Emergent rewrites in knot theory and logic

[video] [js slides]

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I explain in what sense new graph rewrite systems emerge from given ones, with two examples:

- the emergence of the R3 (Reidemeister 3) rewrite from R1, R2 and some uniform continuity assumptions, and relations to curvature,
- the emergence of the beta rewrite in lambda calculus from the shuffle rewrite and relations to the commutativity of the addition of vectors in the tangent space of a manifold

From sub-riemannian geometry to emergent algebras

A riemannian manifold (X,g) is a length metric space (X,d) by Hopf-Rinow thm.

Problem 1: recover (X,g) from (X,d).

- (1935, A. Wald) problem 1 for 2-dim manifolds.
- (1948, A.D. Alexandrov) a metric notion of (sectional) curvature + smoothness solves 2-dim manifolds.
- (1982, A.D. Alexandrov) conjecture that the same is true for n-dim manifolds.
- (1998, I.G. Nikolaev) proves (Alexandrov conjecture) for n-dim manifolds.

but (1996, M. Gromov) asks for a solution of

Problem 2: recover sub-riemannian (X,D,g) from (X,d).

- (X,D,g) sub-riemannian if D completely non-integrable distribution and g a metric on D.
- by Hopf-Rinow (X,D,g) is a length metric space (X,d), with d the CC distance.

Sub-riemannian spaces are weird! (except when riemannian)

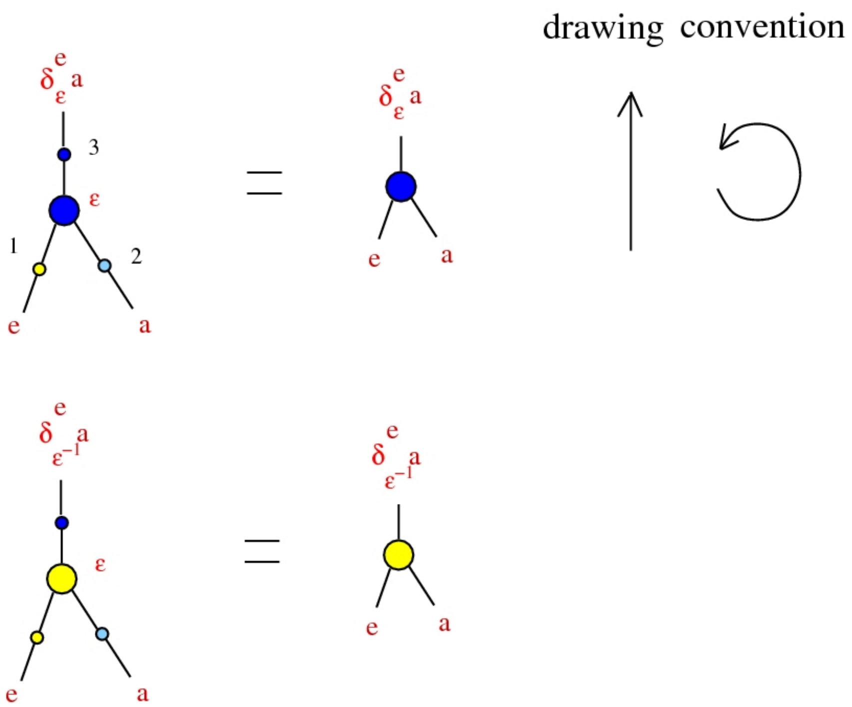
- metric (Hausdorff) dimension > topological dimension
- not Alexandrov spaces

- tangent spaces are nilpotent (Carnot) groups, not vector spaces
- Carnot groups have a peculiar differential calculus (Pansu derivative)

Sub-riemannian spaces (techniques) are useful!

- (1981, M. Gromov) finitely generated groups of polynomial growth same as those groups which have nilpotent subgroups of finite index
- (1989, P. Pansu) proves Rademacher thm for Carnot groups, which implies a short proof for Margulis-Mostow rigidity
- (2006, J.R. Lee, A. Naor) counter-example to Goemans-Linial conjecture by using the Heisenberg group as a SR space
- (2010, E. Hrushovski) (2011, E. Breuillard, B. Green, T. Tao) Approximate groups are essentially Carnot groups

Drawing conventions



Emergent algebras

Used in: A characterization of sub-riemannian spaces as length dilatation structures constructed via coherent projections a solution of the problem of intrinsic characterization of sub-riemannian manifolds posed by M. Gromov, 1996.

introduced as algebras in arXiv:0907.1520, as a λ calculus in arXiv:1807.02058.

emergent algebra:

– uniform space X

– with a family of idempotent right quasigroups

$$e \circ_\varepsilon a = \delta^e_\varepsilon a \qquad e \bullet_\varepsilon a = \delta^e_{\varepsilon^{-1}} a$$

(R1) $a \circ_\varepsilon a = a \bullet_\varepsilon a = a$

(R2) $a \circ_\varepsilon (a \bullet_\varepsilon b) = a \bullet_\varepsilon (a \circ_\varepsilon b) = b$

$$a \circ_\varepsilon (a \circ_\mu b) = a \circ_{\varepsilon\mu} b \qquad a \circ_1 b = b$$

– such that the uniform limits exist as $\varepsilon \longrightarrow 0$

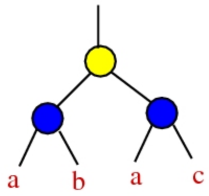
$$e \circ_\varepsilon a \longrightarrow e$$

$$(a \circ_\varepsilon b) \bullet_\varepsilon (a \circ_\varepsilon c) \longrightarrow \Delta^a(b, c)$$

as $\varepsilon \longrightarrow 0$ the following converge:

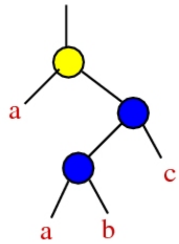
– approximate difference

$$\Delta^a_\varepsilon(b, c) = (a \circ_\varepsilon b) \bullet_\varepsilon (a \circ_\varepsilon c)$$



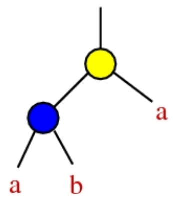
– approximate sum

$$\Sigma^a_\varepsilon(b, c) = a \bullet_\varepsilon ((a \circ_\varepsilon b) \circ_\varepsilon c)$$



– approximate inverse

$$inv^a_\varepsilon b = (a \circ_\varepsilon b) \bullet_\varepsilon a$$



examples:

– riemannian manifold $a \circ_{\varepsilon} b = \exp_a(\varepsilon \log_a b)$

$\Delta^a_{\varepsilon}(b,c) \longrightarrow \exp_a(-\log_a b + \log_a c)$

$\Sigma^a_{\varepsilon}(b,c) \longrightarrow \exp_a(\log_a b + \log_a c)$

$inv^a_{\varepsilon} b \longrightarrow \exp_a(-\log_a b)$

– Lie group $a \circ_{\varepsilon} b = a \exp(\varepsilon \log(a^{-1} b))$

$\Delta^a_{\varepsilon}(b,c) \longrightarrow a \exp(-\log(a^{-1}b) + \log(a^{-1}c))$

$\Sigma^a_{\varepsilon}(b,c) \longrightarrow a \exp(\log(a^{-1}b) + \log(a^{-1}c))$

$inv^a_{\varepsilon} b \longrightarrow a \exp(-\log(a^{-1}b))$

more interesting examples:

– sub–riemannian manifold $a \circ_{\varepsilon} b = \exp_a(\delta_{\varepsilon} \log_a b)$

$\Delta^a_{\varepsilon}(b,c) \longrightarrow \exp_a((\log_a b)^{-1} * (\log_a c))$

$\Sigma^a_{\varepsilon}(b,c) \longrightarrow \exp_a((\log_a b) * (\log_a c))$ Carnot group

$inv^a_{\varepsilon} b \longrightarrow \exp_a((\log_a b)^{-1})$

– group with dilations $a \circ_{\varepsilon} b = a\, \delta_{\varepsilon}(a^{-1} b)$

$\Delta^a_{\varepsilon}(b,c) \longrightarrow a((b^{-1}a) * (a^{-1}c))$

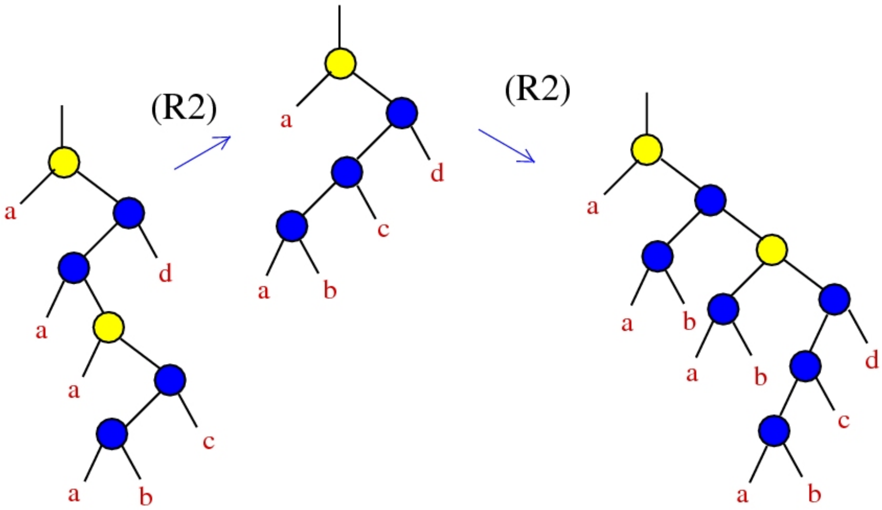
$\Sigma^a_{\varepsilon}(b,c) \longrightarrow a((a^{-1}b) * (a^{-1}c))$ conical group

$inv^a_{\varepsilon} b \longrightarrow a\, b^{-1} a$

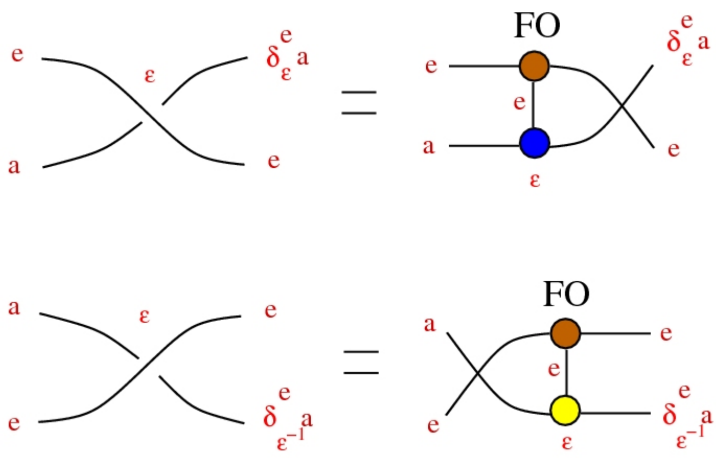
the properties of new operations emerge as $\varepsilon \longrightarrow 0$

example: associativity of sum

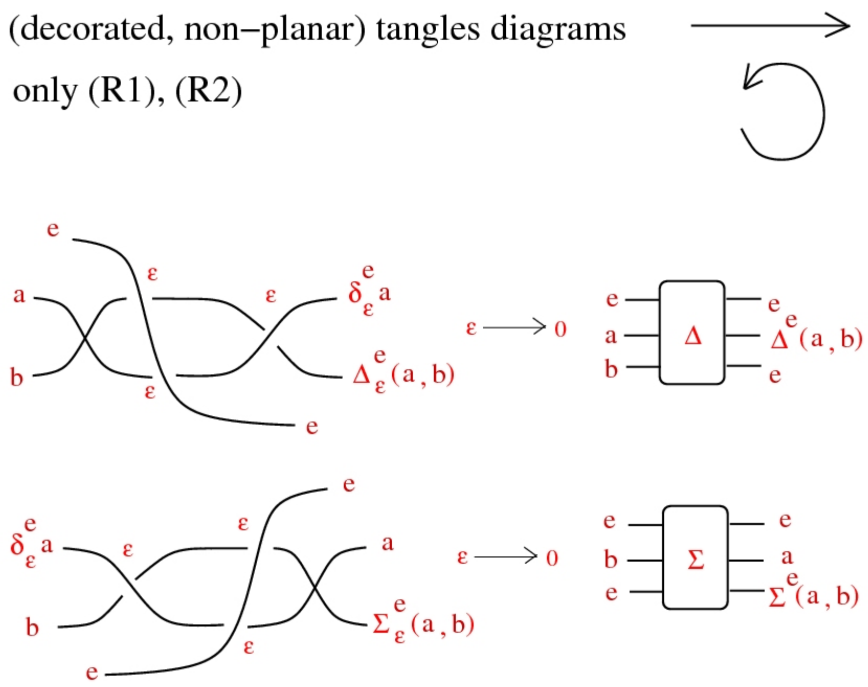
$$\Sigma^a_\varepsilon(\Sigma^a_\varepsilon(b,c),d) = \cdots = \Sigma^a_\varepsilon(b,\Sigma^{a \circ_\varepsilon b}_\varepsilon(c,d))$$



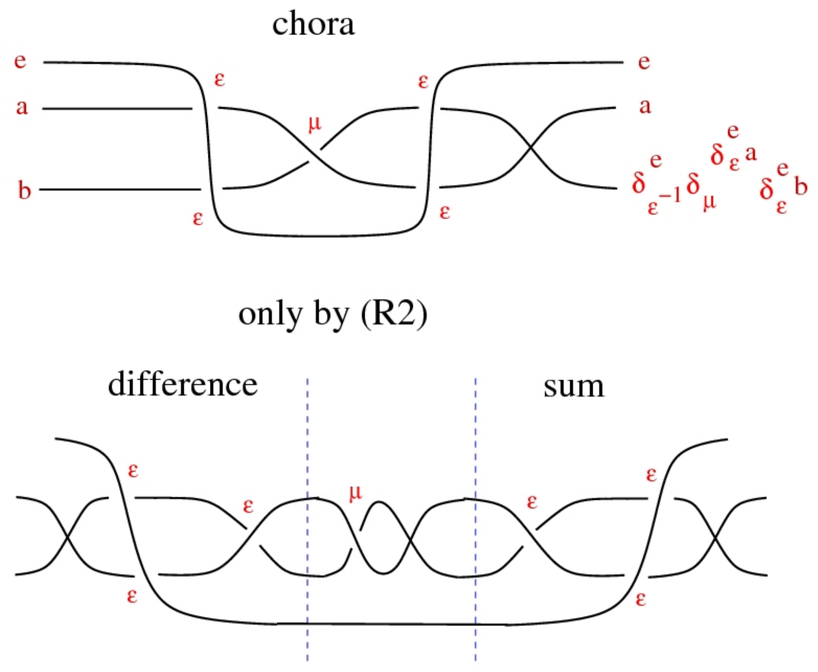
Drawing conventions 2

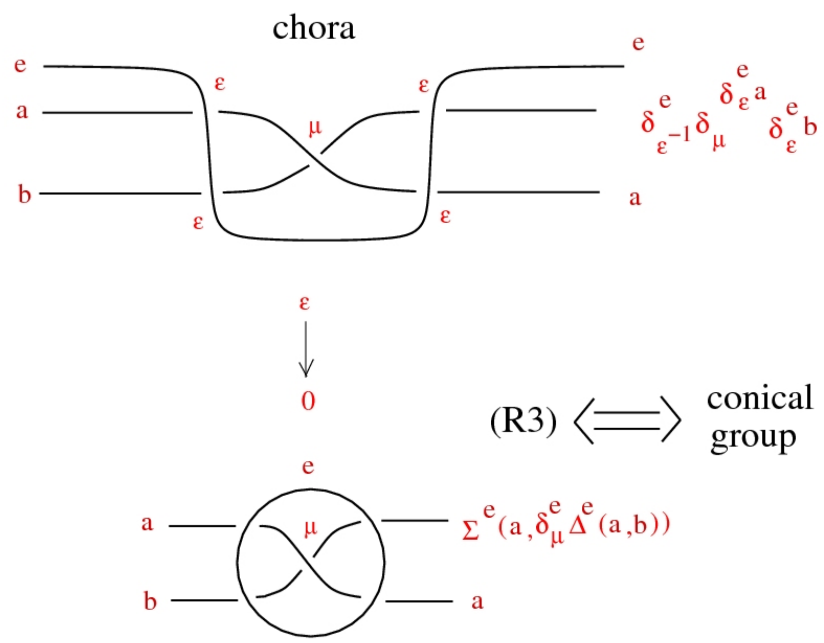


Tangles


(decorated, non–planar) tangles diagrams
only (R1), (R2)
ε ⟶ 0
Δ
Σ

Chora


chora
only by (R2)
difference
sum



Conical groups

groups with:

an action by automorphisms

a · (x * y) = (a · x) * (a · y)

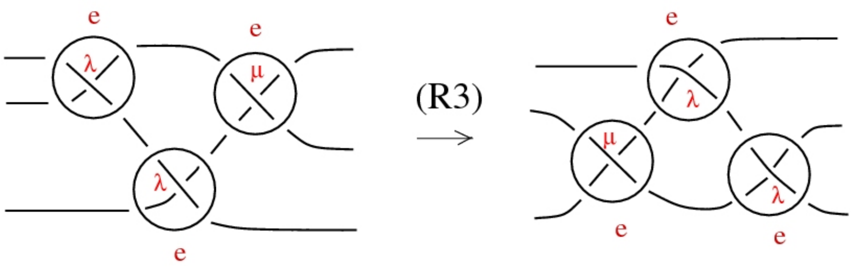
which are uniformly contractive:

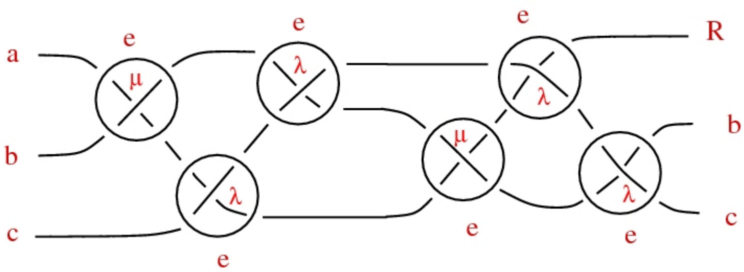
ε → 0, ε · x → e, uniformly wrt x in compact set

Examples

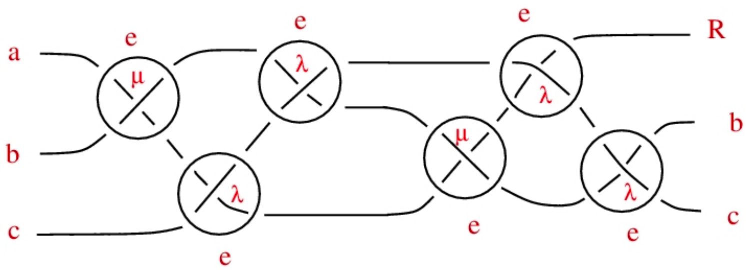
- (normed) finite dim vector space X, * = +, Γ = (0, ∞)
- Heisenberg groups:
  - - take (H, <,>) complex Hilbert space, X = H × R
  - (x,u) * (y,v) = (x + y, u + v + (1/2) Im <x,y>)
  - - distribution D from left translate of H in X, is completely nonintegrable
  - - metric on D from Re <,>
  - - Γ = C \ {0}
  - a · (x,u) = (a x , |a|² u)
- or just any Carnot group

Tangles





Rescale!



– rescale (X,d) at e and scale ε , get the space Y_ε

– rescale this at $\delta^e_\varepsilon c$ and scale λ , get the space Y^λ_ε

– compare the Gromov–Hausdorff distance between:

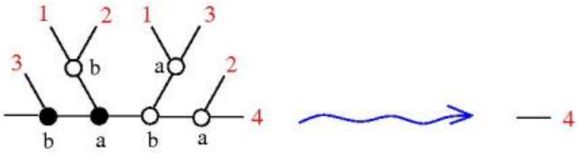
  – the rescale at $\delta^e_\varepsilon b$ and scale μ in Y_ε

  – the rescale at $\delta_\lambda^{\delta^e_\varepsilon c} \; \delta^e_\varepsilon b$ and scale μ in Y^λ_ε

Commutative emergent algebras

The operation * is commutative iff any of the following:

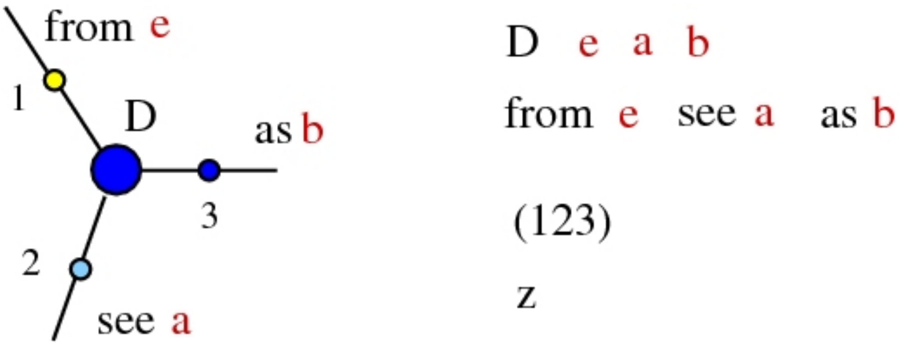
- - we can do the shuffle trick:



-
- - rays are semigroups, uniformly
- $\forall$ a, b $\in \Gamma$, $\exists$ a+b $\in \Gamma$, $(a \cdot x) * (b \cdot x) = (a+b) \cdot x$
- - barycentric condition: $\forall$ a $\in \Gamma$, $\exists$ 1-a $\in \Gamma$

Let's denote the 3 ports of a dilation node as:

port 1: "from", port 2: "see", port 3: "as"



The operation * is commutative iff all the 6 permutations of ports are also dilations, with coefficients from the anharmonic group: Pure See!

Lambda calculus

(1936, A. Church) Untyped λ calculus is a term rewrite system

Terms:

- variable: x, y, z, ...
- term:
  - - variable
  - - A B where A, B terms (application)
  - - λx.A where x var, A term (abstraction)

Term rewrite rule:

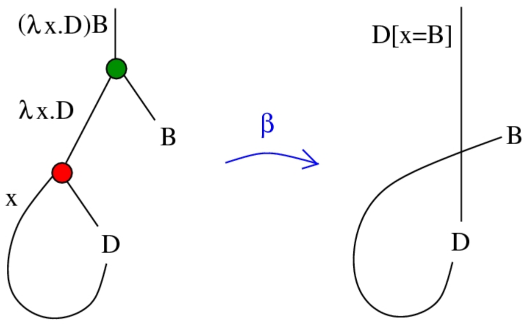
- β-reduction: (λx.D)B → D[x=B]

(1936, A. Church) Pure λ calculus is a term rewrite system

Term rewrite rule:

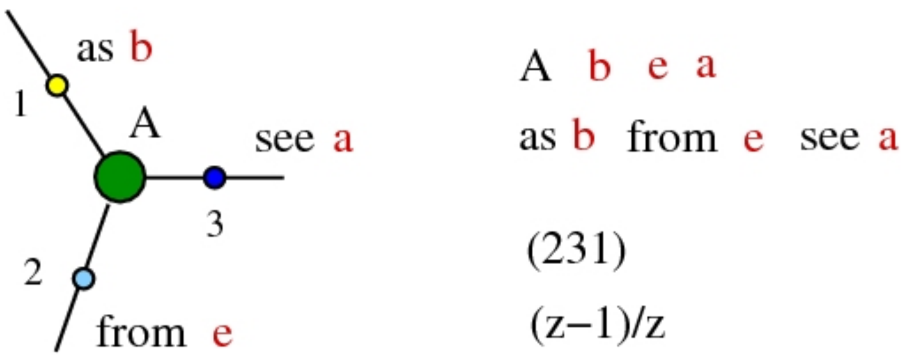
- β-reduction: (λx.D)B → D[x=B]

(1971, C.P. Wadsworth, 1990, J. Lamping) graph rewrite system



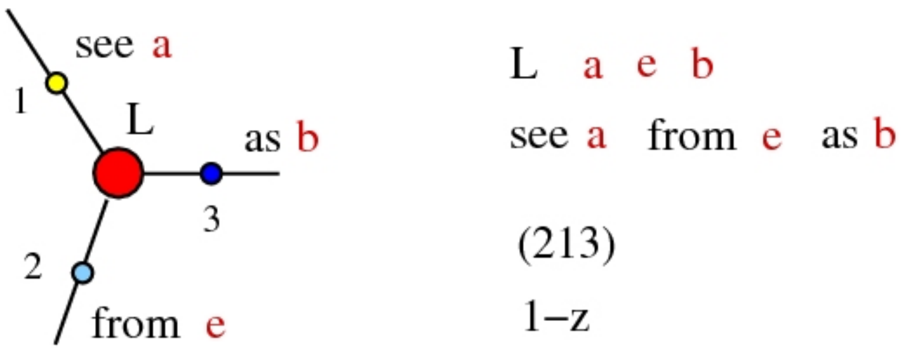
Emergent lambda

Aplication can be seen as:



(all the 6 permutations of ports are also dilations, with coefficients from the anharmonic group: Pure See!)

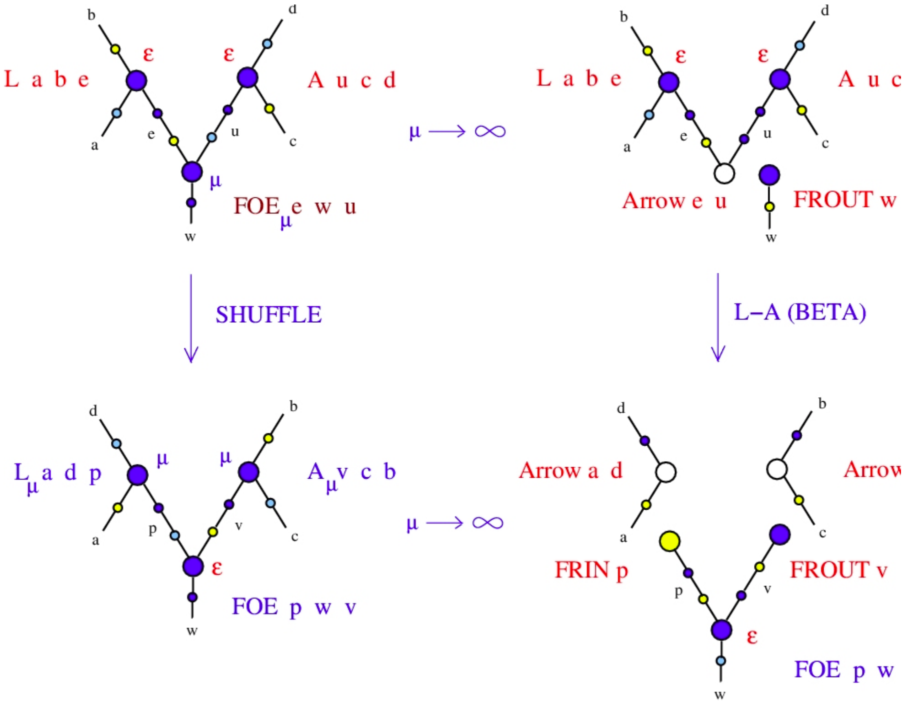
Abstraction can be seen as:



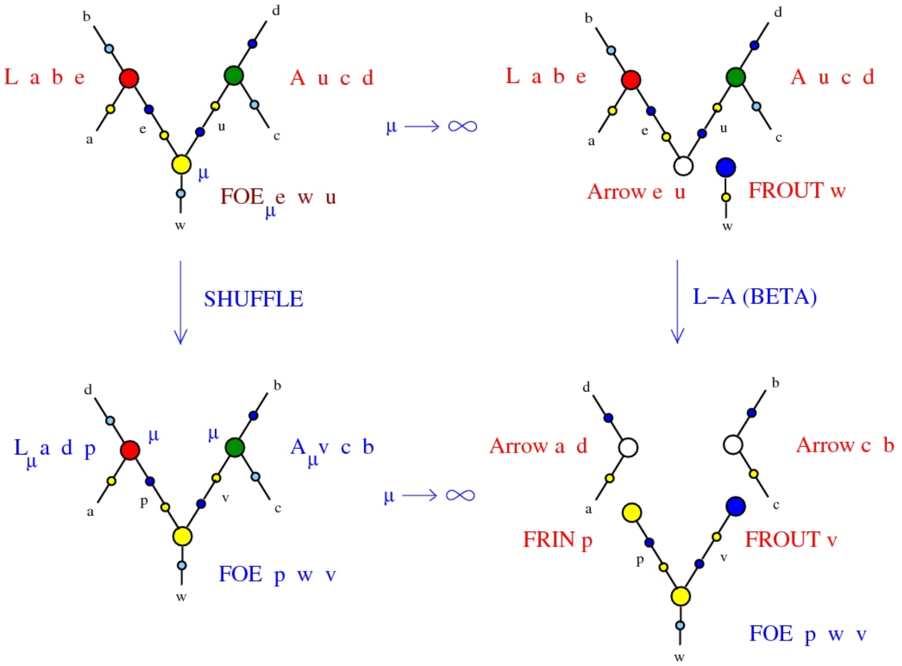
(all the 6 permutations of ports are also dilations, with coefficients from the anharmonic group: Pure See!)

β rewrite from shuffle

only with dilation nodes:



the same with [chemlambda v2](#) nodes:



THANK YOU!

More:

- Pure See
- Graph rewrites, from emergent algebras to chemlambda
- Chemlambda