MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

Math Energy

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Solving Matrix Equations via Empirical Gramians

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About

In system theory, the so-called system Gramian matrices are operators encoding certain properties of an underlying input-output system. Usually, these system Gramians are computed as solutions to matrix equations, such as the Lyapunov equation and Sylvester equation. This means, the solution to certain matrix equations coincides with these system Gramians. Now, if the system Gramians are computable by other means than matrix equations, they still represent solutions to matrix equations. Empirical Gramians are such an alternative for system Gramian computation, which are based on their system-theoretic operator definition, and practically obtained via quadrature. This contribution explores the connection between matrix equations, system Gramians and empirical Gramians, and proposes empirical Gramians as potential solver for matrix equations.

Continuous-Time Linear Time-Invariant System

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

Discrete-Time Linear Time-Invariant System

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$

Lyapunov Equation

Lyapunov Equation $\xrightarrow{\text{rrSVD}}$ Factor RHS $\overset{\text{IbP}}{\Leftrightarrow}$ Controllability Gramian $\overset{\int}{\approx}$ Empirical Controllability Gramian $\overset{\text{tSVD}}{\approx}$ Low Rank Empirical Gramian

$$AW + WA^\intercal = -Y \xrightarrow{\text{rrSVD}} AW + WA^\intercal = -BB^\intercal \overset{\text{IbP}}{\Leftrightarrow} W = \int_0^\infty e^{At}BB^\intercal e^{A^\intercal t}\,dt \overset{\int}{\approx} \widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} X_m(A,B)X_m(A,B)^\intercal \overset{\text{tSVD}}{\approx} \widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(A,B)^\intercal$$

Sylvester Equation

Sylvester Equation $\xrightarrow{\text{rrSVD}}$ Factor RHS $\overset{\text{IbP}}{\Leftrightarrow}$ Cross Gramian $\overset{\int}{\approx}$ Empirical Cross Gramian $\overset{\text{tSVD}}{\approx}$ Low Rank Empirical Gramian

$$AW + WF = -Y \xrightarrow{\text{rrSVD}} AW + WF = -BH \overset{\text{IbP}}{\Leftrightarrow} W = \int_0^\infty e^{At}BHe^{Ft}\,dt \overset{\int}{\approx} \widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} X_m(A,B)X_m(F,H)^\intercal \overset{\text{tSVD}}{\approx} \widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(F,H)^\intercal$$

(Symmetric) Stein Equation

Stein Equation $\xrightarrow{\text{rrSVD}}$ Factor RHS $\Leftrightarrow$ Controllability Gramian $\overset{\int}{\approx}$ Empirical Controllability Gramian $\overset{\text{tSVD}}{\approx}$ Low Rank Empirical Gramian

$$AWA^\intercal - W = -Y \xrightarrow{\text{rrSVD}} AWA^\intercal - W = -BB^\intercal \Leftrightarrow W = \sum_{k=0}^{\infty} A^k BB^\intercal (A^\intercal)^k \overset{\int}{\approx} \widetilde{W} = \frac{1}{M}\sum_{m=1}^{M} X_m(A,B)X_m(A,B)^\intercal \overset{\text{tSVD}}{\approx} \widetilde{W} = \frac{1}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(A,B)^\intercal$$

Riccati Equation

Riccati Equation $\xrightarrow{\text{rrSVD}}$ Factor RHS $\rightarrow$ Newton Iteration via Lyapunov Equations

$$AW + WA^\intercal = -Y - WZW \xrightarrow{\text{rrSVD}} AW + WA^\intercal = -BB^\intercal - WC^\intercal CW \rightarrow (A - C^\intercal CW_k)W_{k+1} + W_{k+1}(A - C^\intercal CW_k)^\intercal = -(BB^\intercal + W_k C^\intercal CW_k),\ W_0 = 0$$

More Matrix Equations

- Generalized Lyapunov Equation: $AWE^\intercal + EWA^\intercal = BB^\intercal$
- Generalized Sylvester Equation: $AWE + EWA = BC$
- Generalized Stein Equation: $A^T WA + EWE^\intercal = BB^\intercal$
- Cross-Riccati Equation: $AW + WA = -BC - WBCW$

Generalized Continuous-Time Linear Time-Invariant System

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

Generalized Discrete-Time Linear Time-Invariant System

$$Ex_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$

Notes

- Choice of time-stepping solver is crucial.
- Right-hand-sides need to be low-rank.
- Time varying systems relate to matrix differential equation: $\dot{x}(t) = A(t)x(t) + B(t)u(t) \leftrightarrow \dot{X}(t) = A(t)X(t) + X(t)A(t)^\intercal + B(t)B(t)^\intercal$

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