Convergence Properties of a Sequential Regression Multiple Imputation Algorithm: Supplementary Material

Jian Zhu¹, Trivellore E. Raghunathan²

Department of Biostatistics, School of Public Health, University of Michigan, Ann Arbor

Proof of Theorem 3

Proof: As in Theorem 1, we first prove the theorem for exactly specified model sequences, where $m_i(y_i | y_{[-i]}, \theta_i = \psi_i) = f_i(y_i | y_{[-i]}, \psi_i)$ for $i = 1, \dots, p$. We need to show that for each regression model, the asymptotic score equation $\tilde{s}_i(\psi_i | \psi_{[-i]})$ holds.

Denoting $u_i(y_i | y_{[-i]}) = \partial \log(m_i(y_i | y_{[-i]}, \theta_i)) / \partial \theta_i$ and n_i the sample size of each missingness group, then the asymptotic function for the i th model given $\theta_{[-i]}^* = \psi_{[-i]}$ is as

$$\tilde{s}_i(\theta_i | \psi_{[-i]}) = \tilde{s}_i(\theta_i | R = 0) + \sum_{j \neq i} \tilde{s}_i(\theta_i | \psi_j, R = j),$$

where

$$\tilde{s}_i(\theta_i | R = 0) = \int \cdots \int u_i(y_i | y_{[-i]}) n_0 f(y_1, \dots, y_p | R = 0) dy_1 \cdots dy_p,$$

and for $j \neq i$,

$$\begin{aligned} \tilde{s}_i(\theta_i | \psi_j, R = j) &= \int \cdots \int u_i(y_i | y_{[-i]}) n_j m_j(y_j | y_{[-j]}, \psi_j) f(y_{[-j]} | R = j) dy_1 \cdots dy_p \\ &= \int \cdots \int u_i(y_i | y_{[-i]}) n_j f(y_1, \dots, y_p | R = j) dy_1 \cdots dy_p. \end{aligned}$$

Since the missingness is ignorable, we have

$$\sum_{j=0, j \neq i}^p n_j f(y_1, \dots, y_p | R = j) = (n - n_i) f(y_1, \dots, y_p | R \neq i) = (n - n_i) m_i(y_i | y_{[-i]}, \psi_i) f(y_{[-i]} | R \neq i),$$

and the asymptotic function can be rewritten as below:

$$\tilde{s}_i(\theta_i | \psi_{[-i]}) = (n - n_i) \int \cdots \int u_i(y_i | y_{[-i]}) m_i(y_i | y_{[-i]}, \psi_i) f(y_{[-i]} | R \neq i) dy_1 \cdots dy_p,$$

and it is easy to show that the asymptotic score equation holds:

$$\tilde{s}_i(\psi_i | \psi_{[-i]}) = (n - n_i) \int \cdots \int u_i(y_i | y_{[-i]}) m_i(y_i | y_{[-i]}, \psi_i) f(y_{[-i]} | R \neq i) dy_1 \cdots dy_p \Big|_{\theta_i = \psi_i} = 0.$$

¹jianzhu@umich.edu

²teraghu@umich.edu

Therefore, as $n, t \rightarrow \infty$, $\theta_i^{(t)} \rightarrow \psi_i$, which leads to that $m_i(y_i | y_{[-i]}, \theta_i^{(t)}) \rightarrow f_i(y_i | y_{[-i]}, \psi_i)$, for $i = 1, \dots, p$.

We now prove the theorem for validly specified model sequences with extra terms compared to the true conditional densities. As in Theorem 2, suppose that without loss of any generality we introduce a parameterization $\theta_i = (\zeta_i, \xi_i)$ such that $m_i(y_i | y_{[-i]}, \zeta_i = \psi_i, \xi_i = 0) = f_i(y_i | y_{[-i]}, \psi_i)$. We need to show that $\{\theta_i^* = (\psi_i, 0), i = 1, \dots, p\}$ is the convergent point of the asymptotic iterative algorithm.

Given that $\theta_j^* = (\psi_j, 0)$ for $j \neq i$,

$$\tilde{s}_i(\theta_i | \theta_{[-i]}^*) = (n - n_i) \int \cdots \int u_i(y_i | y_{[-i]}) f_i(y_i | y_{[-i]}, \psi_i) f(y_{[-i]} | R \neq i) dy_1 \cdots dy_p.$$

As in Theorem 2, to find the solution to $\tilde{s}_i(\theta_i | \theta_{[-i]}^*) = 0$ is equivalent to minimize $\int \cdots \int \log[f_i(y_i | y_{[-i]}, \psi_i)/m_i(y_i | y_{[-i]}, \theta_i)] f_i(y_i | y_{[-i]}, \psi_i) f(y_{[-i]} | R \neq i) dy_1 \cdots dy_p$. Since the relative entropy has non-negative values and its minimum 0 is reached if and only if $m_i(y_i | y_{[-i]}, \theta_i = (\psi_i, 0)) = f_i(y_i | y_{[-i]}, \psi_i)$. Therefore, the asymptotic score equation $\tilde{s}_i(\theta_i | \theta_{[-i]}^*) = 0$ holds at $(\theta_1^*, \dots, \theta_p^*)$ for any $i = 1, \dots, p$.

Proof of Theorem 4

Proof: As in Theorem 2, we first apply the joint model $p(y_1, \dots, y_p | \theta)$ to analyze the incomplete data to determine the convergent point. Denote $\tilde{Q}(\theta | \theta^{(t-1)})$ the expected log-likelihood from the Expectation-Maximization algorithm, where $\theta = (\theta_1, \dots, \theta_p) \in \Theta_C$. For $i = 1, \dots, p$, denote $\theta_{M,[-i]}$ the parameter of the marginal joint model of $Y_{[-i]}$ from the joint model, then because of the separability of marginal parameters, $\theta_{M,[-i]}$ is distinctive from θ_i and $\theta = (\theta_i, \theta_{M,[-i]})$ is a parameterization of the joint model. Since $\theta^* = (\theta_1^*, \dots, \theta_p^*)$ is the maximum likelihood estimate, $\partial \tilde{Q}(\theta | \theta^*) / \partial \theta|_{\theta=\theta^*} = 0$. On the other hand, because marginal separability ensures that

$$\partial \log p(y_1, \dots, y_p | \theta_i, \theta_{M,[-i]}) / \partial \theta_i = \partial (\log(m_i(y_i | y_{[-i]}, \theta_i))) / \partial \theta_i = u_i(y_i | y_{[-i]}),$$

we have

$$\begin{aligned} \partial \tilde{Q}(\theta_i, \theta_{M,[-i]}^* | \theta^*) / \partial \theta_i &= \int \cdots \int u_i(y_i | y_{[-i]}) n_0 f(y_1, \dots, y_p | R = 0) dy_1 \cdots dy_p \\ &+ \int \cdots \int u_i(y_i | y_{[-i]}) \sum_{j=1}^p [n_j m_j(y_j | y_{[-j]}, \theta_j^*) f(y_{[-j]} | R = j)] dy_1 \cdots dy_p \\ &= \tilde{s}_i(\theta_i | \theta_{[-i]}^*) \\ &+ n_i \int \cdots \int u_i(y_i | y_{[-i]}) m_i(y_i | y_{[-i]}, \theta_i^*) f(y_{[-i]} | R = i) dy_1 \cdots dy_p. \end{aligned}$$

Therefore, the asymptotic score equations hold at θ^* as for all $i = 1, \dots, p$,

$$\begin{aligned}
& \tilde{s}_i(\theta_i^* \mid \theta_{[-i]}^*) \\
= & \left[\partial \tilde{Q}(\theta_i, \theta_{M, [-i]}^* \mid \theta^*) / \partial \theta_i - n_i \int \cdots \int u_i(y_i \mid y_{[-i]}) m_i(y_i \mid y_{[-i]}, \theta_i^*) f(y_{[-i]} \mid R = i) dy_1 \cdots dy_p \right] \Big|_{\theta_i^*} \\
= & 0.
\end{aligned}$$

Therefore, as $n, t \rightarrow \infty$, $\theta_i^{(t)} \rightarrow \theta_i^*$, which leads to that $m_i(y_i \mid y_{[-i]}, \theta_i^{(t)}) \rightarrow p_i(y_i \mid y_{[-i]}, \theta_i^*)$, for $i = 1, \dots, p$.