Computational techniques for kinematically constrained convection models

Introduction

In order to mix plate kinematic constraints with mantle dynamics we need to be able to apply plate-like boundary conditions to convection models in which the penalty for the model failing to satisfy the condition can be weighted according to the uncertainty in the kinematic reconstruction. Constraints typically include the near-rigidity of oceanic plate interiors and prescribing observed plate motions.

An additional requirement is a kinematically-plausible dynamic model for the deformation at plate boundaries and regions where there is poor observational constraints on the kinematics.

Numerically, the application of the rigidity constraint can be achieved by either penalizing the strain-rate in the plate interior or by prescribing that the velocities must be consistent with a rigid body motion. Plate reconstruction information is imposed upon the rigid plates by an additional penalty on the mismatch between the convection model plate velocities and those from the reconstructed plate motions.

Here we address the issue of solving the typical systems of equations which emerge when such constraints are applied, and which are present when appropriate constitutive models for the plate boundaries are included.

The context for our investigation is the Particle-in-Cell Finite Element Method described in Moresi et al. (2002, 2003). This has the capacity to include the relevant constitutive behaviour (e.g. strain-softening plasticity) expressed with a standard Finite Element formulation in which the constraint problem can be formulated.

Constrained Flow – template problem

The archetypal problem is Stokes flow

$$\nabla \cdot \boldsymbol{\sigma} - \nabla \boldsymbol{p} = g\rho\hat{\boldsymbol{z}} \qquad (1)$$

subject to various constraints including, for example, incompressibility

$$\nabla \cdot \boldsymbol{u} = 0 \qquad (2)$$

and strongly varying material properties (viscosity or an equivalent parameter which acts as a penalty on the plate-rigidity constraint)

$$\sigma_{ij} = \eta(x,y,z)\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - p\delta_{ij} \qquad (3)$$

Discrete Problem

An important issue in the large-scale constrained modelling problem is choosing a preconditioned iterative method. While there are many possible approaches, thus we would like to experiment to determine which strategy is optimal.

To provide flexibility in defining both the iterative method and a suitable preconditioner, we prefer to define the discrete counterpart to each operator in (1) as an individual matrix. The resulting system of equations is then given as a block matrix –

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \qquad (4)$$

Here, $\mathbb{B}$ represents the discrete gradient operator and $\mathbb{C} = \mathbb{B}^T$, the discrete divergence. In the general case, each of $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ may themselves be block matrices.

The constraint system above is indefinite due to the zero $(2, 2)$ block. As a consequence, a common operator for use in either defining a preconditioner or operator for a Krylov subspace method is the Schur complement, $\mathbb{S} = \mathbb{C}\mathbb{A}^{-1}\mathbb{B}$.

Preconditioners can be constructed from the manipulating the block matrices in (4) (in some manner) or through a block preconditioner

$$\hat{\mathcal{A}} = \begin{pmatrix} \hat{\mathbb{A}}_{11} & \mathbb{B} \\ 0 & \hat{\mathbb{A}}_{22} \end{pmatrix} \qquad (5)$$

which has been shown to effective when applied as a right preconditioner to (4). See Elman et al. (2002), Elman (2002).

We implemented arbitrary sized block vectors, matrices and preconditioner objects within PETSc Balay et al. (1997), with a matrix free representation of the Schur complement and other preconditioners for $\mathbb{S}$.

The acid test

To demonstrate the benefits of using a block framework, we compare several solution strategies applied to a number of Stokes flow models. The reference solver upon which we seek to improve is the segregated approach employed by Citcom as define in Moresi and Solomatov (1995).

We ran three simple test examples to guage the ability of a very large range of solvers to address typical challenges: flow models driven by internal density contrasts with: 1) exponentially varying vertical gradient in viscosity of $\Delta\eta = 10^6$; 2) exponentially varying horizontal viscosity gradient ($\Delta\eta = 10^6$); and 3) a sinking, dense ball of viscosity contrast 10^8

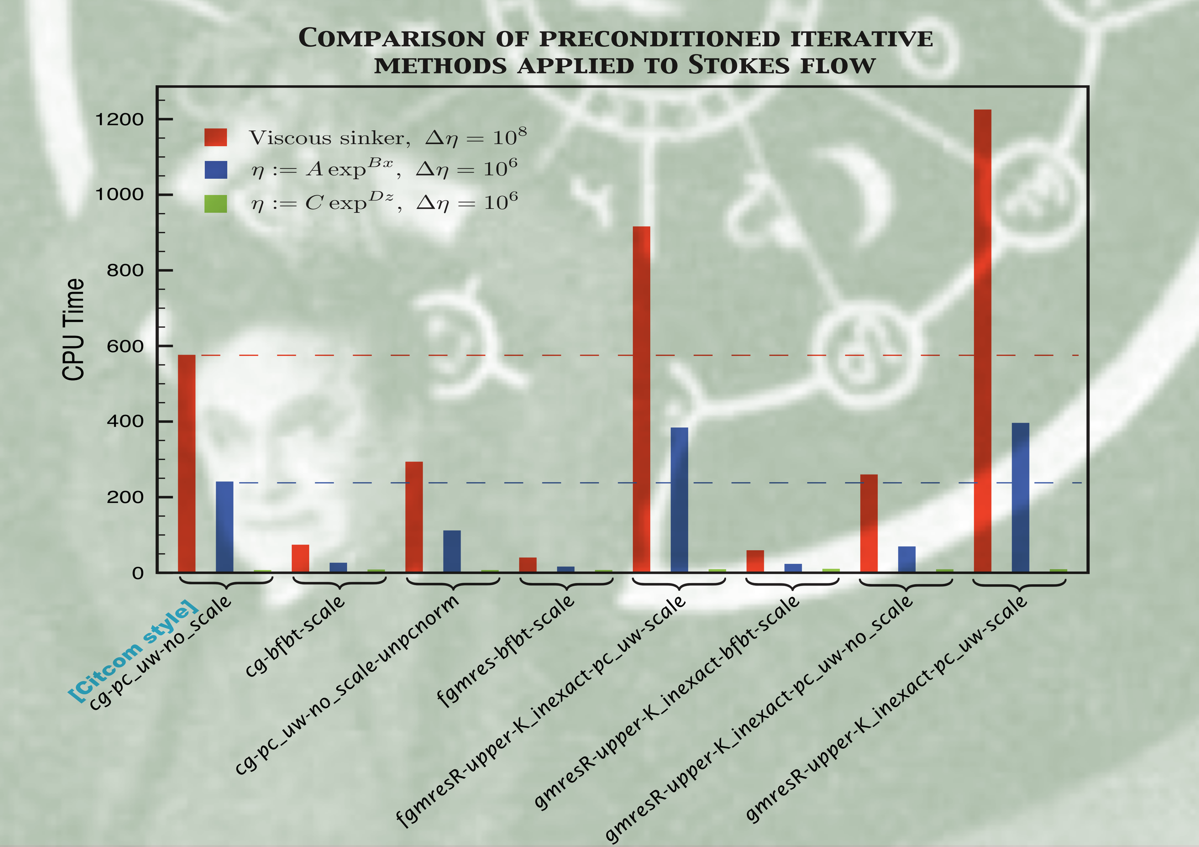

Figure 1 – The qualifying round: which solvers can beat Citcom out-of-the-box ? Any which come in under the qualifying time for the

And the winner is …

The BFBt preconditioners, particularly when combined with GMRES, beat the Citcom solver strategy in the simple tests by a respectable margin. In the slab subduction test they are considerably more robust. Importantly, their convergence is uniform to at small tolerances where the Citcom scheme falls off in efficiency quite early.

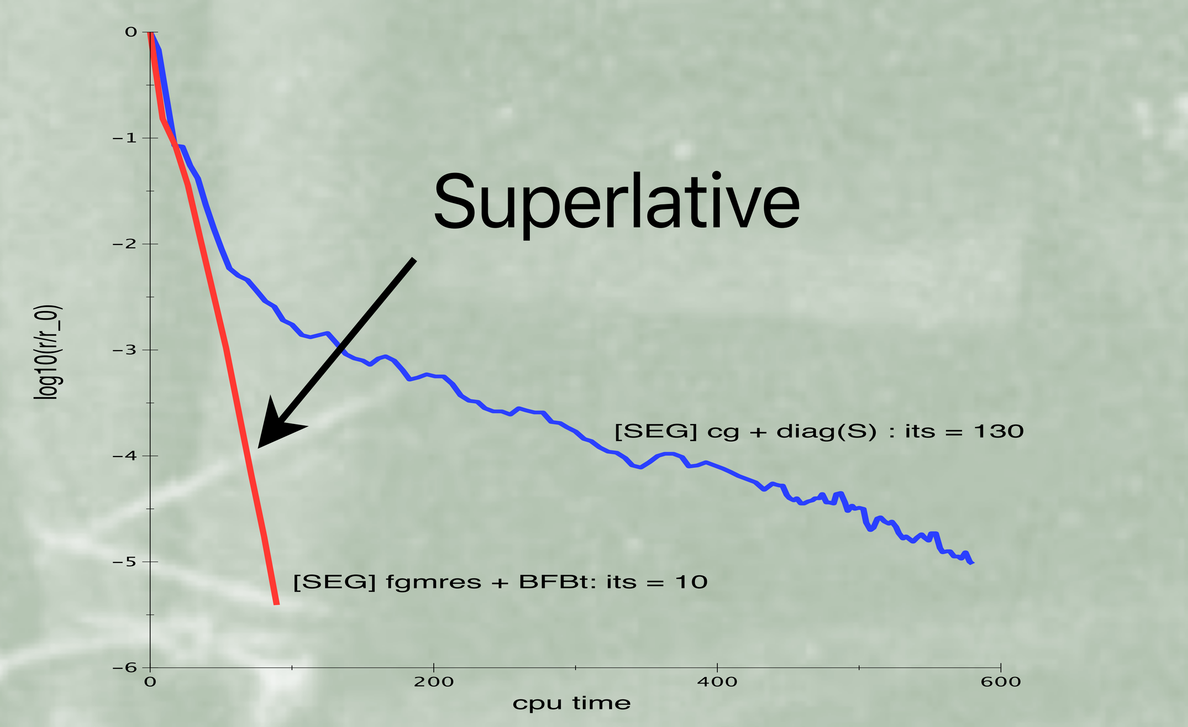

Figure 2 – How well does the best alternative preconditioning strategy stack up against Citcom ? The convergence for the slab-subduction challenge shows very reliable residual reduction without

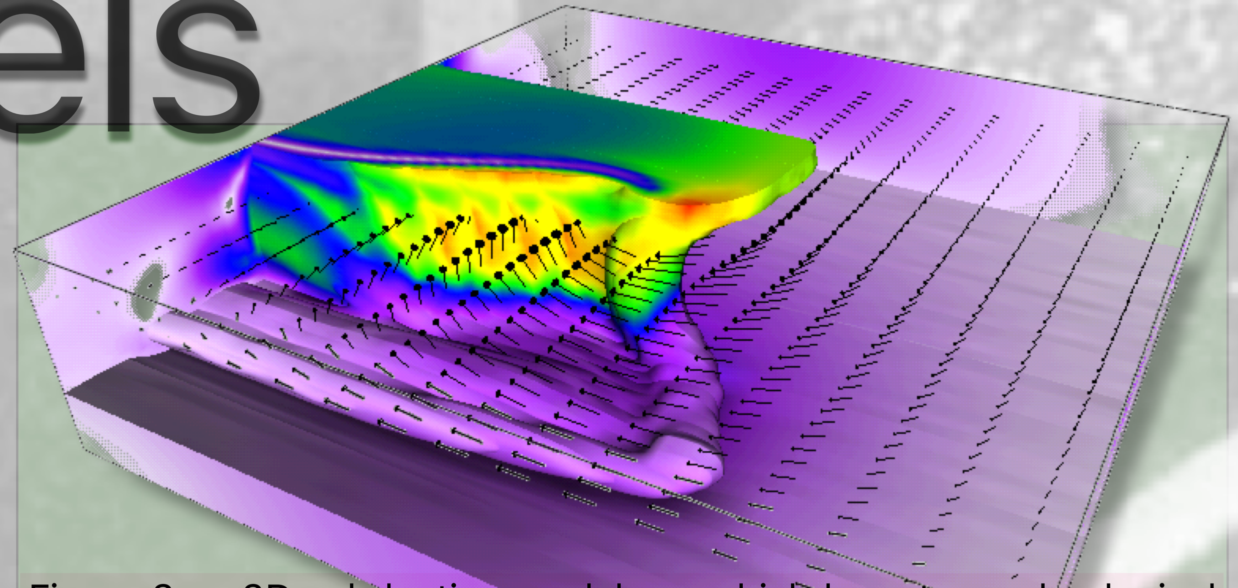

Figure 3– a 3D subduction model run which has strong rheological variations due to natural layering and the development of localization. The material is constrained to be incompressible.

In Summary

We regard a preconditioner as optimal if the iterative method used exhibits convergence rates independent of the discretisation parameter h, and independent of the constitutitve behaviour. These stringent requirements are tough to satisfy under all circumstances. From a suite of Stokes flow derived numerical experiments in which grid resolution and viscosity contrast were our free parameters, we observe the BFBt preconditioner satisfies our definition of an optimal preconditioner. The commonly used choice of $\hat{S} = \mathrm{diag}(G^T)(\mathrm{diag}(K))^{-1}G$, demonstrates a sensitivity to both the grid resolution and the viscosity contrast.

The development of generic block structure objects enables us to easily define coupled systems arsing from the discretisation of PDEs. In addition, block based preconditioners can also be defined with minimal effort. Togther, we envisage that these tools will enable improved efficiency in solving simple constrained problems such as Stokes flow, and more complex constrained systems such as coupled mantle flow, plate kinematic problem.

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