Supplemental Material

Optical chirality exhibited by two axially propagating electromagnetic waves in counter-rotations

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This Supplemental Material consists of 3 Parts with 12 Sections as summarized below.

Part I.	FUNDAMENTALS	S1. Fundamental Formulas S2. Formulas for Rotations S3. Helmholtz Equations S4. Bessel Family
Part II.	ENERGY and CHIRALTY	S5. Analysis of Energy Density S6. Analysis of Optical Chirality S7. Additional Computations
Part III	ELECTRODYNAMICS	S8 to S12

The purpose of this Supplemental Material is to keep as detailed as possible all the intermediate steps leading to what are presented in the main text. It is because one is likely to make mistakes in dealing with complex parameters and variables and their complex conjugates. In particular, Section S3 for deriving the Helmholtz equations looks obvious, but it is included here to show how the axial propagations and the azimuthal rotations affect the derivation processes in altering various intermediate steps.

This Supplemental Material makes our manuscript self-contained as far as the derivation of all the analytical formulas presented in the main text is concerned. Readers do not need to refer even to the authors' previous publications.

Part I. FUNDAMENTALS

S1. Fundamental Formulas

Our configuration is shown in Fig. 1 of the main text, where the transverse-electric (TE) and transverse-magnetic (TM) waves are characterized respectively by the non-zero axial electric and magnetic fields in the axial z -direction. The final hybrid set of the electromagnetic (EM) waves is assumed to propagate into the positive z -direction, while undergoing rotations on the in-plane xy -plane. Fig. 1 of the main text displays both Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) . Both electric field vector $\vec{E}$ and magnetic field vector $\vec{H}$ are assumed to depend on (r, θ, z) . The field vectors can be decomposed into two orthogonal modes in the cylindrical coordinates: (i) the transverse-electric (TE) mode with a non-zero axial electric field so that $\vec{E}_{TE} = (0, 0, E_z)$ and $\vec{H}_{TE} = (H_r, H_\theta, 0)$, and (ii) the transverse-magnetic (TM) mode characterized by a non-zero axial magnetic field so that $\vec{E}_{TM} = (E_r, E_\theta, 0)$ and $\vec{H}_{TM} = (0, 0, H_z)$.

The interior is separated from exterior by the interface located at $r = R$. The superscripts "<" and ">" denote the interior and exterior, respectively. These two superscripts are singly denoted by "< / >". It looks like a diamond, but it denotes in fact "<" for $r < R$ or ">" for $r > R$. For instance, the electric permittivity is singly denoted by $\varepsilon^\diamond$, whereas the refractive indices are by $n^\diamond$. For the sake of simplicity, we fix that $n^< = 1$ (vacuum) and $n^> = 2$ in obtaining all the numerical results presented in the main text, thereby referring to a hollow cylinder carved out of a dielectric medium of infinite extent. Furthermore, let us define the ratio of the refractive indices for the lossless dielectric media in the interior and exterior.

$$N = \begin{cases} 1, & r < R \\ n^</n^>, & r > R \end{cases} \quad (\text{S1.1})$$

Consider the Maxwell's equations $c^{-1}(\partial/\partial t)(\varepsilon\vec{E}) = \nabla \times \vec{H}$ and $-c^{-1}(\partial/\partial t)(\mu\vec{H}) = \nabla \times \vec{E}$ without any space charges.

$$\begin{cases} \frac{1}{c} \frac{\partial}{\partial t} (\varepsilon \vec{E}) = \nabla \times \vec{H} \\ \frac{1}{c} \frac{\partial}{\partial t} (\mu \vec{H}) = -\nabla \times \vec{E} \end{cases} \quad (\text{S1.2})$$

Here, c is the speed of light, and t is time. Assuming that all the field variables undergo time-periodic oscillations according to $\exp(-i\omega t)$, Eq. (S1.2) is reduced as follows.

$$\begin{cases} i\mu \vec{H} = \nabla_k \times \vec{E} \\ i\varepsilon \vec{E} = -\nabla_k \times \vec{H} \end{cases} \quad (\text{S1.3})$$

Here, the free-space wave number k , scaled gradient operator ∇_k , and scaled radial coordinate ρ are defined as follows.

$$k \equiv \frac{\omega}{c}, \quad \nabla_k \equiv \frac{1}{k} \nabla, \quad \rho \equiv kr. \quad (\text{S1.4})$$

For time-oscillatory and axially propagating field variables, all the field variables follow the proportionality $\exp[i(k_z z - \omega t)]$ for both rotational cases. Purely for simplicity, we take $k_z > 0$ for $\omega > 0$. The Maxwell's equations are then fully expanded in the cylindrical coordinates (r, θ, z) with their respective unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$.

$$\begin{cases} i\mu \vec{H} = \nabla_k \times \vec{E} \\ -i\varepsilon \vec{E} = \nabla_k \times \vec{H} \end{cases} \quad (\text{S1.5})$$

$$\begin{aligned} &= \left(\frac{1}{\rho} \frac{\partial E_z}{\partial \theta} - \frac{1}{k} \frac{\partial E_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{1}{k} \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial \rho} \right) \hat{e}_\theta + \left(\frac{\partial E_\theta}{\partial \rho} - \frac{1}{\rho} \frac{\partial E_r}{\partial \theta} \right) \hat{e}_z \\ &= \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \theta} - \frac{1}{k} \frac{\partial H_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{1}{k} \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \hat{e}_\theta + \left(\frac{\partial H_\theta}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_r}{\partial \theta} \right) \hat{e}_z \end{aligned}$$

Component-wise, we arrive at the following four relationships for (H_r, H_θ) and (E_r, E_θ) respectively in terms of (E_r, E_θ, E_z) and (H_r, H_θ, H_z) .

$$\begin{cases} TE: & H_r = -i \frac{1}{\mu \rho} \frac{\partial E_z}{\partial \theta} - \frac{k_z}{\mu k} E_\theta, \quad H_\theta = \frac{k_z}{\mu k} E_r + i \frac{1}{\mu} \frac{\partial E_z}{\partial \rho} \\ TM: & E_r = i \frac{1}{\varepsilon \rho} \frac{\partial H_z}{\partial \theta} + \frac{k_z}{\varepsilon k} H_\theta, \quad E_\theta = -\frac{k_z}{\varepsilon k} H_r - i \frac{1}{\varepsilon} \frac{\partial H_z}{\partial \rho} \end{cases} \quad (S1.6)$$

For $k_z = 0$ specifying no axial propagations, Eq. (S1.6) reduces to the following much simpler one.

$$\begin{cases} TE: & H_r = -i \frac{1}{\mu \rho} \frac{\partial E_z}{\partial \theta}, \quad H_\theta = i \frac{1}{\mu} \frac{\partial E_z}{\partial \rho} \\ TM: & E_r = i \frac{1}{\varepsilon \rho} \frac{\partial H_z}{\partial \theta}, \quad E_\theta = -i \frac{1}{\varepsilon} \frac{\partial H_z}{\partial \rho} \end{cases} \quad (S1.7)$$

In the case of Eq. (S1.7) for $k_z = 0$, the two sets of the in-plane components (H_r, H_θ) and (E_r, E_θ) are expressed in terms of the respective axial field components E_z and H_z . In comparison, Eq. (S1.6) does not lend itself to such a simple separation, owing to the TE-TM couplings. Without any further assumptions, we cannot process Eq. (S1.6) any further.

S2. Formulas for Rotations

For this reason, let us examine the two distinct cases of the relative rotations. By the co-rotations, we mean that both TE and TM waves follows the factor $\exp[i(m\theta - \omega t)]$ so that both waves rotate counter-clockwise for $m\omega > 0$. Here, m is the azimuthal mode index (AMI). In comparison, we mean by the counter-rotation that the TE and TM waves follow respectively the factors $\exp[i(-m\theta - \omega t)]$ and $\exp[i(m\theta - \omega t)]$. As a result, the TE waves rotate clockwise in the azimuthal direction for the counter-rotation, whereas the TM waves rotate counter-clockwise. All the exponential factors taken together,

$$\begin{cases} \text{co:} & \begin{cases} TE: & \exp(im\theta + ik_z z - i\omega t) \\ TM: & \exp(im\theta + ik_z z - i\omega t) \end{cases} \\ \text{counter:} & \begin{cases} TE: & \exp(-im\theta + ik_z z - i\omega t) \\ TM: & \exp(im\theta + ik_z z - i\omega t) \end{cases} \end{cases} \quad (S2.1)$$

In order to reduce the number of intermediate steps, let us handle both rotational cases by a single expression.

$$\begin{cases} \frac{\vec{E}}{\sqrt{\mu}} = \frac{1}{\sqrt{1+|q|^2}} (f_r e^{im\theta}, f_\theta e^{im\theta}, q f_z e^{\pm im\theta}) \\ \frac{\vec{H}}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{1+|q|^2}} (q h_r e^{\pm im\theta}, q h_\theta e^{\pm im\theta}, h_z e^{im\theta}) \end{cases} \quad (\text{S2.2})$$

Here, the upper and lower signs refer respectively to the co- and counter-rotational cases.

Let us introduce several parameters and normalized variables.

$$\begin{cases} \bar{k}_z \equiv \frac{k_z}{n^{</>k}}, \quad \tilde{k} \equiv \sqrt{1-\bar{k}_z^2} \\ K \equiv \frac{\bar{k}_z}{\tilde{k}} = \frac{\bar{k}_z}{\sqrt{1-\bar{k}_z^2}} = \frac{k_z}{\sqrt{(n^{</>k})^2 - k_z^2}}, \end{cases} \quad (\text{S2.3})$$

$$\begin{cases} \bar{\rho} \equiv n^{</>\rho} \rho \equiv n^{</>k} r \\ \tilde{\rho} \equiv \sqrt{1-\bar{k}_z^2} \bar{\rho} \equiv n^{</>\rho} \sqrt{1-\left(\frac{k_z}{n^{</>k}}\right)^2} (kr) \equiv r \sqrt{(n^{</>k})^2 - k_z^2}. \end{cases} \quad (\text{S2.4})$$

Although the full notation $(n^{</>}, \bar{k}_z^{</>}, \tilde{\rho}^{</>}, \tilde{R}^{</>})$ is more proper depending on the interior and exterior, its short-hand notation $(n, \bar{k}_z, \tilde{\rho}, \tilde{R})$ will be sometimes employed in the analytic formulas for simplicity.

At one extreme $k_z = 0$, waves are incident normally onto the cylinder surface. In contrast, at the other extreme $k_z = k$, waves are incident along the cylinder surface. In short, we require $0 < \bar{k}_z < 1$ for our axially propagating waves. We restrict ourselves now to the axially propagating waves with $0 < \bar{k}_z^2 < 1$, whereas the axially evanescent waves with $1 < \bar{k}_z^2$ can be taken up in another publication. Notice further that $\sqrt{\mu} = 1$ and $\sqrt{\varepsilon} = n$ for non-

magnetic dielectric media.

Now that we exchanged the azimuthal derivative with the AMI $i \times m$. As a result, the dimensional Eq. (S1.6) becomes as follows via Eq. (S2.2).

$$\begin{cases} TE: & h_r = \pm \frac{m}{\bar{\rho}} f_z - \bar{k}_z f_\theta, \quad h_\theta = \bar{k}_z f_r + i \frac{df_z}{d\bar{\rho}} \\ TM: & f_r = -\frac{m}{\bar{\rho}} h_z + \bar{k}_z h_\theta, \quad f_\theta = -\bar{k}_z h_r - i \frac{dh_z}{d\bar{\rho}} \end{cases} \quad (S2.5)$$

Here, the sign $\pm$ for h_r refer to the co- and counter-rotational cases, respectively. In other

words, $h_r = -(m/\bar{\rho})f_z - (k_z/n^\pm k)f_\theta$ for the counter-rotational case instead of

$h_r = +(m/\bar{\rho})f_z - (k_z/n^\pm k)f_\theta$ for the co-rotational case.

For $k_z = 0$, Eq. (S2.5) reduces to the following much simpler one.

$$\begin{cases} TE: & h_r = \pm \frac{m}{\bar{\rho}} f_z, \quad h_\theta = i \frac{df_z}{d\bar{\rho}} \\ TM: & f_r = -\frac{m}{\bar{\rho}} h_z, \quad f_\theta = -i \frac{dh_z}{d\bar{\rho}} \end{cases} \quad (S2.6)$$

Note again the difference for the TE waves that $h_r = (m/\bar{\rho})f_z$ for the co-rotational case,

whereas $h_r = -(m/\bar{\rho})f_z$ for the counter-rotational case.

We need to take caution in taking the complex conjugates. For instance, we need for Eq.

(S2.5) operations like $h_\theta^* = [\bar{k}_z f_r + i(df_z/d\bar{\rho})]^* = \bar{k}_z f_r^* - i(df_z^*/d\bar{\rho})$ for $h_\theta = i(df_z/d\bar{\rho})$.

Similarly, $f_\theta^* = -\bar{k}_z h_r^* + i(dh_z^*/d\bar{\rho})$ for $f_\theta = -\bar{k}_z h_r - i(dh_z/d\bar{\rho})$. Meanwhile, $|f_z|^2 \equiv f_z^* f_z$

and $|h_z|^2 \equiv h_z^* h_z$ for the respective intensities (the squared magnitudes).

There are now six variables (f_r, f_θ, f_z) and (h_r, h_θ, h_z) for the four relations in Eqs. (S2.5).

Now, we rearrange the terms in Eq. (S2.5) so that we can apply the method of matrix

solutions.

$$\begin{pmatrix} h_r + \bar{k}_z f_\theta = \pm \frac{m}{\bar{\rho}} f_z \\ \bar{k}_z h_r + f_\theta = -i \frac{dh_z}{d\bar{\rho}} \end{pmatrix}, \quad (\text{S2.7})$$

$$\begin{pmatrix} h_\theta - \bar{k}_z f_r = i \frac{df_z}{d\bar{\rho}} \\ \bar{k}_z h_\theta - f_r = \frac{m}{\bar{\rho}} h_z \end{pmatrix}. \quad (\text{S2.8})$$

The solutions to these two sets of 2-by-2 matrix equations are as follows with $\tilde{k} \equiv \sqrt{1 - \bar{k}_z^2}$ defined in Eq. (S2.3).

$$\begin{cases} h_r = \begin{vmatrix} 1 & \bar{k}_z \\ \bar{k}_z & 1 \end{vmatrix}^{-1} \begin{vmatrix} \pm \frac{m}{\bar{\rho}} f_z & \bar{k}_z \\ -i \frac{dh_z}{d\bar{\rho}} & 1 \end{vmatrix} = \frac{\pm \frac{m}{\bar{\rho}} f_z + i \bar{k}_z \frac{dh_z}{d\bar{\rho}}}{1 - \bar{k}_z^2} = \pm \frac{1}{\tilde{k}^2} \frac{m}{\bar{\rho}} f_z + i \frac{\bar{k}_z}{\tilde{k}^2} \frac{dh_z}{d\bar{\rho}} \\ f_\theta = \begin{vmatrix} 1 & \bar{k}_z \\ \bar{k}_z & 1 \end{vmatrix}^{-1} \begin{vmatrix} 1 & \pm \frac{m}{\bar{\rho}} f_z \\ \bar{k}_z & -i \frac{dh_z}{d\bar{\rho}} \end{vmatrix} = \frac{-i \frac{dh_z}{d\bar{\rho}} \mp \bar{k}_z \frac{m}{\bar{\rho}} f_z}{1 - \bar{k}_z^2} = -i \frac{1}{\tilde{k}^2} \frac{dh_z}{d\bar{\rho}} \mp \frac{\bar{k}_z}{\tilde{k}^2} \frac{m}{\bar{\rho}} f_z \end{cases}, \quad (\text{S2.9})$$

$$\begin{cases} h_\theta = \begin{vmatrix} 1 & -\bar{k}_z \\ \bar{k}_z & -1 \end{vmatrix}^{-1} \begin{vmatrix} i \frac{df_z}{d\bar{\rho}} & -\bar{k}_z \\ \frac{m}{\bar{\rho}} h_z & -1 \end{vmatrix} = \frac{-i \frac{df_z}{d\bar{\rho}} + \bar{k}_z \frac{m}{\bar{\rho}} h_z}{-1 + \bar{k}_z^2} = i \frac{1}{\tilde{k}^2} \frac{df_z}{d\bar{\rho}} - \frac{\bar{k}_z}{\tilde{k}^2} \frac{m}{\bar{\rho}} h_z \\ f_r = \begin{vmatrix} 1 & -\bar{k}_z \\ \bar{k}_z & -1 \end{vmatrix}^{-1} \begin{vmatrix} 1 & i \frac{df_z}{d\bar{\rho}} \\ \bar{k}_z & \frac{m}{\bar{\rho}} h_z \end{vmatrix} = \frac{\frac{m}{\bar{\rho}} h_z - i \bar{k}_z \frac{df_z}{d\bar{\rho}}}{-1 + \bar{k}_z^2} = -\frac{1}{\tilde{k}^2} \frac{m}{\bar{\rho}} h_z + i \frac{\bar{k}_z}{\tilde{k}^2} \frac{df_z}{d\bar{\rho}} \end{cases}. \quad (\text{S2.10})$$

Here, we remark particularly that $\tilde{k}^{-2}$ appears in each relation instead just of $\tilde{k}^{-1}$. As a reference, we find the factor $1 + \tilde{k}^2$ for two axially counter-propagating waves instead of the

present factor $1 - \tilde{k}^2$ for two axially co-propagating waves. This issue will be treated in yet another publication.

In the end, we succeeded in expressing the four in-plane components (f_r, f_θ) and (h_r, h_θ) in terms of the two axial components (f_z, h_z) . Summarizing,

$$\begin{cases} (TE): & h_r = \frac{1}{\tilde{k}^2} \left(\pm \frac{m}{\tilde{\rho}} f_z + i \bar{k}_z \frac{dh_z}{d\tilde{\rho}} \right), \quad h_\theta = \frac{1}{\tilde{k}^2} \left(i \frac{df_z}{d\tilde{\rho}} - \bar{k}_z \frac{m}{\tilde{\rho}} h_z \right) \\ (TM): & f_r = \frac{1}{\tilde{k}^2} \left(-\frac{m}{\tilde{\rho}} h_z + i \bar{k}_z \frac{df_z}{d\tilde{\rho}} \right), \quad f_\theta = \frac{1}{\tilde{k}^2} \left(-i \frac{dh_z}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} f_z \right) \end{cases} \quad (S2.11)$$

As a consequence, the TE and TM modes are coupled together, thus disallowing the separation into the field variables depending solely on either f_z or h_z . In other words, all the four in-plane field variables are dependent simultaneously on (f_z, h_z) . Following the conventional usage in the cylindrical coordinates, we continue to classify them as the TE and TM waves, though.

S3. Helmholtz Equations

In deriving the well-known Helmholtz equations from the Maxwell's equations, we will work in the Cartesian coordinates first. We will then revert the thus-derived relations onto the cylindrical (polar) coordinates.

For this purpose, let us introduce the following scaled coordinates in conformity to the scaled cylindrical coordinates in Eq. (2.4).

$$(\bar{x}, \bar{y}, \bar{z}) \equiv n^{</>k}(x, y, z). \quad (S3.1)$$

For the dielectric media without space charges, we have two divergence-free (solenoidal) fields $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{H} = 0$, which are expanded as follows in the Cartesian coordinates.

$$\frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} + iq\bar{k}_z f_z = 0, \quad q\left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}}\right) + i\bar{k}_z h_z = 0. \quad (\text{S3.2})$$

We remark in particular the appearance of the coupling coefficient q in this equation, since we employed the field variables expressed in Eq. (S2.2).

Alternatively, Eq. (S3.2) is written in the polar coordinates.

$$\begin{cases} \frac{df_r}{d\bar{\rho}} \pm \frac{im}{\bar{\rho}} f_\theta + iq\bar{k}_z f_z = 0 \\ \frac{df_r}{d\bar{\rho}} \pm \frac{im}{\bar{\rho}} f_\theta + iq\frac{\bar{k}_z}{k} f_z = 0 \end{cases}, \quad \begin{cases} q\left(\frac{dh_r}{d\bar{\rho}} + \frac{im}{\bar{\rho}} h_\theta\right) + i\bar{k}_z h_z = 0 \\ q\left(\frac{dh_r}{d\bar{\rho}} + \frac{im}{\bar{\rho}} h_\theta\right) + i\frac{\bar{k}_z}{k} h_z = 0 \end{cases}. \quad (\text{S3.3})$$

Here, the sign $\pm$ refer to the co- and counter-rotational cases, respectively.

Consider once more the field variables with the factor $\exp[i(k_z z - \omega t)]$, which applies equally well to both rotational cases. Let us reduce the Maxwell's equations in Eq. (S1.3) to the following in the Cartesian coordinates.

$$\begin{cases} i(qh_x \hat{e}_x + qh_y \hat{e}_y + h_z \hat{e}_z) \\ = \left(q\frac{\partial f_z}{\partial \bar{y}} - i\bar{k}_z f_y\right) \hat{e}_x + \left(i\bar{k}_z f_x - q\frac{\partial f_z}{\partial \bar{x}}\right) \hat{e}_y + \left(\frac{\partial f_y}{\partial \bar{x}} - \frac{\partial f_x}{\partial \bar{y}}\right) \hat{e}_z \\ -i(f_x \hat{e}_x + f_y \hat{e}_y + qf_z \hat{e}_z) \\ = \left(\frac{\partial h_z}{\partial \bar{y}} - iq\bar{k}_z h_y\right) \hat{e}_x + \left(iq\bar{k}_z h_x - \frac{\partial h_z}{\partial \bar{x}}\right) \hat{e}_y + q\left(\frac{\partial h_y}{\partial \bar{x}} - \frac{\partial h_x}{\partial \bar{y}}\right) \hat{e}_z \end{cases}. \quad (\text{S3.4})$$

Here, we made use of Eq. (S2.3) for the wave numbers. Component-wise, Eq. (S3.4) reads

$$\begin{cases} TE: & qh_x = -iq\frac{\partial f_z}{\partial \bar{y}} - \bar{k}_z f_y, \quad qh_y = \bar{k}_z f_x + iq\frac{\partial f_z}{\partial \bar{x}}, \quad f_z = i\left(\frac{\partial h_y}{\partial \bar{x}} - \frac{\partial h_x}{\partial \bar{y}}\right) \\ TM: & f_x = i\frac{\partial h_z}{\partial \bar{y}} + q\bar{k}_z h_y, \quad f_y = -q\bar{k}_z h_x - i\frac{\partial h_z}{\partial \bar{x}}, \quad h_z = -i\left(\frac{\partial f_y}{\partial \bar{x}} - \frac{\partial f_x}{\partial \bar{y}}\right) \end{cases}. \quad (\text{S3.5})$$

In general, the coupling coefficient q springs up in many places in a mixed manner.

For $k_z = 0$, Eq. (S3.5) reduces to the following much simpler one without an explicit dependence on q , thus meaning a separation among various modes.

$$\left\{ \begin{array}{l} TE: h_x = -i \frac{\partial f_z}{\partial y}, h_y = i \frac{\partial f_z}{\partial x}, f_z = i \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \right) \\ TM: f_x = i \frac{\partial h_z}{\partial y}, f_y = -i \frac{\partial h_z}{\partial x}, h_z = -i \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \end{array} \right. \quad (S3.6)$$

In each set of the preceding relations, we substitute the first two into the last relation. As a result, we first arrive at

$$\left\{ \begin{array}{l} qf_z = iq \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \right) = iq \frac{\partial h_y}{\partial x} - iq \frac{\partial h_x}{\partial y} \\ \quad = i \frac{\partial}{\partial x} \left(\bar{k}_z f_x + iq \frac{\partial f_z}{\partial x} \right) - i \frac{\partial}{\partial y} \left(-iq \frac{\partial f_z}{\partial y} - \bar{k}_z f_y \right) \\ h_z = -i \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) = -i \frac{\partial f_y}{\partial x} + i \frac{\partial f_x}{\partial y} \\ \quad = -i \frac{\partial}{\partial x} \left(-q \bar{k}_z h_x - i \frac{\partial h_z}{\partial x} \right) + i \frac{\partial}{\partial y} \left(i \frac{\partial h_z}{\partial y} + q \bar{k}_z h_y \right) \end{array} \right. , \quad (S3.7)$$

Further manipulations lead to the following two Helmholtz-like equations of an almost same form.

$$\left\{ \begin{array}{l} qf_z = i \bar{k}_z \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) - q \frac{\partial^2 f_z}{\partial x^2} - q \frac{\partial^2 f_z}{\partial y^2} \\ h_z = iq \bar{k}_z \left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} \right) - \frac{\partial^2 h_z}{\partial x^2} - \frac{\partial^2 h_z}{\partial y^2} \end{array} \right. , \quad (S3.8)$$

$$\begin{cases} \frac{\partial^2 f_z}{\partial \bar{x}^2} + \frac{\partial^2 f_z}{\partial \bar{y}^2} + f_z = i \frac{\bar{k}_z}{q} \left(\frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} \right) \\ \frac{\partial^2 h_z}{\partial \bar{x}^2} + \frac{\partial^2 h_z}{\partial \bar{y}^2} + h_z = i q \bar{k}_z \left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}} \right) \end{cases} \quad (\text{S3.9})$$

With the help of the divergence-free (solenoidal) conditions in Eq. (S3.2), Eq. (S3.9) gets down to

$$\begin{cases} \frac{\partial^2 f_z}{\partial \bar{x}^2} + \frac{\partial^2 f_z}{\partial \bar{y}^2} + (1 - \bar{k}_z^2) f_z = 0 \\ \frac{\partial^2 h_z}{\partial \bar{x}^2} + \frac{\partial^2 h_z}{\partial \bar{y}^2} + (1 - \bar{k}_z^2) h_z = 0 \end{cases} \quad (\text{S3.10})$$

As it should, the coupling coefficient q disappears from this equation, thus giving rise to two decoupled Helmholtz equations for the respective axial components.

Based on the previously introduced radial coordinate $\bar{\rho} \equiv n^{</>kr}$ in Eq. (S2.4), we obtain from Eq. (S3.1) the following relation among the scaled coordinates in the cylindrical coordinates.

$$\begin{cases} (\bar{x}, \bar{y}) \equiv \bar{\rho} (\cos \theta, \sin \theta) \\ (\tilde{x}, \tilde{y}) \equiv \tilde{\rho} (\cos \theta, \sin \theta) \end{cases} \quad (\text{S3.11})$$

Consequently, Eq. (S3.10) takes the following concise form.

$$\begin{cases} \frac{\partial^2 f_z}{\partial \tilde{x}^2} + \frac{\partial^2 f_z}{\partial \tilde{y}^2} + f_z = 0 \\ \frac{\partial^2 h_z}{\partial \tilde{x}^2} + \frac{\partial^2 h_z}{\partial \tilde{y}^2} + h_z = 0 \end{cases} \quad (\text{S3.12})$$

Converting these equations into the polar coordinates, we obtain the following Helmholtz equations in the scaled radial coordinate $\tilde{\rho}$.

$$\begin{cases} \frac{d^2 f_z}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} + \left(1 - \frac{m^2}{\tilde{\rho}^2}\right) f_z = 0 \\ \frac{d^2 h_z}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{dh_z}{d\tilde{\rho}} + \left(1 - \frac{m^2}{\tilde{\rho}^2}\right) h_z = 0 \end{cases}. \quad (\text{S3.13})$$

We learn that the different signs in the factor $\exp(\pm im\theta)$ for the co- and counter-rotational cases does not affect the eventual Helmholtz equations, which contain only the squared value m^2 for the AMLs.

S4. Bessel Family

Consulting the scaled parameters and variables introduced in Eqs. (S2.3) and (S2.4), let us define the two radii as seen from the viewpoint of $\tilde{\rho}$.

$$\tilde{R}_\rho^{</>} \equiv n^{</>} k R \sqrt{1 - \left(\frac{k_z}{n^{</>} k}\right)^2} = R \sqrt{(n^{</>} k)^2 - k_z^2}. \quad (\text{S4.1})$$

Let us further define two radial functions.

$$\begin{cases} F_m^{<}(\tilde{\rho}) \equiv \frac{J_m(\tilde{\rho})}{J_m(\tilde{R}_\rho^{<})}, & \begin{cases} \tilde{\rho} < \tilde{R}_\rho^{<} \\ r < R \end{cases} \\ F_m^{>}(\tilde{\rho}) \equiv \frac{H_m^{(1)}(\tilde{\rho})}{H_m^{(1)}(\tilde{R}_\rho^{>})}, & \begin{cases} \tilde{\rho} > \tilde{R}_\rho^{>} \\ r > R \end{cases} \end{cases}. \quad (\text{S4.2})$$

Here, $J_m(\tilde{\rho})$ and $H_m^{(1)}(\tilde{\rho})$ are the Bessel function of first kind and Hankel function of the first kind, respectively. Overall, the arguments of the Bessel functions are contracted from $\bar{\rho}$ to $\tilde{\rho}$. The two functions $F_m^{<}(\tilde{\rho})$ and $F_m^{>}(\tilde{\rho})$ are singly denoted by $F_m^{</>}(\tilde{\rho})$.

Moreover, according to the defining Eq. (S4.2) for the axial fields, the argument $\tilde{\rho}$ should be properly interpreted as follows.

$$\begin{cases} F_m^<(\tilde{\rho}) \equiv F_m^<(\tilde{\rho}^<) \equiv F_m^<\left(r\sqrt{(n^<k)^2 - k_z^2}\right) \\ F_m^>(\tilde{\rho}) \equiv F_m^>(\tilde{\rho}^>) \equiv F_m^>\left(r\sqrt{(n^>k)^2 - k_z^2}\right) \end{cases} \quad (\text{S4.3})$$

The solutions to the axial profiles of the axial components are expressed as follows.

$$\begin{cases} f_z(\tilde{\rho}) = F_m^{</>}(\tilde{\rho}) \\ h_z(\tilde{\rho}) = NF_m^{</>}(\tilde{\rho}) \end{cases} \quad (\text{S4.4})$$

These solutions have been constructed to satisfy both continuity relations $(E_z)_{R-}^< = (E_z)_{R+}^>$ and $(H_z)_{R-}^< = (H_z)_{R+}^>$ across the thin layer at $\rho = R_\rho \equiv kR$. It is because of the normalization scheme $E_z \propto f_z$ and $H_z \propto h_z/n$ exploited in Eq. (S2.2).

Thanks to these normalization schemes in (S4.4), it is specified that

$F_m^<(\rho = R_\rho) = 1 = F_m^>(\rho = R_\rho)$. In addition, $F_m^{</>}(\tilde{\rho}) = F_{-m}^{</>}(\tilde{\rho})$, although $J_m(\tilde{\rho}) = (-1)^m J_{-m}(\tilde{\rho})$ and $H_m^{(1)}(\tilde{\rho}) = (-1)^m H_{-m}^{(1)}(\tilde{\rho})$. This fact $F_m^{</>}(\tilde{\rho}) = F_{-m}^{</>}(\tilde{\rho})$ will be repeatedly employed in dealing with the counter-rotational case. We will also utilize the notation that $F_m^{</>,*} \equiv (F_m^{</>})^*$.

In addition, we define the first-order gradient function $G_m^{</>}(\tilde{\rho})$ in the radial direction.

$$G_m^{</>}(\tilde{\rho}) \equiv \frac{d}{d\tilde{\rho}} \ln[F_m^{</>}(\tilde{\rho})] = \frac{1}{F_m^{</>}(\tilde{\rho})} \frac{d}{d\tilde{\rho}} F_m^{</>}(\tilde{\rho}). \quad (\text{S4.5})$$

This form reminds us of the log-potential gradient often discussed in electrodynamics with source terms. In order to compute $G_m^{</>}(\tilde{\rho})$, we have to evaluate the respective derivatives of the Bessel functions of first kind and the Hankel functions of the first kind. For the Bessel functions, the recursion relations are

$$\begin{cases} \frac{dJ_0(\tilde{\rho})}{d\tilde{\rho}} = -J_1(\tilde{\rho}) \\ \frac{dJ_m(\tilde{\rho})}{d\tilde{\rho}} = J_{m-1}(\tilde{\rho}) - \frac{m}{\tilde{\rho}} J_m(\tilde{\rho}), \quad m > 0 \end{cases} \quad (\text{S4.6})$$

The Hankel functions follow a similar set of rules for derivatives. In addition, we need to pay a special attention to the azimuthally stationary case with $m = 0$ in the first relation of the preceding equation.

Figure S1. The radial profiles of the first-order gradient function $G_m^{</>}(\tilde{\rho})$ with the cylindrical interface located at $R_\rho \equiv kR = 1$ and for $m = 0, 1, 2, 3, 4$. (a) Real parts. (b) Imaginary parts.

Figure S1 presents the radial profiles of $G_m^{</>}(\tilde{\rho})$ plotted against $\rho \equiv kr \equiv \omega r/c$ with $R_\rho \equiv kR = 1$. In particular, the azimuthally stationary state with $m = 0$ is drawn in a solid black curve. Figures S1(a) and S1(b) display $\text{Re}(G_m^{</>})$ and $\text{Im}(G_m^{</>})$, respectively. In particular, the fact that $\text{Im}(G_m^{</>}) = 0$ in the interior will play a dominant role in characterizing various wave properties.

For a generic complex variable A , we make use of a set of the complex-variable identities among others.

$$\left\{ \begin{array}{l} \operatorname{Re}(iA) = -\operatorname{Im}(A) \\ \operatorname{Im}(iA) = \operatorname{Re}(A) \\ A + A^* = 2\operatorname{Re}(A) \\ \operatorname{Im}(A^*) = -\operatorname{Im}(A) \end{array} \right. \quad (\text{S4.7})$$

Part II. ENERGY and CHIRALITY

S5. Analysis of Energy Density

The energy density is defined as follows.

$$w \equiv \begin{cases} \frac{1}{4\mu} |\vec{E}|^2 + \frac{1}{4\varepsilon} |\vec{H}|^2 \\ (2n^{</>})^{-2} (\varepsilon |\vec{E}|^2 + \mu |\vec{H}|^2) \end{cases}. \quad (\text{S5.1})$$

The second relation comes from the first one by employing the definition $n^2 = (\mu\varepsilon)^{-1}$ for dielectric media. Notice here the symmetric contributions of the electric and magnetic fields. In fact, some scaling factors have been omitted in arriving at this form of energy density so that both CGS and SI units are handled in this same form. Furthermore, the dividing factor $(2n^{</>})^2 \equiv \frac{1}{4} \varepsilon^{</>}$ for non-magnetic media accounts for the different refractive indices of the interior and exterior. Note of course that $w \geq 0$ for any parameter sets.

We expand Eq. (S5.1) fully as follows via Eq. (S2.2).

$$w = \frac{1}{4} \frac{1}{1+|q|^2} \left[\left| (f_r^* e^{-im\theta}, f_\theta^* e^{-im\theta}, q^* f_z^* e^{im\theta}) \right|^2 + \left| (qh_r e^{-im\theta}, qh_\theta e^{-im\theta}, h_z e^{im\theta}) \right|^2 \right], \quad (\text{S5.2})$$

$$w = \frac{1}{4} \frac{1}{1+|q|^2} (|f_r|^2 + |f_\theta|^2 + |q|^2 |f_z|^2 + |q|^2 |h_r|^2 + |q|^2 |h_\theta|^2 + |h_z|^2). \quad (\text{S5.3})$$

By Eq. (S2.11), Eq. (S5.3) is cast into the following formula in terms of (f_z, h_z) .

$$w = \frac{1}{4} \frac{1}{1+|q|^2} \frac{1}{\tilde{k}^4} \left(\left| -\frac{m}{\tilde{\rho}} h_z + i\bar{k}_z \frac{df_z}{d\tilde{\rho}} \right|^2 + \left| -i \frac{dh_z}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} f_z \right|^2 + |q|^2 \tilde{k}^4 |f_z|^2 \right. \\ \left. + |q|^2 \left| \pm \frac{m}{\tilde{\rho}} f_z + i\bar{k}_z \frac{dh_z}{d\tilde{\rho}} \right|^2 + |q|^2 \left| i \frac{df_z}{d\tilde{\rho}} - \bar{k}_z \frac{m}{\tilde{\rho}} h_z \right|^2 + \tilde{k}^4 |h_z|^2 \right), \quad (\text{S5.4})$$

Since $\tilde{k} = 1$ for $k_z = 0$, Eq. (S5.4) reduces to the following much simpler one for axially

standing waves.

$$w_m^{</>} = \frac{1}{4} \frac{1}{1+|q|^2} \left(\frac{m^2}{\tilde{\rho}^2} |h_z|^2 + \left| \frac{dh_z}{d\tilde{\rho}} \right|^2 + |q|^2 |f_z|^2 + |q|^2 \frac{m^2}{\tilde{\rho}^2} |f_z|^2 + |q|^2 \left| \frac{df_z}{d\tilde{\rho}} \right|^2 + |h_z|^2 \right). \quad (\text{S5.5})$$

Through the solutions in Eqs. (S4.2) and (S4.4), Eq. (S5.4) gets down to

$$w_m^{</>} = \frac{1}{4} \frac{1}{1+|q|^2} \frac{1}{\tilde{k}^4} \times \left(\left| -\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 + \left| -i N \frac{dF_m^{</>}}{d\tilde{\rho}} + \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 + |q|^2 \tilde{k}^4 |F_m^{</>}|^2 \right. \\ \left. + |q|^2 \left| \pm \frac{m}{\tilde{\rho}} F_m^{</>} + i N \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 + |q|^2 \left| i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 + N^2 \tilde{k}^4 |F_m^{</>}|^2 \right). \quad (\text{S5.6})$$

Via Eq. (4.7) for complex variables and other minor manipulations, let us consider Eq. (S5.6) term by term. First,

$$\left| -\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 = \left(-\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \left(-\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right)^* \\ = \left(-\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \left(-\frac{m}{\tilde{\rho}} N F_m^{</>,*} - i \bar{k}_z \frac{dF_m^{</>,*}}{d\tilde{\rho}} \right) \\ = \frac{m^2}{\tilde{\rho}^2} N^2 |F_m^{</>}|^2 + \bar{k}_z^2 \left| \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 + i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>} \frac{dF_m^{</>,*}}{d\tilde{\rho}} - i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \quad (\text{S5.7}) \\ = |F_m^{</>}|^2 \left(\frac{m^2}{\tilde{\rho}^2} N^2 + \bar{k}_z^2 \left| G_m^{</>} \right|^2 \right) - 2 \text{Re} \left(i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \\ = |F_m^{</>}|^2 \left(\frac{m^2}{\tilde{\rho}^2} N^2 + \bar{k}_z^2 \left| G_m^{</>} \right|^2 \right) + 2 \text{Im} \left(\frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right)$$

Therefore, there are two quadratic terms and one interference term as follows.

$$\left| -\frac{m}{\tilde{\rho}} N F_m^{</>} + i \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 \\ = |F_m^{</>}|^2 \left(\frac{m^2}{\tilde{\rho}^2} N^2 + \bar{k}_z^2 \left| G_m^{</>} \right|^2 \right) + 2 \frac{m}{\tilde{\rho}} N \bar{k}_z |F_m^{</>}|^2 \text{Im} \left(G_m^{</>} \right) \quad (\text{S5.8})$$

Second,

$$\begin{aligned}
\left| -iN \frac{dF_m^{</>}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 &= \left(-iN \frac{dF_m^{</>}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right) \left(-iN \frac{dF_m^{</>}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right)^* \\
&= \left(-iN \frac{dF_m^{</>}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right) \left(iN \frac{dF_m^{</>,*}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>,*} \right) \\
&= N^2 \left| \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 + \frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 |F_m^{</>}|^2 \pm i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \mp i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \frac{dF_m^{</>,*}}{d\tilde{\rho}} \quad . \quad (S5.9) \\
&= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 + N^2 |G_m^{</>}|^2 \right) |F_m^{</>}|^2 \pm 2 \operatorname{Re} \left(i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \\
&= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 + N^2 |G_m^{</>}|^2 \right) |F_m^{</>}|^2 \mp 2 \operatorname{Im} \left(\bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right)
\end{aligned}$$

Once again, there are two quadratic terms and one interference term as follows.

$$\begin{aligned}
\left| -iN \frac{dF_m^{</>}}{d\tilde{\rho}} \mp \bar{k}_z \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 \\
= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 + N^2 |G_m^{</>}|^2 \right) |F_m^{</>}|^2 \mp 2 \bar{k}_z N \frac{m}{\tilde{\rho}} |F_m^{</>}|^2 \operatorname{Im} (G_m^{</>}) \quad . \quad (S5.10)
\end{aligned}$$

Third,

$$\begin{aligned}
\left| \pm \frac{m}{\tilde{\rho}} F_m^{</>} + iN \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 &= \left(\pm \frac{m}{\tilde{\rho}} F_m^{</>} + iN \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \left(\pm \frac{m}{\tilde{\rho}} F_m^{</>} + iN \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right)^* \\
&= \left(\pm \frac{m}{\tilde{\rho}} F_m^{</>} + iN \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \left(\pm \frac{m}{\tilde{\rho}} F_m^{</>,*} - iN \bar{k}_z \frac{dF_m^{</>,*}}{d\tilde{\rho}} \right) \\
&= \frac{m^2}{\tilde{\rho}^2} |F_m^{</>}|^2 + \bar{k}_z^2 N^2 \left| \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 \mp i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>} \frac{dF_m^{</>,*}}{d\tilde{\rho}} \pm i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \quad . \quad (S5.11) \\
&= |F_m^{</>}|^2 \left(\frac{m^2}{\tilde{\rho}^2} + \bar{k}_z^2 N^2 |G_m^{</>}|^2 \right) \pm 2 \operatorname{Re} \left(i \frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \\
&= |F_m^{</>}|^2 \left(\frac{m^2}{\tilde{\rho}^2} + \bar{k}_z^2 N^2 |G_m^{</>}|^2 \right) \mp 2 \operatorname{Im} \left(\frac{m}{\tilde{\rho}} N \bar{k}_z F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right)
\end{aligned}$$

Once more, there are two quadratic terms and one interference term as follows.

$$\begin{aligned} & \left| \pm \frac{m}{\tilde{\rho}} F_m^{</>} + i N \bar{k}_z \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 \\ &= \left| F_m^{</>} \right|^2 \left(\frac{m^2}{\tilde{\rho}^2} + \bar{k}_z^2 N^2 \left| G_m^{</>} \right|^2 \right) \mp 2 \frac{m}{\tilde{\rho}} N \bar{k}_z \left| F_m^{</>} \right|^2 \text{Im} \left(G_m^{</>} \right) \end{aligned} \quad (\text{S5.12})$$

Fourth,

$$\begin{aligned} & \left| i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 = \left(i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right) \left(i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right)^* \\ &= \left(i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right) \left(-i \frac{dF_m^{</>,*}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \right) \\ &= \left| \frac{dF_m^{</>}}{d\tilde{\rho}} \right|^2 + \frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 N^2 \left| F_m^{</>} \right|^2 - i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} + i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \frac{dF_m^{</>,*}}{d\tilde{\rho}} , \quad (\text{S5.13}) \\ &= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 N^2 + \left| G_m^{</>} \right|^2 \right) \left| F_m^{</>} \right|^2 - 2 \text{Re} \left(i \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \\ &= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 N^2 + \left| G_m^{</>} \right|^2 \right) \left| F_m^{</>} \right|^2 + 2 \text{Im} \left(\bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>,*} \frac{dF_m^{</>}}{d\tilde{\rho}} \right) \end{aligned}$$

Again, there are two quadratic terms and one interference term as follows.

$$\begin{aligned} & \left| i \frac{dF_m^{</>}}{d\tilde{\rho}} - \bar{k}_z N \frac{m}{\tilde{\rho}} F_m^{</>} \right|^2 \\ &= \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 N^2 + \left| G_m^{</>} \right|^2 \right) \left| F_m^{</>} \right|^2 + 2 \bar{k}_z N \frac{m}{\tilde{\rho}} \left| F_m^{</>} \right|^2 \text{Im} \left(G_m^{</>} \right) \end{aligned} \quad (\text{S5.14})$$

In all the above four items, we have extracted the common factor $\left| F_m^{</>} \right|^2$ for convenience.

Instead, we obtained the log-gradient factor $G_m^{</>}$ in many places.

As a result, the energy density in Eq. (S5.6) becomes

$$w_m^{</>} = \frac{1}{4} \frac{1}{1+|q|^2} \frac{1}{\tilde{k}^4} \times \left[\begin{aligned} & \left| F_m^{</>} \right|^2 \left(\frac{m^2}{\tilde{\rho}^2} N^2 + \bar{k}_z^2 \left| G_m^{</>} \right|^2 \right) + 2 \frac{m}{\tilde{\rho}} N \bar{k}_z \left| F_m^{</>} \right|^2 \text{Im}(G_m^{</>}) \\ & + \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 + N^2 \left| G_m^{</>} \right|^2 \right) \left| F_m^{</>} \right|^2 \mp 2 \bar{k}_z N \frac{m}{\tilde{\rho}} \left| F_m^{</>} \right|^2 \text{Im}(G_m^{</>}) \\ & + |q|^2 \left| F_m^{</>} \right|^2 \left(\frac{m^2}{\tilde{\rho}^2} + \bar{k}_z^2 N^2 \left| G_m^{</>} \right|^2 \right) \mp |q|^2 2 \frac{m}{\tilde{\rho}} N \bar{k}_z \left| F_m^{</>} \right|^2 \text{Im}(G_m^{</>}) \\ & + |q|^2 \left(\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 N^2 + \left| G_m^{</>} \right|^2 \right) \left| F_m^{</>} \right|^2 + |q|^2 2 \bar{k}_z N \frac{m}{\tilde{\rho}} \left| F_m^{</>} \right|^2 \text{Im}(G_m^{</>}) \\ & + |q|^2 \tilde{k}^4 \left| F_m^{</>} \right|^2 + N^2 \tilde{k}^4 \left| F_m^{</>} \right|^2 \end{aligned} \right]. \quad (\text{S5.15})$$

Here, we collect the terms multiplied by N^2 and the other terms multiplied by $|q|^2$ as follows.

$$w_m^{</>} = \frac{1}{4} \frac{1}{1+|q|^2} \frac{\left| F_m^{</>} \right|^2}{\tilde{k}^4} \times \left\{ \begin{aligned} & \left[\frac{m^2}{\tilde{\rho}^2} (N^2 + |q|^2) + \bar{k}_z^2 (1 + N^2 |q|^2) \left| G_m^{</>} \right|^2 \right] \\ & + \left[\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 (1 + N^2 |q|^2) + (N^2 + |q|^2) \left| G_m^{</>} \right|^2 \right] + (N^2 + |q|^2) \tilde{k}^4 \\ & + 2 (\mp 1 + 1 \mp |q|^2 + |q|^2) \frac{m}{\tilde{\rho}} N \bar{k}_z \text{Im}(G_m^{</>}) \end{aligned} \right\}. \quad (\text{S5.16})$$

Considering the signs $\mp$ in the above equation depending on the rotational directions, we obtain a very distinct set of energy density by following the steps as below.

$$\begin{aligned}
w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z) &= \frac{1}{4} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \\
&\times \left\{ \left[\frac{m^2}{\tilde{\rho}^2} (N^2 + |q|^2) + \bar{k}_z^2 (1 + N^2 |q|^2) |G_m^{</>}|^2 \right] \right. \\
&+ \left[\frac{m^2}{\tilde{\rho}^2} \bar{k}_z^2 (1 + N^2 |q|^2) + (N^2 + |q|^2) |G_m^{</>}|^2 \right] + (N^2 + |q|^2) \tilde{k}^4 \Bigg\} \\
&+ 4 \frac{m}{\tilde{\rho}} N \bar{k}_z \text{Im}(G_m^{</>}) \times \begin{cases} 0 \\ 1+|q|^2 \end{cases}
\end{aligned} \tag{S5.17}$$

In other words,

$$\begin{aligned}
w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z) &= \frac{1}{4} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \\
&\times \left\{ \left(\bar{k}_z^2 + \bar{k}_z^2 N^2 |q|^2 + N^2 + |q|^2 \right) \left(\frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) + (N^2 + |q|^2) \tilde{k}^4 \right. \\
&+ 4 \frac{m}{\tilde{\rho}} N \bar{k}_z \text{Im}(G_m^{</>}) \times \begin{cases} 0 \\ 1+|q|^2 \end{cases} \Bigg\}
\end{aligned} \tag{S5.18}$$

We then separate out the two terms as follows.

$$\begin{aligned}
w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z) &= \frac{1}{4} \frac{1}{1+|q|^2} |F_m^{</>}|^2 \\
&\times \left[\frac{1}{\tilde{k}^4} \left(\bar{k}_z^2 + \bar{k}_z^2 N^2 |q|^2 + N^2 + |q|^2 \right) \left(\frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) + (N^2 + |q|^2) \right] \\
&+ \frac{m}{\tilde{\rho}} N \frac{\bar{k}_z}{\tilde{k}^4} |F_m^{</>}|^2 \text{Im}(G_m^{</>}) \times \begin{cases} 0 \\ 1 \end{cases}
\end{aligned} \tag{S5.19}$$

It is noticeable that the interference term (the last term in dark green color of the above equation) involving $\text{Im}(G_m^{</>})$ survives only in the counter-rotational case. Due to this interference term, the energy density in Eq. (S5.19) is not separable for the counter-rotating case.

However, the energy density is still positive. We may include all the parameters in the expression for the energy density such that $w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z)$. In addition, the portion

$|F_m^{</>}|^2 \left(m^2 \tilde{\rho}^{-2} + |G_m^{</>}|^2 \right)$ corresponds to the transverse phase gradient, which is representative of the orbital momentum density.

Let us examine several characteristics of energy density **in the interior**. First, $N = 1$. In addition, $\text{Im}(G_m^<) = 0$, because $G_m^<(\tilde{\rho})$ is real owing to the property of the Bessel functions. Hence, Eq. (5.19) becomes independent of the phase of q . In contrast, $\text{Im}(G_m^>) \neq 0$ in the exterior, since the Hankel functions are complex in general. Consequently, for both rotational cases, Eq. (S5.19) are reduced as follows in the interior.

$$\begin{aligned}
 w_m^<(\tilde{\rho}; N, q, \bar{k}_z) &= \frac{1}{4} \frac{1}{1+|q|^2} |F_m^<|^2 \times \\
 &\quad \left[\left(1+|q|^2 \right) + \frac{1}{\tilde{k}^4} \left(\bar{k}_z^2 + \bar{k}_z^2 |q|^2 + 1+|q|^2 \right) \left(\frac{m^2}{\tilde{\rho}^2} + |G_m^<|^2 \right) \right] \\
 &= \frac{1}{4} |F_m^<|^2 \left[1 + \frac{1}{\tilde{k}^4} (\bar{k}_z^2 + 1) \left(\frac{m^2}{\tilde{\rho}^2} + |G_m^<|^2 \right) \right] \\
 &= \frac{1}{4} |F_m^<|^2 \left[1 + \frac{1+\bar{k}_z^2}{(1-\bar{k}_z^2)^2} \left(\frac{m^2}{\tilde{\rho}^2} + |G_m^<|^2 \right) \right]
 \end{aligned} \tag{S5.20}$$

Here, notice that $G_m^< = \text{Re}(G_m^<) \neq 0$ in the interior. In addition, we utilized the definition $\tilde{k}^2 \equiv 1 - \bar{k}_z^2$ in Eq. (S2.3). Consequently, the energy density takes the same formula for both rotational cases.

Eq. (S5.20) can be expressed as follows for both rotational cases.

$$K \equiv \frac{1+\bar{k}_z^2}{1-\bar{k}_z^2}, \tag{S5.21}$$

$$w_m^< = \frac{1}{4} |F_m^<|^2 \left(1 + K \frac{m^2}{\tilde{\rho}^2} \right). \tag{S5.22}$$

As a result, the rotational effect implied by $m^2/\tilde{\rho}^2$ is strongly modified through the factor K in Eq. (S5.21). Notice that $K(\bar{k}_z^2)$ increases sharply uniformly with $\bar{k}_z^2$, so is $w_m^{<}$ except for the implicit effect of $\bar{k}_z$ on $|F_m^{<}|^2$. From our preliminary study on the counter-propagating waves (viz., an hybrid of upward- and downward-propagating waves), the factor K appears again in the energy density. This sharp increase or decrease in the two colliding waves provides the basis for optical traps.

For the axially non-propagating case with $\bar{k}_z = 0$ and $\tilde{k}^2 = 1$, w is separable into w_{TE} and w_{TM} for the TE and TM modes.

$$\begin{aligned} w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z = 0) &= \frac{1}{4} \frac{N^2 + |q|^2}{1 + |q|^2} \left(1 + |G_m^{</>}|^2 + \frac{m^2}{\tilde{\rho}^2} \right) |F_m^{</>}|^2 \\ &= \frac{|q|^2}{1 + |q|^2} w_{TE}^{</>}(\tilde{\rho}; N, q = \infty, \bar{k}_z = 0) + \frac{1}{1 + |q|^2} w_{TM}^{</>}(\tilde{\rho}; N, q = 0, \bar{k}_z = 0) \end{aligned} \quad (S5.23)$$

Here, the energy density for the TE waves is given by

$$w_{TE}^{</>} = \frac{1}{4} \left(1 + |G_m^{</>}|^2 + m^2 \tilde{\rho}^{-2} \right) |F_m^{</>}|^2. \quad (S5.24)$$

Besides, $w_{TM}^+ = N^2 w_{TE}^+$, and w remains the same for both rotating cases.

The case with $m = 1$, viz., the minimum azimuthal mode index (AMI) offers us a particularly interesting feature. Consider hence Eq. (S5.19) with $m = 1$ for the counter-rotational case.

$$\begin{aligned} w_m^{</>}(\tilde{\rho}; N, q, \bar{k}_z) &= \frac{1}{4} \frac{1}{1 + |q|^2} |F_1^{</>}|^2 \\ &\times \left[\frac{1}{\tilde{k}^4} \left(\bar{k}_z^2 + \bar{k}_z^2 N^2 |q|^2 + N^2 + |q|^2 \right) \left(\frac{1}{\tilde{\rho}^2} + |G_1^{</>}|^2 \right) + \left(N^2 + |q|^2 \right) \right] \\ &+ \frac{1}{\tilde{\rho}} N \frac{\bar{k}_z}{\tilde{k}^4} |F_1^{</>}|^2 \text{Im}(G_1^{</>}) \end{aligned} \quad (S5.25)$$

With $m = 1$, the second term for possesses a minimal non-zero interference in its magnitude.

S6. Analysis of Optical Chirality

The optical chirality is defined as follows.

$$C \equiv -\frac{1}{2n} \text{Im}(\vec{E}^* \cdot \vec{H}). \quad (\text{S6.1})$$

For time-oscillatory fields, we had the Faraday's law $i\mu\vec{H} = \nabla_k \times \vec{E}$ in Eq. (S1.3). Therefore,

we can rewrite the above as either $C \equiv (2n\mu)^{-1} \text{Im} \left[i\vec{E}^* \cdot (\nabla_k \times \vec{E}) \right]$ or

$C \equiv (2n\mu)^{-1} \text{Re} \left[\vec{E}^* \cdot (\nabla_k \times \vec{E}) \right]$. At any rate, the above is fully expressed as follows via the combined fields in Eq. (S2.2).

$$\begin{aligned} [\text{co}]: C &= -\frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[\begin{pmatrix} f_r^* e^{-im\theta}, f_\theta^* e^{-im\theta}, q^* f_z^* e^{-im\theta} \end{pmatrix} \cdot \begin{pmatrix} qh_r e^{im\theta}, qh_\theta e^{im\theta}, h_z e^{im\theta} \end{pmatrix} \right] \\ [\text{counter}]: C &= -\frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[\begin{pmatrix} f_r^* e^{-im\theta}, f_\theta^* e^{-im\theta}, q^* f_z^* e^{im\theta} \end{pmatrix} \cdot \begin{pmatrix} qh_r e^{-im\theta}, qh_\theta e^{-im\theta}, h_z e^{im\theta} \end{pmatrix} \right]. \end{aligned} \quad (\text{S6.2})$$

As usual, we combine these two into a single equation.

$$C = -\frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[\begin{pmatrix} f_r^* e^{-im\theta}, f_\theta^* e^{-im\theta}, q^* f_z^* e^{\mp im\theta} \end{pmatrix} \cdot \begin{pmatrix} qh_r e^{\pm im\theta}, qh_\theta e^{\pm im\theta}, h_z e^{im\theta} \end{pmatrix} \right]. \quad (\text{S6.3})$$

Here, the upper and lower signs refer respectively to the co- and counter-rotational cases when referring to Eq. (S2.2).

Upon scalar multiplications, the exponential factors $e^{\pm im\theta}$ are cancelled in many places except a factor $e^{-2im\theta}$ so that

$$\begin{aligned}
[\text{co}]: C &= -\frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[(q f_r^* h_r + q f_\theta^* h_\theta + q^* f_z^* h_z) \right] \\
&= \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q (-f_r^* h_r - f_\theta^* h_\theta + h_z^* f_z) \right] \\
[\text{counter}]: C &= -\frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[(q e^{-2im\theta} f_r^* h_r + q e^{-2im\theta} f_\theta^* h_\theta + q^* e^{2im\theta} f_z^* h_z) \right] \\
&= \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q e^{-2im\theta} (-f_r^* h_r - f_\theta^* h_\theta + h_z^* f_z) \right].
\end{aligned} \tag{S6.4}$$

Notice that the azimuthal angle survives explicitly as $e^{-2im\theta}$ for the counter-rotational case because of the relative angle $2m\theta$, whereas it does not explicitly show up for the co-rotational case. We combine thus the two cases into a single expression, while making use of Eq. (S4.7) for complex variables.

$$C = \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q \Theta^\pm (e^{-2im\theta}) (-f_r^* h_r - f_\theta^* h_\theta + h_z^* f_z) \right]. \tag{S6.5}$$

Here, a special azimuthal function is introduced as follows.

$$\Theta^\pm (e^{-2im\theta}) = \begin{cases} \Theta^+ (e^{-2im\theta}) \equiv 1 & : [\text{co}] \\ \Theta^- (e^{-2im\theta}) \equiv e^{-2im\theta} & : [\text{counter}] \end{cases}. \tag{S6.6}$$

For convenience, let us introduce an intermediate function.

$$F \equiv -f_r^* h_r - f_\theta^* h_\theta + h_z^* f_z. \tag{S6.7}$$

By Eq. (S2.11), Eq. (S6.7) is cast into the following formula only in terms of the two axial components (f_z, h_z) .

$$\begin{aligned}
F &= \begin{bmatrix} -\left(-\frac{1}{\tilde{k}^2} \frac{m}{\tilde{\rho}} h_z^* - i \frac{\bar{k}_z}{\tilde{k}^2} \frac{df_z^*}{d\tilde{\rho}}\right) \left(\pm \frac{1}{\tilde{k}^2} \frac{m}{\tilde{\rho}} f_z + i \frac{\bar{k}_z}{\tilde{k}^2} \frac{dh_z}{d\tilde{\rho}}\right) \\ -\left(i \frac{1}{\tilde{k}^2} \frac{dh_z^*}{d\tilde{\rho}} \mp \frac{\bar{k}_z}{\tilde{k}^2} \frac{m}{\tilde{\rho}} f_z^*\right) \left(i \frac{1}{\tilde{k}^2} \frac{df_z}{d\tilde{\rho}} - \frac{\bar{k}_z}{\tilde{k}^2} \frac{m}{\tilde{\rho}} h_z\right) + \tilde{h}_z^* f_z \end{bmatrix} \\
&= \frac{1}{\tilde{k}^4} \times \begin{bmatrix} \left(\frac{m}{\tilde{\rho}} h_z^* + i \bar{k}_z \frac{df_z^*}{d\tilde{\rho}}\right) \left(\pm \frac{m}{\tilde{\rho}} f_z + i \bar{k}_z \frac{dh_z}{d\tilde{\rho}}\right) \\ + \left(-i \frac{dh_z^*}{d\tilde{\rho}} \pm \bar{k}_z \frac{m}{\tilde{\rho}} f_z^*\right) \left(i \frac{df_z}{d\tilde{\rho}} - \bar{k}_z \frac{m}{\tilde{\rho}} h_z\right) + \tilde{k}^4 \tilde{h}_z^* f_z \end{bmatrix} \quad (S6.8) \\
&= \frac{1}{\tilde{k}^4} \times \begin{bmatrix} \pm \frac{m^2}{\tilde{\rho}^2} h_z^* f_z \mp \bar{k}_z^2 \frac{m^2}{\tilde{\rho}^2} h_z f_z^* - \bar{k}_z^2 \frac{df_z^*}{d\tilde{\rho}} \frac{dh_z}{d\tilde{\rho}} + \frac{dh_z^*}{d\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} + \tilde{k}^4 \tilde{h}_z^* f_z \\ + i \bar{k}_z \frac{m}{\tilde{\rho}} \left(h_z^* \frac{dh_z}{d\tilde{\rho}} \pm f_z \frac{df_z^*}{d\tilde{\rho}} + h_z \frac{dh_z^*}{d\tilde{\rho}} \pm f_z^* \frac{df_z}{d\tilde{\rho}}\right) \end{bmatrix}
\end{aligned}$$

By Eq. (S2.11), the optical chirality in Eq. (S6.5) is cast into the following formula only in terms of (f_z, h_z) .

$$\begin{aligned}
C &= \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q \Theta^\pm (e^{-2im\theta}) F \right] \\
&= \frac{1}{2} \frac{1}{1+|q|^2} \frac{1}{\tilde{k}^4} \times \quad (S6.9) \\
&\quad \text{Im} \left\{ \begin{aligned} &q \Theta^\pm (e^{-2im\theta}) \times \\ &\left[\left(\tilde{k}^4 \pm \frac{m^2}{\tilde{\rho}^2} \right) \tilde{h}_z^* f_z \mp \bar{k}_z^2 \frac{m^2}{\tilde{\rho}^2} h_z f_z^* - \bar{k}_z^2 \frac{df_z^*}{d\tilde{\rho}} \frac{dh_z}{d\tilde{\rho}} + \frac{dh_z^*}{d\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} \right] \\ &+ 2i \bar{k}_z \frac{m}{\tilde{\rho}} \text{Re} \left(h_z^* \frac{dh_z}{d\tilde{\rho}} \pm f_z \frac{df_z^*}{d\tilde{\rho}} \right) \end{aligned} \right\}
\end{aligned}$$

In addition, we introduce the "co-rotational chirality" σ and "counter-rotational chirality" χ .

$$\sigma \equiv \frac{2 \text{Im}(q)}{1+|q|^2}, \quad \chi \equiv \frac{2 \text{Im}[q \exp(-2im\theta)]}{1+|q|^2}. \quad (S6.10)$$

Through the fields $f_z(\tilde{\rho}) = F_m^\diamond(\tilde{\rho})$ and $h_z(\tilde{\rho}) = NF_m^\diamond(\tilde{\rho})$ in Eq. (S4.4), Eq. (S6.9) reduces as follows.

$$C = \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{\diamond}|^2}{\tilde{k}^4} \times \text{Im} \left\{ q \Theta^{\pm} \left(e^{-2im\theta} \right) \times \left[\left(\tilde{k}^4 \pm \frac{m^2}{\tilde{\rho}^2} \right) N \mp \bar{k}_z^2 \frac{m^2}{\tilde{\rho}^2} N - \bar{k}_z^2 N |G_m^{\diamond}|^2 + N |G_m^{\diamond}|^2 \right] + 2i\bar{k}_z \frac{m}{\tilde{\rho}} \text{Re} \left(N^2 G_m^{\diamond} \pm G_m^{\diamond} \right) \right] \right\}, \quad (\text{S6.11})$$

Once more, there are two quadratic terms and one interference term as follows

$$C = \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \times \text{Im} \left\{ q \Theta^{\pm} \left(e^{-2im\theta} \right) \times \left[\left(\tilde{k}^4 \mp \bar{k}_z^2 \frac{m^2}{\tilde{\rho}^2} \pm \frac{m^2}{\tilde{\rho}^2} \right) N + N \left(1 - \bar{k}_z^2 \right) |G_m^{</>}|^2 \right] + 2i\bar{k}_z \left(N^2 \pm 1 \right) \frac{m}{\tilde{\rho}} \text{Re} \left(G_m^{</>} \right) \right] \right\}. \quad (\text{S6.12})$$

Through $\tilde{k} \equiv \sqrt{1 - \bar{k}_z^2}$ in Eq. (S2.3), we obtain $\tilde{k}^2 = 1 - \bar{k}_z^2$ and $\bar{k}_z^2 = 1 - \tilde{k}^2$. Therefore,

$$C = \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \times \text{Im} \left\{ q \Theta^{\pm} \left(e^{-2im\theta} \right) \times \left[\left(\left(1 - \bar{k}_z^2 \right)^2 \mp \bar{k}_z^2 \frac{m^2}{\tilde{\rho}^2} \pm \frac{m^2}{\tilde{\rho}^2} \right) N + N \left(1 - \bar{k}_z^2 \right) |G_m^{</>}|^2 \right] + 2i\bar{k}_z \left(N^2 \pm 1 \right) \frac{m}{\tilde{\rho}} \text{Re} \left(G_m^{</>} \right) \right] \right\}. \quad (\text{S6.13})$$

$$C = \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \times \text{Im} \left\{ q \Theta^{\pm} \left(e^{-2im\theta} \right) \times \left[N \left(1 - \bar{k}_z^2 \right) \left(1 - \bar{k}_z^2 \pm \frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) \right] + 2i\bar{k}_z \left(N^2 \pm 1 \right) \frac{m}{\tilde{\rho}} \text{Re} \left(G_m^{</>} \right) \right] \right\}$$

When comparing both formulas, the difference lies in the two factors: (i) $\pm m^2 \tilde{\rho}^{-2}$ (something like centrifugal and centripetal forces), and (ii) ± 1 in $N^2 \pm 1$. Here, the upper and lower signs refer respectively to the co- and counter-rotational cases. In addition, the other difference between q and $qe^{-2im\theta}$ for Eq. (S6.10) can be singly denoted by $q \exp(-im\theta \pm im\theta)$.

Let us consider the last equation separately for the co- and counter-rotational cases. For the co-rotational case, by making use of Eq. (S4.7) for complex variables, we ascertain the following.

$$\begin{aligned}
 [co]: \quad C &= \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \times \\
 &\quad \text{Im} \left\{ q \left[N(1-\bar{k}_z^2) \left(1-\bar{k}_z^2 + \frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) + 2i\bar{k}_z (N^2+1) \frac{m}{\tilde{\rho}} \text{Re}(G_m^{</>}) \right] \right\}. \quad (S6.14) \\
 &= \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{1-\bar{k}_z^2} \times \left[\begin{aligned} &N \left(1-\bar{k}_z^2 + \frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) \text{Im}(q) \\ &+ \frac{2\bar{k}_z}{1-\bar{k}_z^2} (N^2+1) \frac{m}{\tilde{\rho}} \text{Re}(G_m^{</>}) \text{Re}(q) \end{aligned} \right]
 \end{aligned}$$

For the counter-rotational case,

$$\begin{aligned}
 [counter]: \quad C &= \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{\tilde{k}^4} \times \\
 &\quad \text{Im} \left\{ qe^{-2im\theta} \left[\begin{aligned} &N(1-\bar{k}_z^2) \left(1-\bar{k}_z^2 - \frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) \\ &+ 2i\bar{k}_z (N^2-1) \frac{m}{\tilde{\rho}} \text{Re}(G_m^{</>}) \end{aligned} \right] \right\}. \quad (S6.15) \\
 &= \frac{1}{2} \frac{1}{1+|q|^2} \frac{|F_m^{</>}|^2}{1-\bar{k}_z^2} \left[\begin{aligned} &N \left(1-\bar{k}_z^2 - \frac{m^2}{\tilde{\rho}^2} + |G_m^{</>}|^2 \right) \text{Im}(qe^{-2im\theta}) \\ &+ \frac{2\bar{k}_z}{1-\bar{k}_z^2} (N^2-1) \frac{m}{\tilde{\rho}} \text{Re}(G_m^{</>}) \text{Re}(qe^{-2im\theta}) \end{aligned} \right]
 \end{aligned}$$

The optical chirality C lends itself to a further interpretation. To this goal, let us denote

$q \equiv |q| \exp(i\phi)$. Hence, $\exp(i\phi) = 1$ refers to the axial components of the electric and magnetic fields being in phase, whereas $\exp(i\phi) = -1$ denotes the two in anti-phase. On the other hand, the two are out of phase for $\exp(i\phi) = \pm i$, which are alternatively called “of quadrature”. As a result, the real and imaginary parts of the factor $qe^{-2im\theta}$ in the above equation can be respectively written as follows.

$$\begin{cases} \operatorname{Re}(qe^{-2im\theta}) = |q|\cos(\phi - 2m\theta) \\ \operatorname{Im}(qe^{-2im\theta}) = |q|\sin(\phi - 2m\theta) \end{cases}. \quad (\text{S6.16})$$

Therefore, these even and odd modes imply standing waves in the respective azimuthal directions. Interestingly enough, this separation into both odd and even modes holds true with the optical circulator based on the counter-rotating waves.

For this particular counter-rotational case, notice that $N^2 = 1$ in the interior, where the following simpler form prevails.

$$\left(\begin{matrix} \text{counter} \\ r < R \end{matrix} \right): C^< = \frac{1}{2} \frac{\operatorname{Im}(qe^{-2im\theta})}{1 + |q|^2} |F_m^<|^2 \left(1 - \bar{k}_z^2 - \frac{m^2}{\tilde{\rho}^2} + |G_m^<|^2 \right). \quad (\text{S6.17})$$

This equation holds true also when $N = 1$ for the whole space, namely, when both interior and exterior are made of the same dielectric materials

For $k_z = 0$ with $\tilde{k} = 1$ from (S2.3), we have

$$C = \frac{1}{2} \frac{1}{1 + |q|^2} \operatorname{Im} \left\{ q \Theta^\pm(e^{-2im\theta}) \times \left[\left(1 \pm \frac{m^2}{\tilde{\rho}^2} \right) h_z^* f_z + \frac{dh_z^*}{d\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} \right] \right\}, \quad (\text{S6.18})$$

Hence, Eq.s (S6.14) and (S6.15) reduce to the following much simpler ones.

$$\begin{aligned}
[co]: \quad C &= \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q \left(\frac{m^2}{\tilde{\rho}^2} h_z^* f_z + \frac{dh_z^*}{d\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} + h_z^* f_z \right) \right] \\
[counter]: \quad C &= \frac{1}{2} \frac{1}{1+|q|^2} \text{Im} \left[q e^{-2im\theta} \left(-\frac{m^2}{\tilde{\rho}^2} f_z^* h_z + \frac{dh_z^*}{d\tilde{\rho}} \frac{df_z}{d\tilde{\rho}} + f_z^* h_z \right) \right].
\end{aligned} \tag{S6.19}$$

From the fields $f_z(\tilde{\rho}) = F_m^{</>}(\tilde{\rho})$ and $h_z(\tilde{\rho}) = N F_m^{</>}(\tilde{\rho})$ in Eq. (S4.4), we obtain the following for the co-rotational case.

$$[co]: \quad C = \frac{1}{4} \frac{2 \text{Im}(q)}{1+|q|^2} N |F_m^{</>}|^2 \left(1 + |G_m^{</>}|^2 + \frac{m^2}{\tilde{\rho}^2} \right). \tag{S6.20}$$

In other words,

$$[co]: \quad C = \frac{1}{4} \sigma Y_m^{</>}, \quad Y_m^{</>} = N |F_m^{</>}|^2 \left(1 + |G_m^{</>}|^2 + \frac{m^2}{\tilde{\rho}^2} \right). \tag{S6.21}$$

As expected, the chirality coefficient is positive for all $\tilde{\rho}$, namely, $Z_m^{</>} > 0$. In addition, C does not explicitly depend on θ . Most importantly, $Y_m^{</>}(\tilde{\rho})$ is proportional to the energy density $w(\tilde{\rho})$ in Eq. (S5.20) with $\bar{k}_{z0} = 0$ in its radial dependence.

For the counter-rotational case,

$$[counter]: \quad C = \frac{1}{4} \chi Z_m^{</>}, \quad Y_m^{</>} \equiv N |F_m^{</>}|^2 \left(1 + |G_m^{</>}|^2 - \frac{m^2}{\tilde{\rho}^2} \right). \tag{S6.22}$$

Here, χ is the counter-rotational chirality defined in Eq. (S6.10). As expected, the azimuthal term $-m^2/\tilde{\rho}^2$ in $Z_m^{</>}$ for this counter-rotational case changes its sign, in comparison to $+m^2/\tilde{\rho}^2$ in $Y_m^{</>}$ of Eq. (S6.21) for the co-rotational case.

S7. Additional Computations

As regards device performances, here presented are a couple of additional results from

parametric studies. Based on Fig. 7 of the main text for the counter-rotational case, let us compare the results from both rotational cases.

Figure S2 shows both $(|C|/w)_{avg}$ and η_{avg} for the same prescribed data $R_p \equiv kR = 5$, $m = 1$, and $q = 2^{-1/2}(1+i)$. Both are plotted against k_{z0} on the linear scale drawn below the display box as well as the inverse-linear scale for the frequency $(ck_z)^{-1}\omega$ shown above the display box. The superscript “+−” on both $(|C|/w)_{avg}^{+-}$ and η_{avg}^{+-} refers to the counter-rotational case, which we have examined in Fig. 7 of the main text. Both curves are reproduced in Fig. S2 for comparison purposes. In comparison, the superscript “++” on both $(|C|/w)_{avg}^{++}$ and η_{avg}^{++} refer to the co-rotational case that we added in Fig. S2.

Figure S2. Comparison of both rotational cases for the averaged optical chirality per energy density $(|C|/w)_{avg}$ and the efficiency η_{avg} . Both are plotted against k_{z0} (the lower horizontal scale) and its inverse $(ck_z)^{-1}\omega$ (the upper horizontal scale). The data is $kR = 5$, $m = 1$, and $q = 2^{-1/2}(1+i)$.

Notice that the curves for $(|C|/w)_{avg}$ and η_{avg} presented in Fig. 7 are not denoted respectively as $(|C|/w)_{avg}^{+-}$ and η_{avg}^{+-} in Fig. S2 (both being in the same colors, although the latter is in a dashed curve). The horizontal thick arrow below the k_{z0} -axis in Fig. S2 reproduces the forbidden k_{z0} -bandgap as mentioned for Figs. 5 and 7 of the main text for the counter-rotational case, because of the mixed chirality over a given one-quarter pie. In a similar way, we draw another thick horizontal arrow in a darker grey color just below the curve for η_{avg}^{++} , which indicates the forbidden k_{z0} -bandgap for the co-rotational case.

Overall, there are minimum features in both $(|C|/w)_{avg}^{++}$ and η_{avg}^{++} when plotted against k_{z0} for the co-rotational case. It is in contrast to the monotonic decrease in $(|C|/w)_{avg}^{+-}$ and the maximum feature exhibited by η_{avg}^{+-} . We indicated a k_{z0} -range for η_{avg}^{++} by the horizontal arrow in the light grey color, which may be suitable for operating actual devices.

Figure S3. Device performance for the co-rotational case. (a) The averaged optical chirality per energy density $(|C|/w)_{avg}$, and (b) the efficiency η_{avg} . The coupling coefficient q is varied as indicated on each sub-figure. The common data is $kR=5$ and $m=1$.

For the co-rotational case, let us now make a variation in the complex coupling coefficient q . Figure S3(a) shows $(|C|/w)_{avg}$, whereas Fig. S3(b) displays η_{avg} for the common data of $kR=5$ and $m=1$. The coupling coefficient q is varied as indicated on each panel among $q=1$, $q=2^{-1/2}(1+i)$, $q=i$, and $q=2^{-1/2}(1+i)$ so that $|q|=1$ for all the four choices. Hence, $(|C|/w)_{avg}$ and η_{avg} imply respectively $(|C|/w)_{avg}^{++}$ and η_{avg}^{++} in terms of the notation employed for Fig. S2. Notice hence that the curves for $q=2^{-1/2}(1+i)$ shown in Fig. S2 can be found also in Fig. S3.

Figure S3(a) shows that the curve of $(|C|/w)_{avg}$ for $q=1$ is monotonically increasing with k_{z0} , whereas the curve of $(|C|/w)_{avg}$ for $q=i$ is monotonically decreasing with k_{z0} in an approximate manner. In comparison, both curves for $q=2^{-1/2}(1+i)$ and $q=2^{-1/2}(1+i)$ exhibit roughly minimum features with respect to k_{z0} . The discrepancy between the two becomes largest at $k_{z0} \approx 0.7$. Although the respective k_{z0} -bandgap can be documented, they are not presented here.

PART III. ELECTRODYNAMICS

Figure S4 shows that two-dimensional electron gas (2DEG) resides within the thin but finite-thickness layer. Our goal in this subsection is to see how the Helmholtz equations are modified in the presence of the 2DEG. By this way, we can derive the functions A and B respectively in Eqs. (S8.1) and (S8.2).

This part is divided further into five sections, thus being the longest part in this Supplementary Material. Yet, the end result of this part is incomplete, thus leaving us with issues suitable for further studies in the future.

S8. Homogeneous Conditions

Let us start with the jump conditions across the interface at $r = R$.

$$\left(\frac{dE_z}{d\rho} \right)_R^> - \left(\frac{dE_z}{d\rho} \right)_R^< = A. \quad (\text{S8.1})$$

$$\frac{1}{\varepsilon^>} \left(\frac{dH_z}{d\rho} \right)_{R+}^> - \frac{1}{\varepsilon^<} \left(\frac{dH_z}{d\rho} \right)_{R-}^< = B. \quad (\text{S8.2})$$

Consider Eq. (S8.1) first by setting $A = 0$.

$$\left(\frac{dE_z}{d\rho}\right)_R^> = \left(\frac{dE_z}{d\rho}\right)_R^<. \quad (\text{S8.3})$$

In terms of the wave fields in Eq. (S2.2) and the scaled coordinates in Eq. (S2.4), this equation provides the following relation in terms of the normalized field variables.

$$n^>\left(\frac{df_z}{d\tilde{\rho}^>}\right)_{R+} = n^<\left(\frac{df_z}{d\tilde{\rho}^<}\right)_{R-}. \quad (\text{S8.4})$$

Via Eqs. (S1.1) and (S4.4), Eq. (S8.4) is further cast as follows.

$$\begin{cases} \frac{n^>}{n^<}\left(\frac{dF_m^>}{d\tilde{\rho}^>}\right)_{R+} = \left(\frac{dF_m^<}{d\tilde{\rho}^<}\right)_{R-} \\ n^>\left(\frac{dF_m^>}{d\tilde{\rho}^>}\right)_{R+} = n^<\left(\frac{dF_m^<}{d\tilde{\rho}^<}\right)_{R-} \end{cases}. \quad (\text{S8.5})$$

Finally, the refractive-index ratio reappears here. Second, consider Eq. (S8.2) by setting $B=0$.

$$\frac{1}{\varepsilon^>}\left(\frac{dH_z}{d\rho}\right)_{R+} = \frac{1}{\varepsilon^<}\left(\frac{dH_z}{d\rho}\right)_{R-}. \quad (\text{S8.6})$$

In terms of the wave fields in Eq. (S2.2) and the scaled coordinates in Eq. (S2.4), this equation provides the following relation in terms of the normalized field variables.

$$\left(\frac{dh_z}{d\tilde{\rho}^>}\right)_{R+} = \left(\frac{dh_z}{d\tilde{\rho}^<}\right)_{R-}. \quad (\text{S8.7})$$

Note that Eq. (S8.7) is free from the permittivity factor in its explicit form. It turns out that Eq. (S8.7) is identical to Eq. (S8.4), when employing Eqs. (S1.1) and (S4.4).

The question is now reduced to: "Do the wave fields in Eq. (S2.2) satisfy the condition in Eq. (S8.4) or (S8.7)?" In order to answer this question, Eq. (S8.5) is put into the following form

through the first-order gradient function $G_m^{\lessgtr}(\tilde{\rho})$ in Eq. (S4.5).

$$\frac{n^>}{n^<} \left(F_m^> G_m^> \right)_{R+} = \left(F_m^< G_m^< \right)_{R-}. \quad (\text{S8.8})$$

Because $\left(F_m^> \right)_{R+} = \left(F_m^< \right)_{R-}$ from the continuity in Eq. (S4.2), we end up with the following.

$$\frac{n^>}{n^<} \left(G_m^> \right)_{R+} = \left(G_m^< \right)_{R-}. \quad (\text{S8.9})$$

From Fig. S1, the complex-valued condition in Eq. (S8.9) is never satisfied for the following two facts.

$$\begin{cases} \text{Re}\left(G_m^<\right)_{R-} < 0, \text{Re}\left(G_m^>\right)_{R+} > 0 \\ \text{Im}\left(G_m^<\right)_{R-} = 0, \text{Im}\left(G_m^>\right)_{R+} \neq 0 \end{cases}. \quad (\text{S8.10})$$

In conclusion of this subsection, we need to simultaneously have $A \neq 0$ and $B \neq 0$ respectively for Eqs. (S8.1) and (S8.2).

This difficulty stems from enforcing the continuity in both electric and magnetic fields in the radial direction, which is physically sound but causes all the ensuing mathematical nuisances. Let us see what happens when we remove these constraints and hence let both dimensional fields variables take discontinuities across the thin layer. It is equivalent to setting

$$\begin{cases} \left(\frac{dE_z}{d\rho} \right)_R^> - \left(\frac{dE_z}{d\rho} \right)_R^< = 0 \\ \frac{1}{\varepsilon^>} \left(\frac{dH_z}{d\rho} \right)_{R+}^> - \frac{1}{\varepsilon^<} \left(\frac{dH_z}{d\rho} \right)_{R-}^< = 0 \end{cases}. \quad (\text{S8.11})$$

In other words, in terms of the normalized field variables,

$$\begin{cases} n^>\left(\frac{df_z}{d\tilde{\rho}}\right)_R = n^<\left(\frac{df_z}{d\tilde{\rho}}\right)_R \\ \left(\frac{dh_z}{d\tilde{\rho}}\right)_{R+} = \left(\frac{dh_z}{d\tilde{\rho}}\right)_{R-} \end{cases} \quad (\text{S8.12})$$

As a result, the field solutions in Eq. (S4.4) should be altered as follows.

$$\begin{cases} f_z(\tilde{\rho}) = NF_m^>(\tilde{\rho}) \\ h_z(\tilde{\rho}) = F_m^>(\tilde{\rho}) \end{cases} \quad (\text{S8.13})$$

Consequently, these solutions turn out to allow for the continuities in both electric and magnetic field variables in dimensional terms. It is because the preceding equation is equivalent to the following across the thin layer at $\rho = R_\rho \equiv kR$.

$$\begin{cases} n^<(E_z)_{R-} = n^>(E_z)_{R+} \\ n^>(H_z)_{R-} = n^<(H_z)_{R+} \end{cases} \quad (\text{S8.14})$$

S9. Electrodynamics with Space Charges

For generic three-dimensional wave propagations in the presence of space charges, consider the Maxwell's equations.

$$\begin{cases} \frac{1}{c} \frac{\partial \sigma}{\partial t} + q \nabla \cdot \vec{J} = 0 \\ \nabla \cdot (\epsilon^{EG} \vec{E}) = \sigma \\ \frac{1}{c} \frac{\partial (\epsilon^{EG} \vec{E})}{\partial t} + q \vec{J} = \nabla \times \vec{H} \\ -\frac{1}{c} \frac{\partial (\mu^{EG} \vec{H})}{\partial t} = \nabla \times \vec{E} \end{cases} \quad (\text{S9.1})$$

Here, σ is the space charge. Although it is traditionally denoted by ρ , it is changed here in order to avoid a confusion with the radial coordinate. Besides, $\vec{J}$ is the normalized space-current density vector representing the free movement of such space charges. The

introduction of the coupling coefficient q in the terms $q\nabla\Box\vec{J}$ and $q\vec{J}$ as the multiplying factor is specific to our hybrid mode with the TE and TM waves. Its advantage will become clear in the course of the subsequent developments.

Meanwhile, ε^{EG} is the permittivity of the electron gas. Assuming no scattering loss for inter-electron collisions, ε^{EG} is real and negative, namely,

$$\varepsilon^{EG} < 0. \quad (\text{S9.2})$$

The permeability μ^{EG} will be set to unity for non-magnetic electron gas. In addition, we define the conventional electric-field displacement vector $\vec{D} \equiv \varepsilon^{EG} \vec{E}$, which is just a redefinition of the electric-field vector.

We can verify the legitimacy of these relations by taking the divergence operators on the third relation in Eq. (S9.1) in addition to its second relation.

$$\begin{cases} \frac{1}{c} \frac{\partial \nabla\Box(\varepsilon^{EG} \vec{E})}{\partial t} + q\vec{J} = \nabla\Box(\nabla \times \vec{H}) \\ \frac{1}{c} \frac{\partial \sigma}{\partial t} + q\nabla\Box\vec{J} = \nabla\Box(\nabla \times \vec{H}) = 0 \end{cases}. \quad (\text{S9.3})$$

As a result, we have successfully recovered the first relation of Eq. (S9.1).

Now, in place of $\vec{J}$, we substitute the space-charge vector $\vec{U}$ for convenience.

$$\begin{cases} \vec{J} = -\frac{1}{c} \frac{\partial(\varepsilon^{EG} \vec{U})}{\partial t} \\ \frac{1}{c} \frac{\partial \sigma}{\partial t} + q\nabla\Box\vec{J} = 0 \Rightarrow \frac{1}{c} \frac{\partial \sigma}{\partial t} + q\nabla\Box \left[-\frac{1}{c} \frac{\partial(\varepsilon^{EG} \vec{U})}{\partial t} \right] = 0. \\ \Rightarrow \frac{\partial \sigma}{\partial t} = q \frac{\partial \nabla\Box(\varepsilon^{EG} \vec{U})}{\partial t} \Rightarrow \sigma = q\nabla\Box(\varepsilon^{EG} \vec{U}) \end{cases}. \quad (\text{S9.4})$$

Here, $\vec{U}$ is distinct from the conventionally defined electric-field displacement vector

$\vec{D} \equiv \varepsilon^{EG} \vec{E}$. Physically speaking, $\vec{U}$ means a charge displacement density as seen from $\sigma = q \nabla \cdot (\varepsilon \vec{U})$. We will see later that $\sigma = q \nabla \cdot (\varepsilon \vec{U})$ leads to a coupling between the components of $\vec{U}$, the complication of which compensates for the previously mentioned simplification obtained by the multiplying factor q in $q \nabla \cdot \vec{J} = 0$ and $q \vec{J}$.

The Maxwell's equations in Eq. (S9.3) is thus modified as follows.

$$\begin{cases} \frac{1}{c} \frac{\partial (\varepsilon \vec{E})}{\partial t} + q \vec{J} = \frac{1}{c} \frac{\partial (\varepsilon \vec{E})}{\partial t} - q \frac{1}{c} \frac{\partial (\varepsilon \vec{U})}{\partial t} = \nabla \times \vec{H} \\ \frac{1}{c} \frac{\partial (\mu \vec{H})}{\partial t} = -\nabla \times \vec{E} \end{cases} \quad (\text{S9.5})$$

S10. Helmholtz Equations in the Cartesian Coordinates with Space Charges

Our goal in this subsection is to see how the Helmholtz equations are modified in the presence of the space charges in the Cartesian coordinates. Via the normalized variables and operators defined in Eq. (S1.4) in addition to the common spectral factor $\exp(-i\omega t)$, Eq. (S9.5) is cast as follows.

$$\begin{cases} -i\varepsilon \vec{E} + i\varepsilon q \vec{U} = \nabla_k \times \vec{H} \\ i\mu \vec{H} = \nabla_k \times \vec{E} \end{cases} \quad (\text{S10.1})$$

As before, we are to work in the Cartesian coordinates to arrive at modified Helmholtz equations. To this goal, let us take a divergence of each relation in Eq. (S10.1) to obtain

$$\begin{cases} -i\varepsilon \nabla_k \cdot (\vec{E} - q \vec{U}) = \nabla_k \cdot (\nabla_k \times \vec{H}) = 0 \\ i\mu \nabla_k \cdot \vec{H} = \nabla_k \cdot (\nabla_k \times \vec{E}) = 0 \end{cases} \quad (\text{S10.2})$$

Here, we utilized the properties of the vector differential operators $\nabla_k \cdot (\nabla_k \times \vec{H}) = 0$ and $\nabla_k \cdot (\nabla_k \times \vec{E}) = 0$.

With the fields provided in Eq. (S2.2) and the propagation factor $\exp[i(k_z z - \omega t)]$, Eq.

(S10.2) are component-wise expanded as follows in the Cartesian coordinates in terms of the normalized field variables.

$$\begin{cases} \frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} + iq\bar{k}_z f_z = q \left(\frac{\partial U_x}{\partial \bar{x}} + \frac{\partial U_y}{\partial \bar{y}} \right) + iq\bar{k}_z U_z \\ q \left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}} \right) + i\bar{k}_z h_z = 0 \end{cases} \quad (\text{S10.3})$$

Here, notice the scaled coordinates $(\bar{x}, \bar{y}, \bar{z}) \equiv n^{\diamond} k(x, y, z)$ in Eq. (S3.1). Here, the coupling constant q seems to render the above two relations coupled together.

Likewise, the second and first Maxwell's equations in Eq. (S10.1) are cast as follows in terms of the normalized field variables.

$$\begin{cases} i(qh_x \hat{e}_x + qh_y \hat{e}_y + h_z \hat{e}_z) \\ = \left(q \frac{\partial f_z}{\partial \bar{y}} - i\bar{k}_z f_y \right) \hat{e}_x + \left(i\bar{k}_z f_x - q \frac{\partial f_z}{\partial \bar{x}} \right) \hat{e}_y + \left(\frac{\partial f_y}{\partial \bar{x}} - \frac{\partial f_x}{\partial \bar{y}} \right) \hat{e}_z \\ -i(f_x \hat{e}_x + f_y \hat{e}_y + qf_z \hat{e}_z) + iq(U_x \hat{e}_x + U_y \hat{e}_y + U_z \hat{e}_z) \\ = \left(\frac{\partial h_z}{\partial \bar{y}} - iq\bar{k}_z h_y \right) \hat{e}_x + \left(iq\bar{k}_z h_x - \frac{\partial h_z}{\partial \bar{x}} \right) \hat{e}_y + q \left(\frac{\partial h_y}{\partial \bar{x}} - \frac{\partial h_x}{\partial \bar{y}} \right) \hat{e}_z \end{cases} \quad (\text{S10.4})$$

Notice here the cancellations of the common factor $\varepsilon \equiv n^2$ on both sides of the second relation in the preceding equation. Component-wise, Eq. (S10.4) becomes

$$\left\{ \begin{array}{l}
TE : \left\{ \begin{array}{l}
qh_x = -iq \frac{\partial f_z}{\partial \bar{y}} - \bar{k}_z f_y \\
qh_y = \bar{k}_z f_x + iq \frac{\partial f_z}{\partial \bar{x}} \\
f_z - \textcolor{red}{q}U_z = i \left(\frac{\partial h_y}{\partial \bar{x}} - \frac{\partial h_x}{\partial \bar{y}} \right)
\end{array} \right. \\
\\
TM : \left\{ \begin{array}{l}
f_x - \textcolor{red}{q}U_x = i \frac{\partial h_z}{\partial \bar{y}} + q\bar{k}_z h_y \\
f_y - \textcolor{red}{q}U_y = -q\bar{k}_z h_x - i \frac{\partial h_z}{\partial \bar{x}} \\
h_z = -i \left(\frac{\partial f_y}{\partial \bar{x}} - \frac{\partial f_x}{\partial \bar{y}} \right)
\end{array} \right.
\end{array} \right. \quad (S10.5)$$

In general, the coupling coefficient q springs up in many places in a mixed manner. The additional terms due to the space charge are indicated in red color. In each set of relations of the above equation, we substitute the first two into the last relation. As a result,

$$\left\{ \begin{array}{l}
qf_z = iq \left(\frac{\partial h_y}{\partial \bar{x}} - \frac{\partial h_x}{\partial \bar{y}} \right) + qU_z = iq \frac{\partial h_y}{\partial \bar{x}} - iq \frac{\partial h_x}{\partial \bar{y}} + qU_z \\
= i \frac{\partial}{\partial \bar{x}} \left(\bar{k}_z f_x + iq \frac{\partial f_z}{\partial \bar{x}} \right) - i \frac{\partial}{\partial \bar{y}} \left(-iq \frac{\partial f_z}{\partial \bar{y}} - \bar{k}_z f_y \right) + qJ_z \\
h_z = -i \left(\frac{\partial f_y}{\partial \bar{x}} - \frac{\partial f_x}{\partial \bar{y}} \right) = -i \frac{\partial f_y}{\partial \bar{x}} + i \frac{\partial f_x}{\partial \bar{y}} \\
= -i \frac{\partial}{\partial \bar{x}} \left(-q\bar{k}_z h_x - i \frac{\partial h_z}{\partial \bar{x}} + J_y \right) + i \frac{\partial}{\partial \bar{y}} \left(i \frac{\partial h_z}{\partial \bar{y}} + q\bar{k}_z h_y + U_x \right)
\end{array} \right. \quad (S10.6)$$

Further manipulating Eq. (S10.6) leads to two Helmholtz-like equations of an almost same form as follows.

$$\left\{ \begin{array}{l}
qf_z = i\bar{k}_z \left(\frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} \right) - q \frac{\partial^2 f_z}{\partial \bar{x}^2} - q \frac{\partial^2 f_z}{\partial \bar{y}^2} + qU_z \\
h_z = iq\bar{k}_z \left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}} \right) - \frac{\partial^2 h_z}{\partial \bar{x}^2} - \frac{\partial^2 h_z}{\partial \bar{y}^2} - i \left(\frac{\partial U_y}{\partial \bar{x}} - \frac{\partial U_x}{\partial \bar{y}} \right)
\end{array} \right. \quad (S10.7)$$

If rearranged, the above is transformed into

$$\begin{cases} \frac{\partial^2 f_z}{\partial \bar{x}^2} + \frac{\partial^2 f_z}{\partial \bar{y}^2} + f_z = i \frac{\bar{k}_z}{q} \left(\frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} \right) + U_z \\ \frac{\partial^2 h_z}{\partial \bar{x}^2} + \frac{\partial^2 h_z}{\partial \bar{y}^2} + h_z = iq \bar{k}_z \left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}} \right) - i \left(\frac{\partial U_y}{\partial \bar{x}} - \frac{\partial U_x}{\partial \bar{y}} \right) \end{cases} \quad (\text{S10.8})$$

Physically these relations appear correct, since the electric current U_z in the axial direction affects the electric field in the first relation, whereas the axial component for the curl of the electric currents $(\partial U_y / \partial \bar{x} - \partial U_x / \partial \bar{y})$ affects the magnetic field in the second relation. In the meantime, the divergence conditions in Eq. (S10.3) now read as follows.

$$\begin{cases} i \frac{\bar{k}_z}{q} \left(\frac{\partial f_x}{\partial \bar{x}} + \frac{\partial f_y}{\partial \bar{y}} \right) = \bar{k}_z^2 f_z + i \bar{k}_z \left(\frac{\partial U_x}{\partial \bar{x}} + \frac{\partial U_y}{\partial \bar{y}} \right) - \bar{k}_z^2 U_z \\ iq \bar{k}_z \left(\frac{\partial h_x}{\partial \bar{x}} + \frac{\partial h_y}{\partial \bar{y}} \right) = \bar{k}_z^2 h_z \end{cases} \quad (\text{S10.9})$$

With the help of Eq. (S10.9), Eq. (S10.8) gets down to

$$\begin{cases} \frac{\partial^2 f_z}{\partial \bar{x}^2} + \frac{\partial^2 f_z}{\partial \bar{y}^2} + (1 - \bar{k}_z^2) f_z = i \bar{k}_z \left(\frac{\partial U_x}{\partial \bar{x}} + \frac{\partial U_y}{\partial \bar{y}} \right) + (1 - \bar{k}_z^2) U_z \\ \frac{\partial^2 h_z}{\partial \bar{x}^2} + \frac{\partial^2 h_z}{\partial \bar{y}^2} + (1 - \bar{k}_z^2) h_z = -i \left(\frac{\partial U_y}{\partial \bar{x}} - \frac{\partial U_x}{\partial \bar{y}} \right) \end{cases} \quad (\text{S10.10})$$

In the absence of space charges, we recover Eq. (S3.10), because the right-hand sides of the above equation disappear. What is nice with the form of Eq. (S10.10) is the fact that the coupling coefficient q does not show up. This fact corroborate the advantage in previously setting $\vec{J} = -c^{-1}(\partial/\partial t)(\epsilon \vec{U})$ in Eq. (S9.4). In particular, the multiplying factor q asserts that the current density is related to the TE mode, which confirms in turn the well-known fact in electromagnetic theory that the electric current is associated only with the TE field but not with the TM field.

However, as mentioned before concerning Eq. (S9.4), we have the following mixed relation among the electric-field components.

$$\sigma = q\nabla \cdot (\epsilon \vec{U}) \Rightarrow \sigma = qk\epsilon n \left(\frac{\partial U_x}{\partial \bar{x}} + \frac{\partial U_y}{\partial \bar{y}} + i\bar{k}_z U_z \right). \quad (\text{S10.11})$$

This is certainly a partial differential equation in solving for (U_x, U_y, U_z) when a spatial distribution $\sigma(\bar{x}, \bar{y}, t)$ were ever specified. In fact, σ is not normally specifiable, but it should be solved self-consistently in combination with an additional constraining equation. This issue belong to the full-scope electrodynamics of course. For instance, the electron density σ is dependent on the fields $\vec{E}$ and $\vec{H}$ through quantum mechanics.

S11. Helmholtz Equations in the Cylindrical Coordinates with Space Charges

Let us transform Eq. (S10.10) into its counterpart the cylindrical coordinates. For this purpose, let us introduce the in-plane divergence as follows.

$$\begin{aligned} \nabla_{k\perp} \cdot (U_x \hat{e}_x + U_y \hat{e}_y) &\equiv \frac{\partial U_x}{\partial \bar{x}} + \frac{\partial U_y}{\partial \bar{y}} \\ &= \nabla_{k\perp} \cdot (U_r \hat{e}_r + U_\theta \hat{e}_\theta) \equiv \frac{\partial U_r}{\partial \bar{\rho}} + \frac{1}{\bar{\rho}} \frac{\partial U_\theta}{\partial \theta}. \end{aligned} \quad (\text{S11.1})$$

Likewise, let us introduce the in-plane Laplacian and axial curl operators as follows.

$$\begin{cases} \nabla_{k\perp}^2 f_z \equiv \frac{\partial^2 f_z}{\partial \bar{x}^2} + \frac{\partial^2 f_z}{\partial \bar{y}^2} \equiv \frac{\partial^2 f_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial f_z}{\partial \bar{\rho}} + \frac{1}{\bar{\rho}^2} \frac{\partial^2 f_z}{\partial \theta^2} \\ \nabla_{k\perp}^2 h_z \equiv \frac{\partial^2 h_z}{\partial \bar{x}^2} + \frac{\partial^2 h_z}{\partial \bar{y}^2} \equiv \frac{\partial^2 h_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial h_z}{\partial \bar{\rho}} + \frac{1}{\bar{\rho}^2} \frac{\partial^2 h_z}{\partial \theta^2} \end{cases}. \quad (\text{S11.2})$$

$$\begin{aligned} \left[\nabla_{k\perp} \times (U_x \hat{e}_x + U_y \hat{e}_y) \right]_z &\equiv \frac{\partial U_y}{\partial \bar{x}} - \frac{\partial U_x}{\partial \bar{y}} \\ &= \left[\nabla_{k\perp} \times (U_r \hat{e}_r + U_\theta \hat{e}_\theta) \right]_z \equiv \frac{\partial U_\theta}{\partial \bar{\rho}} - \frac{1}{\bar{\rho}} \frac{\partial U_r}{\partial \theta}. \end{aligned} \quad (\text{S11.3})$$

Here, we have already performed all the operations in both coordinates via the exponential factors in Eqs. (S2.1) and the field variables in Eq. (S2.2).

Now, we apply the azimuthal mode index (AMI) in Eq. (S2.1) to Eqs. (S11.1) and (S11.3) as follows.

$$\begin{cases} \nabla_{k\perp}[(U_r \hat{e}_r + U_\theta \hat{e}_\theta)] = \frac{\partial U_r}{\partial \bar{\rho}} \pm \frac{im}{\bar{\rho}} U_\theta \\ \left[\nabla_{k\perp} \times (U_r \hat{e}_r + U_\theta \hat{e}_\theta) \right]_z = \frac{\partial U_\theta}{\partial \bar{\rho}} \mp \frac{im}{\bar{\rho}} U_r \end{cases} \quad (\text{S11.4})$$

Of course, we applied $\exp(\pm im\theta)$ to all three components of the electric current according to the rotation of the TE waves in Eq. (S2.2).

In a similar way, (f_z, h_z) in Eq. (S11.2) are treated according to the rotational modes of the TE and TM waves, respectively.

$$\begin{cases} \nabla_{k\perp}^2 f_z = \frac{\partial^2 f_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial f_z}{\partial \bar{\rho}} - \frac{m^2}{\bar{\rho}^2} f_z \\ \nabla_{k\perp}^2 h_z = \frac{\partial^2 h_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial h_z}{\partial \bar{\rho}} - \frac{m^2}{\bar{\rho}^2} h_z \end{cases} \quad (\text{S11.5})$$

However, it turns out that rotational operators acted twice lead to the same sign.

In the end, the two Helmholtz equations in Eq. (S10.10) take the following forms in the cylindrical coordinates.

$$\begin{cases} \frac{\partial^2 f_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial f_z}{\partial \bar{\rho}} - \frac{m^2}{\bar{\rho}^2} f_z + (1 - \bar{k}_z^2) f_z = i \bar{k}_z \left(\frac{\partial U_r}{\partial \bar{\rho}} \pm \frac{im}{\bar{\rho}} U_\theta \right) + (1 - \bar{k}_z^2) U_z \\ \frac{\partial^2 h_z}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial h_z}{\partial \bar{\rho}} - \frac{m^2}{\bar{\rho}^2} h_z + (1 - \bar{k}_z^2) h_z = -i \left(\frac{\partial U_\theta}{\partial \bar{\rho}} \mp \frac{im}{\bar{\rho}} U_r \right) \end{cases} \quad (\text{S11.6})$$

By the scaled radial coordinates in Eq. (S2.4), Eq. (S11.6) becomes

$$\begin{cases} \frac{\partial^2 f_z}{\partial \tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{\partial f_z}{\partial \tilde{\rho}} - \frac{m^2}{\tilde{\rho}^2} f_z + f_z = i \frac{\bar{k}_z}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_r}{\partial \tilde{\rho}} \pm \frac{im}{\tilde{\rho}} U_\theta \right) + U_z \\ \frac{\partial^2 h_z}{\partial \tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{\partial h_z}{\partial \tilde{\rho}} - \frac{m^2}{\tilde{\rho}^2} h_z + h_z = -i \frac{1}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) \end{cases} \quad (\text{S11.7})$$

Let us define the following modified Helmholtz operator in the cylindrical coordinates.

$$\tilde{\nabla}_{H,m}^2 \equiv \frac{\partial^2}{\partial \tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{\rho}} - \frac{m^2}{\tilde{\rho}^2} + f_z. \quad (\text{S11.8})$$

Eq. (S11.7) is then cast as follows.

$$\begin{cases} \tilde{\nabla}_{H,m}^2 f_z = i \frac{\bar{k}_z}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_r}{\partial \tilde{\rho}} \pm \frac{im}{\tilde{\rho}} U_\theta \right) + U_z \\ \tilde{\nabla}_{H,m}^2 h_z = -i \frac{1}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) \end{cases} \quad (\text{S11.9})$$

For reference, we obtain the following much simpler set for $\bar{k}_z = 0$ from Eq. (S11.9).

$$\begin{cases} \tilde{\nabla}_{H,m}^2 f_z = U_z \\ \tilde{\nabla}_{H,m}^2 h_z = -i \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) \end{cases} \quad (\text{S11.10})$$

As a result, the axially non-propagating hybrid waves still need an additional account for the vector $\vec{U}$ of the charge displacement density.

It is important to notice from Eq. (S2.4) that the scaled radial coordinates become complex variables. Hence, it is proper not to define the refractive index for the electron gas but to continue to employ ε^{EG} instead.

$$\begin{cases} \bar{\rho} \equiv \sqrt{\varepsilon^{EG}} \rho \equiv \sqrt{\varepsilon^{EG}} kr \\ \tilde{\rho} \equiv \sqrt{1 - \bar{k}_z^2} \bar{\rho} \equiv n^\diamond \sqrt{1 - \frac{k_z^2}{\varepsilon^{EG} k^2}} (kr) \equiv r \sqrt{\varepsilon^{EG} k^2 - k_z^2} \end{cases} \quad (\text{S11.11})$$

As a result, $1 - \bar{k}_z^2 > 0$ always.

S12. Pill-Box Limits in the Annular Thin Layer

By referring to Fig. S2, let us integrate Eq. (S11.7) across the annular thin layer in the radial direction.

$$\begin{cases} \int_{R-}^{R+} \left[\frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{\rho}} \left(\tilde{\rho} \frac{\partial f_z}{\partial \tilde{\rho}} \right) - \frac{m^2}{\tilde{\rho}^2} f_z + f_z = i \frac{\bar{k}_z}{\sqrt{1 - \bar{k}_z^2}} \left(\frac{\partial U_r}{\partial \tilde{\rho}} \pm \frac{im}{\tilde{\rho}} U_\theta \right) + U_z \right] \tilde{\rho} d\tilde{\rho} \\ \int_{R-}^{R+} \left[\frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{\rho}} \left(\tilde{\rho} \frac{\partial h_z}{\partial \tilde{\rho}} \right) - \frac{m^2}{\tilde{\rho}^2} h_z + h_z = -i \frac{1}{\sqrt{1 - \bar{k}_z^2}} \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) \right] \tilde{\rho} d\tilde{\rho} \end{cases} \quad (\text{S12.1})$$

By integrating out the terms involving the radial gradients, we get

$$\begin{cases} \left(\tilde{\rho} \frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R+}^{EG} - \left(\tilde{\rho} \frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R-}^{EG} + \int_{R-}^{R+} \left[\left(1 - \frac{m^2}{\tilde{\rho}^2} \right) f_z = i \frac{\bar{k}_z}{\sqrt{1 - \bar{k}_z^2}} \left(\frac{\partial U_r}{\partial \tilde{\rho}} \pm \frac{im}{\tilde{\rho}} U_\theta \right) + U_z \right] \tilde{\rho} d\tilde{\rho} \\ \left(\tilde{\rho} \frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R+}^{EG} - \left(\tilde{\rho} \frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R-}^{EG} + \int_{R-}^{R+} \left[\left(1 - \frac{m^2}{\tilde{\rho}^2} \right) h_z = -i \frac{1}{\sqrt{1 - \bar{k}_z^2}} \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) \right] \tilde{\rho} d\tilde{\rho} \end{cases} \quad (\text{S12.2})$$

This set is the equation to solve in combination with the equations for the electromagnetic waves in both interior and exterior.

Next, both left-hand sides in the above equation can be approximated by the condition that the layer thickness is much smaller than the radius of the layer position. Hence,

$$\left\{ \begin{aligned} \left(\frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R+}^{EG} - \left(\frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R-}^{EG} &= \frac{1}{\tilde{R}} \int_{R-}^{R+} \left[i \frac{\bar{k}_z}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_r}{\partial \tilde{\rho}} \pm \frac{im}{\tilde{\rho}} U_\theta \right) + U_z - \left(1 - \frac{m^2}{\tilde{\rho}^2} \right) f_z \right] \tilde{\rho} d\tilde{\rho} \\ \left(\frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R+}^{EG} - \left(\frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R-}^{EG} &= \frac{1}{\tilde{R}} \int_{R-}^{R+} \left[-i \frac{1}{\sqrt{1-\bar{k}_z^2}} \left(\frac{\partial U_\theta}{\partial \tilde{\rho}} \mp \frac{im}{\tilde{\rho}} U_r \right) - \left(1 - \frac{m^2}{\tilde{\rho}^2} \right) h_z \right] \tilde{\rho} d\tilde{\rho} \end{aligned} \right. \quad (S12.3)$$

Across the interface between the interior and the thin layer and across the other interface between the thin layer and the exterior, we have the following relations in the absence of any space charges.

$$\left\{ \begin{aligned} \left(\frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R-}^< &= \left(\frac{\partial f_z}{\partial \tilde{\rho}} \right)_{R-}^{EG} \\ \left(\frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R+}^{EG} &= \left(\frac{\partial h_z}{\partial \tilde{\rho}} \right)_{R+}^> \end{aligned} \right. \quad (S12.4)$$

Combining Eqs. (S12.3) and (S12.4) and reverting to the dimensional variables, we finally end up with

$$\left\{ \begin{aligned} \left(\frac{dE_z}{d\rho} \right)_{R+}^> - \left(\frac{dE_z}{d\rho} \right)_{R-}^< &= A \\ \frac{1}{\varepsilon^>} \left(\frac{dH_z}{d\rho} \right)_{R+}^> - \frac{1}{\varepsilon^<} \left(\frac{dH_z}{d\rho} \right)_{R-}^< &= B \end{aligned} \right. \quad (S12.5)$$

This point needs further clarification, since we can devise a combination of clockwise and counter-clockwise waves for $\vec{U}$. In this case, $\vec{U}$ can be expanded as follows.

$$\vec{U} \equiv \vec{U}^+ + \vec{U}^-. \quad (S12.6)$$

Here, the superscripts $+$ and $-$ refer respectively to the co and counter-rotating components of $\vec{U}$.