Generalized root graded Lie algebras

Thesis submitted for the degree of
Doctor of Philosophy
at the
University of Leicester

by
Hogar Mohammed Yaseen
Department of Mathematics
University of Leicester
2018

This thesis is dedicated to the memory of my father who has physically left me but his advice and memories live with me.

Acknowledgements

First of all, my deepest gratitude is to my supervisor, Dr Alexander Baranov for his insightful comments, encouragement and patience. His guidance and deep insights helped me throughout my time as his student. I do really appreciate his scientific guidance and that he was able to meet me weekly. I count myself fortunate to have had such a supervisor and will remain indebted for his advice.

I would like to express my special appreciation and thanks to Ministry of Higher Education and Scientific Research, Kurdistan Regional Government-Iraq for giving me the opportunity to do my PhD abroad and for their financial support. It would have been impossible for me had they not given me this opportunity.

I would like to thank the Department of Mathematics in the University of Leicester for creating such a friendly environment.

Many thanks go to my colleagues in Michael Atiyah building for their friendship and help, especially, Mohammed and Hasan.

A special thank goes to my beloved wife, Sozan, for her love, patience and support throughout my PhD journey and my life in general.

Abstract

Generalized root graded Lie algebras

Hogar M Yaseen

Let $\mathfrak{g}$ be a non-zero finite-dimensional split semisimple Lie algebra with root system Δ. Let Γ be a finite set of integral weights of $\mathfrak{g}$ containing Δ and $\{0\}$. We say that a Lie algebra L over $\mathbb{F}$ is *generalized root graded*, or more exactly $(\Gamma,\mathfrak{g})$*-graded*, if L contains a semisimple subalgebra isomorphic to $\mathfrak{g}$, the $\mathfrak{g}$-module L is the direct sum of its weight subspaces L_α ($\alpha \in \Gamma$) and L is generated by all L_α with $\alpha \neq 0$ as a Lie algebra. If $\mathfrak{g}$ is the split simple Lie algebra and $\Gamma = \Delta \cup \{0\}$ then L is said to be *root-graded*. Let $\mathfrak{g} \cong sl_n$ and

$$\Theta_n = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}$$

where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the set of weights of the natural sl_n-module. Then a Lie algebra L is $(\Theta_n, \mathfrak{g})$-graded if and only if L is generated by $\mathfrak{g}$ as an ideal and the $\mathfrak{g}$-module L decomposes into copies of the adjoint module, the natural module V, its symmetric and exterior squares S^2V and $\wedge^2 V$, their duals and the one dimensional trivial $\mathfrak{g}$-module.

In this thesis we study properties of generalized root graded Lie algebras and focus our attention on (Θ_n, sl_n)-graded Lie algebras. We describe the multiplicative structures and the coordinate algebras of (Θ_n, sl_n)-graded Lie algebras, classify these Lie algebras and determine their central extensions.

Contents

Chapter 1

Introduction

Throughout the thesis, the ground field $\mathbb{F}$ is of characteristic zero, $\mathfrak{g}$ is a non-zero split finite dimensional semisimple Lie algebra over $\mathbb{F}$ with root system Δ and Γ is a finite set of integral weights of $\mathfrak{g}$. Following [6], we say that a Lie algebra L over $\mathbb{F}$ is $(\Gamma, \mathfrak{g})$-*graded*, or simply Γ-*graded*, if L contains a subalgebra isomorphic to $\mathfrak{g}$, the $\mathfrak{g}$-module L is the direct sum of its weight subspaces L_α ($\alpha \in \Gamma$) and L is generated by all L_α with $\alpha \neq 0$ as a Lie algebra (see also Definition 3.0.1). Unless otherwise stated, we assume that $\mathfrak{g}$ is the grading subalgebra of the $(\Gamma, \mathfrak{g})$-graded L. If $\mathfrak{g}$ is the split simple Lie algebra and $\Gamma = \Delta \cup \{0\}$ then L is said to be *root-graded*. If $\Gamma = BC_n \cup \{0\}$ and $\mathfrak{g}$ is of type B_n, C_n or D_n, then L is BC_n-graded. Let $\mathfrak{g} \cong sl_n$ and

$$\Theta_n = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}$$

where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the set of weights of the natural sl_n-module. The aim of this thesis is to describe the multiplicative structures and the coordinate algebras of (Θ_n, sl_n)-graded Lie algebras, classify these Lie algebras and determine their central extensions.

1.1 Overview

Root graded Lie algebras were introduced by Berman and Moody in 1992 to study toroidal Lie algebras and Slodowy intersection matrix algebras. However, this concept appeared previously in Seligman's study of simple Lie algebras [46]. Root graded Lie algebras of simply-laced finite root systems were classified up to central isogeny by Berman and Moody in [22]. The case of double-laced finite root systems was settled by Benkart and Zelmanov [20]. Neher [44] described Lie algebras graded by 3-graded root systems. This gives an alternative classification of root-graded Lie algebras since most root systems are

3-graded (more precisely, a root system is 3-graded if and only if it does not have an irreducible component of type E_8, F_4 or G_2).

Non-reduced systems BC_n were considered by Allison, Benkart and Gao [4] (for $n \geq 2$) and by Benkart and Smirnov [18] (for $n = 1$). It became clear at that time that this notion can be generalized further by considering Lie algebras graded by finite weight systems.

A *central extension* of a Lie algebra L is a pair $(\tilde{L}, \pi)$ consisting of a Lie algebra $\tilde{L}$ and a surjective Lie algebra homomorphism $\pi : \tilde{L} \to L$ whose kernel lies in the center of $\tilde{L}$. A *cover* or *covering* of L is a central extension $(\tilde{L}, \pi)$ of L with $\tilde{L}$ perfect, i.e., $\tilde{L} = [\tilde{L}, \tilde{L}]$. A homomorphism of central extensions from the central extension $f : K \to L$ to the central extension $f' : K' \to L$ is a Lie algebra homomorphism $g : K \to K'$ satisfying $f = f' \circ g$. A central extension $U : K \to L$ is a *universal central extension*, if there exists a unique homomorphism from K to any other central extension $\tilde{K}$ of L. Any perfect Lie algebra L has a universal central extension which is also perfect, called a *universal covering algebra* of L and any two universal covering algebras of L are isomorphic [32]. Two perfect Lie algebras L_1 and L_2 are said to be *centrally isogenous* if they have the same universal covering algebra (up to isomorphism). Central extensions of root graded Lie algebras in terms of the homology of its coordinate algebra were determined and described up to isomorphism by Allison, Benkart and Y. Gao in [3] and [4]. Derivations and invariant forms of these Lie algebras were described by Benkart in [10]. Their centroids (the spaces of L-module endomorphisms χ of L: $\chi([x,y]) = [x, \chi(y)]$ for all $x, y \in L$) were determined by Benkart and Neher [17]. Gao studied involutive Lie algebras graded by finite root systems and classified the fixed point subalgebras up to central isogeny [31]. Yousofzadeh studied the subalgebras of fixed points of root graded Lie algebras for certain classes of automorphisms of finite order [55]. Bhargava and Gao studied $(BC_r, \mathfrak{g})$-graded intersection matrix algebras where $\mathfrak{g}$ is of type B_r $(r \geq 3)$ [25]. Manninga, Neher and Salmasian studied representations of a root-graded Lie algebra L which are integrable as representations of the grading semisimple subalgebra $\mathfrak{g}$ and whose weights are bounded by some dominant weight [39].

Finite-dimensional semisimple Lie algebras were generalised in many ways. For example, one could try to generalize their presentation given by Serre's Theorem. In this way one obtains, for example, the Kac-Moody algebras or Slodowy's generalized intersection matrix algebras. Root-graded or more generally $(\Gamma, \mathfrak{g})$-graded Lie algebras can be considered as another reasonable generalization of semisimple Lie algebras.

The $(\Gamma, \mathfrak{g})$-graded Lie algebras form an important class of infinite dimensional Lie algebras. Due to their uniform structure, it is possible to describe their multiplicative structure and classify them in terms of their coordinate algebras. Apart from split semisimple

Lie algebras, there are other well known classes of $(\Gamma, \mathfrak{g})$-graded Lie algebras, such as affine Kac–Moody algebras [36], isotropic finite-dimensional simple Lie algebras [46], the intersection matrix Lie algebras introduced by Slodowy [48], derived algebras of affine Lie algebras, extended affine Lie algebras (EALAs) [1], the twisted affine algebras, toroidal Lie algebras, Tits-Kantor-Koecher Lie algebra (see Example 2.2.5), etc. Every extended affine Lie algebra has an ideal called the core, which is a root-graded (or BC_r-graded) Lie algebra. Classifying the extended affine Lie algebras amounts to determining the coordinate algebra, derivations and central extensions of the core (see [1], [23], [24] and [51] for those EALAs which correspond to the reduced root systems).

Another motivation comes from [43], where Neeb applied $(\Gamma, \mathfrak{g})$-graded Lie algebras in a topological setting of locally convex Lie algebras to study some classes of Lie algebras arising in mathematical physics, operator theory, and geometry. This brings some geometric flavor to the theory because the coordinatization theorems for $(\Gamma, \mathfrak{g})$-graded Lie algebras are very similar in nature to certain coordinatization results in synthetic geometry [43]. Muller, Neeb and Seppaunen introduced and studied (weakly) root graded Banach–Lie algebras and corresponding Lie groups as natural generalizations of groups like $GL_n(A)$ for Banach algebras A [42].

Root decompositions also play a crucial role in the classification of the finite dimensional complex simple Lie superalgebras (see [35]). Lie superalgebras graded by the root systems of the finite-dimensional basic classical simple Lie superalgebras $A(m,n)$, $A(n,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$; $(\alpha \neq 0, -1)$, $G(3)$, and $F(4)$ were classified up to central isogeny by Benkart and Elduque [11–13, 15]. Lie superalgebras graded by $P(n)$ and $Q(n)$ were classified by Martinez and Zelmanov [40]. Lie superalgebras graded by locally finite root supersystems were studied by Yousofzadeh [58, 59].

There were several attempts to generalize root graded Lie algebras. Neher switched from fields of characteristic zero to rings containing $\frac{1}{6}$ and working with locally finite root systems instead of finite [44]. He also considered Lie algebras graded by infinite root systems of type $A-D$. Welte in her PhD thesis described the universal central extensions of Lie algebras graded by the root systems of type A with rank at least 2 and of type C defined over commutative associative unital rings [50]. Yoshii [52] studied so-called *predivision* (Δ, G)-graded Lie algebras. These are Δ-graded Lie algebras with additional compatible grading by an abelian group G. He introduced the notion of a root system extended by an abelian group G and showed that (Δ, G)-graded Lie algebras have such root systems. As a special case of division (Δ, G)-graded Lie algebras, Yoshii introduced and studied Lie G-tori [19, 53, 54]. Yousofzadeh studied Lie algebras graded by irreducible locally finite root systems [56, 57]. Elduque [28] and Draper and Elduque [27] related root gradings with fine grading. This notion was extended further by Nervi to the case where $\mathfrak{g}$ is an affine Kac-Moody algebra and Δ is the (infinite) root system of an affine Kac-Moody algebra. She gave the complete classification of all affine Kac–Moody algebras graded by affine root systems [45]. Messaoud and Rousseau studied Kac-Moody Lie algebras graded by Kac-Moody root systems [41].

Shi introduced groups graded by finite root systems which can be thought of as natural generalizations of Steinberg and Chevalley groups over rings [47]. Ershov, Jaikin-Zapirain, Kassabov [30] and Ershov, Jaikin-Zapirain, Kassabov and Zhang [29] studied the class of groups satisfying property T and graded by root systems.

There were several attempts to classify Γ-graded Lie algebras for systems Γ larger than Δ. This includes the BC_n-graded Lie algebras mentioned above. Certain weight-graded Lie algebras were considered by Neeb in [43] (with $\Gamma\setminus\{0\}$ a finite irreducible root system and Δ a sub-root system of $\Gamma\setminus\{0\}$). Let $\mathfrak{g}=sl_n$ and $\Gamma_V=\Delta\cup V\cup\{0\}$ where $\Delta=A_{n-1}$ and V is the set of weights of the natural and conatural $\mathfrak{g}$-modules. Bahturin and Benkart [5] (for $n>3$) and Benkart and Elduque [14] (for $n=3$) described the multiplicative structure of the $(\Gamma_V,\mathfrak{g})$-graded Lie algebras. Note that a Lie algebra is $(\Gamma_V,\mathfrak{g})$-graded if and only if it decomposes as a $\mathfrak{g}$-module into (possibly infinitely many) copies of the adjoint, natural, conatural and trivial modules. We believe that the set Γ_V can be enlarged further by adding the weights of the symmetric and exterior squares of the natural and conatural modules. Recall that we denote the corresponding set of weights by Θ_n. Note that a Lie algebra L is $(\Theta_n,\mathfrak{g})$-graded if and only if L is generated by $\mathfrak{g}$ as an ideal and the $\mathfrak{g}$-module L decomposes into copies of the adjoint module (we will denote it by the same letter $\mathfrak{g}$), the natural module V, its symmetric and exterior squares S^2V and $\wedge^2V$, their duals and the one dimensional trivial $\mathfrak{g}$-module (see Proposition 3.2.2). Thus, by collecting isotypic components, we get the following decomposition of the $\mathfrak{g}$-module L:

$$L=(\mathfrak{g}\otimes A)\oplus(V\otimes B)\oplus(V'\otimes B')\oplus(S\otimes C)\oplus(S'\otimes C')\oplus(\Lambda\otimes E)\oplus(\Lambda'\otimes E')\oplus D \quad (1.1.1)$$

where A,B,B',C,C',E,E' are vector spaces,

$$\begin{aligned}&\mathfrak{g}:=V(\omega_1+\omega_{n-1}),\quad V:=V(\omega_1),\quad V':=V(\omega_{n-1}),\\ &S:=V(2\omega_1),\quad S':=V(2\omega_{n-1}),\quad \Lambda:=V(\omega_2),\quad \Lambda':=V(\omega_{n-2})\end{aligned}$$

and D is the sum of the trivial $\mathfrak{g}$-modules.

Note that the Θ_n-graded Lie algebras did appear in the literature previously in various contexts. Finite dimensional Θ_n-graded Lie algebras and their representations were studied in [8, 9]. It was also proved in [6, 4.3] that a simple locally finite Lie algebra is

Θ_n-graded if and only if it is of diagonal type.

1.2 Outline of methods and summary of results

Let L be a $(\Gamma, \mathfrak{g})$-graded Lie algebra and let Δ be the root system of $\mathfrak{g}$. Then L is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules and there is one possible isotypic component for each dominant weight in Γ. By collecting isotypic components, we get the following decomposition of the $\mathfrak{g}$-module L.

1. If $\Gamma \setminus \{0\} = \Delta = A_n, D_n, E_6, E_7$ or E_8 where $n \geqq 2$, then the $\mathfrak{g}$-module L decomposes into (possibly infinitely many) copies of adjoint modules (modules isomorphic to $\mathfrak{g}$) and one dimensional trivial $\mathfrak{g}$-modules [22].

2. If $\Gamma \setminus \{0\} = \Delta = B_n, C_n, F_4$ or G_2, then the $\mathfrak{g}$-module L is a direct sum of adjoint modules, little adjoint modules (whose highest weight is the highest short root) and one dimensional trivial $\mathfrak{g}$-modules [20].

3. If $\Gamma \setminus \{0\}$ is a finite irreducible root system and Δ is a sub-root system of $\Gamma \setminus \{0\}$, then there are at most three isotypic components, corresponding to the adjoint module, little adjoint module and the one dimensional trivial $\mathfrak{g}$-module [43].

4. If $\Gamma \setminus \{0\} = BC_n$ and $\Delta = B_n, C_n, D_n$ $(n \geq 2)$, then there are four isotypic components, corresponding to the modules $V(2\omega_1)$, $V(\omega_2)$, $V(\omega_1)$ and $V(0)$, except in the case $\Delta = D_2$ where there are five [4].

5. If $\Gamma = \Theta_n$ and $\Delta = A_{n-1}$ $(n \geq 5)$ then the $\mathfrak{g}$-module L is a direct sum of copies of $\mathfrak{g}$, V, V', S, S', Λ, Λ' and T (see Proposition 3.2.2). This makes 8 possible components, which increases the complexity of the problem considerably in comparison with the case of root-graded Lie algebras.

We will need the following notation to describe our classification of Θ_n-graded Lie algebras. Recall that every Θ_n-graded Lie algebra L is decomposed as in (1.1.1). Since $\mathfrak{g}$ is a $\mathfrak{g}$-submodule of $\mathfrak{g} \otimes A$, there exists a distinguished element 1 of A such that $\mathfrak{g} = \mathfrak{g} \otimes 1$. Define by $\mathfrak{g}^+ := \{x \in \mathfrak{g} \mid x^t = x\}$ and $\mathfrak{g}^- := \{x \in \mathfrak{g} \mid x^t = -x\}$ the subspaces of symmetric and skew-symmetric matrices in $\mathfrak{g}$, respectively. Then the component $\mathfrak{g} \otimes A$ in (1.1.1) can be decomposed further as

$$\mathfrak{g} \otimes A = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \otimes A = (\mathfrak{g}^+ \otimes A^-) \oplus (\mathfrak{g}^- \otimes A^+)$$

where A^- and A^+ are two copies of the vector space A. We denote by $a^\pm$ the image of $a \in A$ in $A^\pm$. Denote

$$\mathfrak{a} := A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E' \quad \text{and} \quad \mathfrak{b} := \mathfrak{a} \oplus B \oplus B'.$$

Our main goal of classification of Θ_n-graded Lie algebras L is achieved in the following steps.

1. The determination of the finite-dimensional irreducible $\mathfrak{g}$-modules whose weights relative to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are in Θ_n (Proposition 3.2.2).

2. The proof of the complete reducibility of L as a $\mathfrak{g}$-module (Lemma 3.1.2).

3. The computation of all non-zero $\mathfrak{g}$-module homomorphism spaces $\mathrm{Hom}_{\mathfrak{g}}(X \otimes Y, Z)$ where $X, Y, Z \in \{\mathfrak{g}, V, V', S, \Lambda, S', \Lambda', T\}$, see (3.4.3).

4. The determination of the system of products on the components of the $\mathfrak{g}$-module decomposition of L induced by multiplication in L, see (3.4.4).

5. Description of the "coordinate" algebra $\mathfrak{b}$ of L (Theorem 4.2.9).

6. We define a centerless algebra $\mathscr{L}(\mathfrak{b})$ and show that it is an Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$, see Theorem 5.2.5. Instead of proving directly that $\mathscr{L}(\mathfrak{b})$ satisfies the Jacoby identity (which is quite lengthy), we construct an explicit example of an Θ_n-graded Lie algebra $\mathfrak{u}$ such that $\mathfrak{u}$ modulo its center is isomorphic to $\mathscr{L}(\mathfrak{b})$, see Example 5.2.3.

7. We show that if $\mathfrak{b}$ is the coordinate algebra of L then L is a cover of $\mathscr{L}(\mathfrak{b})$ (Theorem 5.2.5).

8. We show that L is uniquely determined (up to central isogeny) by its coordinate algebra $\mathfrak{b} := \mathfrak{a} \oplus \mathscr{B}$ where $\mathscr{B} := B \oplus B'$ and L is centrally isogenous to the Θ_n-graded unitary Lie algebra $\mathfrak{u}$ of the hermitian form $\xi := w\perp - \chi$ on the $\mathfrak{a}$-module $\mathfrak{a}^n \oplus \mathscr{B}$ where $w : \mathfrak{a}^n \times \mathfrak{a}^n \to \mathfrak{a}$ is a non degenerate bilinear form on $\mathfrak{a}^n$ and $\chi : \mathscr{B} \times \mathscr{B} \to \mathfrak{a}$ is a hermitian form over $\mathfrak{a}$ (Proposition 5.2.4 and Theorem 5.2.6). This completes the classification of Θ_n-graded Lie algebras up to central extensions in the case when $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold.

9. We find the universal central extension $\widehat{\mathscr{L}(\mathfrak{b})}$ of $\mathscr{L}(\mathfrak{b})$ and show that its center is $\mathrm{HF}(\mathfrak{b})$, the full skew-dihedral homology group of $\mathfrak{b}$ (Theorem 5.3.7). We prove that every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is isomorphic to $\mathscr{L}(\mathfrak{b}, X) =$

$\widehat{\mathscr{L}(\mathfrak{b})}/X$ for some subspace X of $\mathrm{HF}(\mathfrak{b})$ which classifies the Θ_n-graded Lie algebras up to isomorphism (Theorem 5.3.8).

Chapters 3 and 4 consist mainly of joint work with Alexander Baranov [7]. Chapter 5 contains some results in joint work with Alexander Baranov. We are now ready to state our main results.

In Chapter 2 we review main concepts and results of the theory of Lie algebras graded by finite root systems. This chapter is organized as follows. First we recall the multiplicative structures and coordinate algebras of Lie algebras graded by finite reduced root systems (Section 2.1). Then we consider some examples (Section 2.2) and state recognition theorem for these Lie algebras (Section 2.3). In Section 2.4 we review Lie algebras graded by non-reduced systems BC_n ($n \geq 2$).

In Chapter 3 we study general properties of generalized root graded Lie algebras and we describe the multiplicative structures of (Θ_n, sl_n)-graded Lie algebras. The coordinate algebra of (Θ_n, sl_n)-graded Lie algebra and its properties are analyzed in Chapter 4.

In Section 3.1 we establish general properties of weight-graded Lie algebras. In particular, we prove that every finite-dimensional perfect Lie algebra is (Γ, sl_2)-graded for some Γ, see Theorem 3.1.9. In Section 3.2 we discuss the similarities between the Θ_n-graded and BC_n-graded Lie algebras by showing that every Θ_n-graded Lie algebra is BC_r-graded with $r = \lfloor \frac{n}{2} \rfloor$ and every BC_n-graded Lie algebra is Θ_n-graded, see Theorems 3.2.4 and 3.2.6. This means that some results about the structure of Θ_n-graded Lie algebras can be derived from those proved in BC_r-contexts [4, 18]. However note that our approach gives a "finer" multiplicative and coordinate algebra structure on L as we have more components in the decomposition of L (see Remark 3.2.5).

Let L be Θ_n-graded and let $\mathfrak{g} \cong sl_n$ be the grading subalgebra of L. Then we have decomposition (1.1.1). Recall that

$$\mathfrak{a} := A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E' \quad \text{and} \quad \mathfrak{b} := \mathfrak{a} \oplus B \oplus B'.$$

We are going to show that the product in L induces an algebra structure on both $\mathfrak{a}$ and $\mathfrak{b}$. Moreover, $\mathfrak{a}$ is associative if $n \geq 7$ or $n = 5, 6$ and the following conditions on multiplication in L hold:

$$\begin{aligned} &[\Lambda \otimes E, \Lambda \otimes E] = [\Lambda' \otimes E', \Lambda' \otimes E'] = 0 \text{ for } n = 6; \\ &[\Lambda \otimes E, (\Lambda \otimes E) \oplus (V \otimes B)] = [\Lambda' \otimes E', (\Lambda' \otimes E') \oplus (V' \otimes B')] = 0 \text{ for } n = 5. \end{aligned} \tag{1.2.1}$$

Note that the conditions (1.2.1) automatically hold for $n \geq 7$ (see Table 3.4.2) and for

BC_n-graded (considered as Θ_n-graded) Lie algebras with $n \geq 5$ (see Proposition 3.2.7). These conditions appear only because of irregularities in the tensor product decompositions of the specified modules for small ranks, see Remark 3.4.4. We do not consider the case of $n \leq 4$ in this thesis because of additional technicalities (e.g. $\Lambda \cong \Lambda'$ for A_3 and $\Lambda \cong V'$ and $\Lambda' \cong V$ for A_2, so we have less summands in the decomposition (1.1.1)), this is the subject of our further research.

Suppose that $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold. We prove that there exists a system of products (see Formulae (3.4.4)) on the components of the decomposition (1.1.1) which is compatible with the product in L and induces an algebra structure on both $\mathfrak{a}$ and $\mathfrak{b}$ satisfying the following properties.

(i) $\mathfrak{a}$ is a unital associative subalgebra of $\mathfrak{b}$ with identity element 1^+ and with involution whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E'$ and $A^- \oplus C \oplus C'$, respectively, see Theorems 4.1.3 and 4.1.6.

(ii) $\mathfrak{b}$ is a unital algebra with identity element 1^+ and with an involution η whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E' \oplus B \oplus B'$ and $A^- \oplus C \oplus C'$, respectively, see Theorem 4.2.1 and Proposition 4.2.2.

(iii) $B \oplus B'$ is an associative $\mathfrak{a}$-bimodule with a hermitian form χ with values in $\mathfrak{a}$. More exactly, for all $\beta_1, \beta_2 \in B \oplus B'$ and $\alpha \in \mathfrak{a}$ we have $\chi(\beta_1, \beta_2) = \beta_1\beta_2$, $\chi(\alpha\beta_1, \beta_2) = \alpha\chi(\beta_1, \beta_2)$, $\eta(\chi(\beta_1, \beta_2)) = \chi(\beta_2, \beta_1)$ and $\chi(\beta_1, \alpha\beta_2) = \chi(\beta_1, \beta_2)\eta(\alpha)$, see Propositions 4.2.4 and 4.2.6.

(iv) $\mathscr{A} := A^- \oplus A^+$ is a unital associative subalgebra of $\mathfrak{a}$ and $C \oplus E$, $C' \oplus E'$, B and B' are $\mathscr{A}$-bimodules, see Corollaries 4.1.4, 4.1.5 and 4.2.5.

(v) D acts by derivations on $\mathfrak{b}$, see Propositions 4.2.7 and 4.2.8.

Let $e_1 = \frac{1^+ + 1^-}{2}$ and $e_2 = \frac{1^+ - 1^-}{2}$. Consider the subspaces $A_1 = span\{a^+ + a^- \mid a \in A\}$ and $A_2 = span\{a^+ - a^- \mid a \in A\}$. In Section 4.3 we show that e_1 and e_2 are orthogonal idempotents with $e_1 + e_2 = 1^+$ and $\eta(e_1) = e_2$ where η is the involution of the coordinate algebra $\mathfrak{b}$. We also show that A_1 and A_2 are 2-sided ideals of the algebra $\mathscr{A}$ with identity elements e_1 and e_2, respectively. Moreover, we prove that the associative algebra $\mathfrak{a}$ has the following realization by 2×2 matrices with entries in the components of $\mathfrak{a}$:

$$\mathfrak{a} \cong \begin{bmatrix} A_1 & C \oplus E \\ C' \oplus E' & A_2 \end{bmatrix}.$$

In Chapter 5 we classify Θ_n-graded Lie algebras in the case when $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold. The chapter is organized as follows. First we study basic properties of central extensions of $(\Gamma, \mathfrak{g})$-graded Lie algebras. We show all Lie algebras in a given isogeny class are Γ-graded if one of them is, and all have isomorphic weight spaces for non-zero weights. We also show that for every central extension $(\tilde{L}, \pi)$ of a $(\Gamma, \mathfrak{g})$-graded Lie algebra L with kernel E, there is lifting of the grading subalgebra $\mathfrak{g}$ to a subalgebra of $\tilde{L}$ and L can be lifted to a subspace L of $\tilde{L}$ which contains the given $\mathfrak{g}$ so that the corresponding 2-cocycle satisfies $\lambda(\mathfrak{g}, L) = 0$ (see Section 5.1). Then we focus our attention to (Θ_n, sl_n)-graded Lie algebras. We define a centerless algebra $\mathscr{L}(\mathfrak{b})$ and show that it is Θ_n-graded with coordinate algebra $\mathfrak{b}$ and any Θ_n-graded Lie algebra L with coordinate algebra $\mathfrak{b}$ is a cover of the centerless Lie algebra $\mathscr{L}(\mathfrak{b})$. Then we show that every Θ_n-graded Lie algebra L is uniquely determined (up to central isogeny) by its coordinate algebra $\mathfrak{b}$. In Section 5.2 we show that L is centrally isogenous to the explicitly constructed Θ_n-graded unitary Lie algebra $\mathfrak{u}$ of the hermitian form $\xi = w\perp - \chi$ on the $\mathfrak{a}$-module $\mathfrak{a}^n \oplus \mathscr{B}$. This completes the classification of Θ_n-graded Lie algebras up to central extensions. In Section 5.3 we find the universal central extension $\widehat{\mathscr{L}(\mathfrak{b})}$ of $\mathscr{L}(\mathfrak{b})$ and show that its center is $\mathrm{HF}(\mathfrak{b})$, the full skew-dihedral homology group of $\mathfrak{b}$. We prove that every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is isomorphic to $\mathscr{L}(\mathfrak{b}, X) = \widehat{\mathscr{L}(\mathfrak{b})}/X$ for some subspace X of $\mathrm{HF}(\mathfrak{b})$, which classifies the Θ_n-graded Lie algebras up to isomorphism.

At the end of the chapter we relate the Θ_n-graded Lie algebras to the quasiclassical Lie algebras (see Definition 5.4.5) by showing that every $(\Xi_n, sl_n)-$graded Lie algebra with

$$\Xi_n := \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \le i \ne j \le n\} \subset \Theta_n$$

is centrally isogenous to a quasiclassical Lie algebra (see Section 5.4).

1.3 Notation

- For convenience of the reader we mostly follow notations of [3, 4] whenever possible.

- $\mathfrak{g}$ is a non-zero split finite dimensional semisimple Lie algebra over a field $\mathbb{F}$ of characteristic zero with root system Δ. Unless otherwise stated we assume that $\mathfrak{g}$ is the grading subalgebra of $(\Gamma, \mathfrak{g})$-graded L.

- We denote
$$\Theta_n = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\varepsilon_i \mid 1 \le i \ne j \le n\}$$
where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the set of weights of the natural sl_n-module. We fix a base

$\Pi = \{\varepsilon_i - \varepsilon_{i+1} \text{ for } i = 1,2,\cdots,n-1\}$ of simple roots for the root system

$$A_{n-1} = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}.$$

- Θ_n^+ is the set of dominant weights in Θ_n. Thus,

$$\Theta_n^+ = \{\varepsilon_1 - \varepsilon_n,\ \varepsilon_1, -\varepsilon_n,\ 2\varepsilon_1,\ -2\varepsilon_n,\ \varepsilon_1 + \varepsilon_2,\ -\varepsilon_{n-1} - \varepsilon_n,\ 0\}.$$

- We say that a Lie algebra is Θ_n-graded if it is $(\Theta_n, \mathfrak{g})$-graded with $\mathfrak{g} \cong sl_n$.

- Let $\mathfrak{g}$ be a split simple Lie algebra of type A_n, B_n, C_n or D_n. We use the following representation of the simple roots α_k $(1 \le k \le n)$ and fundamental weights ω_k $(1 \le k \le n)$ of $\mathfrak{g}$ in terms of ε's, see [26],

$$\alpha_k = \begin{cases} \varepsilon_n & \text{if } k = n\ (B) \\ 2\varepsilon_n & \text{if } k = n\ (C) \\ \varepsilon_{n-1} + \varepsilon_n & \text{if } k = n\ (D) \\ \varepsilon_k - \varepsilon_{k+1} & \text{otherwise,} \end{cases}$$
$$\omega_k = \begin{cases} \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n) & \text{if } k = n\ (B \text{ or } D) \\ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{n-1} - \varepsilon_n) & \text{if } k = n-1\ (D) \\ \varepsilon_1 + \ldots + \varepsilon_k & \text{otherwise.} \end{cases}$$

$V_{\mathfrak{g}}(\lambda)$ (or simply $V(\lambda)$) is the simple $\mathfrak{g}$-module of highest weight λ; $V_{\mathfrak{g}} := V_{\mathfrak{g}}(\omega_1)$ (or simply V) is the natural $\mathfrak{g}$-module; if M is a $\mathfrak{g}$-module then M' denotes its dual and $\mathscr{W}(M)$ is the set of weights of M.

- If $\mathfrak{g}$ is of type A_{n-1}, we will use the following notations for the $\mathfrak{g}$-modules below:

$$\mathfrak{g} := V(\omega_1 + \omega_{n-1}),\ V := V(\omega_1),\ S := V(2\omega_1),\ \Lambda := V(\omega_2) \text{ and } T := V(0).$$

Note that $V' \cong V(\omega_{n-1})$, $S' \cong V(2\omega_{n-1})$ and $\Lambda' \cong V(\omega_{n-2})$.

- $\ll \mathfrak{g} \gg_L$ (or simply $\ll \mathfrak{g} \gg$) is the ideal generated by $\mathfrak{g}$ in L

- Γ is a finite set of integral weights of $\mathfrak{g}$.

- Let x and y be $n \times n$ matrices. We will use the following products:

$$[x, y] = xy - yx,$$

$$x \circ y = xy + yx - \frac{2}{n}\operatorname{tr}(xy)I,$$
$$x \diamond y = xy + yx,$$
$$(x \mid y) = \frac{1}{n}\operatorname{tr}(xy).$$

- If $\mathfrak{g} = sl_n$, we denote by $\mathfrak{g}^+ := \{x \in \mathfrak{g} \mid x^t = x\}$ and $\mathfrak{g}^- := \{x \in \mathfrak{g} \mid x^t = -x\}$ the subspaces of symmetric and skew-symmetric matrices in $\mathfrak{g}$, respectively. Then the component $\mathfrak{g} \otimes A$ can be decomposed further as

$$\mathfrak{g} \otimes A = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \otimes A = (\mathfrak{g}^+ \otimes A^-) \oplus (\mathfrak{g}^- \otimes A^+)$$

  where A^- and A^+ are two copies of the vector space A.

- If A is an associative algebra with involution σ (of the first kind) over $\mathbb{F}$ then $sym(A)$ (resp. $skew(A)$) denotes the set of symmetric elements (resp. skew-symmetric elements) of A with respect to σ.

- $A^{(-)}$ denotes the Lie algebra of an associative algebra A with the Lie bracket defined by $[x,y] = xy - yx$ for all $x, y \in A$ where xy is the usual multiplication of A and $A^{(1)}$ denotes the derived subalgebra of $A^{(-)}$.

- $M_n(A)$ the algebra of $n \times n$ matrices over A and $gl_n(A) = M_n(A)^{(-)}$ denote the Lie algebra of $n \times n$ matrices over A.

- $sl_n(A) = \{x \in gl_n(A) \mid \operatorname{tr} x \in [A,A]\}$.

- M_n the algebra of all $n \times n$-matrices over $\mathbb{F}$ and $E_{i,j}$ denote the matrix units.

- gl_n the general linear algebra and sl_n denote the special linear algebra over $\mathbb{F}$.

- sp_{2n} the symplectic Lie algebra and so_m ($m = 2n+1$ or $2n$) denote the orthogonal Lie algebra over $\mathbb{F}$.

- Let L be an Θ_n-graded Lie algebra and

$$L = (\mathfrak{g}^+ \otimes A^-) \oplus (\mathfrak{g}^- \otimes A^+) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D$$

  see (1.1.1). We identify the $\mathfrak{g}$-modules V and V' with the space $\mathbb{F}^n$ of column vectors with the following actions:

$$x.v = xv \quad \text{for } x \in sl_n,\ v \in V,$$

$$x.v' = -x^t v' \text{ for } x \in sl_n,\ v' \in V'.$$

We identify S and S' (resp. Λ and Λ') with symmetric (resp. skew-symmetric) $n \times n$ matrices. Then S, S', Λ and Λ' are $\mathfrak{g}$-modules under the actions:

$$\begin{aligned}
x.s &= xs + sx^t \text{ for } x \in sl_n,\ s \in S,\\
x.\lambda &= x\lambda + \lambda x^t \text{ for } x \in sl_n,\ \lambda \in \Lambda,\\
x.s' &= -s'x - x^t s' \text{ for } x \in sl_n,\ s' \in S,\\
x.\lambda' &= -\lambda' x - x^t \lambda' \text{ for } x \in sl_n,\ \lambda' \in \Lambda'.
\end{aligned}$$

Denote $\mathscr{A} = A^- \oplus A^+$, $\mathscr{B} := B \oplus B'$, $\mathfrak{a} := \mathscr{A} \oplus C \oplus E \oplus C' \oplus E'$ and $\mathfrak{b} := \mathfrak{a} \oplus \mathscr{B}$. The products on the components of L induces an algebra structure on both $\mathfrak{a}$ and $\mathfrak{b}$.

- We show that $\mathfrak{a}$ is an associative algebra with involution γ with respect to multiplication defined as follows:

$$\alpha_1\alpha_2 := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2}$$

  for all homogeneous $\alpha_1, \alpha_2 \in \mathfrak{a}$ with the products $[\ ]$ and $\circ$ given by Table 4.1.1. Note that $[\alpha_1, \alpha_2] = \alpha_1\alpha_2 - \alpha_2\alpha_1$ and $\alpha_1 \circ \alpha_2 = \alpha_1\alpha_2 + \alpha_2\alpha_1$.

- It can be shown that all products $(\beta_1, \beta_2)_Z$ with $\beta_1, \beta_2 \in B \oplus B'$ or $\beta_1, \beta_2 \in \mathfrak{a}$ are either symmetric or skew-symmetric. This is why we will write $(\beta_1 \circ \beta_2)_Z$ or $[\beta_1, \beta_2]_Z$, respectively, instead of $(\beta_1, \beta_2)_Z$. For $\alpha \in \mathfrak{a}$ and $\beta \in B \oplus B'$ we will write $\alpha\beta$ (resp. $\beta\alpha$) instead of $(\alpha, \beta)_Z$ (resp. $(\beta, \alpha)_Z$) (see Table 3.4.5 and Section 4.2). Let $b \in B$ and $b' \in B$. We define $b\alpha := \gamma(\alpha)b$ and $\alpha b' := b'\gamma(\alpha)$. We show that $B \oplus B'$ is an $\mathfrak{a}$-bimodule.

- Let $b_1, b_2 \in B$ and $b'_1, b'_2 \in B'$. We define

$$\begin{aligned}
b_1b_2 &:= \frac{[b_1, b_2]_C}{2} + \frac{(b_1 \circ b_2)_E}{2}, & b'_1b'_2 &:= \frac{[b'_1, b'_2]_{C'}}{2} + \frac{(b'_1 \circ b'_2)_{E'}}{2},\\
bb' &:= \frac{[b, b']_{A^-}}{2} + \frac{(b \circ b')_{A^+}}{2}, & b'b &:= -\frac{[b, b']_{A^-}}{2} + \frac{(b \circ b')_{A^+}}{2}.
\end{aligned}$$

  Then $\mathfrak{b} = \mathfrak{a} \oplus B \oplus B'$ is an algebra with multiplication extending that on $\mathfrak{a}$ (see Table 4.2.1) .

- $\mathrm{Der}_*(\mathfrak{b}) := \{d \in \mathrm{Der}(\mathfrak{b}) \mid dX \subseteq X \text{ for } X = A^+, A^-, B, \cdots, E'\}$.

- $D_{\mathfrak{b},\mathfrak{b}} = span\{D_{\alpha,\beta} \mid \alpha,\beta \in \mathfrak{b}\}$ where $D_{\alpha,\beta} := \langle \alpha,\beta \rangle$ for $\alpha,\beta \in \mathfrak{b}$ ($\langle , \rangle$ is a surjective map from $\mathfrak{b} \otimes \mathfrak{b}$ to D, see (4.2.5)).

- Let I be the subspace of $\mathfrak{b} \otimes \mathfrak{b}$ spanned by the elements

$$\begin{aligned}
&\alpha \otimes \beta + \beta \otimes \alpha, \\
&\gamma\alpha \otimes \beta + \beta\gamma \otimes \alpha + \alpha\beta \otimes \gamma, \\
&x \otimes y
\end{aligned}$$

where $\alpha,\beta \in \mathfrak{b}$ and $x \in X$ and $y \notin X'$ with $X = B, C, E$ or $x \in A^+$ and $y \in A^-$. Denote $\{\mathfrak{b},\mathfrak{b}\} = \mathfrak{b} \otimes \mathfrak{b}/I$ (resp. $\prec \mathfrak{b},\mathfrak{b} \succ = \{\mathfrak{b},\mathfrak{b}\}/X$) with product $\{\alpha,\beta\} = \alpha \otimes \beta + I$ (resp. $\prec \alpha,\beta \succ = \{\alpha,\beta\} + X$). Denote

$$\begin{aligned}
\mathscr{L}(\mathfrak{b}) &:= (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \ldots \oplus (\Lambda' \otimes E') \oplus D_{\mathfrak{b},\mathfrak{b}}, \\
\widehat{\mathscr{L}(\mathfrak{b})} &:= (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \ldots \oplus (\Lambda' \otimes E') \oplus \{\mathfrak{b},\mathfrak{b}\}, \\
\mathscr{L}(\mathfrak{b},X) &:= (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \ldots \oplus (\Lambda' \otimes E') \oplus \prec \mathfrak{b},\mathfrak{b} \succ,
\end{aligned}$$

see (5.3.3) and Theorems 5.2.5 and 5.3.7.

Chapter 2

Lie algebras graded by finite root systems

In this chapter we review main concepts and results of the theory of Lie algebras graded by finite root systems. Root graded Lie algebras were introduced by Berman and Moody in 1992 to study toroidal Lie algebras and Slodowy intersection matrix algebras [22]. However, this concept appeared previously in Seligman's study of finite-dimensional isotropic simple Lie algebras [46]. He described the multiplicative structure of these Lie algebras and he constructed a model for them.

Recall that any perfect Lie algebra L has a universal central extension which is also perfect, called a *universal covering algebra* of L and any two universal covering algebras of L are isomorphic [32]. Two perfect Lie algebras L_1 and L_2 are said to be *centrally isogenous* if they have the same universal covering algebra (up to isomorphism). One can easily check that every root graded Lie algebra L is perfect (see Theorem 5.1.1). Let (U,ψ) be the universal covering algebra of L. Then U is $(\Delta,\mathfrak{g})$-graded if and only if L is $(\Delta,\mathfrak{g})$-graded (see Theorem 5.1.2). For that reason, root graded Lie algebras of simply-laced finite root systems were classified up to central isogeny by Berman and Moody [22]. In the case of double-laced finite root systems this was finalized by Benkart and Zelmanov [20]. Non-reduced systems BC_n were considered by Allison, Benkart and Gao [4] (for $n \geq 2$) and by Benkart and Smirnov [18] (for $n = 1$). Also, central extensions of these Lie algebras in terms of the homology of its coordinate algebra were determined and described up to isomorphism by Allison, Benkart and Y. Gao [3].

The chapter is organized as follows. First we recall the multiplicative structures and coordinate algebras of Lie algebras graded by finite reduced root systems (Section 2.1). Then we consider some examples (Section 2.2) and state recognition theorem for these Lie algebras (Section 2.3). In Section 2.4 we review Lie algebras graded by non-reduced systems BC_n ($n \geq 2$).

2.1 Root graded Lie algebras

A subalgebra $\mathfrak{h}$ of a Lie algebra L is called a *Cartan subalgebra* if it is nilpotent and self-normalising. A Cartan subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra is said to be *splitting* if the characteristic roots of every $\mathrm{ad}\,h$, $h \in \mathfrak{h}$, are in the base field. A Lie algebra L is called *split* if it contains a splitting Cartan subalgebra [34]. If the base field $\mathbb{F}$ is algebraically closed, then every Cartan subalgebra is a splitting Cartan subalgebra. We start with the definition of Lie algebras graded by finite reduced root systems.

Definition 2.1.1. Recall that a Lie algebra L over a field $\mathbb{F}$ of characteristic zero is *graded by the (reduced) root system* Δ (or is Δ-*graded*) if

($\Delta1$) L contains as a subalgebra a finite-dimensional split simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

($\Delta2$) $L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha$ where $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$;

($\Delta3$) $L_0 = \sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}]$.

The condition ($\Delta2$) in the definition of a Δ-graded Lie algebra can be replaced by:

($\Delta2$)$'$ As a $\mathfrak{g}$-module L is a direct sum of adjoint modules (modules isomorphic to $\mathfrak{g}$), little adjoint modules whose highest weight is the highest short root, or one-dimensional $\mathfrak{g}$-modules; the latter being contained in L_0 [20, 22].

Let L be a Lie algebra graded by the (reduced) root system Δ with grading subalgebra $\mathfrak{g}$ of type Δ. The multiplicative structure and the coordinate algebra of L is obtained as follows.

(1) $\Delta = A_{n-1}$ with $n \geq 3$ ([22] and [3, 4.14]). Note that the Lie algebra L in this case is also Θ_n-graded, so $L \cong (\mathfrak{g} \otimes A) \oplus D$ with the same multiplication as in (3.4.4) with $B = B' = C = C' = E = E = \{0\}$. Here A is an associative (if $n \geq 4$) or alternative (if $n = 3$) algebra over $\mathbb{F}$ and D is the sum of trivial $\mathfrak{g}$-modules (acting by derivations on A).

(2) $\Delta = E_r$ ($r = 6,7,8$) or $\Delta = A_1$ ([22] and [3, 2.34]). Then there is a commutative associative algebra A (or Jordan algebra A if $\Delta = A_1$) over $\mathbb{F}$ such that $L \cong (\mathfrak{g} \otimes A) \oplus D$,

with

$$[x\otimes a, d] = x\otimes ad,$$
$$[x\otimes a, y\otimes a'] = [x,y]\otimes aa' + (x \mid y)\langle a, a'\rangle$$

where $x,y\in\mathfrak{g}$, $a,a'\in A$ and $d,\langle a,a'\rangle\in D$.

(3) $\Delta = B_n$, C_n, or D_n with $n\geq 2$ [20]. Note that L is also BC_n-graded, so $L = (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus(V\otimes C)\oplus D$ (except in the case $\Delta = D_2$ where there are five components) and Theorem 2.4.4 can be used to describe the multiplicative structures and the coordinate algebras of L with

$$\begin{aligned} B &= \{0\} && \text{if } \Delta \text{ is of type } B_n,\\ C &= \{0\} && \text{if } \Delta \text{ is of type } C_n,\\ B = C &= \{0\} && \text{if } \Delta \text{ is of type } D_n. \end{aligned}$$

(4) $\Delta = F_4, G_2$ [20], see Theorems 2.3.5 and 2.3.4.

2.2 Examples of root graded Lie algebras

Example 2.2.1. Let A be an associative commutative $\mathbb{F}$-algebra with unit 1 and let $\mathfrak{g}$ be a split simple Lie algebra of type Δ. Then $L = \mathfrak{g}\otimes A$ is a $(\Delta, \mathfrak{g}\otimes 1)$-graded Lie algebra with respect to the bracket

$$[x\otimes a, y\otimes b] = [x,y]\otimes ab$$

for all $x,y\in\mathfrak{g}$ and $a,b\in A$. More generally, any perfect central extension of $\mathfrak{g}\otimes A$ is also $(\Delta,\mathfrak{g}\otimes 1)$-graded. The universal covering algebra of $\mathfrak{g}\otimes A$ is a generalization of the affine Kac-Moody algebra determined by $\mathfrak{g}$ [20, 0.5].

Example 2.2.2. Seligman showed that any finite-dimensional isotropic (i.e. containing ad-nilpotent elements) simple Lie algebra L over a field of characteristic zero is either Δ-graded or BC_r-graded [46].

Example 2.2.3. Let $L = \mathfrak{g}_1\oplus\mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of L isomorphic to sl_n and let $\mathfrak{g}$ be the diagonal subalgebra of L isomorphic to sl_n. Then L is $(A_{n-1},\mathfrak{g})$-graded. Note that L is not $(A_{n-1},\mathfrak{g}_i)$-graded as it fails to satisfy condition $(\Delta 3)$ in the definition of root graded Lie algebras. We identify the $\mathfrak{g}$-module L with $sl_n\otimes A$ where $A = span\{e_1, e_2\}$. The Lie algebra structure of L gives the following multiplication on $sl_n\otimes A$:

$$[x\otimes e_i, y\otimes e_i] = [x,y]\otimes e_i,$$
$$[x\otimes e_1, y\otimes e_2] = 0,$$

$$[x\otimes 1, y\otimes e_i] = [x\otimes (e_1+e_2), y\otimes e_i] = [x,y]\otimes e_i,$$

for $x,y\in sl_n$ and $i=1,2$. Then A becomes a unital associative algebra with multiplication $e_ie_j = \delta_{i,j}e_i$ $(i,j=1,2)$ and e_1+e_2 is the identity element of A, so $A\cong \mathbb{F}\oplus\mathbb{F}$ (the sum of two ideals).

Example 2.2.4. Let $L = sl_{n+k}$ and let $\mathfrak{g}$ be the copy of sl_n in the northwest corner. We consider the adjoint action of $\mathfrak{g}$ on L. Then the $\mathfrak{g}$-module L decomposes into k copies of the natural module $V=\mathbb{F}^n$, k copies of the dual module $V'=\mathrm{Hom}(V,\mathbb{F})$, an adjoint module $\mathfrak{g}$ and one dimensional trivial $\mathfrak{g}$-modules in its southeast corner. Then

$$L = \mathfrak{g}\oplus V^{\oplus k}\oplus V'^{\oplus k}\oplus D$$

where D is the sum of the trivial sl_n-modules. As a result, we may write

$$L = \mathfrak{g}\oplus (V\otimes B)\oplus (V'\otimes B')\oplus D$$

where $B\cong B'\cong \mathbb{F}^k$. Then L is $(A_{n-1},\mathfrak{g})$-graded. Bahturin and Benkart [5] (for $n>3$) and Benkart and Elduque [14] (for $n=3$) described the multiplicative structure of this type of Lie algebras.

Note that the Lie algebra L in Example 2.2.4 is also (A_{n+k-1},L)-graded. This shows that Lie algebras can be root graded in different ways.

Example 2.2.5. [4] (1) Affine Lie algebras (or more precisely their derived algebras) which have realization as

$$\mathfrak{g}^{aff} = (\mathfrak{g}\otimes\mathbb{F}[t^{\pm 1}])\oplus\mathbb{F}z$$

where $\mathbb{F}[t^{\pm}]$ is the algebra of Laurent polynomials in t over $\mathbb{F}$ and $\mathbb{F}z$ is a one dimensional (non split) center, are Δ-graded.

(2) Toroidal Lie algebras, which can be realized as

$$\mathfrak{g}^{aff} = (\mathfrak{g}\otimes\mathbb{F}[t_1^{\pm 1},\cdots,t_n^{\pm 1}])\oplus Z$$

where Z is an infinite dimensional non-split center, are Δ-graded.

(3) The twisted affine algebras

$$(\mathfrak{g}\otimes F[t^{\pm 2}])\oplus (W\otimes tF[t^{\pm 2}])\oplus Fz \quad (\Delta = B_r, C_r, F_4)$$

and their toroidal counterparts are graded by the root system of $\mathfrak{g}$.

(4) The Tits-Kantor-Koecher Lie algebra

$$K(A) = (sl_2 \otimes A) \oplus [L_A, L_A]$$

of a unital Jordan algebra A where L_A denotes left multiplication by $a \in A$, is graded by $\Delta = A_1$.

2.3 Recognition theorem of root graded Lie algebras

In this section we recall so-called recognition theorems for root graded Lie algebras proved by Berman and Moody [22] for simply laced case and by Benkart and Zelmanov [20] for double laced case. To state the theorems in a unified way we mainly use [3] and [20] as our source. Recall that two perfect Lie algebras L_1 and L_2 are said to be *centrally isogenous* if they have the same universal covering algebra (up to isomorphism).

Theorem 2.3.1 (Recognition theorem for type A_n and D_n)**.** *[22] Let L be a Lie algebra over $\mathbb{F}$ graded by a simply-laced finite root system Δ of rank $n \geqq 2$.*

(a) If $\Delta = D_n, n \geqq 4$ or if $\Delta = E_6, E_7, E_8$, then there exists a commutative associative unital $\mathbb{F}$-algebra A such that L is centrally isogenous with $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is the split simple Lie algebra with root system Δ.

(b) If $\Delta = A_n, n \geqq 3$, then there exists a unital associative $\mathbb{F}$-algebra A such that L is centrally isogenous with $e_{n+1}(A)$ where $e_{n+1}(A)$ is the ideal of $gl_{n+1}(A)$ generated by the elements $aE_{i,j}$, $a \in A$ and $i \neq j$.

(c) If $\Delta = A_2$, then L is centrally isogenous with Steinberg Lie algebra $st_3(A)$, where A is a unital alternative $\mathbb{F}$-algebra.

Theorem 2.3.2 (Recognition theorem for type B_n)**.** *[20] Let L be a Lie algebra over $\mathbb{F}$ graded by B_n for $n \geqq 3$. Then there exists a unital, commutative, associative $\mathbb{F}$-algebra A and an A-module B having a symmetric A-bilinear form $(,) : B \times B \to A$ such that L is centrally isogenous with the Lie algebra*

$$T(J(V)/\mathbb{F}, J(B)/A) = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D_{J(B),J(B)}$$

where V is $(2n+1)$-dimensional $\mathbb{F}$-vector space with a nondegenerate symmetric bilinear form (the defining representation for B_n), $\mathfrak{g}$ is the set of skew-symmetric transformations on V relative to the form on V, and $D_{J(B),J(B)}$ is the Lie algebra of inner derivations on the Jordan algebra $J(B) = A \oplus B$.

Theorem 2.3.3 (Recognition theorem for type C_n)**.** *[20] Let L be a Δ-graded Lie algebra over $\mathbb{F}$.*

(a) If $\Delta = C_n, n \geqq 4$, then there exists a unital, associative algebra A with an involution $* : A \to A$ such that L is centrally isogenous with the algebra $sp_{2n}(A,*)$ of symplectic $(2n) \times (2n)$ matrices over A.

(b) If $\Delta = C_3$, then L is centrally isogenous with the symplectic Steinberg algebra $st\,sp_6(A,*)$, where A is an alternative involutive algebra whose symmetric elements, $\{a \in A \mid a^* = a\}$, lie in the associative center of A.

(c) If $\Delta = C_2$, then L is centrally isogenous with a Tits-Kantor-Koecher construction of a unital Jordan algebra J which contains the Jordan algebra of symmetric 2×2 matrices, and the identity of J lies in this subalgebra.

(d) If $\Delta = C_1 = A_1$, then L is centrally isogenous with a Tits-Kantor-Koecher construction of a unital Jordan algebra J.

Theorem 2.3.4 (Recognition theorem for type G_2)**.** *[20] Let L be a G_2-graded Lie algebra over $\mathbb{F}$. Assume $\mathfrak{g}$ is the split simple Lie algebra of type G_2, which we identify with (inner) derivations of the 8-dimensional alternative algebra $\mathscr{O}$ of split octonions over $\mathbb{F}$. Then there exist a unital commutative associative $\mathbb{F}$-algebra A and a Jordan algebra $\mathfrak{a}$ over A with a normalized trace satisfying the Cayley-Hamilton trace identity $ch_3(x) = 0$ of degree 3 such that L is centrally isogenous with the Lie algebra*

$$T(\mathscr{O}/\mathbb{F}, \mathfrak{a}/A) = (\mathfrak{g} \otimes A) \oplus (\mathscr{O}_0 \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$$

where B is the set of trace zero elements in $\mathfrak{a}$ and $D_{\mathfrak{a},\mathfrak{a}}$ is the Lie algebra of inner derivations of $\mathfrak{a}$.

Theorem 2.3.5 (Recognition theorem for type F_4)**.** *[20] Let L be an F_4-graded Lie algebra over $\mathbb{F}$. Assume $\mathfrak{g}$ is the split simple Lie algebra of type F_4, which we identify with the (inner) derivation algebra of the split exceptional 27-dimensional Jordan algebra $\mathscr{J}$ over $\mathbb{F}$. Then there exist a unital commutative associative $\mathbb{F}$-algebra A and an alternative algebra $\mathfrak{a}$ over A with a normalized trace satisfying the Cayley-Hamilton trace identity $ch_2(x) = 0$ of degree 2 such that L is centrally isogenous with the Lie algebra*

$$T(\mathscr{J}/A, \mathfrak{a}/\mathbb{F}) = (\mathfrak{g} \otimes A) \oplus (\mathscr{J}_0 \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$$

where B is the set of trace zero elements in $\mathfrak{a}$ and $D_{\mathfrak{a},\mathfrak{a}}$ is the Lie algebra of inner derivations of $\mathfrak{a}$.

2.4 BC_r-graded Lie algebras

Lie algebras graded by non-reduced root systems BC_r ($r \geq 2$) were classified by Allison, Benkart, and Gao [4]. The grading subalgebra $\mathfrak{g}$ is a simple Lie algebra of type B_r,C_r or D_r. Thus

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta_X} \mathfrak{g}_\alpha$$

where Δ_X is a root system of type $X = B, C$ or D. Let Δ_{BC} denotes the system BC_r. Recall that

$$\begin{aligned}
\Delta_B &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\varepsilon_i \mid i = 1, 2, .., r\}, \\
\Delta_C &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm 2\varepsilon_i \mid i = 1, 2, .., r\}, \\
\Delta_D &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq r\} \\
\Delta_{BC} &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\varepsilon_i, \pm 2\varepsilon_i \mid i = 1, 2, .., r\}
\end{aligned}$$

in terms of ε's (see Bourbaki [26]).

Definition 2.4.1. A Lie algebra L over a field F of characteristic zero is *graded by the root system* BC_r or is BC_r*-graded* if:

(1) L contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_X} \mathfrak{g}_\alpha$ whose root system relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Δ_X, $X = B, C$ or D.

(2) $L = \bigoplus_{\alpha \in \Delta_{BC} \cup \{0\}} L_\alpha$ where $L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}\}$ for $\alpha \in \Delta_{BC}$.

(3) $L_0 = \sum_{\alpha \in \Delta_{BC}} [L_\alpha, L_{-\alpha}]$.

Example 2.4.2. Any Lie algebra which is graded by a finite root system of type B_r, C_r, or D_r is also BC_r-graded with grading subalgebra of type B_r, C_r, or D_r, respectively. For such a Lie algebra L, the space $L_\mu = \{0\}$ for all μ not in Δ_B, Δ_C or Δ_D, respectively.

Remark 2.4.3. For $r \geq 2$, a BC_r-graded Lie algebra L with grading subalgebra $\mathfrak{g}$ is a direct sum of the modules $\mathfrak{g} = V(2\omega_1)$, $\mathfrak{s} = V(\omega_2)$, $V = V(\omega_1)$ and $V(0)$, except in the case $\Delta_{BC} = D_2$ where there are five isotypic components [4]. The components can be parametrized by subspaces A, B, C, and D so that $L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D$, where D is the centralizer of $\mathfrak{g}$ in L. Let $n = \dim V$, so $n = 2r$ or $2r + 1$. Since V is a natural $\mathfrak{g}$-module, the algebra $\mathfrak{g}$ is defined by a non degenerate $\mathfrak{g}$-invariant bilinear form $(|)$ on V which is symmetric of maximal Witt index or is skew-symmetric. Set $\rho = 1$ if the form is symmetric, and $\rho = -1$ if it is skew-symmetric, so that

$$(v \mid u) = \rho\,(u \mid v) \text{ for all } u, v \in V.$$

Then

$$\mathfrak{g} = \{x \in \mathrm{End}_F(V) \mid (xu \mid v) = -(u \mid xv) \text{ for all } u, v \in V\},$$

$$\mathfrak{s} = \{s \in \mathrm{End}_F(V) \mid (su \mid v) = (u \mid sv) \text{ for all } u, v \in V \text{ and } \mathrm{tr}(s) = 0\},$$

and $\mathfrak{g}$ is a split simple Lie algebra. When

(1) $n = 2r+1$ and $\rho = 1$, then $\mathfrak{g}$ has type B_r.

(2) $n = 2r$ and $\rho = -1$, then $\mathfrak{g}$ has type C_r.

(3) $n = 2r$ and $\rho = 1$, then $\mathfrak{g}$ has type D_r.

Theorem 2.4.4 (Multiplicative structure and coordinate algebra for type BC_r)**.** *[4] Suppose that L is a BC_r-graded Lie algebra for $r \geq 3$ with grading subalgebra $\mathfrak{g}$ (not of type D_3) over $\mathbb{F}$. Then there exists an $\mathbb{F}$-algebra $\mathfrak{a}$ with involution η having symmetric elements A and skew symmetric elements B relative to η, an $\mathfrak{a}$-module C, an $\mathfrak{a}$-sesquilinear form $\chi(\,,\,)$ on C so that*

(a) $\mathfrak{a}$ is associative unless $r = 3$ and $\mathfrak{g}$-has type C_3 in which case $\mathfrak{a}$ is alternative and A is contained in the nucleus (associative center) of $\mathfrak{a}$;

(b) C is an associative $\mathfrak{a}$-module and $\chi(\,,\,)$ is hermitian (skew-hermitian) if the form on V is symmetric (skew-symmetric);

(c) $L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D$ and we may suppose that there exist commutative products

$a \otimes a' \mapsto a \circ a' \in A \qquad b \otimes b' \mapsto b \circ b' \in A$

anti commutative products

$a \otimes a' \mapsto [a, a'] \in B \qquad b \otimes b' \mapsto [b, b'] \in B \qquad a \otimes a' \mapsto \langle a, a' \rangle \in D \qquad b \otimes b' \mapsto \langle b, b' \rangle \in D$

products

$a \otimes b \mapsto [a, b] \in A \qquad a \otimes b \mapsto a \circ b \in B \qquad a \otimes c \mapsto a.c \in C \qquad b \otimes c \mapsto b.c \in C$

$c \otimes c' \mapsto c \star c' = \rho c' \star c \in A \quad c \otimes c' \mapsto c \diamond c' = -\rho c' \diamond c \in B \quad c \otimes c' \mapsto \langle c, c' \rangle = -\rho \langle c', c \rangle \in D$

$d \otimes a \mapsto da \in A \qquad d \otimes b \mapsto db \in B \qquad d \otimes c \mapsto dc \in C$

so that the multiplication in L is given as follows. For all $x, y \in \mathfrak{g}$, $s, u \in \mathfrak{s}$, $a \in A$, $b \in B$, $c \in C$, $d \in D$,

$$[x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2} a \circ a' + x \circ y \otimes \frac{1}{2}[a, a'] + \mathrm{tr}(xy)\langle a, a' \rangle$$

$$[x \otimes a, s \otimes b] = x \circ s \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2} a \circ b = -[s \otimes b, x \otimes a],$$

$$[s \otimes b, t \otimes b'] = [s, t] \otimes \frac{1}{2} b \circ b' + s \circ t \otimes \frac{1}{2}[b, b'] + \mathrm{tr}(st)\langle b, b' \rangle,$$

$$[x \otimes a, u \otimes c] = xu \otimes a.c = -[u \otimes c, x \otimes a],$$

$$[s \otimes b, u \otimes c] = su \otimes b.c = -[u \otimes c, s \otimes b],$$

$$
\begin{aligned}
&[u\otimes c, v\otimes c] = \gamma_{u,v}\otimes c\star c' + \sigma_{u,v}\otimes c\diamond c' + (u \mid v)\langle c, c'\rangle,\\
&[d, x\otimes a] = x\otimes da = -[x\otimes a, d],\\
&[d, s\otimes b] = s\otimes db = -[s\otimes b, d],\\
&[d, u\otimes c] = u\otimes dc = -[u\otimes c, d],\\
&[d_1, d_2] = d_1 d_2 - d_2 d_1,
\end{aligned}
$$

where

$$
\begin{aligned}
c\star c' &= \frac{1}{2}(\chi(c,c') + \eta(\chi(c,c'))),\\
c\diamond c' &= \frac{1}{2}(\chi(c,c') - \eta(\chi(c,c'))),\\
\gamma_{u,v}(w) &= \frac{1}{2}((v \mid w)u - (w \mid u)v),\\
\gamma_{u,v}(w) &= \frac{1}{2}((v \mid w)u + (w \mid u)v) - \frac{1}{2n}\operatorname{tr}((v \mid w)u + (w \mid u)v)I.
\end{aligned}
$$

Moreover,

$$D = \langle \mathfrak{b}, \mathfrak{b}\rangle = \langle A, A\rangle + \langle B, B\rangle + \langle C, C\rangle,$$

and $\langle \beta, \beta'\rangle\beta'' = D_{\beta,\beta'}\beta''$ *for all* $\beta, \beta', \beta'' \in \mathfrak{b}$,*where* $D_{\beta,\beta'}\beta'' \in \mathrm{Der}_*(\mathfrak{b})$ *is defined by*

$$
\begin{aligned}
D_{\alpha,\alpha'}\alpha'' &= \frac{[[\alpha,\alpha'] + [\eta(\alpha),\eta(\alpha')],\alpha''] + 3(\alpha,\alpha'',\alpha') + 3(\eta(\alpha),\alpha'',\eta(\alpha'))}{2n},\\
D_{\alpha,\alpha'}c &= \frac{([\alpha,\alpha'] + [\eta(\alpha),\eta(\alpha')])c}{2n},\\
D_{c,c'}\alpha &= \frac{\rho}{n}[c\diamond c', \alpha],\\
D_{c,c'}c'' &= \frac{1}{2}(\eta(\chi(c',c'').c - \chi(c'',c).c') + \frac{\rho}{2n}(\chi(c,c') - \eta(\chi(c,c')).c''),
\end{aligned}
$$

for all $\alpha, \alpha', \alpha'' \in \mathfrak{a}$, $c, c', c'' \in C$ *and* $D_{\mathfrak{a},C} = D_{C,\mathfrak{a}} = (0)$.

Theorem 2.4.5 (Recognition theorem for type BC_r)**.** *[4] Suppose that* $r \geq 3$ *and* $\mathfrak{g}$ *does not have type* C_3 *or* D_3*. A Lie algebra* L *is a* BC_r*-graded Lie algebra with grading subalgebra* $\mathfrak{g}$ *if and only if there exist an associative algebra* $\mathfrak{a}$ *with involution, an* $\mathfrak{a}$*-module* C *so that* L *is centrally isogenous to the* BC_r*-graded unitary Lie algebra of the* ρ*-hermitian form* $\xi = w\perp - \rho\chi$ *on the* $\mathfrak{a}$*-module* $\mathfrak{a}^n \oplus \mathscr{B}$ *(see [4, Example 1.23]).*

Chapter 3

Generalized root graded Lie algebras

We start with the general definition of Lie algebras graded by finite weight systems.

Definition 3.0.1. [6] Let Δ be a root system and let Γ be a finite set of integral weights of Δ containing Δ and $\{0\}$. A Lie algebra L is called $(\Gamma,\mathfrak{g})$-*graded* (or simply Γ-*graded*) if

($\Gamma 1$) L contains as a subalgebra a non-zero finite-dimensional split semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

($\Gamma 2$) $L = \bigoplus_{\alpha\in\Gamma} L_\alpha$ where $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$;

($\Gamma 3$) $L_0 = \sum_{\alpha,-\alpha\in\Gamma\setminus\{0\}} [L_\alpha, L_{-\alpha}]$.

The subalgebra $\mathfrak{g}$ is called the *grading subalgebra* of L. A Lie algebra L is called $(\Gamma,\mathfrak{g})$-*pregraded* if it satisfies ($\Gamma 1$) and ($\Gamma 2$) (but not necessarily ($\Gamma 3$)). Note that the condition ($\Gamma 2$) yields $[L_\mu, L_\nu] \subseteq L_{\mu+\nu}$ if $\mu+\nu \in \Gamma$ and $[L_\mu, L_\nu] = 0$ otherwise. We denote by $\ll \mathfrak{g} \gg_L$ (or simply $\ll \mathfrak{g} \gg$) the ideal generated by $\mathfrak{g}$ in L. Note that a $(\Gamma,\mathfrak{g})$-pregraded Lie algebra L is $(\Gamma,\mathfrak{g})$-graded if and only if $\ll \mathfrak{g} \gg = L$, see Proposition 3.1.3.

3.1 Basic properties of Γ-graded Lie algebras

The following is well-known (see for example [6, Lemma 4.2]).

Lemma 3.1.1. *Let $\mathfrak{g}$ be a split simple subalgebra of a Lie algebra L. Assume that a Lie algebra L is $(\Gamma,\mathfrak{g})$-pregraded. Then the space*

$$I = \bigoplus_{\alpha\in\Gamma\setminus\{0\}} L_\alpha + \sum_{\alpha_1,-\alpha\in\Gamma\setminus\{0\}} [L_\alpha, L_{-\alpha}]$$

is a non-zero Γ-graded ideal of L. In particular, if L is simple then it is Γ-graded.

Lemma 3.1.2. *Let L be a Lie algebra containing a non-zero split semisimple subalgebra $\mathfrak{g}$. Then L is $(\Gamma,\mathfrak{g})$-pregraded for some finite set Γ if and only if there exists a finite set Q of dominant weights of $\mathfrak{g}$ such that L is the direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weights are in Q, i.e. as a $\mathfrak{g}$-module,*

$$L \cong \bigoplus_{\lambda\in Q} V(\lambda)\otimes W_\lambda$$

for some vector spaces W_λ (the vector space W_λ indexes the copies of $V(\lambda)$ and the $\mathfrak{g}$-action is given by

$$x.(v_\lambda\otimes w_\lambda) = [x, v_\lambda\otimes w_\lambda] = x.v_\lambda\otimes w_\lambda$$

for $x\in\mathfrak{g}$, $v_\lambda\in V(\lambda)$ and $w_\lambda\in W_\lambda$).

Proof. The "if" part is obvious with Γ being the union of the weights of the modules $V(\lambda)$, $\lambda\in Q$.

For the converse, it is enough to show that every finite-dimensional subspace U of L is contained in a finite-dimensional $\mathfrak{g}$-submodule M of L. Indeed, by enlarging if necessary one can assume that U is a weighted subspace. Let $\{u_1,\dots,u_k\}$ be a basis of U consisting of weight vectors. It is enough to show that each u_i belongs to a a finite-dimensional $\mathfrak{g}$-submodule M_i of L. Put $M_i = U(\mathfrak{g})u_i$. Following [13, Lemma 2.2], once we fix an ordering of the roots of $\mathfrak{g}$, there is a triangular decomposition $\mathfrak{g} = \mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ ($\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$) and

$$M_i = U(\mathfrak{g})u_i = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)u_i.$$

But $\dim(U(\mathfrak{n}^+)u_i)$ is finite, since $\dim U(\mathfrak{n}^+)_\nu$ is finite for any $\nu\in\mathbb{Z}\Delta$, and L has only finitely many $\mathfrak{h}$-weight spaces. Also, $U(\mathfrak{h})U(\mathfrak{n}^+)u_i = U(\mathfrak{n}^+)u_i$ because the action of $U(\mathfrak{h})$ is diagonalizable, and again $\dim(U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)u_i)$ is finite by the same weight argument as above.

□

Proposition 3.1.3. *Let $\mathfrak{g}$ be a split simple subalgebra of a Lie algebra L and suppose L is $(\Gamma,\mathfrak{g})$-pregraded. Then the following are equivalent.*

(1) L is Γ-graded.

(2) $L_0 = \sum_{\alpha,-\alpha\in\Gamma\setminus\{0\}} [L_\alpha, L_{-\alpha}]$.

(3) $L = \bigoplus_{\alpha\in\Gamma\setminus\{0\}} L_\alpha + \sum_{\alpha,-\alpha\in\Gamma\setminus\{0\}} [L_\alpha, L_{-\alpha}]$.

(4) $\ll \mathfrak{g} \gg = L$.

Proof. $(1) \Leftrightarrow (2)$ and $(2) \Leftrightarrow (3)$ follows from Definition 3.0.1.

$(3) \Rightarrow (4)$: Note that $L_\alpha = [\mathfrak{h}, L_\alpha] \subseteq \ll \mathfrak{g} \gg$ for all $\alpha \neq 0$. Since $\ll \mathfrak{g} \gg$ is a subalgebra of L, we have $L = \bigoplus_{\alpha \in \Gamma \setminus \{0\}} L_\alpha + \sum_{\alpha, -\alpha \in \Gamma \setminus \{0\}} [L_\alpha, L_{-\alpha}] \subseteq \ll \mathfrak{g} \gg \subseteq L$, so $\ll \mathfrak{g} \gg = L$.

$(4) \Rightarrow (3)$: By Lemma 3.1.1, $\bigoplus_{\alpha \in \Gamma \setminus \{0\}} L_\alpha + \sum_{\alpha, -\alpha \in \Gamma \setminus \{0\}} [L_\alpha, L_{-\alpha}]$ is an ideal of L containing $\mathfrak{g}$, so (4) implies (3). □

Corollary 3.1.4. *Let $\mathfrak{g}$ be a split simple finite dimensional subalgebra of a simple Lie algebra L and let Γ be the set of all weights of the $\mathfrak{g}$-module L. Suppose L is $(\Gamma, \mathfrak{g})$-pregraded. Then L is $(\Gamma, \mathfrak{g})$-graded.*

Proposition 3.1.5. *Suppose L is $(\Gamma_1, \mathfrak{g}_1)$-graded and $\mathfrak{g}_1$ is $(\Gamma_2, \mathfrak{g}_2)$-graded. Then L is $(\Gamma_3, \mathfrak{g}_2)$-graded where Γ_3 is the set of all weights of the $\mathfrak{g}_2$-module L.*

Proof. We only need to check the condition $(\Gamma 3)$ of the definition, $(\Gamma 1)$ and $(\Gamma 2)$ being obvious. By Lemma 3.1.3, $\ll \mathfrak{g}_1 \gg_L = L$ and $\ll \mathfrak{g}_2 \gg_{\mathfrak{g}_1} = \mathfrak{g}_1$, so

$$\ll \mathfrak{g}_2 \gg_L = \ll \ll \mathfrak{g}_2 \gg_{\mathfrak{g}_1} \gg_L = \ll \mathfrak{g}_1 \gg_L = L.$$

Using Lemma 3.1.3 again we get $(\Gamma 3)$, as required. □

Lemma 3.1.6. *Let L_i be $(\Gamma_i, \mathfrak{g}_i)$-graded for $i = 1, 2$. Suppose that $\mathfrak{g}_1 \cong \mathfrak{g}_2$. Then $L_1 \oplus L_2$ is $(\Gamma_1 \cup \Gamma_2, \mathfrak{g})$-graded for some subalgebra $\mathfrak{g}$ isomorphic to $\mathfrak{g}_1$.*

Proof. Let $L = L_1 \oplus L_2$ and let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ $(i = 1, 2)$ be any isomorphism. Denote

$$\mathfrak{g} = \{x + f(x) \mid x \in \mathfrak{g}_1\}.$$

Then $\mathfrak{g}$ is a subalgebra of L isomorphic to $\mathfrak{g}_1$. We claim that L is $(\Gamma_1 \cup \Gamma_2, \mathfrak{g})$-graded. By Lemma 3.1.2, L_1 and L_2 are the direct sums of finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weights are in Γ_1 and Γ_2, respectively. Note that $\Gamma_1 \cup \Gamma_2$ is finite and $L = \bigoplus_{\alpha \in \Gamma_1 \cup \Gamma_2} L_\alpha$, so $(\Gamma 2)$ holds. It remains to prove $(\Gamma 3)$, or equivalently, that $\ll \mathfrak{g} \gg_L = L$. By Proposition 3.1.3, $\ll \mathfrak{g}_1 \gg_{L_1} = L_1$ and $\ll \mathfrak{g}_2 \gg_{L_2} = L_2$. We have $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i] = [\mathfrak{g}_i, \mathfrak{g}] \subseteq \ll \mathfrak{g} \gg$, so

$$L = L_1 \oplus L_2 = \ll \mathfrak{g}_1 \gg_{L_1} \oplus \ll \mathfrak{g}_2 \gg_{L_2} \subseteq \ll \mathfrak{g} \gg.$$

Therefore, $\ll \mathfrak{g} \gg_L = L$. □

Lemma 3.1.7. *Let S be a finite-dimensional simple Lie algebra and let $\mathfrak{g}$ be a non-zero split semisimple subalgebra of S. Then S is $(\Gamma, \mathfrak{g})$-graded where Γ is the set of all weights of the $\mathfrak{g}$-module S.*

Proof. This follows from Lemma 3.1.1. □

Lemma 3.1.8. *Every non-zero finite-dimensional split semisimple Lie algebra is (Γ, sl_2)-graded for some Γ.*

Proof. Let L be a non-zero finite-dimensional split semisimple Lie algebra. Then, $L = S_1 \oplus S_2 \oplus \cdots \oplus S_k$ where S_i are split simple ideals. Note that each S_i is (Γ, sl_2)-graded (just fix any subalgebra $\mathfrak{g}_i \cong sl_2$ of S_i and use Lemma 3.1.7). It remains to apply Lemma 3.1.6. □

Theorem 3.1.9. *Let L be finite-dimensional perfect Lie algebra L and let Q be a Levi subalgebra of L over an algebraically closed field of characteristic zero. Then*

(1) *L is (Γ_1, Q)-graded for some Γ_1.*

(2) *L is (Γ, sl_2)-graded for some Γ.*

Proof. (1) let R be the solvable radical of L. Then $L = Q \oplus R$. Note that L is (Γ_1, Q)-pregraded where Γ_1 is the set of weights of the Q-module L. Since R is solvable,

$$L/\ll Q\gg = (\ll Q\gg + R)/\ll Q\gg \cong R/(\ll Q\gg \cap R)$$

is solvable. But $L/\ll Q\gg$ is perfect, so $L/\ll Q\gg = \{0\}$ and $L = \ll Q\gg$. By Proposition 3.1.3, L is (Γ_1, Q)-graded.

(2) This follows from Lemma 3.1.8 and Proposition 3.1.5. □

3.2 Θ_n-graded and BC_n-graded Lie algebras

In this section we discuss the relationship between Θ_n-graded and BC_n-graded Lie algebras. Let $\mathfrak{g}$ be a split simple Lie algebra of classical type A_n, B_n, C_n or D_n. Throughout this thesis, $\{\omega_1, \ldots, \omega_n\}$ is the set of the fundamental weights of $\mathfrak{g}$; $V_{\mathfrak{g}}(\omega)$ (or simply $V(\omega)$) denotes the highest weight $\mathfrak{g}$-module of weight ω; $V_{\mathfrak{g}} := V_{\mathfrak{g}}(\omega_1)$ (or simply V) is the natural $\mathfrak{g}$-module; if M is a $\mathfrak{g}$-module then M' is its dual and $\mathscr{W}(M)$ is the set of weights of M. If $\mathfrak{g}$ is of type A_{n-1}, we will use the following notations for the $\mathfrak{g}$-modules below:

$$\mathfrak{g} := V(\omega_1 + \omega_{n-1}),\ V := V(\omega_1),\ S := V(2\omega_1),\ \Lambda := V(\omega_2) \text{ and } T := V(0).$$

Note that $V' \cong V(\omega_{n-1})$, $S' \cong V(2\omega_{n-1})$ and $\Lambda' \cong V(\omega_{n-2})$.

Recall that a Lie algebra L is $(\Gamma, \mathfrak{g})$-pregraded if it satisfies $(\Gamma 1)$ and $(\Gamma 2)$ of Definition 3.0.1. It is easy to see that BC_n-pregraded Lie algebras have the following decomposition, see for example [4, 2.5].

Proposition 3.2.1. *Let L be a Lie algebra and let $\mathfrak{b}$ be a split simple subalgebra of L of type type B_n, C_n ($n \geq 2$) or D_n ($n \geq 3$). Then L is $(BC_n \cup \{0\}, \mathfrak{b})$-pregraded if and only if the $\mathfrak{b}$-module L is a direct sum of copies of $V_{\mathfrak{b}}(2\omega_1)$, $V_{\mathfrak{b}}(\omega_2)$, $V_{\mathfrak{b}}(\omega_1)$ and $V_{\mathfrak{b}}(0)$.*

A similar decomposition exists for Θ_n-pregraded Lie algebras.

Proposition 3.2.2. *Let L be a Lie algebra and let $\mathfrak{g}$ be a subalgebra of L isomorphic to sl_n. Then L is $(\Theta_n, \mathfrak{g})$-pregraded if and only if the $\mathfrak{g}$-module L is a direct sum of copies of $\mathfrak{g}$, V, V', S, S', Λ, Λ' and T.*

Proof. We only need to prove the "only if" part, the "if" part being obvious. Suppose L is $(\Theta_n, \mathfrak{g})$-graded. Then by Lemma 3.1.2, L is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. Note that only the following dominant weights appear in Θ_n:

$$\omega_1 + \omega_{n-1},\ \omega_1,\ \omega_{n-1},\ 2\omega_1,\ 2\omega_{n-1},\ \omega_2,\ \omega_{n-2},\ 0$$

where $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. They are the highest weights of the modules $\mathfrak{g}$, V, V', S, S', Λ, Λ' and T, respectively. $\square$

Suppose L is $(\Theta_n, \mathfrak{g})$-graded. By collecting isomorphic summands of L into isotypic components, we may assume that there are vector spaces $A, B, B', C, C'E, E'$ such that

$$L \cong (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D \quad (3.2.1)$$

where D is the sum of the trivial $\mathfrak{g}$-modules (and also the centralizer of $\mathfrak{g}$ in L).

Remark 3.2.3. Recall that $\mathscr{W}(M)$ denotes the set of weights of a $\mathfrak{g}$-module M and M' denotes the dual of M.

(1) Let $\mathfrak{k}$ be a simple Lie algebra of type type B_r, C_r or D_r and let

$$\Gamma_{\mathfrak{k}} := \mathscr{W}((T \oplus V_{\mathfrak{k}}) \otimes (T \oplus V_{\mathfrak{k}})).$$

Then $\Gamma_{\mathfrak{k}} = BC_r \cup \{0\}$.

(2) Let $\mathfrak{g}$ be a simple Lie algebra of type A_{n-1} and let

$$\Gamma_{\mathfrak{g}} := \mathscr{W}((T \oplus V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}}) \otimes (T \oplus V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}})).$$

Then $\Gamma_{\mathfrak{g}} = \Theta_n$.

(3) Let $\mathfrak{g} \cong sl_n$ and let $\mathfrak{k} \cong so_n$ be a naturally embedded subalgebra of $\mathfrak{g}$. Then $V_{\mathfrak{g}} \downarrow \mathfrak{k} \cong V_{\mathfrak{k}}$, $V'_{\mathfrak{g}} \downarrow \mathfrak{k} \cong V_{\mathfrak{k}}$ and

$$\Gamma_{\mathfrak{g}} \downarrow \mathfrak{k} = \mathscr{W}((T \oplus V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}}) \otimes (T \oplus V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}}) \downarrow \mathfrak{k}) = \mathscr{W}((T \oplus V_{\mathfrak{k}}) \otimes (T \oplus V_{\mathfrak{k}})) = \Gamma_{\mathfrak{k}}.$$

(4) Let $\mathfrak{k} \cong so_{2n+1}$, so_{2n} or sp_{2n} and let $\mathfrak{g} \cong sl_n$ be a naturally embedded subalgebra of $\mathfrak{k}$. Then $V_{\mathfrak{k}} \downarrow \mathfrak{g} \cong V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}}$ (or $V_{\mathfrak{g}} \oplus V'_{\mathfrak{g}} \oplus T$ if $\mathfrak{k} \cong so_{2n+1}$) and $\Gamma_{\mathfrak{k}} \downarrow \mathfrak{g} = \Gamma_{\mathfrak{g}}$.

Theorem 3.2.4. *Let $n \geq 2$ and $r = \lfloor \frac{n}{2} \rfloor$. Then every Θ_n-graded Lie algebra is BC_r-graded.*

Proof. Suppose L is $(\Theta_n, \mathfrak{g})$-graded. Let $\mathfrak{k} \cong so_n$ be a naturally embedded subalgebra of $\mathfrak{g} \cong sl_n$. Note that the rank of $\mathfrak{k}$ is $r = \lfloor \frac{n}{2} \rfloor$ and sl_n is $(BC_r \cup \{0\}, \mathfrak{k})$-graded. By Proposition 3.1.5, we only need to show that the set of all weights of the $\mathfrak{k}$-module L is a subset of $BC_r \cup \{0\}$. Using Remark 3.2.3, we get

$$\mathscr{W}(L \downarrow \mathfrak{k}) = \mathscr{W}(L \downarrow \mathfrak{g}) \downarrow \mathfrak{k} \subseteq \Theta_n \downarrow \mathfrak{k} = \Gamma_{\mathfrak{g}} \downarrow \mathfrak{k} = \Gamma_{\mathfrak{k}} = BC_r \cup \{0\},$$

as required. □

Remark 3.2.5. Suppose L is $(\Theta_n, \mathfrak{g})$-graded ($n \geq 5$). Let $\mathfrak{k} \cong so_n$ be a naturally embedded subalgebra of $\mathfrak{g} \cong sl_n$. As shown in the proof of Theorem 3.2.4, the algebra L is BC_r-graded with respect to the grading subalgebra $\mathfrak{k}$ with $r = \lfloor \frac{n}{2} \rfloor$. The general theory of BC_r-graded Lie algebras gives multiplication structure of L in terms of $\mathfrak{k}$-decomposition components. We are going to show that the multiplication structure of L as an $(\Theta_n, \mathfrak{g})$-graded algebra is "finer" and more specific. Let $V_{\mathfrak{k}}(\lambda)$ denote the simple $\mathfrak{k}$-module with highest weight λ. We have

$$\begin{aligned}
&V_{\mathfrak{g}}(\omega_1) \downarrow_{\mathfrak{k}} \cong V_{\mathfrak{g}}(\omega_n) \downarrow_{\mathfrak{k}} \cong V_{\mathfrak{k}}, \\
&V_{\mathfrak{g}}(2\omega_1) \downarrow_{\mathfrak{k}} \cong V_{\mathfrak{g}}(2\omega_n) \downarrow_{\mathfrak{k}} \cong \mathfrak{s} + T, \\
&V_{\mathfrak{g}}(\omega_2) \downarrow_{\mathfrak{k}} \cong V_{\mathfrak{g}}(\omega_{n-1}) \downarrow_{\mathfrak{k}} \cong \mathfrak{k}, \\
&V_{\mathfrak{g}}(\omega_1 + \omega_n)) \downarrow_{\mathfrak{k}} \cong \mathfrak{k} + \mathfrak{s}
\end{aligned} \tag{3.2.2}$$

where $T = V_{\mathfrak{k}}(0)$, $\mathfrak{k} = V_{\mathfrak{k}}(\omega_2)$, $\mathfrak{s} = V_{\mathfrak{k}}(2\omega_1)$ and $V_{\mathfrak{k}} = V_{\mathfrak{k}}(\omega_1)$. By combining (3.2.1) and (3.2.2), we can rewrite L as a $\mathfrak{k}$-module as follows:

$$L = (\mathfrak{k} \otimes (A \oplus E \oplus E')) \oplus (\mathfrak{s} \otimes (A \oplus C \oplus C')) \oplus (V_{\mathfrak{k}} \otimes (B \oplus B')) \oplus D' \tag{3.2.3}$$

where $D' = (T \otimes (C \oplus C')) \oplus D$. Set $\mathfrak{a} = \mathfrak{A} \oplus \mathfrak{B}$ where $\mathfrak{A} = A \oplus E \oplus E'$ and $\mathfrak{B} = A \oplus C \oplus C'$. Then $\mathfrak{a} = \mathfrak{A} + \mathfrak{B}$ is an associative algebra with involution $*$ given by $a^* = a$ and $b^* = -b$

for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (see [4]). If we wish to calculate the product $[\mathfrak{k}\otimes E, \mathfrak{k}\otimes E]$ in L using BC_r-grading structure then we can only say that

$$[\mathfrak{k}\otimes E, \mathfrak{k}\otimes E] \subseteq (\mathfrak{k}\otimes(A\oplus E\oplus E'))\oplus(\mathfrak{s}\otimes(A\oplus C\oplus C'))\oplus D'.$$

On the other hand, Θ_n-grading structure (see Table 3.4.2) implies that

$$[\mathfrak{k}\otimes E, \mathfrak{k}\otimes E] \subseteq [\Lambda\otimes E, \Lambda\otimes E] = 0.$$

Similarly, in BC_r case we have

$$[\mathfrak{k}\otimes E, \mathfrak{k}\otimes E'] \subseteq (\mathfrak{k}\otimes(A\oplus E\oplus E'))\oplus(\mathfrak{s}\otimes(A\oplus C\oplus C'))\oplus D'$$

and in Θ_n case we have

$$[\mathfrak{k}\otimes E, \mathfrak{k}\otimes E'] \subseteq [\Lambda\otimes E, \Lambda'\otimes E'] \subseteq (\mathfrak{g}\otimes A)\oplus D = (\mathfrak{k}\otimes A)\oplus(\mathfrak{s}\otimes A)\oplus D.$$

Theorem 3.2.6. *Let L be BC_r-graded for some integer $r \geq 2$. Then L is Θ_r-graded.*

Proof. Suppose L is BC_r-graded with grading subalgebra $\mathfrak{k}$ of type B_r, C_r, or D_r. Let $\mathfrak{g} \cong sl_r$ be a naturally embedded subalgebra of $\mathfrak{k}$. It is easy to see that $\mathfrak{k}$ is $(\Theta_r, \mathfrak{g})$-graded. By Proposition 3.1.5, we only need to show that the set of all weights of the $\mathfrak{g}$-module L is a subset of Θ_r. Using Remark 3.2.3, we get

$$\mathscr{W}(L\downarrow\mathfrak{g}) = \mathscr{W}(L\downarrow\mathfrak{k})\downarrow\mathfrak{g} \subseteq BC_r\cup\{0\}\downarrow\mathfrak{g} = \Gamma_{\mathfrak{k}}\downarrow\mathfrak{g} = \Gamma_{\mathfrak{g}} = \Theta_r,$$

as required. □

Proposition 3.2.7. *Let L be BC_n-graded for some integer $n \geq 5$. Then L is Θ_n-graded and the conditions (1.2.1) hold.*

Proof. Suppose that L is BC_n-graded with a grading subalgebra $\mathfrak{k}$. Let $\mathfrak{g} \cong sl_n$ be a naturally embedded subalgebra of $\mathfrak{k}$ as in the proof of Theorem 3.2.6. Then L is Θ_n-graded and we need to check the conditions (1.2.1). We will assume that $\mathfrak{k}$ is of type C_n (the cases B_n and D_n are proved similarly). We have the following decomposition of the $\mathfrak{k}$-module L:

$$L = (\mathfrak{k}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus\mathfrak{v}\otimes C\oplus D$$

where $\mathfrak{k} \cong V_{\mathfrak{k}}(2\omega_1)$, $\mathfrak{s} \cong V_{\mathfrak{k}}(\omega_2)$ and $\mathfrak{v} \cong V_{\mathfrak{k}}(\omega_1)$. The restrictions of the $\mathfrak{k}$-modules $\mathfrak{k}$, $\mathfrak{s}$ and

$\mathfrak{v}$ to $\mathfrak{g}$ are decomposed as follows:

$$\mathfrak{k} = \mathfrak{g} \oplus S \oplus S', \quad \mathfrak{s} = \mathfrak{g} \oplus \Lambda \oplus \Lambda', \quad \mathfrak{v} = V \oplus V'. \tag{3.2.4}$$

Therefore we have the following decomposition of the $\mathfrak{g}$-module L:

$$\begin{aligned} L &= (\mathfrak{g} \oplus S \oplus S') \otimes A \oplus (\mathfrak{g} \oplus \Lambda \oplus \Lambda') \otimes B \oplus (V \oplus V') \otimes C \oplus D \\ &= (\mathfrak{g} \otimes (A \oplus B)) \oplus (S \otimes A) \oplus (S' \otimes A) \oplus (\Lambda \otimes B) \oplus (\Lambda' \otimes B) \oplus (V \otimes C) \oplus (V' \otimes C) \oplus D, \end{aligned}$$

Fix the standard matrix presentations of the algebra $\mathfrak{k} \cong sp_{2n}$ and its modules $\mathfrak{s}$ and $\mathfrak{v}$ as in [4]. Then $\mathfrak{g}$ is identified with the subalgebra $\{diag(X, -X^t) \mid X \in sl_n\}$ of $\mathfrak{k}$. Let K_n denotes the set of skew-symmetric $n \times n$ matrices. Then the components Λ, V and their duals in the decompositions (3.2.4) have the following matrix shapes:

$$\begin{aligned} \Lambda &= \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y \in K_n \right\}, \quad \Lambda' = \left\{ \begin{pmatrix} 0 & 0 \\ Y' & 0 \end{pmatrix} \mid Y' \in K_n \right\}, \\ V &= \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} \mid v \in \mathbb{F}^n \right\}, \quad V' = \left\{ \begin{pmatrix} 0 \\ v' \end{pmatrix} \mid v' \in \mathbb{F}^n \right\}. \end{aligned}$$

Let $\lambda_1 \otimes b_1, \lambda_2 \otimes b_2 \in \Lambda \otimes B$ and $u \otimes c \in V \otimes C$. Using Formulae in [4, (2.8)] and the fact that $\Lambda \otimes B \subseteq \mathfrak{s} \otimes B$ and $V \otimes C \subseteq \mathfrak{v} \otimes C$ we get

$$\begin{aligned} [\lambda_1 \otimes b_1, \lambda_2 \otimes b_2] &= (\lambda_1 \circ \lambda_2) \otimes \frac{[b_1, b_2]}{2} + [\lambda_1, \lambda_2] \otimes \frac{b_1 \circ b_2}{2} + \mathrm{tr}(\lambda_1 \lambda_2) \langle b_1, b_2 \rangle, \\ [u \otimes c, \lambda \otimes b] &= -\lambda u \otimes c \cdot b = -[\lambda \otimes b, u \otimes c]. \end{aligned}$$

Note that $\lambda_1 \circ \lambda_2 = [\lambda_1, \lambda_2] = \lambda_1 \lambda_2 = 0$ and $\lambda u = 0$. Substituting these values in the formulae above we get $[\Lambda \otimes B, \Lambda \otimes B] = [\Lambda \otimes B, V \otimes C] = 0$. Similarly, we get $[\Lambda' \otimes B', \Lambda' \otimes B] = [\Lambda' \otimes B', V' \otimes C'] = 0$, as required. $\square$

3.3 Examples of Θ_n-graded Lie algebras

Example 3.3.1. As discussed previously (see Theorems 3.2.6), every BC_n-graded Lie algebra ($n \geq 2$) is Θ_n-graded.

Example 3.3.2. Any Lie algebra which is (A_{n-1}, sl_n)-graded is also Θ_n-graded. For such a Lie algebra, the space $L_\alpha = \{0\}$ for all α not in A_{n-1}.

Example 3.3.3. Let $L = sl_{2n+1}$ and $\mathfrak{g} = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & -x^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in sl_n \right\} \subset L$. We consider the adjoint action of $\mathfrak{g}$ on L. We have the following decomposition of the $\mathfrak{g}$-module L:

$$L = \mathfrak{g} \oplus \mathfrak{g}' \oplus V_1 \oplus V_2 \oplus V_1' \oplus V_2' \oplus S \oplus S' \oplus \Lambda \oplus \Lambda' \oplus D$$

where $D = \left\{ \begin{bmatrix} t_1 I_n & 0 & 0 \\ 0 & t_2 I_n & v \\ 0 & 0 & -n(t_1+t_2) \end{bmatrix} \mid t_1, t_2 \in \mathbb{F} \right\}$ is the sum of the trivial $\mathfrak{g}$-modules and

$$\mathfrak{g}' = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in sl_n \right\} \cong \mathfrak{g} \cong V(\omega_1 + \omega_{n-1})$$

$$V_1 = \left\{ \begin{bmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid v \in \mathbb{F}^n \right\} \cong V_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & v^t & 0 \end{bmatrix} \mid v \in \mathbb{F}^n \right\} \cong V(\omega_1),$$

$$V_1' = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \mid v \in \mathbb{F}^n \right\} \cong V_2' = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v^t & 0 & 0 \end{bmatrix} \mid v \in \mathbb{F}^n \right\} \cong V(\omega_{n-1}),$$

$$S = \left\{ \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = x^t \right\} \cong V(2\omega_1),$$

$$S' = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = x^t \right\} \cong V(2\omega_{n-1}),$$

$$\Lambda = \left\{ \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = -x^t \right\} \cong V(\omega_2),$$

$$\Lambda' = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = -x^t \right\} \cong V(\omega_{n-2}),$$

as $\mathfrak{g}$-modules. Then L is $(\Theta_n, \mathfrak{g})$-graded.

Example 3.3.4. Let $L = \mathfrak{g} \oplus R$ where $R = \mathrm{Rad} L$ and $\mathfrak{g}$ is a simple submodule of L isomorphic to sl_n. Suppose $[R,R]=0$ and $R \cong V(w)$ as a $\mathfrak{g}$-module. Then L is $(\Theta_n,\mathfrak{g})$-graded if and only if $w \in \Theta_n$.

3.4 Multiplication in Θ_n-graded Lie algebras, $n \geq 5$

Recall that $\Theta_n = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}$ were $\{\varepsilon_1,\ldots,\varepsilon_n\}$ is the set of weights of the natural sl_n-module. We denote by Θ_n^+ the set of dominant weights in Θ_n and the corresponding simple sl_n-modules. Thus,

$$\Theta_n^+ = \{\omega_1+\omega_{n-1} = \varepsilon_1-\varepsilon_n,\ \omega_1=\varepsilon_1,\ \omega_{n-1}=-\varepsilon_n,\\ 2\omega_1=2\varepsilon_1,\ 2\omega_{n-1}=-2\varepsilon_n,\ \omega_2=\varepsilon_1+\varepsilon_2,\ \omega_{n-2}=-\varepsilon_n-\varepsilon_{n-1},\ 0\}.$$

These are the highest weights of the modules $\mathfrak{g}$, V, V', S, S', Λ, Λ' and T, respectively.

We fix a base

$$\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } i = 1,2,\cdots,n-1\}$$

of simple roots for the root system

$$A_{n-1} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq n+1\}.$$

Let L be an Θ_n-graded Lie algebra and let $\mathfrak{g}$ be the grading subalgebra of L of type $\Delta = A_{n-1}$ with $n \geqq 5$. We identify $\mathfrak{g}$ with the matrix algebra sl_n. By Proposition 3.2.2 the $\mathfrak{g}$-module L is a direct sum of copies of $\mathfrak{g}$, V, V', S, S', Λ, Λ' and T. By collecting isomorphic summands of L into isotypic components, we may assume that there are vector spaces $A,B,B',C,C'E,E'$ such that

$$L \cong (\mathfrak{g}\otimes A)\oplus(V\otimes B)\oplus(V'\otimes B')\oplus(S\otimes C)\oplus(S'\otimes C')\oplus(\Lambda\otimes E)\oplus(\Lambda'\otimes E')\oplus D.$$

Alternatively, these spaces can also be viewed as the corresponding $\mathfrak{g}$-mod Hom-spaces: $A = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},L)$, $B = \mathrm{Hom}_{\mathfrak{g}}(V,L)$, etc, so for each simple $\mathfrak{g}$-module M, the space $M \otimes \mathrm{Hom}_{\mathfrak{g}}(M,L)$ is canonically identified with the M-isotypic component of L via the evaluation map

$$M \otimes \mathrm{Hom}_{\mathfrak{g}}(M,L) \rightharpoondown L,\ m\otimes\varphi \mapsto \varphi(m). \tag{3.4.1}$$

Definition 3.4.1. (1) We identify the $\mathfrak{g}$-modules V and V' with the space $\mathbb{F}^n$ of column vectors with the following actions:

$$x.v = xv \text{ for } x \in sl_n,\ v \in V,$$

$$x.v' = -x^t v' \text{ for } x \in sl_n,\ v' \in V'.$$

(2) We identify S and S' (resp. Λ and Λ') with symmetric (resp. skew-symmetric) $n \times n$ matrices. Then S, S', Λ and Λ' are $\mathfrak{g}$-modules under the actions:

$$\begin{aligned}
&x.s = xs + sx^t \text{ for } x \in sl_n,\ s \in S,\\
&x.\lambda = x\lambda + \lambda x^t \text{ for } x \in sl_n,\ \lambda \in \Lambda,\\
&x.s' = -s'x - x^t s' \text{ for } x \in sl_n,\ s' \in S,\\
&x.\lambda' = -\lambda' x - x^t \lambda' \text{ for } x \in sl_n,\ \lambda' \in \Lambda'.
\end{aligned}$$

Since the subalgebra $\mathfrak{g}$ of L is a $\mathfrak{g}$-submodule, there exists a distinguished element 1 of A such that $\mathfrak{g} = \mathfrak{g} \otimes 1$. In particular,

$$[x \otimes 1, y \otimes b] = x.y \otimes b. \tag{3.4.2}$$

where $x \otimes 1$ is in $\mathfrak{g} \otimes 1$, $y \otimes b$ belongs to one of the components in (3.2.1), and $x.y$ is as in Definition 3.4.1.

Let $\Theta(M)$ be the Θ-component of M, i.e. the sum of all simple submodules of M with highest weights in Θ_n^+. In order to describe multiplication in L we need to calculate first the Θ-components of the tensor products of the modules in Θ_n^+. Most of the decompositions are easily derived from stability results in [21, Cor. 6.22 and 7.2] (for larger ranks) with a computer program (such as LiE) used to verify the small rank cases. These are the following (full) decompositions:

$$\begin{aligned}
V(\omega_1) \otimes V(\omega_1) &= V(2\omega_1) \oplus V(\omega_2),\\
V(\omega_{n-1}) \otimes V(\omega_{n-1}) &= V(2\omega_{n-1}) \oplus V(\omega_{n-2}),\\
V(\omega_1) \otimes V(2\omega_1) &= V(\omega_1 + \omega_2) \oplus V(3\omega_1),\\
V(\omega_{n-1}) \otimes V(2\omega_{n-1}) &= V(\omega_{n-1} + \omega_{n-2}) \oplus V(3\omega_{n-1}),\\
V(\omega_1) \otimes V(\omega_{n-2}) &= V(\omega_1 + \omega_{n-2}) \oplus V(\omega_{n-1}),\\
V(\omega_{n-1}) \otimes V(\omega_2) &= V(\omega_{n-1} + \omega_2) \oplus V(\omega_1),\\
V(2\omega_1) \otimes V(2\omega_1) &= V(4\omega_1) \oplus V(2\omega_1 + \omega_2) \oplus V(2\omega_2),\\
V(2\omega_{n-1}) \otimes V(2\omega_{n-1}) &= V(4\omega_{n-1}) \oplus V(2\omega_{n-1} + \omega_{n-2}) \oplus V(2\omega_{n-2}),\\
V(2\omega_1) \otimes V(\omega_{n-2}) &= V(2\omega_1 + \omega_{n-2}) \oplus V(\omega_1 + \omega_{n-1}),\\
V(2\omega_{n-1}) \otimes V(\omega_2) &= V(\omega_2 + 2\omega_{n-1}) \oplus V(\omega_1 + \omega_{n-1}),\\
V(\omega_2) \otimes V(\omega_2) &= V(2\omega_2) \oplus V(\omega_1 + \omega_3) \oplus V(\omega_4),
\end{aligned}$$

$$
\begin{aligned}
V(\omega_{n-2}) \otimes V(\omega_{n-2}) &= V(2\omega_{n-2}) \oplus V(\omega_2 + \omega_{n-3}) \oplus V(\omega_{n-4}), \\
V(\omega_2) \otimes V(\omega_1) &= V(\omega_1 + \omega_2) \oplus V(\omega_3), \\
V(\omega_{n-2}) \otimes V(\omega_{n-1}) &= V(\omega_{n-3}) \oplus V(\omega_{n-1} + \omega_{n-2}), \\
V(2\omega_1) \otimes V(\omega_2) &= V(2\omega_1 + \omega_2) \oplus V(\omega_1 + \omega_3), \\
V(2\omega_{n-1}) \otimes V(\omega_{n-2}) &= V(2\omega_{n-1} + \omega_{n-2}) \oplus V(\omega_{n-1} + \omega_{n-3}).
\end{aligned}
$$

Note that the modules V, V', Λ, and Λ' are minuscule i.e. their weights form a single W-orbit where W is the Weyl group, so the following lemma can be used.

Lemma 3.4.2. *[38, Cor.3.5] For two dominant weights λ, μ such that $V(\mu)$ is minuscule, we have the decomposition*

$$
V(\lambda) \otimes V(\mu) \cong \bigoplus_{\substack{\omega \in W/W_\mu : \\ \lambda + \omega\mu \text{ is dominant}}} V(\lambda + \omega\mu)
$$

with each summand occurring with multiplicity 1, where $W_\mu := \{\omega \in W \mid \omega\mu = \mu\}$ is the isotropy group of μ. Moreover, the number of irreducible components in $V(\lambda) \otimes V(\mu)$ is equal to the cardinality $W_\lambda \backslash W / W_\mu$.

This lemma gives us 8 more decompositions:

$$
\begin{aligned}
V(\omega_1 + \omega_{n-1}) \otimes V(\omega_2) &= V(\omega_1 + \omega_2 + \omega_{n-1}) \oplus V(\omega_3 + \omega_{n-1}) \oplus V(2\omega_1) \oplus V(\omega_2), \\
V(\omega_1 + \omega_{n-1}) \otimes V(\omega_{n-2}) &= V(\omega_{n-1} + \omega_{n-2} + \omega_1) \oplus V(\omega_{n-3} + \omega_1) \oplus V(2\omega_{n-1}) \oplus V(\omega_{n-2}), \\
V(2\omega_1) \otimes V(\omega_{n-1}) &= V(2\omega_1 + \omega_{n-1}) \oplus V(\omega_1), \\
V(2\omega_{n-1}) \otimes V(\omega_1) &= V(\omega_1 + 2\omega_{n-1}) \oplus V(\omega_{n-1}), \\
V(\omega_1 + \omega_{n-1}) \otimes V(\omega_1) &= V(2\omega_1 + \omega_{n-1}) \oplus V(\omega_2 + \omega_{n-1}) \oplus V(\omega_1), \\
V(\omega_1 + \omega_{n-1}) \otimes V(\omega_{n-1}) &= V(\omega_1 + 2\omega_{n-1}) \oplus V(\omega_1 + \omega_{n-2}) \oplus V(\omega_{n-1}), \\
V(\omega_1) \otimes V(\omega_{n-1}) &= V(\omega_1 + \omega_{n-1}) \oplus V(0), \\
V(\omega_{n-2}) \otimes V(\omega_2) &= V(\omega_2 + \omega_{n-2}) \oplus V(\omega_1 + \omega_{n-1}) \oplus V(0),
\end{aligned}
$$

Seligman [46, A-2]) found the following decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ for $n > 4$:

$$
\begin{aligned}
V(\omega_1 + \omega_{n-1}) \otimes V(\omega_1 + \omega_{n-1}) = {} & V(2\omega_1 + 2\omega_{n-1}) \oplus V(2\omega_1 + \omega_{n-2}) \oplus V(\omega_2 + 2\omega_{n-1}) \\
& \oplus V(\omega_2 + \omega_{n-2}) \oplus 2V(\omega_1 + \omega_{n-1}) \oplus V(0).
\end{aligned}
$$

It remains to find the decompositions of $\mathfrak{g} \otimes S$, $\mathfrak{g} \otimes S'$ and $S \otimes S'$. We will only calculate the Θ-components. It is well known that the only possible $V(\nu)$ which can occur as summands of $V(\lambda)\otimes V(\mu)$ are those with $\nu = \lambda + \mu_1$ for some μ_1 in the set of weights of $V(\mu)$ [33, p.142]. The following lemma gives a bit more precise information on multiplicities.

Lemma 3.4.3. *[38, Proposition 3.2] Let λ, μ be two dominant weights. Then any component $V(\nu)$ of $V(\lambda)\otimes V(\mu)$ is of the form $\nu = \lambda + \mu_1$ for some μ_1 in the set of weights of $V(\mu)$. Moreover, its multiplicity $m^{\nu}_{\lambda,\mu} \leq \dim V(\mu)_{\mu_1}$.*

By Lemma 3.4.3, to calculate $\Theta(V(\lambda)\otimes V(\mu))$ we need to find all dominant weights $\nu \in \Theta_n$ such that $\nu = \lambda + \mu_1$ for some μ_1 in the set of weights of $V(\mu)$. All these possibilities are listed in the table below. Note that we have $V(\mu) = S$ or S', so all weight spaces of $V(\mu)$ are 1-dimensional and the corresponding modules $V(\nu)$ appear in the decomposition with multiplicity at most 1. On the other hand, in the list (3.4.3) below we explicitly construct all these summands $V(\nu)$, so their multiplicities are exactly 1.

λ	μ	$\nu = \lambda + \mu_1 \in \Theta_n^+$	$\Theta(V(\lambda)\otimes V(\mu))$
$\varepsilon_1 - \varepsilon_n$	$2\varepsilon_1$	$2\varepsilon_1 = (\varepsilon_1 - \varepsilon_n) + (\varepsilon_1 + \varepsilon_n)$ $\varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_n) + (\varepsilon_2 + \varepsilon_n)$	$\Theta(\mathfrak{g}\otimes S) = S + \Lambda$
$\varepsilon_1 - \varepsilon_n$	$\varepsilon_1 + \varepsilon_2$	$-2\varepsilon_n = (\varepsilon_1 - \varepsilon_n) + (-\varepsilon_1 - \varepsilon_n)$ $-\varepsilon_{n-1} - \varepsilon_n = (\varepsilon_1 - \varepsilon_n) + (-\varepsilon_1 - \varepsilon_{n-1})$	$\Theta(\mathfrak{g}\otimes S') = S' + \Lambda'$
$2\varepsilon_1$	$-2\varepsilon_{n-1}$	$\varepsilon_1 - \varepsilon_n = (2\varepsilon_1) + (-\varepsilon_1 - \varepsilon_n)$ $0 = (2\varepsilon_1) + (-2\varepsilon_1)$	$\Theta(S\otimes S') = \mathfrak{g} + T$

Table 3.4.1:

To summarize, in Tables 3.4.2-3.4.4 below and Remark 3.4.4 we describe Θ-components of all tensor product decompositions for the modules in Θ_n^+ ($n \geq 3$). If the cell in row X and column Y contains Z this means that $\Theta(X \otimes Y) \cong Z$.

$\otimes$	$\mathfrak{g}$	S	Λ	S'	Λ'	V	V'
$\mathfrak{g}$	$\mathfrak{g}+\mathfrak{g}+T$	$S+\Lambda$	$S+\Lambda$	$S'+\Lambda'$	$S'+\Lambda'$	V	V'
S	$S+\Lambda$	0	0	$\mathfrak{g}+T$	$\mathfrak{g}$	0	V
Λ	$S+\Lambda$	0	0	$\mathfrak{g}$	$\mathfrak{g}+T$	0	V
S'	$S'+\Lambda'$	$\mathfrak{g}+T$	$\mathfrak{g}$	0	0	V'	0
Λ'	$S'+\Lambda'$	$\mathfrak{g}$	$\mathfrak{g}+T$	0	0	V'	0
V	V	0	0	V'	V'	$S+\Lambda$	$\mathfrak{g}+T$
V'	V'	V	V	0	0	$\mathfrak{g}+T$	$S'+\Lambda'$

Table 3.4.2: Θ-component of tensor product decompositions for sl_n ($n \geq 7$)

Remark 3.4.4. For $n = 5,6$ all the decompositions are the same as in Table 3.4.2 except in addition we have $\Theta(\Lambda \otimes \Lambda) = \Lambda'$ and $\Theta(\Lambda' \otimes \Lambda') = \Lambda$ for sl_6 and $\Theta(\Lambda \otimes \Lambda) = V'$, $\Theta(\Lambda \otimes V) = \Lambda'$, $\Theta(\Lambda' \otimes \Lambda') = V$ and $\Theta(\Lambda' \otimes V') = \Lambda$ for sl_5.

Note that $\Lambda \cong \Lambda'$ for sl_4 and $\Lambda \cong V'$ and $\Lambda' \cong V$ for sl_3 so we have the following decompositions.

$\otimes$	$\mathfrak{g}$	S	$\Lambda \cong \Lambda'$	S'	V	V'
$\mathfrak{g}$	$\mathfrak{g}+\mathfrak{g}+T$	$S+\Lambda$	$S+\Lambda$	$S'+\Lambda$	V	V'
S	$S+\Lambda$	0	$\mathfrak{g}$	$\mathfrak{g}+T$	0	V
Λ	$S+\Lambda$	$\mathfrak{g}$	$\mathfrak{g}+T$	$\mathfrak{g}$	V'	V
S'	$S'+\Lambda$	$\mathfrak{g}+T$	$\mathfrak{g}$	0	V'	0
V	V	0	V'	V'	$S+\Lambda$	$\mathfrak{g}+T$
V'	V'	V	V	0	$\mathfrak{g}+T$	$S'+\Lambda$

Table 3.4.3: Θ-component of tensor product decompositions for sl_4

$\otimes$	$\mathfrak{g}$	S	S'	$V \cong \Lambda'$	$V' \cong \Lambda$
$\mathfrak{g}$	$\mathfrak{g}+\mathfrak{g}+T$	$S+V'$	$S'+V$	$S'+V$	$S+V'$
S	$S+V'$	S'	$\mathfrak{g}+T$	$\mathfrak{g}$	V
S'	$S'+V$	$\mathfrak{g}+T$	S	V'	$\mathfrak{g}$
V	$S'+V$	$\mathfrak{g}$	V'	$S+V'$	$\mathfrak{g}+T$
V'	$S+V'$	V	$\mathfrak{g}$	$\mathfrak{g}+T$	$S'+V$

Table 3.4.4: Θ-component of tensor product decompositions for sl_3

Let L be an Θ_n-graded Lie algebra and let $\mathfrak{g}$ be the grading subalgebra of L. Suppose that $n \geq 7$ or $n = 5,6$ and the conditions (1.2.1) hold. In (3.4.3) we list bases for all non-zero $\mathfrak{g}$-module homomorphism spaces $\mathrm{Hom}_{\mathfrak{g}}(X \otimes Y, Z)$ where $X,Y,Z \in \{\mathfrak{g}, V, V', S, \Lambda, S', \Lambda', T\}$ and X and Y are both non-trivial. Note that all of them are 1-dimensional except the first one (which is 2-dimensional).

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) &= span\{x \otimes y \mapsto xy - yx,\ x \otimes y \mapsto xy + yx - \frac{2}{n}\mathrm{tr}(xy)I\}, \qquad (3.4.3)\\
\mathrm{Hom}_{\mathfrak{g}}(V \otimes V', \mathfrak{g}) &= span\{u \otimes v' \mapsto uv'^t - \frac{\mathrm{tr}(uv'^t)}{n} I\},\\
\mathrm{Hom}_{\mathfrak{g}}(S \otimes \Lambda', \mathfrak{g}) &= span\{s \otimes \lambda' \mapsto s\lambda'\},\\
\mathrm{Hom}_{\mathfrak{g}}(S' \otimes \Lambda, \mathfrak{g}) &= span\{s' \otimes \lambda \mapsto s'\lambda\},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(\Lambda \otimes \Lambda', \mathfrak{g}) &= span\{\lambda \otimes \lambda' \mapsto \lambda\lambda' - \frac{\mathrm{tr}(\lambda\lambda')}{n} I\},\\
\mathrm{Hom}_{\mathfrak{g}}(S \otimes S', \mathfrak{g}) &= span\{s \otimes s' \mapsto ss' - \frac{\mathrm{tr}(ss')}{n} I\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V, V) &= span\{x \otimes v \mapsto xv\},\\
\mathrm{Hom}_{\mathfrak{g}}(\Lambda \otimes V', V) &= span\{\lambda \otimes v' \mapsto \lambda v'\},\\
\mathrm{Hom}_{\mathfrak{g}}(S \otimes V', V) &= span\{s \otimes v' \mapsto sv\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V', V') &= span\{x \otimes v' \mapsto xv'\},\\
\mathrm{Hom}_{\mathfrak{g}}(S' \otimes V, V') &= span\{s' \otimes v \mapsto s'v\},\\
\mathrm{Hom}_{\mathfrak{g}}(\Lambda' \otimes V', V') &= span\{\lambda' \otimes v' \mapsto \lambda' v'\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes S, S) &= span\{x \otimes s \mapsto xs + sx^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(V \otimes V, S) &= span\{u \otimes v \mapsto uv^t + vu^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \Lambda, S) &= span\{x \otimes \lambda \mapsto x\lambda - \lambda x^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(S' \otimes \mathfrak{g}, S') &= span\{s' \otimes x \mapsto s'x + x^t s'\},\\
\mathrm{Hom}_{\mathfrak{g}}(V' \otimes V', S') &= span\{u' \otimes v' \mapsto u'v'^t + v'u'^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(\Lambda' \otimes \mathfrak{g}, S') &= span\{\lambda' \otimes x \mapsto \lambda' x - x^t \lambda'\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \Lambda, \Lambda) &= span\{x \otimes \lambda \mapsto x\lambda + \lambda x^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes S, \Lambda) &= span\{x \otimes s \mapsto xs - sx^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(V \otimes V, \Lambda) &= span\{u \otimes v \mapsto uv^t - vu^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(\Lambda' \otimes \mathfrak{g}, \Lambda) &= span\{\lambda' \otimes x \mapsto \lambda' x + x^t \lambda'\},\\
\mathrm{Hom}_{\mathfrak{g}}(S' \otimes \mathfrak{g}, \Lambda') &= span\{s' \otimes x \mapsto s'x - x^t s'\},\\
\mathrm{Hom}_{\mathfrak{g}}(V' \otimes V', \Lambda') &= span\{u' \otimes v' \mapsto u'v'^t - v'u'^t\},\\
\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, T) &= span\{x_1 \otimes x_2 \mapsto \frac{1}{n}\mathrm{tr}(x_1 x_2)\},\\
\mathrm{Hom}_{\mathfrak{g}}(V' \otimes V, T) &= span\{v^t \otimes u \mapsto \frac{1}{n}\mathrm{tr}(uv^t)\},\\
\mathrm{Hom}_{\mathfrak{g}}(S \otimes S', T) &= span\{s \otimes s' \mapsto \frac{1}{n}\mathrm{tr}(ss')\},\\
\mathrm{Hom}_{\mathfrak{g}}(\Lambda \otimes \Lambda', T) &= span\{\lambda \otimes \lambda' \mapsto \frac{1}{n}\mathrm{tr}(\lambda\lambda')\}.
\end{aligned}
$$

We claim that the Lie algebra structure on the decomposition (3.2.1) induces certain bilinear maps among the spaces $A, B, B', C, C', E, E', D$. Indeed, denote the irreducible modules and the corresponding spaces by $M_1, \ldots, M_8$ and $H_1, \ldots, H_8$, respectively. Then $L = \bigoplus_{i=1}^{8} M_i \otimes H_i$ and $H_i = \mathrm{Hom}_{\mathfrak{g}}(M_i, L)$, see (3.4.1). The Lie product on L can be identified with an element of $\mathrm{Hom}_{\mathfrak{g}}(L \otimes L, L)$. Since any homomorphisms between non-isomorphic irreducible $\mathfrak{g}$-modules are zero, the product is actually an element of $\mathrm{Hom}_{\mathfrak{g}}(\Theta(L\otimes L), L)$ where $\Theta(L\otimes L)$ is the sum of all irreducible $\mathfrak{g}$-submodules of $L\otimes L$ isomorphic to one of $M_1,\ldots,M_8$. The $\mathfrak{g}$-module $L\otimes L$ is decomposed as $L\otimes L = \bigoplus_{i,j=1}^{8} M_i\otimes M_j\otimes H_i\otimes H_j$ and the Θ-component of $L\otimes L$ can be found as

$$\Theta(L\otimes L) = \bigoplus_{k=1}^{8} M_k\otimes \mathrm{Hom}_{\mathfrak{g}}(L\otimes L, M_k) = \bigoplus_{k=1}^{8} M_k\otimes \left(\oplus_{i,j=1}^{8} M_{ij}^k\otimes H_i\otimes H_j\right)$$

where $M_{ij}^k = \mathrm{Hom}_{\mathfrak{g}}(M_i\otimes M_j, M_k)$. Then the Lie bracket on L is an element μ of the space

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(\Theta(L\otimes L), L) &= \bigoplus_{k=1}^{8}\mathrm{Hom}_{\mathbb{F}}\left(\oplus_{i,j=1}^{8} M_{ij}^k\otimes H_i\otimes H_j, H_k\right)\\
&= \bigoplus_{i,j,k=1}^{8}\mathrm{Hom}_{\mathbb{F}}\left(M_{ij}^k\otimes H_i\otimes H_j, H_k\right).\\
&= \bigoplus_{i,j,k=1}^{8}\mathrm{Hom}_{\mathbb{F}}\left(M_{ij}^k, \mathrm{Hom}_{\mathbb{F}}(H_i\otimes H_j, H_k)\right)
\end{aligned}$$

Denote by $\{b_1^{kij}, b_2^{kij}\ldots\}$ the basis of the space $\mathrm{Hom}_{\mathfrak{g}}(M_i\otimes M_j, M_k)$ as in (3.4.3). Then there exist unique elements $\chi_1^{kij}, \chi_2^{kij},\ldots$ in $\mathrm{Hom}_{\mathbb{F}}(H_i\otimes H_j, H_k)$ (the images of $b_1^{kij}, b_2^{kij}\ldots$) which correspond to multiplication μ on L. These elements $\chi_s^{kij}\in\mathrm{Hom}_{\mathbb{F}}(H_i\otimes H_j, H_k)$ are the claimed bilinear maps $H_i\times H_j\to H_k$.

In Table 3.4.5, if the cell in row X and column Y contains Z, this means that there is a bilinear map $X\otimes Y\to Z$ given by $x\otimes y\mapsto (x,y)_Z$. For simplicity of notation, we will write dy instead of $(d,y)_D$ if $X=Z=D$ and we will write $\langle x,y\rangle$ instead of $(x,y)_D$ if $X,Y\neq D$ and $Z=D$. In the case $X=Y=Z=A$, we have two bilinear products $a_1\otimes a_2\mapsto a_1\circ a_2$ and $a_1\otimes a_2\mapsto [a_1,a_2]$ for $a_1,a_2\in A$. Note that some of the cells are empty. The corresponding products $X\otimes Y\to Z$ will be defined later by extending the existing maps $Y\otimes X\to Z$. This will make the table symmetric.

.	A	B	B'	C	C'	E	E'	D
A	$(A,\circ,[\,]),D$	B		C,E		C,E		
B		C,E	A,D	0		0		
B'	A		C',E'	B	0	B	0	
C		0		0	A,D	0	A	
C'	C',E'	B'	0		0	A	0	
E		0		0		0	A,D	
E'	C',E'	B'	0		0		0	
D	A	B	B'	C	C'	E	E'	D

Table 3.4.5: Bilinear products

Let x and y be $n \times n$ matrices. We will use the following products:

$$\begin{aligned}
[x,y] &= xy - yx, \\
x \circ y &= xy + yx - \frac{2}{n}\mathrm{tr}(xy)I, \\
x \diamond y &= xy + yx, \\
(x \mid y) &= \frac{1}{n}\mathrm{tr}(xy).
\end{aligned}$$

Following the methods in [4, 20, 22, 46] (see also our argument below) and using the results of (3.4.3), Tables 3.4.2 and 3.4.5 and Remark 3.4.4, we may suppose that the multiplication in L is given as follows. For all $x,y \in sl_n$, $u,v \in V$, $u',v' \in V'$, $s \in S$, $\lambda \in \Lambda$, $s' \in S'$, $\lambda' \in \Lambda'$ and for all $a,a_1,a_2 \in A$, $b,b_1,b_2 \in B$, $b',b'_1,b'_2 \in B'$, $c \in C$, $c' \in C'$, $e \in E$, $e' \in E'$ and $d,d_1,d_2 \in D$,

$$\begin{aligned}
&[x \otimes a_1, y \otimes a_2] = (x \circ y) \otimes \frac{[a_1,a_2]}{2} + [x,y] \otimes \frac{a_1 \circ a_2}{2} + (x \mid y)\langle a_1,a_2\rangle, \qquad (3.4.4)\\
&[u \otimes b, v' \otimes b'] = (uv'^t - \frac{\mathrm{tr}(uv'^t)}{n}I) \otimes (b,b')_A + \frac{2}{n}\mathrm{tr}(uv'^t)\langle b,b'\rangle = -[v' \otimes b', u \otimes b],\\
&[s \otimes c, s' \otimes c'] = (ss' - (s \mid s')I) \otimes (c,c')_A + (s \mid s')\langle c,c'\rangle = -[s' \otimes c', s \otimes c],\\
&[\lambda \otimes e, \lambda' \otimes e'] = (\lambda\lambda' - (\lambda \mid \lambda')I) \otimes (e,e')_A + (\lambda \mid \lambda')\langle e,e'\rangle = -[\lambda' \otimes e', \lambda \otimes e],\\
&[u \otimes b_1, v \otimes b_2] = (uv^t + vu^t) \otimes \frac{(b_1,b_2)_C}{2} + (uv^t - vu^t) \otimes \frac{(b_1,b_2)_E}{2},\\
&[u' \otimes b'_1, v' \otimes b'_2] = (u'v'^t + v'u'^t) \otimes \frac{(b'_1,b'_2)_{C'}}{2} + (u'v'^t - v'u'^t) \otimes \frac{(b'_1,b'_2)_{E'}}{2},\\
&[x \otimes a, s \otimes c] = (xs + sx^t) \otimes \frac{(a,c)_C}{2} + (xs - sx^t) \otimes \frac{(a,c)_E}{2} = -[s \otimes c, x \otimes a],
\end{aligned}$$

$$
\begin{aligned}
&[x\otimes a,\lambda\otimes e]=(x\lambda+\lambda x^t)\otimes\frac{(a,e)_E}{2}+(x\lambda-\lambda x^t)\otimes\frac{(a,e)_C}{2}=-[\lambda\otimes e,x\otimes a],\\
&[s'\otimes c',x\otimes a]=(s'x+x^ts')\otimes\frac{(c',a)_{C'}}{2}+(s'x-x^ts')\otimes\frac{(c',a)_{E'}}{2}=-[x\otimes a,s'\otimes c'],\\
&[\lambda'\otimes e',x\otimes a]=(\lambda'x+x^t\lambda')\otimes\frac{(e',a)_{E'}}{2}+(\lambda'x-x^t\lambda')\otimes\frac{(e',a)_{C'}}{2}=-[x\otimes a,\lambda'\otimes e'],\\
&[s\otimes c,\lambda'\otimes e']=s\lambda'\otimes(c,e')_A=-[\lambda'\otimes e',s\otimes c],\\
&[s'\otimes c',\lambda\otimes e]=s'\lambda\otimes(c',e)_A=-[\lambda\otimes e,s'\otimes c'],\\
&[x\otimes a,u\otimes b]=xu\otimes(a,b)_B=-[u\otimes b,x\otimes a],\\
&[s'\otimes c',u\otimes b]=s'u\otimes(c',b)_{B'}=-[u\otimes b,s'\otimes c'],\\
&[\lambda'\otimes e',u\otimes b]=\lambda'u\otimes(e',b)_{B'}=-[u\otimes b,\lambda'\otimes e'],\\
&[u'\otimes b',x\otimes a]=x^tu'\otimes(b',a)_{B'}=-[x\otimes a,u'\otimes b'],\\
&[u'\otimes b',s\otimes c]=su'\otimes(b',c)_B=-[s\otimes c,u'\otimes b'],\\
&[u'\otimes b',\lambda\otimes e]=-\lambda u'\otimes(b',e)_B=-[\lambda\otimes e,u'\otimes b'],\\
&[d,x\otimes a]=x\otimes da=-[x\otimes a,d],\\
&[d,u\otimes b]=u\otimes db=-[u\otimes b,d],\\
&[d,s\otimes c]=s\otimes dc=-[s\otimes c,d],\\
&[d,\lambda\otimes e]=\lambda\otimes de=-[\lambda\otimes e,d],\\
&[d,s'\otimes c']=s'\otimes dc'=-[s'\otimes c',d],\\
&[d,u'\otimes b']=u'\otimes db'=-[u'\otimes b',d],\\
&[d,\lambda'\otimes e']=\lambda'\otimes de'=-[\lambda'\otimes e',d],\\
&[d_1,d_2]\in D,
\end{aligned}
$$

All other products of the homogeneous components of the decomposition (3.2.1) are zero.

Following the methods in [4, 20], we present a sample argument for the existence of these maps. Let $\{a_i \mid i\in I\}$ and $\{d_s \mid s\in S\}$ be bases of the vector spaces A and D, respectively. Fix any a_i and a_j of A. Then for all $x,y\in\mathfrak{g}$, write

$$[x\otimes a_i,y\otimes a_j]=\sum_{k\in I}\varepsilon_{i,j}^k(x,y)\otimes a_k+\sum_{s\in S}\eta_{i,j}^s(x,y)d_s+r(x,y)$$

where $r(x,y)$ is the sum of projections of $[x\otimes a_i,y\otimes a_j]$ to other isotypic components. It is easy to see that the maps $\varepsilon_{i,j}^k:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ and $\eta_{i,j}^s:\mathfrak{g}\times\mathfrak{g}\to T$ are bilinear and induce $\mathfrak{g}$-module homomorphisms $\varepsilon_{i,j}^k:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ and $\eta_{i,j}^s:\mathfrak{g}\otimes\mathfrak{g}\to T$. By (3.4.3)

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}) = span\{x\otimes y \mapsto xy-yx,\ x\otimes y \mapsto xy+yx-\frac{2}{n}\mathrm{tr}(xy)I\},$$
$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},T) = span\{x_1\otimes x_2 \mapsto \frac{1}{n}\mathrm{tr}(x_1x_2)\},$$

and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},M)=0$ for all other types of irreducible submodules M of L, so

$$\begin{aligned}\varepsilon_{i,j}^k(x,y) &= \xi_{i,j}^k[x,y]+\delta_{i,j}^k x\circ y,\\ \eta_{i,j}^s(x,y) &= \eta_{i,j}^s(x\mid y),\\ r(x,y) &= 0.\end{aligned}$$

As a result,

$$\begin{aligned}[x\otimes a_i,y\otimes a_j] &= \sum_{k\in I}(\xi_{i,j}^k[x,y]+\delta_{i,j}^k x\circ y)\otimes a_k+\sum_{s\in S}\eta_{i,j}^s(x\mid y)d_s\\ &= [x,y]\otimes\sum_{k\in I}\xi_{i,j}^k a_k + x\circ y\otimes\sum_{k\in I}\delta_{i,j}^k a_k+(x\mid y)\sum_{s\in S}\eta_{i,j}^s d_s.\end{aligned}$$

These expressions in A depend only on i and j not on $x,y\in\mathfrak{g}$, and so we define

$$\begin{aligned}a_i\circ a_j &:= 2\sum_k \xi_{i,j}^k a_k\\ [a_i,a_j] &:= 2\sum_k \delta_{i,j}^k\\ \langle a_i,a_j\rangle &:= \sum_{s\in S}\eta_{i,j}^s d_s\end{aligned}$$

for all $i,j\in I$ (The factor of 2 is just for notational convenience later on). These maps can be extended to give $\mathbb{F}$-bilinear mappings $\circ : A\otimes A\to A$, $[\,,\,]: A\otimes A\to A$ and $\langle\,,\,\rangle : A\otimes A\to D$ such that

$$[x\otimes a_i,y\otimes a_j] = [x,y]\otimes\frac{a_i\circ a_j}{2}+x\circ y\otimes\frac{[a_i,a_j]}{2}+(x\mid y)\langle a_i,a_j\rangle$$

for all $x,y\in\mathfrak{g}$, $a_i,a_j\in A$. Since D is the centralizer of $\mathfrak{g}$, it is a subalgebra of L. The product

$$[d,\mathfrak{g}\otimes a_i]\subseteq\mathfrak{g}\otimes A \text{ for all } d\in D.$$

Hence

$$[d,x\otimes a_i]=\sum_{k\in I}\varepsilon_i^k(d,x)\otimes a_k.$$

Note that $\varepsilon_i^k(\,,\,) \in \mathrm{Hom}_{\mathfrak{g}}(\mathbb{F}d \otimes \mathfrak{g}, \mathfrak{g})$ and so $\varepsilon_i^k(d,x) = \varepsilon_i^k x$ for some $\varepsilon_i^k \in \mathbb{F}$ for each i. Thus,

$$[d, x \otimes a_i] = \sum_{k \in I} \varepsilon_i^k x \otimes a_k = x \otimes \sum_{k \in I} \varepsilon_i^k a_k.$$

Setting $da_i = \sum_{k \in I} \varepsilon_i^k a_k$ and extending gives an action of D on A. One can show that A is a D-module and $\langle A, A \rangle$ is an ideal of D (by using the Jacoby identity for d, $x \otimes a$, $y \otimes b$ and for d, d', $x \otimes a$ where $d, d' \in D$ and $a, b \in A$).

Chapter 4

The coordinate algebra of a Θ_n-graded Lie algebra, $n \geq 5$

Let L be an Θ_n-graded Lie algebra and let $\mathfrak{g} \cong sl_n$ be the grading subalgebra of L. Assume that $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold. Let $\mathfrak{g}^{\pm} = \{x \in sl_n \mid x^t = \pm x\}$. Then

$$\mathfrak{g} \otimes A = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \otimes A = (\mathfrak{g}^+ \otimes A) \oplus (\mathfrak{g}^- \otimes A) = (\mathfrak{g}^+ \otimes A^-) \oplus (\mathfrak{g}^- \otimes A^+) \tag{4.0.1}$$

where $A^{\pm}$ is a copy of the vector space A. Recall that we identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1$ where 1 is a distinguished element of A. We denote by $a^{\pm}$ the image of $a \in A$ in the space $A^{\pm}$.

In Chapter 3 we described the multiplicative structures of Θ_n-graded Lie algebras. In this chapter we describe the coordinate algebras of these Lie algebras. Denote

$$\mathfrak{a} := A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E' \quad \text{and} \quad \mathfrak{b} := \mathfrak{a} \oplus B \oplus B'.$$

We show that the product in L induces an algebra structure on both $\mathfrak{a}$ and $\mathfrak{b}$. In Section 4.1 we prove that $\mathfrak{a}$ is a unital associative subalgebra of $\mathfrak{b}$ with involution whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E'$ and $A^- \oplus C \oplus C'$. In Section 4.2 we prove that $\mathfrak{b}$ is a unital algebra with an involution η whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E' \oplus B \oplus B'$ and $A^- \oplus C \oplus C'$. It is also shown that $B \oplus B'$ is an associative $\mathfrak{a}$-bimodule with a hermitian form χ with values in $\mathfrak{a}$. More exactly, for all $\beta_1, \beta_2 \in B \oplus B'$ and $\alpha \in \mathfrak{a}$ we have $\chi(\beta_1, \beta_2) = \beta_1\beta_2$, $\chi(\alpha\beta_1, \beta_2) = \alpha\chi(\beta_1, \beta_2)$, $\eta(\chi(\beta_1, \beta_2)) = \chi(\beta_2, \beta_1)$ and $\chi(\beta_1, \alpha\beta_2) = \chi(\beta_1, \beta_2)\eta(\alpha)$. In Section 4.3 we show that the associative algebra $\mathfrak{a}$ has the following realization by 2×2 matrices with entries in the components of $\mathfrak{a}$:

$$\mathfrak{a} \cong \begin{bmatrix} A_1 & C \oplus E \\ C' \oplus E' & A_2 \end{bmatrix}.$$

4.1 Unital associative algebra $\mathfrak{a}$

We are going to define Lie and Jordan multiplication on $\mathfrak{a}$ by extending the bilinear products given in Table 4.1.1 in a natural way. It can be shown that all products $(\alpha_1, \alpha_2)_Z$ with $\alpha_1, \alpha_2 \in \mathfrak{a}$ are either symmetric or skew-symmetric. This is why we will write $(\alpha_1 \circ \alpha_2)_Z$ or $[\alpha_1, \alpha_2]_Z$, respectively, instead of $(\alpha_1, \alpha_2)_Z$. The aim of this section is to show that $\mathfrak{a}$ is an associative algebra with respect to the new multiplication given by

$$\alpha_1 \alpha_2 := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2}.$$

Remark 4.1.1. In this remark we rewrite some of the products in (3.4.4) in terms of symmetric and skew-symmetric elements. Note that every $x \in \mathfrak{g}$ is uniquely decomposed as $x = x^+ + x^-$ where $x^+ = \frac{x+x^t}{2} \in \mathfrak{g}^+$ and $x^- = \frac{x-x^t}{2} \in \mathfrak{g}^-$.

(a) Let $x_1^+ \otimes a_1^-, x_2^+ \otimes a_2^- \in \mathfrak{g}^+ \otimes A^-$ and $x_1^- \otimes a_1^+, x_2^- \otimes a_2^+ \in \mathfrak{g}^- \otimes A^+$. Since

$$[x \otimes a_1, y \otimes a_2] = x \circ y \otimes \frac{[a_1, a_2]}{2} + [x, y] \otimes \frac{a_1 \circ a_2}{2} + (x \mid y)\langle a_1, a_2 \rangle,$$

and $(x_1^+ \mid x_1^-) = \frac{1}{n}\operatorname{tr}(x_1^+ x_1^-) = 0$ we have

$$[x_1^+ \otimes a_1^-, x_2^+ \otimes a_2^-] = x_1^+ \circ x_2^+ \otimes \frac{[a_1^-, a_2^-]_{A^-}}{2} + [x_1^+, x_2^+] \otimes \frac{(a_1^- \circ a_2^-)_{A^+}}{2} + (x_1^+ \mid x_2^+)\langle a_1^-, a_2^- \rangle,$$
$$[x_1^- \otimes a_1^+, x_2^- \otimes a_2^+] = x_1^- \circ x_2^- \otimes \frac{[a_1^+, a_2^+]_{A^-}}{2} + [x_1^-, x_2^-] \otimes \frac{(a_1^+ \circ a_2^+)_{A^+}}{2} + (x_1^- \mid x_2^-)\langle a_1^+, a_2^+ \rangle,$$
$$[x_1^+ \otimes a_1^-, x_1^- \otimes a_1^+] = x_1^+ \diamond x_1^- \otimes \frac{[a_1^-, a_1^+]_{A^+}}{2} + [x_1^+, x_1^-] \otimes \frac{(a_1^- \circ a_1^+)_{A^-}}{2}.$$

(b) Let $s \otimes c \in S \otimes C$ and $\lambda \otimes e \in \Lambda \otimes E$. Since

$$[x \otimes a, s \otimes c] = (xs + sx^t) \otimes \frac{(a,c)_C}{2} + (xs - sx^t) \otimes \frac{(a,c)_E}{2} \text{ and}$$
$$x^+ s + s(x^+)^t = x^+ s + s x^+ = x^+ \circ s,$$
$$x^+ s - s(x^+)^t = x^+ s - s x^+ = [x^+, s],$$
$$x^- s + s(x^-)^t = x^- s - s x^- = [x^-, s],$$
$$x^- s - s(x^-)^t = x^- s + s x^- = x^- \circ s,$$

we obtain

$$[x^+ \otimes a^-, s \otimes c] = x^+ \diamond s \otimes \frac{[a^-, c]_C}{2} + [x^+, s] \otimes \frac{(a^- \circ c)_E}{2},$$
$$[x^- \otimes a^+, s \otimes c] = x^- \diamond s \otimes \frac{[a^+, c]_E}{2} + [x^-, s] \otimes \frac{(a^+ \circ c)_C}{2}.$$

Since

$$[x \otimes a, \lambda \otimes e] = (x\lambda + \lambda x^t) \otimes \frac{(a,e)_E}{2} + (x\lambda - \lambda x^t) \otimes \frac{(a,e)_C}{2} \text{ and}$$
$$x^+\lambda + \lambda(x^+)^t = x^+\lambda + \lambda x^+ = x^+ \circ \lambda,$$
$$x^+\lambda - \lambda(x^+)^t = x^+\lambda - \lambda x^+ = [x^+, \lambda],$$
$$x^-\lambda + \lambda(x^-)^t = x^-\lambda - \lambda x^- = [x^-, \lambda],$$
$$x^-\lambda - \lambda(x^-)^t = x^-\lambda + \lambda x^- = x^- \circ \lambda,$$

we get

$$[x^+ \otimes a^-, \lambda \otimes e] = x^+ \diamond \lambda \otimes \frac{[a^-, e]_E}{2} + [x^+, \lambda] \otimes \frac{(a^- \circ e)_C}{2},$$
$$[x^- \otimes a^+, \lambda \otimes e] = x^- \diamond \lambda \otimes \frac{[a^+, e]_C}{2} + [x^-, \lambda] \otimes \frac{(a^+ \circ e)_E}{2}.$$

(c) Let $s' \otimes c' \in S' \otimes C'$ and $\lambda' \otimes e' \in \Lambda' \otimes E'$. Since

$$[s' \otimes c', x \otimes a] = (s'x + x^t s') \otimes \frac{(c', a)_{C'}}{2} + (s'x - x^t s') \otimes \frac{(c', a)_{E'}}{2} \text{ and}$$
$$s'x^+ + (x^+)^t s' = s' \circ x^+,\ s'x^+ - (x^+)^t s' = [s', x^+],$$
$$s'x^- + (x^-)^t s' = [s', x^-],\ s'x^- - (x^-)^t s' = s' \circ x^-,$$

we get

$$[s' \otimes c', x^+ \otimes a^-] = s' \diamond x^+ \otimes \frac{[c', a^-]_{C'}}{2} + [s', x^+] \otimes \frac{(c' \circ a^-)_{E'}}{2},$$
$$[s' \otimes c', x^- \otimes a^+] = s' \diamond x^- \otimes \frac{[c', a^+]_{E'}}{2} + [s', x^-] \otimes \frac{(c' \circ a^+)_{C'}}{2}.$$

Since

$$[\lambda' \otimes e', x \otimes a] = (\lambda' x + x^t \lambda') \otimes \frac{(e', a)_{E'}}{2} + (\lambda' x - x^t \lambda') \otimes \frac{(e', a)_{C'}}{2} \text{ and}$$
$$\lambda' x^+ + (x^+)^t \lambda' = \lambda' \circ x^+,\ \lambda' x^+ - (x^+)^t \lambda' = [\lambda', x^+],$$

$$\lambda' x^- + (x^-)^t \lambda' = [\lambda', x^-],\ \lambda' x^- - (x^-)^t \lambda' = \lambda' \circ x^-,$$

we have

$$[\lambda' \otimes e', x^+ \otimes a^-] = \lambda' \diamond x^+ \otimes \frac{[e', a^-]_{E'}}{2} + [\lambda', x^+] \otimes \frac{(e' \circ a^-)_{C'}}{2},$$
$$[\lambda' \otimes e', x^- \otimes a^+] = \lambda' \diamond x^- \otimes \frac{[e', a^+]_{C'}}{2} + [\lambda', x^-] \otimes \frac{(e' \circ a^+)_{E'}}{2}.$$

(d) For any $x \otimes a \in \mathfrak{g} \otimes A$, $x \otimes a = \frac{(x+x^t)}{2} \otimes a + \frac{(x-x^t)}{2} \otimes a \in \mathfrak{g}^+ \otimes A + \mathfrak{g}^- \otimes A$. Since

$$[s \otimes c, s' \otimes c'] = (ss' - (s \mid s')I) \otimes (c, c')_A + (s \mid s')\langle c, c'\rangle, \text{ and}$$
$$ss' - (s \mid s')I + (ss' - (s \mid s')I)^t = s \circ s',$$
$$ss' - (s \mid s')I - (ss' - (s \mid s')I)^t = [s, s'],$$

we get

$$[s \otimes c, s' \otimes c'] = s \circ s' \otimes \frac{[c, c']_{A^-}}{2} + [s, s'] \otimes \frac{(c \circ c')_{A^+}}{2} + (s \mid s')\langle c, c'\rangle.$$

Since

$$[\lambda \otimes e, \lambda' \otimes e'] = (\lambda\lambda' - (\lambda \mid \lambda')I) \otimes (e, e')_A + (\lambda \mid \lambda')\langle e, e'\rangle \text{ and}$$
$$\lambda\lambda' - (\lambda \mid \lambda')I + (\lambda\lambda' - (\lambda \mid \lambda')I)^t = \lambda \circ \lambda',$$
$$\lambda\lambda' - (\lambda \mid \lambda')I - (\lambda\lambda' - (\lambda \mid \lambda')I)^t = [\lambda, \lambda'],$$

we obtain

$$[\lambda \otimes e, \lambda' \otimes e'] = \lambda \circ \lambda' \otimes \frac{[e, e']_{A^-}}{2} + [\lambda, \lambda'] \otimes \frac{(e \circ e')_{A^+}}{2} + (\lambda \mid \lambda')\langle e, e'\rangle.$$

Since $[s \otimes c, \lambda' \otimes e'] = s\lambda' \otimes (c, e')_A$ and $s\lambda' + (s\lambda')^t = [s, \lambda']$, $s\lambda' - (s\lambda')^t = s \circ \lambda'$, we get

$$[s \otimes c, \lambda' \otimes e'] = s \diamond \lambda' \otimes \frac{[c, e']_{A^+}}{2} + [s, \lambda'] \otimes \frac{(c \circ e')_{A^-}}{2}.$$

Since $[s' \otimes c', \lambda \otimes e] = s'\lambda \otimes (c', e)_A$ and $\lambda' s + (\lambda' s)^t = [\lambda', s]$, $(\lambda' s) - (s\lambda')^t = \lambda' \diamond s$, we have

$$[s' \otimes c', \lambda \otimes e] = s' \diamond \lambda \otimes \frac{[c', e]_{A^+}}{2} + [s', \lambda] \otimes \frac{(c' \circ e)_{A^-}}{2}.$$

The mappings $\alpha \otimes \beta \mapsto (\alpha \circ \beta)_{Z_1}$ and $\alpha \otimes \beta \mapsto [\alpha, \beta]_{Z_2}$ can be extended to $Y \otimes X$ in a consistent way by defining $(\beta \circ \alpha)_{Z_1} = (\alpha \circ \beta)_{Z_1}$ and $[\beta, \alpha]_{Z_2} = -[\alpha, \beta]_{Z_2}$. In Table 4.1.1

below, if the cell in row X and column Y contains $(Z_1, \circ)$, and $(Z_2, [\,])$ this means that there is a symmetric bilinear map $X \times Y \to Z_1$, given by $\alpha \otimes \beta \mapsto (\alpha \circ \beta)_{Z_1}$ and a skew symmetric bilinear map $X \times Y \to Z_2$, given by $\alpha \otimes \beta \mapsto [\alpha, \beta]_{Z_2}$ $(\alpha \in X, \beta \in Y)$.

.	A^+	A^-	C	E	C'	E'
A^+	$(A^+, \circ)$ $(A^-, [\,])$	$(A^-, \circ)$ $(A^+, [\,])$	$(C, \circ)$ $(E, [\,])$	$(E, \circ)$ $(C, [\,])$	$(C', \circ)$ $(E, [\,])$	$(E', \circ)$ $(C', [\,])$
A^-	$(A^-, \circ)$ $(A^+, [\,])$	$(A^+, \circ)$ $(A^-, [\,])$	$(E, \circ)$ $(C, [\,])$	$(C, \circ)$ $(E, [\,])$	$(E', \circ)$ $(C', [\,])$	$(C', \circ)$ $(E', [\,])$
C	$(C, \circ)$ $(E, [\,])$	$(E, \circ)$ $(C, [\,])$	0	0	$(A^+, \circ)$ $(A^-, [\,])$	$(A^-, \circ)$ $(A^+, [\,])$
E	$(E, \circ)$ $(C, [\,])$	$(C, \circ)$ $(E, [\,])$	0	0	$(A^-, \circ)$ $(A^+, [\,])$	$(A^+, \circ)$ $(A^-, [\,])$
C'	$(C', \circ)$ $(E, [\,])$	$(E', \circ)$ $(C', [\,])$	$(A^+, \circ)$ $(A^-, [\,])$	$(A^-, \circ)$ $(A^+, [\,])$	0	0
E'	$(E', \circ)$ $(C', [\,])$	$(C', \circ)$ $(E', [\,])$	$(A^-, \circ)$ $(A^+, [\,])$	$(A^+, \circ)$ $(A^-, [\,])$	0	0

Table 4.1.1: Products of homogeneous components of $\mathfrak{a}$

We are going to show that $\mathfrak{a} = A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E'$ is an associative algebra with respect to multiplication defined as follows:

$$\alpha_1 \alpha_2 := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2}$$

for all homogeneous $\alpha_1, \alpha_2 \in \mathfrak{a}$ with the products $[\,]$ and $\circ$ given by Table 4.1.1. Note that $[\alpha_1, \alpha_2] = \alpha_1 \alpha_2 - \alpha_2 \alpha_1$ and $\alpha_1 \circ \alpha_2 = \alpha_1 \alpha_2 + \alpha_2 \alpha_1$.

From Table 4.1.1 and the formulas in Remark 4.1.1, we deduce the following.

Lemma 4.1.2. *Let α_1, α_2 and α_3 be homogeneous elements of $\mathfrak{a}$. Then*

$$[z_1 \otimes \alpha_1, z_2 \otimes \alpha_2] \quad = \quad z_1 \circ z_2 \otimes \frac{[\alpha_1, \alpha_2]}{2} + [z_1, z_2] \otimes \frac{\alpha_1 \circ \alpha_2}{2} + (z_1 \mid z_2) \langle \alpha_1, \alpha_2 \rangle,$$

if $\alpha_1, \alpha_2 \in X = A^\pm$ or $\alpha_1 \in X$ and $\alpha_2 \in X'$ with $X = C, E$. In all other cases we have

$$[z_1 \otimes \alpha_1, z_2 \otimes \alpha_2] \quad = \quad z_1 \diamond z_2 \otimes \frac{[\alpha_1, \alpha_2]}{2} + [z_1, z_2] \otimes \frac{\alpha_1 \circ \alpha_2}{2}.$$

Theorem 4.1.3. *$\mathfrak{a} = A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E'$ is an associative algebra with identity element 1^+.*

Proof. It will be shown in Proposition 4.2.2 that 1^+ is the identity element of a larger algebra $\mathfrak{b}$ containing $\mathfrak{a}$ as a subalgebra. Therefore we only need to prove the associativity. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{a}$. We need to show that $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$. By linearity, we can assume that α_1, α_2 and α_3 are homogeneous. Set

$$z_1 = E_{1,2} + \varepsilon_1 E_{2,1},\; z_2 = E_{2,3} + \varepsilon_2 E_{3,2} \text{ and } z_3 = E_{3,4} + \varepsilon_3 E_{4,3} \text{ where } \varepsilon_i = \pm 1.$$

The signs of each ε_i can be chosen in such a way that $z_i \otimes \alpha_i$ belongs to the corresponding homogeneous component of L. Note that $\operatorname{tr}(z_i z_j) = 0$, for all $i \neq j$. Hence by Lemma 4.1.2, we have

$$[z_i \otimes \alpha_i, z_j \otimes \alpha_j] = z_i \diamond z_j \otimes \frac{[\alpha_i, \alpha_j]}{2} + [z_i, z_j] \otimes \frac{\alpha_i \circ \alpha_j}{2}.$$

Consider the Jacoby identity for $z_1 \otimes \alpha_1, z_2 \otimes \alpha_2, z_3 \otimes \alpha_3$:

$$[z_1 \otimes \alpha_1, [z_2 \otimes \alpha_2, z_3 \otimes \alpha_3]] = [[z_1 \otimes \alpha_1, z_2 \otimes \alpha_2], z_3 \otimes \alpha_3] + [z_2 \otimes \alpha_2, [z_1 \otimes \alpha_1, z_3 \otimes \alpha_3]].$$

Using Lemma 4.1.2 yields

$$\begin{aligned}
&[z_1, [z_2, z_3]] \otimes \frac{\alpha_1 \circ (\alpha_2 \circ \alpha_3)}{2} + z_1 \diamond [z_2, z_3] \otimes \frac{[\alpha_1, \alpha_2 \circ \alpha_3]}{4} \\
&+ [z_1, (z_2 \diamond z_3)] \otimes \frac{\alpha_1 \circ [\alpha_2, \alpha_3]}{4} + z_1 \diamond (z_2 \diamond z_3) \otimes \frac{[\alpha_1, [\alpha_2, \alpha_3]]}{4} \\
&= [[z_1, z_2], z_3] \otimes \frac{(\alpha_1 \circ \alpha_2) \circ \alpha_3}{4} + ([z_1, z_2] \diamond z_3) \otimes \frac{[\alpha_1 \circ \alpha_2, \alpha_3]}{4} + [z_1 \diamond z_2, z_3] \otimes \frac{[\alpha_1, \alpha_2] \circ \alpha_3}{4} \\
&+ (z_1 \circ z_2) \circ z_3 \otimes \frac{[[\alpha_1, \alpha_2], \alpha_3]}{4} + [z_2, [z_1, z_3]] \otimes \frac{\alpha_2 \circ (\alpha_1 \circ \alpha_3)}{4} + z_2 \diamond [z_1, z_3] \otimes \frac{[\alpha_2, \alpha_1 \circ \alpha_3]}{4} \\
&+ [z_2, (z_1 \diamond z_3)] \otimes \frac{\alpha_2 \circ [\alpha_1, \alpha_3]}{4} + z_2 \diamond (z_1 \diamond z_3) \otimes \frac{[\alpha_2, [\alpha_1, \alpha_3]]}{4}.
\end{aligned} \tag{4.1.1}$$

Note that

$$\begin{aligned}
z_1 \diamond (z_2 \diamond z_3) &= E_{1,4} + \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
[z_1, (z_2 \diamond z_3)] &= E_{1,4} - \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
z_1 \diamond [z_2, z_3] &= E_{1,4} - \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
[[z_1, z_2], z_3] &= E_{1,4} + \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
(z_1 \diamond z_2) \diamond z_3 &= E_{1,4} + \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
[z_1 \diamond z_2, z_3] &= E_{1,4} - \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1}, \\
[z_1, z_2] \diamond z_3 &= E_{1,4} - \varepsilon_1\varepsilon_2\varepsilon_3 E_{4,1},
\end{aligned}$$

$$[z_2,[z_1,z_3]] = z_2 \diamond (z_1 \diamond z_3) = [z_2,(z_1 \diamond z_3)] = z_2 \diamond [z_1,z_3] = 0.$$

Now (4.1.1) becomes

$$\begin{aligned}
&(E_{1,4}+\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes\alpha_1\circ(\alpha_2\circ\alpha_3)+(E_{1,4}-\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes[\alpha_1,\alpha_2\circ\alpha_3]\\
&+(E_{1,4}-\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes\alpha_1\circ[\alpha_2,\alpha_3]+(E_{1,4}+\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes[\alpha_1,[\alpha_2,\alpha_3]]\\
&=(E_{1,4}+\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes(\alpha_1\circ\alpha_2)\circ\alpha_3+(E_{1,4}-\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes[\alpha_1\circ\alpha_2,\alpha_3]\\
&+(E_{1,4}-\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes[\alpha_1,\alpha_2]\circ\alpha_3+(E_{1,4}+\varepsilon_1\varepsilon_2\varepsilon_3E_{4,1})\otimes[[\alpha_1,\alpha_2],\alpha_3].
\end{aligned}$$

By collecting the coefficients of $E_{1,4}$ we get

$$\begin{aligned}
&\alpha_1\circ(\alpha_2\circ\alpha_3)+[\alpha_1,\alpha_2\circ\alpha_3]+\alpha_1\circ[\alpha_2,\alpha_3]+[\alpha_1,[\alpha_2,\alpha_3]]\\
&=(\alpha_1\circ\alpha_2)\circ\alpha_3+[\alpha_1\circ\alpha_2,\alpha_3]+[\alpha_1,\alpha_2]\circ\alpha_3+[[\alpha_1,\alpha_2],\alpha_3],
\end{aligned}$$

or equivalently $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$, as required. □

From Theorem 4.1.3 and tensor product decompositions for sl_n ($n \geq 4$), we deduce the following

Corollary 4.1.4. *$\mathscr{A} = A^- \oplus A^+$ is an associative subalgebra of $\mathfrak{a}$ with identity element 1^+.*

Corollary 4.1.5. *$C \oplus E$ and $C' \oplus E'$ are $\mathscr{A}$-bimodules.*

Theorem 4.1.6. *The linear transformation $\gamma : \mathfrak{a} \to \mathfrak{a}$ defined by*

$$\gamma(a^-) = -a^-, \gamma(a^+) = a^+, \gamma(c) = -c, \gamma(e) = e, \gamma(c') = -c', \gamma(e') = e',$$

is an antiautomorphism of order 2 of the algebra $\mathfrak{a}$.

Proof. We need only to check that $\gamma(xy) = \gamma(y)\gamma(x)$ for all homogeneous x and y in $\mathfrak{a}$:

$$\begin{aligned}
\gamma(a_1^+a_2^+) &= \gamma\Big(\frac{[a_1^+,a_2^+]}{2}+\frac{a_1^+\circ a_2^+}{2}\Big) = -\frac{[a_1^+,a_2^+]}{2}+\frac{a_1^+\circ a_2^+}{2} = a_2^+a_1^+ = \gamma(a_2^+)\gamma(a_1^+),\\
\gamma(a_1^-a_2^-) &= \gamma\Big(\frac{[a_1^-,a_2^-]}{2}+\frac{a_1^-\circ a_2^-}{2}\Big) = -\frac{[a_1^-,a_2^-]}{2}+\frac{a_1^-\circ a_2^-}{2} = a_2^-a_1^- = \gamma(a_2^-)\gamma(a_1^-),\\
\gamma(a_1^+a_2^-) &= \gamma\Big(\frac{[a_1^+,a_2^-]}{2}+\frac{a_1^+\circ a_2^-}{2}\Big) = \frac{[a_1^+,a_2^-]}{2}-\frac{a_1^+\circ a_2^-}{2} = (-a_2^-)a_1^+ = \gamma(a_2^-)\gamma(a_1^+),\\
\gamma(a_1^-a_2^+) &= \gamma\Big(\frac{[a_1^-,a_2^+]}{2}+\frac{a_1^-\circ a_2^+}{2}\Big) = \frac{[a_1^-,a_2^+]}{2}-\frac{a_1^-\circ a_2^+}{2} = a_2^+(-a_1^-) = \gamma(a_2^+)\gamma(a_1^-),\\
\gamma(a^-c) &= \gamma\Big(\frac{[a^-,c]_C}{2}+\frac{(a^-\circ c)_E}{2}\Big) = -\frac{[a^-,c]_C}{2}+\frac{(a^-\circ c)_E}{2} = ca^- = \gamma(c)\gamma(a^-),
\end{aligned}$$

$$\gamma(a^-e) = \gamma(\frac{[a^-,e]_E}{2} + \frac{(a^-\circ e)_C}{2}) = \frac{[a^-,e]_E}{2} - \frac{(a^-\circ e)_C}{2} = e(-a^-) = \gamma(e)\gamma(a^-),$$
$$\gamma(a^+c) = \gamma(\frac{[a^+,c]_E}{2} + \frac{(a^+\circ c)_C}{2}) = \frac{[a^+,c]_E}{2} - \frac{(a^+\circ c)_C}{2} = (-c)a^+ = \gamma(c)\gamma(a^+),$$
$$\gamma(a^+e) = \gamma(\frac{[a^+,e]_C}{2} + \frac{(a^+\circ e)_E}{2}) = -\frac{[a^+,e]_C}{2} + \frac{(a^+\circ e)_E}{2} = ea^+ = \gamma(e)\gamma(a^+),$$
$$\gamma(c'a^-) = \gamma(\frac{[c',a^-]_{C'}}{2} + \frac{(c'\circ a^-)_{E'}}{2}) = -\frac{[a^-,c']_{C'}}{2} + \frac{(a^-\circ c')_{E'}}{2} = a^-c' = \gamma(a^-)\gamma(c'),$$
$$\gamma(e'a^-) = \gamma(\frac{[e',a^-]_{E'}}{2} + \frac{(e'\circ a^-)_{C'}}{2}) = \frac{[e',a^-]_{E'}}{2} - \frac{(e'\circ a^-)_{C'}}{2} = (-a^-)e' = \gamma(a^-)\gamma(e'),$$
$$\gamma(c'a^+) = \gamma(\frac{[c',a^+]_{E'}}{2} + \frac{(c',a^+)_{C'}}{2}) = \frac{[c',a^+]_{E'}}{2} - \frac{(c',a^+)_{C'}}{2} = a^+(-c') = \gamma(a^+)\gamma(c'),$$
$$\gamma(e'a^+) = \gamma(\frac{[e',a^+]_{C'}}{2} + \frac{(e'\circ a^+)_{E'}}{2}) = -\frac{[e',a^+]_{C'}}{2} + \frac{(e'\circ a^+)_{E'}}{2} = a^+e' = \gamma(a^+)\gamma(e'),$$
$$\gamma(cc') = \gamma(\frac{[c,c']_{A^-}}{2} + \frac{(c\circ c')_{A^+}}{2}) = -\frac{[c,c']_{A^-}}{2} + \frac{(c\circ c')_{A^+}}{2} = c'c = \gamma(c')\gamma(c),$$
$$\gamma(ee') = \gamma(\frac{[e,e']_{A^-}}{2} + \frac{(e\circ e')_{A^+}}{2}) = -\frac{[e,e']_{A^-}}{2} + \frac{(e\circ e')_{A^+}}{2} = e'e = \gamma(e')\gamma(e),$$
$$\gamma(ce') = \gamma(\frac{[c,e']_{A^+}}{2} + \frac{(c\circ e')_{A^-}}{2}) = \frac{[c,e']_{A^+}}{2} - \frac{(c\circ e')_{A^-}}{2} = e'(-c) = \gamma(e')\gamma(c),$$
$$\gamma(c'e) = \gamma(\frac{[c',e]_{A^+}}{2} + \frac{(c'\circ e)_{A^-}}{2}) = \frac{[c',e]_{A^+}}{2} - \frac{(c'\circ e)_{A^-}}{2} = (-c')e = \gamma(e)\gamma(c').$$

□

4.2 Coordinate algebra $\mathfrak{b}$

Define
$$\mathfrak{b} := \mathfrak{a}\oplus B\oplus B' = A^+\oplus A^-\oplus C\oplus E\oplus C'\oplus E'\oplus B\oplus B'.$$

The aim of this section is to show that $\mathfrak{b}$ is an algebra with identity 1^+ with respect to the multiplication extending that on $\mathfrak{a}$ given in Table 4.2.1. It can be shown that all products $(\beta_1,\beta_2)_Z$ with $\beta_1,\beta_2\in B\oplus B'$ are either symmetric or skew-symmetric. This is why we will write $(\beta_1\circ\beta_2)_Z$ or $[\beta_1,\beta_2]_Z$, respectively, instead of $(\beta_1,\beta_2)_Z$. For $\alpha\in\mathfrak{a}$ and $\beta\in B\oplus B'$ we will write $\alpha\beta$ (resp. $\beta\alpha$) instead of $(\alpha,\beta)_Z$ (resp. $(\beta,\alpha)_Z$). Let $b\in B$ and $b'\in B$. We define $b\alpha := \gamma(\alpha)b$ and $\alpha b' := b'\gamma(\alpha)$. We will show that $B\oplus B'$ is an $\mathfrak{a}$-bimodule.

Recall that

$$x\otimes a = \frac{(x+x^t)}{2}\otimes a + \frac{(x-x^t)}{2}\otimes a \in \mathfrak{g}^+\otimes A + \mathfrak{g}^-\otimes A.$$

Let $u\otimes b\in V\otimes B$ and $v'\otimes b'\in V'\otimes B'$. We need the following formula from (3.4.4):

$$[u\otimes b, v'\otimes b'] = (uv'^t - \frac{\operatorname{tr}(uv'^t)}{n}I)\otimes (b,b')_A + \frac{2\operatorname{tr}(uv'^t)}{n}\langle b,b'\rangle.$$

By splitting $(b,b')_A$ into symmetric and skew-symmetric parts and using the equations

$$\begin{aligned}
(uv'^t - \frac{\operatorname{tr}(uv'^t)}{n}I) + (uv'^t - \frac{\operatorname{tr}(uv'^t)}{n}I)^t &= uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n}I,\\
(uv'^t - \frac{\operatorname{tr}(uv'^t)}{n}I) - (uv'^t - \frac{\operatorname{tr}(uv'^t)}{n}I)^t &= uv'^t - v'u^t,
\end{aligned}$$

we get

$$\begin{aligned}
[u\otimes b, v'\otimes b'] = (uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n}I)\otimes \frac{[b,b']_{A^-}}{2} + \\
(uv'^t - v'u^t)\otimes \frac{(b\circ b')_{A^+}}{2} + \frac{2\operatorname{tr}(uv'^t)}{n}\langle b,b'\rangle.
\end{aligned} \tag{4.2.1}$$

Let $b,b_1,b_2\in B$ and $b',b'_1,b'_2\in B'$. Using (3.4.4) and (4.2.1) we get

$$\begin{aligned}
[u\otimes b_1, v\otimes b_2] &= (uv^t + vu^t)\otimes \frac{[b_1,b_2]_C}{2} + (uv^t - vu^t)\otimes \frac{(b_1\circ b_2)_E}{2},\\
[u'\otimes b'_1, v'\otimes b'_2] &= (u'v'^t + v'u'^t)\otimes \frac{[b'_1,b'_2]_{C'}}{2} + (u'v'^t - v'u'^t)\otimes \frac{(b'_1\circ b'_2)_{E'}}{2},\\
[u\otimes b, v'\otimes b'] &= (uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n}I)\otimes \frac{[b,b']_{A^-}}{2} + \\
&\quad (uv'^t - v'u^t)\otimes \frac{(b\circ b')_{A^+}}{2} + \frac{2\operatorname{tr}(uv'^t)}{n}\langle b,b'\rangle.
\end{aligned} \tag{4.2.2}$$

We define

$$\begin{aligned}
b_1b_2 &:= \frac{[b_1,b_2]_C}{2} + \frac{(b_1\circ b_2)_E}{2}, & b'_1b'_2 &:= \frac{[b'_1,b'_2]_{C'}}{2} + \frac{(b'_1\circ b'_2)_{E'}}{2},\\
bb' &:= \frac{[b,b']_{A^-}}{2} + \frac{(b\circ b')_{A^+}}{2}, & b'b &:= -\frac{[b,b']_{A^-}}{2} + \frac{(b\circ b')_{A^+}}{2}.
\end{aligned}$$

Then $\mathfrak{b} = \mathfrak{a}\oplus B\oplus B'$ is an algebra with multiplication extending that on $\mathfrak{a}$. The following table describes the products of homogeneous elements of $\mathfrak{b}$ (use Table 4.1.1 for the products on $\mathfrak{a}$).

$\cdot$	$A^+ + A^-$	$C+E$	$C'+E'$	B	B'
$A^+ + A^-$	$A^+ + A^-$	$C+E$	$C'+E'$	B	B'
$C+E$	$C+E$	0	$A^+ + A^-$	0	B
$C'+E'$	$C'+E'$	$A^+ + A^-$	0	B'	0
B	B	0	B'	$(E,\circ)$ $(C,[\,])$	$(A^+,\circ)$ $(A^-,[\,])$
B'	B'	B	0	$(A^+,\circ)$ $(A^-,[\,])$	$(E',\circ)$ $(C',[\,])$

Table 4.2.1: Products in $\mathfrak{b}$

Theorem 4.2.1. *The linear transformation $\eta : \mathfrak{b} \to \mathfrak{b}$ defined by $\eta(\alpha) = \gamma(\alpha)$, $\eta(b) = b$ and $\eta(b') = b'$ for all $\alpha \in \mathfrak{a}$, $b \in B$ and $b' \in B'$ is an antiautomorphism of order 2 of the algebra $\mathfrak{b}$.*

Proof. In Theorem 4.1.6, we showed that $\eta(xy) = \eta(y)\eta(x)$ for all x and y in $\mathfrak{a}$. Let $b, b_1, b_2 \in B, b', b_1', b_2' \in B'$ and $\alpha \in \mathfrak{a}$. We have

$$\begin{aligned}
\eta(b_1b_2) &= \eta\Big(\frac{[b_1,b_2]_C + (b_1 \circ b_2)_E}{2}\Big) = \frac{-[b_1,b_2]_C + (b_1 \circ b_2)_E}{2} = b_2b_1 = \eta(b_2)\eta(b_1),\\
\eta(b_1'b_2') &= \eta\Big(\frac{[b_1',b_2']_{C'} + (b_1' \circ b_2')_{E'}}{2}\Big) = \frac{-[b_1',b_2']_{C'} + (b_1' \circ b_2')_{E'}}{2} = b_2'b_1' = \eta(b_2')\eta(b_1'),\\
\eta(bb') &= \eta\Big(\frac{[b,b']_{A^-} + (b \circ b')_{A^+}}{2}\Big) = \frac{-[b,b']_{A^-} + (b \circ b')_{A^+}}{2} = b'b = \eta(b')\eta(b),\\
\eta(\alpha b) &= \alpha b = b\eta(\alpha) = \eta(b)\eta(\alpha),\\
\eta(b'\alpha) &= b'\alpha = \eta(\alpha)b' = \eta(\alpha)\eta(b').
\end{aligned}$$

Using these properties and Theorem 4.1.6 we deduce that η is an antiautomorphism of order 2 of the algebra $\mathfrak{b}$. $\square$

Proposition 4.2.2. *1^+ is the identity element of $\mathfrak{b}$.*

Proof. Let $a^\pm \in A^\pm$, $b \in B$, $b' \in B'$, $c \in C$, $c' \in C'$, $e \in E$ and $e' \in E'$. Recall that we identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1$ where 1 is a distinguished element of A and 1^+ is the image of 1 in A^+. Using (3.4.4) and (3.4.2) we get

$$\begin{aligned}
[x_1^-, x_2^-] \otimes a^+ &= [x_1^-, x_2^-] \otimes \frac{(1^+ \circ a^+)_{A^+}}{2} + [x_1^-, x_2^-] \otimes \frac{[1^+, a^+]_{A^-}}{2} + (x_1^- \mid x_2^-)\langle 1^+, a^+\rangle,\\
[x_1^-, x_1^+] \otimes a^- &= [x_1^-, x_1^+] \otimes \frac{(1^+ \circ a^-)_{A^-}}{2} + x_1^- \circ x_1^+ \otimes \frac{[1^+, a^-]_{A^-}}{2},
\end{aligned}$$

$$[x^-,s]\otimes c=[x^-,s]\otimes\frac{(1^+\circ c)_C}{2}+x^-\circ s\otimes\frac{[1^+,c]_E}{2},$$
$$[x^-,\lambda]\otimes e=[x^-,\lambda]\otimes\frac{(1^+\circ e)_E}{2}+x^-\circ\lambda\otimes\frac{[1^+,e]_C}{2},$$
$$[x^-,s']\otimes c'=[x^-,s']\otimes\frac{(1^+\circ c')_{C'}}{2}+x^-\circ s'\otimes\frac{[1^+,c']_{E'}}{2},$$
$$[x^-,\lambda']\otimes e'=[x^-,\lambda']\otimes\frac{(1^+\circ e')_{E'}}{2}+x^-\circ\lambda'\otimes\frac{[1^+\circ e']_{C'}}{2},$$

and $xu\otimes 1^+.b=xu\otimes b$, $x^t u'\otimes 1^+.b'=x^t u'\otimes b'$. This implies that $\frac{(1^+\circ a^+)_{A^+}}{2}=a^+$, $\frac{[1^+,a^+]_{A^-}}{2}=0$, $\frac{(1^+\circ a^-)_{A^-}}{2}=a^-$, $\frac{[1^+,a^-]_{A^-}}{2}=0$, $\frac{(1^+\circ c)_C}{2}=c$, $\frac{[1^+,c]_E}{2}=0$, $\frac{(1^+\circ e)_E}{2}=e$, $\frac{[1^+,e]_C}{2}=0$, $\frac{(1^+\circ c')_{C'}}{2}=c'$, $\frac{[1^+,c']_{E'}}{2}=0$, $\frac{(1^+\circ e')_{E'}}{2}=e'$, $\frac{[1^+\circ e']_{C'}}{2}=0$, $1^+.b=b$ and $1^+.b'=b'$. Combining these properties and the fact that $\circ$ is symmetric, $[\,,]$ is skew symmetric and $\eta(1^+)=1^+$, we see that 1^+ is the identity element of $\mathfrak{b}$. $\square$

Using (3.4.4) and Table 4.2.1, we deduce the following.

Lemma 4.2.3. *Let $b\in B$, $b'\in B'$ and $\alpha\in\mathfrak{a}$. Then*

$$[z\otimes\alpha,u\otimes b]=zu\otimes\alpha b=-[u\otimes b,z\otimes\alpha],$$
$$[u'\otimes b',z\otimes\alpha]=z^t u'\otimes b'\alpha=-[z\otimes\alpha,u'\otimes b'].$$

Proposition 4.2.4. *$B\oplus B'$ is an $\mathfrak{a}$-bimodule.*

Proof. Let $b\in B, b'\in B'$ and let α_1,α_2 be homogeneous elements in $\mathfrak{a}$. Set

$$z_1=E_{1,2}+\varepsilon_1E_{2,1},\ z_2=E_{2,3}+\varepsilon_2E_{3,2}\text{ and }u=u'=e_3\text{ where }\varepsilon_i=\pm1.$$

Then $[z_1,z_2]=E_{1,3}-\varepsilon_1\varepsilon_2E_{3,1}$, $z_1\circ z_2=E_{1,3}+\varepsilon_1\varepsilon_2E_{3,1}$, $z_1z_2=E_{1,3}$ and $(z_1|z_2)=0$.

First we are going to show that $(\alpha_1\alpha_2)b=\alpha_1(\alpha_2b)$. Consider the Jacoby identity for $z_1\otimes\alpha_1,z_2\otimes\alpha_2,u\otimes b$:

$$[z_1\otimes\alpha_1,[z_2\otimes\alpha_2,u\otimes b]]=[[z_1\otimes\alpha_1,z_2\otimes\alpha_2],u\otimes b]+[z_2\otimes\alpha_2,[z_1\otimes\alpha_1,u\otimes b]].$$

Using Lemmas 4.2.3 and 4.1.2 we get

$$z_1(z_2u)\otimes\alpha_1(\alpha_2b)-(z_1\circ z_2)u\otimes\frac{[\alpha_1,\alpha_2]}{2}b-[z_1,z_2]u\otimes\frac{\alpha_1\circ\alpha_2}{2}b=0. \tag{4.2.3}$$

Substituting matrix units, we get that

$$e_1 \otimes (\alpha_1(\alpha_2 b) - \frac{[\alpha_1,\alpha_2]}{2} b - \frac{\alpha_1 \circ \alpha_2}{2} b) = 0,$$

so $\alpha_1(\alpha_2 b) = \frac{[\alpha_1,\alpha_2]}{2} b + \frac{\alpha_1\circ\alpha_2}{2} b = (\alpha_1\alpha_2)b$, as required.

Now we are going to show that $(b'\alpha_2)\alpha_1 = b'(\alpha_2\alpha_1)$. Consider the Jacoby identity for $z_1\otimes\alpha_1, z_2\otimes\alpha_2, u'\otimes b'$:

$$[z_1\otimes\alpha_1,[z_2\otimes\alpha_2,u'\otimes b']] = [[z_1\otimes\alpha_1,z_2\otimes\alpha_2],u'\otimes b'] + [z_2\otimes\alpha_2,[z_1\otimes\alpha_1,u'\otimes b']].$$

Using Lemmas 4.1.2 and 4.2.3 we get

$$(z_2z_1)^t u' \otimes (b'\alpha_2)\alpha_1 = -(z_1\circ z_2)^t u' \otimes b'\frac{[\alpha_1,\alpha_2]}{2} - [z_1,z_2]^t u' \otimes b'\frac{\alpha_1\circ\alpha_2}{2}.$$

Substituting matrix units, we get that

$$\varepsilon_1\varepsilon_2 e_1 \otimes (b'\alpha_2)\alpha_1 = -\varepsilon_1\varepsilon_2 e_1 \otimes b'\frac{[\alpha_1,\alpha_2]}{2} + b'\frac{\alpha_1\circ\alpha_2}{2},$$

so $(b'\alpha_2)\alpha_1 = b'(\alpha_2\alpha_1)$, as required. It remains to show $b(\alpha_1\alpha_2) = (b\alpha_1)\alpha_2$ and $(\alpha_1\alpha_2)b' = \alpha_1(\alpha_2 b')$. We have

$$\begin{aligned} b(\alpha_1\alpha_2) &= \eta(\eta(\alpha_1\alpha_2)\eta(b)) = \eta((\eta(\alpha_2)\eta(\alpha_1))\eta(b)) \\ &= \eta(\eta(\alpha_2)(\eta(\alpha_1)\eta(b))) = \eta(\eta(\alpha_2)\eta((b\alpha_1))) = (b\alpha_1)\alpha_2. \end{aligned}$$

Similarly, we get $(\alpha_1\alpha_2)b' = \alpha_1(\alpha_2 b')$, as required. □

Note that both B and B' are invariant under multiplication by $\mathscr{A}$, see Table 4.2.1, so we get the following.

Corollary 4.2.5. *B and B' are $\mathscr{A}$-bimodules.*

Proposition 4.2.6. *Let $\chi(\beta_1,\beta_2) := \beta_1\beta_2$ for all $\beta_1,\beta_2 \in B\oplus B'$. Then χ is a hermitian form on the $\mathfrak{a}$-bimodule $B\oplus B'$ with values in $\mathfrak{a}$. More exactly, for all $\alpha\in\mathfrak{a}$ and $\beta_1,\beta_2\in B\oplus B'$ we have*

(i) $\chi(\alpha\beta_1,\beta_2) = \alpha\chi(\beta_1,\beta_2)$,
(ii) $\eta(\chi(\beta_1,\beta_2)) = \chi(\beta_2,\beta_1)$,
(iii) $\chi(\beta_1,\alpha\beta_2) = \chi(\beta_1,\beta_2)\eta(\alpha)$.

Proof. (i) We need to show that $(\alpha\beta_1)\beta_2 = \alpha(\beta_1\beta_2)$ for all homogeneous β_1, β_2 in $B\oplus B'$ and $\alpha\in\mathfrak{a}$. Set $z = E_{1,2}+\varepsilon E_{2,1}$, $u_1 = u_1' = e_1$ and $u_2 = u_2' = e_3$ where $\varepsilon = \pm 1$. Let

$b_1, b_2 \in B$ and $b'_1, b'_2 \in B'$. First we are going to show that $\alpha(b_1b_2) = (\alpha b_1)b_2$. Consider the Jacoby identity for $z\otimes\alpha$, $u_1\otimes b_1$, $u_2\otimes b_2$:

$$[z\otimes\alpha,[u_1\otimes b_1,u_2\otimes b_2]] = [[z\otimes\alpha,u_1\otimes b_1],u_2\otimes b_2]+[u_1\otimes b_1,[z\otimes\alpha,u_2\otimes b_2]].$$

Using (4.2.2) and Lemma 4.2.3 we get

$$[z\otimes\alpha,(E_{1,3}+E_{3,1})\otimes\frac{[b_1,b_2]_C}{2}+[z\otimes\alpha,(E_{1,3}-E_{3,1})\otimes\frac{(b_1\circ b_2)_E}{2}] = [\varepsilon e_2\otimes\alpha b_1,u_2\otimes b_2]. \tag{4.2.4}$$

By using Lemma 4.1.2 and (4.2.2), we get

$$\begin{aligned}
&(\varepsilon E_{2,3}+\varepsilon E_{3,2})\otimes\frac{[\alpha,[b_1,b_2]_C]}{2}+(\varepsilon E_{2,3}-\varepsilon E_{3,2})\otimes\frac{\alpha\circ[b_1,b_2]_C}{2}\\
&+(\varepsilon E_{2,3}+\varepsilon E_{3,2})\otimes\frac{[\alpha,(b_1\circ b_2)_E]}{2}+(\varepsilon E_{2,3}-\varepsilon E_{3,2})\otimes\frac{\alpha\circ(b_1\circ b_2)_E}{2}\\
&=(\varepsilon E_{2,3}+\varepsilon E_{3,2})\otimes\frac{[\alpha b_1,b_2]}{2}+(\varepsilon E_{2,3}-\varepsilon E_{3,2})\otimes\frac{\alpha b_1\circ b_2}{2}
\end{aligned}$$

By collecting the coefficients of $E_{2,3}$, we get:

$$\frac{[\alpha,[b_1,b_2]_C]+\alpha\circ[b_1,b_2]_C}{2}+\frac{[\alpha,(b_1\circ b_2)_E]+\alpha\circ(b_1\circ b_2)_E}{2}=\frac{[\alpha b_1,b_2]+\alpha b_1\circ b_2}{2},$$

or equivalently $\alpha(b_1b_2) = (\alpha b_1)b_2$, as required.

Similarly, one can show that $\alpha(b_1b'_2) = (\alpha b_1)b'_2$ (by using the Jacoby identity for $z\otimes\alpha, u_1\otimes b_1, u'_2\otimes b'_2$).

By using the Jacoby identity for $z\otimes\alpha$, $u'_1\otimes b'_1$, $u'_2\otimes b'_2$ and similar calculations we get $b'_2(b'_1\alpha) = (b'_2b'_1)\alpha$. By applying the involution η to both sides and using the fact that η is identity on both B and B', we get $(\eta(\alpha)b'_1)b'_2 = \eta(\alpha)(b'_1b'_2)$, or equivalently $(\alpha b'_1)b'_2 = \alpha(b'_1b'_2)$, as required.

By using the Jacoby identity for $z\otimes\alpha, u_1\otimes b_1, u'_2\otimes b'_2$ we get $(b_2b'_1)\alpha = b_2(b'_1\alpha)$. By applying η we get $\eta(\alpha)(b'_1b_2) = (\eta(\alpha)b'_1)b_2$, or equivalently $\alpha(b'_1b_2) = (\alpha b'_1)b_2$, as required.

(ii) We only need to check this for homogeneous elements. We have

$$\begin{aligned}
\eta(\chi(b_1,b_2)) &= \eta(\frac{[b_1,b_2]_C+(b_1\circ b_2)_E}{2}) = \frac{-[b_1,b_2]_C+(b_1\circ b_2)_E}{2} = \chi(b_2,b_1),\\
\eta(\chi(b'_1,b'_2)) &= \eta(\frac{[b'_1,b'_2]_{C'}+(b'_1\circ b'_2)_{E'}}{2}) = \frac{-[b'_1,b'_2]_{C'}+(b'_1\circ b'_2)_{E'}}{2} = \chi(b'_2,b'_1),\\
\eta(\chi(b_1,b'_1)) &= \eta(\frac{[b_1,b'_1]_{A^-}+(b_1\circ b'_1)_{A^+}}{2}) = \frac{-[b_1,b'_1]_{A^-}+(b_1\circ b'_1)_{A^+}}{2} = \chi(b'_1,b_1),
\end{aligned}$$

$$\eta(\chi(b_1',b_1)) = \eta(\frac{[b_1',b_1]_{A^-}+(b_1'\circ b_1)_{A^+}}{2}) = \frac{-[b_1',b_1]_{A^-}+(b_1'\circ b_1)_{A^+}}{2} = \chi(b_1,b_1'),$$

as required.

(iii) Using (i) and (ii), we get

$$\chi(\beta_1,\alpha\beta_2) = \eta(\chi(\alpha\beta_2,\beta_1)) = \eta(\alpha\chi(\beta_2,\beta_1)) = \eta(\chi(\beta_2,\beta_1))\eta(\alpha) = \chi(\beta_1,\beta_2)\eta(\alpha).$$

□

The mapping $\langle,\rangle : X\otimes X' \to D$ with $X = B,C,E$ can be extended to $X'\otimes X$ in a consistent way by defining $\langle x',x\rangle := -\langle x,x'\rangle$. Let $X,Y \in \{A^+,A^-,B,B',C,C',E,E'\}$. Recall also the maps $\langle,\rangle : A^\pm\otimes A^\pm \to D$ described previously (see Remark 4.1.1(a)). For the convenience, we extend the mappings to the whole space $\mathfrak{b}$ by defining the remaining $\langle X,Y\rangle$ to be zero. Hence

$$\langle\mathfrak{b},\mathfrak{b}\rangle = \langle A^+,A^+\rangle + \langle A^-,A^-\rangle + \langle B,B'\rangle + \langle C,C'\rangle + \langle E,E'\rangle.$$

It follows from condition $(\Gamma 3)$ in Definition 3.0.1 that

$$D = \langle\mathfrak{b},\mathfrak{b}\rangle. \tag{4.2.5}$$

Proposition 4.2.7. *Let α_1,α_2 and α_3 be homogeneous elements in $\mathfrak{b}$ with $\langle\alpha_1,\alpha_2\rangle \neq 0$. Then*

$$\langle\alpha_1,\alpha_2\rangle\alpha_3 = \begin{cases} \frac{[\alpha_1,\alpha_2]_{A^-}\alpha_3}{2} + \frac{n((\alpha_3\alpha_2)\alpha_1-(\alpha_3\alpha_1)\alpha_2)}{2} & \textit{if } \alpha_1,\alpha_2,\alpha_3 \in B\oplus B', \\ [[\alpha_1,\alpha_2]_{A^-},\alpha_3] & \textit{if } \alpha_1,\alpha_2,\alpha_3 \in \mathfrak{a}, \\ [\alpha_1,\alpha_2]_{A^-}\alpha_3 & \textit{if } \alpha_1,\alpha_2 \in \mathfrak{a},\ \alpha_3 \in B\oplus B', \\ \frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{2} & \textit{if } \alpha_1 \in B,\ \alpha_2 \in B',\ \alpha_3 \in \mathfrak{a}. \end{cases}$$

Proof. Since $\langle\alpha_1,\alpha_2\rangle \neq 0$, we need to consider only the following cases:

Case 1: $\alpha_1,\alpha_2,\alpha_3 \in \mathfrak{a}$. Consider the Jacoby identity for $z_1\otimes\alpha_1$, $z_2\otimes\alpha_2$, $z_3\otimes\alpha_3$:

$$[z_1\otimes\alpha_1,[z_2\otimes\alpha_2,z_3\otimes\alpha_3]] = [[z_1\otimes\alpha_1,z_2\otimes\alpha_2],z_3\otimes\alpha_3] + [z_2\otimes\alpha_2,[z_1\otimes\alpha_1,z_3\otimes\alpha_3]].$$

Let $z_1 = z_2 = E_{1,2}+\varepsilon_1 E_{2,1}$ and $z_3 = E_{2,3}+\varepsilon_2 E_{3,2}$ where $\varepsilon_i = \pm 1$. Using Lemma 4.1.2 we get

$$[z_1,[z_2,z_3]]\otimes\frac{\alpha_1\circ(\alpha_2\circ\alpha_3)}{4} + z_1\circ[z_2,z_3]\otimes\frac{[\alpha_1,\alpha_2\circ\alpha_3]}{4}$$

$$
\begin{aligned}
&+[z_1,(z_2\circ z_3)]\otimes\frac{\alpha_1\circ[\alpha_2,\alpha_3]}{4}+z_1\circ(z_2\circ z_3)\otimes\frac{[\alpha_1,[\alpha_2,\alpha_3]]}{4}\\
&=[[z_1,z_2],z_3]\otimes\frac{(\alpha_1\circ\alpha_2)_{A^+}\circ\alpha_3}{4}+[z_1,z_2]\circ z_3\otimes\frac{[(\alpha_1\circ\alpha_2)_{A^+},\alpha_3]}{4}\\
&(z_1\mid z_2)z_3\otimes\langle\alpha_1,\alpha_2\rangle\alpha_3+[z_1\circ z_2,z_3]\otimes\frac{[\alpha_1,\alpha_2]_{A^-}\circ\alpha_3}{4}+(z_1\circ z_2)\circ z_3\otimes\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{4}\\
&+[z_2,[z_1,z_3]]\otimes\frac{\alpha_2\circ(\alpha_1\circ\alpha_3)}{4}+z_2\circ[z_1,z_3]\otimes\frac{[\alpha_2,\alpha_1\circ\alpha_3]}{4}\\
&+[z_2,(z_1\circ z_3)]\otimes\frac{\alpha_2\circ[\alpha_1,\alpha_3]}{4}+z_2\circ(z_1\circ z_3)\otimes\frac{[\alpha_2,[\alpha_1,\alpha_3]]}{4}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
[z_1,[z_2,z_3]]&=\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2},\\
z_1\circ(z_2\circ z_3)&=\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2},\\
[z_1,z_2\circ z_3]&=\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2},\\
z_1\circ[z_2,z_3]&=\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2},\\
(z_1\circ z_2)\circ z_3&=2\frac{(n-4)}{n}(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2}),\\
[z_1\circ z_2,z_3]&=2(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2}),\\
[[z_1,z_2],z_3]&=[z_1,z_2]\circ z_3=0,\\
[z_2,[z_1,z_3]]&=\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2},\\
z_2\circ(z_1\circ z_3)&=\varepsilon_1E_{2,3}+\varepsilon_1,\varepsilon_2E_{3,2},\\
[z_2,(z_1\circ z_3)]&=\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2},\\
z_2\circ[z_1,z_3]&=\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2}.
\end{aligned}
\tag{4.2.6}
$$

Now (4.2.6) becomes

$$
\begin{aligned}
&(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{\alpha_1\circ(\alpha_2\circ\alpha_3)}{4}+(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{[\alpha_1,\alpha_2\circ\alpha_3]}{4}\\
&+(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{\alpha_1\circ[\alpha_2,\alpha_3]}{4}+(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{[\alpha_1,[\alpha_2,\alpha_3]]}{4}\\
&=2(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{[\alpha_1,\alpha_2]_{A^-}\circ\alpha_3}{4}+2\frac{(n-4)}{n}(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2})\\
&\otimes\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{4}+\frac{2\varepsilon_1}{n}(E_{2,3}+\varepsilon_2E_{3,2})\otimes\langle\alpha_1,\alpha_2\rangle\alpha_3+(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2})\\
&\otimes\frac{\alpha_2\circ(\alpha_1\circ\alpha_3)}{4}+(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{[\alpha_2,\alpha_1\circ\alpha_3]}{4}+(\varepsilon_1E_{2,3}-\varepsilon_1\varepsilon_2E_{3,2})\\
&\otimes\frac{\alpha_2\circ[\alpha_1,\alpha_3]}{4}+(\varepsilon_1E_{2,3}+\varepsilon_1\varepsilon_2E_{3,2})\otimes\frac{[\alpha_2,[\alpha_1,\alpha_3]]}{4}.
\end{aligned}
$$

By collecting the coefficients of $E_{2,3}$ we get

$$\begin{aligned}&\frac{\alpha_1\circ(\alpha_2\circ\alpha_3)}{4}+\frac{[\alpha_1,\alpha_2\circ\alpha_3]}{4}+\frac{\alpha_1\circ[\alpha_2,\alpha_3]}{4}+\frac{[\alpha_1,[\alpha_2,\alpha_3]]}{4}\\ &=\frac{[\alpha_1,\alpha_2]_{A^-}\circ\alpha_3}{2}+\frac{(n-4)}{n}\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{2}+\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3\\ &+\frac{\alpha_2\circ(\alpha_1\circ\alpha_3)}{4}+\frac{[\alpha_2,\alpha_1\circ\alpha_3]}{4}+\frac{\alpha_2\circ[\alpha_1,\alpha_3]}{4}+\frac{[\alpha_2,[\alpha_1,\alpha_3]]}{4}.\end{aligned}$$

Since $\mathfrak{a}$ is an associative algebra (see Theorem 4.1.3) we obtain

$$\langle\alpha_1,\alpha_2\rangle\alpha_3=[[\alpha_1,\alpha_2]_{A^-},\alpha_3],$$

as required.

Case 2: $\alpha_1,\alpha_2\in\mathfrak{a}$ and $\alpha_3\in B\oplus B'$. First assume that $\alpha_3\in B$ and consider the Jacoby identity for $z_1\otimes\alpha_1$, $z_2\otimes\alpha_2$, $u\otimes\alpha_3$:

$$[z_1\otimes\alpha_1,[z_2\otimes\alpha_2,u\otimes\alpha_3]]=[[z_1\otimes\alpha_1,z_2\otimes\alpha_2],u\otimes\alpha_3]+[z_2\otimes\alpha_2,[z_1\otimes\alpha_1,u\otimes\alpha_3]].$$

Using Lemmas 4.2.3 and 4.1.2 we get

$$\begin{aligned}z_1(z_2u)\otimes\alpha_1(\alpha_2\alpha_3)-(z_1\circ z_2)u\otimes\frac{[\alpha_1,\alpha_2]_{A^-}}{2}\alpha_3-\frac{1}{2}[z_1,z_2]u\otimes(\alpha_1\circ\alpha_2)_{A^+}\alpha_3\\ -u\otimes\frac{\mathrm{tr}(z_1z_2)}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3-z_2(z_1u)\otimes\alpha_2(\alpha_1\alpha_3)=0.\end{aligned}$$

Set $z_1=z_2=E_{1,2}+\varepsilon_1E_{2,1}$ and $u=e_1$ with $\varepsilon_1=\pm1$. We get

$$\varepsilon_1e_1\otimes(\alpha_1(\alpha_2\alpha_3)+(-2+\frac{4}{n})\frac{[\alpha_1,\alpha_2]_{A^-}}{2}\alpha_3-\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3-\alpha_2(\alpha_1\alpha_3))=0,$$

so $\alpha_1(\alpha_2\alpha_3)-(2-\frac{4}{n})\frac{[\alpha_1,\alpha_2]_{A^-}}{2}\alpha_3-\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3-\alpha_2(\alpha_1\alpha_3)=0$. Since $[\alpha_1,\alpha_2]_{A^-}\alpha_3=\alpha_1(\alpha_2\alpha_3)-\alpha_2(\alpha_1\alpha_3)$, we get

$$\langle\alpha_1,\alpha_2\rangle\alpha_3=[\alpha_1,\alpha_2]_{A^-}\alpha_3,$$

as required. Similarly, one can show that $\langle\alpha_1,\alpha_2\rangle\alpha_3=[\alpha_1,\alpha_2]_{A^-}\alpha_3$ for $\alpha_1,\alpha_2\in\mathfrak{a}$ and $\alpha_3\in B'$.

Case 3: $\alpha_1\in B,\alpha_2\in B'$ and $\alpha_3\in\mathfrak{a}$. Consider the Jacoby identity for $u\otimes\alpha_1$, $u'\otimes\alpha_2$, $z\otimes\alpha_3$:

$$[u\otimes\alpha_1,[u'\otimes\alpha_2,z\otimes\alpha_3]]\quad=\quad[[u\otimes\alpha_1,u'\otimes\alpha_2],z\otimes\alpha_3]+[u'\otimes\alpha_2,[u\otimes\alpha_1,z\otimes\alpha_3]].$$

Set $u = e_1$, $u' = e_1$ and $z = E_{1,2} + \varepsilon E_{2,1}$ with $\varepsilon = \pm 1$. Using (4.2.2), Lemmas 4.2.3 and 4.1.2 we get

$$\begin{aligned}
&(E_{2,1}+E_{1,2})\otimes\frac{[\alpha_1,\alpha_2\alpha_3]}{2}+(E_{1,2}-E_{2,1})\otimes\frac{\alpha_1\circ(\alpha_2\alpha_3)}{2}\\
&=((E_{1,2}+\varepsilon E_{2,1})-\frac{2}{n}(E_{1,2}+\varepsilon E_{2,1}))\otimes\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{2}\\
&+(E_{1,2}-\varepsilon E_{2,1})\otimes\frac{[\alpha_1,\alpha_2]_{A^-}\circ\alpha_3}{2}+(E_{1,2}+\varepsilon E_{2,1})\otimes\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3\\
&+\varepsilon(E_{2,1}+E_{1,2})\otimes\frac{[\alpha_3\alpha_1,\alpha_2]}{2}+\varepsilon(E_{2,1}-E_{1,2})\otimes\frac{(\alpha_3\alpha_1)\circ\alpha_2}{2}.
\end{aligned}$$

By collecting the coefficients of $E_{1,2}$ we get

$$\begin{aligned}
&\frac{[\alpha_1,\alpha_2\alpha_3]}{2}+\frac{\alpha_1\circ(\alpha_2\alpha_3)}{2}=\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{2}+\frac{[\alpha_1,\alpha_2]_{A^-}\circ\alpha_3}{2}\\
&-\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{n}+\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3+\varepsilon\frac{[\alpha_3\alpha_1,\alpha_2]}{2}-\varepsilon\frac{(\alpha_3\alpha_1)\circ\alpha_2}{2},
\end{aligned}$$

or equivalently,

$$\alpha_1(\alpha_2\alpha_3)=[\alpha_1,\alpha_2]_{A^-}\alpha_3-\varepsilon\alpha_2(\alpha_3\alpha_1)-\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{n}+\frac{2}{n}\langle\alpha_1,\alpha_2\rangle\alpha_3,$$

Since

$$[\alpha_1,\alpha_2]_{A^-}\alpha_3=(\alpha_1\alpha_2-\alpha_2\alpha_1)\alpha_3=(\alpha_1\alpha_2)\alpha_3-(\alpha_2\alpha_1)\alpha_3,$$

and $(\alpha_1\alpha_2)\alpha_3=\alpha_1(\alpha_2\alpha_3)$, $(\alpha_2\alpha_1)\alpha_3=\alpha_2(\eta(\alpha_3)\alpha_1)=-\varepsilon\alpha_2(\alpha_3\alpha_1)$ (Using Proposition 4.2.6) we obtain

$$\langle\alpha_1,\alpha_2\rangle\alpha_3=\frac{[[\alpha_1,\alpha_2]_{A^-},\alpha_3]}{2},$$

as required.

Case 4: $\alpha_1\in B,\alpha_2\in B'$ and $\alpha_3\in B$. Consider the Jacoby identity for $v\otimes\alpha_3$, $u'\otimes\alpha_2$, $u\otimes\alpha_1$:

$$[v\otimes\alpha_3,[u'\otimes\alpha_2,u\otimes\alpha_1]] \;=\; [[v\otimes\alpha_3,u'\otimes\alpha_2],u\otimes\alpha_1]+[u'\otimes\alpha_2,[v\otimes\alpha_3,u\otimes\alpha_1]].$$

Taking $v=e_2$, $u'=e_1$ and $u=e_1$. Using (4.2.2) we get

$$\begin{aligned}
&[e_2\otimes\alpha_3,(2E_{11}-\frac{2}{n}I)\otimes\frac{[\alpha_2,\alpha_1]_{A^-}}{2}+\frac{2}{n}\langle\alpha_2,\alpha_1\rangle]\\
&=[(E_{2,1}+E_{1,2})\otimes\frac{[\alpha_3,\alpha_2]_{A^-}}{2}+(E_{2,1}-E_{1,2})\otimes\frac{(\alpha_3\circ\alpha_2)_{A^+}}{2},e_1\otimes\alpha_1]
\end{aligned}$$

$$+[e_1\otimes\alpha_2,(E_{2,1}+E_{1,2})\otimes\frac{[\alpha_3,\alpha_1]_C}{2}+(E_{2,1}-E_{1,2})\otimes\frac{(\alpha_3\circ\alpha_1)_E}{2}].$$

Using (3.4.4) and Lemma 4.2.3 we get

$$e_2\otimes(\frac{[\alpha_2,\alpha_1]_{A^-}\alpha_3}{n}-\frac{2}{n}\langle\alpha_2,\alpha_1\rangle\alpha_3)=$$
$$e_2\otimes(\frac{[\alpha_3,\alpha_2]_{A^-}}{2}\alpha_1+\frac{(\alpha_3\circ\alpha_2)_{A^+}}{2}\alpha_1-\frac{[\alpha_3,\alpha_1]_C}{2}\alpha_2-\frac{(\alpha_3\circ\alpha_1)_E}{2}\alpha_2),$$

so,

$$\begin{aligned}\frac{[\alpha_2,\alpha_1]_{A^-}\alpha_3}{n}-\frac{2}{n}\langle\alpha_2,\alpha_1\rangle\alpha_3=&\frac{[\alpha_3,\alpha_2]_{A^-}}{2}\alpha_1+\frac{(\alpha_3\circ\alpha_2)_{A^+}}{2}\alpha_1\\&-\frac{[\alpha_3,\alpha_1]_C}{2}\alpha_2-\frac{(\alpha_3\circ\alpha_1)_E}{2}\alpha_2.\end{aligned}$$

We conclude that

$$\langle\alpha_2,\alpha_1\rangle\alpha_3=\frac{[\alpha_2,\alpha_1]_{A^-}\alpha_3}{2}+\frac{n((\alpha_3\alpha_1)\alpha_2-(\alpha_3\alpha_2)\alpha_1)}{2},$$

or equivalently,

$$\langle\alpha_1,\alpha_2\rangle\alpha_3=\frac{[\alpha_1,\alpha_2]_{A^-}\alpha_3}{2}+\frac{n((\alpha_3\alpha_2)\alpha_1-(\alpha_3\alpha_1)\alpha_2)}{2}.$$

Case 6: $\alpha_1\in B,\alpha_2\in B'$ and $\alpha_3\in B'$. This is proved similarly to Case 5 by setting $v'=e_2$, $u'=e_1$ and $u=e_1$ and considering the Jacoby identity for $v'\otimes\alpha_3$, $u'\otimes\alpha_2$, $u\otimes\alpha_1$. □

Proposition 4.2.8. *(1) $[d,\langle\alpha,\beta\rangle]=\langle d\alpha,\beta\rangle+\langle\alpha,d\beta\rangle$ for all $\alpha,\beta\in\mathfrak{b}$ and $d\in D$.*

(2) $\langle A^+,A^+\rangle$, $\langle A^-,A^-\rangle$, $\langle B,B'\rangle$, $\langle C,C'\rangle$ and $\langle E,E'\rangle$ are ideals of the Lie algebra D.

(3) D acts by derivations on $\mathfrak{b}$ and leaves all subspaces $A^+,A^-,B,\ldots,E'$ invariant.

Proof. Let $\alpha=a_1^++a_1^-+b_1+b_1'+c_1+c_1'+e_1+e_1'$ and $\beta=a_2^++a_2^-+b_2+b_2'+c_2+c_2'+e_2+e_2'$ be the decompositions of α and β into homogeneous parts. By considering Jacobi identities for the following 5 triples,

$(i)\ d,x_1^+\otimes a_1^-,x_2^+\otimes a_2^-;$

$(ii)\ d,x_1^-\otimes a_1^+,x_2^-\otimes a_2^+;$

$(iii)\ d,u\otimes b_i,v'\otimes b_j';$

$(iv)\ d,s\otimes c,s'\otimes c';$

$(v)\ d,\lambda\otimes e,\lambda'\otimes e';$

we get the following equations, respectively,

$$
\begin{aligned}
{[d,\langle a_1^-,a_2^-\rangle]} &= \langle da_1^-,a_2^-\rangle + \langle a_1^-,da_2^-\rangle, \\
{[d,\langle a_1^+,a_2^+\rangle]} &= \langle da_1^+,a_2^+\rangle + \langle a_1^+,da_2^+\rangle, \\
{[d,\langle b_i,b_j'\rangle]} &= \langle db_i,b_j'\rangle + \langle b_i,db_j'\rangle, \\
{[d,\langle c_i,c_j'\rangle]} &= \langle dc_i,c_j'\rangle + \langle c_i,dc_j'\rangle, \\
{[d,\langle e_i,e_j'\rangle]} &= \langle de_i,e_j'\rangle + \langle e_i,de_j'\rangle.
\end{aligned} \tag{4.2.7}
$$

and

$$
\begin{aligned}
d(a_1^-a_2^-) &= (da_1^-)a_2^- + a_1^-(da_2^-), \\
d(a_1^+a_2^+) &= (da_1^+)a_2^+ + a_1^+(da_2^+), \\
d(b_ib_j') &= (db_i)b_j' + b(db_j'), \\
d(c_ic_j') &= (dc_i)c_j' + c_i(dc_j'), \\
d(e_ie_j') &= (de_i)e_j' + e_i(de_j'),
\end{aligned} \tag{4.2.8}
$$

where $i,j=1,2$. We illustrate this by considering the case (i). By applying Jacobi identity to d, $x_1^+\otimes a_1^-$, $x_2^+\otimes a_2^-$, we get

$$
[d,[x_1^+\otimes a_1^-,x_2^+\otimes a_2^-]] = [[d,x_1^+\otimes a_1^-],x\otimes a_2^-] + [x_1^+\otimes a_1^-,[d,x_2^+\otimes a_2^-]]
$$

Using (3.4.4) and Lemma 4.1.2 we get

$$
\begin{aligned}
& x_1^+\circ x_2^+\otimes d\frac{[a_1^-,a_2^-]_{A^-}}{2} + [x_1^+,x_2^+]\otimes d\frac{(a_1^-\circ a_2^-)_{A^+}}{2} + (x_1^+\mid x_2^+)[d,\langle a_1^-,a_2^-\rangle] \\
&\quad = x_1^+\circ x_2^+\otimes\frac{[da_1^-,a_2^-]_{A^-}}{2} + [x_1^+,x_2^+]\otimes\frac{(da_1^-\circ a_2^-)_{A^+}}{2} + (x_1^+\mid x_2^+)\langle da_1^-,a_2^-\rangle \\
&\quad + x_1^+\circ x_2^+\otimes\frac{[a_1^-,da_2^-]_{A^-}}{2} + [x_1^+,x_2^+]\otimes\frac{(a_1^-\circ da_2^-)_{A^+}}{2} + (x_1^+\mid x_2^+)\langle a_1^-,da_2^-\rangle.
\end{aligned} \tag{4.2.9}
$$

Then

$$
\begin{aligned}
& x_1^+\circ x_2^+\otimes d\frac{[a_1^-,a_2^-]_{A^-}}{2} + [x_1^+,x_2^+]\otimes d\frac{(a_1^-\circ a_2^-)_{A^+}}{2} = x_1^+\circ x_2^+\otimes\frac{[da_1^-,a_2^-]_{A^-}}{2} + \\
& [x_1^+,x_2^+]\otimes\frac{(da_1^-\circ a_2^-)_{A^+}}{2} + x_1^+\circ x_2^+\otimes\frac{[a_1^-,da_2^-]_{A^-}}{2} + [x_1^+,x_2^+]\otimes\frac{(a_1^-\circ da_2^-)_{A^+}}{2}
\end{aligned} \tag{4.2.10}
$$

and

$$
(x_1^+\mid x_2^+)[d,\langle a_1^-,a_2^-\rangle] = (x_1^+\mid x_2^+)(\langle da_1^-,a_2^-\rangle + \langle a_1^-,da_2^-\rangle). \tag{4.2.11}
$$

When $x_1^+ = x_2^+ = E_{1,2} + E_{2,1}$, we have $\mathrm{tr}(x_1^+ x_2^+) = 1$. Hence (4.2.11) is equivalent to

$$[d, \langle a_1^-, a_2^- \rangle] = \langle da_1^-, a_2^- \rangle + \langle a_1^-, da_2^- \rangle,$$

When $x_1^+ = E_{1,2} + E_{2,1}$ and $x_2^+ = E_{2,3} + E_{3,2}$, we have $[x_1^+, x_2^+] = E_{1,3} + E_{3,1}$ and $x_1^+ \circ x_2^+ = E_{1,3} + E_{3,1}$. Hence (4.2.10) is equivalent to:

$$\begin{aligned} d\Big(\frac{[a_1^-, a_2^-]_{A^-}}{2} + \frac{(a_1^- \circ a_2^-)_{A^+}}{2}\Big) = \Big(&\frac{[da_1^-, a_2^-]_{A^-}}{2} + \frac{(da_1^- \circ a_2^-)_{A^+}}{2}\Big) \\ &+ \Big(\frac{[a_1^-, da_2^-]_{A^-}}{2} + \frac{(a_1^- \circ da_2^-)_{A^+}}{2}\Big), \end{aligned}$$

or equivalently, $d(a_1^- a_2^-) = (da_1^-)a_2^- + a_1^-(da_2^-)$, as in equation (4.2.7).

By combining the equations (4.2.7) we get

$$[d, \langle \alpha, \beta \rangle] = \langle d\alpha, \beta \rangle + \langle \alpha, d\beta \rangle,$$

for all $d \in D$ and $\alpha, \beta \in \mathfrak{b}$. This implies that the subspaces $\langle A^+, A^+ \rangle$, $\langle A^-, A^- \rangle$, $\langle B, B' \rangle$, $\langle C, C' \rangle$ and $\langle E, E' \rangle$ are ideals in D. The equations (4.2.8) show that d acts by derivation. Similarly, one can show that D acts by derivations on $\mathfrak{b}$. Using Proposition 4.2.7 and Tables 4.1.1 and 4.2.1 we get the action of D leaves all subspaces $A^+, A^-, B, \ldots, E'$ invariant as required. □

The above results can be summarized as follows.

Theorem 4.2.9 (The structure theorem for Θ_n-graded Lie algebras)**.** *Let L be an Θ_n-graded Lie algebra and let $\mathfrak{g} \cong sl_n$ be the grading subalgebra of L. Suppose that $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold. Then*

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D$$

with multiplication given by (3.4.4) where A, B, B', C, C', E, E' are vector spaces and D is the sum of the trivial $\mathfrak{g}$-modules. Define by $\mathfrak{g}^+ := \{x \in \mathfrak{g} \mid x^t = x\}$ and $\mathfrak{g}^- := \{x \in \mathfrak{g} \mid x^t = -x\}$ the subspaces of symmetric and skew-symmetric matrices in $\mathfrak{g}$, respectively. Then the component $\mathfrak{g} \otimes A$ can be decomposed further as

$$\mathfrak{g} \otimes A = (\mathfrak{g}^+ \oplus \mathfrak{g}^-) \otimes A = (\mathfrak{g}^+ \otimes A^-) \oplus (\mathfrak{g}^- \otimes A^+)$$

where A^- and A^+ are two copies of the vector space A. Denote

$$\mathfrak{a} := A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E' \quad \text{and} \quad \mathfrak{b} := \mathfrak{a} \oplus B \oplus B'.$$

Then the product in L induces an algebra structure on both $\mathfrak{a}$ and $\mathfrak{b}$ satisfying the following properties.

(i) $\mathfrak{a}$ is a unital associative subalgebra of $\mathfrak{b}$ with involution whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E'$ and $A^- \oplus C \oplus C'$, respectively, see Theorems 4.1.3 and 4.1.6.

(ii) $\mathfrak{b}$ is a unital algebra with an involution η whose symmetric and skew-symmetric elements are $A^+ \oplus E \oplus E' \oplus B \oplus B'$ and $A^- \oplus C \oplus C'$, respectively, see Theorem 4.2.1 and Proposition 4.2.2.

(iii) $B \oplus B'$ is an associative $\mathfrak{a}$-bimodule with a hermitian form χ with values in $\mathfrak{a}$. More exactly, for all $\beta_1, \beta_2 \in B \oplus B'$ and $\alpha \in \mathfrak{a}$ we have $\chi(\beta_1, \beta_2) = \beta_1\beta_2$, $\chi(\alpha\beta_1, \beta_2) = \alpha\chi(\beta_1, \beta_2)$, $\eta(\chi(\beta_1, \beta_2)) = \chi(\beta_2, \beta_1)$ and $\chi(\beta_1, \alpha\beta_2) = \chi(\beta_1, \beta_2)\eta(\alpha)$, see Propositions 4.2.4 and 4.2.6.

(iv) $\mathscr{A} := A^- \oplus A^+$ is a unital associative subalgebra of $\mathfrak{a}$ and $C \oplus E$, $C' \oplus E'$, B and B' are $\mathscr{A}$-bimodules, see Corollaries 4.1.4, 4.1.5 and 4.2.5.

(v) D acts by derivations on $\mathfrak{b}$, see Propositions 4.2.7 and 4.2.8.

4.3 Matrix realization of the algebra $\mathfrak{a}$

Recall that $\mathfrak{g} \otimes A = \mathfrak{g}^+ \otimes A^- \oplus \mathfrak{g}^- \otimes A^+$ where $\mathfrak{g}^\pm = \{x \in sl_n \mid x^t = \pm x\}$ and $A^\pm$ is a copy of the vector space A. We identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1$ where 1 is a distinguished element of A. We denote by $a^\pm$ the image of $a \in A$ in the space $A^\pm$. Recall that $\mathscr{A} = A^+ \oplus A^-$ is an associative algebra (for $n \geq 4$) with identity element 1^+. Consider the subspaces $A_1 = span\{a^+ + a^- \mid a \in A\}$ and $A_2 = span\{a^+ - a^- \mid a \in A\}$. Then $\mathscr{A} = A_1 \oplus A_2$ as a vector space. In this section we show that A_1 and A_2 are 2-sided ideals of the algebra $\mathscr{A}$ and that the associative algebra $\mathfrak{a}$ has the following realization by 2×2 matrices with entries in the components of $\mathfrak{a}$:

$$\mathfrak{a} \cong \begin{bmatrix} A_1 & C \oplus E \\ C' \oplus E' & A_2 \end{bmatrix}.$$

We start with the following observation.

Lemma 4.3.1. *For all $a^\pm \in A^\pm$, $c \in C$, $c' \in C'$, $e \in E$, $e' \in E'$, $b \in B$, $b' \in B'$ we have*

(1) $1^-.a^- = a^+ = a^-.1^-$ *and* $1^-.a^+ = a^- = a^+.1^-$;

(2) $c = 1^-.c = -c.1^-$ *and* $e = 1^-.e = -e.1^-$;
(3) $c' = c'.1^- = -1^-.c'$ *and* $e' = e'.1^- = -1^-.e'$;
(4) $b = 1^- b$ *and* $b' = b'.1^-$.

Proof. Let $x^{\pm}, x_1^{\pm}, x_2^{\pm} \in \mathfrak{g}^{\pm}$. Using (3.4.4), we get

$$\begin{aligned}
[x_1^+ \otimes 1^-, x_2^+ \otimes a^-] &= [x_1^+, x_2^+] \otimes a^+,\\
[x_1^+ \otimes 1^-, x_1^- \otimes a^+] &= [x_1^+, x_1^-] \otimes a^-,\\
[x^+ \otimes 1^-, s \otimes c] &= x^+ \diamond s \otimes c,\\
[x^+ \otimes 1^-, \lambda \otimes e] &= x^+ \diamond \lambda \otimes e,\\
[s' \otimes c', x^+ \otimes 1^-] &= s' \diamond x^+ \otimes c',\\
[\lambda' \otimes e', x^+ \otimes 1^-] &= \lambda' \diamond x^+ \otimes e',\\
[x^+ \otimes 1^-, u \otimes b] &= x^+ u \otimes b,\\
[u' \otimes b', x^+ \otimes 1^-] &= x^t u' \otimes b'
\end{aligned}$$

Using these properties and the formulas in Remark 4.1.1, we get

$$\begin{aligned}
[x^+, x_2^+] \otimes a^+ &= x^+ \circ x_2^+ \otimes \frac{[1^-, a^-]_{A^-}}{2} + [x^+, x_2^+] \otimes \frac{(1^- \circ a^-)_{A^+}}{2} + (x^+ \mid x_2^+)\langle 1^-, a^- \rangle,\\
[x^+, x_1^-] \otimes a^- &= x^+ \diamond x_1^- \otimes \frac{[1^-, a^+]_{A^+}}{2} + [x^+, x_1^-] \otimes \frac{(1^- \circ a^+)_{A^-}}{2},\\
x^+ \diamond s \otimes c &= x^+ \diamond s \otimes \frac{[1^-, c]_C}{2} + [x^+, s] \otimes \frac{(1^- \circ c)_E}{2},\\
x^+ \diamond \lambda \otimes e &= x^+ \diamond \lambda \otimes \frac{[1^-, e]_E}{2} + [x^+, \lambda] \otimes \frac{(1^- \circ e)_C}{2},\\
s' \diamond x^+ \otimes c' &= s' \diamond x^+ \otimes \frac{[c', 1^-]_{C'}}{2} + [s', x^+] \otimes \frac{(c' \circ 1^-)_{E'}}{2},\\
\lambda' \diamond x^+ &= \lambda' \diamond x^+ \otimes \frac{[e', 1^-]_{E'}}{2} + [\lambda', x^+] \otimes \frac{(e' \circ 1^-)_{C'}}{2},\\
x^+ u \otimes b &= x^+ u \otimes 1^- b,\\
x^t u' \otimes b' &= x^t u' \otimes b'.1^-,
\end{aligned}$$

so

$$\begin{aligned}
a^+ &= \frac{(1^- \circ a^-)_{A^+}}{2}, \ \frac{[1^-, a^-]_{A^-}}{2} = 0, & a^- &= \frac{(1^- \circ a^+)_{A^-}}{2}, \ \frac{[1^-, a^+]_{A^+}}{2} = 0,\\
c &= \frac{[1^-, c]_C}{2}, \ \frac{(1^- \circ c)_E}{2} = 0, & e &= \frac{[1^-, e]_E}{2}, \ \frac{(1^- \circ e)_C}{2} = 0,\\
c' &= \frac{[c', 1^-]_{C'}}{2}, \ \frac{(c' \circ 1^-)_{E'}}{2} = 0, & e' &= \frac{[e', 1^-]_{E'}}{2}, \ \frac{(e' \circ 1^-)_{C'}}{2}.
\end{aligned}$$

$$b = 1^- b, \qquad b' = b'.1^-.$$

This implies (1)-(4) as required. □

Proposition 4.3.2. *Let $e_1 = \frac{1^+ + 1^-}{2}$ and $e_2 = \frac{1^+ - 1^-}{2}$. Then the following hold.*

(1) e_1 and e_2 are orthogonal idempotents with $e_1 + e_2 = 1^+$ and $\eta(e_1) = e_2$.

(2) Let $\mathfrak{a} = e_1\mathfrak{a}e_1 \oplus e_1\mathfrak{a}e_2 \oplus e_2\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2$ be the Peirce decomposition of $\mathfrak{a}$. Then $e_1\mathfrak{a}e_1 = A_1$, $e_1\mathfrak{a}e_2 = C \oplus E$, $e_2\mathfrak{a}e_1 = C' \oplus E'$, and $e_2\mathfrak{a}e_2 = A_2$.

(3) A_1 and A_2 are 2-sided ideals of $\mathscr{A} = A_1 \oplus A_2$.

(4) e_i is the identity of A_i.

(5) $\eta(A_1) = A_2$.

(6) $B = \mathscr{B}e_2$ and $B' = \mathscr{B}e_1$.

(7) $A_1 \cong A$ and $A_2 \cong A^{op}$ (the opposite algebra of A) as algebras.

Proof. (1)-(6) This is easy to check using Lemma 4.3.1 and properties of the Peirce decomposition.

(7) Define the map $\varphi : A \to A_1$ by $\varphi(a) = \frac{a^+ + a^-}{2}$ where $a \in A$. Note that this map is well defined and bijective. It remains only to check that φ is an algebra homomorphism. Let $a, b \in A$. Then

$$\begin{aligned}
\varphi(ab) &= \varphi\left(\frac{a \circ b}{2} + \frac{[a,b]}{2}\right) \\
&= \varphi\left(\frac{a \circ b}{2}\right) + \varphi\left(\frac{[a,b]}{2}\right) \\
&= \left(\frac{a \circ b}{4}\right)^+ + \left(\frac{a \circ b}{4}\right)^- + \left(\frac{[a,b]}{4}\right)^+ + \left(\frac{[a,b]}{4}\right)^- \\
&= \frac{a^+a^+ + a^+a^- + a^-a^+ + a^-a^-}{4} \\
&= \left(\frac{a^+ + a^-}{2}\right)\left(\frac{a^+ + a^-}{2}\right) \\
&= \varphi(a)\varphi(b),
\end{aligned}$$

so φ is a homomorphism. Thus, $A_1 \cong A$ and $A_2 = \eta(A_1) \cong A^{op}$, as required. □

Using Peirce decomposition of $\mathfrak{a}$ as in Proposition 4.3.2 we immediately get the following.

Proposition 4.3.3. *The associative algebra $\mathfrak{a}$ has the following realization by 2×2 matrices with entries in the components of $\mathfrak{a}$:*

$$\mathfrak{a} \cong \begin{bmatrix} A_1 & C \oplus E \\ C' \oplus E' & A_2 \end{bmatrix}.$$

In particular,

$$A^+ \cong \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{bmatrix} \mid a_1 \in A_1 \right\} \quad (a^+ \mapsto \begin{bmatrix} \frac{a^+ + a^-}{2} & 0 \\ 0 & \frac{a^+ - a^-}{2} \end{bmatrix}),$$
$$A^- \cong \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & -\eta(a_1) \end{bmatrix} \mid a_1 \in A_1 \right\} \quad (a^- \mapsto \begin{bmatrix} \frac{a^+ + a^-}{2} & 0 \\ 0 & \frac{-a^+ + a^-}{2} \end{bmatrix}).$$

Let A be an associative algebra with involution σ (of the first kind) over F. Recall that A becomes a Lie algebra $A^{(-)}$ under the Lie bracket $[x,y] = xy - yx$. Let $sym(A)$ (resp. $skew(A)$) denotes the set of symmetric elements (resp. skew-symmetric elements) of A with respect to σ. Then, $skew(A)$ is a Lie subalgebra of $A^{(-)}$. The following is well known.

Lemma 4.3.4. *Let A_1 and A_2 be two associative algebras with involutions σ_1 and σ_2, respectively. Then $A = A_1 \otimes A_2$ is an associative algebra with involution $\sigma = \sigma_1 \otimes \sigma_2$. Moreover, we have*

(1) $sym(A) = sym(A_1) \otimes sym(A_2) \oplus skew(A_1) \otimes skew(A_2)$.

(2) $skew(A) = skew(A_1) \otimes sym(A_2) \oplus sym(A_1) \otimes skew(A_2)$.

Proof. It is easy to see that the *RHS* of the equation (1) (resp. (2)) is a subspace of $sym(A)$ (resp. $skew(A)$). It remains to note that

$$\begin{aligned} A_1 \otimes A_2 &= (sym(A_1) \oplus skew(A_1)) \otimes (sym(A_2) \oplus skew(A_2)) \\ &= sym(A_1) \otimes sym(A_2) \oplus skew(A_1) \otimes skew(A_2) \\ &\quad \oplus skew(A_1) \otimes sym(A_2) \oplus sym(A_1) \otimes skew(A_2). \end{aligned}$$

□

Lemma 4.3.5. *Define $\sigma : \mathfrak{a} \to \mathfrak{a}$ by*

$$\sigma(\begin{bmatrix} a_1 & c+e \\ c'+e' & a_2 \end{bmatrix}) = \begin{bmatrix} \eta(a_2) & \eta(c+e) \\ \eta(c'+e') & \eta(a_1) \end{bmatrix}$$

Then σ is an involution on $\mathfrak{a}$ and

$$\begin{aligned} sym(\mathfrak{a}) &= \left\{ \begin{bmatrix} a_1 & e \\ e' & \eta(a_1) \end{bmatrix} \mid a_1 \in A_1, e \in E, e' \in E' \right\}, \\ skew(\mathfrak{a}) &= \left\{ \begin{bmatrix} a_1 & c \\ c' & -\eta(a_1) \end{bmatrix} \mid a_1 \in A_1, c \in C, c' \in C' \right\}. \end{aligned}$$

Chapter 5

Central extensions of Θ_n-graded Lie algebras, $n \geq 5$

The aim of this chapter is to classify Θ_n-graded Lie algebras up to isomorphism in the case when $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold.

The chapter is organized as follows. First we study basic properties of central extensions of $(\Gamma, \mathfrak{g})$-graded Lie algebras. We show all Lie algebras in a given isogeny class are Γ-graded if one of them is, and all have isomorphic weight spaces for nonzero weights. We also show that for every central extension $(\tilde{L}, \pi)$ of a $(\Gamma, \mathfrak{g})$-graded Lie algebra $L = \bigoplus_{\mu \in Q} V(\mu) \otimes W_\mu$ with kernel $\mathbb{E}$, there is lifting of the grading subalgebra $\mathfrak{g}$ of L to a subalgebra of $\tilde{L}$ and L can be lifted to a subspace L of $\tilde{L}$ which contains the given $\mathfrak{g}$ so that the corresponding 2-cocycle satisfies $\zeta(\mathfrak{g}, L) = 0$. Moreover, there exists an $\mathbb{F}$-bilinear map $\varepsilon : W \times W \to \mathbb{E}$ on the space $W := \bigoplus_{\mu \in Q \setminus \{0\}} W_\mu$ with $\varepsilon(W_\mu, W_\nu) = 0$ whenever $V(\mu) \not\cong V(\nu)'$, such that

$$\zeta(u_\mu \otimes w_\mu, v_\nu \otimes w_\nu) = \pi(u_\mu, u_\nu)\varepsilon(w_\mu, w_\nu)$$

for all $u_\mu \otimes w_\mu \in V(\mu) \otimes W_\mu$ and $u_\nu \otimes w_\nu \in V(\nu) \otimes W_\nu$ (see Section 5.1). We will use these properties to compute universal central extensions in Section 5.3. Then we focus our attention to (Θ_n, sl_n)-graded Lie algebras. First we define a centerless algebra $\mathscr{L}(\mathfrak{b})$ and show that it is Θ_n-graded with coordinate algebra $\mathfrak{b}$. It is also shown that any Θ_n-graded Lie algebra L with coordinate algebra $\mathfrak{b}$ is a cover of the centerless Lie algebra $\mathscr{L}(\mathfrak{b})$. Then we show that every Θ_n-graded Lie algebra L is uniquely determined (up to central isogeny) by its "coordinate" algebra $\mathfrak{b}$ and we show that L is centrally isogenous to the explicitly constructed Θ_n-graded unitary Lie algebra $\mathfrak{u}$ of the hermitian form $\xi =$

$w\perp - \chi$ on the $\mathfrak{a}$-module $\mathfrak{a}^n \oplus \mathscr{B}$ (see Section5.2). This completes the classification of Θ_n-graded Lie algebras up to central extensions. In Section 5.3 we find the universal central extension $\widehat{\mathscr{L}(\mathfrak{b})}$ of $\mathscr{L}(\mathfrak{b})$ and show that its center is $\mathrm{HF}(\mathfrak{b})$, the full skew-dihedral homology group of $\mathfrak{b}$. We prove that every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is isomorphic to $\mathscr{L}(\mathfrak{b}, X) = \widehat{\mathscr{L}(\mathfrak{b})}/X$ for some subspace X of $\mathrm{HF}(\mathfrak{b})$, which classifies the Θ_n-graded Lie algebras up to isomorphism.

At the end of this chapter we discuss the similarities between the Θ_n-graded Lie algebras and quasiclassical Lie algebras by showing that every $(\Xi_n, sl_n)-$graded Lie algebra with

$$\Xi_n = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\} \subset \Theta_n$$

is centrally isogenous to a quasiclassical Lie algebra (see Section 5.4).

For convenience of the reader we mostly follow notations of [3, 4] whenever possible.

5.1 Central extensions of $(\Gamma, \mathfrak{g})$-graded Lie algebras

Recall that a *central extension* of a Lie algebra L is a pair $(\tilde{L}, \pi)$ consisting of a Lie algebra $\tilde{L}$ and a surjective Lie algebra homomorphism $\pi : \tilde{L} \to L$ whose kernel lies in the center of $\tilde{L}$. A *cover* or *covering* of L is a central extension $(\tilde{L}, \pi)$ of L with $\tilde{L}$ perfect, i.e., $\tilde{L} = [\tilde{L}, \tilde{L}]$. A homomorphism of central extensions from the central extension $f : K \to L$ to the central extension $f' : K' \to L$ is a Lie algebra homomorphism $g : K \to K'$ satisfying $f = f' \circ g$. A central extension $U : K \to L$ is a *universal central extension*, if there exists a unique homomorphism from K to any other central extension $\tilde{K}$ of L. A Lie algebra L is said to be centrally closed if (L, Id) is a universal central extension of L.

Central extensions of Lie algebras graded by finite root systems in terms of the homology of its coordinate algebra were determined and described up to isomorphism by Allison, Benkart and Y. Gao in [3] and [4]. The same technique can be used to describe central extensions of $(\Gamma, \mathfrak{g})$-graded Lie algebras.

Theorem 5.1.1. *Let L be a $(\Gamma, \mathfrak{g})$-graded Lie algebra. Then L is perfect.*

Proof. We need to show $L \subseteq [L, L]$, i.e. $L_\alpha \subseteq [L, L]$ for all $\alpha \in \Gamma$. By condition $(\Gamma 3)$ in Definition 3.0.1, $L_0 \subseteq [L, L]$. Suppose now that $\alpha \in \Gamma \setminus \{0\}$. Then there exists $h \in H$ such that $\alpha(h) \neq 0$ so for all $x \in L_\alpha$,

$$[h, x] = \alpha(h)x \text{ and } x = [\alpha(h)^{-1}h, x] \in [L_0, L_\alpha].$$

Thus, $L_\alpha \subseteq [L_0, L_\alpha]$, as required. $\square$

Recall that any perfect Lie algebra L has a universal central extension which is also perfect, called a *universal covering algebra* of L. Any two universal covering algebras of L are isomorphic [32]. Therefore every Γ-graded Lie algebra has a universal covering algebra. We need the following simple generalization of [22, Proposition 1.24].

Theorem 5.1.2. *Let L be a $(\Gamma,\mathfrak{g})$-graded Lie algebra and let (U,ψ) be the universal covering algebra of L. Then U is graded by Γ and $\psi\,|_{U_\alpha} U_\alpha \to L_\alpha$ is an isomorphism for all $\alpha\in\Gamma\setminus\{0\}$. In particular* $\operatorname{Ker}\psi\subset U_0$.

Proof. This is similar to the proof of [22, Proposition 1.24]. It is well known that $\mathfrak{g}$ is centrally closed [16]. Thus the central extension $\psi:\psi^{-1}(\mathfrak{g})\to\mathfrak{g}$ splits and we may view $\mathfrak{g}$ as a subalgebra of U. In particular, $\mathfrak{h}$ is a subalgebra of U. We define

$$\tilde{U}_\alpha := \psi^{-1}(L_\alpha),\ \alpha\in\Gamma,$$
$$U_\alpha := \begin{cases} [\tilde{U}_\alpha,\mathfrak{h}], & \alpha\in\Gamma\setminus\{0\},\\ \tilde{U}_0, & \alpha=0.\end{cases}$$

We are going to show that U_α is exactly the α-weight space for $\operatorname{ad}_U\mathfrak{h}$. For all $k,h\in\mathfrak{h}$ and $x\in\tilde{U}_\alpha$,

$$[k,[x,h]] = [[k,x],h] = \alpha(k)[x+v,h] = \alpha(k)[x,h]$$

for some $v\in\operatorname{Ker}\psi$. This proves that U_α is a subspace of the α-weight space for $\operatorname{ad}_U\mathfrak{h}$, $\alpha\in\Gamma\setminus\{0\}$. It follows that for $\alpha\in\Gamma\setminus\{0\}$, $U_\alpha\cap\operatorname{Ker}\psi=\{0\}$, and hence $\psi\,|_{U_\alpha} U_\alpha\to L_\alpha$ is an isomorphism for all $\alpha\in\Gamma\setminus\{0\}$. Now we are going to show that $U=\sum\limits_{\alpha\in\Delta}U_\alpha+U_0$. Let $x\in U$ and write $x=\sum_{\alpha\in\Gamma}\tilde{x}_\alpha$ where $\tilde{x}_\alpha\in\tilde{U}_\alpha$. Let $\alpha\in\Gamma\setminus\{0\}$. Fix any $h\in\mathfrak{h}$ such that $\alpha(h)\neq 0$. We claim that

$$\tilde{x}_\alpha-\alpha(h)^{-1}[h,\tilde{x}_\alpha]\in\operatorname{Ker}\psi\subset\tilde{U}_0=U_0.$$

Indeed,

$$\psi(\tilde{x}_\alpha-\alpha(h)^{-1}[h,\tilde{x}_\alpha]) = \psi(\tilde{x}_\alpha)-\alpha(h)^{-1}[h,\psi(\tilde{x}_\alpha)] = \psi(\tilde{x}_\alpha)-\psi(\tilde{x}_\alpha)=0$$

as $\psi(\tilde{x}_\alpha)\in L_\alpha$. Thus, we may rewrite x as $\sum_{\alpha\in\Gamma}x_\alpha$ where $x_\alpha\in U_\alpha$. It follows that

$$U=U_0+\sum_{\alpha\in\Gamma}U_\alpha.$$

Now

$$U_0 = \tilde{U}_0 = \psi^{-1}\left(\sum_{\alpha\in\Gamma\setminus\{0\}}[L_\alpha,L_{-\alpha}]\right) = \sum_{\alpha\in\Gamma\setminus\{0\}}[\tilde{U}_\alpha,\tilde{U}_{-\alpha}] + \operatorname{Ker}\psi.$$

Since $\psi\mid_{U_\alpha}: U_\alpha \to L_\alpha$ is an isomorphism of vector spaces for all $\alpha\in\Gamma\setminus\{0\}$, we have $\tilde{U}_\alpha = U_\alpha + \operatorname{Ker}\psi$ so

$$U_0 = \sum_{\alpha\in\Gamma\setminus\{0\}}[U_\alpha,U_{-\alpha}] + \operatorname{Ker}\psi. \tag{5.1.1}$$

This proves that U_0 is a 0-eigenspace for $\mathfrak{h}$. Now we see that U_α is exactly the α-eigenspace for $\mathfrak{h}$ for all $\alpha\in\Gamma$ and hence $\left[U_\alpha,U_\beta\right]\subseteq U_{\alpha+\beta}$, whenever $\alpha+\beta\in\Gamma$. Thus, U is $(\Gamma,\mathfrak{g})$-pregraded. It remains to show that $U_0\subseteq\sum_{\alpha\in\Gamma\setminus\{0\}}[U_\alpha,U_{-\alpha}]$.

Since $U=[U,U]$, we have

$$U_0 = \sum_{\alpha\in\Gamma\setminus\{0\}}[U_\alpha,U_{-\alpha}] + [U_0,U_0].$$

But by (5.1.1),

$$[U_0,U_0] = \sum_{\alpha,\beta\in\Gamma\setminus\{0\}}[[U_\alpha,U_{-\alpha}],[U_\beta,U_{-\beta}]] \subseteq \sum_{\gamma\in\Gamma\setminus\{0\}}[U_\gamma,U_{-\gamma}].$$

Hence

$$U_0 = \sum_{\alpha\in\Gamma\setminus\{0\}}[U_\alpha,U_{-\alpha}].$$

This proves that U is $(\Gamma,\mathfrak{g})$-graded, as required. □

Corollary 5.1.3. *(1) Let (U,ψ) be the universal covering algebra of L. Then U is $(\Gamma,\mathfrak{g})$-graded if and only if L is $(\Gamma,\mathfrak{g})$-graded.*

(2) All Lie algebras in a given isogeny class are Γ-graded if one of them is, and all have isomorphic weight spaces for non-zero weights.

Lemma 5.1.4. *Suppose that $\pi:\tilde{L}\to L$ is a central extension of a $(\Gamma,\mathfrak{g})$-graded Lie algebra L with kernel $\mathbb{E}$. Then there is lifting of the grading subalgebra $\mathfrak{g}$ of L to a subalgebra of $\tilde{L}$. Moreover, L can be lifted to a subspace L of $\tilde{L}$ which contains the given $\mathfrak{g}$ so that the corresponding 2-cocycle satisfies $\zeta(\mathfrak{g},L)=0$.*

Proof. We use the same method as in [4, 3.1-3.4]. Since $\mathbb{F}$ is a field, we can lift L to a subspace of $\tilde{L}$ which is mapped isomorphically to L by π if we identify this subspace of $\tilde{L}$ with L. We have $\tilde{L}=L\oplus\mathbb{E}$ and the multiplication on $\tilde{L}$ is given by

$$[f,g]^{\tilde{}} = [f,g]+\zeta(f,g), f,g\in L,$$

where $[f,g]$ denotes the product in L and $\zeta \colon L \times L \to \mathbb{E}$ is the corresponding 2-cocycle. Thus ζ is a bilinear mapping satisfying

$$\begin{aligned} &(i)\ \zeta(f,g) = -\zeta(\mathfrak{g}, f) \\ &(ii)\ \zeta([f,g],h) + \zeta([g,h],f) + \zeta([h,f],\mathfrak{g}) = 0, \end{aligned} \tag{5.1.2}$$

for all $f,g,h \in L$. The subalgebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \zeta(\mathfrak{g},\mathfrak{g})$, is a finite-dimensional $\mathfrak{g}$-module under the action $x.\ w = [x,w]\tilde{}$, which can be readily seen from the calculation

$$[x,y].w = [[x,y],w]\tilde{} = [[x,y]\tilde{},w]\tilde{} = [x,[y,w]\tilde{}]\tilde{} - [y,[x,w]\tilde{}]\tilde{} = x.(y.w) - y.(x.w).$$

By complete reducibility of finite-dimensional $\mathfrak{g}$-modules (see Lemma 3.1.2), there must exist a $\mathfrak{g}$-complement $\tilde{\mathfrak{g}} = \mathfrak{g}' \oplus \zeta(\mathfrak{g},\mathfrak{g})$ to the $\mathfrak{g}$-submodule $\zeta(\mathfrak{g},\mathfrak{g})$. Then each $y \in \mathfrak{g}$ has a unique expression $y = y' + e_y$, where $y' \in \mathfrak{g}'$ and $e_y \in \zeta(\mathfrak{g},\mathfrak{g})$. For $x, y \in \mathfrak{g}$,

$$[x,y] = [x,y]' + e_{[x,y]},$$

is one such expression, while $[x,y] = [x,y']\tilde{} - \zeta(x,y)$ is yet another since $\mathfrak{g}'$ is a $\mathfrak{g}$-submodule. Therefore

$$[x,y]' = [x,y']\tilde{} = [x',y']\tilde{},$$

which shows that $\mathfrak{g}'$ is a subalgebra of $\tilde{L}$ and that the map $\mathfrak{g} \to \mathfrak{g}'$ $(y \mapsto y')$ is a Lie algebra isomorphism.

Using Lemma 3.1.2, we get $L \cong \bigoplus_{\mu \in Q} V(\mu) \otimes W_\mu$ for some vector spaces W_μ where Q is the set of dominant weights of $\mathfrak{g}$. Let $\{w_\mu^j \mid j \in J_\mu\}$ be a basis of W_μ. Then L is the direct sum of the finite-dimensional $\mathfrak{g}$-modules

$$M \in \mathfrak{M} := \{V(\mu) \otimes w_\mu^j \mid \mu \in Q,\ j \in J_\mu\}.$$

For such a module $M \neq \mathfrak{g} \otimes 1$ consider the following $\mathfrak{g}$-submodule of $\tilde{L}$:

$$\tilde{M} = M \oplus \zeta(\mathfrak{g}, M),$$

with $\mathfrak{g}$-action given by $x.w = [x,w]\tilde{}$. This can be viewed as a $\mathfrak{g}'$-module where $x'.w = [x',w]\tilde{} = [x,w]\tilde{}$for all $x' \in g'$ and $w \in M^{\sim}$. The submodule $\zeta(\mathfrak{g},M)$ has a $\mathfrak{g}'$-complement,

$$\tilde{M} = M' \oplus \zeta(\mathfrak{g}, M).$$

Thus for each $m \in M$ there exist unique elements $m' \in M', e_m \in \zeta(\mathfrak{g}, M)$ with $m = m' + e_m$. Let $L' =^{def} \sum_{M \in \mathfrak{M}} M'$. Then $\tilde{L} = L' \oplus \mathbb{E}$.

Suppose $\pi_1 : \tilde{L} \to L'$ and $\pi_2 : \tilde{L} \to \mathbb{E}$ are the projections onto the summands, and define

$$[w', z']_1 = \pi_1([w', z'\tilde{]}),$$
$$\zeta'(w', z') = \pi_2([w', z'\tilde{]}),$$

for $w', z' \in L'$. We claim $(L', [,]_1)$ is a Lie algebra isomorphic to L. Indeed,

$$[[w', z']_1, t']_1 = \pi_1([[w', z']_1, t'\tilde{]}) = \pi_1([[w', z'\tilde{]}).$$

It is clear that cyclically permuting w', z', t' and summing will give 0. Now assume that $m \in M$ and $n \in N$ where M, N are two (possibly equal) modules in $\mathfrak{M}$, and write $m = m' + e_m$ and $n = n' + e_n$. Let $M_r,\ r \in \mathfrak{R}$, be an enumeration of the modules in $\mathfrak{M}$. Then the calculation

$$\begin{aligned}
[m', n'\tilde{]} = [m, n\tilde{]} &= [m, n] + \zeta(m, n) \\
&= \sum_{r \in \mathfrak{R}} f_r + \zeta(m, n) \\
&= \sum_{r \in \mathfrak{R}} f'_r + \sum_{r \in \mathfrak{R}} e_{f_r} + \zeta(m, n).
\end{aligned}$$

Thus

$$[m', n']_1 = \sum_{r \in \mathfrak{R}} f'_r,$$
$$\zeta'(m', n') = \sum_{r \in \mathfrak{R}} e_{f_r} + \zeta(m, n),$$

where $[m, n] = \sum_{r \in \mathfrak{R}} f_r$ and $f_r \in M_r$ for all $r \in \mathfrak{R}$. Hence the map $L \to L'$, $f \mapsto f' = \pi_1(f)$ can be seen to be an isomorphism of Lie algebras.

Finally, it is clear that $\zeta'(,)$ is a 2-cocycle on L' with values in $\mathbb{E}$. Moreover, it has the property
$$\zeta'(x', z') = \pi_2([x', z'\tilde{]}) = 0 \text{ for all } x' \in \mathfrak{g}', z' \in L'.$$

Since L' is a $\mathfrak{g}'$-submodule. Thus by replacing L with L', we see that L can be lifted to a subspace L of $\tilde{L}$ so that the corresponding 2-cocycle satisfies $\zeta(\mathfrak{g}, L) = 0$, as required. □

Let V be an irreducible $\mathfrak{g}$-module and let V' be its dual. Let $\pi : V \times V' \to \mathbb{F}$ be any non-degenerate $\mathfrak{g}$-invariant bilinear form. Note that π is unique up to a scalar multiple as $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V',\mathbb{F})\cong\mathrm{Hom}_{\mathfrak{g}}(V,V)\cong\mathbb{F}$. Set $\pi(V,W)=0$ if V and W are irreducible and $W\not\cong V'$.

Theorem 5.1.5. *Let L be a $(\Gamma,\mathfrak{g})$-graded Lie algebra and $L=\bigoplus_{\mu\in Q}V(\mu)\otimes W_\mu$ for some vector spaces W_μ. Assume that $\tilde{L}=L\oplus\mathbb{E}$ is a central extension of L determined by the 2-cocycle $\zeta(\,,\,)\colon L\times L\to\mathbb{E}$ with $\zeta(\mathfrak{g},L)=0$. Then,*

(1) $V(\mu)$ and $V(\nu)$ ($\mu,\nu\in Q$) are orthogonal relative to $\zeta(\,,\,)$ whenever $V(\mu)\not\cong V(\nu)'$ as $\mathfrak{g}$-modules;

(2) there exists an $\mathbb{F}$-bilinear map $\varepsilon:W\times W\to\mathbb{E}$ on the space $W:=\bigoplus_{\mu\in Q\setminus\{0\}}W_\mu$ with $\varepsilon(W_\mu,W_\nu)=0$ whenever $V(\mu)\not\cong V(\nu)'$, such that

$$\zeta(u_\mu\otimes w_\mu,v_\nu\otimes w_\nu)=\pi(u_\mu,u_\nu)\varepsilon(w_\mu,w_\nu)$$

for all $u_\mu\otimes w_\mu\in V(\mu)\otimes W_\mu$ and $u_\nu\otimes w_\nu\in V(\nu)\otimes W_\nu$.

Proof. (1) Let $\{w_\mu^j\mid j\in J_\mu\}$ be a basis of W_μ and let

$$M,N\in\mathfrak{M}:=\{V(\mu)\otimes w_\mu^j\mid \mu\in Q,\ j\in J_\mu\}.$$

Assume $\{e_k\mid k\in K\}$ is a basis for $\mathbb{E}$. The 2-cocycle $\zeta(\,,\,)$ induces an $\mathbb{F}$-linear transformation $\zeta_k:M\otimes N\to\mathbb{F}$, obtained from reading off the coefficient of e_k in $\zeta(m,n)$. It follows from the 2-cocycle condition, $\zeta([x,m],n)+\zeta([m,n],x)+\zeta([n,x],m)=0$, and the assumption that $\zeta(\mathfrak{g},L)=0$, that the map ζ_k is a $\mathfrak{g}$-module homomorphism. But since M and N are irreducible $\mathfrak{g}$-modules and $\mathrm{Hom}_{\mathfrak{g}}(M\otimes N,\mathbb{F})\cong\mathrm{Hom}_{\mathfrak{g}}(M',N)$, $\zeta_k\neq 0$ implies $M'\cong N$. Thus $\zeta(M,N)\neq 0$ implies $M'\cong N$, as required.

(2) Fix $w_\mu^i\in W_\mu$ and $w_\nu^j\in W_\nu$. The mapping $u_\mu\otimes u_\nu\to\zeta_k(u_\mu\otimes w_\mu^i,u_\nu\otimes w_\nu^j)$ determines a $\mathfrak{g}$-module homomorphism from $V(\mu)\otimes V(\nu)$ to $\mathbb{F}$. By Part (1) we can assume that $V(\mu)\cong V(\nu)'$ (otherwise the mapping ζ_k is zero). Then this mapping must be a multiple of the form π, i.e. $\zeta_k(u_\mu\otimes w_\mu^i,u_\nu\otimes w_\nu^j)=\eta_{\mu,\nu}^k\pi(u_\mu,u_\nu)$ for some $\eta_{\mu,\nu}^k\in\mathbb{F}$. Define $\varepsilon_k:W_\mu\times W_\nu\to\mathbb{F}$ by first setting $\varepsilon_k(w_\mu^i,w_\nu^j)=\eta_{\mu,\nu}^k$ and extending this bilinearly. Then

$$\zeta_k(u_\mu\otimes w_\mu^i,u_\nu\otimes w_\nu^j)=\pi(u_\mu,u_\nu)\varepsilon_k(w_\mu^i,w_\nu^j)$$

for all $w_\mu^i\in W_\mu$, $w_k^j\in W_\nu$ and $u_\mu\in V(\mu)$, $u_\nu\in V(\nu)$. As a result,

$$\begin{aligned}\zeta(u_\mu\otimes w_\mu^i,u_\nu\otimes w_\nu^j)&=\sum_{k\in K}\zeta_k(u_\mu\otimes w_\mu^i,u_\nu\otimes w_\nu^j))e_k\\&=\pi(u_\mu,u_\nu)\sum_{k\in K}\varepsilon_k(w_\mu^i,w_\nu^j)e_k\end{aligned}$$

$$= \pi(u_\mu, u_\nu)\varepsilon(w_\mu^i, w_\nu^j)$$

where $\varepsilon(w_\mu^i, w_\nu^j) := \sum_{k\in K} \varepsilon_k(w_\mu^i, w_\nu^j)e_k \in \mathbb{E}$. Thus we get a map $\varepsilon : V(\mu)\otimes V(\nu) \to \mathbb{E}$ such that

$$\zeta(u_\mu \otimes w_\mu^i, u_\nu \otimes w_\nu^j) = \pi(u_\mu, u_\nu)\varepsilon(w_\mu^i, w_\nu^j).$$

We extend the mappings to the whole space $W\times W$ by defining $\varepsilon(w_\mu^i, w_k^j) = 0$ for all $V(\mu) \ncong V(\nu)$ and $w_\mu^i \in W_\mu$, $w_k^j \in W_\nu$. We obtain an $\mathbb{F}$-bilinear map taking $W\times W$ to $\mathbb{E}$, as required. □

Theorem 5.1.6. *Assume that $\tilde{L} = L\oplus\mathbb{E}$ is a central extension of the Θ_n-graded Lie algebra $L = (\mathfrak{g}\otimes A)\oplus(V\otimes B)\oplus\cdots\oplus(\Lambda'\otimes E')\oplus D$ determined by the 2-cocycle $\zeta(\,,\,)$: $L\times L \to \mathbb{E}$ with $\zeta(\mathfrak{g}, L) = 0$. Then,*

(1) $V(\mu)$ and $V(\nu)$ ($\mu,\nu\in\Theta_n^+$) are orthogonal relative to $\zeta(\,,\,)$ whenever $V(\mu)\ncong V(\nu)'$ as $\mathfrak{g}$-modules;

(2) there exists a 2-cocycle $\varepsilon : \mathfrak{b}\times\mathfrak{b}\to\mathbb{E}$ on the algebra $\mathfrak{b}$ with $\varepsilon(W_\mu, W_\nu) = 0$ whenever $V(\mu)\ncong V(\nu)'$ such that

$$\begin{array}{lll}
(a) & \zeta(x^\pm\otimes a_1^\mp, y^\pm\otimes a_2^\mp) & = \mathrm{tr}(x^\pm y^\pm)\varepsilon(a_1^\mp, a_2^\mp) \\
(b) & \zeta(s\otimes c, s'\otimes c') & = \mathrm{tr}(ss')\varepsilon(c,c') \\
(c) & \zeta(\lambda\otimes e, \lambda'\otimes e') & = \mathrm{tr}(\lambda\lambda')\varepsilon(e,e') \\
(d) & \zeta(v\otimes b, v'\otimes b') & = \mathrm{tr}(vv'^t)\varepsilon(b,b') \\
(e) & \zeta(d, \langle\beta,\beta'\rangle) & = \varepsilon(d\beta,\beta')+\varepsilon(\beta,d\beta') = -\zeta(\langle\beta,\beta'\rangle, d),
\end{array} \tag{5.1.3}$$

for all $x,y\in\mathfrak{g}$, $v\in V$, $v'\in V'$, $s\in S$, $\lambda\in\Lambda$, $s'\in S'$, $\lambda'\in\Lambda'$ and for all $a_1^\mp, a_2^\mp\in A^\mp$, $b\in B$, $b'\in B'$, $c\in C$, $c'\in C'$, $e\in E$, $e'\in E'$, $\beta,\beta'\in\mathfrak{b}$ and $d\in D$.

Proof. This is similar to [4, Proposition 5.33] and [4, Thereom 3.7]. Let $W := A\oplus C\oplus E\oplus C'\oplus E'\oplus B\oplus B'$. In Theorem 5.1.5, we show that there exists an $\mathbb{F}$-bilinear map ε an $\mathbb{F}$-bilinear map taking $W\times W$ to $\mathbb{E}$ and

$$\begin{array}{lll}
(a) & \zeta(x\otimes a_1, y\otimes a_2) & = \mathrm{tr}(xy)\varepsilon(a_1,a_2) \\
(b) & \zeta(s\otimes c, s'\otimes c') & = \mathrm{tr}(ss')\varepsilon(c,c') \\
(c) & \zeta(\lambda\otimes e, \lambda'\otimes e') & = \mathrm{tr}(\lambda\lambda')\varepsilon(e,e') \\
(d) & \zeta(v\otimes b, v'\otimes b') & = \mathrm{tr}(vv'^t)\varepsilon(b,b')
\end{array}$$

for all $x,y\in\mathfrak{g}$, $v\in V$, $v'\in V'$, $s\in S$, $\lambda\in\Lambda$, $s'\in S'$, $\lambda'\in\Lambda'$ and for all $a_1,a_2\in A$, $b\in B$, $b'\in B'$, $c\in C$, $c'\in C'$, $e\in E$, $e'\in E'$, $\beta,\beta'\in\mathfrak{b}$ and $d\in D$. Since $(x^+\mid x^-) = 0$ for all $x^\pm\in\mathfrak{g}^\pm$, we can extend the mapping ε to the algebra $\mathfrak{b} = A^+\oplus A^-\oplus C\oplus E\oplus C'\oplus E'\oplus B\oplus B'$

by defining $\varepsilon(a_1^+,a_2^-)=\varepsilon(a_1^-,a_2^+)=0$ and $\varepsilon(a_1^\pm,a_2^\pm)=\varepsilon(a_1,a_2)$ for all $a_1,a_2\in A$. Thus, we obtain an $\mathbb{F}$-bilinear map taking $\mathfrak{b}\times\mathfrak{b}$ to $\mathbb{E}$.

It remains to show that $\varepsilon(\ ,\ )$ is a 2- cocycle of $\mathfrak{b}$ and

$$\zeta(d,\langle\beta,\beta'\rangle)=\varepsilon(d\beta,\beta')+\varepsilon(\beta,d\beta')=-\zeta(\langle\beta,\beta'\rangle,d),$$

for all $\beta,\beta'\in\mathfrak{b}$ and $d\in D$. Applying the cocycle relation $\zeta([f,g],h)+\zeta([g,h],f)+\zeta([h,f],\mathfrak{g})=0$ and using the orthogonality of some of the components, we determine that $\varepsilon(\ ,\ )$ is a 2- cocycle of $\mathfrak{b}$. We illustrate these calculations by considering homogeneous elements α_1,α_2 and α_3 in $\mathfrak{a}$. Set

$$z_1=E_{1,2}+\varepsilon_1E_{2,1},\ z_2=E_{2,3}+\varepsilon_2E_{3,2}\text{ and }z_3=E_{3,1}+\varepsilon_3E_{1,3}\text{ where }\varepsilon_i=\pm1.$$

The sign of each ε_i can be chosen in such a way that $z_i\otimes\alpha_i$ belongs to the corresponding homogeneous component of L. Note that $\mathrm{tr}(z_iz_j)=0$ for all $i\neq j$. Hence by Lemma 4.1.2, we have

$$[z_i\otimes\alpha_i,z_j\otimes\alpha_j]=z_i\diamond z_j\otimes\frac{[\alpha_i,\alpha_j]}{2}+[z_i,z_j]\otimes\frac{\alpha_i\circ\alpha_j}{2}.$$

Then from (5.1.2) with $z_1\otimes\alpha_1$, $z_2\otimes\alpha_2$, $z_1\otimes\alpha_3$, we obtain

$$\begin{aligned}([z_1,z_2]\mid z_3)\varepsilon(\alpha_1\circ\alpha_2,\alpha_3)&+(z_1\diamond z_2\mid z_3)\varepsilon([\alpha_1,\alpha_2],\alpha_3)\\&+([z_2,z_3]\mid z_2)\varepsilon([\alpha_2,\alpha_3],\alpha_1)+(z_2\diamond z_3\mid z_2)\varepsilon([\alpha_2,\alpha_3],\alpha_1)\\&\quad+([z_3,z_1]\mid z_2)\varepsilon(\alpha_3\circ\alpha_1,\alpha_2)+(z_3\diamond z_1\mid z_2)\varepsilon([\alpha_3,\alpha_1],\alpha_2)=0\end{aligned}$$

Using the fact that $(z\mid y)=\frac{1}{n}\mathrm{tr}(zy)$, it is easy to verify that the form is associative relative to the "$\diamond$" product, (i.e. $(z\diamond y\mid z)=(z\mid y\diamond z)$ holds for all $x,y,z\in\mathfrak{g}\cup S\cup S'\cup\Lambda\cup\Lambda'$), and also relative to the commutator product. Thus,

$$\begin{aligned}([z_1,z_2]\mid z_3)(\varepsilon(\alpha_1\circ\alpha_2,\alpha_3)+\varepsilon(\alpha_2\circ\alpha_3,\alpha_1)+\varepsilon(\alpha_3\circ\alpha_1,\alpha_2))\\+(z_1\diamond z_2\mid z_3)(\varepsilon([\alpha_1,\alpha_2],\alpha_3)+\varepsilon([\alpha_2,\alpha_3],\alpha_1)+\varepsilon([\alpha_3,\alpha_1],\alpha_2))=0.\end{aligned}\tag{5.1.4}$$

Note that $\varepsilon_1\varepsilon_2\varepsilon_3=\pm1$ and

$$[z_1,z_2]z_3=E_{11}-\varepsilon_1\varepsilon_2\varepsilon_3E_{33}.$$
$$(z_1\diamond z_2)z_3=E_{11}+\varepsilon_1\varepsilon_2\varepsilon_3E_{33}.$$

If $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, then

$$\varepsilon([\alpha_1,\alpha_2],\alpha_3)+\varepsilon([\alpha_2,\alpha_3],\alpha_1)+\varepsilon([\alpha_3,\alpha_1],\alpha_2)=0 \tag{5.1.5}$$

and we have four cases: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = -1$; $\varepsilon_1 = \varepsilon_2 = -1$ and $\varepsilon_3 = 1$; $\varepsilon_1 = \varepsilon_3 = -1$ and $\varepsilon_2 = 1$. In each of these cases $\varepsilon(\alpha_1\circ\alpha_2,\alpha_3) = \varepsilon(\alpha_2\circ\alpha_3,\alpha_1) = \varepsilon(\alpha_3\circ\alpha_1,\alpha_2) = 0$ (see Table 4.1.1), so

$$\varepsilon(\alpha_1\circ\alpha_2,\alpha_3)+\varepsilon(\alpha_2\circ\alpha_3,\alpha_1)+\varepsilon(\alpha_3\circ\alpha_1,\alpha_2)=0 \tag{5.1.6}$$

as well. Adding equations (5.1.5) and (5.1.6) gives the desired 2-cocycle condition.

If $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, then

$$\varepsilon(\alpha_1\circ\alpha_2,\alpha_3)+\varepsilon(\alpha_2\circ\alpha_3,\alpha_1)+\varepsilon(\alpha_3\circ\alpha_1,\alpha_2)=0 \tag{5.1.7}$$

and we have four cases: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$; $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$; $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = -1$. In each of these cases $\varepsilon([\alpha_1,\alpha_2],\alpha_3) = \varepsilon([\alpha_2,\alpha_3],\alpha_1) = \varepsilon([\alpha_3,\alpha_1],\alpha_2) = 0$ (see Table 4.1.1), so

$$\varepsilon([\alpha_1,\alpha_2],\alpha_3)+\varepsilon([\alpha_2,\alpha_3],\alpha_1)+\varepsilon([\alpha_3,\alpha_1],\alpha_2)=0 \tag{5.1.8}$$

as well. Adding equations (5.1.7) and (5.1.8) gives the desired 2-cocycle condition.

To prove (e), consider the 2-cocycle relation (5.1.2) for the elements $x_1\otimes\alpha_1$, $x_2\otimes\alpha_1$, d and use Lemma 4.1.2. □

Proposition 5.1.7. *$\langle,\rangle : \mathfrak{b}\times\mathfrak{b}\to D$ is a surjective 2-cocycle.*

Proof. Linearity of the bracket of L lead to $\langle,\rangle$ is an $\mathbb{F}$-bilinear map. Let α,β,γ and δ be homogeneous elements in $\mathfrak{b}$. From anti-commutativity of the bracket and the fact that

$$\operatorname{tr}(xy) = \operatorname{tr}(yx),$$
$$\operatorname{tr}(uv'^t) = \operatorname{tr}(v'u^t),$$

for all $n\times n$ matrices x and y and $v\in V$ and $v'\in V'$, we deduce that $\langle\alpha,\beta\rangle = -\langle\beta,\alpha\rangle$ for all $\alpha,\beta\in\mathfrak{b}$. It only remains to show that $\langle,\rangle$ satisfies the Jacoby identity, which can be proved by making various choices of $z_1\otimes\alpha$, $z_2\otimes\beta$, $z_3\otimes\gamma \in (\mathfrak{g}\otimes A)\cup(V\otimes B)\cup(V'\otimes B')\cup(S\otimes C)\cup(S'\otimes C')\cup(\Lambda\otimes E)\cup(\Lambda'\otimes E')$ and calculating the corresponding Jacoby identity. As illustration, consider $\alpha = a^- \in A^-$, $\beta = b' \in B'$, $\gamma = b \in B$. We get

$$[z \otimes a^-, [v' \otimes b', u \otimes b] = [[z \otimes a^-, v' \otimes b'], u \otimes b] + [v' \otimes b', [z \otimes a^-, u \otimes b]].$$

Using (3.4.4) and Lemma 4.2.3 we get

$$\frac{z \circ (uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n} I)}{4} \otimes [a^-, [b, b']_{A^-}]_{A^-} + \frac{[z, uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n} I]}{4}$$
$$\otimes (a^- \circ [b, b']_{A^-})_{A^+} + \frac{(z \mid (uv'^t + v'u^t - \frac{2\operatorname{tr}(uv'^t)}{n} I))}{2} \langle a^-, [b, b']_{A^-} \rangle$$
$$= -\frac{(u(z^t u')^t + z^t u' u^t - \frac{2\operatorname{tr}(uv'^t)}{n} I)}{2} \otimes [b, b'a^-]_{A^-} - \frac{(u(z^t u')^t - z^t u' u^t)}{2} \otimes (b \circ b'a^-)_{A^+}$$
$$- \frac{2\operatorname{tr}(u(z^t u')^t}{n} \langle b, b'a^- \rangle - \frac{(zuv'^t + v'(zu)^t - \frac{2\operatorname{tr}(zuv'^t)}{n} I)}{2} \otimes [a^- b, b']_{A^-}$$
$$- \frac{(zuv'^t - v'(zu)^t)}{2} \otimes (a^- b \circ b')_{A^+} - \frac{2\operatorname{tr}(zuv'^t)}{n} \langle a^- b, b' \rangle.$$

Then $\langle \mathfrak{b}, \mathfrak{b} \rangle$-component of the Jacobi identity gives

$$\operatorname{tr}(z(uv'^t)(\langle a^-, \frac{[b, b']_{A^-}}{2} \rangle + \langle b, b'a^- \rangle + \langle b', a^- b \rangle) = 0.$$

Choosing u, v' and z such that, $\operatorname{tr}(z(uv'^t) \neq 0$ (for example $u_1 = e_1$, $u' = e_2$ and $z = (E_{1,2} + E_{2,1})$), we get $\langle a^-, \frac{[b,b']_{A^-}}{2} \rangle + \langle b, b'a^- \rangle + \langle b', a^- b \rangle = 0$. Since $\langle (b \circ b')_{A^+}, a^- \rangle = 0$, we obtain $\langle a^-, bb' \rangle + \langle b, b'a^- \rangle + \langle b', a^- b \rangle = 0$. Thus, $\langle , \rangle$ is a 2-cocyclic map. In (4.2.5), we showed that $D = \langle \mathfrak{b}, \mathfrak{b} \rangle$. Therefore $\langle , \rangle$ is a surjective 2-cocycle as required. □

5.2 Classification of Θ_n-graded Lie algebras, $n \geq 5$

We define a centerless algebra $\mathscr{L}(\mathfrak{b})$ and show that it is Θ_n-graded with coordinate algebra $\mathfrak{b}$. Instead of proving directly that $\mathscr{L}(\mathfrak{b})$ satisfies the Jacoby identity (which is quite lengthy), we construct an explicit example of a Θ_n-graded Lie algebra $\mathfrak{u}$ such that $\mathfrak{u}$ modulo its center is isomorphic to $\mathscr{L}(\mathfrak{b})$, see Example 5.2.3. It is also shown that any Θ_n-graded Lie algebra L with coordinate algebra $\mathfrak{b}$ is a cover of the centerless Lie algebra $\mathscr{L}(\mathfrak{b})$. We show that every Θ_n-graded Lie algebra L is uniquely determined (up to central isogeny) by its coordinate algebra $\mathfrak{b}$ and L is centrally isogenous to the Θ_n-graded unitary Lie algebra $\mathfrak{u}$ of the hermitian form $\xi = w\perp - \chi$ on the $\mathfrak{a}$-module $\mathfrak{a}^n \oplus \mathscr{B}$ (Proposition 5.2.4 and Theorem 5.2.6).

Definition 5.2.1. [2, 2.2] Let A be an associative algebra with involution η. A map $\xi : X \times X \to A$ is called a *hermitian form* over A if X is a right A-module and $\xi : X \times X \to A$ is a bi-additive map such that

$$\begin{aligned} \xi(xa, y) &= \eta(a)\xi(x, y), \\ \xi(x, ya) &= \xi(x, y)a, \\ \xi(y, x) &= \eta(\xi(x, y)), \end{aligned}$$

for $a \in A$ and $x, y \in X$. If Y is an A-submodule of X, then

$$Y^{\perp} := \{x \in X \mid \xi(x, y) = 0 \text{ for all } y \in Y\}$$

is also an A-submodule of X. The form ξ is said to be *nondegenerate* if $X^{\perp} = 0$.

Definition 5.2.2. [2, 4.1.1] Let A be an associative algebra with involution. Suppose that $\xi : X \times X \to A$ is a hermitian form over A. Let

$$\mathfrak{U}(X, \xi) = \{T \in \mathrm{End}_A(X) \mid \xi(T(u), v) + \xi(u, T(v)) = 0,\ \forall u, v \in X\}$$

Then $\mathfrak{U}(X, \xi)$ is a Lie subalgebra of $\mathrm{End}_A(X)$, and we say that $\mathfrak{U}(X, \xi)$ is the *unitary* Lie algebra of ξ.

Example 5.2.3. Let $\mathfrak{a}$ be any associative algebra with involution η, identity element 1^+ and two orthogonal idempotents e_1 and e_2 such that $1^+ = e_1 + e_2$ and $e_2 = \eta(e_1)$ and let $\mathscr{B}$ be any unital associative right $\mathfrak{a}$-module with a hermitian form χ with values in $\mathfrak{a}$. Put $\eta_{\mathscr{B}} = I$. Define $\beta_1.\beta_2 = \chi(\beta_1, \beta_2)$ for all $\beta_1, \beta_2 \in B \oplus B'$. Then $\mathfrak{b} = \mathfrak{a} \oplus B \oplus B'$ is a (non-associative) algebra with multiplication extending that on $\mathfrak{a}$. For every $n \geq 5$, we are going to explicitly construct a Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b} = \mathfrak{a} \oplus \mathscr{B}$.

We start with the Peirce decomposition

$$\mathfrak{a} = e_1\mathfrak{a}e_1 \oplus e_1\mathfrak{a}e_2 \oplus e_2\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2.$$

Note that $\eta(e_1\mathfrak{a}e_1) = e_2\mathfrak{a}e_2$ and both $e_1\mathfrak{a}e_2$ and $e_2\mathfrak{a}e_1$ are η-invariant. Define

$$\begin{aligned} A^+ = sym(e_1\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2),\ A^- = skew(e_1\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2),\ B = \mathscr{B}e_2, \quad B' = \mathscr{B}e_1. \\ E = sym(e_1\mathfrak{a}e_2), \quad C = skew(e_1\mathfrak{a}e_2), \quad E' = sym(e_2\mathfrak{a}e_1), \quad C' = skew(e_2\mathfrak{a}e_1), \end{aligned}$$

Thus, we have $\mathfrak{a} = A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E'$ and $\mathscr{B} = B \oplus B'$. The right $\mathfrak{a}$-module $\mathscr{B}$

can be regarded as a left $\mathfrak{a}$-module by means of the action $\alpha.\beta = \beta\eta(\alpha)$ for $\alpha \in \mathfrak{a}$ and $\beta \in \mathscr{B}$.

Since $\mathfrak{a}$ is a right $\mathfrak{a}$-module under right multiplication, $\mathfrak{a}^n$ ($n \times 1$ column vectors with entries in $\mathfrak{a}$) is also a right $\mathfrak{a}$-module. Let $w : \mathfrak{a}^n \times \mathfrak{a}^n \to \mathfrak{a}$ be a non degenerate bilinear form on $\mathfrak{a}^n$ defined by

$$w(\alpha_1, \alpha_2) = \eta(\alpha_1)^t \alpha_2$$

where $\alpha_1, \alpha_2 \in \mathfrak{a}^n$. Let $\xi : (\mathfrak{a}^n \oplus \mathscr{B}) \times (\mathfrak{a}^n \oplus \mathscr{B}) \to \mathfrak{a}^n \oplus \mathscr{B}$ be a bilinear form on $\mathfrak{a}^n \oplus \mathscr{B}$ defined by

$$\xi(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2) = w(\alpha_1, \alpha_2) - \chi(\beta_1, \beta_2)$$

where $\beta_1, \beta_2 \in \mathscr{B}$ and $\alpha_1, \alpha_2 \in \mathfrak{a}^n$. Then

$$\mathfrak{U} = \mathfrak{U}(X, \xi) = \{T \in \mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n \oplus \mathscr{B}) \mid \xi(T(u), v) + \xi(u, T(v)) = 0,\ \forall u, v \in \mathfrak{a}^n \oplus \mathscr{B}\}$$

is a Lie subalgebra of $\mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n \oplus \mathscr{B})$ under the commutator $[T, T'] = TT' - T'T$, called the *unitary Lie algebra of the hermitian form* $\xi = w\perp - \chi$. We can identify $\mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n \oplus \mathscr{B})$ in a natural way with the algebra of 2×2 matrices:

$$\begin{bmatrix} \mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n) & \mathrm{Hom}_{\mathfrak{a}}(\mathscr{B}, \mathfrak{a}^n) \\ \mathrm{Hom}_{\mathfrak{a}}(\mathfrak{a}^n, \mathscr{B}) & \mathrm{End}_{\mathfrak{a}}(\mathscr{B}) \end{bmatrix}$$

whose components have the following realizations:

$M_n(\mathfrak{a}) \cong \mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n)$ via the map $M \mapsto \hat{M}(\alpha \mapsto M\alpha)$.

$(\mathscr{B}^*)^n \cong \mathrm{Hom}_{\mathfrak{a}}(\mathscr{B}, \mathfrak{a}^n)$ where $\mathscr{B}^* = \mathrm{End}_{\mathfrak{a}}(\mathscr{B}, \mathfrak{a})$ via the map

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \mapsto \hat{\lambda}(\beta \mapsto \begin{bmatrix} \lambda_1\beta \\ \vdots \\ \lambda_n\beta \end{bmatrix}).$$

$(\mathscr{B}^n)^t \cong \mathrm{Hom}_{\mathfrak{a}}(\mathfrak{a}^n, \mathscr{B})$ via the map

$$\beta^t = \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix} \mapsto \hat{\beta}^t(\alpha \mapsto \beta^t\alpha).$$

Elements of $\mathfrak{a}^n \oplus \mathscr{B}$ can be viewed as column vectors $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \beta \end{bmatrix}$, where $\alpha_1, \cdots, \alpha_n \in \mathfrak{a}$ and

$\beta \in \mathscr{B}$ and elements of $\mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n \oplus \mathscr{B})$ can be regarded as matrices

$$\begin{bmatrix} & & & \lambda_1 \\ & M & & \vdots \\ & & & \lambda_n \\ \beta_1 & \cdots & \beta_n & N \end{bmatrix}$$

where $M \in M_n(\mathfrak{a})$, $\beta_1, \cdots, \beta_n \in \mathscr{B}$, $\lambda_1, \cdots, \lambda_n \in \mathscr{B}^* := \mathrm{Hom}_{\mathfrak{a}}(\mathscr{B}, \mathfrak{a})$ and $N \in \mathrm{End}_{\mathfrak{a}}(\mathscr{B})$. The action of $\mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n + \mathscr{B})$ on $\mathfrak{a}^n \oplus \mathscr{B}$ is by left multiplication, and composition in $\mathrm{End}_{\mathfrak{a}}(\mathfrak{a}^n \oplus \mathscr{B})$ is matrix multiplication. The elements of $M_n(\mathfrak{a})$ are linear combinations of the elements $E_{i,j}\alpha$ $(1 \leq i, j \leq n)$, but the multiplication in $M_n(\mathfrak{a})$ is given by

$$(E_{i,j}\alpha)(E_{r,s}\alpha') = \delta_{j,r}E_{i,s}\alpha\alpha'.$$

We define $\chi_c : \mathscr{B} \to \mathfrak{a}$ by $\chi_c(c') = \chi(c, c')$ and for

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in (\mathscr{B}^*)^n, \text{ set } \chi_{\underline{\lambda}} = \begin{bmatrix} \chi_{\lambda_1} \\ \vdots \\ \chi_{\lambda_n} \end{bmatrix}.$$

Let $\begin{bmatrix} M & Y \\ X & N \end{bmatrix} \in \mathfrak{U}$ and $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \in \mathfrak{a}^n \oplus \mathscr{B}$. Then

$$\begin{aligned}
0 &= \xi(\begin{bmatrix} M & Y \\ X & N \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}) + \xi(\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{bmatrix} M & Y \\ X & N \end{bmatrix}\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}) \\
&= \xi(\begin{pmatrix} M\alpha_1 + Y\beta_1 \\ X\alpha_1 + N\beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}) + \xi(\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} M\alpha_2 + Y\beta_2 \\ X\alpha_2 + N\beta_2 \end{pmatrix}) \\
&= w(M\alpha_1 + Y\beta_1, \alpha_2) - \chi(X\alpha_1 + N\beta_1, \beta_2) + w(\alpha_1, M\alpha_2 + Y\beta_2) - \chi(\beta_1, X\alpha_2 + N\beta_2) \\
&= \eta(M\alpha_1 + Y\beta_1)^t\alpha_2 + \eta(\alpha_1)^t(M\alpha_2 + Y\beta_2) - \chi(X\alpha_1 + N\beta_1, \beta_2) - \chi(\beta_1, X\alpha_2 + N\beta_2) \\
&= \eta(M\alpha_1)^t\alpha_2 + \eta(Y\beta_1)^t\alpha_2 + \eta(\alpha_1)^t(M\alpha_2) + \eta(\alpha_1)^t(Y\beta_2) \\
&- \chi(X\alpha_1, \beta_2) - \chi(N\beta_1, \beta_2) - \chi(\beta_1, X\alpha_2) - \chi(\beta_1, N\beta_2).
\end{aligned}$$

We deduce that

(1) $\eta(M\alpha_1)^t\alpha_2 + \eta(\alpha_1)^t(M\alpha_2) = 0$. We get, $\eta(M)^t + M = 0$.

(2) $\chi(N\beta_1, \beta_2) + \chi(\beta_1, N\beta_2) = 0$.

(3) $\eta(Y\beta_1)^t\alpha_2 = \chi(\beta_1, X\alpha_2)$.

(4) $\eta(\alpha_1)^t(Y\beta_2) - \chi(X\alpha_1, \beta_2) = w(\alpha_1, Y\beta_2) - \chi(X\alpha_1, \beta_2) = 0.$

Fix $X = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ and $Y = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$. By (3), we have $\eta(Y\beta_1)^t\alpha_2 = \beta_1(X\alpha_2)$ where $\alpha_2 \in \mathfrak{a}^n$ and $\beta_1 \in \mathscr{B}$. Hence $\eta(\begin{bmatrix} \lambda_1\beta_1 \\ \vdots \\ \lambda_n\beta_1 \end{bmatrix})^t\alpha_2 = \beta_1(\begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}\alpha_2)$, so

$$\begin{bmatrix} \lambda_1\beta_1 & \cdots & \lambda_n\beta_1 \end{bmatrix} = \begin{bmatrix} \eta(\beta_1\gamma_1) & \cdots & \eta(\beta_1\gamma_n) \end{bmatrix} = \begin{bmatrix} \gamma_1\beta_1 & \cdots & \gamma_n\beta_1 \end{bmatrix}.$$

Therefore $\lambda_i\beta_1 = \gamma_i\beta_1 = \chi(\gamma_i, \beta_1)$. It follows from the nondegeneracy of w that for any $X \in (\mathscr{B}^n)^t \cong \mathrm{Hom}_{\mathfrak{a}}(\mathfrak{a}^n, \mathscr{B})$, there is a unique $Y \in (\mathscr{B}^*)^n \cong \mathrm{Hom}_{\mathfrak{a}}(\mathscr{B}, \mathfrak{a}^n)$ satisfying (3). Moreover, when $X = (\underline{\beta})^t$ in (3), then $Y = \chi_{\underline{\beta}}$. With these convention, we have

$$\mathfrak{U} = \left\{ \begin{bmatrix} M & \chi_\beta \\ \beta^t & N \end{bmatrix} \mid M \in M_n(\mathfrak{a}),\ (\eta M)^t + M = 0,\ \beta \in \mathscr{B}^n,\ N \in \mathfrak{U}(\chi) \right\},$$

where

$$\mathfrak{U}(\chi) = \{N \in \mathrm{End}_{\mathfrak{a}}(\mathscr{B}) \mid \chi(N\beta, \beta') + \chi(\beta, N\beta') = 0\ \forall \beta, \beta' \in \mathscr{B}\}$$

is the unitary Lie algebra of χ. Recall that $1^+ = e_1 + e_2$. Put $1^- = e_1 - e_2$. Let

$$\begin{aligned} \overline{\mathfrak{g}} &= \left\{ \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \mid M \in M_n \otimes span\{1^+, 1^-\} \text{ and } (\eta M)^t + M = 0 \right\} \\ &= \left\{ \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \mid M \in sym(M_n) \otimes 1^- \oplus skew(M_n) \otimes 1^+ \right\}. \end{aligned}$$

By Lemma 4.3.4, the map $\eta : M_n \otimes \mathfrak{a} \to M_n \otimes \mathfrak{a}$ given by $\sigma(x \otimes \alpha) = x^t \otimes \eta(\alpha)$ is an involution of the algebra $M_n \otimes \mathfrak{a} \cong M_n(\mathfrak{a})$. We have

$$skew(M_n \otimes \mathfrak{a}) = sym(M_n) \otimes skew(\mathfrak{a}) \oplus skew(M_n) \otimes sym(\mathfrak{a})$$

where $skew(\mathfrak{a}) = A^- \oplus C \oplus C'$ and $sym(\mathfrak{a}) = A^+ \oplus E \oplus E'$ with respect to η. Note that $sym(M_n) \otimes 1^- \oplus skew(M_n) \otimes 1^+$ is a Lie subalgebra of $skew(M_n \otimes \mathfrak{a})$ and it is isomorphic to gl_n. (The corresponding isomorphism $\varphi : gl_n \to \overline{\mathfrak{g}}$ is given by

$$\varphi(x) = \begin{bmatrix} (x + x^t) \otimes \frac{(e_1 - e_2)}{2} \oplus (x - x^t) \otimes \frac{(e_1 + e_2)}{2} & 0 \\ 0 & 0 \end{bmatrix}).$$

Put

$$\mathfrak{g} = [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}] \cong sl_n.$$

Let $\mathfrak{h} = \begin{bmatrix} H \otimes 1^- & 0 \\ 0 & 0 \end{bmatrix}$ where H is the set of diagonal matrices of sl_n. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{U}$ has the following weight spaces with respect to the adjoint action of $\mathfrak{h}$:

$$\mathfrak{U}_{\varepsilon_i - \varepsilon_j} = \left\{ \begin{bmatrix} E_{i,j} \otimes e_1 \alpha e_1 + E_{j,i} \otimes e_2 \alpha e_2 & 0 \\ 0 & 0 \end{bmatrix} \mid \alpha \in \mathfrak{a} \right\}, \; 1 \leq i \neq j \leq n;$$

$$\mathfrak{U}_{\varepsilon_i + \varepsilon_j} = \left\{ \begin{bmatrix} E_{i,j} \otimes (c+e) - E_{j,i} \otimes \eta(c+e) & 0 \\ 0 & 0 \end{bmatrix} \mid (c+e) \in C+E \right\}, \; 1 \leq i, j \leq n;$$

$$\mathfrak{U}_{-\varepsilon_i - \varepsilon_j} = \left\{ \begin{bmatrix} E_{i,j} \otimes (c'+e') - E_{j,i} \otimes \eta(c'+e') & 0 \\ 0 & 0 \end{bmatrix} \mid (c'+e') \in C'+E' \right\}, \; 1 \leq i, j \leq n;$$

$$\mathfrak{U}_{\varepsilon_i} = \left\{ \begin{bmatrix} 0 & v_i \otimes b \\ (v_i)^t \otimes b & 0 \end{bmatrix} \mid v \in V, \; b \in B \right\}, \; 1 \leq i \leq n;$$

$$\mathfrak{U}_{-\varepsilon_i} = \left\{ \begin{bmatrix} 0 & v'_i \otimes b' \\ (v'_i)^t \otimes b' & 0 \end{bmatrix} \mid v' \in V', \; b' \in B' \right\}, 1 \leq i \leq n;$$

$$\mathfrak{U}_0 = \left\{ \begin{bmatrix} (E_{i,i} - E_{i+1,i+1}) \otimes a^- & 0 \\ 0 & 0 \end{bmatrix} \mid a^- \in A^-, \; i = 1, 2, \cdots, n-1 \right\}$$
$$\cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \mid N \in \mathfrak{U}(\chi) \right\}.$$

In general, the Lie algebra $\mathfrak{U}$ is not Θ_n-graded since it may fail to satisfy Condition $(\Gamma 3)$ in Definition 3.0.1. To obtain a Θ_n-graded Lie algebra we need to pass to the subalgebra $\mathfrak{u}$ of $\mathfrak{U}$ generated by the weight spaces $\mathfrak{U}_\alpha$ corresponding to no zero weights $\alpha \in \Theta_n$. Then,

$$\mathfrak{u} = \bigoplus_{\alpha \in \Theta_n \setminus \{0\}} \mathfrak{U}_\alpha \bigoplus \sum_{\alpha \in \Theta_n \setminus \{0\}} [\mathfrak{U}_\alpha, \mathfrak{U}_{-\alpha}],$$

and $\mathfrak{u}$ is a Θ_n-graded Lie algebra with grading subalgebra $\mathfrak{g}$. Note that

$$\mathfrak{u}_0 = \mathfrak{u} \cap \mathfrak{U}_0 \text{ and } \mathfrak{u}_\alpha = \mathfrak{U}_\alpha \text{ for } \alpha \in \Theta_n \setminus \{0\}.$$

We call $\mathfrak{u}$ the Θ_n-*graded unitary Lie algebra of* $\xi = w\perp - \chi$.

Proposition 5.2.4. *Let $n \geq 5$ and let $\mathfrak{a}$ and $\mathscr{B}$ be as in Example 5.2.3. Let $\mathfrak{u}$ be the Θ_n-*

graded unitary Lie algebra of the hermitian form $\xi = w \perp -\chi$ *on the* $\mathfrak{a}$*-module* $\mathfrak{a}^n \oplus \mathscr{B}$. *Then* $\mathfrak{u}$ is Θ_n*-graded with coordinate algebra* $\mathfrak{b}$.

Proof. In Example 5.2.3 we showed that $\mathfrak{u}$ is Θ_n-graded. It only remains to show that $\mathfrak{u}$ has coordinate algebra $\mathfrak{b} = \mathfrak{a} \oplus \mathscr{B}$. Recall that

$$A^+ = sym(e_1\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2),\ A^- = skew(e_1\mathfrak{a}e_1 \oplus e_2\mathfrak{a}e_2),\ B = \mathscr{B}e_2, \quad B' = \mathscr{B}e_1.$$
$$E = sym(e_1\mathfrak{a}e_2), \quad C = skew(e_1\mathfrak{a}e_2), \quad E' = sym(e_2\mathfrak{a}e_1), \quad C' = skew(e_2\mathfrak{a}e_1),$$

We adopt the notation of Example 5.2.3. In particular, $\mathfrak{U}$ is the unitary Lie algebra of the form ξ, and $\mathfrak{u}$ is the subalgebra of $\mathfrak{U}$ generated by the weight spaces $\mathfrak{U}_\alpha$ for $\alpha \in \Theta_n$, and $\mathfrak{g} = [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$ where

$$\overline{\mathfrak{g}} = \left\{ \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \mid M \in sym(M_n) \otimes 1^- \oplus skew(M_n) \otimes 1^+ \right\},$$

$1^+ = e_1 + e_2$ and $1^- = e_1 - e_2$. Identify $M \otimes \alpha \in M_n \otimes \mathfrak{a}$ with $\begin{pmatrix} M\otimes\alpha & 0 \\ 0 & 0 \end{pmatrix}$ (resp. $P \in \mathrm{End}_{\mathfrak{a}}(\mathscr{B})$ with $\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ and $v\otimes\beta$ with $\begin{pmatrix} 0 & v\otimes\beta \\ v^t\otimes\beta & 0 \end{pmatrix}$ where $v \in V$ and $\beta \in \mathscr{B}$). As $\mathfrak{g}$-modules, $\mathfrak{g} \otimes A$, $V \otimes B$, $V' \otimes B'$, $S \otimes C$, $S' \otimes C'$, $\Lambda \otimes E$ and $\Lambda' \otimes E'$ are generated by highest weight vectors corresponding to non-zero weights. Hence, these modules are contained in L. Then, with the above identifications, we have

$$\mathfrak{u} = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus \ldots \oplus (\Lambda' \otimes E') \oplus (D_U \cap \mathfrak{u})$$

where $D_U = \begin{bmatrix} I\otimes A^- & 0 \\ 0 & U(\chi) \end{bmatrix}$ is the centralizer of $\mathfrak{g}$ in L. We have a standard Lie bracket on $\mathfrak{u}$:

$$[x\otimes\alpha, y\otimes\beta] = (x\otimes\alpha)(y\otimes\beta) - (y\otimes\beta)(x\otimes\alpha) = xy\otimes\alpha\beta - yx\otimes\beta\alpha.$$

We claim that $\mathfrak{u}$ has coordinate algebra $\mathfrak{b}$. Define $[\alpha_1, \alpha_2] = \alpha_1\alpha_2 - \alpha_2\alpha_1$ and $\alpha_1 \circ \alpha_2 = \alpha_1\alpha_2 + \alpha_2\alpha_1$ for $\alpha_1\alpha_2 \in \mathfrak{b}$. Note that for $x, y \in sl_n$, $u, v \in V$, $u', v' \in V'$, $s \in S$, $\lambda \in \Lambda$, $s' \in S'$, $\lambda' \in \Lambda'$ and for $a^\pm, a_1^\pm, a_2^\pm \in A^\pm$, $b, b_1, b_2 \in B$, $b', b_1', b_2' \in B'$, $c \in C$, $c' \in C'$, $e \in E$, $e' \in E'$ and $d, d_1, d_2 \in D$, we have

$$[x_1^+ \otimes a_1^-, x_2^+ \otimes a_2^-] = \begin{bmatrix} x_1^+ x_2^+ \otimes a_1^- a_2^- - x_2^+ x_1^+ \otimes a_2^- a_1^- & 0 \\ 0 & 0 \end{bmatrix} = x_1^+ \circ x_2^+ \otimes [a_1^-, a_2^-]_{A^-}$$
$$+ [x_1^+, x_2^+] \otimes \frac{(a_1^- \circ a_2^-)_{A^+}}{2} + (x_1^- \mid x_2^-) \begin{bmatrix} I \otimes [a_1^-, a_2^-]_{A^-} & 0 \\ 0 & 0 \end{bmatrix}.$$

Indeed, $a_1^-, a_2^- \in e_1 \mathfrak{a} e_1 \oplus e_2 \mathfrak{a} e_2$, so $[a_1^-, a_2^-], a_1^- \circ a_2^- \in e_1 \mathfrak{a} e_1 \oplus e_2 \mathfrak{a} e_2$. Since $\eta([a_1^-, a_2^-]) = -[a_1^-, a_2^-]$ and $\eta(a_1^- \circ a_2^-) = a_1^- \circ a_2^-$, we have $[a_1^-, a_2^-] \in A^-$ and $a_1^- \circ a_2^- \in A^+$. Then

$$x_1^+ x_2^+ \otimes a_1^- a_2^- - x_2^+ x_1^+ \otimes a_2^- a_1^- = (x_1^+ x_2^+ - x_2^+ x_1^+) \otimes \frac{[a_1^-, a_2^-]_{A^-}}{2}$$
$$+ (x_1^+ x_2^+ + x_2^+ x_1^+ - \frac{2}{n} \operatorname{tr}(x_1^+ x_2^+) I) \otimes \frac{(a_1^- \circ a_2^-)_{A^+}}{2} + (x_1^+ \mid x_2^+)[a_1^-, a_2^-]_{A^-}.$$

Similarly,

$$[x_1^- \otimes a_1^+, x_2^- \otimes a_2^+] = x_1^- \circ x_2^- \otimes \frac{[a_1^+, a_2^+]_{A^-}}{2} + [x_1^-, x_2^-] \otimes \frac{(a_1^+ \circ a_2^+)_{A^+}}{2} + \begin{bmatrix} (x_1^- \mid x_2^-) I \otimes [a_1^+, a_2^+] & 0 \\ 0 & 0 \end{bmatrix},$$
$$[x_1^+ \otimes a_1^-, x_1^- \otimes a_1^+] = x_1^+ \diamond x_1^- \otimes \frac{[a_1^-, a_1^+]_{A^+}}{2} + [x_1^+, x_1^-] \otimes \frac{(a_1^- \circ a_1^+)_{A^-}}{2}.$$

Note that

$$[v \otimes b, v' \otimes b'] = \begin{bmatrix} v(v')^t \otimes bb' - v' v^t \otimes b' b & 0 \\ 0 & (v)^t v' \otimes [b, b']_{A^-} \end{bmatrix} = v' \circ v \otimes \frac{[b, b']_{A^-}}{2}$$
$$+ [v', v] \otimes \frac{(b \circ b')_{A^+}}{2} + \operatorname{tr}(v(v')^t) \begin{bmatrix} \frac{1}{n} I \otimes [b, b']_{A^-} & 0 \\ 0 & 1 \otimes [b, b']_{A^-} \end{bmatrix}.$$

Indeed, $b \in \mathscr{B} e_2$ and $b' \in \mathscr{B} e_1$, so $[b, b'], b \circ b' \in e_1 \mathfrak{a} e_1 \oplus e_2 \mathfrak{a} e_2$. Since $\eta([b, b']) = -[b, b']$ and $\eta(b \circ b') = b \circ b'$, we have $[b, b'] \in A^-$ and $b \circ b' \in A^+$. Then

$$v(v')^t \otimes bb' - v' v^t \otimes b' b = (v(v')^t - v' v^t) \otimes \frac{[b, b']_{A^-}}{2} + (v(v')^t + v' v^t - \frac{2}{n} \operatorname{tr}(v' v^t) I)$$
$$\otimes \frac{(b \circ b')_{A^+}}{2} + \frac{1}{n} \operatorname{tr}(v' v^t) \frac{[b, b']_{A^-}}{2},$$

Similarly, one can show that

$$
\begin{aligned}
[s\otimes c, s'\otimes c'] &= s\circ s'\otimes \frac{[c,c']_{A^-}}{2} + [s,s']\otimes \frac{(c\circ c')_{A^+}}{2} + \begin{bmatrix} (s\mid s')I\otimes [c,c']_{A^-} & 0 \\ 0 & 0 \end{bmatrix},\\
[\lambda\otimes e, \lambda'\otimes e'] &= \lambda\circ \lambda'\otimes \frac{[e,e']_{A^-}}{2} + [\lambda,\lambda']\otimes \frac{(e\circ e')_{A^+}}{2} + \begin{bmatrix} (\lambda\mid \lambda)I\otimes [e,e']_{A^-} & 0 \\ 0 & 0 \end{bmatrix},\\
[u\otimes b_1, v\otimes b_2] &= (uv^t + vu^t)\otimes \frac{[b_1,b_2]_C}{2} + (uv^t - vu^t)\otimes \frac{(b_1\circ b_2)_E}{2},\\
[u'\otimes b_1', v'\otimes b_2'] &= (u'v'^t + v'u'^t)\otimes \frac{[b_1',b_2']_{C'}}{2} + (u'v'^t - v'u'^t)\otimes \frac{(b_1'\circ b_2')_{E'}}{2}.\\
[x^+\otimes a^-, s\otimes c] &= x^+\diamond s\otimes \frac{[a^-,c]_C}{2} + [x^+,s]\otimes \frac{(a^-\circ c)_E}{2},\\
[x^-\otimes a^+, s\otimes c] &= x^-\diamond s\otimes \frac{[a^+,c]_E}{2} + [x^-,s]\otimes \frac{(a^+\circ c)_C}{2},\\
[s'\otimes c', x^+\otimes a^-] &= s'\diamond x^+\otimes \frac{[c',a^-]_{C'}}{2} + [s',x^+]\otimes \frac{(c'\circ a^-)_{E'}}{2},\\
[s'\otimes c', x^-\otimes a^+] &= s'\diamond x^-\otimes \frac{[c',a^+]_{E'}}{2} + [s',x^-]\otimes \frac{(c'\circ a^+)_{C'}}{2},\\
[x^+\otimes a^-, \lambda\otimes e] &= x^+\diamond \lambda\otimes \frac{[a^-,e]_E}{2} + [x^+,\lambda]\otimes \frac{(a^-\circ e)_C}{2},\\
[x^-\otimes a^+, \lambda\otimes e] &= x^-\diamond \lambda\otimes \frac{[a^+,e]_C}{2} + [x^-,\lambda]\otimes \frac{(a^+\circ e)_E}{2},\\
[\lambda'\otimes e', x^+\otimes a^-] &= \lambda'\diamond x^+\otimes \frac{[e',a^-]_{E'}}{2} + [\lambda',x^+]\otimes \frac{(e'\circ a^-)_{C'}}{2},\\
[\lambda'\otimes e', x^-\otimes a^+] &= \lambda'\diamond x^-\otimes \frac{[e',a^+]_{C'}}{2} + [\lambda',x^-]\otimes \frac{(e'\circ a^+)_{E'}}{2},\\
[s\otimes c, \lambda'\otimes e'] &= s\diamond \lambda'\otimes \frac{[c,e']_{A^+}}{2} + [s,\lambda']\otimes \frac{(c\circ e')_{A^-}}{2},\\
[s'\otimes c', \lambda\otimes e] &= s'\diamond \lambda\otimes \frac{[c',e]_{A^+}}{2} + [s',\lambda]\otimes \frac{(c'\circ e)_{A^-}}{2}.
\end{aligned}
$$

Since $(x^+)^t = x^+$, $\eta(a^-) = -a^-$, $(v\otimes b)^t(x^+\otimes a^-)^t = v^t(x^+)^t\otimes ba^- = -(x^+v\otimes a^-b)^t$, then

$$[x^+\otimes a^-, v\otimes b] = \begin{bmatrix} 0 & x^+v\otimes a^-b \\ (x^+v)^t\otimes a^-b & 0 \end{bmatrix} = x^+v\otimes a^-b.$$

Similarly,

$$[v'\otimes b', x^+\otimes a^-] = \begin{bmatrix} 0 & (x^+)^t v'\otimes b'a^- \\ ((x^+)^t v')^t\otimes b'a^- & 0 \end{bmatrix} = (x^+)^t v'\otimes b'a^-,$$

$$[s' \otimes c', v \otimes b] = \begin{bmatrix} 0 & s'v \otimes c'b \\ v^t s' \otimes c'b & 0 \end{bmatrix} = s'v \otimes c'b,$$

$$[\lambda' \otimes e', v \otimes b] = \begin{bmatrix} 0 & \lambda' v \otimes e'b \\ v^t \lambda' \otimes be' & 0 \end{bmatrix} = \lambda' v \otimes e'b,$$

$$[s \otimes c, v' \otimes b'] = \begin{bmatrix} 0 & sv' \otimes cb' \\ (v')^t s \otimes cb' & 0 \end{bmatrix} = sv' \otimes cb',$$

$$[\lambda' \otimes e', v \otimes b] = \begin{bmatrix} 0 & \lambda v' \otimes e'b \\ (v')^t \lambda \otimes b'e & 0 \end{bmatrix} = \lambda v' \otimes eb'.$$

So the product on $\mathfrak{b}$ determined by (3.4.4) (see Tables 4.1.1 and 4.2.1) is exactly the given product on $\mathfrak{b}$, as required. □

Define $\mathrm{Der}_*(\mathfrak{b}) := \{d \in \mathrm{Der}(\mathfrak{b}) \mid dX \subseteq X \text{ for } X = A^+, A^-, B, \cdots, E' \}$. Using Proposition 4.2.8, we get $D \subseteq \mathrm{Der}_*(\mathfrak{b})$. Let $\alpha, \beta \in \mathfrak{b}$. Define $D_{\alpha,\beta} := \langle \alpha, \beta \rangle$. Set

$$D_{\mathfrak{b},\mathfrak{b}} = span\{D_{\alpha,\beta} \mid \alpha, \beta \in \mathfrak{b}\}.$$

Theorem 5.2.5. *Let $n \geq 5$ and let $\mathfrak{a}$ and $\mathscr{B}$ be as in Example 5.2.3. Define the algebra*

$$\mathscr{L}(\mathfrak{b}) := (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \cdots \oplus (\Lambda' \otimes E') \oplus D_{\mathfrak{b},\mathfrak{b}}$$

with multiplication as in (3.4.4) with D replaced by $D_{\mathfrak{b},\mathfrak{b}}$ and $\langle \alpha, \beta \rangle$ replaced by $D_{\alpha,\beta}$. Then the following hold.

(1) *$\mathscr{L}(\mathfrak{b}) \cong \mathfrak{u}/Z(\mathfrak{u})$ is a Lie algebra where $Z(\mathfrak{u})$ is the center of $\mathfrak{u}$.*

(2) *$\mathscr{L}(\mathfrak{b})$ is Θ_n-graded with coordinate algebra $\mathfrak{b}$.*

(3) *Every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is a cover of $\mathscr{L}(\mathfrak{b})$.*

Proof. (1) Define $f : \mathfrak{u} \to \mathscr{L}(\mathfrak{b})$ by

$$f(x) = x, \quad \forall x \in (\mathfrak{g} \otimes A) \oplus \cdots \oplus (\Lambda' \otimes E'),$$
$$f(\langle \alpha, \beta \rangle) = D_{\alpha,\beta}, \forall \alpha, \beta \in \mathfrak{b}.$$

It is clear that f is a surjective map. Now we are going to show that f is a Lie algebra homomorphism. We need to check that $f([x,y]) = [f(x), f(y)]$ for all homogeneous $x, y \in \mathfrak{u}$. This is obvious if $x \notin D$ or $y \notin D$. If both $x, y \in D$, we have

$$\begin{aligned} f([\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle]) &= f(\langle D_{\alpha_1,\alpha_2}\beta_1, \beta_2 \rangle + \langle \beta_1, D_{\alpha_1,\alpha_2}\beta_2 \rangle) \\ &= D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2} + D_{\beta_1, D_{\alpha_1,\alpha_2}\beta_2} \end{aligned}$$

$$= [D_{\alpha_1,\alpha_2}, D_{\beta_1,\beta_2}]$$
$$= [f(\langle \alpha_1, \alpha_2 \rangle), f(\langle \beta_1, \beta_2 \rangle)],$$

as required. The center $Z(\mathfrak{u})$ of $\mathfrak{u}$ is equal to $\operatorname{Ker} f$. Thus, $\mathscr{L}(\mathfrak{b}) \cong \mathfrak{u}/Z(\mathfrak{u})$ and so $\mathscr{L}(\mathfrak{b})$ is a Lie algebra.

(2) By construction, it is clear that $\mathscr{L}(\mathfrak{b})$ is Θ_n-graded with coordinate algebra $\mathfrak{b}$.

(3) As in the proof of (1), we can show that every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is isomorphic to $\mathscr{L}(\mathfrak{b})$ modulo its center. Thus, it is a cover of $\mathscr{L}(\mathfrak{b})$. $\square$

Next theorem completes the classification of Θ_n-graded Lie algebras up to central extensions in the case when $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold.

Theorem 5.2.6 (Classification of Θ_n-graded Lie algebras, $n \geq 5$)**.** *A Lie algebra L is $(\Theta_n, \mathfrak{g})$-graded if and only if there exist an associative algebra $\mathfrak{a}$ with involution η, identity element 1^+ and two orthogonal idempotents e_1 and e_2 such that $1^+ = e_1 + e_2$ and $e_2 = \eta(e_1)$, a unital associative right $\mathfrak{a}$-module $\mathscr{B}$ with a hermitian form χ with values in $\mathfrak{a}$ such that L is centrally isogenous to the $(\Theta_n, \mathfrak{g})$-graded unitary Lie algebra $\mathfrak{u}$ of the hermitian form $\xi = w\perp - \chi$ on the right $\mathfrak{a}$-module $\mathfrak{a}^n \oplus \mathscr{B}$ (see Example 5.2.3).*

Proof. The "if" part follows from Proposition 5.2.4. To prove the "only if" suppose that L is a Θ_n-graded Lie algebra with grading subalgebra $\mathfrak{g}$. By Theorem 4.2.9 and Proposition 4.3.2, L has coordinate algebra $\mathfrak{b} = \mathfrak{a} + \mathscr{B}$ with $\mathfrak{a}$ being associative containing two orthogonal idempotents e_1 and e_2 with the above properties. By Proposition 5.2.4, the $(\Theta_n, \mathfrak{g})$-graded unitary Lie algebra $\mathfrak{u}$ has the same coordinate algebra. By Theorem 5.2.5, $L/Z(L') \cong \mathscr{L}(\mathfrak{b}) \cong \mathfrak{u}/Z(\mathfrak{u})$. It follows that L and $\mathfrak{u}$ are centrally isogenous. $\square$

5.3 The universal central extensions of Θ_n-graded Lie algebras, $n \geq 5$

In this section we use the same method as in [3, 4] to compute the universal central extension $\widehat{\mathscr{L}(\mathfrak{b})}$ of $\mathscr{L}(\mathfrak{b})$. We show that for every Θ_n-graded Lie algebra L there is a subspace X of the center of $\widehat{\mathscr{L}(\mathfrak{b})}$ such that L is isomorphic to $\mathscr{L}(\mathfrak{b}, X) = \widehat{\mathscr{L}(\mathfrak{b})}/X$. This finishes the classification of Θ_n-graded Lie algebras up to isomorphism.

Recall that $\operatorname{Der}_*(\mathfrak{b}) := \{d \in \operatorname{Der}(\mathfrak{b}) \mid dX \subseteq X \text{ for } X = A^+, A^-, B, \cdots, E'\}$ and

$$D_{\mathfrak{b},\mathfrak{b}} = span\{D_{\alpha,\beta} \mid \alpha, \beta \in \mathfrak{b}\}$$

where $D_{\alpha,\beta} := \langle \alpha, \beta \rangle$ for $\alpha, \beta \in \mathfrak{b}$ ($\langle , \rangle$ is a surjective map from $\mathfrak{b} \otimes \mathfrak{b}$ to D, see (4.2.5)). The centerless $(\Theta_n, \mathfrak{g})$-graded Lie algebra $\mathscr{L}(\mathfrak{b})$ in Theorem 5.2.5 has Lie bracket defined as follows. For all $x^{\pm}, x_1^{\pm}, x_2^{\pm} \in \mathfrak{g}^{\pm}$, $u, v \in V$, $u', v' \in V'$, $s \in S$, $\lambda \in \Lambda$, $s' \in S'$, $\lambda' \in \Lambda'$ and for all $a^{\pm}, a_1^{\pm}, a_2^{\pm} \in A$, $b, b_1, b_2 \in B$, $b', b_1', b_2' \in B'$, $c \in C$, $c' \in C'$, $e \in E$, $e' \in E'$, $d, \alpha_1, \alpha_2 \in D_{\mathfrak{b},\mathfrak{b}}$,

$$
\begin{aligned}
[u \otimes b, v' \otimes b'] &= (uv'^t + v'u^t - \frac{2\,\mathrm{tr}(uv'^t)}{n} I) \otimes \frac{[b, b']_{A^-}}{2} + \\
&\qquad (uv'^t - v'u^t) \otimes \frac{(b \circ b')_{A^+}}{2} + \frac{2\,\mathrm{tr}(uv'^t)}{n} D_{b,b'} = -[v' \otimes b', u \otimes b], \\
[x_1^+ \otimes a_1^-, x_2^+ \otimes a_2^-] &= x_1^+ \circ x_2^+ \otimes \frac{[a_1^-, a_2^-]_{A^-}}{2} + [x_1^+, x_2^+] \otimes \frac{(a_1^- \circ a_2^-)_{A^+}}{2} + (x_1^+ \mid x_2^+) D_{a_1^-, a_2^-}, \\
[x_1^- \otimes a_1^+, x_2^- \otimes a_2^+] &= x_1^- \circ x_2^- \otimes \frac{[a_1^+, a_2^+]_{A^-}}{2} + [x_1^-, x_2^-] \otimes \frac{(a_1^+ \circ a_2^+)_{A^+}}{2} + (x_1^- \mid x_2^-) D_{a_1^-, a_2^-}, \\
[x_1^+ \otimes a_1^-, x_1^- \otimes a_1^+] &= x_1^+ \diamond x_1^- \otimes \frac{[a_1^-, a_1^+]_{A^+}}{2} + [x_1^+, x_1^-] \otimes \frac{(a_1^- \circ a_1^+)_{A^-}}{2}, \\
[s \otimes c, s' \otimes c'] &= s \circ s' \otimes \frac{[c, c']_{A^-}}{2} + [s, s'] \otimes \frac{(c \circ c')_{A^+}}{2} + (s \mid s') D_{c,c'} = -[s' \otimes c', s \otimes c], \\
[\lambda \otimes e, \lambda' \otimes e'] &= \lambda \circ \lambda' \otimes \frac{[e, e']_{A^-}}{2} + [\lambda, \lambda'] \otimes \frac{(e \circ e')_{A^+}}{2} + (\lambda \mid \lambda') D_{e,e'} = -[\lambda' \otimes e', \lambda \otimes e] \\
[u \otimes b_1, v \otimes b_2] &= (uv^t + vu^t) \otimes \frac{[b_1, b_2]_C}{2} + (uv^t - vu^t) \otimes \frac{(b_1 \circ b_2)_E}{2}, \\
[u' \otimes b_1', v' \otimes b_2'] &= (u'v'^t + v'u'^t) \otimes \frac{[b_1', b_2']_{C'}}{2} + (u'v'^t - v'u'^t) \otimes \frac{(b_1' \circ b_2')_{E'}}{2}, \\
[x^+ \otimes a^-, s \otimes c] &= x^+ \diamond s \otimes \frac{[a^-, c]_C}{2} + [x^+, s] \otimes \frac{(a^- \circ c)_E}{2} = -[s \otimes c, x^+ \otimes a^-], \\
[x^- \otimes a^+, s \otimes c] &= x^- \diamond s \otimes \frac{[a^+, c]_E}{2} + [x^-, s] \otimes \frac{(a^+ \circ c)_C}{2} = [s \otimes c, x^- \otimes a^+], \\
[x^+ \otimes a^-, \lambda \otimes e] &= x^+ \diamond \lambda \otimes \frac{[a^-, e]_E}{2} + [x^+, \lambda] \otimes \frac{(a^- \circ e)_C}{2} = -[\lambda \otimes e, x^+ \otimes a^-], \\
[x^- \otimes a^+, \lambda \otimes e] &= x^- \diamond \lambda \otimes \frac{[a^+, e]_C}{2} + [x^-, \lambda] \otimes \frac{(a^+ \circ e)_E}{2} = -[\lambda \otimes e, x^- \otimes a^+], \\
[s' \otimes c', x^+ \otimes a^-] &= s' \diamond x^+ \otimes \frac{[c', a^-]_{C'}}{2} + [s', x^+] \otimes \frac{(c' \circ a^-)_{E'}}{2} = -[x^+ \otimes a^-, s' \otimes c'], \\
[s' \otimes c', x^- \otimes a^+] &= s' \diamond x^- \otimes \frac{[c', a^+]_{E'}}{2} + [s', x^-] \otimes \frac{(c' \circ a^+)_{C'}}{2} = -[x^- \otimes a^+, s' \otimes c'], \\
[\lambda' \otimes e', x^+ \otimes a^-] &= \lambda' \diamond x^+ \otimes \frac{[e', a^-]_{E'}}{2} + [\lambda', x^+] \otimes \frac{(e' \circ a^-)_{C'}}{2} = -[x^+ \otimes a^-, \lambda' \otimes e'], \\
[\lambda' \otimes e', x^- \otimes a^+] &= \lambda' \diamond x^- \otimes \frac{[e', a^+]_{C'}}{2} + [\lambda', x^-] \otimes \frac{(e' \circ a^+)_{E'}}{2} = -[x^- \otimes a^+, \lambda' \otimes e'],
\end{aligned} \tag{5.3.1}
$$

$$[s\otimes c,\lambda'\otimes e'] = s\diamond\lambda'\otimes\frac{[c,e']_{A^+}}{2} + [s,\lambda']\otimes\frac{(c\circ e')_{A^-}}{2} = -[\lambda'\otimes e',s\otimes c],$$
$$[s'\otimes c',\lambda\otimes e] = s'\diamond\lambda\otimes\frac{[c',e]_{A^+}}{2} + [s',\lambda]\otimes\frac{(c'\circ e)_{A^-}}{2} = -[\lambda\otimes e,s'\otimes c'],$$
$$[x\otimes a,u\otimes b] = xu\otimes ab = -[u\otimes b,x\otimes a],$$
$$[s'\otimes c',u\otimes b] = s'u\otimes c'b = -[u\otimes b,s'\otimes c'],$$
$$[\lambda'\otimes e',u\otimes b] = \lambda'u\otimes e'b = -[u\otimes b,\lambda'\otimes e'],$$
$$[u'\otimes b',x\otimes a] = x^t u'\otimes b'a = -[x\otimes a,u'\otimes b'],$$
$$[u'\otimes b',s\otimes c] = su'\otimes b'c = -[s\otimes c,u'\otimes b'],$$
$$[u'\otimes b',\lambda\otimes e] = -\lambda u'\otimes b'e = -[\lambda\otimes e,u'\otimes b'],$$
$$[d,x\otimes a] = x\otimes da = -[x\otimes a,d],$$
$$[d,u\otimes b] = u\otimes db = -[u\otimes b,d],$$
$$[d,s\otimes c] = s\otimes dc = -[s\otimes c,d],$$
$$[d,\lambda\otimes e] = \lambda\otimes de = -[\lambda\otimes e,d],$$
$$[d,s'\otimes c'] = s'\otimes dc' = -[s'\otimes c',d],$$
$$[d,u'\otimes b'] = u'\otimes db' = -[u'\otimes b',d],$$
$$[d,\lambda'\otimes e'] = \lambda'\otimes de' = -[\lambda'\otimes e',d],$$
$$[d,D_{\alpha_1,\alpha_2}] = D_{d\alpha_1,\alpha_2} + D_{\alpha_1,d\alpha_2}.$$

Proposition 5.3.1. $[D_1,D_2] = D_1D_2 - D_2D_1$ *for all* $D_1,D_2 \in D_{\mathfrak{b},\mathfrak{b}}$.

Proof. Let D_{α_1,β_1}, $D_{\alpha_2,\beta_2} \in D_{\mathfrak{b},\mathfrak{b}}$. We need to show that

$$[D_{\alpha_1,\beta_1},D_{\alpha_2,\beta_2}](\delta) = (D_{\alpha_1,\beta_1}D_{\alpha_2,\beta_2} - D_{\alpha_2,\beta_2}D_{\alpha_1,\beta_1})(\delta),$$

for all $\delta\in\mathfrak{b}$. To prove this, we need to make various choices of α_1, β_1, α_2, β_2 and δ, use Propositions 4.2.8, 4.2.7, 4.2.6 and associativity of $\mathfrak{a}$. As illustration, we demonstrate the case when $\alpha_1,\beta_1,\alpha_2,\beta_2,\delta\in\mathfrak{a}$. We have

$$\begin{aligned}
[D_{\alpha_1,\beta_1},D_{\alpha_2,\beta_2}](\delta) &= D_{D_{\alpha_1,\beta_1}\alpha_2,\beta_2}(\delta) + D_{\alpha_2,D_{\alpha_1,\beta_1}\beta_2}(\delta)\\
&= [[[\alpha_1,\beta_1]_{A^-},\alpha_2],\beta_2]_{A^-},\delta] + [[\alpha_2,[[\alpha_1,\beta_1]_{A^-},\beta_2]]_{A^-},\delta]\\
&= [[[\alpha_1,\beta_1]_{A^-},\alpha_2],\beta_2]_{A^-} + [\alpha_2,[[\alpha_1,\beta_1]_{A^-},\beta_2]]_{A^-},\delta]\\
&= [[[\alpha_1,\beta_1]_{A^-},[\alpha_2,\beta_2]_{A^-}],\delta]\\
&= [[\alpha_1,\beta_1]_{A^-},[[\alpha_2,\beta_2]_{A^-},\delta]] + [[[\alpha_1,\beta_1]_{A^-},\delta],[\alpha_2,\beta_2]_{A^-}]\\
&= [[\alpha_1,\beta_1]_{A^-},[[\alpha_2,\beta_2]_{A^-},\delta]] - [[\alpha_2,\beta_2]_{A^-},[[\alpha_1,\beta_1]_{A^-},\delta]]
\end{aligned}$$

$$= D_{\alpha_1,\beta_1} D_{\alpha_2,\beta_2} - D_{\alpha_2,\beta_2} D_{\alpha_1,\beta_1}(\delta),$$

as required. □

Lemma 5.3.2. $D_{\mathfrak{b},\mathfrak{b}}$ is an ideal in $\mathrm{Der}_*(\mathfrak{b})$.

Proof. This is similar to [4, Lemma 3.6]. In Proposition 4.2.8, we showed that $D_{\mathfrak{b},\mathfrak{b}} X \subseteq X$ for $X = A^+, A^-, B, \cdots, E'$, so it is enough to prove that (1) $D_{\mathfrak{b},\mathfrak{b}} \subseteq \mathrm{Der}_*(\mathfrak{b})$ and (2) $[\psi, D_{\alpha_1,\alpha_2}] = D_{\vartheta\alpha_1,\alpha_2} + D_{\alpha_1,\psi\alpha_2}$, for all $\alpha_1, \alpha_2 \in \mathfrak{b}$ and $\psi \in \mathrm{Der}_*(\mathfrak{b})$. To prove this we make various choices of $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{a} \cup \mathscr{B}$ and calculate the corresponding derivation actions by using Proposition 4.2.7. As an illustration, we consider the case when $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{a}$.

(1) Let $\vartheta = D_{\beta_1,\beta_2} \in D_{\mathfrak{b},\mathfrak{b}}$ where $\beta_1, \beta_2 \in \mathfrak{a}$. Using Proposition 4.2.7 and the associativity of $\mathfrak{a}$, we get

$$\begin{aligned}
\vartheta(\alpha_1)\alpha_2 + \alpha_1\vartheta(\alpha_2) &= [[\beta_1,\beta_2]_{A^-}, \alpha_1]\alpha_2 + \alpha_1[[\beta_1,\beta_2]_{A^-}, \alpha_2] \\
&= ([\beta_1,\beta_2]_{A^-}\alpha_1)\alpha_2 - \alpha_1(\alpha_2[\beta_1,\beta_2]_{A^-}) \\
&= [\beta_1,\beta_2]_{A^-}(\alpha_1\alpha_2) - (\alpha_1\alpha_2)[\beta_1,\beta_2]_{A^-} \\
&= [[\beta_1,\beta_2]_{A^-}, \alpha_1\alpha_2] \\
&= \vartheta(\alpha_1\alpha_2),
\end{aligned}$$

for all $\alpha_1, \alpha_2 \in \mathfrak{a}$, as required.

(2) Let $\psi \in \mathrm{Der}_*(\mathfrak{b})$ and $\alpha_1, \alpha_2 \in \mathfrak{a}$. Let $\delta \in \mathfrak{b}$. We have two cases.

Case 1: $\delta \in \mathfrak{a}$. Using Proposition 4.2.7, the associativity of $\mathfrak{a}$ and $D_{\mathfrak{b},\mathfrak{b}} X \subseteq X$ for $X = A^+, A^-, B, \cdots, E'$ we get

$$\begin{aligned}
[\psi, D_{\alpha_1,\alpha_2}](\delta) &= \psi D_{\alpha_1,\alpha_2}(\delta) - D_{\alpha_1,\alpha_2}\psi(\delta) \\
&= \psi([[\alpha_1,\alpha_2]_{A^-}, \delta]) - [[\alpha_1,\alpha_2]_{A^-}, \psi(\delta)] \\
&= \psi([\alpha_1,\alpha_2]_{A^-}\delta) - \psi(\delta[\alpha_1,\alpha_2]_{A^-}) \\
&- [\alpha_1,\alpha_2]_{A^-}.\psi(\delta) + \psi(\delta)[\alpha_1,\alpha_2]_{A^-} \\
&= \psi([\alpha_1,\alpha_2]_{A^-})\delta + [\alpha_1,\alpha_2]_{A^-}\psi(\delta) - \psi(\delta)[\alpha_1,\alpha_2]_{A^-} \\
&- \delta\psi([\alpha_1,\alpha_2]_{A^-}) - [\alpha_1,\alpha_2]_{A^-}.\psi(\delta) + \psi(\delta)[\alpha_1,\alpha_2]_{A^-} \\
&= \psi([\alpha_1,\alpha_2]_{A^-})\delta - \delta\psi([\alpha_1,\alpha_2]_{A^-}) \\
&= [\psi([\alpha_1,\alpha_2]_{A^-}), \delta] \\
&= [(\psi\alpha_1)\alpha_2 + \alpha_1(\psi\alpha_2) - (\psi\alpha_2)\alpha_1 - \alpha_2(\psi\alpha_1), \delta] \\
&= [(\psi\alpha_1)\alpha_2 - \alpha_2(\psi\alpha), \delta] + [\alpha_1(\psi\alpha_2) - (\psi\alpha_2)\alpha_1, \delta]
\end{aligned}$$

$$
\begin{aligned}
&= [[\psi\alpha_1, \alpha_2]_{A^-}, \delta] + [[\alpha_1, \psi\alpha_2]_{A^-}, \delta] \\
&= D_{\psi\alpha_1,\alpha_2} + D_{\alpha_1,\psi\alpha_2}(\delta).
\end{aligned}
$$

Case 2: $\delta \in B \oplus B'$. Using Proposition 4.2.7, the associativity of $\mathfrak{a}$ and $D_{\mathfrak{b},\mathfrak{b}}X \subseteq X$ for $X = A^+, A^-, B, \cdots, E'$ we get

$$
\begin{aligned}
[\psi, D_{\alpha_1,\alpha_2}](\delta) &= \psi D_{\alpha_1,\alpha_2}(\delta) - D_{\alpha_1,\alpha_2}\psi(\delta) \\
&= \psi([\alpha_1,\alpha_2]_{A^-}\delta) - [\alpha_1,\alpha_2]_{A^-}\psi(\delta) \\
&= \psi([\alpha_1,\alpha_2]_{A^-})\delta + [\alpha_1,\alpha_2]_{A^-}\psi(\delta) - [\alpha_1,\alpha_2]_{A^-}.\psi(\delta) \\
&= \psi([\alpha_1,\alpha_2]_{A^-})\delta \\
&= \psi(\alpha_1\alpha_2)\delta - \psi(\alpha_2\alpha_1)\delta \\
&= ((\psi\alpha_1)\alpha_2 - \alpha_2(\psi\alpha_1) + \alpha_1(\psi\alpha_2) - (\psi\alpha_2)\alpha_1)\delta \\
&= [\psi\alpha_1, \alpha_2]_{A^-}\delta + [\alpha_1, \psi\alpha_2]_{A^-}\delta \\
&= D_{\psi\alpha_1,\alpha_2} + D_{\alpha_1,\psi\alpha_2}(\delta).
\end{aligned}
$$

Then (1) and (2) hold, as required. □

Lemmas 5.1.7 and 5.3.2 and Propositions 4.2.7 and 4.2.8 imply the following.

Proposition 5.3.3. *(1) The space $D_{\mathfrak{b},\mathfrak{b}}$ of inner derivations is an ideal of* $\mathrm{Der}_*(\mathfrak{b})$ *and $D_{\mathfrak{b},\mathfrak{b}}(X) \subseteq X$ for $X = A^+$, A^-, B, $\cdots$, E'.*

(2) The inner derivations satisfy

$$
\begin{aligned}
&D_{\alpha,\beta} + D_{\beta,\alpha} = 0, \\
&D_{\alpha\beta,\gamma} + D_{\beta\gamma,\alpha} + D_{\gamma\alpha,\beta} = 0,
\end{aligned}
$$

for all $\alpha, \beta \in \mathfrak{b}$. Moreover, $D_{x,y} = 0$ if $x \in X$ and $y \notin X'$ with $X = B, C, E$ or $x \in A^+$ and $y \in A^-$.

Let I be the subspace of $\mathfrak{b} \otimes \mathfrak{b}$ spanned by the elements

$$
\begin{aligned}
&\alpha \otimes \beta + \beta \otimes \alpha, \\
&\gamma\alpha \otimes \beta + \beta\gamma \otimes \alpha + \alpha\beta \otimes \gamma, \\
&x \otimes y
\end{aligned}
\tag{5.3.2}
$$

where $\alpha, \beta \in \mathfrak{b}$ and $x \in X$ and $y \notin X'$ with $X = B, C, E$ or $x \in A^+$ and $y \in A^-$. In Propositions 4.2.7 and 4.2.8 we showed that $\mathfrak{b}$ is a $D_{\mathfrak{b},\mathfrak{b}}$-module, so $\mathfrak{b} \otimes \mathfrak{b}$ is a $D_{\mathfrak{b},\mathfrak{b}}$-module.

Thus, the space I is invariant under $D_{\mathfrak{b},\mathfrak{b}}$, and so $\{\mathfrak{b},\mathfrak{b}\}$ is a $D_{\mathfrak{b},\mathfrak{b}}$-module under the induced action:

$$D_{\alpha_1,\alpha_2}.\{\beta_1,\beta_2\} := \{D_{\alpha_1,\alpha_2}\beta_1,\beta_2\} + \{\beta_1, D_{\alpha,\alpha_2}\beta_2\}.$$

Consider the quotient space $\{\mathfrak{b},\mathfrak{b}\} = \mathfrak{b}\otimes\mathfrak{b}/I$ and set $\{\alpha,\beta\} = \alpha\otimes\beta + I$ in $\{\mathfrak{b},\mathfrak{b}\}$. Then the relations in (5.3.2) translate to say

$$\begin{aligned}
&\{\alpha,\beta\} = -\{\beta,\alpha\},\\
&\{\gamma\alpha,\beta\} + \{\beta\gamma,\alpha\} + \{\alpha\beta,\gamma\} = 0,\\
&\{x,y\} = 0.
\end{aligned}$$

The mapping $\mathfrak{b}\otimes\mathfrak{b} \to D_{\mathfrak{b},\mathfrak{b}}$, $\alpha\otimes\beta \mapsto D_{\alpha,\beta}$ has I in its kernel. We define the induced mapping $p : \{\mathfrak{b},\mathfrak{b}\} \to D_{\mathfrak{b},\mathfrak{b}}$ by $\rho(\{\alpha,\beta\}) = D_{\alpha,\beta}$. We have the following.

Proposition 5.3.4. *(1) The space $\{\mathfrak{b},\mathfrak{b}\}$ is a Lie algebra with the multiplication*

$$[\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\}] = \{D_{\alpha_1,\alpha_2}\beta_1,\beta_2\} + \{\beta_1, D_{\alpha_1,\alpha_2}\beta_2\},$$

for all $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathfrak{b}$.

(2) The mapping $\rho : \{\mathfrak{b},\mathfrak{b}\} \to D_{\mathfrak{b},\mathfrak{b}}$ given by $\rho(\{\alpha,\beta\}) = D_{\alpha,\beta}$ is a surjective Lie algebra homomorphism.

Proof. This is similar to [3, 4.8-4.10] and [4, 5.24].

(1) This can be checked by making various choices of $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \in \mathfrak{a}\cup\mathscr{B}$ and calculating the corresponding derivations by using Proposition 4.2.7. As illustration, consider the case when $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \in \mathfrak{a}$. Note that

$$\begin{aligned}
[\{\alpha_1,\alpha_2\}, r_1\{\beta_1,\beta_2\} + r_2\{\gamma_1,\gamma_2\}] &= D_{\alpha_1,\alpha_2}.(r_1\{\beta_1,\beta_2\} + r_2\{\gamma_1,\gamma_2\})\\
&= D_{\alpha_1,\alpha_2}.r_1\{\beta_1,\beta_2\} + D_{\alpha_1,\alpha_2}.r_2\{\gamma_1,\gamma_2\}\\
&= [\{\alpha_1,\alpha_2\}, r_1\{\beta_1,\beta_2\}] + [\{\alpha_1,\alpha_2\}, r_2\{\gamma_1,\gamma_2\}].
\end{aligned}$$

This means that, the bracket is bilinear. Now we are going to show that $\{\mathfrak{b},\mathfrak{b}\}$ satisfies the Jacoby identity.

$$\begin{aligned}
&[\{\alpha_1,\alpha_2\},[\{\beta_1,\beta_2\},\{\gamma_1,\gamma_2\}]] - [\{\beta_1,\beta_2\},[\{\alpha_1,\alpha_2\},\{\gamma_1,\gamma_2\}]]\\
&= [\{\alpha_1,\alpha_2\},\{D_{\beta_1,\beta_2}\gamma_1,\gamma_2\} + \{\gamma_1, D_{\beta_1,\beta_2}\gamma_2\}] -\\
&[\{\beta_1,\beta_2\},\{D_{\alpha_1,\alpha_2}\gamma_1,\gamma_2\} + \{\gamma_1, D_{\alpha_1,\alpha_2}\gamma_2\}]
\end{aligned}$$

$$= [\{\alpha_1, \alpha_2\}, \{\gamma_1, D_{\beta_1,\beta_2}\gamma_2\}] + [\{\alpha_1, \alpha_2\}, \{D_{\beta_1,\beta_2}\gamma_1, \gamma_2\} -$$
$$[\{\beta_1, \beta_2\}, \{D_{\alpha_1,\alpha_2}\gamma_1, \gamma_2\}] - [\{\beta_1, \beta_2\}, \{\gamma_1, D_{\alpha_1,\alpha_2}\gamma_2\}]$$
$$= \{D_{\alpha_1,\alpha_2}\gamma_1, D_{\beta_1,\beta_2}\gamma_2\} + \{\gamma_1, D_{\alpha_1,\alpha_2}D_{\beta_1,\beta_2}\gamma_2\} +$$
$$\{D_{\alpha_1,\alpha_2}D_{\beta_1,\beta_2}\gamma_1, \gamma_2\} + \{D_{\beta_1,\beta_2}\gamma_1, D_{\alpha_1,\alpha_2}\gamma_2\} - \{D_{\beta_1,\beta_2}D_{\alpha_1,\alpha_2}\gamma_1, \gamma_2\} -$$
$$\{D_{\alpha_1,\alpha_2}\gamma_1, D_{\beta_1,\beta_2}\gamma_2\} - \{D_{\beta_1,\beta_2}\gamma_1, D_{\alpha_1,\alpha_2}\gamma_2\} - \{\gamma_1, D_{\beta_1,\beta_2}D_{\alpha_1,\alpha_2}\gamma_2\}$$
$$= \{D_{\alpha_1,\alpha_2}D_{\beta_1,\beta_2}\gamma_1, \gamma_2\} - \{D_{\beta_1,\beta_2}D_{\alpha_1,\alpha_2}\gamma_1, \gamma_2\} +$$
$$\{\gamma_1, D_{\alpha_1,\alpha_2}D_{\beta_1,\beta_2}\gamma_2\} - \{\gamma_1, D_{\beta_1,\beta_2}D_{\alpha_1,\alpha_2}\gamma_2\}$$
$$= \{[D_{\alpha_1,\alpha_2}, D_{\beta_1,\beta_2}]\gamma_1, \gamma_2\} + \{\gamma_1, [D_{\alpha_1,\alpha_2}, D_{\beta_1,\beta_2}]\gamma_2\}$$
$$= \left\{D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2} + D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_1, \gamma_2\right\} + \left\{\gamma_1, D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2} + D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_2\right\}$$
$$= \left\{D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma + D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_1, \gamma_2\right\} + \left\{\gamma_1, D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_2 + D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_2\right\}$$
$$= \left\{D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_1, \gamma_2\right\} + \left\{D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_1, \gamma_2\right\} + \left\{\gamma_1, D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_2\right\} + \left\{\gamma_1, D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_2\right\}$$
$$= \left\{D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_1, \gamma_2\right\} + \left\{\gamma_1, D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}\gamma_2\right\} + \left\{D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_1, \gamma_2\right\} + \left\{\gamma_1, D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2}\gamma_2\right\}$$
$$= [\{D_{\alpha_1,\alpha_2}\beta_1, \beta_2\}, \{\gamma_1, \gamma_2\}] + [\{\beta_1, D_{\alpha_1,\alpha_2}\beta_2\}, \{\gamma_1, \gamma_2\}]$$
$$= [\{D_{\alpha_1,\alpha_2}\beta_1, \beta_2\} + \{\beta_1, D_{\alpha_1,\alpha_2}\beta_2\}, \{\gamma_1, \gamma_2\}]$$
$$[[\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}], \{\gamma_1, \gamma_2\}].$$

It follows that, $\{\mathfrak{b}, \mathfrak{b}\}$ satisfies the Jacoby identity. It remains to prove that the multiplication $[\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}]$ is anti-commutative. We have

$$[\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}] + [\{\beta_1, \beta_2\}, \{\alpha_1, \alpha_2\}]$$
$$= \{D_{\alpha_1,\alpha_2}\beta_1, \beta_2\} + \{\beta_1, D_{\alpha_1,\alpha_2}\beta_2\} + \{D_{\beta_1,\beta_2}\alpha_1, \alpha_2\} + \{\alpha_1, D_{\beta_1,\beta_2}\alpha_2\}$$
$$= D_{\alpha_1,\alpha_2}\beta_1 \otimes \beta_2 + \beta_1 \otimes D_{\alpha_1,\alpha_2}\beta_2 + D_{\beta_1,\beta_2}\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes D_{\beta_1,\beta_2}\alpha_2 + I$$
$$= [[\alpha_1, \alpha_2], \beta_1] \otimes \beta_2 + \beta_1 \otimes [[\alpha_1, \alpha_2], \beta_2] + [[\beta_1, \beta_2], \alpha_1] \otimes \alpha_2 + \alpha_1 \otimes [[\beta_1, \beta_2], \alpha_2] + I$$
$$= [\alpha_1\alpha_2 - \alpha_2\alpha_1, \beta_1] \otimes \beta_2 + \beta_1 \otimes [\alpha_1\alpha_2 - \alpha_2\alpha_1, \beta_2] +$$
$$[\beta_1\beta_2 - \beta_2\beta_1, \alpha_1] \otimes \alpha_2 + \alpha_1 \otimes [\beta_1\beta_2 - \beta_2\beta_1, \alpha_2] + I$$
$$= [\alpha_1, \alpha_2]\beta_1 \otimes \beta_2 - \beta_1[\alpha_1, \alpha_2] \otimes \beta_2 + \beta_1 \otimes [\alpha_1, \alpha_2]\beta_2 - \beta_1 \otimes \beta_2[\alpha_1, \alpha_2] +$$
$$[\beta_1, \beta_2]\alpha_1 \otimes \alpha_2 - \alpha_1[\beta_1, \beta_2] \otimes \alpha_2 + \alpha_1 \otimes [\beta_1, \beta_2]\alpha_2 - \alpha_1 \otimes \alpha_2[\beta_1, \beta_2] + I$$
$$= [\alpha_1, \alpha_2] \otimes [\beta_1, \beta_2] + [\beta_1, \beta_2] \otimes [\alpha_1, \alpha_2] + I = I.$$

Thus, the space $\{\mathfrak{b}, \mathfrak{b}\}$ becomes a Lie algebra under this product.

(2) It is clear that ρ is a surjective map. It remains to show that f is a Lie algebra

homomorphism. Using Lemma 5.3.2, we get

$$\begin{aligned}\rho([\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\}]) &= \rho(\{D_{\alpha_1,\alpha_2}\beta_1,\beta_2\}+\{\beta_1,D_{\alpha_1,\alpha_2}\beta_2\}) \\ &= D_{D_{\alpha_1,\alpha_2}\beta_1,\beta_2}+D_{\beta_1,D_{\alpha_1,\alpha_2}\beta_2} \\ &= [D_{\alpha_1,\alpha_2},D_{\beta_1,\beta_2}] \\ &= [\rho(\{\alpha_1,\alpha_2\}),\rho(\{\beta_1,\beta_2\})].\end{aligned}$$

Thus, ρ is a Lie algebra homomorphism, as required. □

Propositions 4.2.8 and 5.3.4 imply the following.

Proposition 5.3.5. $\mathfrak{b}$ *is a module for the Lie algebra* $\{\mathfrak{b},\mathfrak{b}\}$ *with action defined by* $\{\alpha,\beta\}.\gamma = \rho(\{\alpha,\beta\})\gamma = D_{\alpha,\beta}\gamma$ *for* $\{\alpha,\beta\} \in \{\mathfrak{b},\mathfrak{b}\}$, $\gamma \in \mathfrak{b}$. *This action stabilizes the subspaces* $A^+, A^-, B, \cdots, E'$.

Definition 5.3.6. [4, 5.26] The *full skew-dihedral homology group* of $\mathfrak{b}$ is

$$\mathrm{HF}(\mathfrak{b}) = \ker\rho = \left\{\sum_i \{\alpha_i,\beta_i\} \in \{\mathfrak{b},\mathfrak{b}\} \mid \sum_i D_{\alpha_i,\beta_i} = 0\right\}.$$

Theorem 5.3.7. *Let* $n \geq 5$ *and let* $\mathfrak{a}$ *and* $\mathscr{B}$ *be as in Example 5.2.3. Let*

$$\widehat{\mathscr{L}(\mathfrak{b})} := (\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E')\oplus\{\mathfrak{b},\mathfrak{b}\}$$

be the algebra with multiplication defined by (5.3.1) with $D_{\mathfrak{b},\mathfrak{b}}$ *replaced by* $\{\mathfrak{b},\mathfrak{b}\}$ *and* $D_{\alpha,\beta}$ *replaced by* $\{\alpha,\beta\}$. *Then* $(\widehat{\mathscr{L}(\mathfrak{b})}, f)$ *where* $f : \widehat{\mathscr{L}(\mathfrak{b})} \to \mathscr{L}(\mathfrak{b})$ *is given by*

$$\begin{aligned} f(x) &= x,\ \forall x \in (\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E'), \\ f(\{\alpha,\beta\}) &= D_{\alpha,\beta},\ \forall\{\alpha,\beta\}\in\{\mathfrak{b},\mathfrak{b}\},\end{aligned}$$

is the universal covering algebra of $\mathscr{L}(\mathfrak{b})$ *and the center of* $\widehat{\mathscr{L}(\mathfrak{b})}$ *is* $\mathrm{HF}(\mathfrak{b})$.

Proof. This is similar to [3, Theorem 4.13] and [4, Theorem 5.34]. First, we are going to show that $\widehat{\mathscr{L}(\mathfrak{b})}$ with the above multiplication is a Lie algebra. It is clear that the bracket is bilinear. It remains to check $\widehat{\mathscr{L}(\mathfrak{b})}$ satisfies the Jacobi identity. Observe that if at least 2 of the 3 factors are from $(\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E')$, then the products behave as in $\mathscr{L}(\mathfrak{b})$. The only difference is that the $\{\mathfrak{b},\mathfrak{b}\}$-component of the products involves expressions such as $\{\alpha_1,\alpha_2\}$ rather than D_{α_1,α_2}. But when such a term acts on $\mathfrak{b}$, the action of the two is the same. When all of them belong to $\{\mathfrak{b},\mathfrak{b}\}$, by Proposition 5.3.4, the Jacobi identity hold. When exactly 2 of the 3 factors belongs to $\{\mathfrak{b},\mathfrak{b}\}$ then it is necessary to know that products of the form $[\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\}]$ are represented as $[D_{\alpha_1,\alpha_2},D_{\beta_1,\beta_2}]$, but that is the content of Proposition 5.3.4. As illustration, we consider $\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\} \in \{\mathfrak{a},\mathfrak{a}\}$ and $x\otimes\alpha \in (\mathfrak{g}\otimes A)\oplus(S\otimes C)\oplus(S'\otimes C')\oplus(\Lambda\otimes E)\oplus(\Lambda'\otimes E')$. Using Proposition 4.2.7 and the associativity of $\mathfrak{a}$ we get

$$
\begin{aligned}
[[\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\}],x\otimes\alpha] &= [\{D_{\alpha_1,\alpha_2}\beta_1,\beta_2\}+\{\beta_1,D_{\alpha,\alpha_2}\beta_2\},x\otimes\alpha]\\
&= [\{[[\alpha_1,\alpha_2],\beta_1],\beta_2\},x\otimes\alpha]+[\{\beta_1,[[\alpha_1,\alpha_2],\beta_2]\},x\otimes\alpha]\\
&= x\otimes([[[[\alpha_1,\alpha_2],\beta_1],\beta_2],\alpha]+[[\beta_1,[[\alpha_1,\alpha_2],\beta_2]],\alpha])\\
&= x\otimes[[[\alpha_1,\alpha_2],[\beta_1\beta_2]],\alpha]\\
&= x\otimes([[\alpha_1,\alpha_2],[[\beta_1,\beta_2],\alpha]]+[[[\alpha_1,\alpha_2],\alpha],[\beta_1,\beta_2]])\\
&= [\{\alpha_1,\alpha_2\},x\otimes[[\beta_1,\beta_2],\alpha]]+[x\otimes[[\alpha_1,\alpha_2],\alpha],\{\beta_1,\beta_2\}]\\
&= [\{\alpha_1,\alpha_2\},[\{\beta_1,\beta_2\},x\otimes\alpha]+[[\{\alpha_1,\alpha_2\},x\otimes\alpha],\{\beta_1,\beta_2\}]
\end{aligned}
$$

Therefore $\widehat{\mathscr{L}(\mathfrak{b})}$ with the above multiplication is a Lie algebra. By its construction $\widehat{\mathscr{L}(\mathfrak{b})}$ is graded by the same root system as $\mathscr{L}(\mathfrak{b})$ and it is perfect. In Lemma 5.3.4 we showed that f is a surjective Lie algebra homomorphism and

$$
\ker f = \left\{\sum_i \{\alpha_i,\beta_i\} \in \{\mathfrak{b},\mathfrak{b}\} \mid \sum_i D_{\alpha_i,\beta_i} = 0\right\}.
$$

Thus, $(\widehat{L},f)$ is a central extension of L. We have $\ker f \subseteq Z(\widehat{L})$ and it easy to check that $Z(\widehat{L}) \subseteq \ker f$, so $Z(\widehat{L}) = \ker f = \mathrm{HF}(\mathfrak{b})$, as required.

To see that $f:\widehat{\mathscr{L}(\mathfrak{b})} \to \mathscr{L}(\mathfrak{b})$ is universal, suppose that $f:\widetilde{\mathscr{L}(\mathfrak{b})} \to \mathscr{L}(\mathfrak{b})$ is a central extension of L. By Lemma 5.1.4, we can lift $\mathscr{L}(\mathfrak{b})$ to a subspace of $\widetilde{\mathscr{L}(\mathfrak{b})}$, which we identify with $\mathscr{L}(\mathfrak{b})$, so that the corresponding 2-cocycle satisfies $\zeta(\mathfrak{g},\mathscr{L}(\mathfrak{b}))=0$. Then, by Theorem 5.1.6, we may assume that the corresponding 2-cocycle is obtained from a 2-cocycle ε of $\mathfrak{b}$ as in (5.1.3). The 2-cocycle induces a mapping $\tilde{\varepsilon}:\{\mathfrak{b},\mathfrak{b}\} \to \mathbb{E}$ with $\{\alpha,\beta\} \mapsto \varepsilon(\alpha,\beta) \in E$. Thus, there is a homomorphism $\varphi:\widehat{\mathscr{L}(\mathfrak{b})} \to \widetilde{\mathscr{L}(\mathfrak{b})}$ with

$$
\begin{aligned}
\varphi(x\otimes a) &= x\otimes a,\ \forall x \in (\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E'),\\
\varphi(\{\alpha,\beta\}) &= D_{\alpha,\beta}+\tilde{\varepsilon}(\alpha,\beta),\ \forall\{\alpha,\beta\} \in \{\mathfrak{b},\mathfrak{b}\}.
\end{aligned}
$$

Hence $\widehat{\mathscr{L}(\mathfrak{b})}$ is the universal covering algebra of $\mathscr{L}(\mathfrak{b})$, as required. $\square$

Consider the quotient space $\prec \mathfrak{b},\mathfrak{b} \succ = \{\mathfrak{b},\mathfrak{b}\}/X$ and set $\prec \alpha,\beta \succ = \{\alpha,\beta\}+X$ in $\{\mathfrak{b},\mathfrak{b}\}/X$. Let

$$\mathscr{L}(\mathfrak{b},X) = (\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E')\oplus \prec \mathfrak{b},\mathfrak{b} \succ \tag{5.3.3}$$

be the algebra with multiplication same as $\mathscr{L}(\mathfrak{b})$ with $D_{\alpha,\beta}$ replaced by $\prec \alpha,\beta \succ$. Then we have the following:

Theorem 5.3.8. *(1) $\mathscr{L}(\mathfrak{b},X)$ is a $(\Theta_n,\mathfrak{g})$-graded Lie algebra with coordinate algebra $\mathfrak{b}$.*

(2) Every Θ_n-graded Lie algebra with coordinate algebra $\mathfrak{b}$ is isomorphic to $\mathscr{L}(\mathfrak{b},X)$ for some subspace X of $\mathrm{HF}(\mathfrak{b})$.

Proof. This proof is similar to the proof of [3, Theorem 4.20] and [4, Theorem 5.35]. We need to prove only (2). Denote $L := \mathscr{L}(\mathfrak{b})$. Suppose that $\tilde{L}$ is a $(\Theta_n,\mathfrak{g})$-graded Lie algebra with coordinate algebra $\mathfrak{b}$. By Theorem 5.2.5, $\tilde{L}$ is a cover of L. Since L is $(\Theta_n,\mathfrak{g})$-graded with coordinate algebra $\mathfrak{b}$, by Lemma 5.1.4 we can lift L to a subspace of $\tilde{L}$, which we identify with L, so that the corresponding 2-cocycle satisfies $\zeta(\mathfrak{g},L)=0$. We get $\tilde{L}=L\oplus\mathbb{E}$ where $\mathbb{E}$ is the center of $\tilde{L}$. Let $\pi:\tilde{L}\to L$ be the canonical projection. Then $\pi\mid\mathfrak{g}=\mathrm{id}$ is a monomorphism which we can use to identify $\mathfrak{g}$ with its image in L. By Theorem 5.1.6, we may assume that the 2-cocycle ζ is gotten from a 2-cocycle ε of $\mathfrak{b}$ as in (5.1.3). The 2-cocycle induces a mapping $\tilde{\varepsilon}:\{\mathfrak{b},\mathfrak{b}\}\to\mathbb{E}$ with $\{\alpha,\beta\}\mapsto\varepsilon(\alpha,\beta)\in E$. Thus, there is a homomorphism $\varphi:\widehat{L}\to\tilde{L}$ with

$$\varphi(x\otimes a) = x\otimes a,\ \forall x\in(\mathfrak{g}\otimes A)\oplus\cdots\oplus(\Lambda'\otimes E'),$$
$$\varphi(\{\alpha,\beta\}) = D_{\alpha,\beta}+\tilde{\varepsilon}(\alpha,\beta),\ \forall\{\alpha,\beta\}\in\{\mathfrak{b},\mathfrak{b}\}.$$

Then the homomorphism $\varphi:\widehat{L}\to\tilde{L}$ has the additional property that $\varphi\mid\mathfrak{g}=\mathrm{id}$. Hence, if X is the kernel of φ, then φ induces an isomorphism $\psi:\mathscr{L}(\mathfrak{b},X)\to\tilde{L}$ so that $\psi\mid\mathfrak{g}=\mathrm{id}$. □

Using basic facts about central extension [49] we also have

Theorem 5.3.9. *The natural map $\widehat{\mathscr{L}(\mathfrak{b})}\to\mathscr{L}(\mathfrak{b},X)$ is the universal cover of $\mathscr{L}(\mathfrak{b},X)$, and hence $H_2(\mathscr{L}(\mathfrak{b},X),\mathbb{F})\cong X$.*

5.4 Quasiclassical Lie algebras and Θ_n-graded Lie algebras

Let A be any associative algebra with identity 1 and let $n\geq 2$. We denote by $M_n(A)\cong M_n\otimes A$ the associative algebra of $n\times n$ matrices over A. The corresponding Lie algebra

$M_n(A)^{(-)}$ is denoted $gl_n(A)$ and has the following multiplication:

$$[x\otimes\alpha, y\otimes\beta] = (x\otimes\alpha)(y\otimes\beta) - (y\otimes\beta)(x\otimes\alpha) = xy\otimes\alpha\beta - yx\otimes\beta\alpha.$$

One can check that its derived subalgebra $sl_n(A) := gl_n(A)^{(1)}$ is an A_{n-1}-graded Lie algebra with grading subalgebra $sl_n(F)\otimes 1$ (see for example, [43, Example 1.5] or [39, 5.1]). The following is well known.

Proposition 5.4.1. *The following definitions of $sl_n(A)$ are equivalent.*

(1) $sl_n(A) := gl_n(A)^{(1)}$.

(2) $sl_n(A) = e_n(A)$ *where $e_n(A)$ is the ideal of $gl_n(A)$ generated by the elements $aE_{i,j}$, $r\in A$ and $i\neq j$) (see [22]).*

(3) $sl_n(A) = \{x\in gl_n(A) \mid \operatorname{tr} x\in [A,A]\}$ *(see [44]).*

(4) $sl_n(A) = \operatorname{Ker} T$ *where T is a natural non-comutative trace map*

$$T : gl_n(A) \mapsto A/[A,A], \quad x\mapsto [\sum_{j=1}^{n} x_{jj}]$$

and $[a]$ denotes the class of a in $A/[A,A]$.

Proof. We will only show $(1)\Leftrightarrow(2)$ (the other being obvious). Note that

$$sl_n(A) = gl_n(A)^{(1)} = (sl_n\otimes A)\oplus(I\otimes[A,A])$$

(see for example [43, Example 1.5]). We claim that $e_n(A) = (sl_n\otimes A)\oplus(I\otimes[A,A])$. We have for all $a\in A$, $i\neq j$ and $[a_1,a_2]\in[A,A]$,

$$\begin{aligned}(E_{i,i}-E_{j,j})\otimes a &= [E_{i,j}\otimes a, E_{j,i}\otimes 1]\in e_n(A) \\ E_{i,j}\otimes a &= [E_{i,k}\otimes a, E_{k,j}\otimes 1]\in e_n(A)a,\ k\neq i,j \\ E_{i,i}\otimes[a_1,a_2] &= ([E_{i,j}\otimes a_1, E_{j,i}\otimes a_2] - [E_{i,j}\otimes a_2a_1, E_{j,i}\otimes 1])\in e_n(A)\end{aligned} \tag{5.4.1}$$

Thus $(sl_n\otimes A)\oplus I\otimes[A,A]\subseteq e_n(A)$. Since $gl_n(A)^{(1)} = (sl_n\otimes A)\oplus(I\otimes[A,A])$ (see [43, Example 1.5]), we have $gl_n(A)^{(1)}\subseteq e_n(A)$. From Formulas (5.4.1) we see that $e_n(A)\subseteq gl_n(A)^{(1)}$, as required. □

Definition 5.4.2. The Steinberg Lie algebra $\mathfrak{st}_n(A)$ $(n\geq 3)$ is defined to be the Lie algebra over $\mathbb{F}$ generated by the symbols $X_{ij}(r)$, $1\leq i,j\leq n$, $i\neq j$, $r\in A$, where A is any $\mathbb{F}$-algebra with identity subject to the relations:

(1) $X_{ij}(ar+bs) = aX_{ij}(r) + bX_{ij}(s)$.

(2) $[X_{ij}(r), X_{jk}(A)] = X_{ik}(rA)$ if i, j, k are distinct.

(3) $[X_{ij}(r), X_{kt}(A)] = 0$ if $i \neq t$ and $j \neq k$, for all $a, b \in k$ and for all $r, s \in A$.

Lemma 5.4.3. *[37] Let A be any associative algebra with identity and let $n \geq 3$. Let $\psi : \mathfrak{st}_n(A) \to sl_n(A)$ be the Lie algebra epimorphism such that $\psi(X_{ij}(r)) = E_{i,j}(r)$. Then $(\mathfrak{st}_n(A), \psi)$ is a central extension of $sl_n(A)$ and the kernel of ψ is isomorphic to $HC_1(A)$, the first cyclic homology group of A.*

Theorem 5.4.4. *[22] Let L be an A_{n-1}-graded Lie algebra with coordinate algebra A where $n \geq 4$. Then*

(1) $\mathfrak{st}_n(A)$ is an A_{n-1}-graded Lie algebra with coordinate algebra A such that $\mathfrak{st}_n(A) = \mathfrak{st}_n^0(A) \oplus \sum_{i \neq j} \mathfrak{st}_n^{i,j}(A)$ where $\mathfrak{st}_n^0(A) := \sum_{i \neq j}[X_{ij}(A), X_{ji}(A)]$ and $\mathfrak{st}_n^{i,j}(A) := X_{ij}(A)$.

(2) $\mathfrak{st}_n(A)$ is centrally closed.

(3) $\mathfrak{st}_n(A)$ is the universal covering algebra of L and $sl_n(A)$.

Definition 5.4.5. [9] A Lie algebra L is said to be *quasiclassical* if there exists an associative algebra A with involution such that $L \cong skew(A)^{(1)}$.

Remark 5.4.6. Let A be an associative algebra and let $L = A^{(1)}$. Then L is quasiclassical. Indeed it is easy to see that $L \cong skew(\tilde{A})^{(1)}$ where $\tilde{A} := A \oplus A^{op}$, the direct sum of two ideals, with involution swapping the components.

Corollary 5.4.7. *Let L be an A_{n-1}-graded Lie algebra where $n \geq 4$. Then L is centrally isogenous to a quasiclassical Lie algebra.*

Proof. By Theorem 5.4.4(3), L is centrally isogenous to $sl_n(A)$. It remains to note that $sl_n(A)$ is quasiclassical by Remark 5.4.6. $\square$

Denote by Ξ_n the following set of integral weights of sl_n:

$$\Xi_n = \Gamma((V \oplus V^*)^{\otimes 2}) = \{0, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i, j \leq n\} \subset \Theta_n.$$

We are going to show (Ξ_n, sl_n)-graded Lie algebras are centrally isogenous to quasiclassical Lie algebras for $n \geq 5$.

Example 5.4.8. Let $L = sl_{2n}$ and $\mathfrak{g} = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x^t \end{bmatrix} \mid x \in sl_n \right\} \subset L$. We consider the adjoint action of $\mathfrak{g}$ on L. We have the following decomposition of the $\mathfrak{g}$-module L:

$$L = \mathfrak{g} \oplus \mathfrak{g}' \oplus S \oplus S' \oplus \Lambda \oplus \Lambda' \oplus D$$

where $D = \left\{ \begin{bmatrix} t_1 I_n & 0 \\ 0 & -t_1 I_n \end{bmatrix} \mid t_1 \in \mathbb{F} \right\}$ is a trivial $\mathfrak{g}$-module and

$$\mathfrak{g}' = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^t \end{bmatrix} \mid x \in sl_n \right\} \cong \mathfrak{g} \cong V(\omega_1 + \omega_{n-1}),$$
$$S = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = x^t \right\} \cong V(2\omega_1),$$
$$S' = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = x^t \right\} \cong V(2\omega_{n-1}),$$
$$\Lambda = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = -x^t \right\} \cong V(\omega_2),$$
$$\Lambda' = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \mid x \in M_n(F) \text{ and } x = -x^t \right\} \cong V(\omega_{n-2}),$$

as $\mathfrak{g}$-modules. Thus, L is $(\Xi_n, \mathfrak{g})$-graded.

Define $sym(M_n) := \{x \in M_n \mid x^t = x\}$, $sym_0(M_n) := \{x \in sl_n \mid x^t = x\}$ and $skew(M_n) := \{x \in M_n \mid x^t = -x\}$.

Theorem 5.4.9. *Let L be (Ξ_n, sl_n)-graded. Let $\mathfrak{a} = A^+ \oplus A^- \oplus C \oplus E \oplus C' \oplus E'$ be the coordinate algebra of L with involution η (as in Theorem 4.2.9 with $B = B' = 0$) and let $\mathfrak{U} = M_n \otimes \mathfrak{a}$. Suppose $n \geq 7$ or $n = 5, 6$ and the conditions (1.2.1) hold. Then*

(1) *$\mathfrak{U}$ is an associative algebra with involution $\sigma : x \otimes \alpha \mapsto x^t \otimes \eta(\alpha)$;*

(2) *$skew(\mathfrak{U})^{(1)} = sym_0(M_n) \otimes skew(\mathfrak{a}) \oplus skew(M_n) \otimes sym(\mathfrak{a}) \oplus I \otimes (C \oplus C') \oplus \mathfrak{D}$ where*

$$\mathfrak{D} = I \otimes ([A^-, A^-] \oplus [A^+, A^+] \oplus [C, C'] \oplus [E, E']);$$

(3) *$skew(\mathfrak{U})^{(1)}$ is $(\Xi_n, \mathfrak{g})$-graded with coordinate algebra $\mathfrak{a}$ where $\mathfrak{g} \cong sl_n$;*

(4) *$\widehat{\mathscr{L}(\mathfrak{a})}$ is the universal covering algebra of both L and $skew(\mathfrak{U})^{(1)}$. In particular, all these three algebras are centrally isogenous.*

Proof. (1) This follows from Lemma 4.3.4.

(2) Let $\mathfrak{U} = M_n \otimes \mathfrak{a}$. Recall that $\mathfrak{U}^{(-)}$ is a Lie algebra with multiplication:

$$[x \otimes \alpha, y \otimes \beta] = (x \otimes \alpha)(y \otimes \beta) - (y \otimes \beta)(x \otimes \alpha) = xy \otimes \alpha\beta - yx \otimes \beta\alpha.$$

By Lemma 4.3.4,

$$skew(\mathfrak{U}) = sym(M_n) \otimes skew(\mathfrak{a}) \oplus skew(M_n) \otimes sym(\mathfrak{a}).$$

Denote

$$\mathfrak{L} = sym_0(M_n) \otimes skew(\mathfrak{a}) \oplus skew(M_n) \otimes sym(\mathfrak{a}) \oplus I \otimes (C \oplus C') \oplus \mathfrak{D}$$

where

$$\mathfrak{D} = I \otimes ([A^-,A^-] \oplus [A^+,A^+] \oplus [C,C'] \oplus [E,E']).$$

We need to show that $skew(\mathfrak{U})^{(1)} = \mathfrak{L}$. First we need to prove that $skew(\mathfrak{U})^{(1)}$ contains $\mathfrak{L}$. Note that $x \otimes c = [I \otimes \frac{1^-}{2}, x \otimes c]$, for all $x \in sym(M_n)$ and $c \in C$, so

$$sym(M_n) \otimes C \subset skew(\mathfrak{U})^{(1)}.$$

Similarly, we prove that

$$sym(M_n) \otimes (C \oplus C') \oplus skew(M_n) \otimes (E \oplus E') \subseteq skew(\mathfrak{U})^{(1)}.$$

It remains to check

$$sym_0(M_n) \otimes A^- \oplus skew(M_n) \otimes A^+ \oplus \mathfrak{D} \subseteq skew(\mathfrak{U})^{(1)}.$$

We have for all $a^\pm \in A^\pm$, $i \neq j$ and $(\alpha, \alpha') \in (A^-,A^-) \cup (A^+,A^+) \cup (C,C'] \cup (E,E')$,

$$\begin{aligned} E_{i,j} \otimes a^\pm &= [E_{i,k} \otimes a^\pm, E_{k,j} \otimes 1^+] \in skew(\mathfrak{U})^{(1)},\ k \neq i,j \\ (E_{i,i} - E_{j,j}) \otimes a^- &= [E_{i,j} \otimes a^-, E_{j,i} \otimes 1^+] \in skew(\mathfrak{U})^{(1)}, \\ E_{i,i} \otimes [\alpha, \alpha'] &= ([E_{i,j} \otimes \alpha, E_{j,i} \otimes \alpha'] - [E_{i,j} \otimes \alpha'\alpha, E_{j,i} \otimes 1^+]) \in skew(\mathfrak{U})^{(1)}, \end{aligned} \tag{5.4.2}$$

as required. Now we are going to show that $skew(\mathfrak{U})^{(1)} \subseteq \mathfrak{L}$. Let $x \otimes \alpha$ and $y \otimes \beta$ be homogeneous elements in $sym(M_n) \otimes A^- \oplus skew(M_n) \otimes A^+$. If both $x \otimes \alpha$ and $y \otimes \beta$ belong to $sym(M_n) \otimes A^-$ or $skew(M_n) \otimes A^+$ then

$$[x \otimes \alpha, y \otimes \beta] = x \circ y \otimes \frac{[\alpha, \beta]}{2} + [x,y] \otimes \frac{\alpha \circ \beta}{2} + I \otimes (x \mid y)[\alpha, \beta] \in \mathfrak{L}.$$

Otherwise, $\mathrm{tr}(xy) = 0$ (as the product of a symmetric and a skew symmetric matrices has zero trace) and

$$[x\otimes\alpha, y\otimes\beta] = x\diamond y\otimes\frac{[\alpha,\beta]}{2}+[x,y]\otimes\frac{\alpha\circ\beta}{2}\in\mathfrak{L}.$$

Thus,

$$(sym(M_n)\otimes A^-\oplus skew(M_n)\otimes A^+)^{(1)}\subseteq\mathfrak{L}.$$

Similarly,

$$[sym(M_n)\otimes C\oplus skew(M_n)\otimes E, sym(M_n)\otimes C'\oplus skew(M_n)\otimes E']\subseteq\mathfrak{L}.$$

It is easy to check (using Table 4.1.1) that

$$\begin{aligned}
&[sym(M_n)\otimes A^-\oplus skew(M_n)\otimes A^+, sym(M_n)\otimes(C\oplus C')\oplus skew(M_n)\otimes(E\oplus E')]\\
&\subseteq sym(M_n)\otimes(C\oplus C')\oplus skew(M_n)\otimes(E\oplus E')\subseteq\mathfrak{L},\\
&[sym(M_n)\otimes C\oplus skew(M_n)\otimes E, sym(M_n)\otimes C\oplus skew(M_n)\otimes E]=0,\\
&[sym(M_n)\otimes C'\oplus skew(M_n)\otimes E', sym(M_n)\otimes C'\oplus skew(M_n)\otimes E']=0.
\end{aligned}$$

Thus, $skew(\mathfrak{U})^{(1)}\subseteq\mathfrak{L}$, as required.

(3) Denote $\tilde{\mathfrak{g}} := sym(M_n)\otimes 1^-\oplus skew(M_n)\otimes 1^+$. We claim that $\tilde{\mathfrak{g}}$ is a Lie subalgebra of $skew(\mathfrak{U})$ isomorphic to gl_n. Indeed, since $e_1=\frac{1^++1^-}{2}$ and $e_2=\frac{1^+-1^-}{2}$ are orthogonal idempotents in $\mathscr{A}=A^+\oplus A^-$ (see Proposition 4.3.2) , it is easy to see that the following map $\varphi: gl_n\to\tilde{\mathfrak{g}}$ is a Lie algebra isomorphism:

$$\begin{aligned}
\varphi(x) &= \frac{(x+x^t)}{2}\otimes 1^- + \frac{(x-x^t)}{2}\otimes 1^+\\
&= \frac{(x+x^t)}{2}\otimes(e_1-e_2)+\frac{(x-x^t)}{2}\otimes(e_1+e_2)\\
&= x\otimes e_1+(-x^t)\otimes e_2.
\end{aligned}$$

Put $\mathfrak{g}=\tilde{\mathfrak{g}}^{(1)}\cong sl_n$. We wish to show that $\mathfrak{L}$ is $(\Xi_n,\mathfrak{g})$-graded with coordinate algebra $\mathfrak{a}$. Let $\mathfrak{h}=H\otimes 1^-$ where H is the set of diagonal matrices of sl_n. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{L}$ has the following weight spaces with respect to the adjoint action of $\mathfrak{h}$:

$$\begin{aligned}
\mathfrak{L}_{\varepsilon_i-\varepsilon_j} &= \{E_{i,j}\otimes e_1\alpha e_1+E_{j,i}\otimes e_2\alpha e_2 \mid \alpha\in\mathfrak{a}\}, 1\le i\ne j\le n;\\
\mathfrak{L}_{\varepsilon_i+\varepsilon_j} &= \left\{E_{i,j}\otimes(c+e)-E_{j,i}\otimes\eta(c+e)\mid (c+e)\in C\oplus E\right\}, 1\le i,j\le n;
\end{aligned}$$

$$\mathfrak{L}_{-\varepsilon_i-\varepsilon_j} = \left\{E_{i,j}\otimes(c'+e') - E_{j,i}\otimes\eta(c'+e') \mid (c'+e')\in C'\oplus E'\right\}, 1\le i,j\le n;$$
$$\mathfrak{L}_0 = (H\otimes A^-)\oplus\mathfrak{D}.$$

From the formulas (5.4.2), we see that $\mathfrak{L}_0 = \sum\limits_{\alpha\in\Xi_n\setminus\{0\}}[\mathfrak{L}_\alpha,\mathfrak{L}_{-\alpha}]$. Thus $\mathfrak{L}$ is $(\Xi_n,\mathfrak{g})$-graded. It is easy to check that $\mathfrak{a}$ is the coordinate algebra of $\mathfrak{L}$ (this was also proved in more general case, see Example 5.2.4 and Theorem 5.2.6).

(4) By Theorem 5.2.5, $\mathfrak{L}$ and $skew(\mathfrak{U})^{(1)}$ are covers of $\mathscr{L}(\mathfrak{b})$ and by Theorem 5.3.7 $\widehat{\mathscr{L}(\mathfrak{b})}$ is the universal covering algebra of both of them. □

Corollary 5.4.10. *Let L be $(\Xi_n, sl_n)-$graded. Suppose $n\ge 7$ or $n=5,6$ and the conditions (1.2.1) hold. Then L is centrally isogenous to a quasiclassical Lie algebra.*

References

[1] Bruce Allison. *Extended affine Lie algebras and their root systems*, volume 603. American Mathematical Soc., 1997.

[2] Bruce Allison and Georgia Benkart. Unitary Lie algebras and Lie tori of type BCr, r>2. 2008.

[3] Bruce Allison, Georgia Benkart, and Yun Gao. Central extensions of Lie algebras graded by finite root systems. *Mathematische Annalen*, 316(3):499–527, 2000.

[4] Bruce Allison, Georgia Benkart, and Yun Gao. *Lie Algebras graded by the root systems BCr, r>1*. Number 751. American Mathematical Soc., 2002.

[5] Yuri Bahturin and Georgia Benkart. Some constructions in the theory of locally finite simple Lie algebras. *J. Lie Theory*, 14(1):243–270, 2004.

[6] Yuri Bahturin, Alexander Baranov, and Alexander Zalesski. Simple Lie subalgebras of locally finite associative algebras. *Journal of Algebra*, 281(1):225–246, 2004.

[7] Alexander Baranov and Hogar Yaseen. Lie algebras with prescribed sln decomposition. *arXiv preprint arXiv:1709.03023*, 2017.

[8] Alexander Baranov and Alexander Zalesskii. Plain representations of Lie algebras. *Journal of the London Mathematical Society*, 63(3):571–591, 2001.

[9] Alexander Baranov and Alexander Zalesskii. Quasiclassical Lie algebras. *Journal of Algebra*, 243(1):264–293, 2001.

[10] Georgia Benkart. Derivations and invariant forms of Lie algebras graded by finite root systems. *Canadian Journal of Mathematics*, 50(2):225–241, 1998.

[11] Georgia Benkart and Alberto Elduque. Lie superalgebras graded by the root systems C (n), D (m, n), D (2, 1: α), F (4), G (3). *Canad. Math. Bull*, 45(4):509–524, 2002.

[12] Georgia Benkart and Alberto Elduque. Lie superalgebras graded by the root system A (m, n). *J. Lie Theory*, 13(2):387–400, 2003.

[13] Georgia Benkart and Alberto Elduque. Lie superalgebras graded by the root system B(m, n). *Selecta Mathematica, New Series*, 9(3):313–360, 2003.

[14] Georgia Benkart and Alberto Elduque. Lie algebras with prescribed sl3 decomposition. *Proceedings of the American Mathematical Society*, 140(8):2627–2638, 2012.

[15] Georgia Benkart and Consuelo Martınez. A(n, n)-graded Lie superalgebras. *J. reine angew. Math*, 573:139–156, 2004.

[16] Georgia Benkart and Robert Moody. Derivations, central extensions, and affine Lie algebras. *Algebras Groups Geom*, 3(4):456–492, 1986.

[17] Georgia Benkart and Erhard Neher. The centroid of extended affine and root graded Lie algebras. *Journal of Pure and Applied Algebra*, 205(1):117–145, 2006.

[18] Georgia Benkart and Oleg Smirnov. Lie algebras graded by the root system BC1. *J. Lie theory*, 13(1):91–132, 2003.

[19] Georgia Benkart and Yoji Yoshii. Lie G-tori of symplectic type. *The Quarterly Journal of Mathematics*, 57(4):425–448, 2006.

[20] Georgia Benkart and Efim Zelmanov. Lie algebras graded by finite root systems and intersection matrix algebras. *Inventiones mathematicae*, 126(1):1–45, 1996.

[21] Georgia Benkart, Daniel Britten, and Frank Lemire. *Stability in modules for classical Lie algebras: a constructive approach*, volume 430. American Mathematical Soc., 1990.

[22] Stephen Berman and Robert Moody. Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy. *Inventiones mathematicae*, 108(1): 323–347, 1992.

[23] Stephen Berman, Yun Gao, Yaroslav Krylyuk, and Erhard Neher. The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A2. *Transactions of the American Mathematical Society*, 347(11):4315–4363, 1995.

[24] Stephen Berman, Yun Gao, and Yaroslav S Krylyuk. Quantum tori and the structure of elliptic quasi-simple Lie algebras. *journal of functional analysis*, 135(2):339–389, 1996.

[25] Sandeep Bhargava and Yun Gao. Realizations of BCr-graded intersection matrix algebras with grading subalgebras of type Br, r> 2. *Pacific Journal of Mathematics*, 263(2):257–281, 2013.

[26] Nicolas Bourbaki. *Groupes et Algebres de Lie: Elements de Mathematique*. Hermann, 1968.

[27] Cristina Draper and Alberto Elduque. An overview of fine gradings on simple Lie algebras. *arXiv preprint arXiv:1603.09482*, 2016.

[28] Alberto Elduque. Fine gradings and gradings by root systems on simple Lie algebras. *arXiv preprint arXiv:1303.0651*, 2013.

[29] Mikhail Ershov, Andrei Jaikin-Zapirain, Martin Kassabov, and Zezhou Zhang. Groups graded by root systems and property T. *Proceedings of the National Academy of Sciences*, 111(50):17759–17764, 2014.

[30] Mikhail Ershov, Andrei Jaikin-Zapirain, and Martin Kassabov. *Property T for Groups Graded by Root Systems*, volume 249. American Mathematical Society, 2017.

[31] Yun Gao. Involutive Lie algebras graded by finite root systems and compact forms of im algebras. *Mathematische Zeitschrift*, 223(4):651–672, 1996.

[32] Howard Garland. The arithmetic theory of loop groups. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 52(1):5–136, 1980.

[33] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[34] Nathan Jacobson. *Lie algebras*. Number 10. Courier Corporation, 1979.

[35] Victor Kac. Lie superalgebras. *Advances in mathematics*, 26(1):8–96, 1977.

[36] Victor Kac. Infinite dimensional Lie algebras, cambridge univ. *Pr., UK*, 1990.

[37] Christian Kassel and Jean-Louis Loday. Extensions centrales d algebres de Lie. *Ann. Inst. Fourier (Grenoble)*, 32(4):119–142, 1982.

[38] Shrawan Kumar. Tensor product decomposition. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 1226–1261. World Scientific, 2010.

[39] Nathan Manning, Erhard Neher, and Hadi Salmasian. Integrable representations of root-graded Lie algebras. *Journal of Algebra*, 2017.

[40] Consuelo Martínez and Efim Zelmanov. Lie superalgebras graded by P (n) and Q (n). *Proceedings of the National Academy of Sciences of the United States of America*, 100(14):8130, 2003.

[41] Hechmi Ben Messaoud and Guy Rousseau. Kac-moody Lie algebras graded by kac-moody root systems. *Journal of Lie Theory*, 24:321–350, 2014.

[42] Christoph Müller, Karl-Hermann Neeb, and Henrik Seppänen. Borel–weil theory for root graded banach–Lie groups. *International Mathematics Research Notices*, 2010(5):783–823, 2009.

[43] Karl-Hermann Neeb. *Locally convex root graded Lie algebras*. Fachbereich Mathematik, Technische Universität Darmstadt, 2002.

[44] Erhard Neher. Lie algebras graded by 3-graded root systems and jordan pairs covered by grids. *American Journal of Mathematics*, pages 439–491, 1996.

[45] Josiane Nervi. Affine kac–moody algebras graded by affine root systems. *Journal of Algebra 2014*, 253(1):50–99, 2002.

[46] George Seligman. *Rational methods in Lie algebras*, volume 17. Marcel Dekker Inc, 1976.

[47] Zhiyong Shi. Groups graded by finite root systems. *Tohoku Mathematical Journal, Second Series*, 45(1):89–108, 1993.

[48] Peter Slodowy. Beyond kac-moody algebras and inside. In *Britten, Lemire, and Moody, Eds.,Lie Algebras and Related Topics, Canad. Math. Soc. Conf. Proc. 5*, pages 361–371, 1986.

[49] Charles Weibel. An introduction to homological algebra. number 38 in cambridge studies in advanced mathematics, 1994.

[50] Angelika Welte. Central extension of graded Lie algebras. *arXiv preprint arXiv:1004.4404*, 2010.

[51] Yoji Yoshii. Jordan tori. *Mathematical Reports of the Academy of Sciences*, 18(4): 153–158, 1996.

[52] Yoji Yoshii. Root-graded Lie algebras with compatible grading. *Communications in Algebra*, 29(8):3365–3391, 2001.

[53] Yoji Yoshii. Root systems extended by an abelian group and their Lie algebras. *J. Lie theory*, 14(2):371–394, 2004.

[54] Yoji Yoshii. Lie tori-A simple characterization of extended affine Lie algebras. *Publications of the Research Institute for Mathematical Sciences*, 42(3):739–762, 2006.

[55] Malihe Yousofzadeh. Fixed point subalgebras of root graded Lie algebras. *Osaka Journal of Mathematics*, 46(3):611–643, 2009.

[56] Malihe Yousofzadeh. Structure of root graded Lie algebras. *J. Lie Theory 22*, pages 397–435, 2012.

[57] Malihe Yousofzadeh. Central extensions of root graded Lie algebras. *Publ. Res. Inst. Math. Sci.49 , no. 4*, pages 801–829, 2013.

[58] Malihe Yousofzadeh. Root graded Lie superalgebras. *arXiv preprint arXiv:1507.08060*, 2015.

[59] Malihe Yousofzadeh. Root graded Lie superalgebras which appear as the centerless cores of extended affine Lie superalgebras. *Toyama mathematical journal*, 37:73–86, 2015.