A note on the nonzero spectra of irreducible matrices

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Abstract

In this note we extend the necessary and sufficient conditions of Boyle-Handleman 1991 and Kim-Ormes-Roush 2000 for a nonzero eigenvalue multiset of primitive matrices over $\mathbb{R}_+$ and $\mathbf{Z}_+$, respectively, to irreducible matrices.

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1 Introduction

Denote by $\mathbb{R}^{n\times n} \supset \mathbb{R}_+^{n\times n}$ the algebra of real valued $n \times n$ matrices and the cone of $n \times n$ nonnegative matrices, respectively. For $A \in \mathbb{R}^{n\times n}$ denote by $\Lambda(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$ the eigenvalue multiset of A, i.e. $\det(zI - A) = \prod_{i=1}^n (z - \lambda_i(A))$. An outstanding problem in matrix theory, called NIEP, is to characterize a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ which is an eigenvalue multiset of some $A \in \mathbb{R}^{n\times n}$. Denote by $\rho(\Lambda) := \max\{|\lambda|, \lambda \in \Lambda\}$, and by $\Lambda(r)$ all elements in Λ satisfying $|\lambda| = r \geq 0$. For $\lambda \in \Lambda$ denote by $m(\lambda) \in \mathbb{N}$ the multiplicity of λ in Λ. The obvious necessary conditions for $\Lambda = \Lambda(A)$ for some $A \in \mathbb{R}_+^{n\times n}$ are the trace conditions:

$$s_k(\Lambda) := \sum_{i=1}^{n} \lambda_i^k \geq 0 \text{ for } k = 1, \ldots, \tag{1.1}$$

since $s_k(\Lambda(A)) = \operatorname{tr} A^k$. The following theorem is deduced straightforward from [2, Thm 2]. (See §2.)

Theorem 1.1 *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a multiset of complex numbers. Assume that the inequalities in (1.1) hold except for a finite number values of k. Then*

1. $\bar{\Lambda} = \Lambda$.
2. $\rho(\Lambda) \in \Lambda$.
3. $m(\rho(\Lambda)) \geq m(\lambda)$ *for all* $\lambda \in \Lambda(\rho(\Lambda))$.
4. *Assume that $\rho(\Lambda) > 0$ and let $\{\lambda_1, \ldots, \lambda_p\}$ be all distinct elements of Λ such that $|\lambda_i| = \rho(\Lambda)$ and $m(\lambda_i) = m(\rho(\Lambda))$ for $i = 1, \ldots, p$. Then $\zeta\Lambda(\rho(\Lambda)) = \Lambda(\rho(\Lambda))$ for $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$.*

By considering the diagonal elements of A^k and comparing them with the diagonal elements of A^{km} Loewy and London added the following additional necessary conditions [7].

Theorem 1.2 *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be an eigenvalue multiset of some $A \in \mathbb{R}_+^{n\times n}$. Then in addition to the inequalities (1.1) the following inequalities hold.*

$$s_{km}(\Lambda) \geq \frac{1}{n^{k-1}}(s_m(\Lambda))^k \text{ for } m, k-1 = 1, \ldots. \tag{1.2}$$

In particular,

$$\text{if } s_m(\Lambda) > 0 \text{ then } s_{km}(\Lambda) > 0 \text{ for } k = 2, \ldots. \tag{1.3}$$

The inequalities (1.1) and (1.2) imply that Λ is an eigenvalue multiset of some $A \in \mathbb{R}_+^{n\times n}$ in the following cases: $n = 3$; $n = 4$ and Λ is a multiset of real numbers. For $n = 4$ and nonreal $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the conditions (1.1) and (1.2) are not sufficient [7]. The necessary and sufficient conditions are given in [9]. The inequality $ns_4(\Lambda) \geq (s_2(\Lambda))^2$ in (1.2) can be improved to $(n-1)s_4(\Lambda) \geq (s_2(\Lambda))^2$ if $s_1(\Lambda) = 0$ and n is odd [8].

Definition 1.3 *A multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is called a Frobenius multiset if the following conditions hold.*

1. *$\bar{\Lambda} = \Lambda$.*
2. *$\rho(\Lambda) \in \Lambda$.*
3. *$m(\lambda) = 1$ for each $\lambda \in \Lambda(\rho(\Lambda))$.*
4. *Assume that $\#\Lambda(\rho(\Lambda)) = p$. Then $\zeta\Lambda = \Lambda$ for $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$.*

The Frobenius theorem for irreducible $A \in \mathbb{R}_+^{n\times n}$, i.e. $(I + A)^{n-1}$ is a positive matrix, claims that $\rho(\Lambda(A)) > 0$ and $\Lambda(A)$ is a Frobenius set. In particular, an irreducible $A \in \mathbb{R}_+^{n\times n}$ is primitive, i.e. $A^{(n-1)^2+1}$ is a positive matrix, if and only if $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$, see [3, XII.§5] and [4, §8.5.9].

We say that a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_i \neq 0$ for $i = 1, \ldots, n$, is a nonzero eigenvalue multiset of a nonnegative matrix if there exists an integer $N \geq n$ and $A \in \mathbb{R}_+^{N\times N}$, such that Λ is obtained from $\Lambda(A)$ by removing all zero eigenvalues. The following remarkable theorem was proved by Boyle and Handelman [1]. Namely, a multiset $\Lambda \subset \mathbb{C}\backslash\{0\}$ is a nonzero spectrum of a nonnegative primitive matrix if and only if $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$, and the inequalities (1.1) and (1.3) hold. See the recent proof of Thomas Laffey [6] of a simplified version of this result. The aim of this note is to extend the theorem of Boyle-Handelman to a nonzero eigenvalue multiset of nonnegative irreducible matrices.

Theorem 1.4 *Let Λ be a multiset of nonzero complex numbers. Then Λ is a nonzero eigenvalue multiset of a nonnegative irreducible matrix if and only if Λ is a Frobenius set, and (1.1) and (1.3) hold.*

Similarly, we extend the results of Kim, Ormes and Roush [5] to a nonzero eigenvalue multiset of nonnegative irreducible matrices with integer entries.

2 Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. For $\Lambda = \{0, \ldots, 0\}$ the theorem is trivial. Assume that $\rho(\Lambda) > 0$. Consider the function

$$f_\Lambda(z) = \sum_{i=1}^{n} \frac{1}{1 - \lambda_i z} = \sum_{k=0}^{\infty} s_k(\Lambda) z^k.$$

Assume that $s_k(\Lambda) \geq 0$ for $k > N$. Then by subtracting a polynomial $P(z)$ of degree N at most, we deduce that $f_0(z) := f(z) - P(z)$ has real nonnegative MacLaurin coefficients. So $\overline{f_0(\bar{z})} = f(z)$. Hence $\bar{\Lambda} = \Lambda$. The radius of convergence of this series is $R(f_\Lambda) = \frac{1}{\rho(\Lambda)}$. The principal part of f is $f_1 := \sum_{i, |\lambda_i| = \rho(\Lambda)} \frac{1}{1 - \lambda_i z}$. So $\pi(f_\Lambda(z)) = \{(\lambda_1, m(\lambda_1), 1), \ldots, (\lambda_q, m(\lambda_q), 1)\}$, where $\lambda_1, \ldots, \lambda_q$ are all pairwise distinct elements of $\Lambda(\rho(\Lambda))$, see [2, Dfn. 1]. Then parts 2–4 follow from [2, Thm. 2].

Proof of Theorem 1.4. Assume first that Λ is a nonzero eigenvalue multiset of a nonnegative irreducible matrix. The Frobenius theorem yields that Λ has to be a Frobenius set, and (1.1) and (1.3) hold. Assume now that Λ is a Frobenius set, and (1.1) and (1.3) hold. In view of the Boyle-Handelman theorem it is enough to consider the case

$$\Lambda(\rho(\Lambda)) = \{\rho(\Lambda), \zeta\rho(\Lambda), \ldots, \zeta^{p-1}\rho(\Lambda)\}, \text{ for } \zeta = e^{\frac{2\pi\sqrt{-1}}{p}} \text{ and } 1 < p \in \mathbb{N}. \tag{2.1}$$

Observe first that $s_k(\Lambda) = 0$ if $p \not| k$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be the map $z \mapsto z^p$. Since $\zeta\Lambda = \Lambda$, it follows that for $z \in \Lambda$ with multiplicity $m(z)$ the multiplicity of z^p in $\phi(\Lambda)$ is $pm(z)$. Hence $\phi(\Lambda)$ is a union of p copies of a Frobenius set Λ_1, where $\rho(\Lambda_1) = \rho(\Lambda)^p$ and $\Lambda_1(\rho(\Lambda_1)) = \{\rho(\Lambda_1)\}$. Moreover $s_{kp}(\Lambda) = ps_k(\Lambda_1)$. Hence Λ_1 satisfies the assumptions of the Boyle-Handelman theorem. Thus there exists a primitive matrix $B \in \mathbb{R}_+^{M \times M}$ whose nonzero eigenvalue multiset is Λ_1. Let $A = [A_{ij}]_{i=j=1}^{p}$ be the following nonnegative matrix of order pM.

$$A = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ B & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \end{bmatrix}. \tag{2.2}$$

Then A is irreducible and the nonzero part of eigenvalue multiset $\Lambda(A)$ is Λ. $\square$

3 An extension of Kim-Ormes-Roush theorem

In this section we give necessary and sufficient conditions on a multiset Λ of nonzero complex number to be a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries. Recall the Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$. First $\mu(1) = 1$. Assume that $n > 1$. If n is not square free, i.e. n is divisible by l^2 for some positive integer $l > 1$, then $\mu(n) = 0$. If $n > 1$ is square free, let $\omega(n)$ be the number of distinct primes that divide n. Then $\mu(n) = (-1)^{\omega(n)}$. The following theorem is a generalization of the Kim-Ormes-Roush theorem [5].

Theorem 3.1 *Let Λ be a multiset of nonzero complex numbers. Then Λ is a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries if and only if the following conditions hold.*

1. *Λ is a Frobenius set.*
2. *The coefficients of the polynomial $\prod_{\lambda\in\Lambda}(z-\lambda)$ are integers.*
3. *$t_k(\Lambda) := \sum_{d|k} \mu(\frac{k}{d}) s_d(\Lambda) \geq 0$ for all $k \in \mathbb{N}$.*

The case $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$ is the Kim-Ormes-Roush theorem.

Proof. Assume that Λ is a nonzero spectrum of a nonnegative irreducible matrix with integer entries, i.e. $A \in \mathbf{Z}_+^{N\times N}$. Then part 1 follows from the Frobenius theorem. Since $\det(zI - A)$ has integer coefficients we deduce part 2. It is known that $t_k(\Lambda) = t_k(\Lambda(A))$ is the number of minimal loops of length k in the directed multigraph induced by A, see [1]. Hence part 3 holds.

Suppose that Λ satisfies 1–3. In view of the Kim-Ormes-Roush theorem it is enough to assume the case (2.1). We now use the notations and the arguments of the proof of Theorem 1.4. First $s_k(\Lambda) = 0$ if $p \not| k$. Second $\prod_{\lambda\in\Lambda}(z-\lambda) = \prod_{\kappa\in\Lambda_1}(z^p-\kappa)$. Hence $\prod_{\kappa\in\Lambda_1}(z-\kappa)$ has integer coefficients. A straightforward calculation shows that $t_{pk}(\Lambda) = pt_k(\Lambda_1)$. Hence $t_k(\Lambda_1) \geq 0$. Kim-Ormes-Roush theorem yields the existence of $B \in \mathbf{Z}_+^{M\times M}$ such that Λ_1 is the nonzero eigenvalue multiset of B. Hence Λ is the nonzero eigenvalue set of $A \in \mathbf{Z}_+^{pM\times pM}$ given by (2.2). $\square$

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