Acoustic and inertial modes in planetary-like rotating ellipsoids

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Supplementary Material

ABSTRACT

Finding hydrostatic solutions in ellipsoids is a challenging task. An idealised reference state has been introduced in the main text, which admits simple polynomial expressions in the Cartesian coordinates. We outline a method to obtain more realistic background models, which are compatible with the polynomial algorithm.

1. SINGULARITY FOR A VANISHING DENSITY ON THE BOUNDARY

Clausen & Tilgner (2014) modelled planetary polytropic profiles in ellipsoids by considering power-law profiles in the form

$$\rho_0^* = \rho_c \left[1 - F\right]^{\Lambda}, \quad P_0^* = P_c \left(\rho_0^*/\rho_c\right)^{\Upsilon}, \quad F = (x/a)^2 + (y/b)^2 + (z/c)^2, \qquad \text{(1a–c)}$$

where $[\Lambda, \Upsilon]$ are positive exponents and $[\rho_c, P_c]$ the arbitrary central values. In the stellar context, Υ is referred as the polytropic exponent (Chandrasekhar 1958). The polynomial profile considered in the main text with $\alpha = 1$ is recovered from expressions (1) by setting $\Lambda = 1$ and $\Upsilon = 2$. Reference states (1) are unfortunately incompatible with the polynomial method. The normal component of $\boldsymbol{\xi}$ (at least), obtained from the weighted polynomial Helmholtz decomposition of $\rho_0 \boldsymbol{\zeta}$, diverges when $\rho_0 \to 0$ on the boundary (as first pointed out by Lebovitz 1989). Even if $\boldsymbol{\xi}$ were singular, we could still obtain solutions with finite kinetic energies $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle_{\rho_0} < \infty$. Yet, this approach is not entirely satisfactory to achieve numerical convergence.

To remove the singularity, additional regularity conditions should be enforced on the boundary. In the ellipsoid, Clausen & Tilgner (2014) obtained a wave-like equation that is separable for background states (1), such that the regularity conditions can be easily enforced numerically. However, this approach cannot be implemented with any global polynomial expansions in the Cartesian coordinates.

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2. POLYNOMIAL REFERENCE STATES

2.1. *Hydrostatic state in spheres*

Approximate hydrostatic states, which neglect rotational effects but obey Poisson equation, can be obtained in spheres as follows. We work in dimensional variables for the sake of generality. Given the polynomial density profile of the main text (equation (2.4a) in section 2c), the hydrostatic reference state is given by

$$\nabla^2 \Phi_0^* = 4\pi G\, \rho_0^*, \quad \boldsymbol{\nabla} P_0^* = -\rho_0^* \, \boldsymbol{\nabla} \Phi_0^*, \tag{2a,b}$$

with $\boldsymbol{g}^* = -\boldsymbol{\nabla}\Phi_0^*$ where Φ_0^* is the self-gravitational potential, and G is the gravitational constant. We solve equations (2) in spherical coordinates, denoting $r = r^*/R$ the dimensionless radial coordinate. From the Poisson equation, we get

$$\Phi_0^*(r) = \frac{K_c}{\rho_c}\left(r^2 - \frac{3\alpha}{10} r^4\right), \quad K_c = \frac{2\pi G \rho_c^2 R^2}{3}. \tag{3}$$

Then, we obtain the pressure profile from the hydrostatic equilibrium

$$P_0^*(r) = P_c - K_c\left(r^2 - \frac{4\alpha}{5} r^4 + \frac{\alpha^2}{5} r^6\right) \quad (\alpha \neq 1), \tag{4}$$

with P_c the central pressure and K_c that is now identified as the isentropic bulk modulus at the centre. For Jupiter-like models ($\alpha = 1$), we impose the boundary condition $P_0^*(1) = 0$, yielding $P_c = 2K_c/5 = 4\pi G \rho_c^2 R^2/15$. Hence, profile (4) gives for Jupiter-like models

$$P_0^*(r) = P_c\left(1 - \frac{5r^2}{2} + 2r^4 - \frac{r^6}{2}\right) \quad (\alpha = 1). \tag{5}$$

The speed of sound C_0^* is finally obtained from the Equation of State (EoS)

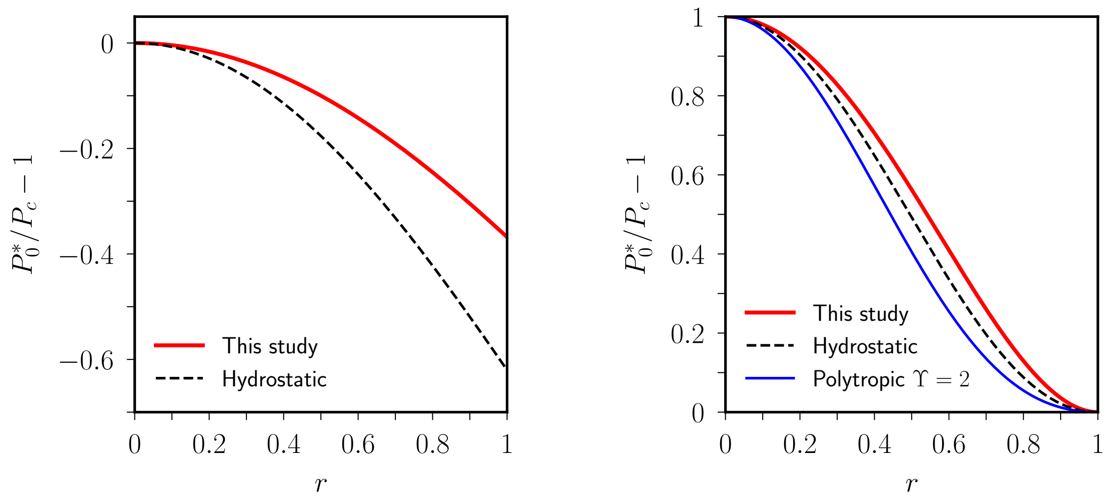
$$C_0^{*2} \boldsymbol{\nabla} \rho_0^* = (1-\Gamma)\, \boldsymbol{\nabla} P_0^* = (1-\Gamma)\, \rho_0^* \, \boldsymbol{g}^*, \tag{6}$$

where the stability parameter Γ measures the extent to which the equilibrium density profile departs from neutral stratification (Pekeris & Accad 1972). We can indeed introduce the Brunt-Väisälä frequency N_0^* defined as

$$N_0^{*2} = \boldsymbol{S}_0^* \boldsymbol{\cdot} \boldsymbol{g}^*, \quad \boldsymbol{S}_0^* = (1/\rho_0^*)\, \boldsymbol{\nabla} \rho_0^* - \boldsymbol{g}^*/C_0^{*2}, \tag{7a,b}$$

where $\boldsymbol{S}_0^*$ is the (vectorial) Schwarzschild discriminant (Dyson & Schutz 1979). Here, formula (7a) reduces to $N_0^{*2} = -\Gamma\, (\boldsymbol{g}^* \boldsymbol{\cdot} \boldsymbol{g}^*)/C_0^{*2}$ from equilibrium (6). Therefore, neutral isentropic interiors (i.e. $N_0^{*2} = 0$) are obtained with $\Gamma = 0$, and stably stratified ones (i.e. $N_0^{*2} > 0$) have $\Gamma < 0$. The speed of sound is then given by

$$C_0^{*2} = (1-\Gamma)\, C_c^2 \left(1 - \frac{8\alpha}{5} r^2 + \frac{3\alpha^2}{5} r^4\right), \tag{8}$$



(a) Earth-like model (b) Jupiter-like model

Figure 1. Pressure profiles as a function of the dimensionless spherical radius r. Red solid line: polytropic-like profile presented in the main text (see equation (2.8) in section 2c). (a) Earth-like model ($\alpha = 0.205$). Dashed black line: Earth-like hydrostatic profile (4). (b) Jupiter-like model ($\alpha = 1$). Dashed black line: Jupiter-like hydrostatic profile (5). Thin blue line: analytical polytropic profile with exponent $\Upsilon = 2$ (Chandrasekhar 1958).

with $C_c = \sqrt{K_c/(\alpha\rho_c)}$ the speed of sound at the centre.

We compare in figure 1 the polytropic-like profile of the main text (see equation (2.4a) in section 2c), with the non-polytropic hydrostatic profile (4) in the sphere. While the density profile is the same in both reference states, the central difference between the two reference states is that the gravity does not obey the Poisson equation in the former case, contrary to the latter reference state (even if the same background density profile is considered). We have also shown the analytical polytropic pressure profile with $\Upsilon = 2$ in figure 1b, obtained from the analytical polytropic density $\rho_0^*(r)/\rho_c = \sin(\pi r)/(\pi r)$ (Chandrasekhar 1958). Departures are more pronounced for the Earth-like model.

2.2. *Ellipsoidal re-scaling*

The above spherical profiles are converted into ellipsoidal profiles by using the geometrical transformation

$$r^2 = (x/a)^2 + (y/b)^2 + (z/c)^2 = F. \tag{9}$$

Then, we obtain the approximate ellipsoidal states

$$\rho_0^* = \rho_c\,(1 - \alpha F), \tag{10a}$$

$$P_0^* = P_c - K_c \left(F - \frac{4\alpha}{5} F^2 + \frac{\alpha^2}{5} F^3 \right), \tag{10b}$$

$$\boldsymbol{g}^* = -\alpha C_c^2\, \boldsymbol{\nabla} F \left(1 - \frac{3\alpha}{5} F \right), \tag{10c}$$

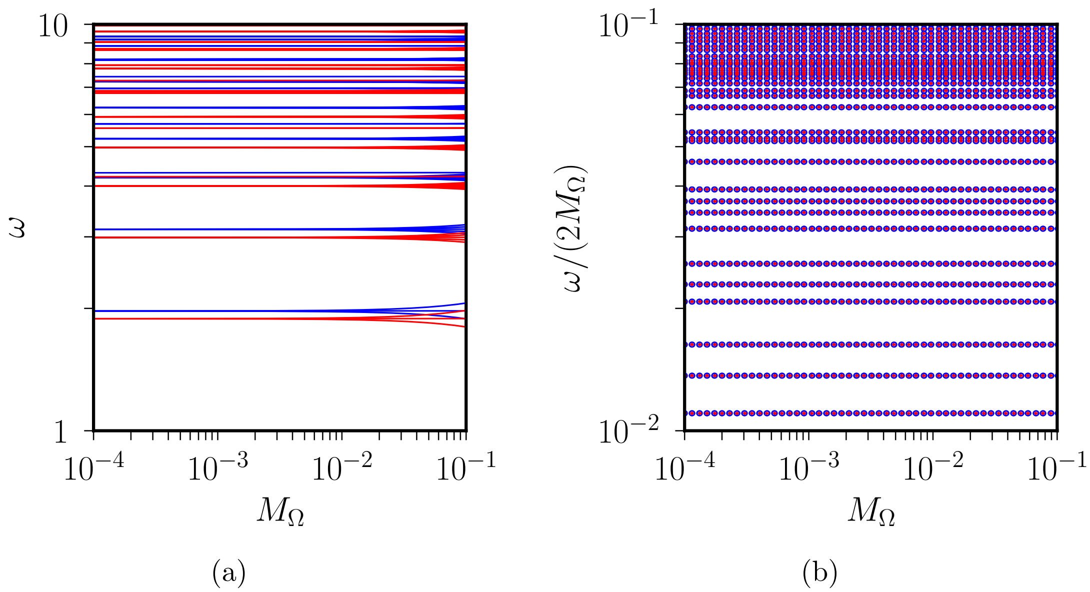


Figure 2. Comparison between the polynomial profiles in the sphere with $\alpha = 0.205$, for the acoustic modes in (a) and the inertial modes in (b). Polynomial solutions with $n = 15$. Blue curves/circles: polytropic-like hydrostatic reference state of the main text (equations (2.8)-(2.9) in section 2c). Red curves/squares: non-polytropic hydrostatic reference state given by expressions (10) with $\Gamma = 0$.

$$C_0^{*2} = (1-\Gamma)\, C_c^2 \left(1 - \frac{8\alpha}{5} F + \frac{3\alpha^2}{5} F^2\right). \tag{10d}$$

Given density profile (10a), ellipsoidal state (10) exactly satisfies the hydrostatic balance $\nabla P_0^* = \rho_0^* \boldsymbol{g}$ and also EoS (6) in the ellipsoid. The second-order terms in α^2 must be considered in expressions (10) for C_0^* to vanish on the boundary (when $\alpha = 1$).

2.3. *Numerical results*

The comparison between the background states in figure 1 may suggest that the compressible effects might be stronger in planetary interiors, because the polytropic-like pressure profile we have considered in the main text actually underestimates the planetary pressure variations across the fluid domain compared to (4). The normal modes obtained with both models are superimposed in figure 2, for an Earth-like model with $\alpha = 0.205$. We find that the acoustic modes are indeed quantitatively modified in frequency but, qualitatively, the modal properties are unchanged (e.g. here the scaling with M_Ω). Conversely, the inertial modes in figure 2b are much less sensitive to the background pressure variations. Actually, the acoustic modes more strongly depend on the background pressure profile (and so on the speed of sound), contrary to the inertial modes that are more sensitive to the background density variations (which are here identical in the two background reference models).

Therefore, we expect that the results presented in the main text remain qualitatively (for the acoustic modes) and quantitatively (for the inertial modes) valid in considering more accurate planetary models (at least for Earth-like conditions).

3. NON-POLYNOMIAL REFERENCE STATES

The global spectral polynomial method can virtually account for non-polynomial reference profiles, as long as the density does not vanish on the boundary. To do so, we could use the change of variables

$$x = a\, r \sin(\theta) \cos(\phi), \quad y = b\, r \sin(\theta) \sin(\phi), \quad z = c\, r \cos(\theta), \tag{11a–c}$$

introducing the spherical-like coordinates (r, θ, ϕ). The elementary volume becomes

$$\mathrm{d}V = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = abc\, r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi. \tag{12}$$

Then, the volume integrals could be computed numerically on the spherical-like mesh, by using efficient spherical methods (Schaeffer 2013). Therefore, the polynomial description could accommodate a wide class of density-pressure profiles, either analytically or numerically.

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