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Experimental certification of an informationally complete quantum measurement in a device-independent protocol: supplementary material

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This document provides supplementary information to "Experimental certification of an informationally complete quantum measurement in a device-independent protocol," https://doi.org/10.1364/OPTICA.377959. In particular, the following will be presented: details on the experimental realization of non-projective measurements, proof of information completeness, results of tomographic reconstruction of Alice's eight local states and details on error estimation.

1. EXPERIMENTAL REALIZATION OF ALICE'S POVM

We will derive in this section the Kraus operators corresponding to the four outcomes of Alice's non-projective measurement (Eq. (4) in main paper). In order to get to them, we shall work with a four dimensional Hilbert space on Alice's side, which includes the usual polarization space (with basis vectors $|H\rangle$ and $|V\rangle$) and the additional path degree of freedom added by the Sagnac interferometer. Referring to Fig. 2 in the main paper, we denote by $|a\rangle$ the mode transmitted by the polarizing beam splitter (PBS) at the entrance of the interferometer, passing through lambda-half wave plate (HWP) H1, and transmitted again by the PBS. Counter-propagating to it, and going through HWP H2, is instead mode $|b\rangle$.

We can then describe any four-dimensional state $|\Psi\rangle_A$ as a vector with basis $\{|H\rangle|a\rangle, |H\rangle|b\rangle, |V\rangle|a\rangle, |V\rangle|b\rangle\}$, where each element refers to one polarization-path combined mode. In this context, a PBS can be described as:

$$U_{PBS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \tag{S1}$$

while HWP, lambda-quarter wave plates (QWP) and phase plates (PP) as:

$$\begin{aligned} U_H(\theta) &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \\ U_Q(\theta) &= \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix}, \\ U_{PP}(\theta) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}. \end{aligned} \tag{S2}$$

The HWPs inside the interferometer act as:

$$U_{H1H2} = U_H(\theta_{H1}) \oplus U_H(\theta_{H2}) = \begin{pmatrix} \cos 2\theta_{H1} & \sin 2\theta_{H1} & 0 & 0 \\ \sin 2\theta_{H1} & -\cos 2\theta_{H1} & 0 & 0 \\ 0 & 0 & \cos 2\theta_{H2} & \sin 2\theta_{H2} \\ 0 & 0 & \sin 2\theta_{H2} & -\cos 2\theta_{H2} \end{pmatrix} \quad \text{(S3)}$$

and the whole interferometer is thus given by $U_{int}(\theta_{H1}, \theta_{H2}) = U_{PBS} U_{H1H2} U_{PBS}$.

After the Sagnac interferometer, each of the two output paths includes a combination of PP, HWP and QWP, PBS and two single photon detectors. Referring to Fig. 2 in the main paper, we can then express the transformations in polarization space for outcomes $j = \{1, 2, 3, 4\}$ in terms of Kraus operators as:

$$\begin{aligned} A_1 &= \langle a| U_{PBS} U_Q(\theta_{Qa}) U_H(\theta_{Ha}) U_{PP}(\theta_{Pa}) |a\rangle \langle a| U_{int} |a\rangle , \\ A_2 &= \langle a| U_{PBS} U_Q(\theta_{Qb}) U_H(\theta_{Hb}) U_{PP}(\theta_{Pb}) |a\rangle \langle b| U_{int} |a\rangle , \\ A_3 &= \langle b| U_{PBS} U_Q(\theta_{Qb}) U_H(\theta_{Hb}) U_{PP}(\theta_{Pb}) |a\rangle \langle b| U_{int} |a\rangle , \\ A_4 &= \langle b| U_{PBS} U_Q(\theta_{Qa}) U_H(\theta_{Ha}) U_{PP}(\theta_{Pa}) |a\rangle \langle a| U_{int} |a\rangle , \end{aligned} \quad \text{(S4)}$$

so that Alice's qubit undergoes the operation $|\psi\rangle_A \to A_j |\psi\rangle_A$, and each of her non-projective measurement operators (in Eq. (4) in the main paper) is described by $A_{4,j} = A_j^\dagger A_j$. As a side note, while a combination of HWP and QWP at each interferometer output would in principle be sufficient, adding the PPs allows for "standard" σ_x and σ_y measurement settings to be used, together with a fixed phase given by the PPs. Moreover, since the relative phase between the interferometer's arms is fixed but unknown, the additional PP simplifies the experimental task of compensating for this additional phase.

Because of the effectively redundant PPs, there are several combinations of settings that lead to optimal violation of Eq. (5). In our experimental realization, we used the following: $\theta_{H1} = 31.32°$, $\theta_{H2} = 13.68°$, $\theta_{Pa} = 45°$, $\theta_{Ha} = 0°$, $\theta_{Qa} = 45°$, $\theta_{Pb} = 135°$, $\theta_{Hb} = 22.5°$, $\theta_{Qb} = 0°$.

2. INFORMATION COMPLETENESS OF 4-OUTCOME POVMS IN DIMENSION 2

We will now show that if a 4-outcome POVM is implemented in dimension 2, then the information retrieved from the quantum system is complete.

Note that a system of dimension 2 contains 3 independent parameters. Thus, in order to show that a 4-outcome POVM obtains these parameters we need to show that it consists of 3 linearly independent operators.

Let us assume that a POVM contains only 2 linearly independent operators. Without loss of generality we can write it in one of the forms:

$$(A, B, \alpha A + \beta B, \mathbb{1} - (1+\alpha)A - (1+\beta)B) , \quad \text{(S5)}$$

or

$$(A, B, \alpha A - \beta B, \mathbb{1} - (1+\alpha)A - (1-\beta)B) , \quad \text{(S6)}$$

with $\alpha, \beta \geq 0$. We now show that POVMs in Eq. (S5) and Eq. (S6) can be expressed as a convex combination of POVMs with at most 3 outcomes.

Indeed, for Eq. (S5) we have:

$$\begin{aligned} &(A, B, \alpha A + \beta B, \mathbb{1} - (1+\alpha)A - (1+\beta)B) \\ &= \frac{1}{(1+\alpha)(1+\beta)} ((1+\alpha)A, (1+\beta)B, 0, R) \\ &+ \frac{\alpha}{(1+\alpha)(1+\beta)} (0, (1+\alpha)A, (1+\beta)B, R) \\ &+ \frac{\beta}{(1+\alpha)(1+\beta)} ((1+\alpha)A, 0, (1+\beta)B, R) \\ &+ \frac{\alpha\beta}{(1+\alpha)(1+\beta)} (0, 0, (1+\alpha)A + (1+\beta)B, R) , \end{aligned} \quad \text{(S7)}$$

where $R = \mathbb{1} - (1+\alpha)A - (1+\beta)B$.

Similarly, for Eq. (S6) we have:

$$\begin{aligned} &(A, B, \alpha A - \beta B, \mathbb{1} - (1+\alpha)A - (1-\beta)B) \\ &= \frac{\alpha^2}{(\alpha+\beta)(1+\alpha)} \left(0, \left(1+\frac{\beta}{\alpha}\right) B, \right. \\ &\qquad \left. \left(1+\frac{1}{\alpha}\right)(\alpha A - \beta B), R' \right) \\ &+ \frac{\alpha}{(\alpha+\beta)(1+\alpha)} \left(\left(1+\frac{1}{\alpha}\right)(\alpha A - \beta B), \right. \\ &\qquad \left. \left(1+\frac{\beta}{\alpha}\right) B, 0, R' \right) \\ &+ \frac{\alpha\beta}{(\alpha+\beta)(1+\alpha)} \left(\left(1+\frac{\beta}{\alpha}\right) B, 0, \right. \\ &\qquad \left. \left(1+\frac{1}{\alpha}\right)(\alpha A - \beta B), R' \right) \\ &+ \frac{\beta}{(\alpha+\beta)(1+\alpha)} ((1+\alpha)A + (1-\beta)B, 0, 0, R') , \end{aligned} \quad \text{(S8)}$$

where $R' = \mathbb{1} - (1+\alpha)A - (1-\beta)B$.

3. DETECTION EFFICIENCY

In order to calculate critical detection efficiency of the modified elegant Bell expression (Eq. (7) in the main paper), we first calculated its local hidden variables bound. To this end we enumerated all possible deterministic strategies. This revealed that the strategy assigning outcomes $+$ for Alice's measurements 1 and 3, $-$ for Alice's measurement 2, 2 for the POVM measurement, $+$ for Bob's measurements 1, 2 and 4 and $-$ for his measurement 3 yields the value 6.1652.

For detection efficiency calculations we assumed that, in post-processing of the experimental data whenever no-click events occur, an outcome from the optimal LHV strategy above is assigned. The critical detection efficiency calculated using the method in Ref. [1] is equal to 0.9439.

4. TOMOGRAPHIC RECONSTRUCTION OF ALICE'S EIGHT LOCAL STATES

In Table S1 we report the eight qubit states, in Bloch vector notation, reconstructed using both standard projective tomography ($\sigma_x, \sigma_y, \sigma_z$), and with single-setting MIC-POVM tomography. These states are the local states of Alice's qubit, conditioned on Bob's measurement settings and outcomes. The pairwise fidelity is also reported.

5. ERROR ESTIMATION

Here we provide a more comprehensive description of the errors considered in the experiment.

Projective tomography	SIC-POVM tomography	Fidelity
$\begin{pmatrix} 0.561 & 0.601 & 0.570 \end{pmatrix}$	$\begin{pmatrix} 0.544 & 0.508 & 0.668 \end{pmatrix}$	$0.995^{+0.004}_{-0.004}$
$\begin{pmatrix} 0.570 & -0.589 & -0.574 \end{pmatrix}$	$\begin{pmatrix} 0.506 & -0.681 & -0.530 \end{pmatrix}$	$0.996^{+0.003}_{-0.004}$
$\begin{pmatrix} -0.572 & 0.525 & -0.630 \end{pmatrix}$	$\begin{pmatrix} -0.572 & 0.525 & -0.630 \end{pmatrix}$	$0.997^{+0.002}_{-0.003}$
$\begin{pmatrix} -0.551 & -0.590 & 0.591 \end{pmatrix}$	$\begin{pmatrix} -0.526 & -0.695 & 0.490 \end{pmatrix}$	$0.995^{+0.002}_{-0.005}$
$\begin{pmatrix} -0.548 & -0.581 & -0.601 \end{pmatrix}$	$\begin{pmatrix} -0.518 & -0.574 & -0.634 \end{pmatrix}$	$0.999^{+0.001}_{-0.001}$
$\begin{pmatrix} -0.577 & 0.611 & 0.541 \end{pmatrix}$	$\begin{pmatrix} -0.658 & 0.586 & 0.474 \end{pmatrix}$	$0.997^{+0.002}_{-0.002}$
$\begin{pmatrix} 0.588 & -0.532 & 0.610 \end{pmatrix}$	$\begin{pmatrix} 0.560 & -0.603 & 0.569 \end{pmatrix}$	$0.998^{+0.001}_{-0.001}$
$\begin{pmatrix} 0.540 & 0.573 & -0.616 \end{pmatrix}$	$\begin{pmatrix} 0.480 & 0.570 & -0.667 \end{pmatrix}$	$0.998^{+0.001}_{-0.002}$

Table S1. Tomographic reconstruction of the states depicted in Fig. 4 in the main text, using both projective and MIC-POVM tomography, and their pairwise fidelities. Uncertainties represent 15.9% and 84.1% quantiles of the respective results' distributions.

A. Counting statistics

Whenever (coincident) events with a constant rate are counted for some amount of time, the distribution of the final amount is in very good approximation Poissonian. We therefore considered all our empirical counts to have an uncertainty equal to their square root, and propagated it in the results. This is, by far, the predominant contribution to the final uncertainties in our experiment, giving errors of the order of $2 \cdot 10^{-3}$ and 10^{-4} on each E_{ab} and $P(a = i, b = +1|x = 4, y = i)$ term, respectively.

B. Motor precision

All measurement wave plates were rotated by motorized mounts controlled by a computer. The step motors have a precision equivalent to 0.02°. This results in errors of the same order of the Poissonian ones. In order to reduce their contribution, each setting was repeated 23 times, therefore decreasing the uncertainties by almost a factor of 5.

C. Detector dark counts

Each of the single photon detectors used in the measurements have dark count rates of about 500 detections per second. The chances of a coincident event stemming from a true detection and a dark count, with the rates used, was as low as 10^{-11}, thus negligible.

D. Higher order down-conversion events

The rate of accidental coincidences $ac_{m,ij}$ coming from multiple down-conversion events in a single pulse, for measurement setting m and detectors (i, j), can be estimated with the formula

$$ac_{m,ij} = \frac{S_{m,i} S_{m,j} \Delta t}{T}, \tag{S9}$$

where $S_{m,k}$ are the total (single) events on detector k during measurement time T, when coincidence windows of length Δt are used. While the resulting rates are fairly minimal (of the order of 10^{-3} events per second), they can still worsen, although slightly, the results obtained. Since the DI certification protocol can, in principle, work even if the state and measurements are not characterized, we chose not to correct our evaluations for this type of error. In the case of the full state tomography, the derived fidelity of 99.6% (and corresponding uncertainty), did not change whether we took accidental counts into consideration or not, due to their very low rate.

REFERENCES

1. M. Czechlewski and M. Pawłowski, "The influence of the choice of post-processing method on Bell inequalities," Phys. Rev. A **97**, 062123 (2018).