0.700	1000	0.705	0.064	0.705	0.039	0.705	0.069
		2000	0.705	0.045	0.703	0.028	0.704	0.043
		500	1.005	0.116	0.996	0.132	1.006	0.118
$\beta_1 =$	1.000	1000	1.000	0.082	0.995	0.098	1.003	0.082
		2000	1.003	0.056	0.998	0.067	1.000	0.058
		500	0.698	0.075	0.698	0.117	0.706	0.098
$\beta_2 =$	0.700	1000	0.702	0.055	0.703	0.083	0.697	0.066
		2000	0.699	0.037	0.701	0.057	0.700	0.044
		500	1.099	0.056	1.103	0.192	1.096	0.080
$\beta_3 =$	1.100	1000	1.101	0.043	1.099	0.140	1.100	0.055
		2000	1.099	0.030	1.099	0.097	1.100	0.040
		500	1.199	0.098	1.222	0.101	1.199	0.098
$\phi =$	1.200	1000	1.198	0.064	1.211	0.068	1.199	0.066
		2000	1.200	0.044	1.205	0.046	1.200	0.044
		500	0.000	0.205	-0.008	0.147	-0.003	0.210
$\rho =$	0.000	1000	-0.001	0.155	0.003	0.109	-0.010	0.153
		2000	-0.004	0.107	0.003	0.079	-0.001	0.104

Table 5. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

			Configuration 1		Configuration 2		Configuration 3	
Parameters		n	Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.611	0.084	1.610	0.222	1.609	0.090
$\gamma_1 =$	1.600	1000	1.605	0.060	1.607	0.156	1.607	0.062
		2000	1.599	0.042	1.603	0.108	1.602	0.043
		500	0.814	0.098	0.825	0.207	0.808	0.126
$\gamma_2 =$	0.800	1000	0.804	0.066	0.803	0.152	0.810	0.086
		2000	0.803	0.044	0.805	0.105	0.805	0.063
		500	0.205	0.081	0.220	0.366	0.210	0.121
$\gamma_3 =$	0.200	1000	0.204	0.062	0.188	0.247	0.200	0.080
		2000	0.200	0.043	0.202	0.172	0.198	0.058
		500	0.713	0.094	0.717	0.059	0.712	0.098
$\gamma_4 =$	0.700	1000	0.705	0.064	0.705	0.039	0.706	0.068
		2000	0.705	0.044	0.703	0.027	0.704	0.044
		500	1.007	0.101	0.996	0.130	1.005	0.104
$\beta_1 =$	1.000	1000	1.000	0.072	0.995	0.097	1.004	0.071
		2000	1.002	0.050	0.998	0.067	1.000	0.051
		500	0.696	0.074	0.698	0.117	0.705	0.096
$\beta_2 =$	0.700	1000	0.702	0.055	0.704	0.084	0.697	0.066
		2000	0.699	0.037	0.701	0.057	0.701	0.044
		500	1.099	0.057	1.103	0.189	1.097	0.081
$\beta_3 =$	1.100	1000	1.101	0.042	1.100	0.139	1.100	0.055
		2000	1.099	0.030	1.099	0.096	1.100	0.040
		500	1.202	0.104	1.222	0.102	1.201	0.107
$\phi =$	1.200	1000	1.199	0.069	1.210	0.068	1.202	0.070
		2000	1.201	0.047	1.204	0.046	1.200	0.048
		500	0.192	0.197	0.193	0.147	0.192	0.199
$\rho =$	0.200	1000	0.195	0.148	0.202	0.105	0.186	0.149
		2000	0.195	0.104	0.200	0.076	0.199	0.100

Table 6. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

			Configuration 1		Configuration 2		Configuration 3	
Parameters		n	Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.612	0.083	1.608	0.220	1.606	0.088
$\gamma_1 =$	1.600	1000	1.605	0.060	1.609	0.157	1.606	0.061
		2000	1.599	0.042	1.605	0.107	1.602	0.044
		500	0.815	0.096	0.822	0.207	0.807	0.128
$\gamma_2 =$	0.800	1000	0.805	0.066	0.801	0.151	0.810	0.086
		2000	0.802	0.043	0.804	0.102	0.805	0.063
		500	0.208	0.081	0.214	0.355	0.210	0.120
$\gamma_3 =$	0.200	1000	0.204	0.061	0.190	0.245	0.199	0.079
		2000	0.200	0.044	0.202	0.169	0.199	0.057
		500	0.712	0.093	0.713	0.056	0.710	0.093
$\gamma_4 =$	0.700	1000	0.707	0.061	0.706	0.040	0.706	0.066
		2000	0.705	0.043	0.704	0.028	0.704	0.042
		500	1.005	0.079	0.995	0.125	1.005	0.086
$\beta_1 =$	1.000	1000	1.000	0.057	0.998	0.093	1.003	0.056
		2000	1.001	0.039	0.999	0.065	1.000	0.040
		500	0.696	0.070	0.699	0.115	0.704	0.093
$\beta_2 =$	0.700	1000	0.701	0.052	0.703	0.083	0.697	0.064
		2000	0.699	0.035	0.701	0.057	0.700	0.043
		500	1.100	0.056	1.102	0.185	1.097	0.080
$\beta_3 =$	1.100	1000	1.101	0.042	1.098	0.136	1.100	0.054
		2000	1.099	0.030	1.097	0.096	1.100	0.039
		500	1.210	0.130	1.226	0.105	1.211	0.131
$\phi =$	1.200	1000	1.205	0.086	1.210	0.070	1.211	0.088
		2000	1.204	0.062	1.205	0.047	1.203	0.060
		500	0.490	0.158	0.496	0.124	0.487	0.168
$\rho =$	0.500	1000	0.493	0.118	0.500	0.086	0.483	0.122
		2000	0.495	0.081	0.502	0.062	0.496	0.078

- **Scenario 2**

We conducted Monte Carlo simulations to evaluate and compare the performance of the maximum likelihood estimators of the parameters of the Heckman-BS model for finite samples and under the absence of exclusion constraint. We set $\beta_1 = 1, \beta_2 = 0.7, \beta_3 = 1.1, \gamma_1 = 1.6, \gamma_2 = 0.8, \gamma_3 = 0.2, \phi = 1.2$ and consider some values for the correlation parameter ρ.

Table 7. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.7$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.607	0.086	1.622	0.190	1.604	0.087
$\gamma_1 =$	1.600	1000	1.603	0.060	1.612	0.134	1.604	0.059
		2000	1.601	0.042	1.602	0.088	1.601	0.040
		500	0.813	0.089	0.793	0.328	0.804	0.117
$\gamma_2 =$	0.800	1000	0.808	0.065	0.800	0.232	0.801	0.081
		2000	0.804	0.046	0.804	0.160	0.802	0.064
		500	0.203	0.023	0.203	0.016	0.206	0.110
$\gamma_3 =$	0.200	1000	0.202	0.016	0.202	0.011	0.196	0.074
		2000	0.201	0.011	0.201	0.007	0.198	0.054
		500	0.914	0.281	0.987	0.160	0.856	0.339
$\beta_1 =$	1.000	1000	0.975	0.147	0.990	0.109	0.794	0.390
		2000	0.996	0.073	0.997	0.078	0.963	0.176
		500	0.742	0.141	0.699	0.192	0.779	0.195
$\beta_2 =$	0.700	1000	0.710	0.071	0.710	0.139	0.809	0.219
		2000	0.701	0.035	0.698	0.095	0.720	0.095
		500	1.111	0.035	1.101	0.009	1.113	0.087
$\beta_3 =$	1.100	1000	1.102	0.018	1.101	0.006	1.126	0.072
		2000	1.100	0.008	1.100	0.004	1.104	0.044
		500	1.227	0.161	1.225	0.117	1.231	0.172
$\phi =$	1.200	1000	1.214	0.107	1.210	0.082	1.211	0.129
		2000	1.207	0.075	1.207	0.057	1.205	0.084
		500	-0.541	0.459	-0.683	0.133	-0.435	0.595
$\rho =$	-0.700	1000	-0.662	0.209	-0.697	0.079	-0.322	0.727
		2000	-0.695	0.060	-0.698	0.056	-0.629	0.312

Table 8. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.609	0.087	1.624	0.195	1.604	0.084
$\gamma_1 =$	1.600	1000	1.605	0.060	1.611	0.136	1.604	0.059
		2000	1.601	0.042	1.601	0.090	1.601	0.040
		500	0.816	0.091	0.796	0.339	0.805	0.115
$\gamma_2 =$	0.800	1000	0.807	0.066	0.799	0.238	0.806	0.081
		2000	0.804	0.047	0.804	0.160	0.805	0.063
		500	0.203	0.023	0.204	0.016	0.207	0.113
$\gamma_3 =$	0.200	1000	0.202	0.016	0.201	0.011	0.197	0.075
		2000	0.201	0.011	0.201	0.008	0.199	0.056
		500	0.959	0.220	0.986	0.172	0.937	0.236
$\beta_1 =$	1.000	1000	0.983	0.137	0.990	0.119	0.942	0.201
		2000	0.998	0.087	0.999	0.084	0.996	0.100
		500	0.723	0.110	0.698	0.198	0.742	0.145
$\beta_2 =$	0.700	1000	0.708	0.067	0.709	0.142	0.731	0.116
		2000	0.701	0.041	0.698	0.098	0.704	0.052
		500	1.106	0.028	1.101	0.011	1.104	0.083
$\beta_3 =$	1.100	1000	1.102	0.016	1.101	0.007	1.108	0.059
		2000	1.100	0.010	1.100	0.005	1.100	0.040
		500	1.206	0.143	1.219	0.110	1.207	0.150
$\phi =$	1.200	1000	1.206	0.096	1.208	0.078	1.204	0.116
		2000	1.203	0.073	1.205	0.054	1.202	0.080
		500	-0.404	0.342	-0.472	0.195	-0.361	0.402
$\rho =$	-0.500	1000	-0.465	0.194	-0.493	0.121	-0.384	0.356
		2000	-0.492	0.106	-0.499	0.084	-0.485	0.129

Table 9. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.611	0.086	1.626	0.202	1.604	0.082
$\gamma_1 =$	1.600	1000	1.605	0.061	1.609	0.137	1.604	0.059
		2000	1.602	0.042	1.602	0.092	1.601	0.040
		500	0.816	0.093	0.795	0.352	0.805	0.119
$\gamma_2 =$	0.800	1000	0.808	0.067	0.800	0.242	0.806	0.083
		2000	0.803	0.048	0.803	0.166	0.805	0.063
		500	0.203	0.023	0.203	0.016	0.208	0.118
$\gamma_3 =$	0.200	1000	0.202	0.016	0.201	0.011	0.199	0.075
		2000	0.201	0.011	0.200	0.008	0.199	0.056
		500	0.996	0.185	0.993	0.173	0.992	0.192
$\beta_1 =$	1.000	1000	0.997	0.128	0.992	0.125	0.996	0.143
		2000	1.003	0.091	1.000	0.086	1.001	0.102
		500	0.710	0.101	0.695	0.201	0.719	0.123
$\beta_2 =$	0.700	1000	0.704	0.068	0.710	0.146	0.705	0.087
		2000	0.700	0.046	0.698	0.099	0.704	0.059
		500	1.103	0.026	1.101	0.011	1.099	0.081
$\beta_3 =$	1.100	1000	1.101	0.016	1.100	0.008	1.102	0.057
		2000	1.100	0.011	1.100	0.005	1.100	0.040
		500	1.176	0.118	1.209	0.096	1.176	0.119
$\phi =$	1.200	1000	1.190	0.074	1.202	0.066	1.184	0.084
		2000	1.194	0.054	1.203	0.047	1.191	0.059
		500	-0.154	0.312	-0.178	0.233	-0.145	0.323
$\rho =$	-0.200	1000	-0.183	0.209	-0.195	0.160	-0.175	0.240
		2000	-0.197	0.149	-0.201	0.114	-0.189	0.169

Table 10. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters		n	Configuration 1		Configuration 2		Configuration 3	
			Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.608	0.085	1.629	0.201	1.607	0.083
$\gamma_1 =$	1.600	1000	1.605	0.062	1.609	0.138	1.605	0.060
		2000	1.601	0.043	1.602	0.095	1.601	0.041
		500	0.812	0.093	0.790	0.345	0.806	0.122
$\gamma_2 =$	0.800	1000	0.809	0.067	0.801	0.242	0.807	0.084
		2000	0.803	0.047	0.801	0.170	0.804	0.063
		500	0.203	0.023	0.204	0.016	0.207	0.121
$\gamma_3 =$	0.200	1000	0.202	0.017	0.201	0.011	0.198	0.076
		2000	0.201	0.011	0.200	0.008	0.199	0.056
		500	1.011	0.175	1.001	0.167	1.016	0.180
$\beta_1 =$	1.000	1000	1.002	0.119	0.995	0.122	1.012	0.134
		2000	1.007	0.086	1.001	0.084	1.005	0.092
		500	0.704	0.098	0.692	0.201	0.708	0.123
$\beta_2 =$	0.700	1000	0.702	0.067	0.709	0.146	0.697	0.087
		2000	0.698	0.048	0.697	0.099	0.702	0.060
		500	1.101	0.026	1.100	0.012	1.097	0.083
$\beta_3 =$	1.100	1000	1.100	0.017	1.100	0.008	1.100	0.057
		2000	1.100	0.012	1.100	0.006	1.099	0.040
		500	1.169	0.118	1.205	0.093	1.165	0.119
$\phi =$	1.200	1000	1.186	0.069	1.201	0.065	1.178	0.079
		2000	1.192	0.049	1.202	0.045	1.188	0.051
		500	0.012	0.305	0.011	0.239	0.007	0.320
$\rho =$	0.000	1000	0.006	0.215	-0.003	0.168	-0.008	0.241
		2000	-0.007	0.163	-0.002	0.121	-0.000	0.175

Table 11. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

			Configuration 1		Configuration 2		Configuration 3	
Parameters		n	Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.608	0.085	1.632	0.206	1.605	0.083
$\gamma_1 =$	1.600	1000	1.604	0.061	1.611	0.138	1.604	0.061
		2000	1.601	0.042	1.602	0.095	1.600	0.041
		500	0.814	0.093	0.790	0.349	0.808	0.121
$\gamma_2 =$	0.800	1000	0.809	0.068	0.798	0.243	0.807	0.084
		2000	0.804	0.046	0.801	0.170	0.803	0.062
		500	0.202	0.023	0.204	0.017	0.206	0.123
$\gamma_3 =$	0.200	1000	0.202	0.017	0.201	0.011	0.198	0.077
		2000	0.201	0.011	0.200	0.008	0.200	0.055
		500	1.030	0.171	1.006	0.161	1.027	0.169
$\beta_1 =$	1.000	1000	1.008	0.109	0.998	0.116	1.016	0.120
		2000	1.008	0.077	1.002	0.079	1.008	0.082
		500	0.692	0.099	0.692	0.204	0.700	0.122
$\beta_2 =$	0.700	1000	0.699	0.067	0.708	0.145	0.693	0.087
		2000	0.697	0.049	0.696	0.099	0.699	0.061
		500	1.098	0.026	1.100	0.011	1.095	0.083
$\beta_3 =$	1.100	1000	1.100	0.017	1.100	0.008	1.099	0.056
		2000	1.099	0.012	1.100	0.005	1.099	0.039
		500	1.175	0.121	1.205	0.096	1.169	0.125
$\phi =$	1.200	1000	1.188	0.075	1.203	0.068	1.184	0.083
		2000	1.195	0.056	1.203	0.046	1.191	0.058
		500	0.165	0.307	0.198	0.234	0.171	0.322
$\rho =$	0.200	1000	0.189	0.213	0.188	0.162	0.173	0.238
		2000	0.187	0.160	0.194	0.116	0.189	0.173

Table 12. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

			Configuration 1		Configuration 2		Configuration 3	
Parameters		n	Mean	RMSE	Mean	RMSE	Mean	RMSE
		500	1.607	0.086	1.624	0.203	1.606	0.082
$\gamma_1 =$	1.600	1000	1.604	0.062	1.609	0.134	1.604	0.060
		2000	1.601	0.042	1.602	0.093	1.599	0.042
		500	0.816	0.091	0.803	0.346	0.808	0.123
$\gamma_2 =$	0.800	1000	0.809	0.067	0.799	0.234	0.807	0.082
		2000	0.804	0.047	0.802	0.161	0.803	0.062
		500	0.203	0.023	0.204	0.017	0.205	0.118
$\gamma_3 =$	0.200	1000	0.202	0.016	0.201	0.011	0.198	0.076
		2000	0.201	0.011	0.201	0.008	0.200	0.056
		500	1.043	0.169	1.008	0.141	1.067	0.217
$\beta_1 =$	1.000	1000	1.012	0.099	0.998	0.099	1.014	0.101
		2000	1.004	0.053	1.001	0.067	1.009	0.075
		500	0.680	0.104	0.691	0.198	0.672	0.138
$\beta_2 =$	0.700	1000	0.694	0.065	0.704	0.140	0.692	0.083
		2000	0.697	0.043	0.697	0.098	0.696	0.059
		500	1.096	0.026	1.099	0.010	1.090	0.084
$\beta_3 =$	1.100	1000	1.099	0.016	1.100	0.007	1.099	0.054
		2000	1.099	0.011	1.100	0.005	1.099	0.039
		500	1.207	0.146	1.215	0.107	1.199	0.151
$\phi =$	1.200	1000	1.206	0.096	1.209	0.076	1.204	0.111
		2000	1.204	0.076	1.205	0.051	1.203	0.082
		500	0.415	0.317	0.485	0.184	0.375	0.396
$\rho =$	0.500	1000	0.469	0.196	0.490	0.119	0.462	0.209
		2000	0.488	0.118	0.496	0.083	0.476	0.158