ON THE COMPLEXITY OF SOME INEVITABLE CLASSES OF SEPARABLE BANACH SPACES.

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ABSTRACT. In this paper, we study the descriptive complexity of some inevitable classes of Banach spaces. Precisely, as shown in [Go], every Banach space either contains a hereditarily indecomposable subspace or an unconditional basis, and, as shown in [FR], every Banach space either contains a minimal subspace or a continuously tight subspace. In these notes, we study the complexity of those inevitable classes as well as the complexity of containing a subspace in any of those classes.

1. INTRODUCTION.

Let $SB = \{X \subset C(\Delta) \mid X \text{ is linear and closed}\}$ be our coding for the separable Banach spaces, and endow SB with the Effros-Borel structure (for definitions, see Section 2). Given a separable Banach space $X \in SB$, it is natural to ask about the complexity of the isomorphism class $\langle X \rangle = \{Y \in SB \mid Y \cong X\}$. Is $\langle X \rangle$ Borel, analytic, coanalytic? This question has only been solved for some very specific Banach spaces. For example, S. Kwapien showed (see [Kw], Proposition 3.1) that a Banach space X is isomorphic to a Hilbert space if, and only if, X has both type and cotype equal 2. As ℓ_2 is the only separable Hilbert space (up to isomorphism), this gives us a characterization of separable Banach spaces which are isomorphic to ℓ_2 in terms of type and cotype. Using this characterization, it is not hard to show that $\langle \ell_2 \rangle$ is Borel (see [B], page 130). In fact, $\langle \ell_2 \rangle$ is the only known example of a Borel isomorphism class. We recall the following problem (see [B], Problem 2.9).

Problem 1. Let $X \in SB$ be a separable Banach space whose isomorphism class $\langle X \rangle$ is Borel. Is X isomorphic to ℓ_2?

For some classical spaces, such as $L_p[0,1]$, with $p \neq 2$, Pelczynski's universal space U, and $C(\Delta)$, it had been shown that their isomorphism classes are analytic non Borel (see [B], page 130 and Theorem 2.3, and [K], Theorem 33.24, respectively).

The problem of classifying Banach spaces up to isomorphism is extremely complex. Indeed, considering the isomorphism relation $\cong$ as a subset of $SB \times SB$, we have that $\cong$ is a complete analytic equivalence relation (see [FLR], Theorem 31). In a more precise sense, separable Banach spaces up to isomorphism may serve as a complete invariant for any other reasonable class of mathematical objects. Therefore, it makes sense to study easier problems such as classifying Banach spaces up to its subspaces. We have then another natural problem: given a Banach space

$X \in SB$, what can we say about the complexity of $C_X = \{Y \in SB \mid X \hookrightarrow Y\}$? This question had been solved by B. Bossard (see [B], Corollary 3.3).

Theorem 2. *(Bossard) Let $X \in SB$. If X is finite dimensional, then C_X is Borel. If X is infinite dimensional, then C_X is complete analytic.*

In [Go2], W. T. Gowers had started with a new classification theory for Banach spaces. Indeed, after Gowers and Maurey had solved the unconditional basis problem, and proved the existence of hereditarily indecomposable Banach spaces (see [GM]), Gowers proved that every Banach space either contains a subspace with an unconditional basis, or a hereditarily indecomposable subspace (see [Go]). Later, in [Go2], Gowers used Ramsey methods in order to refine his dichotomy. Many other dichotomies had been proved by V. Ferenczi and C. Rosendal in [FR], for example, their main dichotomy says that every Banach space either contains a minimal subspace or a continuously tight subspace (see [FR], Theorem 3.13).

In these notes, we study the descriptive complexity of the inevitable classes above as well as the complexity of containing a subspace in any of those inevitable classes. The table below summarizes the lower and upper bounds known for the complexity of those classes. In the table below, an asterisk before the complexity indicates that this bound was already known before this paper. All the other bounds are either computed in this paper or are given by trivial computations.

Classes of Banach spaces	Lower bound	Upper bound
Minimal spaces		Δ_2^1
Containing a minimal subspace	Σ_1^1-hard	$(*) \ \Sigma_2^1$
Continuously tight spaces	Π_1^1-hard	Σ_2^1
Containing a continuously tight subspace	Σ_1^1-hard	Σ_2^1
HI spaces	Π_1^1-hard	Π_1^1
Containing a HI subspace	Σ_1^1-hard	Σ_2^1
Spaces with unconditional basis		Σ_1^1
Containing unconditional basis	Σ_1^1-hard	Σ_1^1

In particular, we completely compute the descriptive complexity of the class of hereditarily indecomposable spaces, and the class of spaces containing an unconditional basis, by showing those classes are complete coanalytic and complete analytic, respectively.

The properties of being a continuously tight space and of having an unconditional basis are more properties about basis then about the spaces themselves. Therefore, it is more natural to ask what is the complexity of the set of continuously tight basis, and the complexity of the set of unconditional basis. We have the following.

Classes of basis	Lower bound	Upper bound
Continuously tight basis	Π_1^1-hard	Σ_2^1
Unconditional basis		Borel

This paper is organized in the following way. In Section 2, we give all the basic background on both descriptive set theory and the theory of Banach spaces necessary for these notes. In Section 3, we prove a general lemma for Banach spaces with an unconditional basis $(x_s)_{s \in \theta}$ indexed by a tree $\theta \in WF$. In Section 4, we work with ℓ_p-Baire sums of basic sequences as a tool to compute lower bounds for

the complexity of classes of Banach spaces. Then, Theorem 9 gives a sufficient condition for the set of separable Banach spaces containing a subspace in a given class to be Σ_1^1-hard (in particular, non Borel).

Section 5 is dedicated to the computation of the complexities in the tables above. We apply Theorem 9 to the class of spaces containing a hereditarily indecomposable subspace (Subsection 5.1, Corollary 10), and to the class of spaces containing a continuously tight subspace (Subsection 5.2, Corollary 11), obtaining a lower bound for the complexity of those classes.

In order to obtain the results above, we rely on the method of ℓ_p-Baire sums of separable Banach spaces (see Lemma 8). However, this method does not allow us to obtain any information about the complexity of the class of spaces (i) containing an unconditional basis, (ii) containing a minimal subspace, or (iii) containing a continuously tight subspace. Hence, in Subsection 5.3, we define a parameterized version of Tsirelson space in order to show that the class of spaces containing a minimal subspace is Σ_1^1-hard. Such parametrization will show that it is impossible to Borel separate the set of separable Banach spaces containing c_0 from the set of Tsirelson-saturated Banach spaces (see Theorem 12). In Subsection 5.4, we use the methods of Subsection 5.3 in order to show that the class of continuously tight spaces is Π_1^1-hard (Theorem 17).

At last, in Subsection 5.6, we follow Argyros presentation of how to construct HI extensions of ground norms ([AT], Chapters II and III) in order to show that the set of hereditarily indecomposable separable Banach spaces is complete coanalytic (Theorem 27). We also obtain that the set of hereditary indecomposable Banach spaces cannot be Borel separated from the set of Banach spaces containing ℓ_1. By the same methods, in Subsection 5.7, we show that the class of Banach spaces containing an unconditional basis is complete analytic (Theorem 29).

Problem 3. For all the classes in the tables above, we have only completely computed the descriptive complexity of the set of HI spaces, and the set of spaces containing an unconditional basis. What are the exact complexities of the remaining classes?

At last, we would like to make a small remark. In [FR], the authors actually present Theorem 1.1 as their main dichotomy, which says that every Banach space contains either a minimal subspace or a tight subspace. This dichotomy does not explicitly say anything about continuously tight spaces. However, their proof for Theorem 1.1 shows that any Banach space X must contain either a minimal Banach subspace or a continuously tight Banach subspace (see [FR], Theorem 3.13).

We have two reasons for choosing to deal with continuously tight spaces instead of tight spaces, (i) we easily get better estimates for the upper bound of continuously tight spaces and spaces containing a continuously tight subspace, (ii) Rosendal had shown (see [R], Appendix) that, for minimal Banach spaces not containing c_0, there exists a notion of being continuously minimal, and this notion coincides with being minimal (see Subsection 5.5).

2. BACKGROUND.

A separable metric space is said to be a *Polish space* if there exists an equivalent metric in which it is complete. A continuous image of a Polish space into another Polish space is called an *analytic* set and a set whose complement is analytic is called *coanalytic.* A measure space $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a

σ-algebra of subsets of X, is called a *standard Borel space* if there exists a Polish topology on this set whose Borel σ-algebra coincides with $\mathcal{A}$. We define Borel, analytic and coanalytic sets in standard Borel spaces by saying that these are the sets that, by considering a compatible Polish topology, are Borel, analytic, and coanalytic, respectively. Observe that this is well defined, i.e., this definition does not depend on the Polish topology itself but on its Borel structure. A function between two standard Borel spaces is called *Borel measurable* if the inverse image of each Borel subset of its codomain is Borel in its domain. We usually refer to Borel measurable functions just as Borel functions. Notice that, if you consider a Borel function between two standard Borel spaces, its inverse image of analytic sets (resp. coanalytic) is analytic (resp. coanalytic) (see [S], Proposition 1.3, page 50).

Given a Polish space X, the set of analytic (resp. coanalytic) subsets of X is denoted by $\Sigma_1^1(X)$ (resp. $\Pi_1^1(X)$). We usually omit X, and simply write Σ_1^1, or Π_1^1. Hence, the terminology Σ_1^1-set (resp. Π_1^1-set) is used to refer to analytic sets (resp. coanalytic sets). We define the *projective hierarchy* by induction of n. We say a subset of a standard Borel space is Π_n^1 if it is the complement of a Σ_n^1-set, and we say that a subset is Σ_{n+1}^1 if it is a Borel image of a Π_n^1-set (see [D], Chapter 1). For each $n \in \mathbb{N}$, we denote by Δ_n^1 the subsets of a standard Borel space which are both Σ_n^1 and Π_n^1, i.e., $\Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1$.

Let X be a standard Borel space. An analytic (resp. coanalytic) subset $A \subset X$ is said to be *complete analytic* (resp. *complete coanalytic*) if for every standard Borel space Y and every analytic subset $B \subset Y$ (resp. coanalytic), there exists a Borel function $f : Y \to X$ such that $f^{-1}(A) = B$. This function is called a *Borel reduction* from B to A, and B is said to be *Borel reducible* to A.

Let X be a standard Borel space. We call a subset $A \subset X$ Σ_n^1-*hard* (resp. Π_n^1-*hard*) if for all standard Borel space Y and all Σ_n^1-set $B \subset Y$ (resp. Π_n^1-set) there exists a Borel reduction from B to A. Therefore, saying that a set $A \subset X$ is Σ_n^1-hard (resp. Π_n^1-hard) means that A is at least as complex as any Σ_n^1-set (resp. Π_n^1-set) in the projective hierarchy. With this terminology we have that $A \subset X$ is Σ_n^1-complete (resp. Π_n^1-complete) if, and only if, A is Σ_n^1-hard (resp. Π_n^1-hard) and Σ_n^1 (resp. Π_n^1).

As there exist analytic non Borel (resp. coanalytic non Borel) sets we have that Σ_1^1-hard (resp. Π_1^1-hard) sets are non Borel. Also, if X is a standard Borel space, $A \subset X$, and there exists a Borel reduction from a Σ_1^1-hard (resp. Π_1^1-hard) subset B of a standard Borel space Y to A, then A is Σ_1^1-hard (resp. Π_1^1-hard). If A is also analytic (resp. coanalytic), then A is Σ_1^1-complete (resp. Π_1^1-complete).

Consider a Polish space X and let $\mathcal{F}(X)$ be the set of all its non empty closed sets. We endow $\mathcal{F}(X)$ with the *Effros-Borel structure*, i.e., the σ-algebra generated by

$$\{F \subset X \mid F \cap U \neq \emptyset\},$$

where U varies among the open sets of X. It can be shown that $\mathcal{F}(X)$, endowed with the Effros-Borel structure, is a standard Borel space (see [K], Theorem 12.6). The following well known lemma (see [K], Theorem 12.13) will be crucial in some of our proofs.

Lemma 4. *(**Kuratowski and Ryll-Nardzewski selection principle**) Let X be a Polish space. There exists a sequence of Borel functions $(S_n)_{n\in\mathbb{N}} : \mathcal{F}(X) \to X$ such that $\{S_n(F)\}_{n\in\mathbb{N}}$ is dense in F, for all closed $F \subset X$.*

In these notes we will only work with separable Banach spaces. We denote the closed unit ball of a Banach space X by B_X. It is well known that every separable Banach space is isometrically isomorphic to a closed linear subspace of $C(\Delta)$ (see [K], page 79), where Δ denotes the Cantor set. Therefore, $C(\Delta)$ is called *universal* for the class of separable Banach spaces and we can code the class of separable Banach spaces, denoting it by SB, by SB $= \{X \subset C(\Delta) \mid X$ is a closed linear subspace of $C(\Delta)\}$. As $C(\Delta)$ is clearly a Polish space we can endow $\mathcal{F}(C(\Delta))$ with the Effros-Borel structure. It can be shown that SB is a Borel set in $\mathcal{F}(C(\Delta))$ and hence it is also a standard Borel space (see [D], Theorem 2.2). It now makes sense to wonder if specific classes of separable Banach spaces are Borel or not (in our coding SB).

Throughout these notes we will denote by $\{S_n\}_{n\in\mathbb{N}}$ the sequence of Borel functions $S_n : \text{SB} \to C(\Delta)$ given by Lemma 4, with $X = C(\Delta)$ (more precisely, the restriction of those functions to SB).

Consider the standard Borel space $C(\Delta)^{\mathbb{N}}$, and let

$$\mathcal{B} = \{(x_j)_{j\in\mathbb{N}} \in C(\Delta)^{\mathbb{N}} \mid (x_j)_{j\in\mathbb{N}} \text{ is a basic sequence}\}.$$

It is easy to see that $\mathcal{B}$ is a Borel set. Therefore, we have that $\mathcal{B}$ is a standard Borel space, and we can code all the basic sequences as elements of $\mathcal{B}$. We can now wonder about the descriptive complexity of some specific classes of basis.

Let $(e_n)_{n\in\mathbb{N}}$ be a basic sequence in a Banach space $X \in \text{SB}$. By a standard Skolem hull construction, there is a countable subfield $\mathbb{F}$ of $\mathbb{R}$ containing the rationals such that for any finite linear combination

$$\lambda_0 e_0 + ... + \lambda_n e_n,$$

with $n \in \mathbb{N}$, and $\lambda_1, ..., \lambda_n \in \mathbb{F}$, we have $\|\lambda_0 e_0 + ... + \lambda_n e_n\| \in \mathbb{F}$. By working with $\mathbb{F}$ instead of $\mathbb{Q}$, we guarantee that any $\mathbb{F}$-linear combination of $(e_n)_{n\in\mathbb{N}}$ can be normalized and remain a $\mathbb{F}$-linear combination of $(e_n)_{n\in\mathbb{N}}$. Clearly, the $\mathbb{F}$-span of $(e_n)_{n\in\mathbb{N}}$ is dense in $\overline{\text{span}}\{e_n\}$.

Let D be the set of normalized blocks of $(e_n)_{n\in\mathbb{N}}$, i.e., the set of all $\lambda_0 e_0 + ... + \lambda_n e_n$, with $n \in \mathbb{N}$, and $\lambda_1, ..., \lambda_n \in \mathbb{F}$. Clearly, D is countable. A *block sequence* of $(e_n)_{n\in\mathbb{N}}$ is a sequence $(y_n)_{n\in\mathbb{N}}$ such that, (i) there exist increasing sequences of natural numbers $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that $p_i \leq q_i < p_{i+1}$, for all $i \in \mathbb{N}$, and (ii) for all $n \in \mathbb{N}$

$$y_n = \sum_{j=p_n}^{q_n} a_j e_j,$$

for some sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers. We define a *finite block sequence* analogously. Denote by $bb(e_n)$ the set of normalized block sequences with the coefficients $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$. Endowing D with the discrete topology, we can see $bb(e_n)$ as a closed subset of $D^{\mathbb{N}}$, so $bb(e_n)$ is a standard Borel space. Denote by $fbb(e_n)$ the set of normalized finite block sequences with the coefficients $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$. Letting

$D^{<\mathbb{N}} = \cup_n D^n$, we can see $fbb(e_n)$ as a closed subset of $D^{<\mathbb{N}}$. Hence, $fbb(e_n)$ is a standard Borel space.

Let X and Y be Banach spaces. We write $X \hookrightarrow Y$ if X can be linearly embedded into Y. If $K > 0$, we write $X \hookrightarrow_K Y$ if X can be K-embedded in Y, i.e., if there exists an embedding $T : X \to Y$ such that $\|T\|\|T^{-1}\| \leq K$. If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in Banach spaces, we write $(x_n)_{n\in\mathbb{N}} \approx (y_n)_{n\in\mathbb{N}}$ if $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(y_n)_{n\in\mathbb{N}}$, i.e., if the map $x_n \mapsto y_n$ induces a linear isomorphism between $\overline{\text{span}}\{x_n\}$ and $\overline{\text{span}}\{y_n\}$. Similarly, if $K > 0$, we write $(x_n)_{n\in\mathbb{N}} \approx_K (y_n)_{n\in\mathbb{N}}$ if the induced isomorphism if a K-isomorphism.

Denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all finite tuples of natural numbers plus the empty set. Given $s = (s_0, ..., s_{n-1}), t = (t_0, ..., t_{m-1}) \in \mathbb{N}^{<\mathbb{N}}$ we say that the length of s is $|s| = n$, $s_{|i} = (s_0, ..., s_{i-1})$, for all $i \in \{1, ..., n\}$, $s_0 = \{\emptyset\}$, $s \preceq t$ iff $n \leqslant m$ and $s_i = t_i$, for all $i \in \{0, ..., n-1\}$, i.e., if t is an extension of s. We define $s < t$ analogously. Define the concatenation of s and t as $s^\frown t = (s_0, ..., s_{n-1}, t_0, ..., t_{m-1})$. A subset T of $\mathbb{N}^{<\mathbb{N}}$ is called a *tree* if $t \in T$ implies $t_{|i} \in T$, for all $i \in \{0, ..., |t|\}$. We denote the set of trees on $\mathbb{N}$ by Tr. A subset I of a tree T is called a segment if I is completely ordered and if $s, t \in I$ with $s \preceq t$, then $l \in I$, for all $l \in T$ such that $s \preceq l \preceq t$. Two segments I_1, I_2 are called completely incomparable if neither $s \preceq t$ nor $t \preceq s$ hold if $s \in I_1$ and $t \in I_2$.

As $\mathbb{N}^{<\mathbb{N}}$ is countable, $2^{\mathbb{N}^{<\mathbb{N}}}$ (the power set of $\mathbb{N}^{<\mathbb{N}}$) is Polish with its standard product topology. If we think about Tr as a subset of $2^{\mathbb{N}^{<\mathbb{N}}}$, it is easy to see that Tr is a G_δ set in $2^{\mathbb{N}^{<\mathbb{N}}}$. Thus, it is also Borel in $2^{\mathbb{N}^{<\mathbb{N}}}$. As Tr is Borel in the Polish space $2^{\mathbb{N}^{<\mathbb{N}}}$, we have that Tr is a standard Borel space. A $\beta \in \mathbb{N}^{\mathbb{N}}$ is called a *branch* of a tree T if $\beta_{|i} \in T$, for all $i \in \mathbb{N}$, where $\beta_{|i}$ is defined analogously as above. We call a tree T *well-founded* if T has no branches and *ill-founded* otherwise, we denote the set of well-founded and ill-founded trees by WF and IF, respectively. It is well known that WF is a complete coanalytic set of Tr, hence IF is complete analytic (see [K], Theorem 27.1).

3. A Lemma.

In this section, we prove a basic lemma that will be essential in many of the main results of this paper. Fix a compatible enumeration of $\mathbb{N}^{<\mathbb{N}}$, i.e., a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{<\mathbb{N}}$ such that $s_n \preceq s_m$ implies $n \leqslant m$ and for all $s \in \mathbb{N}^{<\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $s_n = s$. With this enumeration in mind, if $\theta \in \text{Tr}$, we say that a sequence $(x_s)_{s\in\theta}$ is a basis for a given Banach space X, if $(x_{s_n})_{n\in N}$ is a basis for X, where $N = \{n \in \mathbb{N} \mid s_n \in \theta\}$. We now show the following.

Lemma 5. *Let $\theta \in WF$, and let X be a Banach space with an unconditional basis $(e_s)_{s\in\theta}$. Let Y be an infinite dimensional subspace of X. Then Y contains a basic sequence $(y_k)_{k\in\mathbb{N}}$ equivalent to a semi-normalized block sequence $(x_k)_{k\in\mathbb{N}}$ of $(e_s)_{s\in\theta}$ with completely incomparable supports.*

Before we prove this lemma, let's show a simple lemma that will be important in our proof. We say that an operator $T : X \to Y$ is *strictly singular* if for all infinite dimensional subspace $Z \subset X$, $T_{|Z} : Z \to Y$ is not an embedding.

Lemma 6. *Let $(X_1, \|\cdot\|_1), ..., (X_n, \|\cdot\|_n)$ be Banach spaces, and let $Y \subset \oplus_{i=1}^n X_i$ be an infinite dimensional subspace. Consider the standard projections $P_j : \oplus_{i=1}^n X_i \to X_j$, for all $j \in \{1, ..., n\}$. Then, there exists $j \in \{1, ..., n\}$ such that $P_j : Y \to X_j$ is not strictly singular.*

Proof. Let $X = \oplus_{i=1}^n X_i$. As this is a finite sum, we can assume $X = (\oplus_{j=1}^n X_j)_{\ell_1}$, i.e., if $(x_1, ..., x_n) \in X$, then $\|x\|_X = \sum_j \|x_j\|_j$. Assume towards a contradiction that P_j is strictly singular, for all $j \in \{1, ..., n\}$. By a classic property of strictly singular operators (see [D], Proposition B.5), we know that for all $\varepsilon > 0$ there exists an infinite dimensional subspace $A \subset Y$ such that $\|P_{j|A}\| < \varepsilon$, for all $j \in \{1, ..., n\}$. Pick $x \in A$ of norm one. Then, as $x = (P_1(x), ..., P_n(x))$, we have $\|x\|_X \leqslant n\varepsilon$. By choosing $\varepsilon < 1/n$ we get a contradiction. $\qquad\square$

Proof of Lemma 5. For each $s \in \theta$, let $\Lambda_s = \{\lambda \in \mathbb{N} \mid s^\frown(\lambda) \in \theta\}$, and enumerate each Λ_s, say $\Lambda_s = \{\lambda_s^i \mid i \in \mathbb{N}\}$. For each $s \in \theta$, let $\theta_s = \{\tau \in \theta \mid s \preceq \tau\}$.

For each $s \in \theta$, let $P_s : X \to X$ be the projection of X onto

$$\overline{\text{span}}\{e_\tau \mid s \preceq \tau\}.$$

For each $s \in \theta$, and each $n \in \mathbb{N}$, consider the projections

$$
\begin{aligned}
Q_{s,n} : \quad X \quad &\to \oplus_{i=1}^n P_{s^\frown(\lambda_s^i)}(X) \\
(a_\tau)_{\tau \in \theta} &\to (a_\tau)_{\tau \in \cup_{i=1}^n \theta_{s^\frown(\lambda_s^i)}}.
\end{aligned}
$$

Claim: There exists $s \in \theta$ such that $P_s : Y \to X$ is not strictly singular, but $Q_{s,n} : Y \to \oplus_{i=1}^n P_{s^\frown(\lambda_s^i)}(X)$ is strictly singular, for all $n \in \mathbb{N}$.

Let's assume the claim is true and finish the proof of the lemma. Indeed, if $P_s : Y \to X$ is not strictly singular, we can substitute Y by an infinite dimensional $Z \subset Y$ such that $P_s : Z \to X$ is an isomorphism onto its image. Let $E = P_s(Z)$, and notice that

$$E \subset \overline{\text{span}}\{e_\tau \mid s \preceq \tau\}.$$

By Lemma 6, we can actually assume that $E \subset \overline{\text{span}}\{e_\tau \mid s \prec \tau\}$. Hence, for all $x \in E$, we have that $x = \lim_n Q_{s,n}(x)$.

Claim: There exists a normalized sequence $(y_j)_{j \in \mathbb{N}}$ in E such that $Q_{s,n}(y_j) \to 0$, as $j \to \infty$, $\forall n \in \mathbb{N}$.

Indeed, for all $n \in \mathbb{N}$, there exists a normalized sequence $(y_j^n)_{j \in \mathbb{N}}$ in E such that

$$\|Q_{s,n}(y_j^n)\| < 1/j, \quad \text{for all } j \in \mathbb{N}.$$

Let $(y_j)_{j \in \mathbb{N}}$ be the diagonal sequence of the sequences $(y_j^n)_{j \in \mathbb{N}}$, i.e., $y_j = y_j^j$, for all $j \in \mathbb{N}$. Say M is the unconditional constant of $(e_s)_{s \in \theta}$. Then, $m \leqslant n$ implies $\|Q_{s,m}(x)\| \leqslant M\|Q_{s,n}(x)\|$, for all $x \in E$. Hence, $(y_j)_{j \in \mathbb{N}}$ has the required property.

Say $(\varepsilon_i)_{i \in \mathbb{N}}$ is a sequence of positive real numbers converging to zero. As $Q_{s,n}(x) \to x$, as $n \to \mathbb{N}$, for all $x \in E$, we can pick increasing sequences of natural numbers $(n_k)_{k \in \mathbb{N}}$, and $(l_k)_{k \in \mathbb{N}}$ such that

(i) $\|Q_{s,l_k}(y_{n_k}) - y_{n_k}\|_\theta < \varepsilon_k$, for all $k \in \mathbb{N}$, and
(ii) $\|Q_{s,l_k}(y_{n_{k+1}})\|_\theta < \varepsilon_k$, for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let

$$x_k = Q_{s,l_k}(y_{n_k}) - Q_{s,l_{k-1}}(y_{n_k}).$$

Choosing $(\varepsilon_k)_{k \in \mathbb{N}}$ converging to zero fast enough, we have that $(x_k)_{k \in \mathbb{N}}$ is semi-normalized and, by the principle of small perturbations, that $(y_{n_k})_{k \in \mathbb{N}}$ is equivalent to $(x_k)_{k \in \mathbb{N}}$ (see [AK], Theorem 1.3.9). Clearly, $(x_k)_{k \in \mathbb{N}}$ has completely incomparable supports. As Y contains a sequence equivalent to $(x_k)_{k \in \mathbb{N}}$, the proof is complete.

We now prove our first claim. Suppose the claim is false, i.e., suppose that for all $s \in \theta$ such that $P_s : Y \to X$ is not strictly singular, there exists $n \in \mathbb{N}$ such that $Q_{s,n} : Y \to \oplus_{i=1}^n P_{s^\frown(\lambda_s^i)}(X)$ is not strictly singular. By Lemma 6, if $Q_{s,n} : Y \to \oplus_{i=1}^n P_{s^\frown(\lambda_s^i)}(X)$ is not strictly singular, there exists $m \leqslant n$ such that

$$P_{s^\frown \lambda_s^m} = P_{s^\frown \lambda_s^m} \circ Q_{s,n} : Y \to X$$

is not strictly singular. Therefore, for all $s \in \theta$ such that $P_s : Y \to X$ is not strictly singular, there exists $s' \succ s$ such that $P_{s'} : Y \to X$ is not strictly singular.

Now notice that $P_\emptyset : Y \to X$ is not strictly singular, indeed, $P_\emptyset = Id$. Therefore, by applying the last paragraph ω times, we get a sequence $(s_n)_{n \in \mathbb{N}}$ such that $P_{s_n} : Y \to X$ is not strictly singular, and $s_n \prec s_{n+1}$, for all $n \in \mathbb{N}$. In particular, $s_n \in \theta$, for all n, absurd, as θ is well-founded. $\qquad\square$

4. ℓ_p-BAIRE SUMS.

We now deal with ℓ_p-Baire sums of basic sequences, this tool will be crucial in many of our results in these notes. Fix a basic sequence $\mathcal{E} = (e_n)_{n \in \mathbb{N}}$, and $p \in [1, \infty)$. Let us define a Borel function $\varphi : \mathrm{Tr} \to \mathrm{SB}$ in the following manner. For each $\theta \in \mathrm{Tr}$ and $x = (x(s))_{s \in \theta} \in c_{00}(\theta)$ we define

$$\|x\|_{\mathcal{E},p,\theta} = \sup\Big\{ \Big(\sum_{i=1}^n \big\| \sum_{s \in I_i} x(s) e_{|s|} \big\|_{\mathcal{E}}^p \Big)^{\frac{1}{p}} \mid n \in \mathbb{N},\ I_1, ..., I_n \text{ incomparable}$$

$$\text{segments of } \theta \Big\},$$

where $\|.\|_{\mathcal{E}}$ is the norm of $\overline{\mathrm{span}}\{\mathcal{E}\}$. Define $\varphi_{\mathcal{E},p}(\theta)$ as the completion of $c_{00}(\theta)$ under the norm $\|.\|_{\mathcal{E},p,\theta}$. The space $\varphi_{\mathcal{E},p}(\theta)$ is known as the ℓ_p-*Baire sum* of $\overline{\mathrm{span}}\{\mathcal{E}\}$ (indexed by θ).

Similarly, we define $\|.\|_{\mathcal{E},0,\theta}$ as

$$\|x\|_{\mathcal{E},0,\theta} = \sup\Big\{ \big\| \sum_{s \in I} x(s) e_{|s|} \big\|_{\mathcal{E}} \mid I \text{ segment of } \theta \Big\},$$

and let $\varphi_{\mathcal{E},0}(\theta)$ be the completion of $(c_{00}(\theta), \|.\|_{\mathcal{E},0,\theta})$. We denote by $(e_s)_{s \in \theta}$ the sequence in $c_{00}(\theta)$ such that, for each $\tau \in \theta$, the coordinate $e_s(\tau)$ equals 1 if $s = \tau$ and zero otherwise. Considering a compatible enumeration of $\mathbb{N}^{<\mathbb{N}}$, as in Section 3, the sequence $(e_s)_{s \in \theta}$ is clearly a basis for $\varphi(\theta)$.

Pick $Y \subset C(\Delta)$ such that $\varphi_{\mathcal{E},p}(\mathbb{N}^{<\mathbb{N}})$ is isometrically isomorphic to Y. If we consider the natural isometries of $\varphi_{\mathcal{E},p}(\theta)$ into $\varphi_{\mathcal{E},p}(\mathbb{N}^{<\mathbb{N}})$, we can see $\varphi_{\mathcal{E},p}$ as a

Borel function from Tr into SB. This gives us the following (see [S], Proposition 3.1, page 79).

Proposition 7. *Let $p \in [1, \infty)$ or $p = 0$. Then, the map $\varphi_{\mathcal{E},p} : Tr \to SB$ defined above is Borel.*

The following lemma summarizes the main properties of the ℓ_p-Baire sum that we will need later in these notes.

Lemma 8. *Let $\mathcal{E}$ be a basic sequence. The Borel function $\varphi_{\mathcal{E},p} : Tr \to SB$ defined above has the following properties:*

  (i) *If $\theta \in IF$, then $\varphi_{\mathcal{E},p}(\theta)$ contains $\overline{span}\{\mathcal{E}\}$.*
  (ii) *If $\theta \in WF$, then $\varphi_{\mathcal{E},p}(\theta)$ is ℓ_p-saturated, i.e., every infinite dimensional sub-space of $\varphi_{\mathcal{E},p}(\theta)$ contains an isomorphic copy of ℓ_p.*

The analogous is true for $\varphi_{\mathcal{E},0} : Tr \to SB$, i.e.,

  (i) *If $\theta \in IF$, then $\varphi_{\mathcal{E},0}(\theta)$ contains $\overline{span}\{\mathcal{E}\}$.*
  (ii) *If $\theta \in WF$, then $\varphi_{\mathcal{E},0}(\theta)$ is c_0-saturated, i.e., every infinite dimensional sub-space of $\varphi_{\mathcal{E},0}(\theta)$ contains an isomorphic copy of c_0.*

Proof. If $\theta \in IF$, clearly $\varphi_{\mathcal{E},p}(\theta)$ contains $\overline{span}\{\mathcal{E}\}$. Indeed, let β be a branch of θ, then $\overline{span}\{\mathcal{E}\} \cong \varphi_{\mathcal{E},p}(\beta) \hookrightarrow \varphi_{\mathcal{E},p}(\theta)$, where by $\varphi_{\mathcal{E},p}(\beta)$ we mean $\varphi_{\mathcal{E},p}$ applied to the tree $\{s \in \mathbb{N}^{<\mathbb{N}} \mid s < \beta\}$.

Say $\theta \in WF$, and let E be an infinite dimensional subspace of $\varphi_{\mathcal{E},p}(\theta)$. By Lemma 5, E has a basic sequence equivalent to a semi-normalized block sequence $(x_k)_{k \in \mathbb{N}}$ with completely incomparable supports. It is trivial to check that a semi-normalized block sequence with completely incomparable supports is equivalent to the ℓ_p-basis (resp. c_0-basis). So we are done. $\square$

Let $\mathcal{P} \subset SB$. We say that $\mathcal{P}$ is a *class* of Banach spaces if $\mathcal{P}$ is closed under isomorphism, i.e., for all $X, Y \in SB$, $X \in \mathcal{P}$ and $Y \cong X$ imply $Y \in \mathcal{P}$. We say that a class of Banach spaces $\mathcal{P} \subset SB$ is *pure* if $X \in \mathcal{P}$ implies $Y \in \mathcal{P}$, for all subspace $Y \subset X$. We say a class $\mathcal{P} \subset SB$ is *almost-pure* if for all $X \in \mathcal{P}$ and all infinite dimensional $Y \subset X$ there exists an infinite dimensional subspace $Z \subset Y$ such that $Z \in \mathcal{P}$.

Theorem 9. *Say $\mathcal{P} \subset SB$ is almost-pure and that ℓ_p (reps. c_0) does not embed in any $Y \in \mathcal{P}$, for some $p \in [1, \infty)$. Then $C_{\mathcal{P}} = \{Y \in SB \mid \exists Z \in \mathcal{P}, \ Z \hookrightarrow Y\}$ is Σ_1^1-hard. In particular, the same is true if $\mathcal{P}$ is pure and does not contain ℓ_p (reps. c_0), for some $p \in [1, \infty)$.*

Proof. This is a simple application of Lemma 8. Indeed, let $\mathcal{E}$ be a basis for $C(\Delta)$, and consider the restriction of $\varphi_{\mathcal{E},p}$ to the set of infinite trees, say ITr. It is easy to see that ITr is Borel (see [S], Proposition 1.6, page 72), so $\varphi_{\mathcal{E},p|ITr}$ is a Borel function. By Lemma 8, $\varphi_{\mathcal{E},p|ITr} : ITr \to SB$ is a Borel reduction from IF to $C_{\mathcal{P}}$. Therefore, as IF is Σ_1^1-hard, $C_{\mathcal{P}}$ is Σ_1^1-hard. $\square$

5. Descriptive complexity of the inevitable classes.

5.1. **Spaces containing a hereditarily indecomposable subspace.** In 1991 W. T Gowers and B. Maurey independently solved the unconditional basic sequence problem, i.e., they constructed a Banach space with no unconditional basic sequence (see [GM]). It was noticed by W. B. Johnson that the space constructed

by Gowers and B. Maurey not only had no unconditional basic sequence but was also hereditarily indecomposable.

We say that an infinite dimensional Banach space X is *hereditatily indecomposable* if none of X subspaces can be decomposed as a direct sum of two infinite dimensional subspaces. Clearly, the class HI = $\{X \in \text{SB} \mid X$ is hereditarily indecomposable$\}$ is a pure class and it contains no ℓ_p. Hence, Theorem 9 gives us the following.

Corollary 10. C_{HI} *is Σ_1^1-hard.*

We come back to the class of hereditarily indecomposable spaces in Subsection 5.6, where we show that the set HI is complete coanalytic, and that C_{HI} is at most Σ_2^1.

5.2. **Spaces containing a continuously tight subspace.** In [FR], Ferenczi and Rosendal defined a new class of Banach spaces, the class of continuously tight spaces, and proved many interesting properties about this class. In Theorem 3.13 of [FR], for example, Ferenczi and Rosendal have shown that every Banach space must contain either a minimal subspace or a continuously tight subspace, giving us another dichotomy for Banach spaces.

Denote by $[\mathbb{N}]$ the set of increasing sequences of natural numbers. We can see $[\mathbb{N}]$ as a Borel subset of $\mathbb{N}^{\mathbb{N}}$. A basic sequence $(e_n)_{n \in \mathbb{N}}$ is called *continuously tight* if there exists a continuous function $f : bb(e_n) \to [\mathbb{N}]$ such that, for all block basis $\bar{y} = (y_n)_{n \in \mathbb{N}} \in bb(e_n)$, if we set $I_j = \{m \in \mathbb{N} \mid f(\bar{y})_{2j} \le m \le f(\bar{y})_{2j+1}\}$, then for all infinite set $A \subset \mathbb{N}$,

$$\overline{\text{span}}\{\bar{y}\} \not\hookrightarrow \overline{\text{span}}\{e_n \mid n \notin \cup_{j \in A} I_j\},$$

i.e., $\overline{\text{span}}\{\bar{y}\}$ does not embed into $\overline{\text{span}}\{e_n\}$ avoiding an infinite number of the intervals I_j. A space with a continuously tight basis is called *continuously tight*. The Tsirelson space is an example of a continuously tight Banach space (see [FR], Corollary 4.3).

For a detailed study of continuously tight spaces and other related properties (e.g., tight spaces, tight with constants, tight by range, etc) see [FR]. Let CT = $\{X \in \text{SB} \mid X$ is continuoulsy tight$\}$ be our coding for the class of continuously tight separable Banach spaces.

Corollary 11. C_{CT} *is Σ_1^1-hard.*

Proof. Proposition 3.3 of [FR] says that continuously tight spaces contain no minimal subspaces, and Theorem 3.13 of [FR] says that a space with no minimal subspaces contains a continuously tight subspace. Therefore, CT is an almost-pure class. Also, again by Proposition 3.3 of [FR], we have that no elements of CT contain an isomorphic copy of ℓ_p. Hence, Theorem 9 gives us that C_{CT} is Σ_1^1-hard.   □

We cannot obtain any lower bound for the complexity of the set of continuously tight Banach spaces with the method of ℓ_p-Baire sums. We come back to the class of continuously tight spaces in Subsection 5.4, where we show that the set CT is Π_1^1-hard by using a different method.

5.3. **Spaces containing a minimal subspace.** A Banach space X is called *minimal* if every infinite dimensional subspace of X contains an isomorphic copy of X. We now turn our attention to the following: Although $M = \{X \in SB \mid X$ is minimal$\}$ is clearly a pure class, M contains c_0 and ℓ_p, for all $p \in [1, \infty)$. Therefore, Theorem 9 does not say anything about the complexity of C_M. However, we can use the construction of Tsirelson space by T. Figiel and W. B. Johnson (see [FJ], Section 2) in order to construct a $\varphi : Tr \to SB$ that will solve our problem.

Fix a compatible enumeration of $\mathbb{N}^{<\mathbb{N}}$, i.e., $(s_n)_{n \in \mathbb{N}}$ such that $s_n \preceq s_m$ implies $n \leqslant m$ and for all $s \in \mathbb{N}^{<\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $s_n = s$. This enumeration give us an order on $\mathbb{N}^{<\mathbb{N}}$. With this ordering in mind, we say that $E_0 < E_1$ if $\max E_0 < \min E_1$, for all finite sets $E_0, E_1 \subset \mathbb{N}^{<\mathbb{N}}$. We write $k < E$ if $\{s_k\} < E$.

Given $\theta \in Tr$, $E \subset \theta$, and $x = \sum_{n \in \mathbb{N}} a_{s_n} e_{s_n} \in c_{00}(\theta)$ (for some $(a_{s_n})_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$), we let $Ex = \sum_{s_n \in E} a_{s_n} e_{s_n}$.

We define a Borel function $\varphi : Tr \to SB$ as, for each $\theta \in Tr$ and each $x = (x(s))_{s \in \theta} \in c_{00}(\theta)$, let $(\|.\|_{\theta,m})_{m \in \mathbb{N}}$ be inductively defined by

$$\|x\|_{\theta,0} = \|x\|_0, \text{ and}$$

$$\|x\|_{\theta,m+1} = \max\{\|x\|_0, \frac{1}{2} \max \sum_{i=1}^{k} \|E_i x\|_{\theta,m}\},$$

for all $m \in \mathbb{N}$, where the "inner" maximum above is taken over all $k \in \mathbb{N}$ and all completely incomparable finite sets $(E_i)_{i=1}^{k}$ ($E_i \subset \theta$, for all $i \in \{1, ..., k\}$) such that $k \leqslant E_1 < ... < E_k$ (for the definition of completely incomparable sets of a tree, see Section 2). Exactly as we have for the standard Tsirelson space, we can define a norm $\|.\|_\theta$ as

$$\|x\|_\theta = \lim_{m \to \infty} \|x\|_{\theta,m},$$

for all $x \in c_{00}(\theta)$. We define $\varphi(\theta)$ to be the completion of $c_{00}(\theta)$ under this norm. This norm can be implicitly defined as

$$\|x\|_\theta = \max\{\|x\|_0, \frac{1}{2} \max \sum_{i=1}^{k} \|E_i x\|_\theta\},$$

where the "inner" maximum above is taken over all $k \in \mathbb{N}$ and all completely incomparable finite sets $(E_i)_{i=1}^{k}$ ($E_i \subset \theta$, for all $i \in \{1, ..., k\}$) such that $k \leqslant E_1 < ... < E_k$.

By the universality of $C(\Delta)$ for separable Banach spaces, we can identify $\varphi(\mathbb{N}^{<\mathbb{N}})$ with an isometric copy inside of $C(\Delta)$. As we can identify each $\varphi(\theta)$ with a subspace of $\varphi(\mathbb{N}^{<\mathbb{N}})$ in a natural fashion, we can see φ as a Borel function from Tr to SB (see [S], Proposition 3.1, page 79, for similar arguments).

Theorem 12. *Let $\varphi : Tr \to SB$ be defined as above. Then φ is a Borel function with the following properties*

(i) $c_0 \hookrightarrow \varphi(\theta)$, for all $\theta \in IF$, and
(ii) $\varphi(\theta)$ has the following property if $\theta \in WF$: for every infinite dimensional subspace $E \subset \varphi(\theta)$, there exists a further subspace $F \subset E$ isomorphic to

an infinite dimensional subspace of Tsirelson space, i.e., $\varphi(\theta)$ is Tsirelson-saturated.

In particular, C_M is Σ_1^1-hard, and C_{c_0} cannot be Borel separated from the set of Tsirelson-saturated Banach spaces.

Before proving this theorem, notice the following trivial consequence of Lemma 6.

Lemma 13. *A finite sum of spaces satisfying property (ii) of Theorem 12 still has property (ii).*

Proof of Theorem 12. If $\theta \in \mathrm{IF}$, it is clear that $c_0 \hookrightarrow \varphi(\theta)$. Say $\theta \in \mathrm{WF}$, let us show that every infinite dimensional subspace of $\varphi(\theta)$ contains a subspace isomorphic to an infinite dimensional subspace of Tsirelson's space. Say $E \subset \varphi(\theta)$ is an infinite dimensional subspace.

As $\theta \in \mathrm{WF}$, Lemma 5 gives us that E contains a sequence equivalent to a block sequence $(y_n)_{n \in \mathbb{N}}$ of $\varphi(\theta)$ with completely incomparable supports. We will be done once we prove the following claim.

Claim: $(y_n)_{n \in \mathbb{N}}$ is equivalent to a subsequence of the standard basis of Tsirelson space.

Let $\| \cdot \|_{T,\theta}$ be the standard Tsirelson norm on $c_{00}(\theta)$, i.e.,

$$\|x\|_{T,\theta} = \max\{\|x\|_0, \frac{1}{2}\max \sum_{i=1}^k \|E_i x\|_{T,\theta}\},$$

where the "inner" maximum above is taken over all $k \in \mathbb{N}$ and all finite sets $(E_i)_{i=1}^k$ ($E_i \subset \theta$, for all $i \in \{1, ..., k\}$) such that $k \leqslant E_1 < ... < E_k$. The only difference between $\| \cdot \|_\theta$ and $\| \cdot \|_{T,\theta}$ is that in $\| \cdot \|_{T,\theta}$ we do not have the restriction of $(E_i)_{i=1}^k$ being completely incomparable.

It is clear that the the basis $(e_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of the completion of $(c_{00}(\theta), \| \cdot \|_{T,\theta})$ is equivalent to the standard basis of Tsirelson space. Moreover, the basis $(e_s)_{s \in \theta}$ of the completion of $(c_{00}(\theta), \| \cdot \|_{T,\theta})$ is equivalent to a subsequence of the standard basis of Tsirelson space, if θ is infinite (this because this norm has no dependance on the structure of the tree T). Also, we clearly have

$$(5.1) \qquad \|\sum_{i=1}^k a_i y_i\|_\theta \leqslant \|\sum_{i=1}^k a_i y_i\|_{T,\theta},$$

for all $a_1, ..., a_k \in \mathbb{R}$. Mimicking the proof of Lemma II.1 of [CS] we have the following lemma.

Lemma 14. *Let $(p_n)_{n \in \mathbb{N}}$ be a increasing sequence of natural numbers. Let $(e_{s_n})_{n \in \mathbb{N}}$ be the standard basis of $\varphi(\theta)$. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} b_i e_{s_i}$ (for all $n \in \mathbb{N}$) be a normalized block sequence of $(e_{s_n})_{n \in \mathbb{N}}$ and assume $(y_n)_{n \in \mathbb{N}}$ has completely incomparable supports. Then*

$$\|\sum_{n \in \mathbb{N}} a_n e_{s_{p_n+1}}\|_\theta \leqslant \|\sum_{n \in \mathbb{N}} a_n y_n\|_\theta,$$

for any sequence of scalars $(a_n)_{n \in \mathbb{N}}$.

Proof. It is enough to show that, for any sequence $(a_n)_{n \in \mathbb{N}}$, we have

$$\| \sum_{n \in \mathbb{N}} a_n e_{s_{p_n + 1}} \|_{\theta, m} \leqslant \| \sum_{n \in \mathbb{N}} a_n y_n \|_\theta,$$

for all $m \in \mathbb{N}$. Let us proceed by induction on $m \in \mathbb{N}$. For $m = 0$ the result is clear. Assume the equation above holds for a fixed $m \in \mathbb{N}$. Let $x = \sum_{n \in \mathbb{N}} a_n e_{p_n + 1}$ and $y = \sum_{n \in \mathbb{N}} a_n y_n$. Fix $k \in \mathbb{N}$, and completely incomparable finite sets $(E_n)_{n=1}^k$ such that $k \leqslant E_1 < ... < E_k$. Consider the sum

$$\frac{1}{2} \sum_{i=1}^k \| E_i x \|_{\theta, m}.$$

Since the support of x is contained in $\{p_n + 1 \mid n \in \mathbb{N}\}$, we may assume that

$$E_i \subset \{p_n + 1 \mid n \in \mathbb{N}\},$$

for all $i \in \mathbb{N}$. Applying the inductive hypothesis, we have

$$\frac{1}{2} \sum_{i=1}^k \| E_i x \|_{\theta, m} \leqslant \frac{1}{2} \sum_{i=1}^k \| \sum_{\substack{n \in \mathbb{N} \\ p_n + 1 \in E_i}} a_n y_n \|_\theta$$

As $p_n + 1 \in E_i$ implies $k \leqslant p_n + 1$, and as $(y_n)_{n \in \mathbb{N}}$ has completely incomparable supports, the sum on the right hand side of the equation above is allowed as an "inner" sum in the definition of the norm $\| . \|_\theta$. Therefore, we have

$$\frac{1}{2} \sum_{i=1}^k \| E_i x \|_{\theta, m} \leqslant \| y \|_\theta,$$

for all $k \in \mathbb{N}$, and all completely incomparable finite sets $(E_n)_{n=1}^k$ such that $k \leqslant E_1 < ... < E_k$. Hence, for any sequence of scalars $(a_n)_{n \in \mathbb{N}}$, we have

$$\| \sum_{n \in \mathbb{N}} a_n e_{s_{p_n + 1}} \|_{\theta, m+1} \leqslant \| \sum_{n \in \mathbb{N}} a_n y_n \|_\theta,$$

and we are done. $\square$

Let $(b_n)_{n \in \mathbb{N}}$ be the sequence of scalars such that our block sequence $(y_n)_{n \in \mathbb{N}}$ can be written as $y_n = \sum_{i=p_n + 1}^{p_{n+1}} b_i e_{s_i}$. Then, Lemma 14 gives us that

(5.2) $$\| \sum_{n \in \mathbb{N}} a_n e_{s_{p_n + 1}} \|_\theta \leqslant \| \sum_{n \in \mathbb{N}} a_n y_n \|_\theta,$$

for any sequence of scalars $(a_n)_{n \in \mathbb{N}}$. As the supports of $(y_n)_{n \in \mathbb{N}}$ are completely incomparable we have that

(5.3)
$$\|\sum_{n\in\mathbb{N}}a_ne_{s_{p_n+1}}\|_\theta = \|\sum_{n\in\mathbb{N}}a_ne_{s_{p_n+1}}\|_{T,\theta},$$

for any sequence of scalars $(a_n)_{n\in\mathbb{N}}$. By Proposition II.4 of [CS], we have

(5.4)
$$\frac{1}{18}\|\sum_{n\in\mathbb{N}}a_ny_n\|_{T,\theta} \leqslant \|\sum_{n\in\mathbb{N}}a_ne_{s_{p_n+1}}\|_{T,\theta},$$

for all sequence of scalars $(a_n)_{n\in\mathbb{N}}$. Therefore, putting Equation 5.1, Equation 5.2, Equation 5.3, and Equation 5.4 together, we have that

$$\|\sum_{i=1}^{k}a_ne_{s_{p_n+1}}\|_{T,\theta} \leqslant \|\sum_{i=1}^{k}a_ny_n\|_\theta \leqslant 18\|\sum_{i=1}^{k}a_ne_{s_{p_n+1}}\|_{T,\theta},$$

for all $a_1, ..., a_k \in \mathbb{R}$. Hence, the sequence $(y_n)_{n\in\mathbb{N}}$ as a sequence in $\varphi(\theta)$ is equivalent to the sequence $(e_{s_{p_n+1}})_{n\in\mathbb{N}}$ as a sequence in the Tsirelson space (the completion of $(c_{00}(\theta), \|\cdot\|_{T,\theta})$).

To conclude that $\varphi(\theta)$ contains no minimal subspaces if $\theta \in \mathrm{WF}$, recall that Tsirelson space contains no minimal subspaces (see [CS], Corollary VI.b.6), so we are done. $\qquad\square$

It is easy to see, by simply counting quantifiers, that the set $\mathrm{C_M}$ is at most Σ_3^1, i.e., the Borel image of a set that can be written as the complement of the Borel image of a coanalytic set. However, by using Gowers' theorem, Rosendal was able to find a better upper bound for $\mathrm{C_M}$ (see [R], Appendix).

Theorem 15. *(C. Rosendal)* C_M *is* Σ_2^1.

Problem 16. What is the exact complexity of $\mathrm{C_M}$? Is it Σ_2^1-complete?

In Subsection 5.5, we talk a little bit about Rosendal's proof for $\mathrm{C_M}$ being Σ_2^1. We will notice that Rosendal's proof also gives us that M is at most Δ_2^1.

5.4. **Continuously tight Banach spaces.** We are now capable of giving a lower bound for the complexity of CT, the set of continuously tight spaces.

Recall, a Banach space X with a basis $(x_k)_{k\in\mathbb{N}}$ is said to be *strongly asymptotic* ℓ_p if there exists a function $f : \mathbb{N} \to \mathbb{N}$ and a constant $C > 0$ such that any set of m unit vectors in $\overline{\mathrm{span}}\{x_k \mid k \geq f(m)\}$ with disjoint supports is C-equivalent to the basis of ℓ_p^m. The Tsirelson space with its standard basis is an example of an strongly asymptotic ℓ_1 space (see [CS], Chapter V). Also, a Banach space X is said to be *crudely finitely representable* in a Banach space Y if there exists $M > 0$ such that every finite subspace of X M-embeds into Y.

The proof below is an adaptation of the proof of Proposition 4.2, in [FR].

Theorem 17. *The set of continuously tight spaces CT is* Π_1^1*-hard. Moreover, the set of continuously tight basis, say* $\mathcal{CT}$*, is* Π_1^1*-hard.*

Proof. Let $\varphi : \mathrm{Tr} \to \mathrm{SB}$ be the map in Lemma 12. As c_0 embeds into $\varphi(\theta)$ if $\theta \in \mathrm{IF}$, we only need to show that $\varphi(\theta)$ is continuously tight if $\theta \in \mathrm{WF}$. Indeed, as the map $\theta \in \mathrm{Tr} \mapsto (e_s)_{s\in\theta} \in \mathcal{B}$ is a Borel map, this is enough to prove both assertions of the theorem.

A Banach space X with a basis is said to be *tight with constants* if no Banach space embeds uniformly into its tail subspaces (see [FR], Proposition 4.1). Let's show that $\varphi(\theta)$ is tight with constants, for all $\theta \in \mathrm{WF}$. As spaces which are tight with constants are also continuously tight (see Proposition 18 below) we will be done.

Assume towards a contradiction that, for some $K > 0$, there exists a space Y which K-embeds into all tail subspaces of $\varphi(\theta)$. By Theorem 12, we can assume, by taking a subspace, that Y is generated by a sequence $(y_k)_{k \in \mathbb{N}}$ which is equivalent to a subsequence of the basis of Tsirelson space. Therefore, $(y_k)_{k \in \mathbb{N}}$ is unconditional and strongly asymptotic ℓ_1. Let $C > 0$ and $f : \mathbb{N} \to \mathbb{N}$ be as in the definition of strongly asymptotic ℓ_1 spaces.

By Proposition 1 of [Jo], we have that for all $m \in \mathbb{N}$, there exists $N(m) \in \mathbb{N}$ such that $(y_1, ..., y_m)$ is $2K$-equivalent to a sequence of vectors in the linear span of $N(m)$ disjointly supported unit vectors in any tail of Y. In particular, in the tail $\overline{\mathrm{span}}\{y_k \mid k \geq f(N(m))\}$. Therefore, as Y is strongly asymptotic ℓ_1, we have that $(y_1, ..., y_m)$ $2KC$-embeds into ℓ_1, for all $m \in \mathbb{N}$. So Y is crudely finitely representable in ℓ_1, and therefore Y embeds into L_1 (see [AK], Theorem 11.1.8). Hence, as $(y_k)_{k \in \mathbb{N}}$ is unconditional asymptotic ℓ_1, we have that Y contains ℓ_1 (see [DFKO], Proposition 5), absurd, because Y is a subspace of Tsirelson space. $\square$

Proposition 18. *Let X be a Banach space with basis $(e_n)_{n \in \mathbb{N}}$. Say $(e_n)_{n \in \mathbb{N}}$ is tight with constants (see definition in the proof above). Then $(e_n)_{n \in \mathbb{N}}$ is continuously tight.*

Proof. For this, we will use details of the proof of Proposition 4.1 of [FR]. Precisely, let $X \in \mathrm{SB}$ be a Banach space with a basis $(e_n)_{n \in \mathbb{N}}$ which is tight with constants. For each $L \in \mathbb{N}$, let $c(L) > 0$ be a constant such that if two block sequences of $(e_n)_{n \in \mathbb{N}}$ differ from at most L terms, then they are $c(L)$-equivalent. For each $(y_n)_{n \in \mathbb{N}} \in bb(e_n)$, let us define a sequence $(I_j)_{j \in \mathbb{N}}$ of finite intervals of natural numbers.

By Proposition 4.1 of [FR], for each $K, m \in \mathbb{N}$, there exists an $l > m$ such that

$$(5.5) \qquad \mathrm{span}\{y_n \mid m \leq n \leq l\} \not\hookrightarrow_K \overline{\mathrm{span}}\{e_n \mid n \geq l\}.$$

Let $l_1 \in \mathbb{N}$ be the minimal $l \in \mathbb{N}$ as above, for $m = 1$, and $K = c(1)$. Let $I_1 = [1, l_1]$, where if $a \leq b \in \mathbb{N}$, $[a, b] = \{n \in \mathbb{N} \mid a \leq n \leq b\}$. Assume we had already defined finite intervals $I_1 < ... < I_{j-1} \subset \mathbb{N}$ and numbers $l_1 < ... < l_{j-1} \in \mathbb{N}$. Define l_j as the minimal $l \in \mathbb{N}$ as in (5.5) above, for $m = \max\{I_{j-1}\} + 1$, and $K = j \cdot c(\max\{I_{j-1}\} + 1)$. Let $I_j = [\max\{I_{j-1}\} + 1, l_j]$.

By the proof of Proposition 4.1 of [FR], we have that, for all $K \in \mathbb{N}$,

$$\mathrm{span}\{y_n \mid n \in I_K\} \not\hookrightarrow_K \overline{\mathrm{span}}\{e_n \mid n \notin I_K\}.$$

In particular, for all infinite set $A \subset \mathbb{N}$,

$$\overline{\mathrm{span}}\{y_n\} \not\hookrightarrow \overline{\mathrm{span}}\{e_n \mid n \notin \cup_{j \in A} I_j\}.$$

Hence, we had defined a map $\bar{y} = (y_n) \mapsto (I_j)_{j \in \mathbb{N}}$, and we will be done if this

assignment is continuous, i.e., if there exists a continuous function $f : bb(e_n) \to [\mathbb{N}]$ such that $I_j = [f(\bar{y})_{2j}, f(\bar{y})_{2j+1}]$. For this, we only need to notice that in order to obtain a finite chunk of the sequence of intervals $(I_j)_{j \in \mathbb{N}}$, say $I_1, ..., I_K$, we only need to know a finite chunk of $(y_n)_{n \in \mathbb{N}}$, precisely, $y_1, ..., y_{\max\{I_K\}}$. So we are done. $\square$

Proposition 19. *CT, C_{CT} and $\mathcal{CT}$ are at most Σ_2^1.*

Proof. This is a simple matter of counting quantifiers and the fact that we only quantify over standard Borel spaces in the definition of those three classes. Indeed, for $\mathcal{CT}$, for example, we have

$$(e_n)_{n \in \mathbb{N}} \in \mathcal{CT} \Leftrightarrow \exists \text{ continuous } \ f : bb(e_n) \to [\mathbb{N}], \ \forall (y_n)_{n \in \mathbb{N}} \in bb(e_n),$$
$$\forall \text{ infinite } \ A \subset \mathbb{N}, \ \forall (x_n)_{n \in \mathbb{N}} \in \text{span}\{e_n \mid n \notin \cup_{j \in A} I_j\}^{\mathbb{N}},$$
$$\forall K \in \mathbb{N}, \ (x_n)_{n \in \mathbb{N}} \not\approx_K (y_n)_{n \in \mathbb{N}}.$$

The only quantifier that demands some explanation is "$\exists$ continuous $\ f : bb(e_n) \to [\mathbb{N}]$". For this, let $\mathbb{N}^{[<N]}$ be the set of finite increasing sequence of natural numbers. Then it is easy to see that a continuous function $f : bb(e_n) \to [\mathbb{N}]$ gives us a function $g : fbb(e_n) \to \mathbb{N}^{[<N]}$ such that $g(\bar{y}) \preceq g(\bar{x})$, if $\bar{y} \preceq \bar{x}$, and vice versa, where $fbb(e_n)$ is the set of finite block sequences of $(e_n)_{n \in \mathbb{N}}$ (see Section 2). So, as $fbb(e_n)$ is countable, the space of functions $fbb(e_n) \to \mathbb{N}^{[<N]}$ is a standard Borel space, so we are done. The same arguments work for CT and C_{CT}. $\square$

5.5. **Mininal spaces.** It follows straight forward from the definition of minimal Banach spaces that $M = \{X \in SB \mid X \text{ is minimal}\}$ is Π_2^1. In this subsection we show that M is also Σ_2^1. Hence, M is at most Δ_2^1.

Using Gowers' theorem (see [Go2]), and a corollary of the solution of the distortio problem (see [OS], and [OS2]), Rosendal had shown (see [R], Appendix) that if a Banach space X not containing c_0 contains a minimal subspace then there exists a basic sequence $(e_n)_{n \in \mathbb{N}}$ in X, a block subsequence $(y_n)_{n \in \mathbb{N}} \in bb(e_n)$, and a continuous function $f : bb(y_n) \to \overline{\text{span}}\{e_n\}^{\mathbb{N}}$ such that, for all $\bar{z} = (z_n)_{n \in \mathbb{N}} \in bb(y_n)$, we have

$$\overline{\text{span}}\{f(\bar{z})\} \subset \overline{\text{span}}\{z_n\} \quad \text{and} \quad (e_n)_{n \in \mathbb{N}} \approx f(\bar{z}).$$

By counting quantifiers, the set of Banach spaces satisfying the property above is at most Σ_2^1. Clearly, if a Banach space satisfies the property above, then it contains a minimal subspace, indeed, $\overline{\text{span}}\{y_n\}$ is minimal. As the set of Banach spaces containing c_0 is Σ_1^1 (Σ_1^1-complete actually), this gives us that C_M is at most Σ_2^1.

We now notice that this also gives us an equivalent characterization of minimality. Indeed, Rosendal's result clearly implies that if X is a minimal Banach space not containing c_0 then there exists a basic sequence $(e_n)_{n \in \mathbb{N}}$ in X, a block subsequence $(y_n)_{n \in \mathbb{N}} \in bb(e_n)$, such that

$$X \hookrightarrow \overline{\text{span}}\{y_n\},$$

and there exists a continuous function $f : bb(y_n) \to \overline{\text{span}}\{e_n\}^{\mathbb{N}}$ such that, for all $\bar{z} = (z_n)_{n \in \mathbb{N}} \in bb(y_n)$, we have

$$\overline{\text{span}}\{f(\bar{z})\} \subset \overline{\text{span}}\{z_n\} \quad \text{and} \quad (e_n)_{n\in\mathbb{N}} \approx f(\bar{z}).$$

By counting quantifiers, the set of Banach spaces satisfying the property above is at most Σ_2^1. Notice that if a Banach space satisfies the property above, then it is minimal. Therefore, as the set of minimal Banach spaces containing c_0 is the set of spaces that embed into c_0, and as this set is easily seen to be analytic, we have that M is at most Σ_2^1. As M is also Π_2^1, we have the following.

Proposition 20. *The class of minimal Banach spaces M is at most Δ_2^1.*

5.6. **Hereditarily indecomposable spaces.** Let $\text{HI} = \{X \in \text{SB} \mid X \text{ is hereditarily}$ indecomposable$\}$. In Subsection 5.1, we proved, using the method of ℓ_p-Baire sums (Corollary 10), that C_{HI} is Σ_1^1-hard. However, this method does not allow us to obtain any information about the complexity of HI. In this subsection, we will use a more complex construction in order to compute the complexity of HI.

Precisely, we follow Argyros presentation (see [AT], Chapters II and III) of how to construct HI extensions of ground norms in order to define a Borel function $\varphi : \text{Tr} \to \text{SB}$ such that $\varphi^{-1}(\text{HI}) = \text{WF}_\infty$, where WF_∞ denotes the subset of infinite well-founded trees. This will show that HI is Π_1^1-hard. As a Banach space X is hereditarily indecomposable if, and only if, for all subspaces $Z, W \subset X$ and all $\varepsilon > 0$, there exist $z \in Z$ and $w \in W$ such that $\|z - w\| < \varepsilon\|z + w\|$, we can easily show that HI is coanalytic. Therefore, we will show that HI is complete coanalytic.

Theorem 21. *HI is coanalytic.*

Proof. This is a simple consequence of the fact that $X \in \text{HI}$ if, and only if, for all subspaces $Z, W \subset X$ and all $\varepsilon > 0$, there exists $z \in Z$ and $w \in W$ such that $\|z - w\| < \varepsilon\|z + w\|$. Indeed, notice that

$$X \in \text{HI} \Leftrightarrow \forall Z, W \subset X, \ \forall \varepsilon \in \mathbb{Q}_+, \ \exists n, m \in \mathbb{N}$$
$$\|S_n(Z) - S_m(W)\| < \varepsilon\|S_n(Z) + S_m(W)\|,$$

where "$\forall Z, W \subset X$" means "for all subspaces $Z, W \subset X$". As $\{(Z, W, X) \in \text{SB}^3 \mid Z, W \subset X\}$ is well known to be Borel ([S], see Lemma 1.9, page 73), we are done. $\qquad\square$

This trivially gives us the following upper bound for the complexity of C_{HI}.

Corollary 22. *C_{HI} is Σ_2^1.*

The only thing left to show is that HI is Π_1^1-hard. For this, let us define a special Borel map $\varphi : \text{Tr} \to \text{SB}$ such that $\varphi^{-1}(\text{HI}) = \text{WF}_\infty$. As the construction of such φ will heavily rely on the construction of HI extensions of a ground norm, we tried to be consistent with Argyros notation ([AT], Chapters II and III). We believe this will make the presentation more clear for the reader which is familiar with HI spaces. Therefore, the notation used to denote some spaces in this subsection will be slightly different from the notation chosen in the rest of these notes. We will make sure to point out the differences as they appear though. To start with, we will denote by $\mathfrak{X}(D_G(\theta))$ the resulting space $\varphi(\theta)$.

First, we fix a compatible enumeration for $\mathbb{N}^{<\mathbb{N}}$, say $(s_i)_{i \in \mathbb{N}}$. Let $(e_{s_i})_{i \in \mathbb{N}}$ be the standard unit basis of $c_{00}(\mathbb{N}^{<\mathbb{N}})$.

Let $x = (x(s))_{s \in \mathbb{N}^{<\mathbb{N}}} \in c_{00}(\mathbb{N}^{<\mathbb{N}})$, and $x^* = (x^*(s))_{s \in \mathbb{N}^{<\mathbb{N}}} \in c_{00}(\mathbb{N}^{<\mathbb{N}})$, we define

$$x^*(x) = \sum_{s \in \mathbb{N}^{<\mathbb{N}}} x(s) x^*(s),$$

i.e., the notation "$*$" means that we will consider x^* as a functional on $c_{00}(\mathbb{N}^{<\mathbb{N}})$. For $x^* = (x^*(s))_{s \in \mathbb{N}^{<\mathbb{N}}} \in c_{00}(\mathbb{N}^{<\mathbb{N}})$, we let $\operatorname{supp}(x^*) = \{s \in \mathbb{N}^{<\mathbb{N}} \mid x^*(s) \neq 0\}$.

Let

$$G = \Big\{ \sum_{i=1}^n a_i e_{s_{j_i}} \mid n \in \mathbb{N}, s_{j_i} \in \mathbb{N}^{<\mathbb{N}}, s_{j_1} \prec ... \prec s_{j_n}, |a_i| = 1 \Big\}.$$

The set G is called a *ground set* (for a definition of ground sets see [AT], Definition II.1, page 21). We define Y_G as the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ under the norm

$$\|x\|_G = \sup\{g(x) \mid g \in G\},$$

for all $x \in c_{00}(\mathbb{N}^{<\mathbb{N}})$. For each $\theta \in \mathrm{Tr}$, we let $Y_G(\theta)$ denote the subspace of Y_G generated by $\{e_s \mid s \in \theta\}$. So $Y_G(\mathbb{N}^{<\mathbb{N}}) = Y_G$. We will define $\mathfrak{X}(D_G(\theta))$ as a ground norm extension of $Y_G(\theta)$.

Remark: Notice that, according to the notation of Lemma 8, it is clear that $Y_G(\theta) \cong \varphi_{\mathcal{E},0}(\theta)$, if $\mathcal{E}$ is the standard ℓ_1-basis.

For each $j \in \mathbb{N}$, let $\mathcal{A}_j = \{F \subset \mathbb{N} \mid |F| \leqslant j\}$, where $|F|$ is the cardinality of $F \subset \mathbb{N}$. As in Subsection 5.3, for finite sets $E_1, E_2 \subset \mathbb{N}$, we write $E_1 < E_2$ if $\max E_1 < \min E_2$. Let $g_1, ..., g_n \in c_{00}(\mathbb{N}^{<\mathbb{N}})$. We write $g_1 < ... < g_n$ if $\operatorname{supp}(g_1) < ... < \operatorname{supp}(g_n)$. Let $(g_l)_{l=1}^n$ be a finite sequence in $c_{00}(\mathbb{N}^{<\mathbb{N}})$ such that $g_1 < ... < g_n$, and $m \in \mathbb{N}$, we define the $(\mathcal{A}_n, \frac{1}{m})$-*operation on* $(g_l)_{l=1}^n$ as the functional $g = \frac{1}{m}(g_1 + ... + g_n)$.

Fix two sequences of natural numbers $(m_j)_{j \in \mathbb{N}}$ and $(n_j)_{j \in \mathbb{N}}$ such that $m_1 = 2$, $m_{j+1} = m_j^5$, $n_1 = 4$, and $n_{j+1} = (5n_j)^{s_j}$, where $s_j = \log_2 m_{j+1}^3$.

We now define a norming set $D_G \subset c_{00}(\mathbb{N}^{<\mathbb{N}})$ that will give us the norm we will use to define $\mathfrak{X}(D_G(\theta))$, i.e., $\varphi(\theta)$. In the definition below we use the term "n_{2j-1}-*special sequence*". As this definition will play no role in our proof of the theorem and as it is a really technical definition, we chose to omit it here. The interested reader can find the precise definition of an n_{2j-1}-special sequence in [AT], Chapter III, page 40.

Let $E \subset \mathbb{N}^{<\mathbb{N}}$, and $x^* = (x^*(s))_{s \in \mathbb{N}^{<\mathbb{N}}} \in c_{00}(\mathbb{N}^{<\mathbb{N}})$. We define $Ex^* = (x^*(s))_{s \in E}$.

Definition 23. We define D_G as the minimal subset of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ satisfying

   (i) $G \subset D_G$.
  (ii) D_G is symmetric, i.e., $g \in D_G$ implies $-g \in D_G$.
 (iii) D_G is closed under the restriction of its elements to intervals of $\mathbb{N}^{<\mathbb{N}}$, i.e., if $E \subset \mathbb{N}^{<\mathbb{N}}$ is an interval and $g \in D_G$, then $Eg \in D_G$.
 (iv) D_G is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$-operations, i.e., if $(g_l)_{l=1}^{n_{2j}}$ is a sequence in D_G such that $g_1 < ... < g_{n_{2j}}$, then $g = \frac{1}{m_{2j}}(g_1 + ... + g_{n_{2j}})$ belongs to D_G.

(v) D_G is closed under $(\mathcal{A}_{n_{2j-1}}, \frac{1}{m_{2j-1}})$-operations on special sequences, i.e., for every n_{2j-1}-special sequence $(g_1, ..., g_{n_{2j-1}})$ in D_G, the functional $g = \frac{1}{m_{2j-1}}(g_1 + ... + g_{n_{2j-1}})$ belongs to D_G.

(vi) D_G is rationally convex.

We define a norm on $c_{00}(\mathbb{N}^{<\mathbb{N}})$, as

$$\|x\|_{D_G} = \sup\{g(x) \mid g \in D_G\},$$

for all $x \in c_{00}(\mathbb{N}^{<\mathbb{N}})$. Let $\mathfrak{X}(D_G)$ be the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ under this norm. For each $\theta \in \text{Tr}$, let $\mathfrak{X}(D_G(\theta))$ be the subspace of $\mathfrak{X}(D_G)$ generated by $\{e_s \mid s \in \theta\}$. Therefore, for each $\theta \in \text{Tr}$, we assign a space $\varphi(\theta) = \mathfrak{X}(D_G(\theta))$.

Identify $\mathfrak{X}(D_G(\mathbb{N}^{<\mathbb{N}})) = \mathfrak{X}(D_G)$ with one of its isometric copies in SB. It is clear that the map $\varphi : \text{Tr} \to \text{SB}$ such that $\varphi(\theta) = \mathfrak{X}(D_G(\theta))$, is Borel (see [S], Proposition 3.1, page 79, for similar arguments).

Theorem 24. *Let $\varphi : Tr \to SB$ be the function defined above. Then*

(i) $\varphi(\theta) = \mathfrak{X}(D_G(\theta))$ contains ℓ_1, for all $\theta \in IF$.
(ii) $\varphi(\theta) = \mathfrak{X}(D_G(\theta))$ is hereditarily indecomposable, for all $\theta \in WF_\infty$.

In particular, HI is Π_1^1-hard, and C_{ℓ_1} cannot be Borel separated from HI.

Proof. First, notice that if $\theta \in \text{IF}$ then $\ell_1 \hookrightarrow \varphi(\theta)$. Indeed, on segments $I \subset \theta$, the ℓ_1-norm given by the ground set G is greater than the norm given by D_G. So, if θ has a branch, say β, we have $\mathfrak{X}(D_G(\beta)) \cong \ell_1$.

In order to show the second part of the theorem consider the "identity" map $Id : \mathfrak{X}(D_G(\theta)) \to Y_G(\theta)$. We will show that $Id : \mathfrak{X}(D_G(\theta)) \to Y_G(\theta)$ is strictly singular, for all $\theta \in \text{WF}_\infty$. Once we do that, we will be done by Theorem III.7 of [AT] (page 42). Indeed, it is clear from the proof of Theorem III.7 of [AT], that we have the following.

Theorem 25. *Let G be a ground set in $c_{00}(\mathbb{N}^{<\mathbb{N}})$, and let Y_G, and $\mathfrak{X}(D_G)$ be the spaces obtained as above. If $Z \subset \mathfrak{X}(D_G)$ is an infinite dimensional subspace, and the restriction $Id_{|Z} : Z \to Y_G$ is strictly singular, then Z is hereditarily indecomposable.*

Proposition 26. *Let $\theta \in WF_\infty$. Then $Id : \mathfrak{X}(D_G(\theta)) \to Y_G(\theta)$ is strictly singular.*

Proof. Suppose not. Then there exists an infinite dimensional subspace $Y \subset \mathfrak{X}(D_G(\theta))$ such that $Id_{|Y}$ is an isomorphism with its image. We now look at $Y = Id(Y) \subset Y_G(\theta)$. By Lemma 8 and the remark following the definition of $Y_G(\theta)$, there is a normalized sequence $(y_i)_{i \in \mathbb{N}}$ in $Y \subset Y_{D_G(\theta)}$ which is equivalent to the standard basis of c_0. In particular, there exists $C > 0$ such that

$$\|\sum_{i=1}^n y_i\|_G < C, \text{ for all } n \in \mathbb{N}.$$

Moreover, we can assume, by Lemma 28 below, that there exists a sequence $(g_i)_{i \in \mathbb{N}}$ in G such that $g_i(y_i) > \frac{1}{2}$ and $g_i < g_{i+1}$, for all $i \in \mathbb{N}$, and $|g_i(y_k)| < 2^{-(k+2)}$, for all $i \neq k$. Therefore, by the definition of the norm of $\mathfrak{X}(D_G(\theta))$, we have that

$$\big\|\sum_{i=1}^{n_{2j}} y_i\big\|_{D_G} \geq \frac{1}{m_{2j}}\sum_{i=1}^{n_{2j}} g_i(y_i) + \frac{1}{m_{2j}}\sum_{\substack{i,k=1\\i\neq k}}^{n_{2j}} g_i(y_k)$$

$$\geq \frac{n_{2j}}{2m_{2j}} - \frac{n_{2j}}{m_{2j}}\sum_{k=1}^{\infty} 2^{-(k+2)}$$

$$= \frac{n_{2j}}{4m_{2j}}$$

As $\frac{n_{2j}}{m_{2j}} \to \infty$, as $j \to \infty$, and as $Id_{|Y}$ is an isomorphism, we get a contradiction.   □

The proof of Theorem 24 is now done.   □

Theorem 27. *HI is complete coanalytic.*

In order to prove the result above we made use of the following lemma.

Lemma 28. *Let $\varphi_{\mathcal{E},0} : Tr \to SB$ be as in Lemma 8. If $\theta \in WF$, and $Y \subset \varphi_{\mathcal{E},0}(\theta)$ is infinite dimensional, then there exists a normalized sequence $(y_i)_{i\in\mathbb{N}}$ in Y equivalent to the c_0-basis. Moreover, there exists a sequence $(g_i)_{i\in\mathbb{N}}$ in $G(\theta)$ such that $g_i < g_{i+1}$, for all $i \in \mathbb{N}$, $g_i(y_i) > \frac{1}{2}$, for all $i \in \mathbb{N}$, and $|g_i(y_k)| < 2^{-(k+2)}$, for all $i \neq k$.*

Proof. This can be obtained by a simple modification in the proof of Lemma 5. For completeness, we write the modifications here. Assume all the notation in the proof of Lemma 5. So we have $X = \varphi_{\varepsilon,0}(\theta)$, and $s \in \theta$ is such that $P_s : Y \to X$ is not strictly singular, but $Q_{s,n} : Y \to X$ is strictly singular, for all $n \in \mathbb{N}$. Let $E = P_s(Z)$, where $Z \subset Y$ is a subspace such that $P_s : Z \to X$ is an isomorphism with its image. As in Lemma 5, we can assume that

$$E \subset \overline{\mathrm{span}}\{e_\tau \mid s \prec \tau\}.$$

For each $m \in \mathbb{N}$, let $I_m : X \to X$ be the standard projection over the first m coordinates of the basis $(e_s)_{s\in\theta}$, i.e., $I_m(\sum_{s\in\theta} a_s e_s) = \sum_{i=1}^{m} a_{s_i} e_{s_i}$. Let $(y_k)_{k\in\mathbb{N}}$ be a normalized sequence in E such that $Q_{s,n}(y_k) \to 0$, as $k \to \infty$, for all $n \in \mathbb{N}$ (we showed such sequence exists in the proof of Lemma 5). As $Q_{s,n}(x) \to x$, as $n \to \mathbb{N}$, for all $x \in E$, we have that given a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive real numbers, we can pick increasing sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$, and $(l_k)_{k\in\mathbb{N}}$ such that

(i) $\|Q_{s,l_k}(y_{n_k}) - y_{n_k}\|_\theta < \varepsilon_k$, for all $k \in \mathbb{N}$, and
(ii) $\|Q_{s,l_k}(y_{n_{k+1}})\|_\theta < \varepsilon_k$, for all $k \in \mathbb{N}$.
(iii) $\|I_{m_k}\big(Q_{s,l_k}(y_{n_k}) - Q_{s,l_{k-1}}(y_{n_k})\big) - \big(Q_{s,l_k}(y_{n_k}) - Q_{s,l_{k-1}}(y_{n_k})\big)\|_\theta < \varepsilon_k$, for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let

$$x_k = I_{m_k}\big(Q_{s,l_k}(y_{n_k}) - Q_{s,l_{k-1}}(y_{n_k})\big).$$

Choosing $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to 0 sufficiently fast, we have that $(x_k)_{k\in\mathbb{N}}$ is equivalent to $(y_{n_k})_{k\in\mathbb{N}}$, and, as $(x_k)_{k\in\mathbb{N}}$ has completely incomparable supports, it is easy to see that $(x_k)_{k\in\mathbb{N}}$ is also equivalent to the c_0-basis (as in Lemma 8). Also, by taking

a subsequence if necessary, we can assume that $(x_k)_{k\in\mathbb{N}}$ is a block sequence. Hence, as we can assume $\|x_i\|_\theta > \frac{1}{2}$, for all $i \in \mathbb{N}$, there exists a sequence $(g_i)_{i\in\mathbb{N}}$ in $G(\theta)$ such that $g_i < g_{i+1}$, for all $i \in \mathbb{N}$, and $g_i(x_i) > \frac{1}{2}$, for all $i \in \mathbb{N}$, and $g_i(x_k) = 0$, for all $i \neq k$.

Clearly, we can assume $\operatorname{supp}(g_i) \subset \operatorname{supp}(x_i)$, for all $i \in \mathbb{N}$. Hence, if $(y_i')_{i\in\mathbb{N}}$ is a sequence in Z such that $P_s(y_i') = y_{n_i}$, for all $i \in \mathbb{N}$, we have that $g_i(y_i') = g_i(y_{n_i}) = g_i(x_i)$, for all $i \in \mathbb{N}$. Therefore, $g_i(y_i') > \frac{1}{2}$, for all $i \in \mathbb{N}$.

On the other hand, it is easy to see that $g_i(y_k') = g_i(y_{n_k} - x_k)$, for all $i \neq k$. Hence, as $\|y_{n_k} - x_k\|_\theta < 2\varepsilon_k + \varepsilon_{k-1}$, we can assume that $|g_i(y_k')| < 2^{-(k+2)}$, for all $i \neq k$. Although $(y_k')_{k\in\mathbb{N}}$ is only semi-normalized, it is clear from the proof, that we can assume $(y_k')_{k\in\mathbb{N}}$ is normalized, so we are done. $\qquad\square$

5.7. **Unconditional basis.** A basic sequence $(x_j)_{j\in\mathbb{N}}$ in a Banach space X is unconditional if, and only if, there exists $M > 0$ such that, for all $n \in \mathbb{N}$, for all $a_1, ..., a_n \in \mathbb{Q}$, and all $b_1, ..., b_n \in \mathbb{Q}$ such that $|a_j| \leqslant |b_j|$, for all $j \in \{1, ..., n\}$, we have

$$\|\sum_{j=1}^{k} a_j x_j\| \leqslant \|\sum_{j=1}^{k} b_j x_j\|.$$

Therefore, it is clear that the set $\mathcal{UB} \subset \mathcal{B}$ of unconditional basis is Borel, where $\mathcal{B}$ is our coding for the basic sequences (see Section 2).

We now consider instead of the set of unconditional basis, the set of Banach spaces with an unconditional basis, say UB, and the set of Banach spaces containing an unconditional basis, say C_{UB}. As

$$X \in \mathrm{UB} \Leftrightarrow \exists (x_j)_{j\in\mathbb{N}} \in C(\Delta)^{\mathbb{N}} \text{ such that}$$
$$(x_j)_{j\in\mathbb{N}} \text{ is an unconditional basis for } X,$$

and the condition "$(x_j)_{j\in\mathbb{N}}$ is a basis for X" is easily seen to be Borel, we have that UB is analytic. Analogously, C_{UB} is also analytic.

Let us now give a lower bound for C_{UB}. As hereditarily indecomposable spaces cannot contain an unconditional basis, Theorem 24 gives us the following.

Theorem 29. *The set of separable Banach spaces containing an unconditional basis C_{UB} is Σ_1^1-hard. Moreover, C_{UB} is Σ_1^1-complete.*

Problem 30. Is UB Borel? Is UB complete analytic? What about $S = \{X \in \mathrm{SB} \mid X$ has a Schauder basis$\}$? Similarly, as we have for UB, we can easily see that S is analytic. Is S Borel? Is S complete analytic?

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