On the Module Category of Symmetric Special Multiserial Algebras

Thesis submitted for the degree of
Doctor of Philosophy at the
University of Leicester

Drew Damien Duffield

Department of Mathematics
University of Leicester

July 2017

Abstract

On the Module Category of Symmetric Special Multiserial Algebras

Drew D. Duffield

The module category of an algebra is a major source of study for representation theorists. The indecomposable modules over an algebra and the morphisms between them are of tremendous importance, since these essentially determine the finitely generated module category over the algebra. The Auslander-Reiten quiver is a means of presenting this information.

In this thesis, we focus on the class of symmetric special multiserial algebras. These are a broad class of algebras that include the well-studied subclass of symmetric special biserial algebras. A useful property of these algebras is that they have a decorated hypergraph (with orientation) associated to them, called a Brauer configuration. As well as offering a pictorial presentation of the algebra, many aspects of the representation theory are encoded in the combinatorial data of the hypergraph.

In the first half of this thesis, we show that the Auslander-Reiten quiver of a symmetric special biserial algebra is completely determined by its associated Brauer configuration. Specifically, we can determine the indecomposable modules and the irreducible morphisms belonging to any component of the Auslander-Reiten quiver using only information from the Brauer configuration. We also show the number of certain components and their precise size and shape is entirely determined by the Green walks along the Brauer configuration.

The second half of this thesis, comprising of the last two chapters, is a study on the representation type of symmetric special multiserial algebras. Unlike in the biserial case, not all of these algebras are tame. It is important to know if an algebra is tame or wild, since if it is wild, a classification of the indecomposable modules is considered to be hopeless. In this section of the thesis, we describe which symmetric special multiserial algebras are wild, which we present in terms of the Brauer configuration.

Acknowledgments

Amongst the many people I have to thank, my supervisor, Sibylle Schroll, must surely be the first. The attention and guidance she has given me over the last four years has been of immense help and of monumental importance to me. I must particularly give her my warmest gratitude for taking the time to read through many parts of this thesis and for offering her comments. I also extend my appreciation to the Department of Mathematics at the University of Leicester and the EPSRC for providing the financial support and the resources for my PhD studies.

I have had the pleasure of being surrounded by many great friends and family members over the last few years, all of whom have contributed to the person I am today, and have provided me with much welcome support during my studies. To list them all and provide the words needed to sufficiently express my gratitude towards them would perhaps be a thesis in itself. Thus, I can only mention a very special few.

To my mother, I say thank you for giving me both the drive to pursue my intellectual ambitions and for the constant loving support over all these years. Much of my mathematical ability and sense of self, I owe to you. To my father, I say thank you for encouraging my creativity early on and for sharing your wisdom with me throughout my young and adult life. My sister, Leela, deserves a mention also, for she has been a consistent friend and confidant all this time.

I would like to express my warm appreciation to my current and former housemates Liam Collard, Emilie Quinn and Molly Kerswill, as they have all made my time in Leicester that much more fun and enjoyable. I have come to think of Leicester as not just a place I live, but as home to me, and it is largely because of them that I have come to feel this way. No doubt, they have witnessed my somewhat stressed demeanour lately, and they have shown nothing but kindness and understanding in return. I have also been incredibly lucky to have known the people of the Criterion pub quiz team, many of whom are fellow PhD students. Wednesdays would just not have been the same without them all.

I would like to give a special mention to Kirsty Roulston. Whilst we may have only known each other for a relatively short amount of time, she has made the past few months that little bit brighter and happier for me.

Last, but by no means least, I feel compelled to mention my gratefulness and offer a very personal thank you to Laura Baines-Ball. It is difficult to say just how different things would have turned out had it not been for our meetings each Wednesday. However, it is entirely possible that I may not have made it this far without her.

Contents

List of Figures

Introduction

Within the study of the representation theory of finite dimensional algebras, one of the primary aims is to understand the module category of an algebra. To this end, representation theorists are particularly interested in the indecomposable modules of an algebra along with the morphisms between them. A complete classification of the indecomposable modules and a complete description of the space of module homomorphisms yields a complete understanding of the module category of finitely generated A-modules.

With regards to classifying the indecomposable modules over a finite dimensional algebra, the representation type of the algebra is of fundamental importance. Drozd's famous dichotomy ([21]) shows that an algebra can either be of tame or wild representation type (which becomes a trichotomy if one distinguishes between algebras that are of finite or infinite representation type). For tame algebras, the indecomposable modules in each dimension occur in a finite number of one-parameter families, meaning that there is at least some hope of a classification of the indecomposable modules. On the other hand, the representation theory of any wild algebra is at least as complicated as the representation theory of all finite dimensional algebras. Thus, a classification of the indecomposable modules of a wild algebra is often considered to be hopeless. It is therefore of tremendous importance to know beforehand whether an algebra is tame or wild if one aims to develop a detailed understanding of its representation theory.

For tame algebras, one method for calculating indecomposable modules lies with Auslander-Reiten theory. Auslander-Reiten theory was developed in [8], [9] and [10] by Auslander and Reiten. Given an indecomposable module M, one can calculate the Auslander-Reiten translate of M to calculate a new indecomposable module. In addition, the theory of almost split sequences and irreducible morphisms provides a method by which the morphisms between indecomposable modules can be calculated. One way of presenting this information is the Auslander-Reiten quiver of an algebra. The Auslander-Reiten quiver has indecomposable modules as vertices and irreducible morphisms as arrows. Thus, the Auslander-Reiten quiver of an algebra is essentially a way of presenting the module category of the algebra. This makes the Auslander-Reiten quiver a very powerful tool in representation theory and hence a source of great interest.

In this thesis, we are interested in the module category of a particular class of algebras known as special multiserial algebras. Contained in this class are special biserial algebras, which have been of great interest and study. The representation theory of special biserial algebras is well-understood. For example, special biserial algebras are of tame representation type ([18]). The indecomposable modules of special biserial algebras have been classified ([17], [57]) using the functorial filtration method due to Gel'fand and Ponomarev ([31]) and the morphisms between them have been studied in [17], [20], [41] and [53]. The Auslander-Reiten quiver of special biserial algebras is also well-understood ([17]), particularly for those that are self-injective ([24]) and symmetric ([15], [22], [29]).

Special biserial algebras have been instrumental to many classification problems regarding the representation type of an algebra. For example, they are used in the derived equivalence classification of tame self-injective algebras that are either periodic, standard, or of polynomial growth ([5], [52]). Special biserial algebras also play a role in the classification of blocks of group algebras of tame (and finite) representation type (see for example [23]).

A helpful tool in understanding the representation theory of symmetric special multiserial algebras is the notion of a Brauer configuration. A Brauer configuration is a decorated hypergraph with orientation – that is, a collection of vertices and connected polygons, with a cyclic ordering of the polygons at each vertex. Every Brauer configuration gives rise to a symmetric special multiserial algebra ([35]). Conversely, every symmetric special multiserial algebra can be associated to a Brauer configuration ([36]). This motivates an alternative name for symmetric special multiserial algebras, which some authors instead call Brauer configuration algebras. Brauer configurations algebras are particularly useful, as the representation theory of the algebra is encoded in the combinatorial data of the Brauer configuration.

If every polygon in the Brauer configuration is a 2-gon (or edge), then one obtains a graph called a Brauer graph and an associated algebra called a Brauer graph algebra. Brauer graph algebras coincide with the class of symmetric special biserial algebras ([48], [50]). Since Brauer graph algebras are special biserial, they are of tame representation type. It is also known that those of finite representation type are precisely the Brauer tree algebras, which have been of intense study ever since they arose from the study of the representation theory of finite groups. The definition of a Brauer tree orginated from the study of block algebras of finite groups with cyclic defect (see for example [14]). Brauer graph algebras have since been studied extensively by various authors (see for example in [1], [3], [32], [43], [46], [48]).

As with Brauer configurations, it is possible to read off some of the representation theory of a Brauer graph algebra from its underlying Brauer graph. For example, a useful tool in representation theory is the projective resolution of a module. However, projective resolutions in algebras are difficult to calculate in general. For Brauer graph algebras, one can avoid such calculations and easily read off the projective resolutions of certain modules from the Brauer graph. These are given by Green walks around the Brauer graph, which were first described in detail in [37] for Brauer trees and are shown to hold more generally in [48].

The underlying Brauer graph of a domestic symmetric special biserial algebra has been described in [16]. It is also known that the indecomposable non-projective modules of a Brauer graph algebra are given by either string modules or band modules ([57]). The irreducible morphisms between indecomposable modules are then given by adding or deleting hooks and cohooks to strings ([17], [53]).

With all of this in mind, one may be interested in investigating the Auslander-Reiten quiver of Brauer graph algebras. This is the precisely the aim of Chapter 2 of this thesis, where we investigate precisely what information about the Auslander-Reiten quiver of a Brauer graph algebra we can read off from its underlying Brauer graph.

There has already been extensive work on the Auslander-Reiten quiver of Brauer tree algebras. For example, a complete description of the Auslander-Reiten quiver of Brauer tree algebras, has been given in [29]. In [15], the location of the modules in the stable Auslander-Reiten quiver of a Brauer tree algebra has been described in terms of walks in the Brauer tree. However, the descriptions in both [15] and [29] do not address the case where the algebra is of infinite representation type, and thus, is associated to a Brauer graph that is not a Brauer tree.

In Chapter 2, we use the results of [17] to provide a constructive algorithm for reading off the indecomposable modules and irreducible morphisms of any given Auslander-Reiten component by using nothing other than the Brauer graph. However, if the algebra is representation-infinite, then the Auslander-Reiten quiver will consist of many different components, which can be of various different shapes and sizes. The general shapes for the Auslander-Reiten components of selfinjective special biserial algebras is known due to [24], but it is still possible to say more about the precise size, shape and number of certain components. It is shown towards the end of Chapter 2 that all of this information is determined by the Green walks along the Brauer graph. Essentially, this means the Brauer graph completely determines the Auslander-Reiten quiver of the algebra.

Chapters 3 and 4 investigate the broader class of symmetric special multiserial algebras, which were first introduced in [56] and later investigated in [34], [35] and [36]. They both generalise and contain the class of special biserial algebras, and similar to special biserial algebras, the representation theory is generally controlled by the uniserial modules over the algebra. Unlike biserial algebras though, most special multiserial algebras are wild. Symmetric radical cube zero algebras are a subclass of special multiserial algebras (as shown in [36]), and for these, a classification of finite, tame and wild algebras exists in [13]. However, such a classification does not currently exist for symmetric special multiserial algebras in general. It is precisely the aim of both Chapters 3 and 4 to provide a description of the symmetric special multiserial algebras that are wild.

The author of this thesis believes that the list of wild symmetric special multiserial algebras presented in Chapters 3 and 4 is complete, however a proof to this effect is currently a work in progress. Some work towards providing a complete classification of the tame symmetric special multiserial algebras is in Chapter 3, but there is one case which still needs to be addressed. It is therefore the intention of the author to provide a proof of the complete classification of finite, tame and wild symmetric special multiserial algebras in a forthcoming paper.

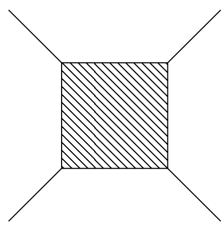
The List of Wild Symmetric Special Multiserial Algebras

The following theorem provides the list of symmetric special multiserial algebras proven to be wild in Chapters 3 and 4. The proof of this theorem is at the end of Chapter 4 in Section 4.3. We refer the reader to Chapter 1 for the relevant preliminaries towards the theorem.

Main Theorem. *Let A be a symmetric special multiserial algebra corresponding to a Brauer configuration χ. Suppose χ satisfies any of the following.*

(a) χ contains an n-gon such that $n > 4$.

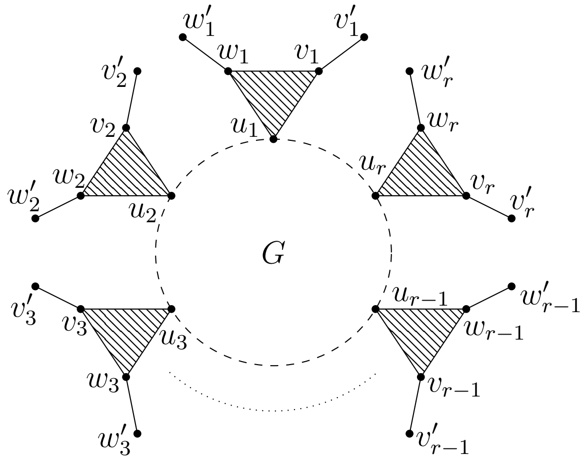
(b) χ contains an n-gon such that $n = 4$ and χ is not of the form



where every vertex has multiplicity one.

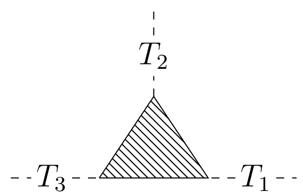
(c) χ contains an n-gon such that $n = 3$ and χ is not of one of the two forms

(i)



where G is a Brauer graph connecting the (not necessarily distinct) vertices $u_1, \dots, u_r$ and $\mathfrak{e}_{v_i} = \mathfrak{e}_{v_i'} = \mathfrak{e}_{w_i} = \mathfrak{e}_{w_i'} = 1$ for all i; or

(ii)



where T_1, T_2 and T_3 are distinct multiplicity-free Brauer trees containing m_1, m_2 and m_3 polygons respectively such that the values of the triple (m_1, m_2, m_3) conform to a column of the following table.

m_1	1	1	1	1	1	2
m_2	2	2	2	2	3	2
m_3	2	3	4	5	3	2

Then A is wild.

We actually show that the converse to (b) and (c)(ii) is also true in Chapter 3. Namely, we show the following.

Theorem. *If A is a tame symmetric special multiserial algebra, then A is at most quadserial. In particular, if A is quadserial and the underlying Brauer configuration of A is of the form given in Case (b) of the Main Theorem, then A is tame.*

Theorem. *Suppose A is a symmetric special triserial algebra whose underlying Brauer configuration is of the form given in Case (c)(ii) of the Main Theorem. Then A is tame.*

This falls short of a complete classification of tame and wild symmetric special multiserial algebras, as we do not show the converse to (c)(i) in the Main Theorem.

Conjecture. *Suppose A is a symmetric special triserial algebra whose underlying Brauer configuration is of the form given in Case (c)(i) of the Main Theorem. Then A is tame.*

Chapter 1

Preliminaries and Notation

We assume the reader has a basic familiarity with the representation theory of algebras, including a knowledge of modules over algebras and representations of quivers. We direct the reader to [7, Chapters I, II, III] and [12] for further reading if this is not the case.

Throughout this thesis, we let K be an algebraically closed field and $Q = (Q_0, Q_1)$ be a finite connected quiver with vertex set Q_0 and arrow set Q_1. We let I be an admissible ideal of the path algebra KQ such that KQ/I is a basic finite dimensional algebra. For any K-algebra A, we denote by $\operatorname{Mod} A$ the module category of A, we denote by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely generated A-modules, and (in the case where A is infinite dimensional) we denote by $\operatorname{fin} A$ the full subcategory of $\operatorname{mod} A$ consisting of finite dimensional A-modules. Unless specified otherwise, all modules considered are right modules, and thus, we typically read paths in a quiver from left to right and compose morphisms between A-modules on the left. We freely use the fact throughout that for a path algebra $A = KQ/I$, the category of K-representations of Q is equivalent to the category of (right) A-modules. Thus, we frequently think of a K-representation as a module and vice versa. We adopt the notation that we compose linear maps in a K-representation on the left. Thus, any linear map $\varphi : K^m \to K^n$ in a K-representation can be written as an $n \times m$ matrix.

By the standard duality D, we mean the standard K-linear dual functor $D =$

$\mathrm{Hom}_K(-, K)$, which assigns to each right A-module M the left A-module $DM = \mathrm{Hom}_K(M, K)$ and to each morphism $h : M \to N$ the morphism $Dh = \mathrm{Hom}_K(h, K) : DN \to DM$. Recall that an algebra A is symmetric if $A \cong DA$ as an A-A-bimodule. Most algebras considered in this thesis are symmetric.

For a right (resp. left) A-module M, we denote $\operatorname{rad} M$ to be the radical of M. That is, the intersection of all maximal right (resp. left) submodules of M. The top of M is given by $\operatorname{top} M = M/\operatorname{rad} M$. By $\operatorname{soc} M$, we mean the socle of M. That is, $\operatorname{soc} M$ is the submodule of M generated by all simple right (resp. left) submodules of M.

Given a module $M \in \operatorname{mod} A$, the projective module $P(M)$ is given by the projective cover of M. Recall that the syzygy operator Ω assigns to a module $M \in \operatorname{mod} A$ the kernel of the projective cover $P(M) \to M$. By $\Omega^n(M)$, we mean the n-th syzygy module of M. Similarly, we mean by $I(M)$ the injective module given by the injective envelope of M and $\Omega^{-1}(M)$ the cosyzygy of M, which is given by the cokernel of the injective envelope $M \to I(M)$. We denote by $\Omega^{-n}(M)$ the n-th cosyzygy of M.

Given an algebra $A = KQ/I$, for any vertex $x \in Q_0$, we denote by $S(x)$ the simple module corresponding to x. By $P(x)$ and $I(x)$ we denote the indecomposable projective and indecomposable injective modules respectively corresponding to the vertex $x \in Q_0$. Specifically, $P(x) = P(S(x))$ and $I(x) = I(S(x))$. In a symmetric algebra, $P(x) = I(x)$ for all $x \in Q_0$, and so we will often use $P(x)$ only to refer to the projective-injective corresponding to $x \in Q_0$.

Given an arrow $\alpha \in Q_1$ we denote the vertex of Q at the source of α by $s(\alpha)$ and the vertex of Q at the target of α by $e(\alpha)$. For any vertex $x \in Q_0$, we denote by ε_x the stationary path in KQ at the vertex x. Thus, for any basic path algebra $A = KQ/I$, the set $\{\varepsilon_x : x \in Q_0\}$ forms a complete set of primitive orthogonal idempotents of A.

We will denote Dynkin ADE diagrams (and quivers that are orientations of Dynkin ADE diagrams) by blackboard bold characters $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_p$. Extended Dynkin ADE diagrams and Euclidean quivers will be denoted by the corresponding blackboard bold characters $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$ and $\widetilde{\mathbb{E}}_p$. The shapes of these quivers may been found in [39].

1.1 Representation Theory Preliminaries

1.1.1 Auslander-Reiten Theory

Understanding the indecomposable modules of an algebra and the morphisms between them is an imporatant aim in representation theory, as it provides a useful insight into the structure of the module category of an algebra. Auslander-Reiten theory, which was first introduced by Maurice Auslander and Idun Reiten in [8], [9] and [10], provides the tools by which we can calculate indecomposable modules and irreducible morphisms for finite dimensional algebras via almost split sequences. We provide a brief review of their work.

Almost Split Sequences

Definition 1.1.1 ([7], IV.1.1)**.** Let A be a finite dimensional algebra over an algebraically closed field K. Let $L, M, N \in \operatorname{mod} A$.

(a) A morphism $g : M \to N$ is called *right almost split* if g does not admit a right inverse and for every morphism $v : Y \to N$ that does not admit a right inverse, there exists $v' : Y \to M$ such that the following diagram is commutative

$$\begin{tikzcd} & Y \arrow[dl, "v'"'] \arrow[d, "v"] \\ M \arrow[r, "g"] & N \end{tikzcd} .$$

(b) A morphism $f : L \to M$ is called *left almost split* if f does not admit a left inverse and for every morphism $u : L \to X$ that does not admit a left inverse, there exists $u' : M \to X$ such that the following diagram is commutative

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[dl, "u'"] \\ X & \end{tikzcd} .$$

(c) A morphism $g : M \to N$ is called *right minimal* if every morphism $g' \in \operatorname{End} M$ such that $gg' = g$ is an automorphism. Similarly, a morphism $f : L \to M$ is called *left minimal* if every morphism $f' \in \operatorname{End} M$ such that $f'f = f$ is an automorphism.

(d) A morphism is said to be *right (resp. left) minimal almost split* if it is both right (resp. left) minimal and right (resp. left) almost split.

Definition 1.1.2 ([11], V.1, p. 144)**.** We say an exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an *almost split sequence* (or *Auslander-Reiten sequence*) if f is left almost split and g is right almost split.

Definition 1.1.3 ([7], IV.1.4)**.** A morphism $g : M \to N$ in $\operatorname{mod} A$ is called an *irreducible morphism* if

(i) g does not admit a left or right inverse and

(ii) if $g = g_2 g_1$ for some $g_1 : M \to X$ and $g_2 : X \to N$ then either g_1 admits a right inverse or g_2 admits a left inverse.

$$\begin{tikzcd} & X \arrow[dr, "g_2"] & \\ M \arrow[ur, "g_1"] \arrow[rr, "g"] & & N \end{tikzcd}$$

Irreducible morphisms are related to minimal almost split morphisms in the following way.

Proposition 1.1.4 ([11], V.5.3)**.** *Let L and N be indecomposable A-modules. Then:*

(i) A morphism $g : M \to N$ is irreducible if and only if there exists a morphism $g' : M' \to N$ such that the induced homomorphism $(g, g') : M \oplus M' \to N$ is right minimal almost split.

(ii) *A morphism $f : L \to M$ is irreducible if and only if there exists a morphism $f' : L \to M'$ such that the induced homomorphism $\binom{f}{f'} : L \to M \oplus M'$ is left minimal almost split.*

Proposition 1.1.5 ([7], [11]). *For an exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0\ ,$$

the following are equivalent:

(a) *The sequence is an almost split sequence.*

(b) *f is left minimal almost split.*

(c) *g is right minimal almost split.*

(d) *L is indecomposable and f is left almost split.*

(e) *N is indecomposable and g is right almost split.*

(f) *L and N are indecomposable, and f and g are irreducible.*

Almost split sequences, if they exist, are unique in the following sense.

Proposition 1.1.6 ([11], V.1.16). *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

and

$$0 \longrightarrow L' \xrightarrow{f'} M' \xrightarrow{g'} N' \longrightarrow 0$$

be two almost split sequences. Then the following are equivalent:

(i) *The two sequences are isomorphic.*

(ii) *$L \cong L'$.*

(iii) *$N \cong N'$.*

Due to our main focus on symmetric algebras, it is of interest to know the form of the almost split sequences involving an indecomposable projective-injective module.

Proposition 1.1.7 ([11], V.5.5)**.** *Let P be an indecomposable projective-injective A-module. Then there exists an almost split sequence*

$$0 \longrightarrow \operatorname{rad} P \longrightarrow \operatorname{rad} P/\operatorname{soc} P \oplus P \longrightarrow P/\operatorname{soc} P \longrightarrow 0\ .$$

The Auslander-Reiten Translate

Almost split sequences can be constructed using the Auslander-Reiten translate, which we will define here. For a finite dimensional algebra A, first consider the A-dual functor $(-)^t = \operatorname{Hom}_A(-, A)$. Then consider the minimal projective presentation of a module $M \in \operatorname{mod} A$

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0\ .$$

By applying the above functor, we obtain an exact sequence

$$0 \longrightarrow M \longrightarrow P_0^t \xrightarrow{p_0^t} P_1^t \xrightarrow{p_1^t} \operatorname{Tr} M \longrightarrow 0\ ,$$

where $\operatorname{Tr} M$ is called the transpose of M and is given by $\operatorname{Coker} p_1^t$. Note that the transpose Tr has the following property.

Proposition 1.1.8 ([7], IV.2.1)**.** *M is projective if and only if $\operatorname{Tr} M = 0$. If M is not projective then $\operatorname{Tr} M$ is indecomposable and $\operatorname{Tr}(\operatorname{Tr} M) \cong M$.*

If we consider the composition of Tr with the standard K-duality functor $D = \operatorname{Hom}_K(-, K)$, then we get the following result.

Proposition 1.1.9 ([11], IV.1.9)**.** *The composition $D\operatorname{Tr} : \underline{\operatorname{mod}}\, A \to \overline{\operatorname{mod}}\, A$ is an equivalence of categories with inverse equivalence $\operatorname{Tr} D : \overline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, A$.*

This motivates the following definition.

Definition 1.1.10 ([7], IV.2.3)**.** We define the *Auslander-Reiten translate* to be the composition $\tau = D\,\mathrm{Tr}$. We define its inverse to be $\tau^{-1} = \mathrm{Tr}\,D$.

Remark 1.1.11. If the algebra A is weakly symmetric, then one can easily show that $\tau = \Omega^2$.

The Auslander-Reiten translations are of central importance due to the following.

Proposition 1.1.12 ([7], IV.3.1)**.**

(a) For any indecomposable non-projective module M, there exists an almost split sequence

$$0 \longrightarrow \tau M \longrightarrow B \longrightarrow M \longrightarrow 0$$

in $\mathrm{mod}\,A$*.*

(b) For any indecomposable non-injective module N, there exists an almost split sequence

$$0 \longrightarrow N \longrightarrow C \longrightarrow \tau^{-1}N \longrightarrow 0$$

in $\mathrm{mod}\,A$*.*

In view of Proposition 1.1.6, we then have the following:

Proposition 1.1.13 ([11], V.1.14)**.** *For an exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0\,,$$

the following are equivalent:

(i) The sequence is an almost split sequence.

(ii) L is isomorphic to τN and f is left almost split.

(iii) N is isomorphic to $\tau^{-1}L$ and g is right almost split.

The Auslander-Reiten Quiver

It is useful to present the information regarding the indecomposable modules of a finite dimensional algebra and the morphisms between them in the form of a quiver.

Definition 1.1.14 ([7], IV.4.6)**.** Let A be a basic and connected finite dimensional algebra. The *Auslander-Reiten quiver* of A is defined to be the quiver Γ_A constructed as follows.

(i) The distinct vertices of Γ_A correspond to the distinct isomorphism classes of indecomposable modules in $\operatorname{mod} A$.

(ii) There exists an arrow $[M] \to [N]$ in Γ_A if and only if there exists an irreducible morphism $M \to N$.

The Auslander-Reiten quiver of an algebra is locally finite at the vertices (that is, there are a finite number of arrows of source or target any given vertex in Γ_A). There are no loops in Γ_A, however it is possible for there to exist multiple arrows between vertices in Γ_A, which is usually expressed by placing a valuation on the arrows of the quiver. This can happen whenever there exists an almost split sequence of the form

$$0 \longrightarrow \tau N \longrightarrow L \oplus M^n \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod} A$, where M is indecomposable and $n > 1$. This rarely happens in the Auslander-Reiten quiver of the algebras we consider for this thesis (symmetric special biserial), and so we do not go into the details of placing valuations on arrows in Γ_A. We instead direct the reader to consult [7] and [11] if they would like further details on the construction of the Auslander-Reiten quiver with valuations.

The Auslander-Reiten quiver actually has the additional structure of being a translation quiver, with the translation defined by the Auslander-Reiten translate τ (see [12, 4.15] for details). Thus, for each almost split sequence

$$0 \longrightarrow \tau N \longrightarrow \bigoplus_{i=1}^{r} M_i \longrightarrow N \longrightarrow 0 \,,$$

where the M_i are pairwise non-isomorphic, we have a mesh in the Auslander-Reiten quiver of the following form.

$$\begin{tikzcd}[row sep=tiny]
 & M_1 \arrow[ddr] & \\
 & M_2 \arrow[dr] & \\
\tau N \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots & N \\
 & M_{r-1} \arrow[ur] & \\
 & M_r \arrow[uur] &
\end{tikzcd}$$

For self-injective algebras, it is often convenient to look at the *stable Auslander-Reiten quiver*, denoted by ${}_s\Gamma_A$. This is the full subquiver of Γ_A given by removing projective-injective objects.

Definition 1.1.15. Define an equivalence relation between indecomposable modules $M, N \operatorname{mod} A$ by $M \sim N$ if and only if there exists a finite (undirected) walk along the arrows of Γ_A between M and N. We call the equivalence classes under the relation $\sim$ the *Auslander-Reiten components* of Γ_A.

It is easy to see that if an algebra A is representation-finite then the Auslander-Reiten quiver of A consists of a single component. However, if the algebra is instead representation-infinite, then Γ_A may consist of many different components, which could be of various different shapes (determined by the Auslander-Reiten translate τ). We present a few examples of possible shapes of Auslander-Reiten components here, which will become relevant later in the thesis.

Examples 1.1.16.

(a) Components of the shape $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ are called *tubes of rank* n and are of the form given in Figure 1.1(a), where the modules along the dashed lines are identified. The modules $M_1, \ldots, M_n$ above are said to sit at the *mouth of the tube*.

(b) Components of the shape $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ are of the form given in Figure 1.1(b), where the modules along the dashed lines are identified at the vertex N and along its translations $\tau^n N$. Informally, one can think of a $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ component as being

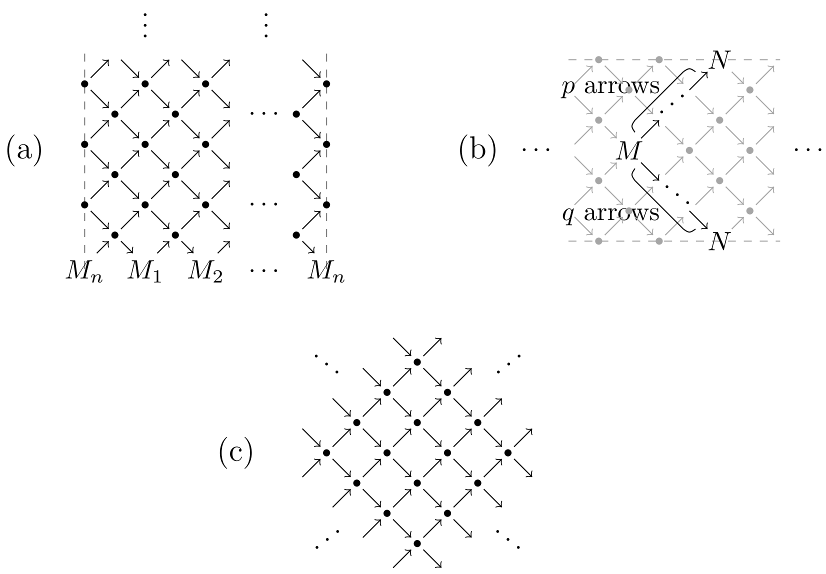


Figure 1.1: Examples of possible shapes of Auslander-Reiten components.

a horizontal tube with a potential τ-shift along the line of identification (if $p \neq q$).

(c) Components of the shape $\mathbb{Z}\mathbb{A}_\infty^\infty$ are of the form given in Figure 1.1(c).

1.1.2 Tilting Complexes

Denote by $\operatorname{proj} A$ the full subcategory of $\operatorname{mod} A$ consisting of projective A-modules. By $K^b(\operatorname{proj} A)$, we mean the bounded homotopy category of chain complexes over $\operatorname{proj} A$. That is, the category whose objects are bounded chain complexes over $\operatorname{proj} A$ and whose morphisms are chain maps modulo homotopy.

We call an object in $K^b(\operatorname{proj} A)$ that has a non-zero term in at most one degree a *stalk complex*. Given an object $T \in K^b(\operatorname{proj} A)$, denote by $\operatorname{add}(T)$ the full subcategory of $K^b(\operatorname{proj} A)$ consisting of direct summands of direct sums of copies of T. We call an object $T \in K^b(\operatorname{proj} A)$ a *tilting complex* if $\operatorname{Hom}(T, T[n]) = 0$ for all $n \neq 0$ and $\operatorname{add}(T)$ generates $K^b(\operatorname{proj} A)$ as a triangulated category.

Theorem 1.1.17 ([46], Theorem 1.1)**.** *Let A and B be finite dimensional algebras. Then A and B are derived equivalent if and only if $B \cong \operatorname{End}_{K^b(\operatorname{proj} A)}(T)$ for some tilting complex $T \in K^b(\operatorname{proj} A)$.*

An example of a tilting complex in a symmetric algebra A is an Okuyama-Rickard complex (c.f. [2], [4], [44]). Let $\varepsilon_1, \ldots, \varepsilon_n$ be complete set of primitive orthogonal idempotents of A. Let E' be a subset of $E = \{1, \ldots, n\}$ and let $\varepsilon = \sum_{i \in E'} \varepsilon_i$. Define T_i to be either the stalk complex with degree zero term $\varepsilon_i A$ if $i \in E'$ or the complex

$$0 \longrightarrow P(\varepsilon_i A \varepsilon A) \xrightarrow{f} \varepsilon_i A \longrightarrow 0$$

if $i \notin E'$, where $P(\varepsilon_i A \varepsilon A)$ is in degree zero and $P(\varepsilon_i A \varepsilon A) \xrightarrow{f} \varepsilon_i A$ is the minimal projective presentation of $\varepsilon_i A / \varepsilon_i A \varepsilon A$. Then the complex $T = \bigoplus_{i \in E} T_i$ is called the *Okuyama-Rickard* tilting complex with respect to E'.

1.1.3 The Representation Type of an Algebra

The representation type of an algebra is of primary importance to representation theorists, as it essentially describes how complicated it is to classify the indecomposable modules over the algebra.

Definition 1.1.18 ([7]). A finite dimensional K-algebra is said to be of *finite representation type* (or is said to be *representation-finite*) if the number of isomorphism classes of indecomposable modules in $\operatorname{mod} A$ is finite. Otherwise A is said to be of *infinite representation type* (or is said to be *representation-infinite*).

Example 1.1.19. Let K be an algebraically closed field and let Q be an orientation of one of the following Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_p$ ($p \in \{6, 7, 8\}$). Then by a theorem of Gabriel ([28]), the (hereditary) path algebra KQ is representation-finite. By [55], the trivial extension of KQ is also representation-finite (see Section 1.5 for the definition of the trivial extension of an algebra).

Classifying the indecomposable modules of a representation-finite algebra is considered to be an easy problem, and thus, the representation theory of representation-finite algebras is well understood. On the opposite end of the spectrum, we have the algebras of wild representation type. To define wild representation type, we need to introduce the notion of a representation embedding, as defined in [51].

Definition 1.1.20 ([51], XIX.1)**.** Let A and B be (not necessarily finite-dimensional) algebras over K and let $\mathcal{A} \subseteq \operatorname{Mod} A$ and $\mathcal{B} \subseteq \operatorname{Mod} B$ be additive full exact subcategories that are closed under direct summands. Let $F : \mathcal{A} \to \mathcal{B}$ be a K-linear functor.

(a) We say F *respects isomorphism classes* if for any modules $M, N \in \mathcal{A}$, we have $FM \cong FN \Rightarrow M \cong N$.

(b) We say F is a *representation embedding* if it is exact, respects isomorphism classes, and maps indecomposable modules in $\mathcal{A}$ to indecomposable modules in $\mathcal{B}$.

Given finite dimensional algebras A and B, a representation embedding $F : \operatorname{mod} A \to \operatorname{mod} B$ induces an injection from the set of isomorphism classes of indecomposable modules in $\operatorname{mod} A$ to the set of isomorphism classes of indecomposable modules in $\operatorname{mod} B$. Thus, the classification of indecomposable modules in $\operatorname{mod} B$ is at least as complicated as the classification of indecomposable modules in $\operatorname{mod} A$. The following observation is also useful.

Lemma 1.1.21 ([51], XIX.1)**.** *Let A and B be (not necessarily finite-dimensional) algebras over K and let $\mathcal{A} \subseteq \operatorname{Mod} A$ be an additive full exact subcategory that is closed under direct summands. Let $F : \mathcal{A} \to \operatorname{Mod} B$ be a K-linear functor. If F is exact and fully faithful, then F is a representation embedding.*

We will call a representation embedding that is fully faithful a *strict representation embedding.*

Definition 1.1.22 ([51], XIX.1)**.** Let B be a finite dimensional K-algebra.

(a) B is said to be of *wild representation type* (or shortly, is said to be *wild*) if for every finite dimensional K-algebra A, there exists a representation embedding $F : \operatorname{mod} A \to \operatorname{mod} B$.

(b) B is said to be of *strictly wild representation type* (or shortly, is said to be *strictly wild*) if for every finite dimensional K-algebra A, there exists an exact fully faithful K-linear functor $F : \operatorname{mod} A \to \operatorname{mod} B$.

Essentially, the above definition says that the problem of classifying the indecomposable modules in a wild algebra contains the problem of classifying the indecomposable modules of all finite dimensional algebras. Thus, gaining a complete understanding of the representation theory of a wild algebra is often considered to be hopeless. Note that an algebra that is strictly wild is wild, but the converse is not necessarily true (for example, wild local algebras, as shown in [38]). We may also observe that for some algebra B of unknown representation type and for some algebra A of wild representation type, it follows that if there exists a representation embedding $F : \operatorname{mod} A \to \operatorname{mod} B$, then B is also wild.

Example 1.1.23. Any acyclic (hereditary) path algebra KQ such that Q is not an orientation of $\mathbb{A}_n$, $\widetilde{\mathbb{A}}_n$, $\mathbb{D}_n$, $\widetilde{\mathbb{D}}_n$, $\mathbb{E}_p$ and $\widetilde{\mathbb{E}}_p$ ($p \in \{6, 7, 8\}$) is strictly wild. There exists a fully faithful representation embedding from KQ into the trivial extension of KQ, so this is also strictly wild.

Consider the algebra $K\langle a_1, a_2\rangle$ and recall that this is the path algebra of the following quiver.

$$Q : a_1 \circlearrowright \bullet \circlearrowleft a_2$$

Whilst this algebra is not finite-dimensional, one can still construct a (strict) representation embedding $\operatorname{Mod} A \to \operatorname{Mod} K\langle a_1, a_2\rangle$ for any finitely generated K-algebra A, which restricts to a (strict) representation embedding $\operatorname{fin} A \to \operatorname{fin} K\langle a_1, a_2\rangle$. This motivates the following alternative and important characterisation of wild (and strictly wild) algebras.

Theorem 1.1.24 ([51], XIX.1)**.** *Let A be a finite dimensional algebra.*

(a) A is wild if and only if there exists a representation embedding functor $F : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$.

(b) A is strictly wild if and only if there exists an exact fully faithful functor $F : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$.

We present here a test for wild algebras due to Crawley-Boevey, where by almost all, we mean all but finitely many.

Theorem 1.1.25 ([19],Theorem D)**.** *Let K be an algebraically closed field. If A is a tame K-algebra, then for each dimension d, $M \cong \tau M$ for almost all indecomposable A-modules of dimension d.*

For some representation-infinite algebras, there may still be some hope in classifying the indecomposable modules over the algebra. This brings us to the definition of tame representation type.

Definition 1.1.26 ([51], XIX.3)**.** A finite dimensional algebra A over an algebraically closed field K is said to be of *tame representation type* (or shortly said to be *tame*) if for each integer $d \geq 1$, there exists a finite number of $K[a]$-A-bimodules $M_1, \ldots, M_{n_d}$ that are finitely generated and free left $K[a]$-modules such that almost all indecomposable A-modules of dimension d are isomorphic to a module of the form $S \otimes_{K[a]} M_i$ for some i and some simple $K[a]$-module S.

Recall that $K[a]$ is the path algebra of the quiver consisting of one vertex and a single arrow a, which is a loop. Also recall that the simple $K[a]$-modules are precisely the modules of dimension 1 with quiver representation (K, φ), where $\varphi : K \to K$ is a linear map such that $\varphi(x) = \lambda x$ for some $\lambda \in K$. Thus, the above definition essentially says that in a tame algebra, almost all indecomposable modules (up to isomorphism) in each dimension occur in a finite number of 1-parameter families.

Remark 1.1.27. Some authors define tame representation type to include finite representation type. We adopt this notation for this thesis.

We now present Drozd's famous tame-wild dichotomy (which becomes a trichotomy if one distinguishes between finite and tame representation type).

Theorem 1.1.28 ([21])**.** *A finite-dimensional algebra is either of tame representation type or wild representation type, but not both.*

It is sometimes of interest to count the number of $K[a]$-A-bimodules in each dimension of a tame algebra A.

Definition 1.1.29 ([51], XIX.3)**.** Let A be an algebra over an algebraically closed field K. For each integer $d \geq 1$, let $\{M_1, \ldots, M_{n_d}\}$ be a set of $K[a]$-A-bimodules from Definition 1.1.26 such that the integer n_d is minimal. Let $\mu_A : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ be a function defined by $\mu_A(d) = n_d$.

(a) We say A is *domestic* if there exists an integer $m \geq 0$ such that $\mu_A(d) \leq m$ for all $d \geq 1$. Specifically, we say A is m-domestic if m is the minimal value such that $\mu_A(d) \leq m$ for all $d \geq 1$.

(b) We say A is of *polynomial growth* if there exists an integer $m \geq 1$ such that $\mu_A(d) \leq d^m$.

All domestic algebras are of polynomial growth.

Example 1.1.30 ([6])**.** The path algebra KQ for any algebraically closed field K and any Euclidean quiver Q of the form $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_p$ ($p \in \{6, 7, 8\}$) is tame (and in fact, domestic). The trivial extension of KQ is also domestic.

We finish this section with an important theorem.

Theorem 1.1.31 ([42],[46])**.** *Let A and B be derived equivalent selfinjective K-algebras. Then A and B have the same representation type.*

1.2 Special Multiserial Algebras

The study of multiserial algebras originates from Nakayama's generalised uniserial algebras, which were later generalised to the class of biserial algebras by Tachikawa ([54]) and Fuller ([27]). In 1983, Skowroński and Waschbüsch ([53]) discussed the subclass of special biserial algebras, which have since been studied extensively. The broader class of special multiserial algebras – the main focus of the third chapter of this thesis – were first introduced in [56] and later investigated in [34], [35] and [36]. They both generalise and contain the class of special biserial algebras.

Central to the definition of a multiserial algebra is the definition of a uniserial module.

Definition 1.2.1 ([7], V.2)**.** A left or right A-module M is called *uniserial* if $\operatorname{rad}^i(M)/\operatorname{rad}^{i+1}(M)$ is simple or zero for all i.

Definition 1.2.2 ([36])**.** Let A be a finite dimensional algebra. We say a left or right A-module is *multiserial* if $\mathrm{rad}(M)$ can be written as a sum of uniserial modules $U_1, \dots, U_n$ such that $U_i \cap U_j$ is simple or zero for all $i \neq j$. We say that an algebra is *multiserial* if A is multiserial as a left and right A-module.

Definition 1.2.3. We call a multiserial algebra A an *n-serial* algebra if for every left or right indecomposable projective A-module P, the module $\mathrm{rad}(P)$ is a sum of at most n uniserial modules $U_1, \dots, U_n$ such that $U_i \cap U_j$ is simple or zero for all $i \neq j$. In particular, if A is an n-serial algebra for $n \in \{1, 2, 3, 4\}$, then we say that A is *uniserial*, *biserial*, *triserial* or *quadserial* respectively.

Biserial algebras are of particular interest of study, as a result of the following.

Theorem 1.2.4 ([18])**.** *Let A be a biserial algebra. Then A is tame.*

Thus, there is at least some hope of classifying the indecomposable modules over a biserial algebra. To date, there is no complete classification for all biserial algebras, but due to the functorial filtration method of Gel'fand and Ponomarev in [31], there is a classification of the indecomposable modules over an algebra that is special biserial ([17], [57]), which we define below. These are given by string and band modules (defined in Section 1.6). The generalisation of the above statement to the multiserial case is not true, since there are numerous examples of multiserial algebras that are wild – indeed, many wild hereditary algebras are multiserial.

In this thesis, we are particularly interested in the subclass of mutliserial algebras known as special multiserial algebras.

Definition 1.2.5 ([36])**.** We say that a finite dimensional algebra A is *special multiserial* if it is Morita equivalent to a quotient KQ/I of a path algebra KQ by an admissible ideal I such that the following property holds.

(S1) For any arrow $\alpha \in Q_1$, there exists at most one arrow $\beta \in Q_1$ and at most one arrow $\gamma \in Q_1$ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Note that special multiserial algebras are multiserial algebras ([36, Corollary 2.4]). The definition of a special biserial/triserial/quadserial algebra is similar.

Definition 1.2.6. We say that a finite dimensional algebra A is *special n-serial* if it is Morita equivalent to a quotient KQ/I of a path algebra KQ by an admissible ideal I which satisfies property (S1), along with the following additional property.

(S2) For any vertex $x \in Q_0$, there are at most n arrows in Q_1 of source x and at most n arrows in Q_1 of target x.

An algebra that is special 2-serial, 3-serial or 4-serial is called a *special biserial*, *triserial* or *quadserial* algebra, respectively.

The definition of special biserial algebras is due to Skowroński and Waschbüsch in [53], where it is shown that special biserial algebras are biserial algebras.

An important class of examples of special multiserial algebras is the subclass of gentle algebras and almost gentle algebras. These are defined as follows.

Definition 1.2.7 ([34])**.** We say that a finite dimensional algebra A is *almost gentle* if it is Morita equivalent to a quotient KQ/I of a path algebra KQ by an admissible ideal I which satisfies property (S1), along with the following additional property.

(G1) I is generated by paths of length two.

Definition 1.2.8 ([45])**.** We say that a finite dimensional algebra A is *gentle* if it is Morita equivalent to a quotient KQ/I of a path algebra KQ by an admissible ideal I which satisfies (S1), (S2) and (G1), along with the following additional property.

(G2) For any arrow $\alpha \in Q_1$, there exists at most one arrow $\beta \in Q_1$ and at most one arrow $\gamma \in Q_1$ such that $\alpha\beta \in I$ and $\gamma\alpha \in I$.

Trivially, one can see that the class of almost gentle algebras contains the class of gentle algebras.

1.3 Locally Embedded Configurations

Configurations (also known as hypergraphs) are a generalisation of graphs. In this thesis, we will be using configurations and graphs extensively, and thus, we will need to review the relevant theory and establish a set of notation which we will use throughout. Many of the ideas presented in this section are based on the work of Green and Schroll in [35]. To define a configuration, we need the notion of a multiset, which we recall here.

Definition 1.3.1. A *multiset* is a pair $(U, \mathfrak{e})$ such that U is a set and $\mathfrak{e} : U \to \mathbb{Z}_{>0}$ is a multiplicity function.

An (unordered) multiset is a generalisation of the concept of an (unordered) set in the sense that a multiset can be viewed as a collection of objects where repetitions are allowed. Thus, we will write a multiset $(\{u_1, \ldots, u_n\}, \mathfrak{e})$ as

$$\{\underbrace{u_1, \ldots, u_1}_{\mathfrak{e}(u_1) \text{ times}}, \ldots, \underbrace{u_n, \ldots, u_n}_{\mathfrak{e}(u_n) \text{ times}}\}.$$

We have the following further definitions, which generalise the relevant notions for sets.

Definition 1.3.2.

(a) The *cardinality* of a multiset $(\{u_1, \ldots, u_n\}, \mathfrak{e})$ is the integer $|(\{u_1, \ldots, u_n\}, \mathfrak{e})| = \sum \mathfrak{e}(u_i)$.

(b) A *submultiset* $(U', \mathfrak{e}')$ of a multiset $(U, \mathfrak{e})$ is a multiset such that $U' \subseteq U$ and $\mathfrak{e}' \leq \mathfrak{e}$.

(c) The *power set* of a multiset $(U, \mathfrak{e})$ is the collection of all possible submultisets of $(U, \mathfrak{e})$, which we denote by $\mathcal{P}(U, \mathfrak{e})$.

We can now proceed with the definition of a configuration

Definition 1.3.3. A *configuration* is a pair $\chi = (\chi_0, \chi_1)$, where χ_0 is a finite set of *vertices* of χ and χ_1 is a finite collection of finite multisets x_i whose elements are vertices in χ_0. We require that $|x_i| \geq 2$ for all $x_i \in \chi_1$. We call a multiset $x \in \chi_1$ a *polygon* of χ. Specifically, we call $x \in \chi_1$ an n-gon if $|x| = n$.

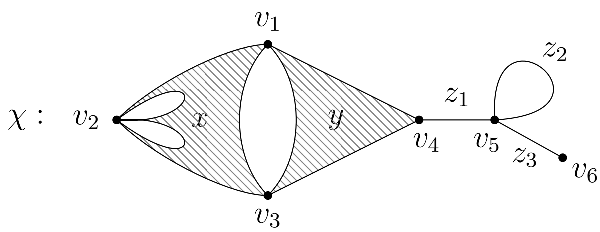
Informally, one can realise a configuration as a generalisation of a graph, where instead of vertices and connected edges, we have vertices and connected polygons.

We typically realise 2-gons as edges in the configuration. If a configuration χ consists entirely of 2-gons, then χ is indeed a graph. We have additional terminology regarding the polygons of a configuration. The following definitions generalise the notions of loops and multiple edges in a graph.

Definition 1.3.4. A polygon x in a configuration χ is said to be *self-folded* if there exists a vertex $v \in x$ such that v occurs more than once in x. In particular, we say that x is *self-folded at the vertex* v. Specifically, if there are m occurrences of v in x, then we say that x is m-self-folded at v. A self-folded 2-gon is called a *loop*.

Definition 1.3.5. Two vertices u and v in a configuration χ are said to have *multiple polygons* between them if there exist at least two polygons $x, y \in \chi_1$ such that $u, v \in x$ and $u, v \in y$.

Example 1.3.6. Let $\chi_0 = \{v_1, \ldots, v_6\}$ and define multisets $x = \{v_1, v_2, v_2, v_2, v_3\}$, $y = \{v_1, v_3, v_4\}$, $z_1 = \{v_4, v_5\}$, $z_2 = \{v_5, v_5\}$ and $z_3 = \{v_5, v_6\}$. Then $\chi = (\chi_0, \chi_1)$, where $\chi_1 = \{x, y, z_1, z_2, z_3\}$, is an example of a configuration. Geometrically, we may present this as follows.



Here, x is a 5-gon that is 3-self-folded at the vertex v_2 and y is a 3-gon. The polygons z_1, z_2 and z_3 are 2-gons (and hence edges in χ). In particular, z_2 is a loop. We can also see that there are multiple polygons between vertices v_1 and v_3.

Let y be a polygon in a configuration χ. We aim to construct a set $\mathcal{G}_y$ from the multiset y. For each vertex $u \in y$ such that u occurs precisely once in y, there exists an element $y^u \in \mathcal{G}_y$. For each $u \in y$ that has precisely $m > 1$ occurances in the multiset y, there exist elements $y^{1,u}, \ldots, y^{m,u} \in \mathcal{G}_y$. Define a set

$$\mathcal{G}_\chi = \{g : g \in \mathcal{G}_y, y \in \chi_1\}.$$

We call the elements of the set $\mathcal{G}_\chi$ *germs of polygons*.

Definition 1.3.7. A *locally embedded configuration* is a tuple $\chi = (\chi_0, \chi_1, \mathcal{G}_\chi)$, where (χ_0, χ_1) is a configuration and $\mathcal{G}_\chi$ is as defined above.

An n-gon x in a locally embedded configuration has precisely n germs of the polygon associated to it. For each polygon $x \in \chi_1$, we define an operation $\bar{\cdot} : \mathcal{G}_x \to \mathcal{P}(\mathcal{G}_x)$ by $\overline{x^v} = \mathcal{G}_x \setminus \{x^v\}$.

In the case where χ is a graph – that is, a configuration where every polygon is a 2-gon (or edge) – we have a specialised set of terminology. In this case, every edge $u \overset{x}{\text{——}} v$ has precisely two germs of polygons/edges x^u and x^v associated to it. We call x^u and x^v the *half-edges* associated to x. Since $\mathcal{G}_x \setminus \{x^v\}$ is a singleton for any half-edge x^v associated to x, the operation $\bar{\cdot}$ can instead be viewed as an involution operation on half-edges, where $\overline{x^v} = x^u$ and $\overline{x^u} = x^v$. If $u = v$ (that is, x is a loop) then we distinguish the two half-edges associated to x by x^v and $\overline{x^v}$.

Sometimes it is useful to count the number of germs of polygons associated to a vertex. For this, we have the following.

Definition 1.3.8. The valency of a vertex v in a locally embedded configuration χ is the number of germs of polygons in χ associated to the vertex v, which we denote by $\mathrm{val}(v)$.

In graph theory, one often considers the paths in a graph. We use paths and cycles extensively in this thesis, so we adapt the notion here to locally embedded configurations.

Definition 1.3.9. A *path of length* n in a locally embedded configuration χ is a sequence

$$p = (v_0, x_1^{v_0}, x_1^{v_1}, v_1, x_2^{v_1}, x_2^{v_2}, v_2, \ldots, v_{n-1}, x_n^{v_{n-1}}, x_n^{v_n}, v_n)$$

of vertices and germs of polygons such that $x_i^{v_i} \in \overline{x_i^{v_{i-1}}}$ for all i (or in the case where χ is a graph, $x_i^{v_i} = \overline{x_i^{v_{i-1}}}$ for all i). We say $x \in \chi_1$ is a polygon in p if there exists a vertex $v \in \chi_0$ such that x^v is a germ of a polygon in p.

Where the context is clear and there are no ambiguities arising from self-folded polygons and multiple polygons between vertices, we will write paths of the form in the above definition as

$$p = v_0 \overset{x_1}{\text{———}} v_1 \overset{x_2}{\text{———}} v_2 \text{———} \cdots \text{———} v_{n-1} \overset{x_n}{\text{———}} v_n .$$

In graph theory, there are the notions of simple paths, cycles, subgraphs, connected graphs and trees. The generalisation of these concepts to locally embedded configurations is as follows.

Definition 1.3.10. Let $\chi = (\chi_0, \chi_1)$ be a locally embedded configuration.

(a) A path p in χ is *simple* if it is non-crossing at polygons and vertices. That is, for any polygon x in p, there are precisely two germs of polygons $x^{v_{i-1}}$ and x^{v_i} in p associated to x, and for any vertices v_i and v_j in p, we have $v_i \neq v_j$ for any $i \neq j$.

(b) A *cycle* of length n in χ is a path

$$c = (v_0, x_1^{v_0}, x_1^{v_1}, v_1, \ldots, v_{n-1}, x_n^{v_{n-1}}, x_n^{v_n}, v_n)$$

in which $v_0 = v_n$. A cycle in χ is said to be *simple* if c satisfies the conditions of a simple path, except for the vertices v_0 and v_n.

(c) We say χ is *connected* if there exists a path between any two vertices of χ.

(d) We say $\chi' = (\chi'_0, \chi'_1)$ is a *subconfiguration* of χ if χ' is a connected configuration such that $\chi'_0 \subseteq \chi_0$ and $\chi'_1 \subseteq \chi_1$.

(e) χ is said to be a *tree* if there exists a unique simple path between any two vertices of χ (or equivalently, there exist no simple cycles in χ).

Example 1.3.11. Let χ be the configuration in Example 1.3.6. We can see that χ is connected, but is not a tree. An example of a simple path in χ is

$$p_1 = (v_1, y^{v_1}, y^{v_4}, v_4, z_1^{v_4}, z_1^{v_5}, v_5, z_3^{v_5}, z_3^{v_6}, v_6).$$

Examples of a paths that are not simple are

$$p_2 = (v_1, y^{v_1}, y^{v_4}, v_4, y^{v_4}, y^{v_3}, v_3),$$

since there are more than two germs of polygons associated to y in p_2, and

$$p_3 = (v_4, z_1^{v_4}, z_1^{v_5}, v_5, z_2^{1,v_5}, z_2^{2,v_5}, v_5),$$

since the vertex v_5 repeats in p_3. Examples of simple cycles in χ are

$$c_1 = (v_1, x^{v_1}, x^{v_3}, v_3, y^{v_3}, y^{v_1}, v_1),$$

$$c_2 = (v_5, z_2^{1,v_5}, z_2^{2,v_5}, v_5).$$

1.4 Brauer Configuration, Brauer Graph and Brauer Tree Algebras

Brauer configuration algebras were defined in [35]. They are defined from a decorated configuration called a Brauer configuration. These are useful, as the representation theory of the algebra is encoded in the combinatorial data of the Brauer configuration. In the special case where the configuration is a graph, we obtain the well-studied classes of Brauer graph and Brauer tree algebras, which have been instrumental to many topics in representation theory.

Definition 1.4.1 ([35])**.** A non-empty, connected, locally embedded configuration $\chi = (\chi_0, \chi_1)$ is called a *Brauer configuration* if we have the following additional structure and properties.

(i) To each $v \in \chi_0$, we equip a cyclic ordering $\mathfrak{o}_v$ of the germs of polygons around v.

(ii) To each $v \in \chi_0$, we assign a strictly positive integer $\mathfrak{e}_v$ called the *multiplicity* of the vertex.

(iii) For any polygon x in χ with $|x| > 2$, there exists no vertex $v \in x$ such that $\mathrm{val}(v) = 1$ and $\mathfrak{e}_v = 1$.

A Brauer configuration may be realised as a local embedding of the polygons around each vertex in the oriented plane. We shall use an anticlockwise cyclic ordering throughout this thesis. The classical notion of Brauer graphs and Brauer trees are also important, so we provide the definition of these here.

Definition 1.4.2 ([12], 4.18.1)**.** Let χ be a Brauer configuration.

(a) We call χ a *Brauer graph* if every polygon of χ is a 2-gon (and thus, χ is a graph).

(b) If χ is a Brauer graph, then we call χ a *Brauer tree* if χ is also a tree and at most one vertex v in χ has multiplicity $\mathfrak{e}_v > 1$.

The following are terms related to the vertices of Brauer configurations, which we will use frequently.

Definition 1.4.3. Let χ be a Brauer configuration.

(a) We say a subconfiguration $\chi' \subseteq \chi$ is *multiplicity-free* if every vertex v in χ' is such that $\mathfrak{e}_v = 1$.

(b) We say a vertex v of χ is *truncated* if $\mathrm{val}(v) = 1$ and $\mathfrak{e}_v = 1$. It follows from the definition of a Brauer configuration that any such vertex is connected to a unique polygon x, which is a 2-gon. We call such a polygon a *truncated edge* of χ.

We will often need to describe which polygons are 'next' to each other in the cyclic ordering around a vertex in a Brauer configuration. For this we have the following terminology.

Definition 1.4.4 ([32], [33])**.** Let x_1^v and x_2^v be germs of polygons at the same vertex in a Brauer configuration.

(a) We say x_2^v is the *successor* to x_1^v if x_2^v directly follows x_1^v in the cyclic ordering at v. We then say that the polygon x_2 is the successor to x_1 at v. From this we obtain a sequence $x_1, x_2, \ldots, x_{\mathrm{val}(v)}$, where each x_i is the successor to x_{i-1}. We call this the *successor sequence* of x_1 at v.

(b) We say x_2^v is the *predecessor* to x_1^v if x_1^v directly follows x_2^v in the cyclic ordering at v, and we say the polygon x_2 is the predecessor to x_1 at v. From this we obtain a (descending) sequence $x_{\mathrm{val}(v)}, \ldots, x_2, x_1$, where each x_i is the predecessor to x_{i-1}. We call this the *predecessor sequence* of x_1 at v.

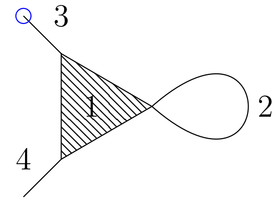
Given a Brauer configuration χ, we construct an algebra A as follows. If χ is a Brauer tree consisting of a single edge and two distinct connected vertices of multiplicity one, then we let $A = KQ/I$, where Q is the quiver consisting of a loop α at a single vertex and I is generated by the relation α^2. Otherwise, we define Q to be the quiver whose vertices are in bijective correspondence with the distinct polygons of χ. If x_2^v is the successor to x_1^v at some non-truncated vertex v in χ, then there exists an arrow $x_1 \to x_2$ in Q. If x is connected to a vertex v such that $\mathrm{val}(v) = 1$ and $\mathfrak{e}_v > 1$, then there exists a loop at x in Q. If $\mathfrak{e}_v = 1$ then no such loop exists. Each non-truncated vertex of χ therefore induces a cycle in Q, and no two such cycles share a common arrow. We denote by $\mathfrak{C}_v$ the cycle of Q up to permutation generated by the non-truncated vertex v in χ. By $\mathfrak{C}_{v,\alpha}$, we denote the permutation of the cycle $\mathfrak{C}_v$ such that the first arrow is α.

We define a set of relations ρ on Q as follows. If x is a truncated edge of χ and $\mathfrak{C}_{v,\gamma_1} = \gamma_1 \ldots \gamma_n$ is the cycle induced by the non-truncated vertex v connected to x with γ_1 of source x, then $(\mathfrak{C}_{v,\gamma_1})^{\mathfrak{e}_v}\gamma_1 \in \rho$. If u and v are (possibly equal) non-truncated vertices connected to the same polygon x and $\mathfrak{C}_{u,\gamma}$ and $\mathfrak{C}_{v,\delta}$ are cycles of source x generated by the respective vertices u and v, then $(\mathfrak{C}_{u,\gamma})^{\mathfrak{e}_u} - (\mathfrak{C}_{v,\delta})^{\mathfrak{e}_v} \in \rho$. Finally, if $\alpha\beta$ is a path of length two in Q such that $\alpha\beta$ is not a subpath of any cycle $\mathfrak{C}_v$ of any non-truncated vertex of χ, then $\alpha\beta \in \rho$. One should note that these relations are not usually minimal.

Definition 1.4.5 ([35])**.** The algebra $A = KQ/I$, where I is the ideal generated by ρ, is called the Brauer configuration algebra associated to χ.

Example 1.4.6. Consider the following Brauer configuration.

$\chi :$



We define χ such that all vertices have multiplicity one, except the circled vertex, which has multiplicity two. The cyclic ordering is induced by walking anticlockwise around each vertex. There is only one truncated edge in χ, which is the edge labelled by 4. The quiver associated to χ is as follows.

$$Q : \begin{tikzcd} & 3 \arrow[loop above, "\delta"] \arrow[d, shift right, "\beta_2"'] & \\ 4 \arrow[r, shift right, "\gamma_2"'] & 1 \arrow[l, shift right, "\gamma_1"'] \arrow[u, shift right, "\beta_1"'] \arrow[r, shift right, "\alpha_1"'] & 2 \arrow[l, shift right, "\alpha_3"'] \arrow[loop right, "\alpha_2"] \end{tikzcd}$$

We also have a set of relations

$$\begin{aligned}\rho = \{&\gamma_2\gamma_1\gamma_2, \alpha_1\alpha_2\alpha_3 - \beta_1\beta_2, \beta_1\beta_2 - \gamma_1\gamma_2, \alpha_2\alpha_3\alpha_1 - \alpha_3\alpha_1\alpha_2, \beta_2\beta_1 - \delta^2, \alpha_3\beta_1, \alpha_3\gamma_1,\\ &\beta_2\alpha_1, \beta_2\gamma_1, \gamma_2\alpha_1, \gamma_2\beta_1, \alpha_2^2, \alpha_1\alpha_3, \beta_1\delta, \delta\beta_2\}.\end{aligned}$$

The Brauer configuration associated to χ is then KQ/I, where I is the ideal generated by ρ.

The terms Brauer configuration algebra and symmetric special multiserial can be used interchangeably, as the next theorem shows. This is an incredibly useful result, since it means that we can use Brauer configurations to describe any symmetric special multiserial algebra. It also allows us to make statements on the representation type of some Brauer configuration algebras.

Theorem 1.4.7 ([36], [48], [50])**.** *An algebra A is a Brauer configuration algebra if and only if it is symmetric special multiserial. In addition, A is a Brauer graph algebra if and only if it is symmetric special biserial.*

Corollary 1.4.8. *Brauer graph algebras are of tame representation type.*

Theorem 1.4.9 ([12], 4.18.4)**.** *Let A be Brauer graph algebra associated to a Brauer graph G. Then A is of finite representation type if and only if A is a Brauer tree.*

1.4.1 The Indecomposable Projective-Injective Modules

Amongst the indecomposable modules over a symmetric algebra, those that are projective-injective play an important role, since they essentially determine the algebra. Thus, it is crucial to have an understanding of the structure of these modules. In the context of Brauer configuration algebras, we begin by making the elementary observation that the indecomposable projective-injective modules of a Brauer configuration algebra $A = KQ/I$ are in bijective correspondence with the number of distinct polygons in the Brauer configuration χ. This follows from the fact that the polygons in χ correspond to the vertices in Q, which in turn correspond to the simple A-modules.

Given a Brauer graph algebra A associated to a Brauer graph G, it is known ([48], for example) that the Loewy structure of any indecomposable projective(-injective) module is given by walking anticlockwise around the non-truncated vertices connected to the associated edge (in accordance with the cyclic ordering and and a number of times with respect to the multiplicity of the vertex), and recording the edges along the walk. It follows from the relations in the definition of a Brauer configuration algebra that this is also true in the multiserial case. We provide in detail the structure of the indecomposable projective-injective modules of a Brauer configuration algebra here, which is based on the results of [35].

So let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose x is a truncated edge of χ and let v be the non-truncated vertex connected to x. Then it follows from the construction of Q that there exists a unique arrow $\alpha_1 \in Q_1$ of source x. Let $m = \mathrm{val}(v)$, $\mathfrak{C}_{v,\alpha_1} = \alpha_1 \dots \alpha_m$ and let $x_1, \dots, x_m$ be the successor sequence of $x = x_1$ at v. Then $s(\alpha_i) = x_i$. Consider the following diagram.

$$
\left.\begin{array}{r}
x_1 \\
\alpha_1 \big\downarrow \\
\vdots \\
\alpha_{m-1} \big\downarrow \\
x_m \\
\alpha_m \big\downarrow \\
\vdots \\
\big\downarrow \\
x_1 \\
\alpha_1 \big\downarrow \\
\vdots \\
\alpha_{m-1} \big\downarrow \\
x_m \\
\alpha_m \big\downarrow \\
x_1
\end{array}\right\} \mathfrak{e}_v m \text{ arrows}
$$

where the arrows α_i each occur $\mathfrak{e}_v$ times. The indecomposable projective module $P(x)$ corresponding to x is as follows. The underling vector space of $P(x)$ is given by replacing each x_i in the above diagram with a copy of K. The action of an arrow $\beta \in Q_1$ on $P(x)$ is induced by the relevant identity maps if $\beta = \alpha_i$ for some $1 \leq i \leq n$, and is zero otherwise. It is easy to see that $P(x)$ is uniserial in this case.

Suppose instead $x = \{v_1, \ldots, v_n\}$ is a polygon in χ that is not a truncated edge (we allow for the possibility that $v_i = v_j$ for some $1 \leq i < j \leq n$). Then each vertex v_i connected to x is non-truncated. Let $x^{v_1}, \ldots, x^{v_n}$ be a complete list of the distinct germs of polygons associated to x. Then it follows from the construction of Q that there exist precisely n arrows $\alpha_{1,1}, \ldots, \alpha_{n,1}$ of source x in Q. For each i, let $m_i = \mathrm{val}(v_i)$, $\mathfrak{C}_{v_i,\alpha_{i,1}} = \alpha_{i,1} \ldots \alpha_{i,m_i}$ and let $x_{i,1}, \ldots, x_{i,m_i}$ be the successor sequence of $x = x_{i,1}$ at the vertex v_i (or more precisely, the successor sequence of polygons corresponding to the successor sequence of x^{v_i}). Consider the following diagram.

$$
\begin{tikzcd}[row sep=small]
 & & x \arrow[dll, "\alpha_{1,1}"'] \arrow[dl, "\alpha_{2,1}"] \arrow[dr, "\alpha_{n,1}"] & & \\
\vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \\
x_{1,m_1} \arrow[d, "\alpha_{1,m_1}"'] & x_{2,m_2} \arrow[d, "\alpha_{2,m_2}"] & & x_{n,m_n} \arrow[d, "\alpha_{n,m_n}"] & \\
\vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \\
x_{1,1} \arrow[d, "\alpha_{1,1}"'] & x_{2,1} \arrow[d, "\alpha_{2,1}"] & & x_{n,1} \arrow[d, "\alpha_{n,1}"] & \\
\vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \\
x_{1,m_1} \arrow[drr, "\alpha_{1,m_1}"'] & x_{2,m_2} \arrow[dr, "\alpha_{2,m_2}"] & & x_{n,m_n} \arrow[dl, "\alpha_{n,m_n}"] & \\
 & & x & &
\end{tikzcd}
$$

where for each $1 \leq i \leq n$ and each $1 \leq j \leq m_i$ the arrows $\alpha_{i,j}$ each occur $\mathfrak{e}_{v_i}$ times. The indecomposable projective module $P(x)$ corresponding to x is as follows. The underling vector space of $P(x)$ is given by replacing each $x_{i,j}$ in the above diagram with a copy of K. The action of an arrow $\beta \in Q_1$ on $P(x)$ is induced by the relevant identity maps if $\beta = \alpha_{i,j}$ for some $1 \leq i \leq n$ and some $1 \leq j \leq m_i$, and is zero otherwise. In this case, $P(x)$ is n-serial.

From this, we can make some elementary observations.

Theorem 1.4.10. *Let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. The following are equivalent.*

(a) A is symmetric special n-serial.

(b) $|x| \leq n$ for all x in χ.

Moreover, if A is symmetric special n-serial then A is n-serial.

Proof. Let A be symmetric special n-serial. Then for each vertex $x \in Q_0$, there are at most n arrows of source x. Every vertex in $x \in Q_0$ corresponds to a polygon in χ, and it follows from the definition of a Brauer configuration algebra that x is the source of precisely $|x|$ arrows in Q. Since there are at most n arrows of source x, $|x| \leq n$ for all x in χ. The converse holds by the same argument. That A is also n-serial follows from the structure of the indecomposable projectives outlined above (and detailed in [35]). □

1.4.2 Projective resolutions in Brauer graph algebras

Let $A = KQ/I$ be a Brauer graph algebra associated to a Brauer graph G. If G is a Brauer tree, then it is well known (see [37]) that the minimal projective resolutions of the simple modules associated to the truncated edges of G are periodic. In particular, the projective modules occurring in the minimal projective resolution are all indecomposable (and thus associated to certain edges in G), and these follow a combinatorial walk around the Brauer tree. This combinatorial walk is often referred to as a Green walk in honour of J. A. Green, who defined the walk for

Brauer trees. In [48], it was later shown that the same result was true for the minimal projective resolutions of the simple modules associated to the truncated edges of a general Brauer graph G and the minimal projective resolutions of the uniserial modules of maximal composition length in the Brauer graph algebra. We summarise these results here.

Definition 1.4.11 ([22],[37]). Let G be a Brauer graph.

(a) We define a *Green walk* around G from an edge x_0 via a vertex v_0 to be a (periodic) sequence $(x_j^{v_j})_{j\in\mathbb{Z}_{\geq 0}}$ of half-edges such that x_i is connected to x_{i+1} via the vertex v_i and $\overline{x_{i+1}^{v_{i+1}}}$ is the successor to $x_i^{v_i}$.

(b) We define a *clockwise Green walk* from x_0 via v_0 to be a similar sequence $(x_j^{v_j})_{j\in\mathbb{Z}_{\geq 0}}$ to (a) that consists of half-edges such that each $\overline{x_{i+1}^{v_{i+1}}}$ is the predecessor to $x_i^{v_i}$.

(c) By a *double-stepped Green walk* of G, we mean a subsequence $(x_{j_k}^{v_{j_k}})_{k\in\mathbb{Z}_{\geq 0}}$ of a (anticlockwise or clockwise) Green walk $(x_j^{v_j})_{j\in\mathbb{Z}_{\geq 0}}$, where $j_k = 2k$.

(d) We say a Green walk (anticlockwise, clockwise and/or double-stepped) is *of length* l if it is of period l – that is, l is the least integer such that $x_i^{v_i} = x_{l+i}^{v_{l+i}}$ for all i.

(e) We say two Green walks $(x_j^{v_j})_{j\in\mathbb{Z}_{\geq 0}}$ and $(y_j^{u_j})_{j\in\mathbb{Z}_{\geq 0}}$ are *distinct* if there exists no integer k such that $x_{i+k}^{v_{i+k}} = y_i^{u_i}$ for all i.

Definition 1.4.12 ([48]). Let A be a Brauer graph algebra associated to a Brauer graph G. Let $\mathcal{M}$ be the set of all simple A-modules whose projective covers are uniserial and all maximal uniserial submodules of indecomposable projective A-modules.

Theorem 1.4.13 ([48]). *Let A be a Brauer graph algebra associated to a Brauer graph G. Let $M \in \mathcal{M}$. Then the minimal projective resolution*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M is periodic and the projective modules occurring are all indecomposable. Moreover, the sequence $(P_i)_{i\in\mathbb{Z}_{\geq 0}}$ correspond to edges in G that follow a Green walk in the Brauer graph.

Example 1.4.14. An example of a Green walk in a Brauer graph is given in Figure 2.4.

1.5 Trivial Extensions of Gentle and Almost Gentle Algebras

Given an algebra A, we can construct a new algebra by taking the trivial extension of A.

Definition 1.5.1. Let K be an algebraically closed field, A be a finite-dimensional associative K-algebra and DA be the standard K-linear dual of A. The trivial extension algebra $T(A)$ is the algebra with underlying K-vector space $A \oplus DA$ and ring multiplication defined by $(a, f)(b, g) = (ab, ag + fb)$.

It is well-known that the trivial extension of an algebra is a symmetric algebra. We also have the following result.

Theorem 1.5.2 ([45], [47], [49])**.** *The trivial extension $T(A)$ of an algebra A is special biserial if and only if A is gentle.*

Since the class of symmetric special biserial algebras coincide with the class of Brauer graph algebras ([48], [50]), we conclude that the trivial extension of a gentle algebra is a Brauer graph algebra. A similar result was proven for the more general setting of almost gentle algebras by Green and Schroll.

Theorem 1.5.3 ([34], [36])**.** *The trivial extension $T(A)$ of an almost gentle algebra A is symmetric special multiserial (and hence, a Brauer configuration algebra).*

The converse statement to the above theorem remains an open problem. Given a Brauer configuration algebra B associated to a multiplicity-free Brauer configuration, we can construct an almost gentle A such that $T(A) = B$. This is achieved by taking an admissible cut of the algebra B, which is defined in [34] (and which is originally based on the definitions in [25] and [26]). We will present the definition here, as this is required later in the thesis.

Definition 1.5.4 ([34])**.** Let $B = KQ/I$ be a Brauer configuration algebra associated to a multiplicity-free Brauer configuration χ. Suppose $\chi_0 = \{v_1, \dots, v_n\}$ and let D be a set consisting of precisely one arrow from each cycle $\mathfrak{C}_{v_i}$. We call D an *admissible cut* of Q and we call $KQ/\langle I \cup D\rangle$ the *cut algebra associated to* D, where $\langle I \cup D\rangle$ is the ideal generated by $I \cup D$.

Theorem 1.5.5 ([34])**.** *Let $B = KQ/I$ be a Brauer configuration algebra associated to a multiplicity-free Brauer configuration. Let D be an admissible cut of Q. Define a quiver Q' by $Q'_0 = Q_0$ and $Q'_1 = Q_1 \setminus D$. Then the cut algebra $KQ/\langle I \cup D\rangle$ is isomorphic to the (basic) algebra $A = KQ'/(I \cap KQ')$. Furthermore, A is almost gentle and $T(A) = B$.*

Note that different admissible cuts of a Brauer configuration algebra B may give rise to many non-isomorphic, non-derived equivalent almost gentle algebras. Thus, there may be many different almost gentle algebras $A_1, \dots, A_m$ such that $T(A_1) = \dots = T(A_m) = B$. In the case where B is a Brauer graph algebra associated to a multiplicity-free Brauer graph G, the cut algebra associated to any admissible cut of B is a gentle algebra ([50]).

1.6 String and Band Modules

Due to the functorial filtration method of Gel'fand and Ponomarev ([31]), the indecomposable modules of special biserial algebras have been classified in [17] and [57]. These have been shown to be precisely the indecomposable projective-injective modules over the algebra, and the string and band modules over the algebra, which we will define here. We follow the definitions of [17] for string and band modules, but in the context of Brauer configuration algebras.

Definition 1.6.1 ([17])**.** Let $A = KQ/I$ be a Brauer configuration algebra. Given an arrow $\alpha \in Q_1$, we define the *formal inverse* of α to be the symbolic arrow α^{-1}

such that $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. Denote the set of formal inverses of all arrows in Q_1 by Q_1^{-1}.

Let χ be the Brauer configuration associated to A and recall that the vertices of Q_0 are in correspondence with the polygons of χ. Then for each arrow $\alpha \in Q_1$, we can consider $s(\alpha)$ and $e(\alpha)$ to be polygons in χ. Further recall that, the arrows of Q_1 correspond to ordered pairs of germs of polygons (x^v, y^v) such that y^v is the successor to x^v. Thus, we can also consider α to be an arrow between two germs of polygons in a Brauer configuration.

Definition 1.6.2. Let $A = KQ/I$ be a Brauer configuration algebra and let $\alpha \in Q_1$. Denote by $\widehat{s}(\alpha)$ the germ of the polygon at the source of α, and by $\widehat{e}(\alpha)$ the germ of the polygon at the target of α. We define $\widehat{s}(\alpha^{-1}) = \widehat{e}(\alpha)$ and $\widehat{e}(\alpha^{-1}) = \widehat{s}(\alpha)$.

In most cases, it is sufficient (and often simpler) to work with polygons instead of germs of polygons. Thus for any symbol $\alpha \in Q_1 \cup Q_1^{-1}$, we are likely to use $s(\alpha)$ and $e(\alpha)$ when it is sufficient to do so. However, wherever there is the potential for ambiguity arising from multiple edges/polygons or loops/self-folded polygons, we will need the additional information that $\widehat{s}$ and $\widehat{e}$ provide, since these eliminate any ambiguities in a general argument. This is essential in Chapter 2 where Green walks are used extensively, since Green walks are defined in terms of half-edges.

String and band modules over an algebra are defined using the notion of a string, which is as follows.

Definition 1.6.3 ([17])**.** Let KQ/I be a path algebra modulo an admissible ideal. We call a word $w = \alpha_1 \dots \alpha_n$, where each symbol $\alpha_i \in Q_1 \cup Q_1^{-1}$, a *string of length* n if w satisfies the following properties.

(i) $\alpha_i \neq \alpha_{i+1}^{-1}$ for all i,

(ii) $e(\alpha_i) = s(\alpha_{i+1})$ for all i, and

(iii) w avoids the relations in I.

We denote the length of w by $|w| = n$. For any string $w = \alpha_1 \dots \alpha_n$, we define $s(w) = s(\alpha_1)$ and $e(w) = e(\alpha_n)$. The inverse of w is defined to be the string $w^{-1} = \alpha_n^{-1} \dots \alpha_1^{-1}$.

There are many different types of strings, which we define below.

Definition 1.6.4 ([17])**.** Let KQ/I be a path algebra modulo an admissible ideal.

(a) We say a string w is a *direct string* if every symbol of w is in Q_1 and we say w is an *inverse string* if every symbol of w is in Q_1^{-1}.

(b) A stationary path ε_x at a vertex $x \in Q_0$ is a string of length zero, which we call a *zero string*. If $w = \varepsilon_x$, then we define $s(w) = x = e(w)$. Zero strings are defined to be both direct and inverse.

(c) A *band* is a cyclic string b such that b^m is a string, but b is not a proper power of any string w.

Note that we have not defined $\widehat{s}(w)$ and $\widehat{e}(w)$, since this is not possible for zero strings.

Let $w = \alpha_1 \dots \alpha_n$ be a string, let $x_0 = s(\alpha_1)$ and for each i, let $x_i = e(\alpha_i)$. From the string w, we obtain an indecomposable module $M(w) \in \operatorname{mod} A$ called a *string module*. The underlying vector space of $M(w)$ is given by replacing each x_i with a copy of the field K. We then say that the action of an arrow $\alpha \in Q_1$ is induced by the relevant identity maps if α or its formal inverse is in w, and is zero otherwise. It follows from the construction of string modules that $M(\varepsilon_x) = S(x)$.

To each band $b = \beta_1 \dots \beta_m$, we obtain an infinite family of indecomposable modules $M(b, n, \phi)$ called *band modules*, where $n \in \mathbb{Z}_{>0}$ and $\phi \in \operatorname{Aut}(K^n)$. We direct the reader to [17] for the full details on the construction of $M(b, n, \phi)$, however we will provide a brief summary here. The underlying vector space of $M(b, n, \phi)$ is given by replacing each vertex of b with a copy of K^n. The action of an arrow in $\gamma \in Q_1$ on $M(b, n, \phi)$ is given by the relevant identity morphism if $\gamma = \beta_i$ or $\gamma = \beta_i^{-1}$ for some $i \neq m$. If we instead have $\gamma = \beta_m$ or $\gamma = \beta_m^{-1}$, then the action of γ on $M(b, n, \phi)$ is ϕ. Otherwise, γ has a zero action on $M(b, n, \phi)$.

1.6.1 Auslander-Reiten Theory for String and Band modules in Symmetric Special Biserial Algebras

Definition 1.6.5 ([17])**.** Let $A = KQ/I$ be a special biserial algebra.

(a) We say a string w *starts on a peak* (resp. *ends on a peak*) if no arrow $\alpha \in Q_1$ exists such that αw (resp. $w\alpha^{-1}$) is a string.

(b) We say w *starts in a deep* (resp. *ends in a deep*) if no arrow $\alpha \in Q_1$ exists such that $\alpha^{-1}w$ (resp. $w\alpha$) is a string.

Definition 1.6.6 ([17])**.** Let $A = KQ/I$ be a special biserial algebra and let w be a string. For some arrow $\alpha \in Q_1$, let u_α and v_α be the unique inverse strings such that $u_\alpha\alpha$ is a string that starts in a deep and αv_α is a string that ends on a peak. See Figure 1.2 for an illustration.

(a) Suppose w does not end on a peak. Then we say the string $w_h = w\alpha^{-1}u_\alpha^{-1}$ is obtained from w by *adding a hook to the end of* w.

(a′) Suppose $w = w_{-h}\alpha^{-1}u_\alpha^{-1}$ for some substring w_{-h}. Then we say w_{-h} is obtained from w by *deleting a hook from the end of* w.

(b) Suppose w does not start on a peak. Then we say the string ${}_hw = u_\alpha\alpha w$ is obtained from w by *adding a hook to the start of* w.

(b′) Suppose $w = u_\alpha\alpha({}_{-h}w)$ for some substring ${}_{-h}w$. Then we say ${}_{-h}w$ is obtained from w by *deleting a hook from the start of* w.

(c) Suppose w does not end in a deep. Then we say the string $w_c = w\alpha v_\alpha$ is obtained from w by *adding a cohook to the end of* w.

(c′) Suppose $w = w_{-c}\alpha v_\alpha$ for some substring w_{-c}. Then we say w_{-c} is obtained from w by *deleting a cohook from the end of* w.

(d) Suppose w does not start in a deep. Then we say the string ${}_cw = v_\alpha^{-1}\alpha^{-1}w$ is obtained from w by *adding a cohook to the start of* w.

(d′) Suppose $w = v_\alpha^{-1}\alpha^{-1}({}_{-c}w)$ for some substring ${}_{-c}w$. Then we say ${}_{-c}w$ is obtained from w by *deleting a cohook from the start of* w.

Note that it is possible for u_α and v_α to be zero strings in the definition above. We caution the reader that our notation for w_h, w_c, ${}_hw$ and ${}_cw$ differ from that

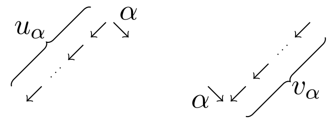


Figure 1.2: The strings $u_\alpha\alpha$ and αv_α from Definition 1.6.6. The strings are read from left to right and formal inverses are considered as going back along an arrow.

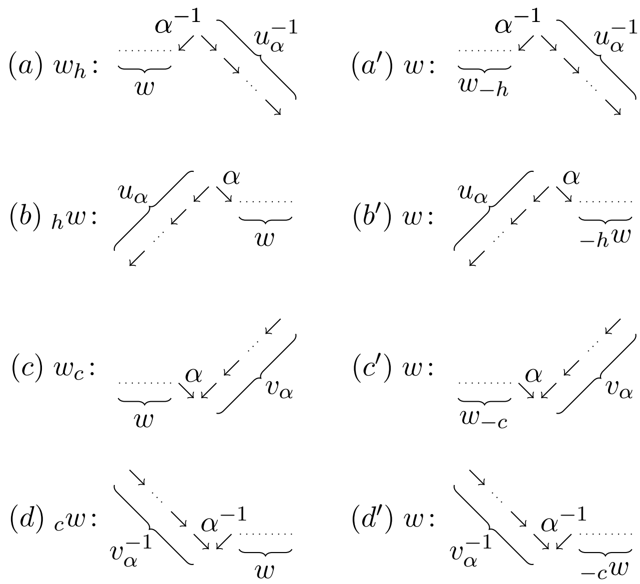


Figure 1.3: The strings in (a)-(d) and (a′)-(d′) of Definition 1.6.6.

presented in other papers on the subject (in particular [17]). We prefer our notation as it allows for greater flexibility when it comes to describing the irreducible morphisms in the Auslander-Reiten quiver of a symmetric special biserial algebra.

Theorem 1.6.7 ([17], [53]). *Let $A = KQ/I$ be a symmetric special biserial algebra and let w be a string. Suppose $M(w)$ is not the radical of some indecomposable projective-injective P. Then the Auslander-Reiten sequence starting in $M(w)$ is*

(a)

$$0 \longrightarrow M(w) \longrightarrow M({}_h w) \oplus M(w_h) \longrightarrow M({}_h w_h) \longrightarrow 0$$

if w does not start or end on a peak;

(b)

$$0 \longrightarrow M(w) \longrightarrow M({}_{-c} w) \oplus M(w_h) \longrightarrow M({}_{-c} w_h) \longrightarrow 0$$

if w starts on a peak but does not end on a peak;

(c)

$$0 \longrightarrow M(w) \longrightarrow M({}_h w) \oplus M(w_{-c}) \longrightarrow M({}_h w_{-c}) \longrightarrow 0$$

if w ends on a peak but does not start on a peak; or

(d)

$$0 \longrightarrow M(w) \longrightarrow M({}_{-c} w) \oplus M(w_{-c}) \longrightarrow M({}_{-c} w_{-c}) \longrightarrow 0$$

if w starts and ends on a peak.

Chapter 2

Auslander-Reiten Components of Brauer Graph Algebras

The material presented in this chapter appears in [22] in a similar format. Here, we concern ourselves with a well-studied subclass of symmetric special multiserial algebras – namely, those that are biserial. Many aspects of the representation theory of this subclass is well-understood, which allows us to go into a great level of detail about the module category of this class of algebras.

Recall that the class of symmetric special biserial algebras coincides with the class of Brauer graph algebras. It is the aim of this chapter to give a specific and detailed account of the Auslander-Reiten quiver of any given symmetric special biserial algebra using only information from its underlying Brauer graph. In effect, we show that all the information about the Auslander-Reiten quiver (such as the shape, size and number of components, along with the indecomposable modules that are situated in them) can be read-off from the Brauer graph.

In Section 2.1, we provide an algorithm for constructing the stable Auslander-Reiten component of a given string module of a Brauer graph algebra using only information from its underlying Brauer graph. This algorithm is of particular importance because it allows us to describe the string combinatorics of the algebra in terms of the Brauer graph. This algorithm thus allows us to prove several results later in the chapter, which relate the number and shape of the components of the

Auslander-Reiten quiver of the algebra to the Brauer graph.

In [24], the Auslander-Reiten components of self-injective special biserial algebras have been described. In particular, any Brauer graph algebra has a finite number of exceptional tubes in the stable Auslander-Reiten quiver. In Section 2.2, we show that the rank and the total number of these tubes is closely related to the distinct Green walks around the Brauer graph. Specifically, we prove the following.

Theorem. *Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let ${}_s\Gamma_A$ be its stable Auslander-Reiten quiver. Then*

(a) there is a bijective correspondence between the exceptional tubes in ${}_s\Gamma_A$ and the distinct double-stepped Green walks along G; and

(b) the rank of an exceptional tube is given by the length of its associated double-stepped Green walk along G.

It is shown in [24] that the Auslander-Reiten components of Brauer graph algebras are strongly related to the growth type of the algebra. The algebra, for example, contains a Euclidean component of the form $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ if and only if the algebra is domestic and of infinite representation type. We use a simple application of the above theorem to show the following results regarding the values of p and q.

Theorem. *Let A be a Brauer graph algebra constructed from a graph G of n edges and suppose ${}_s\Gamma_A$ has a $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ component.*

(a) If A is 1-domestic, then $p+q=2n$.

(b) If A is 2-domestic, then $p+q=n$.

Furthermore, if G is a tree, then $p=q=n$.

Corollary. *For a domestic Brauer graph algebra, if the Brauer graph contains a unique cycle of length l and there are n_1 additional edges on the inside of the cycle and n_2 additional edges along the outside, then the $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ components are given by*

$$p = \begin{cases} l+2n_1 & l \text{ odd,} \\ \frac{l}{2}+n_1 & l \text{ even,} \end{cases} \quad \text{and} \quad q = \begin{cases} l+2n_2 & l \text{ odd,} \\ \frac{l}{2}+n_2 & l \text{ even.} \end{cases}$$

In of Section 2.3, we use the algorithm presented in Section 2.1 to prove results which show how one can determine the specific component containing a given simple or indecomposable projective module from its associated edge in the Brauer graph G.

We first divide the edges of G into two distinct classes, which we call *exceptional* and *non-exceptional edges.* The exceptional edges of a Brauer graph belong to a special class of subtrees of the graph, which we refer to as the *exceptional subtrees* of the Brauer graph. Intuitively, one can think of these subtrees as belonging to parts of the algebra that behave locally as a Brauer tree algebra. We then prove the following regarding the exceptional edges of a Brauer graph.

Theorem. *Let A be a representation-infinite Brauer graph algebra associated to a graph G and let x be an edge in G. Then the simple module and the radical of the indecomposable projective module associated to x belong to exceptional tubes of ${}_s\Gamma_A$ if and only if x is an exceptional edge.*

We also determine when a simple module and the radical of an indecomposable projective module belong to the same Auslander-Reiten component. For exceptional edges, we have the following.

Corollary. *Given an exceptional edge x in a Brauer graph, the simple module and the radical of the indecomposable projective module associated to x belong to the same exceptional tube if and only if we walk over both vertices connected to x in the same double-stepped Green walk.*

We then finally prove a similar result for non-exceptional edges,

Theorem. *Let A be a representation-infinite Brauer graph algebra associated to a Brauer graph G and let $\mathfrak{e}_v$ denote the multiplicity of a vertex v in G. Let x and y be non-exceptional edges of G. Then the simple module associated x is in the same component of ${}_s\Gamma_A$ as the radical of the indecomposable projective module associated to y if and only if either A is 1-domestic or there exists a path*

$$p : u_0 \overset{x_1=x}{\text{——}} v_1 \overset{x_2}{\text{——}} v_2 \text{——} \cdots \text{——} v_{n-2} \overset{x_{n-1}}{\text{——}} v_{n-1} \overset{x_n=y}{\text{——}} u_1$$

of even length in G consisting of non-exceptional edges such that

(i) every edge x_i is not a loop;

(ii) $\mathfrak{e}_{v_i} = 1$ if $x_i \neq x_{i+1}$ and $\mathfrak{e}_{v_i} = 2$ if $x_i = x_{i+1}$;

(iii) x_i and x_{i+1} are the only non-exceptional edges incident to v_i in G.

Whilst this chapter assumes the algebras we are dealing with are symmetric special biserial, many of the results in this chapter should carry over to the weakly symmetric case (and hence, should work for the quantised Brauer graph algebras defined, for example, in [32]). Indeed, the indecomposable modules of a weakly symmetric special biserial algebra are again given by string and band modules (see for example [23]).

2.1 A Constructive Algorithm

It is already known that the non-projective indecomposable modules of a Brauer graph algebra A are given by string and band modules. Given a string module M in a Brauer graph algebra $A = KQ/I$ constructed from a graph G, we wish to be able to read off the (stable) Auslander-Reiten component containing M from G. Since the irreducible morphisms between string modules are given by adding or deleting hooks and cohooks to strings, our algorithm will need to take hooks and cohooks into account.

2.1.1 Presenting strings on a Brauer graph

To achieve the aim of this section, we must first describe a method for presenting strings on Brauer graphs. This technique appears to be well known within the subject area. However, it forms an essential part of this chapter and thus, we will outline the process here.

Let $w = \alpha_1 \ldots \alpha_n$ be a (not necessarily direct) string in A. Since each vertex in G generates a cycle in Q and no two such cycles share a common arrow in Q, we can associate to each arrow (or formal inverse) α_i in the string a vertex in G.

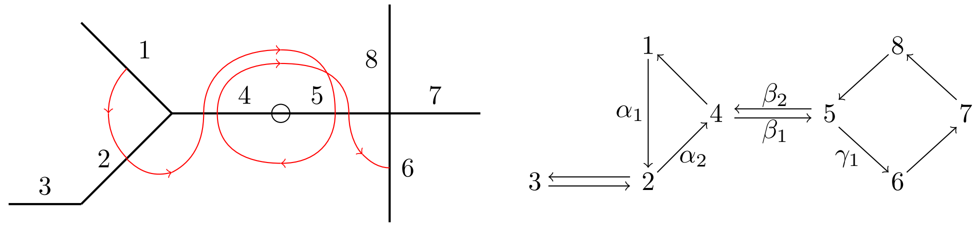


Figure 2.1: A Brauer tree T (left) and its corresponding quiver (right) with the string $w = \alpha_1\alpha_2\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\gamma_1$ drawn on the graph in red. The circled vertex in T has a multiplicity of 2.

Suppose $\alpha_i \in Q_1$ is an arrow belonging to the cycle generated by a vertex v and $s(\alpha_i) = x$ and $e(\alpha_i) = y$. Since x and y are edges in the graph, we realise this arrow on G as an anticlockwise arrow around the vertex v of source x and target y. Similarly, if we instead have $\alpha_i \in Q_1^{-1}$ such that the arrow α_i^{-1} belongs to the cycle generated by v, then we realise this as moving clockwise in G around v from $s(\alpha_i)$ to $e(\alpha_i)$. An example is given in Figure 2.1. Note that if we wish to invert a string, we simply flip the direction of the arrows and formal inverses drawn on the Brauer graph.

We often wish to perform this procedure in reverse. That is, we may wish to draw a sequence (or path) of connected 'arrows' through the edges of G and interpret this as a string in the algebra. To do this, we must describe which of these paths give valid strings. First recall that for a string $w = \alpha_1 \dots \alpha_n$, we require $\alpha_{i+1} \neq \alpha_i^{-1}$. Thus, we cannot draw an arrow anticlockwise around a vertex v from an edge x to an edge y followed by a clockwise formal inverse around v from y to x (and vice versa). Furthermore, if v is a truncated vertex, then v generates no arrows in Q and thus, we cannot draw any arrows around v in the graph.

A string must also avoid the relations in I, so if we have edges $\overset{x}{\text{———}} v \overset{y}{\text{———}} v' \overset{z}{\text{———}}$, we cannot draw an anticlockwise (resp. clockwise) arrow (resp. formal inverse) around v from x to y followed by an anticlockwise (resp. clockwise) arrow (resp. formal inverse) around v' from y to z. Finally, if a vertex v has multiplicity $\mathfrak{e}_v$ and v generates a cycle $\gamma_1 \dots \gamma_m$ in Q, then we generally cannot draw an anticlockwise cycle of arrows $(\gamma_1 \dots \gamma_m)^{\mathfrak{e}_v}$ or a clockwise cycle of formal inverses $(\gamma_m^{-1} \dots \gamma_1^{-1})^{\mathfrak{e}_v}$ around v on the graph. An exception to this rule is the string given by a unise-

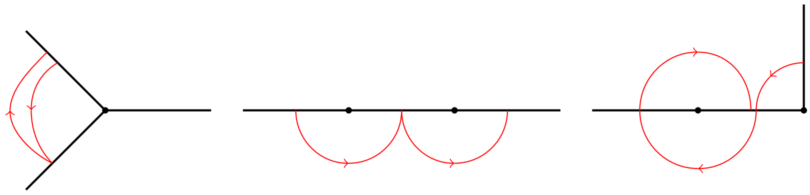


Figure 2.2: Examples of sequences of arrows and formal inverses presented on Brauer graphs that are not valid strings. All vertices are of multiplicity 1.

rial indecomposable projective. However, since we only wish to consider the stable Auslander-Reiten quiver of A, we will ignore this exception. In all other cases, we obtain a valid string. For examples of sequences of arrows and formal inverse presented on the graphs that are not valid strings, see Figure 2.2.

Remark 2.1.1. For any string $\alpha\beta$ of length 2, we have $\widehat{e}(\alpha) = \widehat{s}(\beta)$ if and only if $\alpha\beta$ is direct or inverse. This follows from the fact that $\alpha\beta \in I$ if $\alpha\beta$ is direct/inverse and $\widehat{e}(\alpha) \neq \widehat{s}(\beta)$, since in this case, α does not directly follow β in any cycle $\mathfrak{C}_v$ for any vertex v in G. Conversely, $\alpha = \beta^{-1}$ if $\widehat{e}(\alpha) = \widehat{s}(\beta)$ but $\alpha\beta$ is not direct/inverse.

2.1.2 Maximal direct and inverse strings

To construct hooks and cohooks, we will need to know precisely when a string w ends in a deep or on a peak. Specifically, we have the following.

Lemma 2.1.2. *Let A be a Brauer graph algebra constructed from a graph G and let w be a string in A. Suppose $M(w)$ is non-projective. Then*

(a) w ends in a deep if and only if either

(i) $w = w_0\alpha^{-1}$, where w_0 is a string and $\alpha^{-1} \in Q_1^{-1}$ is a formal inverse such that $e(\alpha^{-1})$ is a truncated edge in G; or

(ii) $w = w_0\gamma^{\mathfrak{e}_v-1}\gamma_1 \dots \gamma_{m-1}$, such that w_0 is a string, $\gamma = \gamma_1 \dots \gamma_m$ is a cycle generated by a vertex v in G, and if w_0 is a zero string then $s(w)$ is not truncated.

(b) w ends on a peak if and only if either

(i) $w = w_0\alpha$, where w_0 is a string and $\alpha \in Q_1$ is an arrow such that $e(\alpha)$ is a truncated edge in G; or

(ii) $w = w_0(\gamma^{-1})^{\mathfrak{e}_v - 1}\gamma_m^{-1} \dots \gamma_2^{-1}$*, such that* w_0 *is a string,* $\gamma = \gamma_1 \dots \gamma_m$ *is a cycle generated by a vertex* v *in* G*, and if* w_0 *is a zero string then* $s(w)$ *is not truncated.*

Proof. (a) ($\Leftarrow$:) We will first prove (i) implies w ends in a deep. So suppose $w = w_0\alpha^{-1}$ and $e(w) = x$, where x is truncated. Then $P(x)$ is uniserial and thus there exists precisely one arrow of source x and precisely one arrow of target x in Q. But this implies the only arrow in Q of source x is α. So there exists no arrow $\beta \in Q_1$ such that $w\beta$ is a string and hence, w ends in a deep.

Now suppose w is instead of the form in (ii). We will show that this also implies w ends in a deep. Let $x = s(\gamma_1)$. The first case to consider is where x is non-truncated. In this case, there exists a relation $\gamma^{\mathfrak{e}_v} - \delta^{\mathfrak{e}_u} \in I$ for some cycle δ in Q generated by a vertex u connected to x. Thus, $\gamma^{\mathfrak{e}_v}$ is not a string. Furthermore, following the definition of a Brauer graph algebra, if $\gamma_i\beta \in I$ for some arrow β, then $\beta \neq \gamma_{i+1}$. Since there are no other relations involving each arrow γ_i of the cycle γ, we conclude $\gamma^{\mathfrak{e}_v - 1}\gamma_1 \dots \gamma_{m-1}$ is a string that ends in a deep.

The other case to consider is where x is truncated. Since w is of the form in (ii), w_0 is not a zero string. Suppose for a contradiction that w does not end in a deep. Then $w\gamma_m$ is a string, since $\gamma_{m-1}\beta \in I$ for any arrow $\beta \neq \gamma_m$. But then $w_0\gamma^{\mathfrak{e}_v}$ is a string for some non-zero string w_0. This is not possible, since $\delta\gamma_1 \in I$ for any $\delta \neq \gamma_m$, and $\gamma_m\gamma^{\mathfrak{e}_v} \in I$. So w must end in a deep.

($\Rightarrow$:) Suppose conditions (i) and (ii) do not hold. If w is a zero string, then w clearly does not end in a deep. So let $w = \alpha_1 \dots \alpha_n$. First consider the case where $\alpha_n \in Q_1^{-1}$. Note that the arrow α_n^{-1} belongs to a cycle $\mathfrak{C}_v$ for some vertex v in G. Since condition (i) does not hold, $P(e(w))$ is biserial and is the source of two distinct arrows in Q, so there exists an arrow β not in $\mathfrak{C}_v$ such that $s(\beta) = e(\alpha_n)$. So clearly $w\beta$ is a string.

Now suppose $\alpha_n \in Q_1$. Let w'' be the direct substring at the end of w such that either $w = w''$ or $w = w'\beta^{-1}w''$ for some substring w' and some $\beta^{-1} \in Q_1^{-1}$. Let γ_1 be the first symbol of w''. Then γ_1 belongs to a cycle of Q generated by some vertex v in G. Suppose $\gamma = \gamma_1 \dots \gamma_m$ is the cycle generated by v. Then $\alpha_n = \gamma_i$ for some

i. Since condition (ii) does not hold, w'' forms a proper subword of $\gamma^{\mathfrak{e}_v-1}\gamma_1\ldots\gamma_{m-1}$ and $i \neq m-1$. Since $\gamma^{\mathfrak{e}_v-1}\gamma_1\ldots\gamma_{m-1}$ is a string, $w\gamma_{i+1}$ is a string and so w does not end in a deep.

(b) The proof is similar to (a). □

We may also describe when a string w presented on G starts in a deep or on a peak – we simply consider the string w^{-1} in the context of the lemma above.

Remark 2.1.3. Suppose x is a non-truncated edge of G connected to a vertex v and $\gamma = \gamma_1\ldots\gamma_n$ is the cycle of Q generated by v with $s(\gamma_1) = x$. Then it follows trivially from the above that a maximal direct string of source x is $\gamma^{\mathfrak{e}_v-1}\gamma_1\ldots\gamma_{n-1}$ and a maximal inverse string of source x is $(\gamma^{-1})^{\mathfrak{e}_v-1}\gamma_n^{-1}\ldots\gamma_2^{-1}$. Furthermore, to each non-truncated edge in G, we can associate two maximal direct strings and two maximal inverse strings – one for each vertex connected to the edge (or in the case of a loop, one for each half-edge associated to the edge).

2.1.3 Hooks and cohooks

Suppose a string w does not end on a peak. If w is a zero string, then there are at most two ways in which we can add a formal inverse to w. The precise number of ways we can do this is determined by the number of arrows of target $e(w)$ in Q. Thus, if $e(w)$ is truncated then there is only one way in which we can add a formal inverse to w. Similarly, if w is not a zero string, it follows from the definition of strings and the fact that A is special biserial that there is only one way to add a formal inverse to the end of w.

Adding a hook to the end of w is by definition a string $w_h = w\alpha^{-1}\beta_1\ldots\beta_n$ that ends in a deep. For any given w, this string is necessarily unique unless w is a zero string arising from a non-truncated edge in the Brauer graph. Let $e(\alpha^{-1})$ be the edge $u \overset{x}{\text{——}} v$ such that $\widehat{e}(\alpha^{-1}) = x^u$. If x is a truncated edge, then $w\alpha^{-1}$ ends in a deep by Lemma 2.1.2(a)(i) and hence $w_h = w\alpha^{-1}$. Otherwise, $\beta_1\ldots\beta_n$ is given by the maximal direct string around the vertex v such that $\widehat{s}(\beta_1) = x^v$, described in Remark 2.1.3.

Similarly, if w does not end in a deep, then $w_c = w\alpha\beta_1^{-1}\ldots\beta_n^{-1}$ and w_c ends on a peak. Let $e(\alpha)$ be the edge $u \overset{x}{\text{——}} v$ such that $\widehat{e}(\alpha) = x^u$. If x is a truncated edge, then $w_c = w\alpha$ by Lemma 2.1.2(b)(i). Otherwise, $\beta_1^{-1} \ldots \beta_n^{-1}$ is given by the maximal inverse string around the vertex v such that $\widehat{s}(\beta_1^{-1}) = x^v$, described in Remark 2.1.3.

Informally, the process of adding or deleting hooks and cohooks to a string w presented on a Brauer graph can be summarised as follows:

(A1) To add a hook to the end of w, we add a clockwise formal inverse to the end of w around a vertex u onto a connected edge $u \overset{x}{\text{——}} v$ and then, if x is not truncated, add a maximal direct string of anticlockwise arrows around the vertex v.

(A2) To add a cohook to the end of w, we add an anticlockwise arrow to the end of w around a vertex u onto a connected edge $u \overset{x}{\text{——}} v$ and then, if x is not truncated, add a maximal inverse string of clockwise formal inverses around the vertex v.

(A3) To delete a hook from the end of w, we delete as many anticlockwise arrows as we can from the end of the string, and then we delete a single clockwise formal inverse.

(A4) To delete a cohook from the end of w, we delete as many clockwise formal inverses as we can from the end of the string, and then we delete a single anticlockwise arrow.

(A5) To add or delete hooks or cohooks from the start of w, we follow (A1)-(A4) with the string w^{-1}.

We further note from the almost split sequences given in Theorem 1.6.7, that given a string module $M = M(w)$, the rays of source and target M in the stable Auslander-Reiten quiver are given by Figure 2.3, where $\tau M(w_+) = M(w'_-)$, $\tau M(w'_+) = M(w_-)$ and

$$w'_- = \begin{cases} {}_{-h}w & \text{if } w \text{ starts in a deep,} \\ {}_{c}w & \text{otherwise,} \end{cases} \qquad w_+ = \begin{cases} w_{-c} & \text{if } w \text{ ends on a peak,} \\ w_h & \text{otherwise,} \end{cases}$$

$$w_- = \begin{cases} w_{-h} & \text{if } w \text{ ends in a deep,} \\ w_c & \text{otherwise,} \end{cases} \qquad w'_+ = \begin{cases} {}_{-c}w & \text{if } w \text{ starts on a peak,} \\ {}_h w & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
\ddots \arrow[dr] & & & & \iddots \\
& M(w'_-) \arrow[dr] & & M(w_+) \arrow[ur] & \\
& & M(w) \arrow[ur] \arrow[dr] & & \\
& M(w_-) \arrow[ur] & & M(w'_+) \arrow[dr] & \\
\iddots \arrow[ur] & & & & \ddots
\end{tikzcd}$$

Figure 2.3: The rays of source and target a given string module in the stable Auslander-Reiten quiver of a Brauer graph algebra.

Combining this information with (A1)-(A5) above, we obtain a constructive algorithm for reading off the stable Auslander-Reiten component containing M from the Brauer graph.

Example 2.1.4. With the Brauer tree algebra given in Figure 2.1, let w be the string such that $M(w) = S(1)$. Then the string presented in Figure 2.1 is the 4th module along the ray of target $S(1)$ in the direction of $M(w_-)$ of Figure 2.3.

Another example is given in Figure 2.8 in Section 2.3. Here $w_0 = w$ and $w_1 = w_+$.

Whilst the above algorithm is for constructing the stable Auslander-Reiten quiver, one can easily obtain the Auslander-Reiten quiver with the indecomposable projectives (which are also injective) using the almost split sequence

$$0 \longrightarrow \operatorname{rad} P \longrightarrow \operatorname{rad} P / \operatorname{soc} P \oplus P \longrightarrow P / \operatorname{soc} P \longrightarrow 0$$

for each indecomposable projective P.

2.2 Components

Throughout this section, we will assume (unless specified otherwise) that our Brauer graph algebras are of infinite representation type. In [24], Erdmann and Skowroński classified the Auslander-Reiten components for self-injective special biserial algebras. For convenience, we shall restate part of their main results here.

Theorem 2.2.1 ([24]). *Let A be a special biserial self-injective algebra. Then the following are equivalent:*

(i) A is representation-infinite domestic.

(ii) A is representation-infinite of polynomial growth.

(iii) There are positive integers m, p, q such that ${}_s\Gamma_A$ is a disjoint union of m components of the form $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$, m components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^p\rangle$, m components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^q\rangle$, and infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau\rangle$.

Theorem 2.2.2 ([24]). *Let $A \cong KQ/I$ be a special biserial self-injective algebra. Then the following are equivalent:*

(i) A is not of polynomial growth.

(ii) (Q, I) has infinitely many bands.

(iii) ${}_s\Gamma_A$ is a disjoint union of a finite number of components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^n\rangle$ with $n > 1$, infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau\rangle$, and infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$.

Note that the n in Theorem 2.2.2(iii) can vary. That is, in a non-domestic self-injective special biserial algebra A, there exist integers $n_1, \ldots, n_r > 1$ such that ${}_s\Gamma_A$ contains components of the form $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^{n_i}\rangle$ for each i. For the purposes of this chapter, we will distinguish between the exceptional tubes of rank 1 – that is, the tubes of rank 1 consisting of string modules, of which there are finitely many – and the homogeneous tubes of rank 1, which consist solely of band modules.

It follows from the above that A is domestic if and only if there are finitely many distinct bands in A. In [16], Bocian and Skowroński described the symmetric special biserial algebras that are domestic. Specifically, those that are 1-domestic are associated to a Brauer graph that is either a tree with precisely two exceptional vertices of multiplicity 2 and with all other vertices of multiplicity 1, or a graph with a unique simple cycle of odd length and with all vertices of multiplicity 1. Those that are 2-domestic are associated to a Brauer graph with a unique simple cycle of even length and with all vertices of multiplicity 1. All other graphs (with the exception of Brauer trees) produce non-domestic algebras.

2.2.1 Exceptional Tubes

We are interested in counting the number of exceptional tubes in ${}_s\Gamma_A$. To do this we must use the results of [48] outlined in Section 1.4.2. Recall that for a Brauer graph algebra A, the set $\mathcal{M}$ consists of all simple modules whose projective covers are uniserial and all maximal uniserial submodules of indecomposable projective A-modules. We also recall from [48] that $\Omega(M) \in \mathcal{M}$ for all $M \in \mathcal{M}$ and the minimal projective resolutions of $M \in \mathcal{M}$ are periodic, which will be required in the proof of the following.

Lemma 2.2.3. *Let A be a Brauer graph algebra. An indecomposable string module M is at the mouth of a tube in the stable Auslander-Reiten quiver ${}_s\Gamma_A$ of A if and only if $M \in \mathcal{M}$.*

Proof. ($\Rightarrow$:) Let M be at the mouth of a tube in ${}_s\Gamma_A$. Then there exists precisely one irreducible morphism $M \to N$ in ${}_s\Gamma_A$ for some indecomposable A-module N. Suppose for a contradiction that M is not uniserial and let $w = \alpha_1 \dots \alpha_m$ be the string such that $M(w) = M$. Then w is not direct (or inverse). If w ends on a peak, then deleting a cohook from the end of w gives a string w_{-c}. Note that in the case where w consists only of an arrow followed by an inverse string, then w_{-c} is a zero string and $M(w_{-c})$ is simple. Similarly, if w starts on a peak then deleting a cohook from the start of w gives a string ${}_{-c}w$ such that $M({}_{-c}w)$ is indecomposable. By the results of [17], the Auslander-Reiten sequence starting in M therefore has two non-projective middle terms. This implies that there are two irreducible morphisms of source M in ${}_s\Gamma_A$ – a contradiction to our assumption that $M(w)$ is not uniserial. It is easy to see that the same contradiction occurs in the cases where w does not start or end on a peak. Therefore w must be direct (or inverse) and M is uniserial.

Suppose M is a simple module corresponding to the stationary path ε_x in Q. Then the injective envelope $I(M)$ must be uniserial, since if $I(M)$ is biserial, then there are two distinct arrows of target x in Q and hence two non-projective middle terms in the Auslander-Reiten sequence starting in M. Thus, if M is a simple module at the mouth of a tube in ${}_s\Gamma_A$ then $M \in \mathcal{M}$.

Now suppose M is not simple. For a contradiction, suppose that $M \notin \mathcal{M}$. Note that this implies that M is not the radical of a uniserial indecomposable projective. Let w be the inverse string such that $M(w) = M$. Then M is not maximal and therefore the inverse string w is not maximal, which implies w does not end on a peak. Hence, the Auslander-Reiten sequence starting in M has an indecomposable middle term $M(w_h)$. If w does not start on a peak, then the there exists another middle term $M({}_hw)$ in the Auslander-Reiten sequence. Otherwise, if w starts on a peak, then ${}_{-c}w = \alpha_2 \dots \alpha_m$. Thus, the Auslander-Reiten sequence starting in M must have two non-projective middle terms by the results of [17] – a contradiction. So $M \in \mathcal{M}$.

($\Leftarrow$:) Suppose $M \in \mathcal{M}$. We will show that the Auslander-Reiten sequence starting in M has precisely one non-projective middle term. If M is the radical of a uniserial indecomposable projective-injective P then the Auslander-Reiten sequence starting in M is of the form

$$0 \longrightarrow M \longrightarrow M/\operatorname{soc}(P) \oplus P \longrightarrow P/\operatorname{soc}(P) \longrightarrow 0$$

and $M/\operatorname{soc}(P)$ is indecomposable since P is uniserial. Thus, there exists precisely one irreducible morphism of source $M/\operatorname{soc}(P)$ in ${}_s\Gamma_A$, as required. So suppose instead $M \in \mathcal{M}$ is not the radical of a uniserial indecomposable projective and let w_0 be the direct string such that $M(w_0) = M$. By [48], $\Omega^{-1}(M) \in \mathcal{M}$. So $\Omega^{-1}(M)$ is associated to a maximal direct string w_1 and $s(w_1) = e(w_0)$, which follows from the fact that A is symmetric and $\Omega^{-1}(M)$ is a maximal uniserial quotient of $I(M)$. Also note that $\Omega^{-1}(M)$ must be non-simple, since this would otherwise contradict our assumption that M is not the radical of a uniserial indecomposable projective. Thus, there exists an arrow $\beta \in Q_1$ such that $s(\beta) = e(w_1)$ and $e(\beta) = e(w_0) = s(w_1)$. Now let w_2 be the (maximal) direct string associated to $\Omega^{-2}(M) \in \mathcal{M}$. Then by a similar argument used for the strings w_0 and w_1, we have $s(w_2) = e(w_1)$. Thus, there exists a string $w_0\beta^{-1}w_2$ and the sequence

$$0 \longrightarrow M(w_0) \xrightarrow{f} M(w_0\beta^{-1}w_2) \xrightarrow{g} M(w_2) \longrightarrow 0$$

is exact. Note that in the case that M (resp. $\Omega^{-2}(M)$) is simple, the string w_0 (resp. w_2) is a zero string. Now $w_0\beta^{-1}w_2$ is obtained from w_0 by adding a hook and $w_0\beta^{-1}w_2$ is obtained from w_2 by adding a cohook. So f and g are irreducible and hence, the above sequence is an Auslander-Reiten sequence. $\square$

To each edge x in a Brauer graph G, we can associate precisely two modules $M_1, M_2 \in \mathcal{M}$. These two modules have the property that $\mathrm{top}(M_1) = \mathrm{top}(M_2) = x$. For each half-edge x^v associated to x and incident to a non-truncated vertex v, we have a string module $M(w) \in \mathcal{M}$ such that $w = (\gamma_1 \dots \gamma_n)^{\mathfrak{e}_v - 1}\gamma_1 \dots \gamma_{n-1}$, where $\mathfrak{C}_{v,\gamma_1} = \gamma_1 \dots \gamma_n$ and $\widehat{s}(\gamma_1) = x^v$. If x is incident to a truncated vertex u, then then $P(x)$ is uniserial and we associate to x^u the module $S(x) \in \mathcal{M}$. Thus, each half-edge in G corresponds (bijectively) to a module in $\mathcal{M}$.

We further recall from [48] that the minimal projective resolution of a module in $\mathcal{M}$ follows a Green walk (described in [37]) around the Brauer graph. Specifically, it follows from [48, Remark 3.6] that if $M \in \mathcal{M}$ corresponds to a half-edge x^v in G (in the sense described above) and the i-th step along a Green walk from $\overline{x^v}$ is a half-edge $\overline{x_i^{v_i}}$, then $\Omega^i(M) \in \mathcal{M}$ corresponds to the half-edge $x_i^{v_i}$.

The relation $M \sim N$ for $M, N \in \mathcal{M} \Leftrightarrow N = \Omega^{2i}(M)$ for some i is an equivalence relation. Thus, we can partition the set $\mathcal{M}$ of a Brauer graph algebra into orbits described by the distinct double-stepped Green walks around the Brauer graph (a similar relation can be constructed for single-stepped Green walks, but this is not of interest to us). Since Brauer graph algebras are symmetric, which implies $\tau = \Omega^2$, a consequence of Lemma 2.2.3 is the following.

Theorem 2.2.4. *Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let ${}_s\Gamma_A$ be its stable Auslander-Reiten quiver. Then*

(a) there is a bijective correspondence between the exceptional tubes in ${}_s\Gamma_A$ and the distinct double-stepped Green walks along G; and

(b) the rank of an exceptional tube is given by the length of its associated double-stepped Green walk along G.

2.2.2 Domestic Brauer Graph Algebras

For Brauer graph algebras that are infinite-domestic, the above theorem allows us to determine the precise shape of the $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ components.

Theorem 2.2.5. *Let A be a Brauer graph algebra constructed from a graph G of n edges and suppose ${}_s\Gamma_A$ has a $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ component.*

(a) If A is 1-domestic, then $p+q=2n$.

(b) If A is 2-domestic, then $p+q=n$.

Furthermore, if G is a tree, then $p=q=n$.

Proof. Suppose A is a 1-domestic Brauer graph algebra constructed from a tree. For any tree, there is only one possible single-stepped Green walk. This walk steps along each edge exactly twice – that is, we walk along both sides of each edge – and thus, the walk is of length $2n$, which is even. This therefore amounts to two distinct double-stepped walks of length n. By Theorem 2.2.4 and [24, Theorem 2.1], $p=q=n$.

Suppose instead we have a Brauer graph consisting solely of a cycle of odd length, say l. Then there are two distinct single-stepped Green walks of odd length l – one along the 'inside' of the cycle and one along the 'outside'. Since Green walks are periodic, this amounts to two distinct double-stepped Green walks of length l. Inserting an additional edge to any vertex in the graph will add another two steps

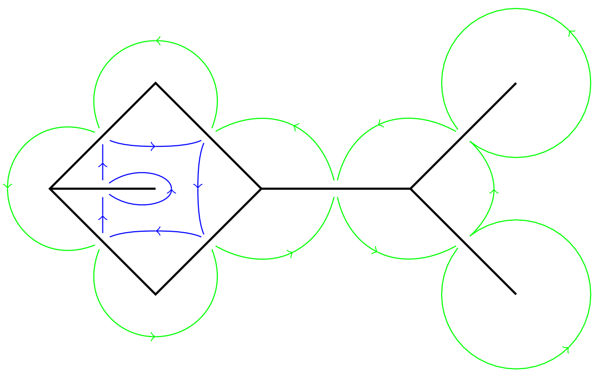


Figure 2.4: A 2-domestic Brauer graph algebra with distinct Green walks on the inside (blue) and outside (green) of the cycle in the Brauer graph.

to one of the two distinct single-stepped Green walks, and thus, an additional two steps along one of the two distinct double-stepped Green walks. Proceeding inductively, the result for (a) follows.

The argument for (b) is similar, except our initial cycle of length l is even. The two distinct single-stepped Green walks are then of even length, and hence each split into a pair of distinct double-stepped Green walks of length $\frac{l}{2}$, making a total of four distinct double-stepped Green walks. Inserting an additional edge to any vertex in the graph will add a single additional step along a pair of the four distinct double-stepped Green walks. Proceeding inductively, the result follows. □

The above proof implies that by distinguishing between the two sides of a cycle (an 'inside' and an 'outside') in a domestic Brauer graph algebra, one can actually read off p and q directly.

Corollary 2.2.6. *For a domestic Brauer graph algebra, if the Brauer graph contains a unique cycle of length l and there are n_1 additional edges on the inside of the cycle and n_2 additional edges along the outside, then the $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ components are given by*

$$p = \begin{cases} l + 2n_1 & l \text{ odd,} \\ \frac{l}{2} + n_1 & l \text{ even,} \end{cases} \quad \text{and} \quad q = \begin{cases} l + 2n_2 & l \text{ odd,} \\ \frac{l}{2} + n_2 & l \text{ even.} \end{cases}$$

Example 2.2.7. In Figure 2.4, we have $l = 4$, $n_1 = 1$ and $n_2 = 3$. So the Auslander-Reiten quiver has two components $\mathbb{Z}\widetilde{\mathbb{A}}_{3,5}$, two components $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^3\rangle$, two components $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^5\rangle$ and two infinite families of homogeneous tubes.

2.3 Components containing Simple and Indecomposable Projective-Injective Modules

There has been some work on the string modules that occur in certain Auslander-Reiten components (for example, in [30]). We are particularly interested in the components containing simple modules and indecomposable projective modules. In this section, we will show that it is possible to determine exactly which simple modules belong to the exceptional tubes of ${}_s\Gamma_A$ by looking at the Brauer graph. Similarly, one can also determine the location in the stable Auslander-Reiten quiver of the radical of a projective P by the edge associated to P in the Brauer graph.

In [17], the irreducible morphisms in ${}_s\Gamma_A$ between string modules is described. In order to consider whether a string module $M(w)$ belongs to an exceptional tube, we will need to look at the rays in the Auslander-Reiten quiver of source and target $M(w)$, of which there are at most two each. If at least one of these rays terminates, then $M(w)$ must belong to an exceptional tube. Otherwise, if all rays are infinite, then $M(w)$ must belong to either a $\mathbb{Z}\widetilde{\mathbb{A}}_{p,q}$ component (if A is domestic) or a $\mathbb{Z}\mathbb{A}_\infty^\infty$ component (if A is non-domestic). Therefore in what follows, we will need to consider the bi-directional sequence $(w_j)_{j\in\mathbb{Z}}$ defined by

$$\begin{aligned} w_{j+1} &= \begin{cases} (w_j)_{-c} & \text{if } w_j \text{ ends on a peak,} \\ (w_j)_h & \text{otherwise,} \end{cases} \\ w_{j-1} &= \begin{cases} (w_j)_{-h} & \text{if } w_j \text{ ends in a deep,} \\ (w_j)_c & \text{otherwise,} \end{cases} \end{aligned} \tag{$*$}$$

with $w_0 = w$. We will refer to this sequence frequently throughout this section of the thesis. Any string module $M(w_i)$ then lies along the line through $M(w)$ in ${}_s\Gamma_A$ given by adding or deleting from the end of w. This is precisely the south west to north east line through $M(w)$ illustrated in Figure 2.3 at the end of Section 2.1. To obtain the line through $M(w)$ in ${}_s\Gamma_A$ given by adding or deleting from the start of w (the north west to south east line through $M(w)$ illustrated in Figure 2.3), we need only consider the sequence $(*)$ with $w_0 = w^{-1}$.

Note that in the case where w_0 is the zero string ε_x at a non-truncated edge $u \overset{x}{\text{---}} v$, there exists an ambiguity in the sequence $(*)$, as there are two possible ways in which we can add a hook (resp. cohook) to w_0. We may either add a hook (resp. cohook) starting with the formal inverse (resp. arrow) α such that $\widehat{s}(\alpha) = x^u$, or with the formal inverse (resp. arrow) α' such that $\widehat{s}(\alpha') = x^v$. In this case, we must define either w_1 or w_{-1}. The following shows that it is sufficient to define only one of these terms.

Lemma 2.3.1. *Let $A = KQ/I$ be a Brauer graph algebra associated to a Brauer graph G, and let x^v be a half-edge associated to an edge x incident to a vertex v in G. Suppose there exists a segment*

$$\cdots \to M(w) \to S(x) \to M(w') \to \cdots$$

of a line L in ${}_s\Gamma_A$ for some strings $w = (\varepsilon_x)_c$ and $w' = (\varepsilon_x)_h$. If the first symbol α of w is such that $\widehat{s}(\alpha) = \overline{x^v}$, then the first symbol β^{-1} of w' is such that $\widehat{s}(\beta^{-1}) = x^v$.

Proof. There exist two irreducible morphisms of source $S(x)$ in ${}_s\Gamma_A$. Suppose $S(x) \to M(w'')$ is an irreducible morphism such that $M(w'') \ncong M(w')$. Then $w'' = {}_h(\varepsilon_x)$ and the morphism $S(x) \to M(w'')$ does not belong to the line L. In particular, $M(w'') = \tau^{-1}M(w)$. Let γ be the last symbol of w'' and let β^{-1} be the first symbol of w'. It is sufficient to show that $\widehat{e}(\gamma) = \overline{x^v}$, since it then follows that $\widehat{s}(\beta^{-1}) = x^v$.

To calculate w'', we will investigate the Auslander-Reiten sequences starting in $M(w)$. Let $y = e(\alpha)$. The string w necessarily ends on a peak. Suppose w also starts on a peak. Then $M(w)$ is a maximal submodule of $P(y)$ (since w is obtained by adding a cohook to a zero string), so $M(w) = \operatorname{rad} P(y)$. Note that $P(y)$ must be biserial, since if $P(y)$ were uniserial then $P(y)$ would be isomorphic to $M(w_0\alpha)$ for some non-zero string w_0 (and hence, our assumption that w starts on a peak would be false). Thus, there exists an Auslander-Reiten sequence

$$0 \to \operatorname{rad} P(y) \to \operatorname{rad} P(y)/\operatorname{soc} P(y) \oplus P(y) \to P(y)/\operatorname{soc} P(y) \to 0.$$

with three middle terms (since $\operatorname{rad} P(y)/\operatorname{soc} P(y)$ is a direct sum of two uniserial modules) and $M(w'') = P(y)/\operatorname{soc} P(y)$. Let $w = \alpha\delta_n^{-1}\dots\delta_1^{-1}$. Then necessarily, $w'' = \delta_{n-1}^{-1}\dots\delta_1^{-1}\delta_0^{-1}\gamma$ for some formal inverse δ_0^{-1} and some arrow γ such that $e(\delta_0^{-1}) = y = s(\gamma)$. Note that by Remark 2.1.1, $\widehat{e}(\alpha) \neq \widehat{s}(\delta_n^{-1})$, $\widehat{e}(\delta_i^{-1}) = \widehat{s}(\delta_{i+1}^{-1})$ and $\widehat{e}(\delta_0^{-1}) \neq \widehat{s}(\gamma)$. Since $\widehat{s}(\delta_n^{-1})$, $\widehat{e}(\delta_0^{-1})$, $\widehat{e}(\alpha)$ and $\widehat{s}(\gamma)$ are all half-edges associated to y (of which there are precisely two) and $e(\gamma) = x = s(\alpha)$, we conclude that $\widehat{e}(\gamma) = \widehat{s}(\alpha) = \overline{x^v}$, as required.

Suppose instead that w does not start on a peak. Then by [17], there exists an Auslander-Reiten sequence

$$0 \to M(w) \to M(w_{-c}) \oplus M({}_hw) \to M({}_hw_{-c}) \to 0.$$

So $M(w'') \cong M({}_hw_{-c})$. Again let γ be the last symbol of w'' and α be the first symbol of w. Then $\gamma\alpha$ is a direct substring of ${}_hw$. Thus, $\widehat{e}(\gamma) = \widehat{s}(\alpha) = \overline{x^v}$ by Remark 2.1.1, as required. The result then follows. □

To prove the results of this section, it will be helpful to track the end of the strings along a ray of source or target a given module in ${}_s\Gamma_A$.

Proposition 2.3.2. *Let $A = KQ/I$ be a Brauer graph algebra associated to a Brauer graph G. Let w_0 be a string such that either w_0 is the zero string ε_x or $w_0 = \alpha_1 \dots \alpha_n$, where $\widehat{e}(\alpha_n) = x^v$ and x^v is a half-edge associated to an edge x connected to a vertex v in G.*

(a) Suppose there exists a ray in ${}_s\Gamma_A$

$$M(w_0) \to M(w_1) \to \dots \to M(w_k) \to \cdots$$

such that w_1 is given by adding or deleting from the end of w_0.

(i) If $\alpha_n \in Q_1^{-1}$ (resp. $\alpha_n \in Q_1$) then the edge $e(w_i)$ corresponds to the half-edge at the i-th step along a clockwise double-stepped Green walk from x^v (resp. $\overline{x^v}$) for all $i \leq k$.

(ii) *If* $w_0 = \varepsilon_x$ *and* $w_1 = \beta^{-1}\gamma_1 \dots \gamma_m$*, where* $\widehat{s}(\beta^{-1}) = x^v$*, then the edge* $e(w_i)$ *corresponds to the half-edge at the* i*-th step along a clockwise double-stepped Green walk from* x^v *for all* $i \leq k$*.*

(b) *Suppose there exists a ray in* ${}_s\Gamma_A$

$$\cdots \to M(w_{-k}) \to \dots \to M(w_{-1}) \to M(w_0)$$

such that w_1 *is given by adding or deleting from the end of* w_0*.*

(i) *If* $\alpha_n \in Q_1$ *(resp.* $\alpha_n \in Q_1^{-1}$*) then the edge* $e(w_i)$ *corresponds to the half-edge at the* i*-th step along a double-stepped Green walk from* x^v *(resp.* $\overline{x^v}$*) for all* $i \leq k$*.*

(ii) *If* $w_0 = \varepsilon_x$ *and* $w_{-1} = \beta\gamma_1^{-1} \dots \gamma_m^{-1}$*, where* $\widehat{s}(\beta) = x^v$*, then the edge* $e(w_{-i})$ *corresponds to the half-edge at the* i*-th step along a double-stepped Green walk from* x^v *for all* $i \leq k$*.*

Proof. (a) Let $x_0^{v_0}$ be a half-edge associated to the edge $x = x_0$ and label the i-th step along a double-stepped clockwise Green walk from $x_0^{v_0}$ as $x_i^{v_i}$. Let α be the last symbol of w_i. Assume that one of the following holds for the string w_i.

(i) $\alpha \in Q_1^{-1}$ and $\widehat{e}(\alpha) = x_i^{v_i}$

(ii) $\alpha \in Q_1$ and $\widehat{e}(\alpha) = \overline{x_i^{v_i}}$

(iii) w_i is the zero string ε_{x_i}

We aim to show that the string w_{i+1} satisfies the analogous properties of (i)-(iii). The proof of this claim requires us to investigate multiple cases related to the end of w_i.

Case 1: $\alpha \in Q_1^{-1}$ and w_i ends on a peak. By Lemma 2.1.2(b)(ii), there must exist a maximal inverse substring $w' = \gamma_1^{-1} \dots \gamma_r^{-1}$ at the end of w_i. Moreover, it follows from the maximality of w' that $\widehat{s}(\gamma_1^{-1}) = y^{v_i}$, where y^{v_i} is the predecessor to $x_i^{v_i}$. Since w_i ends on a peak, w_{i+1} is given by deleting a cohook from the end of w_i. Thus, $w_i = w_{i+1}\beta w'$ for some arrow β (which exists since otherwise $M(w_i) \in \mathcal{M}$

and w_{i+1} would not be defined). Necessarily, $\widehat{s}(\beta) = z^u$, where z^u is the predecessor to $\overline{y^{v_i}}$. This is illustrated in Figure 2.5(a).

The first two steps along a clockwise Green walk from $x_i^{v_i}$ are $\overline{y^{v_i}}$ and $\overline{z^u}$. Thus, $\overline{z^u} = x_{i+1}^{v_{i+1}}$. If w_{i+1} is a zero string, then $e(w_{i+1}) = x_{i+1}$ and hence, $w_{i+1} = \varepsilon_{x_{i+1}}$. So w_{i+1} satisfies property (iii) at the start of the proof. Otherwise, let γ be the last symbol of w_{i+1}. If $\gamma \in Q_1^{-1}$, then we necessarily have $\widehat{e}(\gamma) = \overline{z^u} = x_{i+1}^{v_{i+1}}$. So w_{i+1} satisfies property (i). If $\gamma \in Q_1$, then $\widehat{e}(\gamma) = z^u = \overline{x_{i+1}^{v_{i+1}}}$ and hence, w_{i+1} satisfies property (ii).

Case 2: $\alpha \in Q_1^{-1}$ and w_i does not end on a peak. In this case, w_{i+1} is given by adding a hook to the end of w_i. We first add a formal inverse β^{-1} to the end of w_i. It follows that the arrow β is lies in the cycle $\mathfrak{C}_{v_i}$ and therefore $\widehat{e}(\beta^{-1}) = y^{v_i}$, where y^{v_i} is the predecessor to $x_i^{v_i}$. If the edge y associated to y^{v_i} is truncated then $w_{i+1} = w_i\beta^{-1}$, as illustrated in Figure 2.6(a)(i). In this subcase, the first two steps along a clockwise Green walk from $x_i^{v_i}$ are $\overline{y^{v_i}}$ and y^{v_i} respectively. Thus $x_{i+1}^{v_{i+1}} = y^{v_i}$ and hence, $\widehat{e}(\beta^{-1}) = x_{i+1}^{v_{i+1}}$. So w_{i+1} satisfies property (i) at the start of the proof.

On the other hand, if y is not truncated then $w_{i+1} = w_i\beta^{-1}w'$, where $w' = \gamma_1 \dots \gamma_r$ is a maximal direct string. It follows from the maximality of w' that $\widehat{e}(\gamma_r) = z^u$, where z^u is the predecessor to $\overline{y^{v_i}}$, as illustrated in Figure 2.6(a)(ii). The first two steps along a clockwise Green walk from $x_i^{v_i}$ are $\overline{y^{v_i}}$ and $\overline{z^u}$. Thus $x_{i+1}^{v_{i+1}} = \overline{z^u}$ and hence, $\widehat{e}(\gamma_r) = \overline{x_{i+1}^{v_{i+1}}}$. So w_{i+1} satisfies property (ii).

Case 3: $\alpha \in Q_1$ and w_i ends on a peak. By Lemma 2.1.2(b)(i), $e(w)$ is truncated. Since $w_{i+1} = (w_i)_{-c}$ and there are no formal inverses at the end of w_i, it follows that $w_i = w_{i+1}\alpha$. In this case, $\widehat{s}(\alpha) = y^u$, where y^u is the predecessor to $\overline{x_i^{v_i}}$, as illustrated in Figure 2.5(b). The first two steps along a clockwise Green walk from $x_i^{v_i}$ are $\overline{x_i^{v_i}}$ and $\overline{y^u}$ respectively. Thus $x_{i+1}^{v_{i+1}} = \overline{y^u}$. If w_{i+1} is a zero string, then $e(w_{i+1}) = x_{i+1}$ and hence, $w_{i+1} = \varepsilon_{x_{i+1}}$. So w_{i+1} satisfies property (iii) at the start

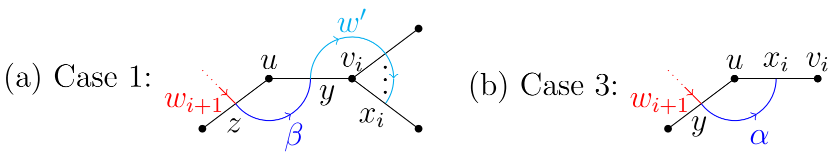


Figure 2.5: Examples of Cases 1 and 3 in the proof of Proposition 2.3.2(a).

of the proof. Otherwise, let γ be the last symbol of w_{i+1}. If $\gamma \in Q_1^{-1}$, then we necessarily have $\widehat{e}(\gamma) = \overline{y^u} = x_{i+1}^{v_{i+1}}$. So w_{i+1} satisfies property (i). If $\gamma \in Q_1$, then $\widehat{e}(\gamma) = y^u = \overline{x_{i+1}^{v_{i+1}}}$ and hence, w_{i+1} satisfies property (ii).

Case 4: $\alpha \in Q_1$ and w_i does not end on a peak. The string combinatorics, illustrated in Figure 2.6(b), is similar to Case 2. The clockwise Green walk from $x_i^{v_i}$ is also identical, so the result follows for this case by similar arguments.

Case 5: w_i is the zero string ε_{x_i}. This is only possible if either $i = 0$ or $w_i = (w_{i-1})_{-c}$. Assume $0 < i < k$. So $w_{i-1} = (w_i)_c = \gamma w'$ and $w_{i+1} = (w_i)_h = \beta^{-1} w''$ for some maximal inverse string w' and some maximal direct string w''. It follows from the string combinatorics detailed in Cases 1 and 3, that $\widehat{s}(\gamma) = \overline{x_i^{v_i}}$. It then follows from Lemma 2.3.1 that $\widehat{s}(\beta^{-1}) = x_i^{v_i}$. The string combinatorics of adding a hook starting with β^{-1} and the clockwise Green walk from $x_i^{v_i}$ is investigated in Case 2. Thus, the result for Case 5 follows by similar arguments to those used in Case 2. If $i = 0$, then w_1 is defined in the proposition statement. The result then follows from setting $x_0^{v_0} = x^v$ and again using similar arguments to those in Case 2.

The Proposition result follows from the inductive step outlined above, since setting $x_0^{v_0} = x^v$ in the cases where $w_0 = \varepsilon_x$ or w_0 ends with $\alpha_n \in Q_1^{-1}$, or setting $x_0^{v_0} = \overline{x^v}$ in the case where w_0 ends with $\alpha_n \in Q_1$ satisfies properties (i)-(iii) at the start of the proof.

(b) The proof is similar to (a). $\square$

Recall that the stable Auslander-Reiten quiver of a Brauer tree algebra is a finite tube (see for example [29]). We will first distinguish between certain edges in the Brauer graph by introducing the notion of *exceptional subtrees* of a graph. The

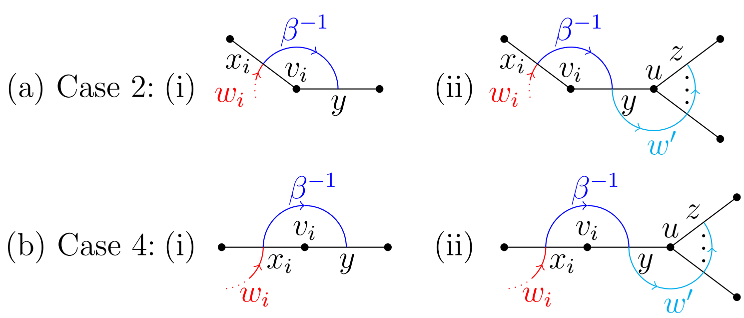


Figure 2.6: Examples of Cases 2 and 4 in the proof of Proposition 2.3.2(a).

motivation behind this definition lies in the fact that these particular subtrees of the Brauer graph have the same local structure of some Brauer tree, and hence, the string combinatorics along these subtrees behave in a similar manner to a Brauer tree algebra with exceptional multiplicity. Since the sequence $(*)$ defined at the start of Section 2.3 terminates (in both directions) in a Brauer tree algebra, one might expect the sequence $(*)$ to terminate (in one direction) for certain modules related to the edges of the exceptional subtrees. This is indeed the case, and we will later show that the simple modules and the radicals of the indecomposable projectives associated to the edges of these subtrees belong to a tube.

Definition 2.3.3. Let G be a Brauer graph that is not a Brauer tree. Consider a subgraph T of G satisfying the following properties:

(i) T is a tree,

(ii) T has a unique vertex v such that the graph $(G \setminus T) \cup \{v\}$ is connected,

(iii) T shares no vertex with any cycle of G, except at perhaps v,

(iv) every vertex of T has multiplicity 1, except for perhaps v.

We will call such a subgraph an *exceptional subtree* of G and the vertex v the *connecting vertex* of T.

Given a graph with exceptional subtrees, we can partition the edges of the graph into two distinct classes.

Definition 2.3.4. An edge of a Brauer graph G is called an *exceptional edge* if it belongs to some exceptional subtree of G. An edge of G is otherwise called a *non-exceptional edge*.

Examples of exceptional subtrees are given in Figure 2.7. The coloured edges of Figure 2.7 are the exceptional edges of G. All others are non-exceptional.

Remark 2.3.5. An exceptional subtree of a Brauer graph can also contain as subgraphs further exceptional subtrees in the following sense. Suppose x is an edge that belongs to an exceptional subtree T of G. Consider the maximal subtree T'

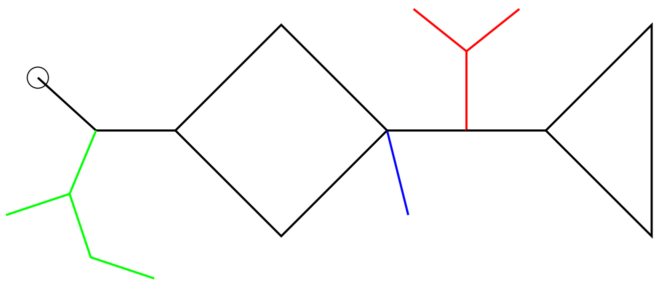


Figure 2.7: Three distinct exceptional subtrees of a Brauer graph, coloured red, green and blue respectively. A vertex represented by a circle has multiplicity $\mathfrak{e} > 1$.

of T connected to (and possibly containing) x via a vertex v such that T' shares no vertex with any non-exceptional edge. Then T' satisfies properties (i)-(iv) of Definition 2.3.3 and is hence exceptional with connecting vertex v.

There is a simple characterisation of the non-exceptional edges of a Brauer graph, as shown by the following.

Lemma 2.3.6. *Let G be a Brauer graph and x be an edge in G. Then x is non-exceptional if and only if it belongs to:*

(i) a cycle, or

(ii) a simple path between two vertices u, v belonging to cycles of G, or

(iii) a simple path between a vertex u belonging to a cycle and a vertex v with $\mathfrak{e}_v > 1$, or

(iv) a simple path between vertices u, v with $\mathfrak{e}_u, \mathfrak{e}_v > 1$.

Proof. If x belongs to a cycle, then either x is a loop or both its vertices belong to a cycle, and hence, x cannot be exceptional. Suppose x instead belongs to simple path

$$p : u \text{———} \cdots \overset{x}{\text{———}} \cdots \text{———} v$$

in G, where u and v satisfy either of (ii)-(iv). Suppose for a contradiction that there exists an exceptional subtree T of G containing x. Then T has at most one vertex v' that belongs to a cycle or has multiplicity $\mathfrak{e}_{v'} > 1$. Thus, T cannot contain both the vertices u and v of p. Consider the subgraph $G' = (G \setminus T) \cup \{v'\}$. It follows that G' is not connected, since T would otherwise contain multiple vertices that belong to a cycle of G, which would result from a simple path which avoids the edge x from p. So no such tree T exists and x is non-exceptional.

For the converse argument, we first assume G is not a Brauer tree, and hence, that there exist edges in G satisfying (i)-(iv). Suppose x is an edge of G that does not belong to any of (i)-(iv). Then at least one vertex connected to x does not belong to a cycle and does not have multiplicity $\mathfrak{e} > 1$. If both vertices satisfy this condition, we note that since G is connected and is not a Brauer tree, there must exist a simple path

$$q : u \text{———} \cdots \text{———} u',$$

where u is a vertex connected to x and u' is a vertex that belongs to a cycle of G or is such that $\mathfrak{e}_{u'} > 1$. Let v be the other vertex connected to x. Since x does not satisfy (ii)-(iv), every path of source v in G not containing x has no vertex belonging to a cycle and has no vertex of multiplicity $\mathfrak{e} > 1$. Since G is finite, the subgraph generated by all such paths of source v is a tree T, which is an exceptional subtree of G. It follows that the subgraph $T' = T \cup \{x\} \cup \{u\}$ of G is also an exceptional subtree of G and u is the connecting vertex of T'. Hence, x is exceptional. $\square$

Remark 2.3.7. It follows from the above that non-exceptional edges in a graph are all connected to each other. Specifically, if G contains at least two non-exceptional edges and x is a non-exceptional edge of G, then there exists a non-exceptional edge y connected to x via a common vertex. It also follows from the above that a non-exceptional edge is never truncated.

We will now need to introduce some technical lemmata, which will be used extensively in the proofs of the main theorems. The first lemma, given below, is used primarily to construct direct or inverse strings through exceptional edges.

Lemma 2.3.8. *Let $A = KQ/I$ be a representation-infinite Brauer graph algebra associated to a Brauer graph G. Let v be a vertex in G and $\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n$ be the cycle $\mathfrak{C}_{v,\alpha_1}$ in Q, where $s(\alpha_1) = x$ and $s(\beta_1) = y$ for some edges x and y in G such that $x \neq y$.*

(a) *Suppose that* $s(\beta_1), \ldots, s(\beta_n)$ *each belong to an exceptional subtree of* G *with connecting vertex* v *and let* w_0 *be a string such that* $e(w_0) = x$ *and* $w_0\beta_n^{-1} \ldots \beta_1^{-1}$ *is a string. Then there exists a ray*

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ *such that* $w_k = w_0\beta_n^{-1} \ldots \beta_1^{-1}$ *and* $|w_i| > |w_0|$ *for all* $0 < i \leq k$.

(b) *Suppose that* $e(\alpha_1), \ldots, e(\alpha_n)$ *each belong to an exceptional subtree of* G *with connecting vertex* v *and let* w_0 *be a string such that* $e(w_0) = x$ *and* $w_0\alpha_1 \ldots \alpha_n$ *is a string. Then there exists a ray*

$$\cdots \to M(w_{-k}) \to \cdots \to M(w_{-1}) \to M(w_0)$$

in ${}_s\Gamma_A$ *such that* $w_{-k} = w_0\alpha_1 \ldots \alpha_n$ *and* $|w_{-i}| > |w_0|$ *for all* $0 < i \leq k$.

Proof. (a) Let (w_j) be the sequence in $(*)$. If w_0 is a zero string, then let $w_1 = \gamma^{-1}\delta_1 \ldots \delta_r$ such that $\widehat{s}(\gamma^{-1}) = x^v$. Otherwise, let α be the last symbol of w_0. Since $w_0\beta_n^{-1}$ is a string, we may conclude that if $\alpha \in Q_1^{-1}$ then $\widehat{e}(\alpha) = x^v$ and if $\alpha \in Q_1$ then $\widehat{e}(\alpha) = \overline{x^v}$. Thus by Proposition 2.3.2, $e(w_i)$ is determined by the i-th step along a double-stepped clockwise Green walk from x^v.

Let T be the exceptional subtree of G with connecting vertex v such that T contains the edges $y_1 = s(\beta_1), \ldots, y_n = s(\beta_n)$. The first k steps along a clockwise double-stepped Green walk from x^v step along the half-edges of T until we reach the $(k+1)$-th step where we then reach a half-edge not in T. So by Proposition 2.3.2, $e(w_i)$ belongs to T for all $i \leq k$. At each step, w_i is of the form $w_0\beta_n^{-1}w_i'$, where w_i' is a string such that each symbol starts and ends at an edge in T. It therefore follows that $|w_i| > |w_0|$ for all $0 < i \leq k$.

We further note that a clockwise Green walk along a tree steps on both half-edges associated to each edge in the tree, so a clockwise double-stepped Green walk steps on precisely one half-edge for each edge in the tree (until we step on a half-edge not in T). One can show that the clockwise double-stepped Green walk from x^v steps along the half-edges $y_n^v, \ldots, y_1^v$. In particular, one can show that y_1^v is the

k-th step along such a walk. Thus, $e(w_k) = y_1 = y$.

Since T is a tree and all vertices of T (except perhaps v) are of multiplicity 1, it is impossible to construct a string ending at y_1 that contains arrows or formal inverses around any vertex of T other than v. Moreover, since each w_i is determined by adding hooks or deleting cohooks from the end of the string and $e(w_i)$ belongs to T for all $i \leq k$, we conclude that $w_k = w_0\beta_n^{-1} \dots \beta_1^{-1}$.

(b) The proof is similar to (a). □

In what follows, it is helpful to define a new form of Green walk, called a *non-exceptional Green walk*. This is a Green walk that ignores exceptional edges.

Definition 2.3.9. By a *non-exceptional Green walk* from a non-exceptional edge x_0 via a vertex v_0, we mean a sequence of half-edges $(x_j^{v_j})_{j \in \mathbb{Z}_{\geq 0}}$, where x_{i+1} is connected to x_i via the vertex v_i and $\overline{x_{i+1}^{v_{i+1}}}$ is the first half-edge in the successor sequence of $x_i^{v_i}$ such that x_{i+1} is non-exceptional. By a *non-exceptional clockwise Green walk* from a non-exceptional edge x_0 via v_0, we mean a similar sequence $(x_j^{v_j})_{j \in \mathbb{Z}_{\geq 0}}$ of half-edges where each $\overline{x_{i+1}^{v_{i+1}}}$ is the first half-edge in the predecessor sequence of $x_i^{v_i}$ such that x_{i+1} is non-exceptional.

The next lemma shows that by skipping certain modules along a ray of source or target a string module $M(w_0)$, one can ignore the effect of exceptional edges when adding hooks or cohooks.

Lemma 2.3.10. *Let $A = KQ/I$ be a representation-infinite Brauer graph algebra constructed from a Brauer graph G. Suppose x_1 and x_2 are non-exceptional edges incident to a vertex v_1 and let $\mathfrak{C}_{v_1,\alpha_1} = \alpha_1 \dots \alpha_m \beta_1 \dots \beta_n$, where $\widehat{s}(\alpha_1) = x_1^{v_1}$ and $\widehat{s}(\beta_1) = x_2^{v_1}$. Suppose w_0 is a string such that $e(w_0) = x_1$.*

(a) Suppose $x_2^{v_2}$ and $x_3^{v_3}$ are the first and second steps along a non-exceptional clockwise Green walk from $x_1^{v_1}$ respectively and suppose $w_0\beta_n^{-1} \dots \beta_1^{-1}$ is a string. Then there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_k = w_0\beta_n^{-1}\dots\beta_1^{-1}w'$, where $w' = \gamma_1\dots\gamma_r$ is the direct string of greatest length such that $\widehat{e}(\gamma_r) = \overline{x_3^{v_3}}$. Furthermore, $e(w_i)$ is exceptional for all $0 < i < k$ and $|w_i| > |w_0|$ for all $0 < i \leq k$.

(b) *Suppose $x_2^{v_2}$ and $x_3^{v_3}$ are the first and second steps along a non-exceptional Green walk from $x_1^{v_1}$ respectively and suppose $w_0\alpha_1\dots\alpha_m$ is a string. Then there exists a ray*

$$\cdots \to M(w_{-k}) \to \cdots \to M(w_{-1}) \to M(w_0)$$

in ${}_s\Gamma_A$ such that $w_{-k} = w_0\alpha_1\dots\alpha_m w'$, where $w' = \gamma_1^{-1}\dots\gamma_r^{-1}$ is the inverse string of greatest length such that $\widehat{e}(\gamma_r^{-1}) = \overline{x_3^{v_3}}$. Furthermore, $e(w_{-i})$ is exceptional for all $0 < i < k$ and $|w_{-i}| > |w_0|$ for all $0 < i \leq k$.

Proof. (a) Let (w_j) be the sequence in $(*)$ with w_0 as defined in the lemma. If w_0 is a zero string, then assume the first symbol α^{-1} of w_1 is such that $\widehat{s}(\alpha^{-1}) = x_1^{v_1}$. Since x_2 is given by a non-exceptional clockwise Green walk from x_1 via v_1, it follows that $s(\beta_i)$ is exceptional for all $i > 1$. Additionally, since $w_0\beta_n^{-1}\dots\beta_2^{-1}$ is a string, Lemma 2.3.8(a) applies and there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_l) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_l = w_0\beta_n^{-1}\dots\beta_2^{-1}$ and $|w_i| > |w_0|$ for all $0 < i \leq l$. It follows from Proposition 2.3.2(a) and the proof of Lemma 2.3.8(a) that $e(w_i)$ is exceptional for all $0 < i \leq l$.

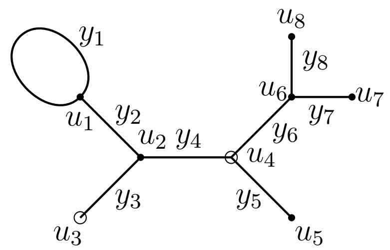
Since $w_l\beta_1^{-1}$ is a string, w_l does not end on a peak and hence, $w_{l+1} = (w_l)_h = w_l\beta_1^{-1}w''$, where $w'' = \gamma_1\dots\gamma_t$ is a maximal direct string. So $|w_{l+1}| > |w_0|$. If $e(w_{l+1})$ is non-exceptional, then $\widehat{e}(\gamma_t) = \overline{x_3^{v_3}}$, as required. So suppose instead $e(w_{l+1})$ is exceptional. Then there exists an integer r such that $\widehat{e}(\gamma_r) = \overline{x_3^{v_3}}$ and $e(\gamma_i)$ is exceptional for all $r < i \leq t$. In particular, the string $w' = \gamma_1\dots\gamma_r$ is the direct string of greatest length such that $\widehat{e}(\gamma_r) = \overline{x_3^{v_3}}$. By Lemma 2.3.8(b), there exists a ray

$$\cdots \to M(w_{l+1}) \to \cdots \to M(w_{k-1}) \to M(w_k)$$

in ${}_s\Gamma_A$ such that $w_k = w_l\beta_1^{-1}w'$ and w_{l+1} is as above. Moreover, $|w_i| > |w_k| > |w_0|$ for all $l+1 \leq i < k$. This ray belongs to the same ray containing $M(w_0)$ and $M(w_l)$, as all changes are made to the end of the string. The result then follows.

(b) The proof is similar to (a). □

Example 2.3.11. Consider the following Brauer graph G, where the circled vertices u_3 and u_4 have a multiplicity of two and all other vertices have multiplicity one.



The exceptional edges of G are y_5, y_6, y_7 and y_8. The non-exceptional edges of G are y_1, y_2, y_3 and y_4.

Let $\mathfrak{C}_{u_2,\beta_1} = \beta_1\beta_2\beta_3$ and $\mathfrak{C}_{u_4,\gamma_1} = \gamma_1\gamma_2\gamma_3$, where $s(\beta_1) = y_2$ and $s(\gamma_1) = y_4$. Consider the string $w_0 = \beta_1^{-1}$. The first two steps along a non-exceptional clockwise Green walk from $y_2^{u_2}$ are $y_4^{u_4}$ and $y_4^{u_2}$ respectively. Since $w_0\beta_3^{-1}$ is a string, Lemma 2.3.10(a) implies there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_k = w_0\beta_3^{-1}\gamma_1\gamma_2\gamma_3$. Moreover, $e(w_i)$ is exceptional for all $0 < i < k$ and $|w_i| > |w_0|$ for all $0 < i \leq k$. One can verify that $k = 5$ in this example, as illustrated in Figure 2.8.

One may also notice from Figure 2.8 the use of Lemma 2.3.8(b) in the proof of Lemma 2.3.10(a) on the ray segment

$$M(w_1) \to \cdots \to M(w_5).$$

The following remark is useful for the next lemma.

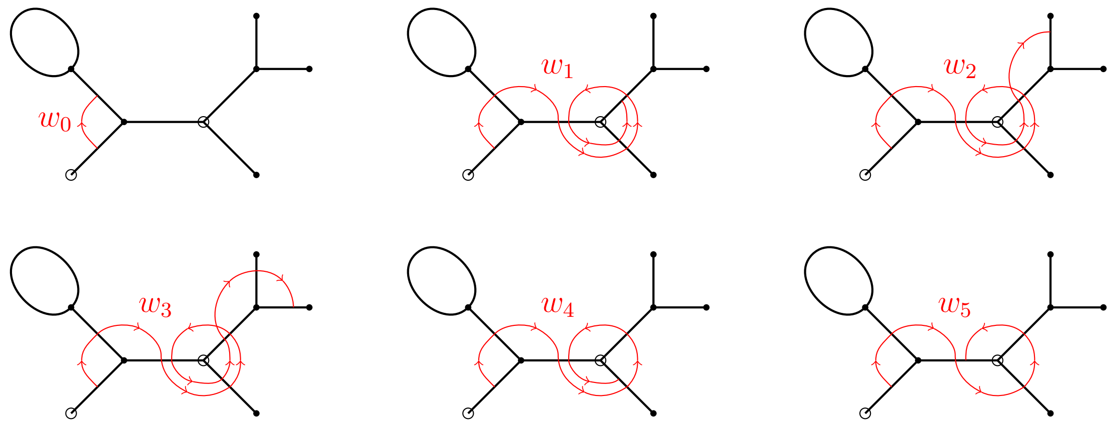


Figure 2.8: The first 5 terms in a sequence of strings given by $(*)$.

Remark 2.3.12. Non-exceptional Green walks are periodic. Thus, one can perform a non-exceptional (clockwise or anticlockwise) Green walk to construct a cycle

$$c: \ v_0 \xrightarrow{x_1} v_1 \xrightarrow{x_2} v_2 \longrightarrow \cdots \longrightarrow v_{m-1} \xrightarrow{x_m} v_0 \ ,$$

of G consisting of non-exceptional-edges such that $x_{i+1}^{v_i}$ is the first half-edge in the predecessor (resp. successor) sequence of $x_i^{v_i}$ such that x_{i+1} is non-exceptional.

Lemma 2.3.13. *Let $A = KQ/I$ be a Brauer graph algebra associated to a Brauer graph G. Let $M(w_0)$ be the string module associated to a string $w_0 = \alpha_1 \dots \alpha_n$.*

(a) (i) If $e(w_0)$ is a non-exceptional edge in G and $\alpha_n \in Q_1$, then the ray in ${}_s\Gamma_A$ of source $M(w_0)$ given by adding or deleting from the end of w_0 is infinite. Furthermore, each module $M(w_i)$ along the ray is such that $|w_i| > |w_0|$ for all $i > 0$.

(ii) If x is a non-exceptional edge in G and $w_0 = \varepsilon_x$, then both rays in ${}_s\Gamma_A$ of source $M(w_0)$ are infinite. Furthermore, $|w| > |w_0|$ for any module $M(w) \ncong M(w_0)$ along any such ray.

(b) (i) If $e(w_0)$ is a non-exceptional edge in G and $\alpha_n \in Q_1^{-1}$, then the ray in ${}_s\Gamma_A$ of target $M(w_0)$ given by adding or deleting from the end of w_0 is infinite. Furthermore, each module $M(w_{-i})$ along the ray is such that $|w_{-i}| > |w_0|$ for all $i > 0$.

(ii) If x is a non-exceptional edge in G and $w_0 = \varepsilon_x$, then both rays in ${}_s\Gamma_A$ of target $M(w_0)$ are infinite. Furthermore, $|w| > |w_0|$ for any module $M(w) \ncong M(w_0)$ along any such ray.

Proof. The proof relies upon the iterative use of Lemma 2.3.10.

(a)(i) Let (w_j) be sequence $(*)$. Let x_1 be the non-exceptional edge such that $e(w_0) = x_1$ and let $x_1^{v_0}$ be a half-edge associated to x_1 such that $\widehat{e}(\alpha_n) = x_1^{v_0}$. We may perform a non-exceptional clockwise Green walk from $x_1^{v_1} = \overline{x_1^{v_0}}$ to construct a (not necessarily simple) cycle c in G of the form given in Remark 2.3.12.

Since $\alpha_n \in Q_1$, there exists a non-zero inverse string $\beta_1^{-1} \dots \beta_r^{-1}$ such that $\widehat{s}(\beta_1^{-1}) = x_1^{v_1}$, $\widehat{e}(\beta_r^{-1}) = x_2^{v_1}$ and $w_0\beta_1^{-1} \dots \beta_r^{-1}$ is a string. Thus by Lemma 2.3.10(a), there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $e(w_k) = x_3$ and $|w_i| > |w_0|$ for all $0 < i \leq k$. Furthermore, the last symbol α of w_k is an arrow such that $\widehat{e}(\alpha) = x_3^{v_2}$.

Since w_k satisfies similar properties to w_0, we may use the above argument iteratively along the cycle c. Thus, the sequence $(*)$ never terminates, and hence, the ray of source $M(w_0)$ given by adding or deleting from the end of w_0 is infinite. Moreover, $|w_i| > |w_0|$ for all $i > 0$.

(a)(ii) Let (w_j) be sequence $(*)$ and let x^v be a half-edge associated to x. There are two rays of source $S(x)$ in ${}_s\Gamma_A$. These are obtained by choosing w_1 such that the first symbol β^{-1} of w_1 is a formal inverse with either $\widehat{s}(\beta^{-1}) = x^v$ or $\widehat{s}(\beta^{-1}) = \overline{x^v}$.

A clockwise non-exceptional Green walk from either x^v or $\overline{x^v}$ induces a cycle of non-exceptional edges similar to that in Remark 2.3.12. The conditions of Lemma 2.3.10(a) are satisfied for any zero string associated to a non-exceptional edge, so similar arguments to (i) show that both possible sequences (w_j) starting with w_0 are infinite and $|w_i| > |w_0|$ for all $i > 0$. Thus, both rays in ${}_s\Gamma_A$ of source $M(w_0)$ are infinite and $|w| > |w_0|$ for any module $M(w) \ncong M(w_0)$ along any such ray.

(b) The proof of (b)(i) and (b)(ii) is similar to (a)(i) and (a)(ii) respectively. $\square$

We now prove the main results of this section.

Theorem 2.3.14. *Let A be a representation-infinite Brauer graph algebra associated to a Brauer graph G and let x be an edge in G. Then the simple module $S(x)$ and the radical of the projective $P(x)$ belong to exceptional tubes of ${}_s\Gamma_A$ if and only if x is an exceptional edge.*

Proof. ($\Rightarrow$:) Suppose x is non-exceptional and consider the simple module $S(x)$. This is associated to the zero string $w = \varepsilon_x$. Thus, Lemma 2.3.13(a)(ii) applies and so both rays of source $S(x)$ in ${}_s\Gamma_A$ are infinite. Hence, $S(x)$ does not belong to a tube.

Now consider the module $\operatorname{rad} P(x)$ and instead let w be the string such that $M(w) = P(x)/\operatorname{soc} P(x) = \tau^{-1} \operatorname{rad} P(x)$. We aim to show that $M(w)$ does not belong to a tube, since it then follows that $\operatorname{rad} P(x)$ does not belong to a tube. Note that $w = w'w''$, where $M(w'), M(w'') \in \mathcal{M}$. That is, w' is a maximal inverse string and w'' is a maximal direct string such that $e(w') = x = s(w'')$.

If $e(w)$ is non-exceptional, then we can apply Lemma 2.3.13(a)(i) to show that the ray of source $M(w)$ given by adding or deleting from the end of the string is infinite. Otherwise if $e(w)$ is exceptional, then suppose $w'' = \beta_1 \dots \beta_n$ and let r be the greatest integer such that $\widehat{e}(\beta_r)$ is the first half-edge associated to a non-exceptional edge in the predecessor sequence from $\widehat{e}(\beta_n)$. Then by Lemma 2.3.8(b), there exists a ray

$$\cdots \to M(w_{-k}) \to \cdots \to M(w_{-1}) \to M(w_0)$$

in ${}_s\Gamma_A$ such that $w_0 = w'\beta_1 \dots \beta_r$ and $w_{-k} = w$. The string w_0 satisfies the conditions of Lemma 2.3.13(a)(i), and therefore the ray in ${}_s\Gamma_A$ of source $M(w_0)$ given by adding or deleting from the end of w_0 is infinite. This is contained within the ray of source $M(w_{-k})$ given by adding or deleting from the end of w_{-k}, and so this ray is also infinite. A similar argument shows that the other ray of source $M(w)$, which is given by adding or deleting from the start of the string, is infinite – we simply use the same arguments with the string w^{-1}.

($\Leftarrow$:) Suppose x is an exceptional edge. If x is truncated, then $P(x)$ is uniserial and $S(x) \in \mathcal{M}$, $\operatorname{rad} P(x) \in \mathcal{M}$. Hence, it follows trivially from Lemma 2.2.3 that $S(x)$ and $\operatorname{rad} P(x)$ belong to an exceptional tube. So suppose instead that x is non-truncated. Let v be a vertex incident to x and consider the maximal exceptional subtree T (Remark 2.3.5) with connecting vertex v. We choose v such that T does not contain x.

To show that $S(x)$ belongs to an exceptional tube, we first note that $\mathfrak{e}_v = 1$ (since x is exceptional). Let $w = \beta_n^{-1} \dots \beta_1^{-1}$ be the maximal inverse string such that $\widehat{s}(\beta_n^{-1}) = x^v$. Then $e(\beta_i^{-1})$ belongs to T and $e(\beta_i^{-1}) \neq x$ for all i. Hence by Lemma 2.3.8(a), there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_0 = \varepsilon_x$ and $w_k = w$. But $M(w_0) = S(x)$ and $M(w_k) \in \mathcal{M}$. Thus, $M(w_k)$ sits at the mouth of an exceptional tube by Lemma 2.2.3 and hence, $S(x)$ belongs to an exceptional tube.

To show that $\operatorname{rad} P(x)$ also belongs to an exceptional tube, let $w' = \gamma_1 \dots \gamma_m$ be the maximal direct string such that $\widehat{e}(\gamma_n) = \overline{x^v}$. Then a similar argument to that used above for $S(x)$ shows that there exists a ray

$$M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_0 = w'$ and $w_k = w'w$. Since $M(w_0) \in \mathcal{M}$ and $M(w_k) = \operatorname{rad} P(x)$, we conclude that $\operatorname{rad} P(x)$ belongs to an exceptional tube. $\square$

The above theorem as stated shows that the modules $S(x)$ and $\operatorname{rad} P(x)$ for an exceptional edge x belong to exceptional tubes of ${}_s\Gamma_A$. However, $S(x)$ and $\operatorname{rad} P(x)$ may not necessarily belong to the same exceptional tube. This is due to the construction in the latter part of the proof, where we show that there exist modules $M_1, M_2 \in \mathcal{M}$ with $M_1 \neq M_2$, such that $S(x)$ and M_1 belong to the same tube and $\operatorname{rad} P(x)$ and M_2 belong to the same tube. We cannot however, guarantee that M_1 and M_2 belong to the same tube. However, we can use Theorem 2.2.4 to describe when $S(x)$ and $\operatorname{rad} P(x)$ do belong to the same exceptional tube of ${}_s\Gamma_A$.

Corollary 2.3.15. *Given an exceptional edge $u \overset{x}{\text{——}} v$ in a Brauer graph, $S(x)$ and $\operatorname{rad} P(x)$ belong to the same exceptional tube if and only if x^u and x^v occur within the same double-stepped Green walk.*

Proof. Let u, v be the vertices connected to x. Let w be the string such that $M(w) = \operatorname{rad} P(x)$. Then $w = w'w''$, where w' is the maximal direct with last symbol α such that $\widehat{e}(\alpha) = x^u$ and w'' is the maximal inverse string with first symbol β such that $\widehat{s}(\alpha) = x^v$. Let $M_1 = M(w'')$ and $M_2 = M(w')$. Then by the construction in the proof of Theorem 2.3.14, $S(x)$ and M_1 belong to the same tube and $\operatorname{rad} P(x)$ and M_2 belong to the same tube. In the case where x is truncated, we actually have $S(x) = M_1$ and $\operatorname{rad} P(x) = M_2$

Recall from Section 2.2.1 that to each half-edge y^t in G, we have a corresponding string module $M = M(w''') \in \mathcal{M}$. If t is a truncated vertex, then $w''' = \varepsilon_y$. Otherwise, w''' is a direct string with first symbol γ such that $\widehat{s}(\gamma) = y^t$. Also recall from [48, Remark 3.6] that if the i-th step along a Green walk from $\overline{y^t}$ is a half-edge $\overline{y_i^{t_i}}$, then $\Omega^i(M) \in \mathcal{M}$ corresponds to the half-edge $y_i^{t_i}$.

Let α' be the first symbol of w' and β' be the last symbol of w''. Then $\widehat{e}(\beta')$ and $\widehat{s}(\alpha')$ correspond to the modules M_1 and M_2 respectively. Let M_1' and M_2' be the modules in $\mathcal{M}$ corresponding to the half-edges x^u and x^v respectively. Since $\overline{\widehat{e}(\beta')}$ is the first step along a Green walk from $\overline{x^u} = x^v$ and $\overline{\widehat{s}(\alpha')}$ is the first step along a Green walk from $\overline{x^v} = x^u$, we have $\Omega(M_1') = M_1$ and $\Omega(M_2') = M_2$. Note that M_1 belongs to the same tube as M_2 if and only if $M_1 = \tau^i M_2 = \Omega^{2i} M_2$ for some i. This is possible if and only if $M_1' = \tau^i M_2'$ for some i. Since M_1' and M_2' correspond to x^u and x^v respectively, it follows that x^u and x^v belong to the same double-stepped Green walk, as required. □

Theorem 2.3.16. *Let A be a representation-infinite Brauer graph algebra and let x and y be (not necessarily distinct) non-exceptional edges of the associated Brauer graph G. Then $S(x)$ and $\operatorname{rad} P(y)$ belong to the same component of ${}_s\Gamma_A$ if and only* *if either A is 1-domestic or there exists a (not necessarily simple) path*

$$p : u_0 \overset{x_1}{\text{——}} v_1 \overset{x_2}{\text{——}} v_2 \text{——} \cdots \text{——} v_{n-2} \overset{x_{n-1}}{\text{——}} v_{n-1} \overset{x_n}{\text{——}} u_1$$

of even length in G consisting of non-exceptional edges such that

(i) $x_1 = x$ *and* $x_n = y$;

(ii) *every edge* x_i *is not a loop;*

(iii) *if* $x_i \neq x_{i+1}$ *then* $\mathfrak{e}_{v_i} = 1$;

(iv) *if* $x_i = x_{i+1}$ *then* $\mathfrak{e}_{v_i} = 2$;

(v) x_i *and* x_{i+1} *are the only non-exceptional edges incident to* v_i *in* G.

Proof. ($\Leftarrow$:) Suppose A is 1-domestic. Then there exists precisely one component of ${}_s\Gamma_A$ that is not a tube. Thus by Theorem 2.3.14, the simple modules and the radicals of the projectives associated to the non-exceptional edges of G all belong to the same component of ${}_s\Gamma_A$. So suppose instead A is not 1-domestic and a path p satisfying the properties (i)-(v) exists. We will show that $S(x_i)$ is in the same component of ${}_s\Gamma_A$ as $\operatorname{rad} P(x_{i+1})$ and $S(x_{i+1})$ is in the same component of ${}_s\Gamma_A$ as $\operatorname{rad} P(x_i)$ for all i. Since p is of even length, it will follow from this that $S(x)$ is in the same component as $\operatorname{rad} P(y)$.

There are two possible cases, which arise from conditions (iii) and (iv) respectively. The proof for both cases is similar. In either case, if there are no exceptional edges incident to v_i, then $S(x_i)$ is a direct summand of $\operatorname{rad} P(x_{i+1})/\operatorname{soc} P(x_{i+1})$. Thus, there exists an irreducible morpism $S(x_i) \to \operatorname{rad} P(x_{i+1})$ and so $S(x_i)$ and $\operatorname{rad} P(x_{i+1})$ belong to the same component of ${}_s\Gamma_A$. Suppose instead there are exceptional edges incident to v_i and let $\mathfrak{C}_{v_i,\gamma_1} = \gamma_1 \dots \gamma_r \delta_1 \dots \delta_t$, where $s(\gamma_1) = x_i$ and $s(\delta_1) = x_{i+1}$. If $x_i = x_{i+1}$ and $\mathfrak{e}_{v_i} = 2$ (condition (iv)) then $t = r$ and $\delta_1 = \gamma_1, \dots, \delta_t = \gamma_r$. Note that (in both cases) the edges $s(\delta_2), \dots, s(\delta_t)$ and $e(\gamma_1), \dots, e(\gamma_{r-1})$ each belong to an exceptional subtree with connecting vertex v_i

and that $x_i \neq s(\delta_2), e(\gamma_{r-1})$. By Lemma 2.3.8(a), there exists a ray

$$R : M(w_0) \to M(w_1) \to \cdots \to M(w_k) \to \cdots$$

in ${}_s\Gamma_A$ such that $w_0 = \varepsilon_{x_i}$ and $w_k = \delta_t^{-1} \ldots \delta_2^{-1}$. Since $(w_k)^{-1}\gamma_1 \ldots \gamma_{r-1}$ is a string, it follows from Lemma 2.3.8(b) that there exists a ray

$$R' : \cdots \to M(w'_{-l}) \to \cdots \to M(w'_{-1}) \to M(w'_0)$$

in ${}_s\Gamma_A$ (which is perpendicular to R) such that $w'_0 = (w_k)^{-1}$ and $w'_{-l} = (w_k)^{-1}\gamma_1 \ldots \gamma_{r-1}$. Now $M(w'_{-l})$ is direct summand of $\operatorname{rad} P(x_{i+1})/\operatorname{soc} P(x_{i+1})$, and thus, there exists an irreducible morphism $\operatorname{rad} P(x_{i+1}) \to M(w'_{-l})$. Hence, $S(x_i) = M(w_0)$ is in the same component of ${}_s\Gamma_A$ as $\operatorname{rad} P(x_{i+1})$, as required. A similar argument shows $S(x_{i+1})$ is in the same component of ${}_s\Gamma_A$ as $\operatorname{rad} P(x_i)$. Since n is even and $x_1 = x$ and $x_n = y$, this implies $S(x)$ and $\operatorname{rad} P(y)$ lie in the same component of ${}_s\Gamma_A$.

($\Rightarrow$:) Suppose $S(x)$ and $\operatorname{rad} P(y)$ lie in the same component of ${}_s\Gamma_A$ and consider the strings w_0 and w'_0 such that $M(w_0) = \operatorname{rad} P(y)$ and $M(w'_0) = S(x)$. Let L be the line in ${}_s\Gamma_A$ through $M(w_0)$ given by adding or deleting from the end of w_0 and let L' be the line in ${}_s\Gamma_A$ through $M(w'_0)$ given by adding or deleting from the start of w'_0. Then L is perpendicular to L' and there exists a module $M(w)$ along L and a module $M(w')$ along L' such that $M(w) = M(w')$. Note that this trivially implies that $w = w'$. We aim to show that this implies that either there exists a path p of the form given in the theorem statement, or G contains precisely one non-exceptional edge, which is a loop (and hence, A is 1-domestic).

First suppose the module at the intersection point of L and L' lies along the ray of target $M(w_0)$. Note that w_0 is of the form $\gamma_1 \ldots \gamma_r \delta_1^{-1} \ldots \delta_t^{-1}$, where $\gamma_1 \ldots \gamma_r$ is maximal direct and $\delta_1^{-1} \ldots \delta_t^{-1}$ is maximal inverse. Let d be the greatest integer such that $e(\delta_d^{-1})$ is non-exceptional and let w_{-k} be the string $\gamma_1 \ldots \gamma_r \delta_1^{-1} \ldots \delta_d^{-1}$. Then it follows from Lemma 2.3.8(a) that there exists a ray

$$M(w_{-k}) \to M(w_{-k+1}) \to \cdots \to M(w_0) \to \cdots$$

in ${}_s\Gamma_A$ such that $|w_{-i}| > |w_{-k}| > 0$ for all $0 \leq i < k$. Suppose $\widehat{s}(\delta_1^{-1}) = y^u$. Then it follows that $e(\delta_d^{-1})$ is the edge associated to the first step along a non-exceptional Green walk from y^u. Using Lemma 2.3.10(b) iteratively starting with the string w_{-k} and the half-edge $\overline{\widehat{e}(\delta_d^{-1})}$, we conclude that if $M(w_{-i})$ is a module along the ray of target $M(w_0)$ such that w_{-i} ends at a non-exceptional edge z, then z belongs to the cycle c of G constructed by performing a non-exceptional Green walk from y^u (described in Remark 2.3.12). In particular, since L and L' intersect at a module along the ray of target $M(w_0)$, there exists a module $M(w_{-i})$ along this ray such that $e(w_{-i}) = x$. Thus, x belongs to c. So perform a non-exceptional Green walk from y^u to construct a path

$$q : u_0 \overset{y_1}{\text{———}} u_1 \text{———} \cdots \text{———} u_{m-2} \overset{y_{m-1}}{\text{———}} u_{m-1} \overset{y_m}{\text{———}} u_m \qquad (\dagger)$$

of even length in G, where $y_m = y$, $u = u_{m-1}$ and $y_1 = x$. Also note that $\widehat{e}(\delta_d^{-1}) = y_{m-1}^{u_{m-1}}$.

Use Lemma 2.3.10(b) iteratively along q starting with w_{-k} until we obtain a ray

$$M(w_{-l}) \to \cdots \to M(w_{-k-1}) \to M(w_{-k}) \to \cdots \to M(w_0)$$

along L such that $e(w_{-l}) = y_1 = x$. Note that it follows from Lemma 2.3.10(b) that $|w_{-i}| > |w_{-k}| > 0$ for all $i > k$. It also follows that w_{-l} is of the form

$$w_{-l} = \gamma_1 \dots \gamma_r w_{m-1}^+ w_{m-2}^- w_{m-3}^+ \dots w_2^- w_1^+,$$

where w_i^- is the direct string of shortest length with first symbol α_i and last symbol β_i such that $\widehat{s}(\alpha_i) = y_{i+1}^{u_i}$ and $\widehat{e}(\beta_i) = y_i^{u_i}$, and w_i^+ is the inverse string of greatest length with first symbol ζ_i^{-1} and last symbol η_i^{-1} such that $\widehat{s}(\zeta_i^{-1}) = y_{i+1}^{u_i}$ and $\widehat{e}(\eta_i^{-1}) = y_i^{u_i}$. Also note that further use of Lemma 2.3.10(b) implies that w_{-l} is a prefix to any string further along the ray of target $M(w_0)$.

Now consider the zero string $w_0' = \varepsilon_x$. We aim to construct the string w_{-l} by adding to the start of w_0', and hence locate $M(w_{-l})$ along the line L'. We note that since w_0' is a zero string and since the last symbol β^{-1} of w_{-l} is a formal inverse,

w_{-l} can only be constructed by adding cohooks to the start of w'_0, and hence, w_{-l} lies along the ray of target $M(w'_0)$. In particular, the last symbol of $w'_{-1} = {}_c(w'_0)$ must be β^{-1}.

Suppose $\widehat{e}(\beta^{-1}) = x^{v_1}$. Let $x_1 = x$ and label the $(i-1)$-th step along a non-exceptional Green walk from $x_1^{v_1}$ by $x_i^{v_i}$. The i-th use of Lemma 2.3.10(b) on w'_0 produces a string w'_{-k_i} along the ray of target $M(w'_0)$ of the form

$$w'_{-k_i} = w'^{+}_{2i} w'^{-}_{2i-1} \dots w'^{+}_{4} w'^{-}_{3} w'^{+}_{2} w'^{-}_{1},$$

where w'^{-}_j is the inverse string of shortest length with first symbol ζ'^{-1}_j and last symbol η'^{-1}_j such that $\widehat{s}(\zeta'^{-1}_j) = x_j^{v_{j-1}}$ and $\widehat{e}(\eta'^{-1}_j) = x_j^{v_j}$, and w'^{+}_j is the direct string of greatest length with first symbol α'_j and last symbol β'_j such that $\widehat{s}(\alpha'_j) = x_{j+1}^{v_j}$ and $\widehat{e}(\beta'_j) = x_j^{v_j}$. Moreover, w_{-k_i} is a suffix of any string w' further along the ray of target $M(w'_0)$.

Since the module $M(w_{-l})$ exists along the ray of target $M(w'_0)$, we may conclude that $w'^{-}_j = w^{+}_j$ and $w'^{+}_j = w^{-}_j$. Thus, $\widehat{s}(\alpha'_j) = \widehat{s}(\alpha_j)$, $\widehat{s}(\zeta'^{-1}_j) = \widehat{s}(\zeta^{-1}_j)$, $\widehat{e}(\beta'_j) = \widehat{e}(\beta_j)$ and $\widehat{e}(\eta'^{-1}_j) = \widehat{e}(\eta^{-1}_j)$. Hence, $x_{j+1}^{v_j} = y_{j+1}^{u_j}$ and $x_j^{v_j} = y_j^{u_j}$. Since each $x_j^{v_j}$ is a step along a non-exceptional Green walk from $x_{j-1}^{v_{j-1}}$ and each $y_{j-1}^{u_{j-1}}$ is a step along a non-exceptional Green walk from $y_j^{u_j}$, this implies that $x_j^{v_j}$ is the first non-exceptional successor to $x_{j-1}^{v_{j-1}}$ and $x_{j-1}^{v_{j-1}}$ is the first non-exceptional successor to $x_j^{v_j}$. This is possible only if there is no half-edge z^{u_i} incident to any u_i in q such that z is non-exceptional and $z^{u_i} \neq y_i^{u_i}, y_{i+1}^{u_i}$. Moreover, $w'^{-}_j = w^{+}_j$ and $w'^{+}_j = w^{-}_j$ only if $\mathfrak{e}_{v_j} = 1$ if $x_j^{v_j} \neq x_{j+1}^{v_j}$ or $\mathfrak{e}_{v_j} = 2$ if $x_j^{v_j} = x_{j+1}^{v_j}$. Thus, either q is a path of the form p in the theorem statement, or q is a path of length 2 along a loop x in G with incident vertex v such that $\mathfrak{e}_v = 1$ and x is the only non-exceptional edge of G. In the latter case, A is 1-domestic, as required.

Next suppose the module at the intersection point of L and L' instead lies along the ray of source $M(w_0) = \operatorname{rad} P(y)$ (along L). Again let $w_0 = \gamma_1 \dots \gamma_r \delta_1^{-1} \dots \delta_t^{-1}$. Note that w_0 ends on a peak, so let $w_1 = (w_0)_{-c} = \gamma_1 \dots \gamma_{r-1}$. Now let d be the greatest integer such that $s(\gamma_d)$ is non-exceptional and let $w_k = \gamma_1 \dots \gamma_{d-1}$ if $d > 1$ or let $w_k = \varepsilon_{s(\gamma_1)}$ if $d = 1$. Then by Lemma 2.3.8(b), there exists a ray in ${}_s\Gamma_A$ of the form

$$\cdots \to M(w_0) \to M(w_1) \to \cdots \to M(w_{k-1}) \to M(w_k).$$

In the case where $w_k = \varepsilon_x$, we note that x and y share a common vertex v. Moreover, it follows from the maximality of the string $\gamma_1 \dots \gamma_r$ that x^v and y^v are the only half-edges incident to v such that their corresponding edges are non-exceptional, and that $\mathfrak{e}_v = 1$ if $x^v \neq y^v$ or $\mathfrak{e}_v = 2$ if $x^v = y^v$. Thus in this special case, x and y belong to a path of length 2 that is either of the form p in the theorem statement, or is either a path along a loop x in G with incident vertex v such that $\mathfrak{e}_v = 1$ and x is the only non-exceptional edge of G.

Otherwise, we note that $e(w_k)$ is non-exceptional and is associated to the first step along a non-exceptional clockwise Green walk from $\widehat{e}(\gamma_r)$. The proof for the case of the intersection point of L and L' belonging to the ray of source $M(w_0)$ is then similar to the proof of the of the case where the intersection point belongs to the ray of target $M(w_0)$. We will summarise the argument. We first construct a path q of the form (†) above by performing a clockwise non-exceptional Green walk from $\widehat{e}(\gamma_r)$, which we label by $y_m^{u_{m-1}}$. We then use Lemma 2.3.10(a) iteratively to produce a string w_l such that $e(w_l) = x$ and $M(w_l)$ is at the intersection of L and L'. It follows that w_l is of the form.

$$w_l = \gamma_1 \dots \gamma_{d-1} w_{m-2}^- w_{m-3}^+ \dots w_2^- w_1^+,$$

where w_i^- is the inverse string of shortest length between $y_{i+1}^{u_i}$ and $y_i^{u_i}$ and w_i^+ is the direct string of greatest length between $y_{i+1}^{u_i}$ and $y_i^{u_i}$. Since the last symbol of w_l is an arrow, w_l lies in the ray (along L') of source $M(w_0')$. Let $x_1 = x$ and label the $(i-1)$-th step along a non-exceptional clockwise Green walk from $x_1^{v_1}$ by $x_i^{v_i}$. The i-th use of Lemma 2.3.10(a) on w_0' produces a string

$$w_{2i}'^+ w_{2i-1}'^- \dots w_2'^+ w_1'^-,$$

where $w_j'^-$ is the inverse string of shortest length between $x_{j+1}^{v_j}$ and $x_j^{v_j}$ and $w_j'^+$ is the direct string of greatest length between $x_{j+1}^{v_j}$ and $x_j^{v_j}$. So $w_j'^- = w_j^+$ and

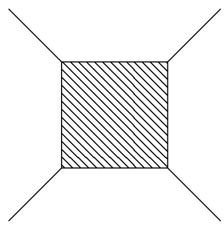
$w_j'^{+} = w_j^{-}$. By similar arguments as presented earlier in the proof, this implies that q is a path of the form p in the theorem statement, or A is 1-domestic. $\square$

Chapter 3

Tame and Wild Symmetric Special Multiserial Algebras

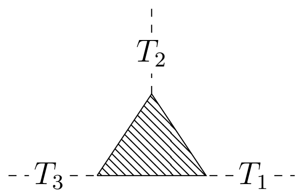
In this chapter, we aim to provide some of examples of tame symmetric special multiserial algebras. Namely, we aim to prove the following two Theorems.

Theorem. *Let A be a Brauer configuration algebra. If A is tame then A is at most quadserial. In particular, A is a tame symmetric special quadserial algebra if and only if A is given by the Brauer configuration*



in which every vertex has multiplicity one.

Theorem. *Let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ is of the form*



where T_1, T_2 and T_3 are distinct multiplicity-free Brauer trees containing m_1, m_2 and m_3 polygons respectively. Suppose further that at least two of m_1, m_2 and m_3 are strictly greater than 1. Then A is tame if and only if the values of the triple (m_1, m_2, m_3) conform to a column of the following table.

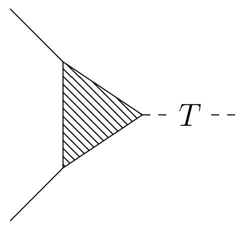
m_1	1	1	1	1	1	2
m_2	2	2	2	2	3	2
m_3	2	3	4	5	3	2

Both classes of tame algebras above are domestic, since they appear in the derived equivalence classification of domestic symmetric algebras in [40]. For the symmetric special multiserial algebras that are representation-finite, we actually obtain two corollaries from the proof of the second theorem above that show the forms of the Brauer configurations associated to these algberas.

Corollary. *Let A be the Brauer configuration algebra associated to the configuration χ in the second theorem above. Then A is of finite representation type if and only if the unordered triple (m_1, m_2, m_3) conforms to a value in the following table.*

m_1	1	1	1
m_2	2	2	2
m_3	2	3	4

Corollary 3.0.1. *Let A be a Brauer configuration algebra associated to a multiplicity-free Brauer configuration of the form*



where T is a (multiplicity-free) Brauer tree. Then A is of finite representation type.

To prove the first theorem, we use the classification of tame symmetric radical cubed zero algebras in [13]. We also use the contrapositve of Boevey's theorem on the Auslander-Reiten components of a tame algebra (Theorem 1.1.25) to show that all symmetric special n-serial algebras with $n \geq 5$ and all other quadserial algebras not of the form in [13] are wild. To achieve the latter goal, we first define two infinite families of indecomposable modules of the same dimension in Section 3.1. The proof in Section 3.2 then relies upon calculating the Auslander-Reiten translate of these indecomposable modules to show that infinitely many of them do not belong to a homogeneous tube in the Auslander-Reiten quiver.

The proof of the second theorem is addressed in Section 3.3. The proof is inspired by Rickard's Brauer star theorem in [46]. We construct a tilting complex similar to the one used by Rickard and calculate the endomorphism algebra of the tilting complex explicitly. We then show that this is isomorphic to a Brauer configuration algebra in which all 2-gons of the Brauer configuration are truncated edges connected to the vertices of a unique 3-gon. That the representation type of this algebra is determined by the number of polygons attached to each vertex of the 3-gon follows from the results of [34], which are summarised in Section 1.5.

3.1 One-Parameter Families of Modules in Symmetric Special Multiserial Algebras

We will be utilising the contrapositive of Theorem 1.1.25 to show an algebra is wild. We restate the theorem here for convenience, where by almost all, we mean all but finitely many.

Theorem ([19],Theorem D)**.** *Let K be an algebraically closed field. If A is a tame K-algebra, then for each dimension d, $M \cong \tau M$ for almost all indecomposable A-modules of dimension d.*

We will need to construct modules that are technically not band modules, but are 'band-like' in the sense that they have a cyclic presentation and form a one-parameter family of indecomposable modules. The motivation for this is that the dimension vector of these one-parameter families of modules is independent of the parameter. Thus, if the family of A-modules M_λ is such a one-parameter family, where $\lambda \in K$ then the algebra A is wild if $\mathbf{dim}(M_\lambda) \neq \mathbf{dim}(\tau M_\lambda)$ for (almost) all

λ, since then $M_\lambda \ncong \tau M_\lambda$ for (almost) all λ. Note that since the algebras we are interested in are symmetric, we have $\tau = \Omega^2$, where $\Omega^2 M_\lambda$ is the second syzygy of M_λ.

We will begin by defining a form of directed graph (with additional structure) that has a cyclic presentation. We will then associate to this directed graph a module, which we will call a *circle module*. However, we caution the reader that unlike band modules, the definitions that follow do not guarantee that circle modules are indecomposable – this is something we must prove on a case by case basis.

Definition 3.1.1. Let $B = (B_0, B_1, \kappa, \Delta)$, where (B_0, B_1) is a finite, connected, directed graph with vertex set B_0 and arrow set B_1, and κ and Δ are maps

$$\kappa : B_0 \to Q_0 : v \mapsto \kappa(v)$$
$$\Delta : B_1 \to Q_1 : a \mapsto \Delta(a).$$

We call B a *circle* provided the following conditions are satisfied.

(B1) The underlying graph of B contains a unique cycle.

(B2) $s(\Delta(a)) = \kappa(u)$ and $e(\Delta(a)) = \kappa(v)$ for any arrow $u \xrightarrow{a} v \in B$

(B3) For any (directed) path

$$v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} v_n$$

in B, $\Delta(a_1) \dots \Delta(a_n)$ is a direct string.

(B4) There exists no connected subtree $B' = (B'_0, B'_1, \kappa', \Delta')$ of B such that B' satisfies (B2) and (B3), B'_0 contains vertices u and v with $\kappa'(u) = \kappa'(v)$, and such that B is equivalent to a connected, directed graph given by glueing n distinct copies of B', where each copy of the vertex u is identified with precisely one copy of the vertex v.

Condition (B4) of the above definition is analogous to the condition that a band is not a proper power of a string. The definition above differs from that of a band in the sense that each vertex may be the source or target of more than two arrows. In addition, we allow for the possibility that there exists a subgraph of B of the form

$$\xrightarrow{a} \cdot \xleftarrow{b} \qquad \text{or} \qquad \xleftarrow{a} \cdot \xrightarrow{b} \tag{$*$}$$

where $\Delta(a) = \Delta(b)$, which is not possible in a band and can possibly give rise to decomposable modules.

To each circle B, we define a family of circle modules $M(B, n, \phi)$, where n is a strictly positive integer and $\phi \in \mathrm{Aut}(K^n)$. The underlying vector space of $M(B, n, \phi)$ is given by replacing each vertex of B with a copy of K^n. We then distinguish an arrow $a \in B_1$ and say that the action of the arrow $\alpha = \Delta(a)$ on $M(B, n, \phi)$ is induced by the automorphism ϕ. The action of any other arrow $\beta \in Q_1 \setminus \{\alpha\}$ on $M(B, n, \phi)$ is induced by the relevant identity morphisms if there exists an arrow $b \in B_1 \setminus \{a\}$ such that $\Delta(b) = \beta$, and is zero otherwise.

For the purposes of this chapter, we are only interested in circle modules of the form $M(B, 1, \phi_\lambda)$ for some circle B and some $\lambda \in K^*$, where $\phi_\lambda : K \to K$ is the automorphism defined by $\phi_\lambda(x) = \lambda x$. As previously stated, the definition of a circle module presented here does not guarantee that we obtain an infinite family of indecomposable modules (unlike bands and band modules). Indeed, there are numerous examples where this is not the case, such as when a circle B is of Euclidean type $\widetilde{\mathbb{A}}_n$ and contains a subgraph as in $(*)$ with $\Delta(a) = \Delta(b)$. Given a specific example of a circle B, we are therefore required to show firstly that $M(B, 1, \phi_\lambda)$ is indecomposable for all $\lambda \in K^*$; and secondly that $M(B, 1, \phi_{\lambda_1}) \ncong M(B, 1, \phi_{\lambda_2})$ for all $\lambda_1 \neq \lambda_2$. For the purposes of readability, when presenting specific examples of circles, we will label the vertices and arrows by their respective images under κ and Δ.

Example 3.1.2. Let A be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ contains a polygon x with $|x| > 3$ and let $\alpha, \beta, \delta, \gamma \in Q_1$ be distinct arrows of source x. The following is an example of a circle.

$$B : \begin{tikzcd} & x \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \arrow[dr, "\gamma"] & & x \arrow[dl, "\gamma"'] \arrow[d, "\delta"] \arrow[dr, "\alpha"] & \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{tikzcd}$$

where the two copies of y_1 are identified. The structure of the module $M_\lambda = M(B, 1, \phi_\lambda)$ is as follows.

$$M_\lambda : \begin{tikzcd} & S(x) \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "1"] & & S(x) \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "\phi_\lambda"] & \\ S(y_1) & S(y_2) & S(y_3) & S(y_4) & S(y_1) \end{tikzcd}$$

We will assume x, y_1, y_2, y_3 and y_4 are pairwise distinct and calculate the space $\mathrm{Hom}_A(M_{\lambda_1}, M_{\lambda_2})$. Let $K(x)$ and $K(y_i)$ denote the underlying K-vector spaces of $S(x)$ and $S(y_i)$ respectively. Then we have the following commutative squares.

$$\text{(i)} \quad \begin{tikzcd} (K(x))^2 \arrow[r, "(1\ \lambda_1)"] \arrow[d, "\varphi_x"] & K(y_1) \arrow[d, "\varphi_{y_1}"] \\ (K(x))^2 \arrow[r, "(1\ \lambda_2)"] & K(y_1) \end{tikzcd} \qquad \text{(ii)} \quad \begin{tikzcd} (K(x))^2 \arrow[r, "(1\ 0)"] \arrow[d, "\varphi_x"] & K(y_2) \arrow[d, "\varphi_{y_2}"] \\ (K(x))^2 \arrow[r, "(1\ 0)"] & K(y_2) \end{tikzcd}$$

$$\text{(iii)} \quad \begin{tikzcd} (K(x))^2 \arrow[r, "(1\ 1)"] \arrow[d, "\varphi_x"] & K(y_3) \arrow[d, "\varphi_{y_3}"] \\ (K(x))^2 \arrow[r, "(1\ 1)"] & K(y_3) \end{tikzcd} \qquad \text{(iv)} \quad \begin{tikzcd} (K(x))^2 \arrow[r, "(0\ 1)"] \arrow[d, "\varphi_x"] & K(y_4) \arrow[d, "\varphi_{y_4}"] \\ (K(x))^2 \arrow[r, "(0\ 1)"] & K(y_4) \end{tikzcd}$$

Squares (ii), (iii) and (iv) imply that φ_x is a matrix

$$\varphi_x = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a = \varphi_{y_2}(1) = \varphi_{y_3}(1) = \varphi_{y_4}(1)$. Square (i) implies that $\varphi_{y_1}(1) = a$ and $a\lambda_1 = a\lambda_2$. Thus, $a \in K$ if $\lambda_1 = \lambda_2$ and $a = 0$ if $\lambda_1 \neq \lambda_2$. From this, we conclude firstly that $\dim \mathrm{End}_A(M_\lambda) = 1$, and so M_λ is indecomposable for all $\lambda \in K^*$; and secondly that $\dim \mathrm{Hom}_A(M_{\lambda_1}, M_{\lambda_2}) = 0$ for all $\lambda_1 \neq \lambda_2$. Thus, $M_{\lambda_1} \ncong M_{\lambda_2}$ for all $\lambda_1 \neq \lambda_2$ and so M_λ describes a 1-parameter family of indecomposable A-modules.

Example 3.1.3. Let B be the circle in Example 3.1.2, except we will now assume x, y_2, y_3 and y_4 are pairwise distinct and $y_1 = x$. Consider the family of modules $M_\lambda = M(B, 1, \phi_\lambda)$. To calculate $\mathrm{Hom}_A(M_{\lambda_1}, M_{\lambda_2})$ we will need to consider the following commutative squares.

(i)
$$\begin{tikzcd}
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \lambda_1 & 0 \end{smallmatrix}\right)"] \arrow[d, "\varphi_x"] & (K(x))^3 \arrow[d, "\varphi_x"] \\
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \lambda_2 & 0 \end{smallmatrix}\right)"] & (K(x))^3
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right)"] \arrow[d, "\varphi_x"] & K(y_2) \arrow[d, "\varphi_{y_2}"] \\
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right)"] & K(y_2)
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 1 & 1 & 0 \end{smallmatrix}\right)"] \arrow[d, "\varphi_x"] & K(y_3) \arrow[d, "\varphi_{y_3}"] \\
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 1 & 1 & 0 \end{smallmatrix}\right)"] & K(y_3)
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right)"] \arrow[d, "\varphi_x"] & K(y_4) \arrow[d, "\varphi_{y_4}"] \\
(K(x))^3 \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right)"] & K(y_4)
\end{tikzcd}$$

The above squares imply that φ_x is a matrix

$$\varphi_x = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ a & b & c \end{pmatrix},$$

where $c = \varphi_{y_2}(1) = \varphi_{y_3}(1) = \varphi_{y_4}(1)$ and $c\lambda_1 = c\lambda_2$. Thus, $c \in K$ if $\lambda_1 = \lambda_2$ and $c = 0$ if $\lambda_1 \neq \lambda_2$. Thus, we conclude that

$$\mathrm{End}_A(M_\lambda) \cong \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ a & b & c \end{pmatrix} \middle| a, b, c \in K \right\}$$

and hence, $\mathrm{End}_A(M_\lambda)$ contains no non-trivial idempotents. So M_λ is indecomposable for all $\lambda \in K^*$. In addition, if $\lambda_1 \neq \lambda_2$ then we note that $M_{\lambda_1} \ncong M_{\lambda_2}$, since then

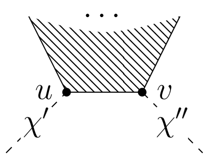
$$\mathrm{Hom}_A(M_{\lambda_1}, M_{\lambda_2}) \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} \middle| a, b \in K \right\}$$

and every $X \in \mathrm{Hom}_A(M_{\lambda_1}, M_{\lambda_2})$ is not invertible. Hence, M_λ describes a 1-parameter family of indecomposable A-modules.

3.2 Symmetric Special n-Serial Algebras with $n \geq 3$

The following result is a particularly useful tool for reducing the number of cases for later results.

Proposition 3.2.1. *Let A be a Brauer configuration algebra associated to a Brauer configuration χ and suppose there exists an n-gon x in χ with $n > 2$. Suppose either x is self-folded or x is locally of the form*



where χ' and χ'' are subconfigurations of χ that both contain a cycle or a vertex of multiplicity strictly greater than 1. Then A is wild.

Proof. Given a simple path of polygons passing through vertices $u_1, \ldots, u_n$ in a Brauer configuration χ, one can construct a string by alternating between the arrows around the vertices u_{2i} and the formal inverses of arrows around the vertices u_{2i+1}. If a Brauer configuration contains two vertices of multiplicity strictly greater than one, a cycle, or a combination of both cycles and vertices of higher multiplicity, then this enables one to construct a band. It follows from the proposition statement that there exists a band b which consists of arrows and formal inverses associated to either a vertex at which x is self-folded, or to the vertices u and v connected to x. Thus there exists a substring $\alpha\beta$ of b such that $\alpha\beta$ is neither direct nor inverse and $e(\alpha) = x = s(\beta)$. This implies that for any band module M associated to b, $S(x)$ is a direct summand of either $\operatorname{top} M$ or $\operatorname{soc} M$. Define a family of band modules M_λ by $M_\lambda = M(b, 1, \phi_\lambda)$, where $\lambda \in K^*$ and $\phi_\lambda : K \to K$ is the automorphism defined by $\phi_\lambda(a) = \lambda a$. Then $S(x)$ is a direct summand of either $\operatorname{top} M_\lambda$ or $\operatorname{soc} M_\lambda$.

Fix a choice of λ and let

$$\operatorname{top} M_\lambda = \bigoplus_{i=1}^{m} S(x_i) \quad \text{and} \quad \operatorname{soc} M_\lambda = \bigoplus_{i=1}^{m} S(x'_i).$$

Then we note that M_λ has the following module structure.

$$
\begin{array}{ccccccccccccc}
 & & S(x_1) & & & & S(x_2) & \cdots & S(x_m) & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow \cdots \swarrow & & \searrow & \\
 U_1 & & & & V_1 \quad U_2 & & & & & & V_m \\
 \swarrow & & & & \searrow \quad \swarrow & & & & & & \searrow \\
 S(x'_m) & & & & S(x'_1) & & & \cdots & & & S(x'_m)
\end{array}
$$

where the two copies of $S(x'_m)$ are identified and U_i and V_i are uniserial modules. Since M_λ is a band module, none of the modules $P(x_i)$ or $P(x'_j)$ are uniserial (otherwise, b would not be a cyclic string). Suppose x_i and x'_j are such that $P(x_i)$ and $P(x'_j)$ are biserial respectively. Then the structure of $P(x_i)$ and $P(x'_j)$ is

$$
P(x_i):\ \begin{array}{ccc}
 & S(x_i) & \\
 \swarrow & & \searrow \\
 U_i & & V_i \\
 \downarrow & & \downarrow \\
 S(x'_{i-1}) & & S(x'_i) \\
 \downarrow & & \downarrow \\
 U'_i & & V'_i \\
 \searrow & & \swarrow \\
 & S(x_i) &
\end{array}
\qquad \text{and} \qquad
P(x'_j):\ \begin{array}{ccc}
 & S(x'_j) & \\
 \swarrow & & \searrow \\
 V'_j & & U'_{i+1} \\
 \downarrow & & \downarrow \\
 S(x_j) & & S(x_{j+1}) \\
 \downarrow & & \downarrow \\
 V_j & & U_{j+1} \\
 \searrow & & \swarrow \\
 & S(x'_j) &
\end{array}
$$

respectively, where U'_i and V'_i are uniserial modules.

Suppose there exists an integer r such that $S(x_r) \cong S(x)$. Then the structure of $P(x_r)$ is of the form

$$
P(x_r):\ \begin{array}{ccccc}
 & & S(x_r) & & \\
 \swarrow & \swarrow & & \searrow & \searrow \\
 U_r & S(y_1) & \cdots & S(y_{n-2}) & V_r \\
 \downarrow & \big\downarrow & & \big\downarrow & \downarrow \\
 S(x'_{r-1}) & & & & S(x'_r) \\
 \downarrow & W_1 & \cdots & W_{n-2} & \downarrow \\
 U'_r & & & & V'_r \\
 \searrow & \searrow & & \swarrow & \swarrow \\
 & & S(x_r) & &
\end{array}
$$

where U_r, U'_r, V_r, V'_r, $W_1, \ldots, W_{n-2}$ are uniserial modules and $y_1, \ldots, y_{n-2}$ are the successors to x at the other vertices connected to x. It follows that

$$
\operatorname{top}\Omega(M_\lambda) \cong \bigoplus_{i=1}^{m} S(x'_i) \oplus \bigoplus_{i=1}^{n-2} (S(y_i))^t,
$$

where t is the number of direct summands in $\operatorname{top} M_\lambda$ that are isomorphic to $S(x)$.

Define the following non-negative integers.

$$u = \sum_{i=1}^{m} \dim(U_i\varepsilon_x), \qquad u' = \sum_{i=1}^{m} \dim(U_i'\varepsilon_x),$$
$$v = \sum_{i=1}^{m} \dim(V_i\varepsilon_x), \qquad v' = \sum_{i=1}^{m} \dim(V_i'\varepsilon_x),$$
$$k = \sum_{i=1}^{n-2} \dim(S(y_i)\varepsilon_x), \qquad w = \sum_{i=1}^{n-2} \dim(W_i\varepsilon_x),$$
$$t = \dim((\operatorname{top} M_\lambda)\varepsilon_x), \qquad s = \dim((\operatorname{soc} M_\lambda)\varepsilon_x).$$

Then we have

$$\begin{aligned}
\dim(M_\lambda\varepsilon_x) &= t + u + v + s,\\
\dim(\Omega(M_\lambda)\varepsilon_x) &= \sum_{i=1}^{m} \dim(P(x_i)\varepsilon_x) - \dim(M_\lambda\varepsilon_x)\\
&= 2t + u + v + 2s + u' + v' + tk + tw - \dim(M_\lambda\varepsilon_x)\\
&= t + s + u' + v' + tk + tw,\\
\dim(\Omega^2(M_\lambda)\varepsilon_x) &= \sum_{i=1}^{m} \dim(P(x_i')\varepsilon_x) + t\sum_{i=1}^{n-2} \dim(P(y_i)\varepsilon_x) - \dim(\Omega(M_\lambda)\varepsilon_x).
\end{aligned}$$

Now

$$\sum_{i=1}^{m} \dim(P(x_i')\varepsilon_x) = 2s + u' + v' + 2t + u + v + sk + sw \text{ and}$$
$$\sum_{i=1}^{n-2} \dim(P(y_i)\varepsilon_x) \geq k + w + n - 2.$$

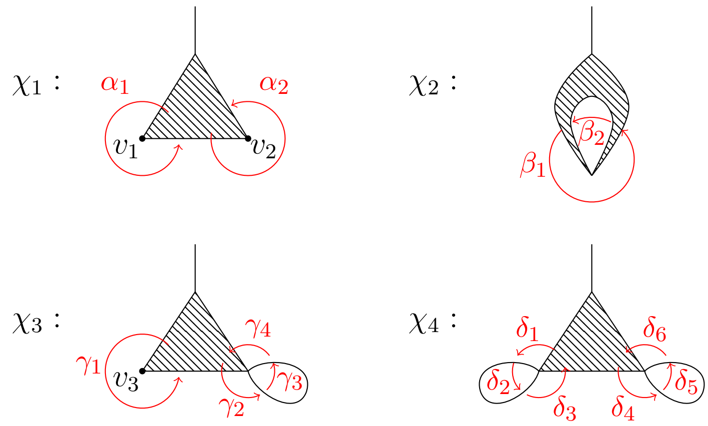
So

$$\begin{aligned}
\dim(\Omega^2(M_\lambda)\varepsilon_x) &\geq t + u + v + s + sk + sw + t(n-2)\\
&\geq \dim(M_\lambda\varepsilon_x) + sk + sw + t(n-2) > \dim(M_\lambda\varepsilon_x)
\end{aligned}$$

if $t > 0$. Thus, if $S(x)$ is a direct summand of $\operatorname{top} M_\lambda$, then $M_\lambda \ncong \Omega^2(M_\lambda) = \tau M_\lambda$. Since M_λ describes an infinite family of non-isomorphic indecomposable modules, we conclude in this case that the algebra A must be wild by the contrapositive of Theorem 1.1.25.

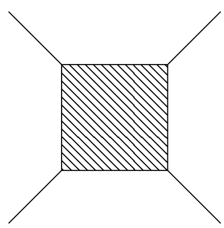
The proof for the case where $S(x)$ is a direct summand of soc M_λ is similar. Since A is symmetric, we simply compute $\Omega^{-2}(M_\lambda) \cong \tau^{-1}M_\lambda$ and note that $M_\lambda \ncong \tau^{-1}M_\lambda$ implies $M_\lambda \ncong \tau M_\lambda$. $\square$

Example 3.2.2. Let A_1, A_2, A_3 and A_4 be Brauer configuration algebras associated to the following Brauer configurations.



where $\mathfrak{e}_{v_i} > 1$ for each $i \in \{1, 2, 3\}$ and all other vertices have multiplicity 1. The following are examples of the bands used in Proposition 3.2.1. For A_1, we have $\alpha_1\alpha_2^{-1}$. For A_2, we have $\beta_1\beta_2^{-1}$. For A_3, we have $\gamma_1\gamma_4^{-1}\gamma_3\gamma_2^{-1}$.For A_4, we have $\delta_1\delta_2^{-1}\delta_3\delta_6^{-1}\delta_5\delta_4^{-1}$.

Theorem 3.2.3. *Let A be a Brauer configuration algebra. If A is tame then A is at most quadserial. In particular, A is a tame symmetric special quadserial algebra if and only if A is given by the Brauer configuration*



in which every vertex has multiplicity one.

Proof. We first note that the quadserial Brauer configuration algebra presented in the theorem is a tame radical cube zero algebra from the classification in [13]. So suppose instead that χ is a Brauer configuration not of the above form and there exists an n-gon x with $n > 3$ in χ. It follows from Proposition 3.2.1 that A is tame only if x is not self-folded at any vertex connected to x. We shall therefore assume that $u \neq v$ for all $u, v \in x$. There are multiple cases to consider in the proof.

Case 1: Suppose there exists a vertex $v \in x$ such that $\mathrm{val}(v) = 1$. Then it follows from the definition of a Brauer configuration that $\mathfrak{e}_v > 1$. We will show that A is wild in this case. We may assume that $\mathfrak{e}_u = 1$ for all vertices $u \neq v$ in χ, and that χ is a tree, since A would otherwise be wild by Proposition 3.2.1. Now choose a 4-tuple (v, u_2, u_3, u_4) of distinct vertices connected to x. Let α be the arrow (which is a loop in the quiver) generated by the vertex v and let β, γ and δ be the arrows of source x such that $e(\beta) = y_2$, $e(\gamma) = y_3$ and $e(\delta) = y_4$ are the successors to x at the vertices u_2, u_3 and u_4 respectively. Let M_λ be the family of circle modules defined in Example 3.1.3.

Note that there are two copies of $S(x)$ in $\mathrm{top}\, M_\lambda$ and one copy of $S(x)$ in $\mathrm{soc}\, M_\lambda$. We further note that $P(x)$ has the following structure

$$\begin{tikzcd}[row sep=small, column sep=tiny]
 & & & & S(x) \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drrr] & & & \\
 & S(x) \arrow[d] & S(y_2) \arrow[d] & S(y_3) \arrow[d] & S(y_4) \arrow[d] & S(z_1) \arrow[d] & \cdots & S(z_m) \arrow[d] \\
P(x): & \vdots \arrow[d] & Y_2 \arrow[ddrr] & Y_3 \arrow[ddr] & Y_4 \arrow[dd] & Z_1 \arrow[ddl] & \cdots & Z_m \arrow[ddlll] \\
 & S(x) \arrow[drrr] & & & & & & \\
 & & & & S(x) & & &
\end{tikzcd}$$

where the Y_i and Z_i are all uniserial modules. It follows from the structure of $P(x)$ and the structure of M_λ that

$$S(x) \oplus \bigoplus_{i=2}^{4} S(y_i) \subseteq \mathrm{top}\, \Omega(M_\lambda).$$

To show that Case 1 is wild, we will calculate the y_2 entry of the dimension vector of $\Omega^2(M_\lambda)$, although the following calculations also hold for any y_i. Since we have assumed that χ is a tree and $\mathfrak{e}_u = 1$ for any vertex $u \neq v$, we have

$\dim(Z_i\varepsilon_{y_2}) = \dim(S(z_i)\varepsilon_{y_2}) = \dim(Y_i\varepsilon_{y_2}) = 0$ for all i. So

$$\begin{aligned}
\dim(\Omega(M_\lambda)\varepsilon_{y_2}) &= 2\dim(P(x)\varepsilon_{y_2}) - \dim(M_\lambda\varepsilon_{y_2}) = 2 - 1 = 1 \\
\dim(\Omega^2(M_\lambda)\varepsilon_{y_2}) &\geq \dim(P(x)\varepsilon_{y_2}) + \sum_{i=2}^{4}\dim(P(y_i)\varepsilon_{y_2}) - \dim(\Omega(M_\lambda)\varepsilon_{y_2}) \\
&\geq 1 + 2 - 1 = 2 > \dim(M_\lambda\varepsilon_{y_2}).
\end{aligned}$$

Thus, $\tau(M_\lambda) = \Omega^2(M_\lambda) \ncong M_\lambda$. Since M_λ describes an infinite family of non-isomorphic indecomposable modules, we conclude in this case that the algebra A must be wild by the contrapositive of Theorem 1.1.25.

Case 2: Now assume that there is no vertex $v \in x$ such that $\mathrm{val}(v) = 1$. Note that in this case, we cannot make the assumption that χ is a tree. Choose a 4-tuple (u_1, u_2, u_3, u_4) of distinct vertices connected to x. Let α, β, γ and δ be the arrows of source x such that $e(\alpha) = y_1$, $e(\beta) = y_2$, $e(\gamma) = y_3$ and $e(\delta) = y_4$ are the successors to x at the vertices u_1, u_2, u_3 and u_4 respectively. Let M_λ be the family of circle modules defined in Example 3.1.2. We have various subcases to consider.

Case 2a: Suppose $|x| > 4$. Then $P(x)$ is of the form

$$P(x): \begin{tikzcd}
 & & & S(x) \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drrr] & & & \\
S(y_1) \arrow[d] & S(y_2) \arrow[d] & S(y_3) \arrow[d] & S(y_4) \arrow[d] & S(z_1) \arrow[d] & \cdots & S(z_m) \arrow[d] \\
Y_1 \arrow[drrr] & Y_2 \arrow[drr] & Y_3 \arrow[dr] & Y_4 \arrow[d] & Z_1 \arrow[dl] & \cdots & Z_m \arrow[dlll] \\
 & & & S(x) & & &
\end{tikzcd}$$

where $m > 0$ and the Y_i and Z_i are all uniserial modules. We note in this case that there are two copies of each $S(z_i)$ in $\mathrm{top}\,\Omega(M_\lambda)$ and that $\mathrm{soc}\,\Omega(M_\lambda) = S(x) \oplus S(x)$. Since $\mathrm{soc}\,P(z_i) = S(z_i)$ for all i, there must exist a copy of $S(z_i)$ in $\mathrm{soc}\,\Omega^2(M_\lambda)$. Thus, $\tau(M_\lambda) \ncong M_\lambda$ for any $\lambda \in K^*$, and so A is wild.

Case 2b: Suppose $|x| = 4$ and that there exists an integer r such that y_r is not uniserial. Then $P(y_r)$ has the following structure.

$$
\begin{tikzcd}[column sep=small]
 & & S(y_r) \arrow[dl] \arrow[dd] \arrow[ddrr] & & \\
 & Y_r \arrow[d] & & & \\
P(y_r): & S(x) \arrow[dr] & V_1 \arrow[d] & \cdots & V_n \arrow[dll] \\
 & & S(y_r) & &
\end{tikzcd}
$$

where $Y_r, V_1, \dots, V_m$ are uniserial. We note that $\operatorname{rad} P(y_r)/\operatorname{soc} P(y_r)$ contains a direct summand which is a uniserial submodule of $P(x)$. This is precisely the module with top isomorphic to $\operatorname{top} Y_r$ and socle isomorphic to $S(x)$ presented in the structure of $P(y_r)$ above. We further note that no other direct summand of $\operatorname{rad} P(y_r)/\operatorname{soc} P(y_r)$ is a submodule of $P(x)$. Since $\operatorname{top} M_\lambda = S(x) \oplus S(x)$ and $S(y_r)$ is a direct summand of $\operatorname{top} \Omega(M_\lambda)$, it follows that for each i, $\operatorname{top} V_i$ is a direct summand of $\operatorname{top} \Omega^2(M_\lambda)$. Thus, $\tau(M_\lambda) \ncong M_\lambda$ for any $\lambda \in K^*$, and so A is wild by the contrapositive of Theorem 1.1.25.

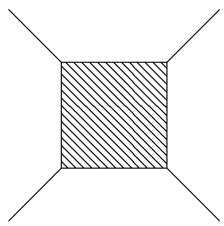
Case 2c: Suppose that $|x| = 4$, that every y_i is such that $P(y_i)$ is uniserial, and that for some r, we have $\operatorname{rad}^3 P(y_r) \neq 0$. We will calculate $\tau^{-1} M_\lambda$ in this case, as it is a simpler calculation. Let v be the vertex connected to x and y_r. Let z be the predecessor to x at v and let $a_i = \dim(Y_i \varepsilon_z)$. Then $a_r \geq 1$. If $a_i > 0$ for some $i \neq r$, then this implies that there exists a polygon connecting two distinct vertices of x, which induces a cycle. This implies A is wild by Proposition 3.2.1, so assume that this is not the case. Similarly, we have $\dim(S(y_i)\varepsilon_z) = 0$ for all $i \neq r$. Note that $\operatorname{soc} \Omega^{-1}(M_\lambda) = S(x) \oplus S(x)$. So

$$
\begin{aligned}
\dim(\Omega^{-1}(M_\lambda)\varepsilon_z) &= \sum_{i=1}^{4} \dim(P(y_i)\varepsilon_z) - \dim(M_\lambda \varepsilon_z) \\
&= 2\dim(S(y_r)\varepsilon_z) + a_r - \dim(S(y_r)\varepsilon_z) \\
&= \dim(S(y_r)\varepsilon_z) + a_r \\
\dim(\Omega^{-2}(M_\lambda)\varepsilon_z) &\geq 2\sum_{i=1}^{4} \dim(P(x)\varepsilon_z) - \dim(\Omega^{-1}(M_\lambda)\varepsilon_z) \\
&\geq 2\left(\dim(S(y_r)\varepsilon_z) + a_r\right) - \left(\dim(S(y_r)\varepsilon_z) + a_r\right) \\
&\geq \dim(S(y_r)\varepsilon_z) + a_r
\end{aligned}
$$

$$\geq \dim(M_\lambda \varepsilon_z) + 1 > \dim(M_\lambda \varepsilon_z).$$

Thus, $\tau^{-1}(M_\lambda) \ncong M_\lambda$ for any $\lambda \in K^*$. Hence, $\tau(M_\lambda) \ncong M_\lambda$ for any $\lambda \in K^*$, and so A is wild by the contrapositive of Theorem 1.1.25.

In conclusion, if A is not wild, then χ does not contain a polygon x with $|x| > 4$. If χ contains a polygon x with $|x| = 4$, then x is not self-folded and no vertex incident to x has valency one. Each vertex must therefore have an edge $y_i \neq x$ incident to it. Moreover, every edge y_i incident to x must be such that $P(y_i)$ is uniserial and $\operatorname{rad}^3 P(y_i) = 0$. Thus, χ is precisely the Brauer configuration



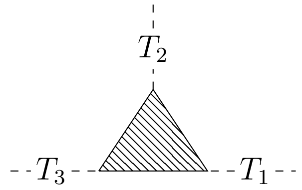
in which every vertex has multiplicity one, which is known to be associated to a tame algebra. $\square$

3.3 Symmetric Special Triserial Algebras Derived Equivalent to the Trivial Extension of a Hereditary Algebra

Throughout this section, we will assume that A is a symmetric special triserial algebra. That is, A is a Brauer configuration algebra associated to a configuration χ such that for any polygon x in χ, we have $|x| \leq 3$.

We aim to prove the following Theorem.

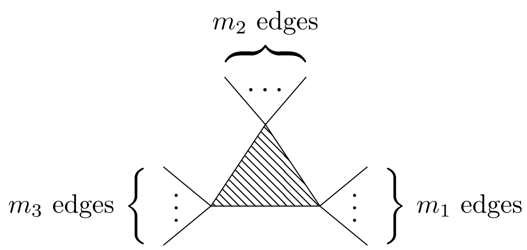
Theorem 3.3.1. *Let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ is of the form*



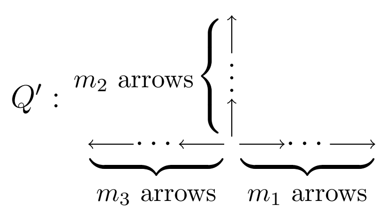
where T_1, T_2 and T_3 are distinct multiplicity-free Brauer trees containing m_1, m_2 and m_3 polygons respectively. Suppose further that at least two of m_1, m_2 and m_3 are strictly greater than 1. Then A is tame if and only if the values of the unordered triple (m_1, m_2, m_3) conform to a column of the following table.

m_1	1	1	1	1	1	2
m_2	2	2	2	2	3	2
m_3	2	3	4	5	3	2

The columns of the above table correspond to the Dynkin and Euclidean diagrams of type $\mathbb{E}_p$ and $\widetilde{\mathbb{E}}_p$ ($p \in \{6, 7, 8\}$). The first step of the proof is to show that any algebra of the above form is derived equivalent to a Brauer configuration algebra $\widetilde{A}$ associated to a Brauer configuration of the form



This is essentially Rickard's Brauer Star Theorem [46, Theorem 4.2] applied to Brauer configuration algebras. By the results of [34], we then know that $\widetilde{A}$ is the trivial extension of a hereditary algebra KQ', where Q' is a quiver of the form



Under the assumption that at least two of m_1, m_2 and m_3 are strictly greater than 1, $\widetilde{A} = T(KQ')$ is tame if and only if the triple (m_1, m_2, m_3) conforms to a column in the table of Theorem 3.3.1, in which case, Q' is an orientation of $\mathbb{E}_p$ or $\widetilde{\mathbb{E}}_p$ ($p \in \{6, 7, 8\}$). In fact, the algebra $\widetilde{A}$ is either of finite representation type [55, Theorem 1.4] or is representation-infinite domestic [6].

Many details of the proof for [46, Theorem 4.2] carry over to the multiserial case. However, for the benefit of the reader, we will run through the full details of the proof here.

3.3.1 Initial assumptions

We will later need to use the results in this section for a slightly broader class of symmetric special triserial algebras. Thus, we will outline precisely what assumptions we are making in the construction of the tilting complex that follows.

Assumption 3.3.2. Let $A = KQ/I$ be any Brauer configuration algebra associated to a Brauer configuration χ. Assume the following.

(i) χ is a tree.

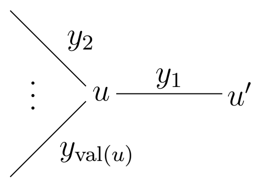
(ii) χ contains precisely one 3-gon and no n-gons with $n > 3$.

(iii) At most one vertex of χ has a multiplicity strictly greater than one.

Assumptions (ii) and (iii) are mainly for the purposes of simplicity, as we do not need to consider other cases. The construction that follows can be adapted to an algebra where assumptions (ii) and (iii) do not hold. We will use these assumptions to prove the following.

Proposition 3.3.3. *Let A be a Brauer configuration algebra associated to a configuration χ satisfying Assumption 3.3.2(i), (ii) and (iii). Let x be the unique 3-gon of χ under Assumption 3.3.2(ii) and suppose χ contains a subtree χ' of the form*

χ' :



such that $y_2, \ldots, y_{\mathrm{val}(u)}$ *are truncated,* $y_1 \neq x$ *and* $\mathfrak{e}_u = \mathfrak{e}_{u'} = 1$. *Let* $y_1, y'_2, \ldots, y'_{\mathrm{val}(u')}$ *be the successor sequence of* y_1 *at* u'. *Then* A *is derived equivalent to a Brauer configuration algebra associated to a Brauer configuration* $\widetilde{\chi}$ *such that*

$$\widetilde{\chi} = \chi \setminus \{y_2, \ldots, y_{\mathrm{val}(u)}\} \cup \{\widetilde{y}_2, \ldots, \widetilde{y}_{\mathrm{val}(u)}\},$$

where $\widetilde{y}_2, \ldots, \widetilde{y}_{\mathrm{val}(u)}$ *are truncated edges connected to* u' *in* $\widetilde{\chi}$, *every vertex in* $\widetilde{\chi}$ *has the same multiplicity as its corresponding vertex in* χ, *and the successor sequence of* y_1 *at* u' *in* $\widetilde{\chi}$ *is*

$$y_1, \widetilde{y}_2, \ldots, \widetilde{y}_{\mathrm{val}(u)}, y'_2, \ldots, y'_{\mathrm{val}(u')}.$$

3.3.2 The maps between indecomposable projective modules

We will begin by investigating the morphisms between the indecomposable projective modules in A. We have the following remark from Rickard, which is a trivial consequence of the multiserial nature of the indecomposable projective modules.

Remark 3.3.4 ([46],Remark 4.1). Let x and y be distinct polygons in a Brauer configuration χ satisfying Assumption 3.3.2.

(a) If x and y have no common vertex, then $\dim_K \mathrm{Hom}_A(P(x), P(y)) = 0$. Otherwise, $\dim_K \mathrm{Hom}_A(P(x), P(y)) = \mathfrak{e}_u$, where u is the (unique) vertex common to x and y.

(b) Suppose x is connected to the vertex v in χ such that $\mathfrak{e}_v > 1$. Then

$$\dim_K \mathrm{End}_A(P(x)) = \mathfrak{e}_v + 1.$$

Otherwise, $\dim_K \mathrm{End}_A(P(x)) = 2$.

It will later be convenient to know the maps between indecomposable projective modules associated to consecutive polygons in the cyclic ordering at any vertex in χ in detail.

Remark 3.3.5. If y is the direct predecessor to x at some vertex in χ, then $S(x) \subseteq \operatorname{top}(\operatorname{rad} P(y))$. Thus, the canonical surjection of $P(x)$ into the maximal uniserial submodule $V \subset P(y)$ with $\operatorname{top} V = S(x)$ is a basis element of $\operatorname{Hom}_A(P(x), P(y))$.

Lemma 3.3.6. *Let A be a Brauer configuration algebra associated to a Brauer configuration χ such that χ is a tree with at most one vertex v' with multiplicity strictly greater than one. Let x_1 be a polygon connected to a non-truncated vertex v in χ. If x_1 is connected to v' then let $v = v'$ (otherwise, v may be any other non-truncated vertex). Let $x_1, \ldots, x_{\operatorname{val}(v)}$ be the successor sequence of x_1 at v. Denote by f_j the basis element of $\operatorname{Hom}(P(x_{j+1}), P(x_j))$ given in Remark 3.3.5. Let $g = f_1 \ldots f_{\operatorname{val}(v)}$.*

(a) $\{\operatorname{id}_{P(x_1)}, g, g^2, \ldots, g^{\mathfrak{e}_v}\}$ is a basis for $\operatorname{End}_A(P(x_1))$.

(b) Let $h \in \operatorname{Hom}_A(P(x_r), P(x_1))$ be the map $h = f_1 \ldots f_{r-1}$. Then

$$\{h, gh, \ldots, g^{\mathfrak{e}_v - 1}h\}$$

is a basis for $\operatorname{Hom}_A(P(x_r), P(x_1))$ when $r \neq 1$.

Proof. First note that f_j is the canonical surjection of $P(x_{j+1})$ into the uniserial submodule $V_j \subseteq \operatorname{rad} P(x_j)$ such that V_j is also a quotient of $\operatorname{rad} P(x_j)$. It follows that $f_{j-1}f_j$ is equivalent to the canonical surjection of $P(x_{j+1})$ into the uniserial submodule $V_{j-1} \subseteq \operatorname{rad}^2 P(x_{j-1})$ such that V_{j-1} is also a quotient of $\operatorname{rad}^2 P(x_{j-1})$. This follows since $\operatorname{top} P(x_j) \subseteq \operatorname{rad} P(x_{j-1})/\operatorname{rad}^2 P(x_{j-1})$, $S(x_{j+1}) \subseteq \operatorname{rad}^2 P(x_{j-1})/\operatorname{rad}^3 P(x_{j-1})$, and $\operatorname{top} P(x_j) \subseteq \operatorname{Coker} f_j$. Using this argument iteratively, we can see that the map $f_{j-n} \ldots f_j$ is equivalent to the canonical surjection of $P(x_{j+1})$ into the uniserial submodule $V_{j-n} \subseteq \operatorname{rad}^{n+1} P(x_{j-n})$ such that V_{j-n} is also a quotient of $\operatorname{rad}^{n+1} P(x_{j-n})$.

(a) Let

$$U_1 \subset U_2 \subset \ldots \subset U_{\mathfrak{e}_v} \subset P(x_1)$$

be the chain of uniserial submodules of $P(x_1)$ such that $\operatorname{top} U_i = S(x_1)$ for all i. Note that $\operatorname{End}_A(P(x_1))$ has a basis $\{\operatorname{id}_{P(x_1)}, b_1, \ldots, b_{\mathfrak{e}_v}\}$, where each b_i is the canonical surjection of $P(x_1)$ into U_i. It follows from the structure of $P(x_1)$ that for each $i \le \mathfrak{e}_v$, $S(x_1)$ is a direct summand of $\operatorname{top}(\operatorname{rad}^{(\mathfrak{e}_v - i+1)\operatorname{val}(v)} P(x_1))$. Thus, $\operatorname{Im} b_i = U_i$ is a quotient of $\operatorname{rad}^{(\mathfrak{e}_v - i+1)\operatorname{val}(v)} P(x_1)$. Hence,

$$b_i = (f_1 \dots f_{\operatorname{val}(v)})^{\mathfrak{e}_v - i+1} = g^{\mathfrak{e}_v - i+1}$$

by the arguments above. So

$$\{\operatorname{id}_{P(x_1)}, b_1, \dots, b_{\mathfrak{e}_v}\} = \{\operatorname{id}_{P(x_1)}, g_1, \dots, g_{\mathfrak{e}_v}\},$$

as required.

(b) Now instead let

$$U_1 \subset U_2 \subset \dots \subset U_{\mathfrak{e}_v} \subset P(x_1)$$

be the chain of uniserial submodules of $P(x_1)$ such that $\operatorname{top} U_i = S(x_r)$ for all i. Then $\operatorname{Hom}_A(P(x_r), P(x_1))$ has a basis $\{b_1, \dots, b_{\mathfrak{e}_v}\}$, where each b_i is the canonical surjection of $P(x_1)$ into U_i. Note that for each $i \le \mathfrak{e}_v$, $S(x_r)$ is a direct summand of $\operatorname{top}(\operatorname{rad}^a P(x_1))$, where $a = (\mathfrak{e}_v - i)\operatorname{val}(v) + r - 1$. So U_i is a quotient of $\operatorname{rad}^a P(x_1)$ and thus,

$$b_i = f_1 \dots f_{r-1}(f_r \dots f_{\operatorname{val}(v)} f_1 \dots f_{r-1})^{\mathfrak{e}_v - i} = (f_1 \dots f_{\operatorname{val}(v)})^{\mathfrak{e}_v - i} f_1 \dots f_{r-1} = g^{\mathfrak{e}_v - i} h,$$

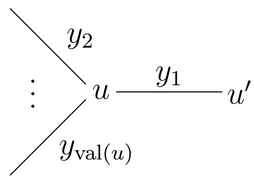
as required. □

Remark 3.3.7. Note that the condition in Lemma 3.3.6 that v is the vertex of higher multiplicity if x_1 is connected to such a vertex is only necessary for part (a) of the Lemma. Part (b) of the lemma applies to any other vertex connected to x_1.

3.3.3 Construction of the tilting complex

Let x be the unique 3-gon of χ under Assumption 3.3.2(ii). Suppose χ contains a subtree χ' of the form

χ' :



such that $y_2, \ldots y_{\mathrm{val}(u)}$ are truncated, $y_1 \neq x$ and $\mathfrak{e}_u = \mathfrak{e}_{u'} = 1$.

For the polygon y_1 in χ', define a stalk complex

$$T(y_1) : 0 \longrightarrow P(y_1) \longrightarrow 0,$$

where $P(y_1)$ is in degree zero. For every other polygon y_i in χ', define a complex

$$T(y_i) : 0 \longrightarrow P(y_1) \xrightarrow{f_i} P(y_i) \longrightarrow 0,$$

where the $P(y_1)$ term is in degree zero. Note that by Remark 3.3.4, such a complex is unique up to isomorphism in $K^b(\mathrm{proj}\, A)$. For every other polygon z in χ (that is, for any polygon z not in χ'), we define a stalk complex

$$T(z) : 0 \longrightarrow P(z) \longrightarrow 0,$$

where $P(z)$ is in degree zero. Then define $T = \bigoplus_{x \in Q_0} T(x)$. Note that for any $i \neq 1$, we have $\varepsilon_{y_i} A \varepsilon_z A = 0$ whenever $z \neq y_j$ for any j. Moreover, $P(y_1) = P(\varepsilon_{y_i} A \varepsilon_{y_1} A)$ for all $i \neq 1$, since each $P(y_i)$ is uniserial. Thus, T is an Okuyama-Rickard tilting complex.

3.3.4 The maps between the direct summands of T

We aim to calculate $\mathrm{End}_{K^b(\mathrm{proj}\, A)}(T)$. For maps between stalk complexes, this can simply be viewed as a map between indecomposable projective modules. We will investigate the morphisms in $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T(y_i), T(y_j))$.

Suppose $j < i$. Then by Lemma 3.3.6(b), any map $f_j \in \mathrm{Hom}_A(P(y_1), P(y_j))$ can be written as a map $f_j = hf_i$, where $f_i \in \mathrm{Hom}_A(P(y_1), P(y_i))$. Thus, given a morphism

$$\begin{tikzcd} 0 \arrow[r] & P(y_1) \arrow[r, "f_i"] \arrow[d, "g_0"] & P(y_i) \arrow[r] \arrow[d, "g_1"] & 0 , \\ 0 \arrow[r] & P(y_1) \arrow[r, "f_j"] & P(y_j) \arrow[r] & 0 \end{tikzcd}$$

we can see that $\dim_K \operatorname{Hom}_{K^b(\operatorname{proj} A)}(T(y_i), T(y_j)) \leq 2$. Namely,

$$\operatorname{Hom}_{K^b(\operatorname{proj} A)}(T(y_i), T(y_j)) = \operatorname{span}\{(\mathrm{id}_{P(y_1)}, h), (t_1, 0)\},$$

where $t_1 \in \operatorname{End}_A(P(y_1))$ maps an element from $\operatorname{top} P(y_1)$ to $\operatorname{soc} P(y_1)$. But $(t_1, 0) \simeq 0$, since any map in $\operatorname{End}_A(P(y_1))$ that factors through f_i is a scalar multiple of t_1 and any map in $\operatorname{Hom}_A(P(y_i), P(y_j))$ that factors through f_j is zero (by the assumption that $\mathfrak{e}_u = 1$). So $\dim_K \operatorname{Hom}_{K^b(\operatorname{proj} A)}(T(y_i), T(y_j)) = 1$.

Now suppose $j > i$. Then for any map $g_1 \in \operatorname{Hom}_A(P(y_i), P(y_j))$, the composition $g_1 f_i$ factors through the map t_1. Such a factorisation is non-trivial, so $g_1 f_i = 0$. So

$$\operatorname{Hom}_{K^b(\operatorname{proj} A)}(T(y_i), T(y_j)) = \operatorname{span}\{(t_1, 0), (0, g_1)\},$$

In addition, there exists a map $h \in \operatorname{Hom}_A(P(y_i), P(y_1))$ such that $hf_i = t_1$ and $g_1 = f_j h$. So $(t_1, 0) \simeq (0, -g_1)$. Hence,

$$\dim_K \operatorname{Hom}_{K^b(\operatorname{proj} A)}(T(y_i), T(y_j)) = 1.$$

Now suppose $i = j$. Then

$$\operatorname{End}_{K^b(\operatorname{proj} A)}(T(y_i)) = \operatorname{span}\{(\mathrm{id}_{P(y_1)}, \mathrm{id}_{P(y_i)}), (t_1, 0), (0, t_i)\},$$

where $t_i \in \operatorname{End}_A(P(y_i))$ maps an element from $\operatorname{top} P(y_i)$ to $\operatorname{soc} P(y_i)$. In fact, $(t_1, 0) \simeq (0, -t_i)$, since there exists a map $h \in \operatorname{Hom}_A(P(y_i), P(y_1))$ such that $t_1 = hf_i$ and $t_i = f_i h$. Moreover, $(t_1, 0) \not\simeq (\mathrm{id}_{P(y_1)}, \mathrm{id}_{P(y_i)})$, since for any morphism $h : P(y_i) \to P(y_1)$, we have $f_i h \neq \lambda \, \mathrm{id}_{P(y_i)}$ for any $\lambda \neq 0$. Thus,

$$\dim_K \operatorname{End}_{K^b(\operatorname{proj} A)}(T(y_i)) = 2.$$

Lemma 3.3.8. *Let $y_1, y'_2, \dots, y'_{\mathrm{val}(u')}$ be the successor sequence of y_1 at u'. For all $1 \leq i < \mathrm{val}(u)$, let $\alpha_i : T(y_{i+1}) \to T(y_i)$ denote the morphism such that the degree zero map is the identity. Let $\alpha_{\mathrm{val}(u)} : T(y'_2) \to T(y_{\mathrm{val}(u)})$ denote the morphism such that the degree zero map is the basis element of $\mathrm{Hom}_A(P(y'_2), P(y_1))$ given in Remark 3.3.5. Finally, for all $2 \leq i \leq \mathrm{val}(u')$, let $\alpha_{\mathrm{val}(u)+i-1} : T(y'_{i+1}) \to T(y'_i)$ denote the morphism such that the degree zero map is the basis element of $\mathrm{Hom}_A(P(y'_{i+1}), P(y'_i))$ given in Remark 3.3.5, where $y'_{\mathrm{val}(u')+1} := y_1$. For any $1 \leq i, j < \mathrm{val}(u) + \mathrm{val}(u')$, consider the vector space $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T(z_i), T(z_j))$, where $z_k = y_k$ if $k \leq \mathrm{val}(u)$, $z_k = y'_{k-\mathrm{val}(u)+1}$ if $\mathrm{val}(u) < k < \mathrm{val}(u) + \mathrm{val}(u')$ and $z_{\mathrm{val}(u)+\mathrm{val}(u')} = z_1 = y_1$.*

(a) For all $j < i$, $\{\alpha_j \alpha_{j+1} \dots \alpha_{i-1}\}$ is a basis for $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T(z_i), T(z_j))$.

(b) For all $j > i$,

$$\{\alpha_j \alpha_{j+1} \dots \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1} \alpha_1 \dots \alpha_{i-1}\}$$

is a basis for $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T(z_i), T(z_j))$.

(c) A basis for $\mathrm{End}_{K^b(\mathrm{proj}\, A)}(T(z_j))$ is

$$\{\mathrm{id}_{T(z_j)}, \alpha_j \alpha_{j+1} \dots \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1} \alpha_1 \dots \alpha_{j-1}\}.$$

Proof. (a) For $\mathrm{val}(u) < j < i \leq \mathrm{val}(u) + \mathrm{val}(u')$, this follows from Lemma 3.3.6(b), since $T(z_i)$ and $T(z_j)$ are stalk complexes. A similar argument for holds for $\mathrm{val}(u) < i < \mathrm{val}(u) + \mathrm{val}(u')$ and $j = \mathrm{val}(u)$ when considering the maps between degree zero terms.

For $1 < j < i \leq \mathrm{val}(u)$, it follows from the reasoning at the start of this subsection that the degree zero map is the identity and the degree -1 map is in the space $\mathrm{Hom}_A(P(i), P(j))$. Thus, the result again follows from Lemma 3.3.6(b) when considering the degree -1 maps. For $1 < i \leq \mathrm{val}(u)$ and $j = 1$. The degree zero map is either the identity map or the map $t_1 : \mathrm{top}\, P(y_1) \to \mathrm{soc}\, P(y_1)$. But by Lemma 3.3.6(a), t_1 factors through a map in $\mathrm{Hom}_A(P(y_1), P(y_i))$. So any morphism in $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T(y_i), T(y_1))$ with degree zero map t_1 is homotopic to zero. If the degree zero map is instead the identity then any morphism in $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(T(y_i), T(y_1))$ is equal to the composition of some morphism in $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(T(y_i), T(y_2))$ with the morphism α_1. By considering Lemma 3.3.6(b) on the degree -1 terms, the result follows.

(b) The arguments used in the proof for (a) form a cycle of maps. The proof for (b) is hence similar.

(c) If $f \in \mathrm{End}_{K^b(\mathrm{proj}\,A)}(T(z_j))$ is a non-identity map, then the degree zero map must be a map from the top to the socle of the projective module in degree zero. By Lemma 3.3.6(a), this is equivalent to a cycle of maps between the indecomposable projective modules corresponding to the polygons around the vertex u'. The result then follows from the proof of (a) and (b). □

Lemma 3.3.9. *Define the following set of pairs of vertices and connected polygons in χ.*

$$Z = \{(v, z) | v \in z \in \chi_1 \textit{ such that } v \textit{ is non-truncated}, v \neq u' \textit{ and } z \neq y_i \textit{ for all } i\}$$

Let $(v, z) \in Z$ and let z' be the successor to z at v. Denote by $\beta_{v,z}$ the morphism in $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(T(z'), T(z))$ whose degree zero map is the basis element of $\mathrm{Hom}_A(P(z'), P(z))$ given in Remark 3.3.5. Then

$$\left\langle \alpha_1, \dots, \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1}, \beta_{v,z} \right\rangle_{(v,z)\in Z} = \mathrm{End}_{K^b(\mathrm{proj}\,A)}(T),$$

where $\alpha_1, \dots, \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1}$ are as in Lemma 3.3.8.

Proof. This is a trivial consequence of both Lemma 3.3.6(b) and Lemma 3.3.8. □

3.3.5 The relations of the endomorphism algebra of T

We will now explicitly calculate the algebra $\mathrm{End}_{K^b(\mathrm{proj}\,A)}(T) = K\widetilde{Q}/\widetilde{I}$. By Lemma 3.3.9, the arrows of $\widetilde{Q}$ are given by the maps $\alpha_1, \dots, \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1}$ and $(\beta_{v,z})_{(v,z)\in Z}$. It remains to calculate the relations that generate $\widetilde{I}$.

Lemma 3.3.10. *Suppose a polygon z of χ is connected to non-truncated vertices $v, v' \notin \{u, u'\}$. Let $z = z_1, \dots, z_{\mathrm{val}(v)}$ and $z = z'_1, \dots, z'_{\mathrm{val}(v')}$ be the successor sequences of z at v and v' respectively. Then*

(a) $(\beta_{v,z_1} \dots \beta_{v,z_{\mathrm{val}(v)}})^{\mathfrak{e}_v} = (\beta_{v',z'_1} \dots \beta_{v',z'_{\mathrm{val}(v')}})^{\mathfrak{e}_{v'}} \neq 0.$

(b) $\beta_{v,z_{\mathrm{val}(v)}}\beta_{v',z'_1} = 0$ *and* $\beta_{v',z'_{\mathrm{val}(v')}}\beta_{v,z_1} = 0.$

If z is instead connected to only one non-truncated vertex v, then

(c) $(\beta_{v,z_1} \dots \beta_{v,z_{\mathrm{val}(v)}})^{\mathfrak{e}_v}\beta_{v,z_1} = 0.$

Proof. (a) Since all morphisms are maps between stalk complexes, this is a trivial consequence of Lemma 3.3.6(a). Namely, $(\beta_{v,z_1} \dots \beta_{v,z_{\mathrm{val}(v)}})^{\mathfrak{e}_v}$ is the morphism such that the degree zero map corresponds to the basis element of $\mathrm{End}_A(P(z_1))$ that maps $\mathrm{top}\, P(z_1)$ to $\mathrm{soc}\, P(z_1)$. The same is true for $(\beta_{v',z'_1} \dots \beta_{v',z'_{\mathrm{val}(v')}})^{\mathfrak{e}_{v'}}$.

(b) This follows from Remark 3.3.4(a), since this corresponds a map between the indecomposable projective modules of two polygons that have no common vertex.

(c) Similar to (a), $(\beta_{v,z_2} \dots \beta_{v,z_{\mathrm{val}(v)}}\beta_{v,z_1})^{\mathfrak{e}_v}$ maps $\mathrm{top}\, P(z_2)$ to $\mathrm{soc}\, P(z_2)$. But $\mathrm{soc}\, P(z_2)$ is in the kernel of the degree zero map of β_{v,z_1}. Thus, the result follows. □

Lemma 3.3.11. *Suppose a polygon z_1 of χ is connected to a non-truncated vertex $v' \in \{u, u'\}$. Suppose z_1 is connected to another non-truncated vertex $v \notin \{u, u', v'\}$ and let $z_1, \dots, z_{\mathrm{val}(v)}$ be the successor sequence of z_1 at v. Suppose $T(z_1)$ is the domain of a map α_r from Lemma 3.3.8 and let*

$$C_r = \alpha_r\alpha_{r+1} \dots \alpha_{\mathrm{val}(u)+\mathrm{val}(u')-1}\alpha_1 \dots \alpha_{r-1}.$$

Then

(a) $(\beta_{v,z_1} \dots \beta_{v,z_{\mathrm{val}(v)}})^{\mathfrak{e}_v} = C_r \neq 0.$

(b) $\alpha_{r-1}\beta_{v,z_1} = 0$ *and* $\beta_{v,z_{\mathrm{val}(v)}}\alpha_r = 0.$

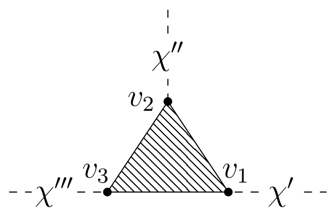
Suppose instead that there is no non-truncated vertex $v \notin \{u, u', v'\}$ connected to z_1. Then *(c)* $C_r \alpha_r = 0$.

Proof. The degree zero map of C_r corresponds to a non-identity map in $\mathrm{End}_A(P(z_1))$. Thus, the degree zero map of C_r maps $\mathrm{top}\, P(z_1)$ to $\mathrm{soc}\, P(z_1)$. It follows from the calculations in Subsection 3.3.4 that if $T(z_1)$ is not a stalk complex, then C_r is equivalent in $K^b(\mathrm{proj}\, A)$ to a morphism in which the map in degree -1 is zero. The proofs to (a), (b) and (c) are then similar to the corresponding proofs in Lemma 3.3.10. □

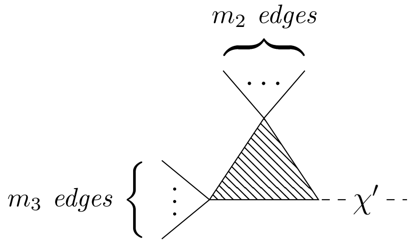
Every arrow of $\widetilde{Q}$ belongs to either a cycle of the form $\beta_{v,z_1} \dots \beta_{v,z_{\mathrm{val}(v)}}$ for some non-truncated vertex v in χ or belongs to the cycle C_r. The only possible paths in $\widetilde{Q}$ that are non-zero are subpaths of the paths in Lemma 3.3.10(a) and Lemma 3.3.11(a). One can see that there are no further relations in $\widetilde{A}$, since otherwise these maps would be zero. Furthermore, these are precisely the relations of the (opposite) Brauer configuration algebra associated the Brauer configuration $\widetilde{\chi}$ in Proposition 3.3.3, thus proving the proposition by Theorem 1.1.17 (since the algebra is symmetric).

3.3.6 The proof of Theorem 3.3.1

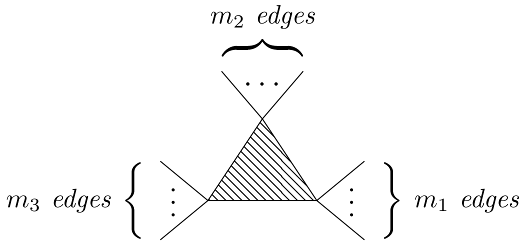
Proposition 3.3.12. *Let A be a Brauer configuration algebra associated to a Brauer configuration χ of the form*



where χ', χ'' and χ''' are subconfigurations of χ. Suppose χ is a tree with precisely one 3-gon and at most one vertex v such that $\mathfrak{e}_v > 1$. If such a vertex exists, then suppose v is in χ'. Suppose further that χ', χ'' and χ''' contain m_1, m_2 and m_3 polygons respectively. Then A is derived equivalent to a Brauer configuration algebra associated to the following Brauer configuration.



where χ' is as in χ. If in particular, χ has no vertex v such that $\mathfrak{e}_v > 1$, then A is derived equivalent to a Brauer configuration algebra associated to the following Brauer configuration.



Proof. Using Proposition 3.3.3 iteratively on the truncated edges attached to non-truncated 2-gons, the result follows. □

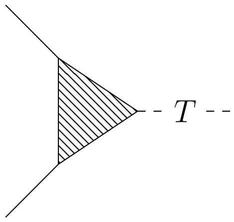
By the reasoning at the start of this section, the above proposition proves Theorem 3.3.1. We actually obtain some further results regarding Brauer configuration algebras of finite representation type.

Corollary 3.3.13. *Let A be the Brauer configuration algebra associated to the configuration χ in Theorem 3.3.1. Then A is of finite representation type if and only if the unordered triple (m_1, m_2, m_3) conforms to a value in the following table.*

m_1	1	1	1
m_2	2	2	2
m_3	2	3	4

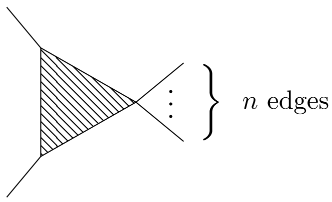
Proof. A is derived equivalent to the trivial extension of a hereditary path algebra KQ', where Q' is an orientation of $\mathbb{E}_p$ for some $p \in \{6, 7, 8\}$. By [55, Theorem 1.4], A is then representation-finite. All other values in the table of Theorem 3.3.1 give rise to representation-infinite algebras by [6]. □

Corollary 3.3.14. *Let A be a Brauer configuration algebra associated to a multiplicity-free Brauer configuration of the form*



where T is a Brauer tree. Then A is of finite representation type.

Proof. Proposition 3.3.12 implies that A is derived equivalent to a Brauer configuration algebra associated to a Brauer configuration of the following form.



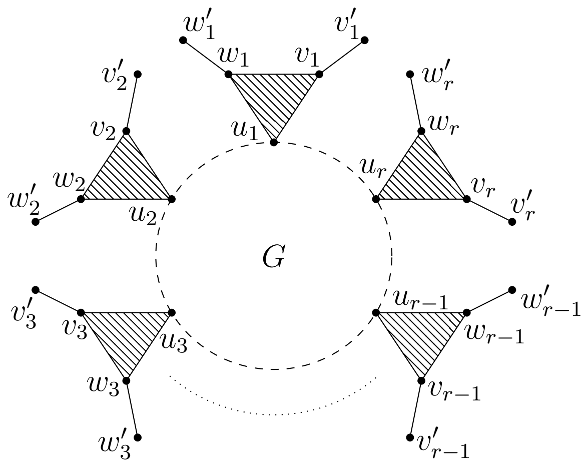
This is the trivial extension of a hereditary path algebra KQ', where Q' is an orientation of $\mathbb{D}_n$. Thus, A is representation-finite by [55, Theorem 1.4]. $\square$

Chapter 4

Wild Symmetric Special Triserial Algebras

In this chapter, we investigate the representation type of symmetric special triserial algebras with the aim of providing a description of the Brauer configuration algebras containing 3-gons that are wild. Ultimately, this leads us to the proof of the Main Theorem at the end of the Introduction. Building on the work of Chapter 3, there is one final class of symmetric special triserial algebras to consider in the main proof.

Theorem. *Let A be a symmetric special triserial algebra associated to a Brauer configuration χ. Suppose χ is not of any form given in Theorem 3.3.1. Suppose χ is also not of the form*



where G is a Brauer graph connecting the (not necessarily distinct) vertices $u_1, \ldots, u_r$ and $\mathfrak{e}_{v_i} = \mathfrak{e}_{v_i'} = \mathfrak{e}_{w_i} = \mathfrak{e}_{w_i'} = 1$ for all i. Then A is wild.

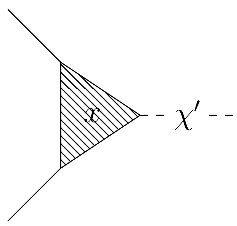
The proof of the above is fairly involved. We begin in Section 4.1 by considering the case where there exists a cycle or vertex of multiplicity strictly greater than one in χ. We then show that every 3-gon in χ must locally be of the same form as each of the 3-gons in the above theorem. To do this, we construct a representation embedding from the category $\operatorname{fin} K\langle a_1, a_2\rangle$ to the module category of a wild hereditary algebra KQ'. We then construct another functor which takes the indecomposable representations of KQ' and, informally speaking, folds them onto the quiver of the Brauer configuration algebra A. The composite of these two functors is then a representation embedding $\operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$, which proves A is wild.

The other case to consider is where we assume that there is more than one 3-gon in χ. This case is addressed in Section 4.2. For this, we show that the Brauer configuration algebra $A = KQ/I$ contains a wild subquiver Q', and we describe a strict representation embedding $\operatorname{mod} KQ' \to \operatorname{mod} A$.

4.1 Brauer Configurations with Cycles or Multiplicities

The aim of this section is to prove the following proposition.

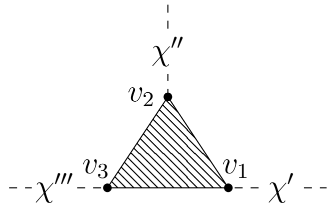
Proposition 4.1.1. *Let A be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ contains a 3-gon x. Suppose further that χ contains a cycle or a vertex v such that $\mathfrak{e}_v > 1$. If χ is not of the form*



where χ' is a subconfiguration of χ and all vertices except those in χ' have multiplicity one, then A is wild.

To prove this, we will explicitly construct a representation embedding H : $\operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$. The construction is fairly involved, so we will provide examples throughout. Before we proceed, we need the following lemma.

Lemma 4.1.2. *Let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ contains a 3-gon x connected to pairwise distinct vertices v_1, v_2 and v_3 that is locally of the form*



where χ', χ'' and χ''' are pairwise disjoint subconfigurations of χ. Suppose further that χ' contains a cycle or a vertex u such that $\mathfrak{e}_u > 1$. Then there exists a string $w = \alpha_1 \dots \alpha_n$ such that $\widehat{s}(\alpha_1) = x^{v_1} = \widehat{e}(\alpha_n)$ and $\alpha_1, \alpha_n \in Q_1$.

Proof. Case 1: Suppose χ' contains a vertex u such that $\mathfrak{e}_u > 1$. If $u = v_1$ then the required string is a direct string of source and target x. Otherwise, since χ is connected, there exists a simple path

$$p : v_1 = u_0 \overset{y_1}{\text{———}} u_1 \overset{y_2}{\text{———}} u_2 \text{———} \cdots \text{———} u_{m-1} \overset{y_m}{\text{———}} u_m = u.$$

between the vertices v_1 and u, consisting of polygons y_i in χ'. Set $y_0 := x$ and let $w_{(i,+)}$ be the direct string of minimal length with first symbol β_i and last symbol γ_i such that $\widehat{s}(\beta_i) = y_i^{u_i}$ and $\widehat{e}(\gamma_i) = y_{i+1}^{u_i}$. Similarly, let $w_{(i,-)}$ be the direct string of minimal length with first symbol δ_i and last symbol ζ_i such that $\widehat{s}(\delta_i) = y_{i+1}^{u_i}$ and $\widehat{e}(\zeta_i) = y_i^{u_i}$. Finally, let w' be the direct string of minimal length with first symbol η and last symbol ξ such that $\widehat{s}(\eta) = y_m^{u_m} = \widehat{e}(\xi)$. If m is of even length, then the required string is

$$w = w_{(0,+)} w_{(1,-)}^{-1} \dots w_{(m-2,+)} w_{(m-1,-)}^{-1} w' w_{(m-1,+)}^{-1} w_{(m-2,-)} \dots w_{(1,+)}^{-1} w_{(0,-)}.$$

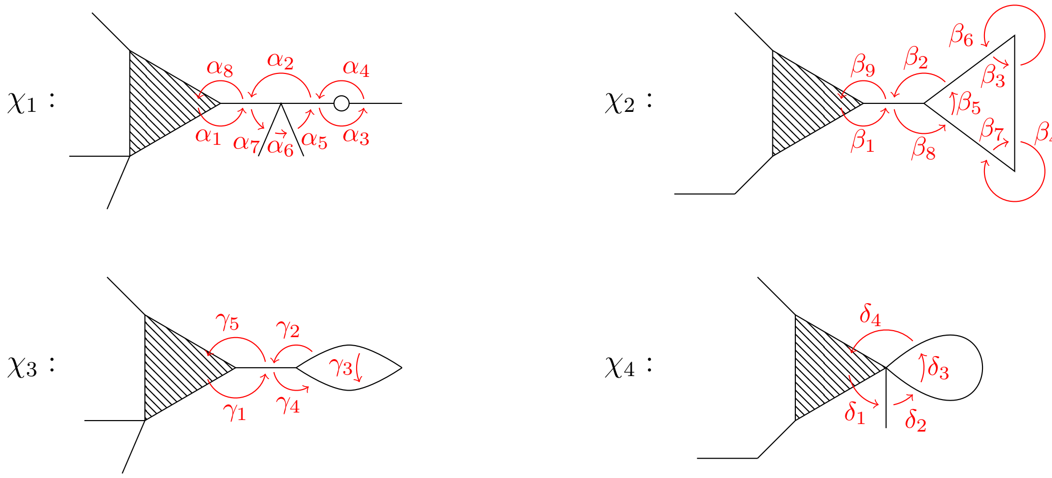
If m is of odd length, then the required string is instead

$$w = w_{(0,+)} w_{(1,-)}^{-1} \dots w_{(m-1,+)} (w')^{-1} w_{(m-1,-)} \dots w_{(1,+)}^{-1} w_{(0,-)}.$$

Case 2: Suppose χ' contains a cycle c. Then there again exists a simple path

p of the same form as in Case 1, where u is a vertex connected to the cycle c. We choose p such that no vertex u_i is a vertex of c for $i < m$. By considering c as a path, similar arguments to those used in Case 1 on the path p show that there exists a string w' (that is neither direct nor inverse) with first symbol $\eta \in Q_1$ and last symbol $\xi \in Q_1$ such that $\widehat{s}(\eta) = y_m^{u_m} = \widehat{e}(\xi)$ – one can show that w' has arrows and formal inverses that start and end on each polygon of c once if c is of even length, and twice if c is of odd length. The required string is then of the same form as in Case 1. $\square$

Examples 4.1.3. Let A_1, A_2, A_3 and A_4 be Brauer configuration algebras associated to the following respective Brauer configurations.



The circled vertex in χ_1 has multiplicity strictly greater than one. We will describe the strings presented in the proof of Lemma 4.1.2 for each algebra. For A_1, we have the string $w_1 = \alpha_1\alpha_2^{-1}\alpha_3\alpha_4\alpha_5^{-1}\alpha_6^{-1}\alpha_7^{-1}\alpha_8$. For A_2, we have the string $w_2 = \beta_1\beta_2^{-1}\beta_3\beta_4^{-1}\beta_5\beta_6^{-1}\beta_7\beta_8^{-1}\beta_9$. For A_3, we have the string $w_3 = \gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_5$. Finally, for A_4, we have the string $w_4 = \delta_1\delta_2\delta_3^{-1}\delta_4$.

Remark 4.1.4. Let v be a vertex in $\chi'' \cup \chi'''$. Then no symbol α_i of the string w constructed in the above proof is such that α_i or α_i^{-1} is an arrow in $\mathfrak{C}_v$.

For the construction that follows, we will make the following additional assumption.

Assumption 4.1.5. Let $A = KQ/I$ be the Brauer configuration algebra and $w =$

$\alpha_1 \ldots \alpha_n$ be the string from Lemma 4.1.2. Assume that both χ'' and χ''' are multiplicity-free trees and that χ''' contains at least two distinct polygons.

Consider the wild acyclic graph

$$\mathbb{X}_n : \quad \begin{tikzcd}[arrows=dash]
 & -3 \arrow[dr, "\beta_1'"] & & & & n+3 \arrow[dl, "\beta_2'"'] & \\
 & & 0 \arrow[r, "\alpha_1'"] & \cdots \arrow[r, "\alpha_n'"] & n \arrow[dr, "\gamma_2'"] & & \\
-2 \arrow[r, "\delta_1'"] & -1 \arrow[ur, "\gamma_1'"] & & & & n+1 \arrow[r, "\delta_2'"] & n+2.
\end{tikzcd} \qquad (n>0)$$

From the graph $\mathbb{X}_n$, we define a quiver Q'. Let Q' be a quiver with vertex set identical to the vertex set of $\mathbb{X}_n$. We define the arrow set of Q' as follows. We say that there exists an arrow $\alpha_i' : i \to i+1$ in Q' whenever $\alpha_i \in Q_1$ and an arrow $\alpha_i' : i \leftarrow i+1$ in Q' whenever $\alpha_i \in Q_1^{-1}$. In addition, we have arrows $\beta_1' : 0 \to -3$, $\gamma_1' : 0 \to -1$, $\beta_2' : n \leftarrow n+3$ and $\gamma_2' : n \leftarrow n+1$ in Q'. If $\operatorname{val}(v_3) > 2$ then we say that there exist arrows $\delta_1' : -1 \to -2$ and $\delta_2' : n+1 \leftarrow n+2$ in Q'. Otherwise if $\operatorname{val}(v_3) = 2$, then we say that there exist arrows $\delta_1' : -1 \leftarrow -2$ and $\delta_2' : n+1 \to n+2$ in Q'.

It is easy to see that KQ' is a finite-dimensional wild hereditary algebra (in fact, strictly wild). We will explicitly describe a fully faithful representation embedding $F : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} KQ'$. Recall that $K\langle a_1, a_2\rangle$ is an infinite-dimensional path algebra associated to a quiver with a single vertex and two loops. Let $M = (M_0, \lambda, \mu)$ be a K-representation of some finite-dimensional $K\langle a_1, a_2\rangle$-module M. That is, M_0 is a finite-dimensional vector space and $\lambda, \mu \in \operatorname{End}_K(M_0)$ are K-linear maps. Define a K-representation $FM = ((FM)_i, \varphi_{\zeta'})_{i \in Q_0', \zeta' \in Q_1'}$ of Q', which is either of the form

$$\begin{tikzcd}
 & M_0^2 & & & & & & M_0^2 \arrow[dl, "\left(\begin{smallmatrix} 1&0\\1&0\\0&1\\0&1 \end{smallmatrix}\right)"'] & \\
 & & M_0^4 \arrow[ul, "\left(\begin{smallmatrix} 1&\lambda&0&0\\0&0&\mu&1 \end{smallmatrix}\right)"'] \arrow[dl, "\left(\begin{smallmatrix} 1&0&0&0\\0&0&1&0 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 1&0&1&0\\0&1&0&0\\0&0&0&1 \end{smallmatrix}\right)"] & M_0^3 \arrow[r, dash, "\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)"] & \cdots \arrow[r, dash, "\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)"] & M_0^3 \arrow[r, "\left(\begin{smallmatrix} 1&0&0\\0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right)"] & M_0^4 & & \\
M_0 & M_0^2 \arrow[l, "\left(\begin{smallmatrix} 0&1 \end{smallmatrix}\right)"'] & & & & & & M_0^2 \arrow[ul, "\left(\begin{smallmatrix} 1&0\\0&0\\0&1\\0&0 \end{smallmatrix}\right)"] & M_0 \arrow[l, "\left(\begin{smallmatrix} 0\\1 \end{smallmatrix}\right)"']
\end{tikzcd}$$

if $\mathrm{val}(v_3) > 2$, or of the form

$$\begin{tikzcd}
 & M_0^2 & & & & & & M_0^2 \arrow[dl, "\left(\begin{smallmatrix}1&0\\1&0\\0&1\\0&1\end{smallmatrix}\right)"'] & \\
 & & M_0^4 \arrow[ul, "\left(\begin{smallmatrix}\lambda&1&0&0\\0&0&1&\mu\end{smallmatrix}\right)"'] \arrow[dl, "\left(\begin{smallmatrix}1&0&0&0\\0&0&1&0\end{smallmatrix}\right)"] \arrow[r, "\left(\begin{smallmatrix}1&0&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)"] & M_0^3 \arrow[r, dash, "\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)"] & \cdots \arrow[r, dash, "\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)"] & M_0^3 \arrow[r, "\left(\begin{smallmatrix}1&0&0\\0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)"] & M_0^4 & & \\
M_0 \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & M_0^2 & & & & & & M_0^2 \arrow[ul, "\left(\begin{smallmatrix}1&0\\0&0\\0&1\\0&0\end{smallmatrix}\right)"] \arrow[r, "\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)"] & M_0
\end{tikzcd}$$

if $\mathrm{val}(v_3) = 2$. If $n = 1$ then we simply have a linear map

$$\varphi_{\alpha_1'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where $\varphi_{\alpha_1'}$ is the linear map corresponding to the arrow $\alpha_1' \in Q_1'$.

Let N be a $K\langle a_1, a_2\rangle$-module with corresponding K-representation (N_0, λ', μ'). Then $f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$ can be viewed as a K-linear map $f_0 : M_0 \to N_0$ such that $\lambda' f_0 = f_0 \lambda$ and $\mu' f_0 = f_0 \mu$. Define $Ff \in \mathrm{Hom}_{KQ'}(FM, FN)$ to be the morphism $Ff = (Ff)_{i \in Q_0'}$, where each $(Ff)_i$ is a block diagonal matrix with diagonal entires f_0. It is easy to verify that this definition satisfies the necessary commutativity relations for Ff to be a genuine morphism of KQ'-modules.

Lemma 4.1.6. *The functor $F : \mathrm{fin}\, K\langle a_1, a_2\rangle \to \mathrm{mod}\, KQ'$ is K-linear, exact and fully faithful. Hence, F is a strict representation embedding.*

Proof. That F is K-linear follows trivially from the definition. It is also easy to see from the definition that given any morphisms $f, g \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$, $Ff = Fg$ if and only if $f = g$. So F is faithful. To show that F is exact, it is sufficient to show that $\mathrm{Ker}\, Ff = F(\mathrm{Ker}\, f)$ and $\mathrm{Im}\, Ff = F(\mathrm{Im}\, f)$ for any $f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$, since given any pair of morphisms $f, g \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$ such that $\mathrm{Ker}\, f = \mathrm{Im}\, g$, we have $\mathrm{Ker}\, Ff = \mathrm{Im}\, Fg$.

We will first calculate $\mathrm{Ker}\, Ff$. Let $\theta : \mathrm{Ker}\, f \to M$ be an inclusion morphism such that $f\theta = 0$. Note that θ is unique up to the universal property. Since each

$(Ff)_i$ is a block diagonal matrix, we have $(\operatorname{Ker} Ff)_i = \operatorname{Ker}((Ff)_i) = (F(\operatorname{Ker} f))_i$ for all i. Moreover, $(F\theta)_i$ is an inclusion morphism such that $(Ff)_i(F\theta)_i = 0$ for all i. Thus, for any arrow $\zeta' : i \to j$ in Q', we have commutative squares of the following form.

$$\begin{tikzcd}
\widetilde{\mu} \circlearrowright \operatorname{Ker} f_0 \circlearrowleft \widetilde{\lambda} \arrow[d, "\theta_0"'] & & \cdots \arrow[r] & (\operatorname{Ker} f_0)^{r_i} \arrow[r, "\widetilde{\varphi}_{\zeta'}"] \arrow[d, "(F\theta)_i = \begin{pmatrix} \theta_0 & & 0 \\ & \ddots & \\ 0 & & \theta_0 \end{pmatrix}"'] & (\operatorname{Ker} f_0)^{r_j} \arrow[r] \arrow[d, "\begin{pmatrix} \theta_0 & & 0 \\ & \ddots & \\ 0 & & \theta_0 \end{pmatrix} = (F\theta)_j"] & \cdots \\
\mu \circlearrowright M_0 \circlearrowleft \lambda \arrow[d, "f_0"'] \arrow[rr, dashed, "F"] & & \cdots \arrow[r] & M_0^{r_i} \arrow[r, "\varphi_{\zeta'}"] \arrow[d, "(Ff)_i = \begin{pmatrix} f_0 & & 0 \\ & \ddots & \\ 0 & & f_0 \end{pmatrix}"'] & M_0^{r_j} \arrow[r] \arrow[d, "\begin{pmatrix} f_0 & & 0 \\ & \ddots & \\ 0 & & f_0 \end{pmatrix} = (Ff)_j"] & \cdots \\
\mu' \circlearrowright N_0 \circlearrowleft \lambda' & & \cdots \arrow[r] & N_0^{r_i} \arrow[r, "\varphi'_{\zeta'}"] & N_0^{r_j} \arrow[r] & \cdots
\end{tikzcd}$$

If the block matrix entry $(\varphi_{\zeta'})_{kl}$ of $\varphi_{\zeta'}$ is an identity map, then it follows that $(\widetilde{\varphi}_{\zeta'})_{kl}$ is also an identity map. Similarly, if $(\varphi_{\zeta'})_{kl} = 0$ then $(\widetilde{\varphi}_{\zeta'})_{kl} = 0$, since θ_0 is an inclusion morphism. If $(\varphi_{\zeta'})_{kl}$ is either the map λ or μ, then we simply note that $\lambda\theta_0 = \theta_0\widetilde{\lambda}$ and $\mu\theta_0 = \theta_0\widetilde{\mu}$. So $(\widetilde{\varphi}_{\zeta'})_{kl}$ is the map $\widetilde{\lambda}$ or $\widetilde{\mu}$ respectively. This is precisely the K-representation $F(\operatorname{Ker} f)$, as required. The proof for showing $\operatorname{Im} Ff = F(\operatorname{Im} f)$ is similar – we simply look at the canonical surjection ξ of M into the image of f and show that $F\xi$ is a surjection into the $\operatorname{Im} Ff$. Thus, the functor F is exact.

It remains to show that F is full. Let $M = (M_0, \lambda, \mu)$ and $N = (N_0, \lambda', \mu')$ and let $F_{M,N} : \operatorname{Hom}_{K\langle a_1,a_2\rangle}(M, N) \to \operatorname{Hom}_{KQ'}(FM, FN)$ be the function defined by $F_{M,N}(f) = Ff$. We will calculate $\operatorname{Hom}_{KQ'}(FM, FN)$ and show that $\operatorname{Im} F_{M,N} = \operatorname{Hom}_{KQ'}(FM, FN)$. We will only give the proof for the case where $\operatorname{val}(v_3) > 2$, as the proof for the other case is similar. Let $\Phi = (\Phi_i)_{i \in Q'_0} \in \operatorname{Hom}_{KQ'}(FM, FN)$. We have the following commutative squares.

$$\text{(i):}\ \begin{tikzcd}
M_0^2 \arrow[r, "(0\ 1)"] \arrow[d, "\Phi_{-1}"'] & M_0 \arrow[d, "\Phi_{-2}"] \\
N_0^2 \arrow[r, "(0\ 1)"] & N_0
\end{tikzcd}
\qquad
\text{(ii):}\ \begin{tikzcd}
M_0^4 \arrow[r, "\begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \end{pmatrix}"] \arrow[d, "\Phi_0"'] & M_0^2 \arrow[d, "\Phi_{-1}"] \\
N_0^4 \arrow[r, "\begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \end{pmatrix}"] & N_0^2
\end{tikzcd}
\qquad
\text{(iii):}\ \begin{tikzcd}
M_0^4 \arrow[r, "\begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix}"] \arrow[d, "\Phi_0"'] & M_0^4 \arrow[d, "\Phi_n"] \\
N_0^4 \arrow[r, "\begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix}"] & N_0^4
\end{tikzcd}$$

$$\text{(iv):}\quad \begin{tikzcd} M_0^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\1&0\\0&1\\0&1\end{smallmatrix}\right)"] \arrow[d, "\Phi_{n+3}"'] & M_0^4 \arrow[d, "\Phi_n"] \\ N_0^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\1&0\\0&1\\0&1\end{smallmatrix}\right)"] & N_0^4 \end{tikzcd} \qquad \text{(v):}\quad \begin{tikzcd} M_0^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&0\\0&1\\0&0\end{smallmatrix}\right)"] \arrow[d, "\Phi_{n+1}"'] & M_0^4 \arrow[d, "\Phi_n"] \\ N_0^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&0\\0&1\\0&0\end{smallmatrix}\right)"] & N_0^4 \end{tikzcd} \qquad \text{(vi):}\quad \begin{tikzcd} M_0 \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \arrow[d, "\Phi_{n+2}"'] & M_0^2 \arrow[d, "\Phi_{n+1}"] \\ N_0 \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & N_0^2 \end{tikzcd}$$

Note that if the length n of the string w used to construct Q' is such that $n > 1$, then square (iii) is obtained by composing the linear maps between the vertices 0 and n of Q'. Square (i) implies that

$$\Phi_{-1} = \begin{pmatrix} f & g \\ 0 & h \end{pmatrix},$$

where $f, g, h \in \mathrm{Hom}_K(M_0, N_0)$ and $h = \Phi_{-2}$. Square (ii) then implies that

$$\begin{array}{llll} (\Phi_0)_{11} = f, & (\Phi_0)_{12} = 0, & (\Phi_0)_{13} = g, & (\Phi_0)_{14} = 0, \\ (\Phi_0)_{31} = 0, & (\Phi_0)_{32} = 0, & (\Phi_0)_{33} = h, & (\Phi_0)_{34} = 0. \end{array}$$

Similarly, square (vi) implies that

$$\Phi_{n+1} = \begin{pmatrix} f' & 0 \\ g' & h' \end{pmatrix},$$

where $f', g', h' \in \mathrm{Hom}_K(M_0, N_0)$ and $h' = \Phi_{n+2}$. Square (v) then implies that

$$\begin{array}{ll} (\Phi_n)_{11} = f', & (\Phi_n)_{13} = 0, \\ (\Phi_n)_{21} = 0, & (\Phi_n)_{23} = 0, \\ (\Phi_n)_{31} = g', & (\Phi_n)_{33} = h', \\ (\Phi_n)_{41} = 0, & (\Phi_n)_{43} = 0. \end{array}$$

By square (iii), we have

$$\begin{pmatrix} f & 0 & g+h & 0 \\ (\Phi_0)_{21} & (\Phi_0)_{22} & (\Phi_0)_{23} & (\Phi_0)_{24} \\ f & 0 & g+h & 0 \\ (\Phi_0)_{41} & (\Phi_0)_{42} & (\Phi_0)_{43} & (\Phi_0)_{44} \end{pmatrix} = \begin{pmatrix} f' & (\Phi_n)_{12} & f' & (\Phi_n)_{14} \\ 0 & (\Phi_n)_{22} & 0 & (\Phi_n)_{24} \\ g'+h' & (\Phi_n)_{32} & g'+h' & (\Phi_n)_{34} \\ 0 & (\Phi_n)_{42} & 0 & (\Phi_n)_{44} \end{pmatrix}.$$

So $f = f'$. In addition, square (iv) implies

$$\begin{aligned} (\Phi_n)_{22} &= f, & (\Phi_n)_{24} &= 0, \\ (\Phi_n)_{42} &= g', & (\Phi_n)_{44} &= h'. \end{aligned}$$

and

$$\Phi_{n+3} = \begin{pmatrix} f & 0 \\ g' & h' \end{pmatrix}.$$

So

$$\Phi_0 = \begin{pmatrix} f & 0 & g & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & g' & 0 & h' \end{pmatrix} \quad \text{and} \quad \Phi_n = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ g' & 0 & h' & 0 \\ 0 & g' & 0 & h' \end{pmatrix}$$

with the relation $f = g + h = g' + h'$. Finally, we consider the commutative square

$$\begin{tikzcd} M_0^4 \arrow[r, "\left(\begin{smallmatrix} 1 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 1 \end{smallmatrix}\right)"] \arrow[d, "\Phi_0"'] & M_0^2 \arrow[d, "\Phi_{-3}"] \\ N_0^4 \arrow[r, "\left(\begin{smallmatrix} 1 & \lambda' & 0 & 0 \\ 0 & 0 & \mu' & 1 \end{smallmatrix}\right)"] & N_0^2 \end{tikzcd}$$

which gives us $g = g' = 0$ and therefore $f = h = h'$. The above square also gives us

$$\Phi_{-3} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

and commutativity relations $\lambda' f = f\lambda$ and $\mu' f = f\mu$. Thus f can also be considered as a morphism in $\mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$. It is easy to see that the commutative squares involving Φ_i for $0 < i < n$ give diagonal matrices with diagonal entries f. Thus, we have shown Φ_i is a block diagonal matrix with diagonal entries $f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M,N)$ for all i. So

$$\mathrm{Im}\, F_{M,N} = \mathrm{Hom}_{KQ'}(FM, FN)$$

and hence, F is a full functor. So F is a strict representation embedding. □

Now that we have a representation embedding $F : \mathrm{fin}\, K\langle a_1, a_2\rangle \to \mathrm{mod}\, KQ'$, we aim to construct a functor $G : \mathrm{mod}\, KQ' \to \mathrm{mod}\, A$ such that the composite functor $H = GF$ is a representation embedding. We give advance notice to the reader that H will not be strict. Thus, we must prove that H is exact, respects isomorphism classes and maps indecomposable $K\langle a_1, a_2\rangle$-modules to indecomposable A-modules

Recall that Q' is defined using the Brauer configuration algebra A associated to a Brauer configuration χ that contains a 3-gon x, and a string $w = \alpha_1 \dots \alpha_n$, which both satisfy Assumption 4.1.5. Define a morphism of quivers $\pi = (\pi_0, \pi_1) : Q' \to Q$. That is, we will define maps $\pi_0 : Q'_0 \to Q_0$ and $\pi_1 : Q'_1 \to Q_1$ such that any arrow $\alpha' : i \to j$ in Q' is mapped to an arrow $\pi_1(\alpha') : \pi_0(i) \to \pi_0(j)$ in Q. Define $\pi_0(0) = x = \pi_0(n)$. Then for all $1 \leq i \leq n$, define $\pi_1(\alpha'_i) = \alpha_i$ if $\alpha_i \in Q_1$ and $\pi_1(\alpha'_i) = \alpha_i^{-1}$ if $\alpha_i \in Q_1^{-1}$. Let β_1, β_2, γ_1 and γ_2 be the distinct arrows of Q such that $\widehat{s}(\beta_1) = x^{v_2}$, $\widehat{e}(\beta_2) = x^{v_2}$, $\widehat{s}(\gamma_1) = x^{v_3}$ and $\widehat{e}(\gamma_2) = x^{v_3}$. Then define $\pi_1(\beta'_1) = \beta_1$, $\pi_1(\beta'_2) = \beta_2$, $\pi_1(\gamma'_1) = \gamma_1$ and $\pi_1(\gamma'_2) = \gamma_2$.

If $\mathrm{val}(v_3) > 2$, then $\gamma'_1\delta'_1$ and $\delta'_2\gamma'_2$ form directed paths in Q'. In this case, we define $\pi_1(\delta'_1)$ to be the unique arrow of Q such that $\pi_1(\gamma'_1)\pi_1(\delta'_1) \notin I$ and define $\pi_1(\delta'_2)$ to be the unique arrow of Q such that $\pi_1(\delta'_2)\pi_1(\gamma'_2) \notin I$. See Figure 4.1(b) for a visual illustration.

Otherwise if $\mathrm{val}(v_3) = 2$, then there exists a polygon $y = \pi_0(e(\gamma'_1)) = \pi_0(s(\gamma'_2))$. Let u be any vertex distinct from v_3 connected to y. Such a choice of vertex u is not necessarily unique (for example, if $\pi_0(e(\gamma'_1))$ is not a 2-gon). However, the proof is not dependent on the choice of the vertex u, and so any choice of u will do. Then let δ_1 and δ_2 be the arrows of Q such that $\widehat{e}(\delta_1) = y^u = \widehat{s}(\delta_2)$ and define $\pi_1(\delta'_1) = \delta_1$ and $\pi_1(\delta'_2) = \delta_2$. See Figure 4.1(a) for a visual illustration.

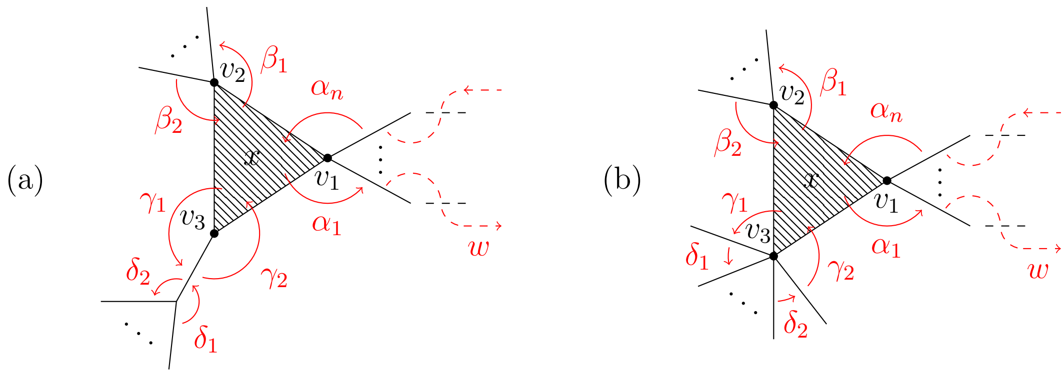


Figure 4.1: The image of the map π in the cases where (a) $\mathrm{val}(v_3) = 2$ and (b) $\mathrm{val}(v_3) > 2$. Primed arrows in Q' are mapped to the corresponding unprimed arrow in the figure. For example, $\pi_1(\alpha'_1) = \alpha_1$.

Remark 4.1.7. We have the following notes on the image of π.

(a) The image of any directed path in Q' under π avoids the relations in I.

(b) For any $i, j \in Q'_0$, define an equivalence relation $i \sim j \Leftrightarrow \pi_0(i) = \pi_0(j)$ and denote the equivalence class of $i \in Q'_0$ by $[i]$. Then for example, we have the following.

$$[-3] = \begin{cases} \{-3, n+3\} & \text{if } \mathrm{val}(v_2) = 2, \\ \{-3\} & \text{otherwise,} \end{cases}$$

$$[0] = [n]$$

$$[n+3] = \begin{cases} \{-3, n+3\} & \text{if } \mathrm{val}(v_2) = 2, \\ \{n+3\} & \text{otherwise.} \end{cases}$$

(c) For any $\zeta', \eta' \in Q'_1$, define an equivalence relation $\zeta' \sim \eta' \Leftrightarrow \pi_1(\zeta') = \pi_1(\eta')$ and denote the equivalence class of $\zeta' \in Q'_1$ by $[\zeta']$. Then by Remark 4.1.4,

$$\beta'_1, \beta'_2, \gamma'_1, \gamma'_2, \delta'_1, \delta'_2 \notin [\alpha'_i] \text{ for any } 1 \leq i \leq n.$$

In addition, since it follows from Assumption 4.1.5 that $\mathrm{val}(v_2), \mathrm{val}(v_3) \geq 2$, we have

$$
\begin{aligned}
&[\beta_1'] = \{\beta_1'\}, && [\beta_2'] = \{\beta_2'\},\\
&[\gamma_1'] = \{\gamma_1'\}, && [\gamma_2'] = \{\gamma_2'\},\\
&[\delta_1'] = \begin{cases} \{\delta_1', \delta_2'\} & \text{if } \operatorname{val}(v_3) = 3,\\ \{\delta_1'\} & \text{otherwise,}\end{cases} && [\delta_2'] = \begin{cases} \{\delta_1', \delta_2'\} & \text{if } \operatorname{val}(v_3) = 3,\\ \{\delta_2'\} & \text{otherwise.}\end{cases}
\end{aligned}
$$

Let $M' = (M_i', \varphi_{\zeta'})_{i\in Q_0', \zeta'\in Q_1'}$ be a representation of the quiver Q' over the field K. Define a representation $GM' = ((GM')_y, \phi_\zeta)_{y\in Q_0, \zeta\in Q_1}$ of the quiver Q over K in the following way. For each $i \in Q_0'$, we say that the vector space M_i' is a direct summand of the vector space $(GM')_{\pi_0(i)}$. If $y \notin \operatorname{Im}\pi_0$, then we define $(GM')_y = 0$. Consider an arrow $\zeta : y \to z$ in Q such that $(GM')_y$ and $(GM')_z$ are non-zero. Suppose

$$(GM')_y = \bigoplus_{\pi_0(i)=y} M_i' \qquad \text{and} \qquad (GM')_z = \bigoplus_{\pi_0(k)=z} M_k'$$

Then the linear map ϕ_ζ is given by a block matrix $((\phi_\zeta)_{k,i})_{k\in R, i\in C}$ with row and column index sets

$$R = \{k : \pi(k) = z\} \qquad \text{and} \qquad C = \{i : \pi(i) = y\}$$

respectively, where $(\phi_\zeta)_{k,i} : M_i' \to M_k'$ is a linear map defined by $\varphi_{\zeta'}$ if $\zeta' : i \to k$ is an arrow in Q' such that $\pi_1(\zeta') = \zeta$, and is zero otherwise. Since the image of any directed path in Q' under π avoids the relations in I, $\phi_\zeta\phi_\eta = 0$ for any path $\zeta\eta \in I$. Thus the representation GM' respects the relations in I and hence corresponds to an A-module.

Remark 4.1.8. We have the following notes on the K-linear maps ϕ_ζ in the K-representation GM'.

(a) Let $\zeta \in \{\pi_1(\beta_j'), \pi_1(\gamma_j') : j = 1, 2\}$. Then by Remark 4.1.7(c), the map ϕ_ζ contains at most one non-zero entry when considered as a block matrix.

(b) For any arrow $\alpha_i' \in Q_1'$ $(1 \le i \le n)$, each row and column of the block matrix

$\phi_{\pi_1(\alpha'_i)}$ contains at most one non-zero entry. This follows from the fact that the arrows $\pi_1(\alpha'_i)$ follow a string of length n.

(c) It follows from Remark 4.1.7(c) that if $\mathrm{val}(v_3) = 3$, then $\phi_{\pi_1(\delta'_1)} = \phi_{\pi_1(\delta'_2)}$ is a diagonal 2×2 block matrix with possibly non-zero diagonal entries. Otherwise, $\phi_{\pi_1(\delta'_1)} \neq \phi_{\pi_1(\delta'_2)}$ and both $\phi_{\pi_1(\delta'_1)}$ and $\phi_{\pi_1(\delta'_2)}$ contain at most one non-zero entry.

Let $f' = (f'_i)_{i \in Q'_0} : M' \to N'$ be a morphism of representations of Q' over K. We define a morphism $Gf' = ((Gf')_y)_{y \in Q_0} : GM' \to GN'$ of representations of Q over K as follows. If $y \notin \operatorname{Im} \pi_0$ (and hence, $(GM')_y = 0$) then $(Gf')_y = 0$. Otherwise, suppose

$$(GM')_y = \bigoplus_{\pi_0(i)=y} M'_i \qquad \text{and} \qquad (GN')_y = \bigoplus_{\pi_0(i)=y} N'_i$$

Then we define $(Gf')_y$ to be a block diagonal matrix $((Gf')_{i,j})_{i,j \in C}$ in which each (i,i)-th diagonal entry is precisely the linear map $f'_i : M'_i \to N'_i$. One can verify that Gf' is a genuine morphism of A-modules by considering commutative squares

$$\begin{tikzcd}
(GM')_y \arrow[r, "\phi_\zeta"] \arrow[d, "(Gf')_y"'] & (GM')_z \arrow[d, "(Gf')_z"] \\
(GN')_y \arrow[r, "\phi'_\zeta"] & (GN')_z
\end{tikzcd}$$

which induce commutativity relations $f'_k(\phi_\zeta)_{k,i} = (\phi'_\zeta)_{k,i} f'_i$ for each arrow $\zeta : y \to z$ in Q. By definition, either $(\phi_\zeta)_{k,i} = (\phi'_\zeta)_{k,i} = 0$, or $(\phi_\zeta)_{k,i} = \varphi_{\zeta'}$ and $(\phi'_\zeta)_{k,i} = \varphi'_{\zeta'}$ for some arrow $\zeta' : i \to k$ in Q'. Since f' is a morphism of KQ'-modules (and hence the commutativity relation $f'_k \varphi_{\zeta'} = \varphi'_{\zeta'} f'_i$ is satisfied), we deduce that Gf' is a morphism of A-modules. Thus, we have defined a functor $G : \operatorname{mod} KQ' \to \operatorname{mod} A$ which maps a KQ'-module M' to an A-module GM' and a morphism $f' : M' \to N'$ to a morphism $Gf' : GM' \to GN'$.

Remark 4.1.9. Given $K\langle a_1, a_2\rangle$-modules M and N and a morphism $f = (f_0) : M \to N$, the linear map $(GFf)_y$ is a block diagonal matrix with diagonal entries f_0 for all $y \in Q_0$.

Example 4.1.10. Consider the Brauer configuration algebra A_4 and string w_4 from

Example 4.1.3. The quiver Q' is the following orientation of $\mathbb{X}_4$.

$$
\begin{tikzcd}
 & -3 & & & & & & 7 \arrow[dl] & \\
 & & 0 \arrow[ul] \arrow[dl] \arrow[r] & 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 & & \\
-2 \arrow[r] & -1 & & & & & & 5 \arrow[ul] \arrow[r] & 6
\end{tikzcd}
$$

Consider the following K-representation of Q'.

$$
\begin{tikzcd}
 & M'_{-3} & & & & & & M'_7 \arrow[dl, "\varphi_{\beta'_2}"'] & \\
 & & M'_0 \arrow[ul, "\varphi_{\beta'_1}"'] \arrow[dl, "\varphi_{\gamma'_1}"] \arrow[r, "\varphi_{\alpha'_1}"] & M'_1 \arrow[r, "\varphi_{\alpha'_2}"] & M'_2 & M'_3 \arrow[l, "\varphi_{\alpha'_3}"'] \arrow[r, "\varphi_{\alpha'_4}"] & M'_4 & & \\
M'_{-2} \arrow[r, "\varphi_{\delta'_1}"] & M'_{-1} & & & & & & M'_5 \arrow[ul, "\varphi_{\gamma'_2}"] \arrow[r, "\varphi_{\delta'_2}"] & M'_6
\end{tikzcd}
$$

Then the A-module GM' is given by the following K-representation.

$$
GM' : \quad
\begin{tikzcd}
 & M'_{-3} \oplus M'_7 \arrow[dr, shift left, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & \varphi_{\beta'_2} \end{smallmatrix}\right)"'] & & & \\
 & & M'_0 \oplus M'_4 \arrow[ul, shift left, "\left(\begin{smallmatrix} \varphi_{\beta'_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[dl, shift left, "\left(\begin{smallmatrix} \varphi_{\gamma'_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[dr, "\left(\begin{smallmatrix} \varphi_{\alpha'_1} & 0 \end{smallmatrix}\right)"] & & M'_2 \oplus M'_3 \arrow[ll, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & \varphi_{\alpha'_4} \end{smallmatrix}\right)"'] \arrow[loop right, "\left(\begin{smallmatrix} 0 & 0 \\ \varphi_{\alpha'_3} & 0 \end{smallmatrix}\right)"] \\
M'_{-2} \oplus M'_6 \arrow[r, shift right, "\left(\begin{smallmatrix} \varphi_{\delta'_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & M'_{-1} \oplus M'_5 \arrow[l, shift right, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & \varphi_{\delta'_2} \end{smallmatrix}\right)"'] \arrow[ur, shift left, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & \varphi_{\gamma'_2} \end{smallmatrix}\right)"'] & & M'_1 \arrow[ur, "\left(\begin{smallmatrix} \varphi_{\alpha'_2} \\ 0 \end{smallmatrix}\right)"'] &
\end{tikzcd}
$$

Given an A-module N' and a morphism $f' = (f'_{-3}, \ldots, f'_7) : M' \to N'$, we have a morphism $Gf' : GM' \to GN'$ of the form

$$
Gf' = \left(\begin{pmatrix} f'_{-3} & 0 \\ 0 & f'_7 \end{pmatrix}, \begin{pmatrix} f'_{-2} & 0 \\ 0 & f'_6 \end{pmatrix}, \begin{pmatrix} f'_{-1} & 0 \\ 0 & f'_5 \end{pmatrix}, \begin{pmatrix} f'_0 & 0 \\ 0 & f'_4 \end{pmatrix}, f'_1, \begin{pmatrix} f'_2 & 0 \\ 0 & f'_3 \end{pmatrix} \right)
$$

Given a $K\langle a_1, a_2 \rangle$-module M associated to the K-representation (M_0, λ, μ), the composite functor $H = GF$ gives us an A-module HM associated to the following K-representation.

$$
HM : \quad
\begin{tikzcd}[column sep=large, row sep=large]
 & M_0^4 \arrow[dr, shift left, "{\left(\begin{smallmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&1 \end{smallmatrix}\right)}"'] & & & \\
 & & M_0^8 \arrow[ul, shift left, "{\left(\begin{smallmatrix} \lambda&1&0&0&0&0&0&0\\ 0&0&1&\mu&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{smallmatrix}\right)}"'] \arrow[dl, shift right, "{\left(\begin{smallmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{smallmatrix}\right)}"'] \arrow[dr, "{\left(\begin{smallmatrix} 1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{smallmatrix}\right)}"] & & M_0^6 \arrow[ll, "{\left(\begin{smallmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{smallmatrix}\right)}"'] \arrow[loop right, "{\left(\begin{smallmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{smallmatrix}\right)}"] \\
M_0^2 \arrow[r, shift right, "{\left(\begin{smallmatrix} 0&0\\ 1&0\\ 0&0\\ 0&0 \end{smallmatrix}\right)}"'] & M_0^4 \arrow[l, shift right, "{\left(\begin{smallmatrix} 0&0&0&0\\ 0&0&0&1 \end{smallmatrix}\right)}"'] \arrow[ur, shift right, "{\left(\begin{smallmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{smallmatrix}\right)}"'] & & M_0^3 \arrow[ur, "{\left(\begin{smallmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right)}"'] &
\end{tikzcd}
$$

Let $N = (N_0, \lambda', \mu')$. Given a morphism $f = (f_0) : M \to N$, we have a morphism $Hf = ((Hf)_y)_{y \in Q_0} : HM \to HN$ such that each $(Hf)_y$ is a block diagonal matrix with diagonal entries f_0.

It is clear that the functor G is not a representation embedding, since it does not respect isomorphism classes. A counterexample is formed with the simple modules S_0' and S_n' associated to the vertices 0 and n of Q', respectively. Clearly, it follows from Remark 4.1.7(b) that we have $GS_0' \cong GS_n'$ but $S_0' \ncong S_n'$. However, the functor G should respect isomorphism classes for all modules $M' \in \operatorname{mod} KQ'$ that do not contain a string module as a direct summand. Indeed, we will show with the following lemmata that G satisfies the necessary properties of a representation embedding under the image of the functor F. That is, we will show that the composite functor $H = GF$ is a representation embedding.

Lemma 4.1.11. *The functor $H = GF : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$ is exact.*

Proof. Let $M = (M_0, \lambda, \mu)$ and $N = (N_0, \lambda', \mu')$ be K-representations of $K\langle a_1, a_2\rangle$-modules M and N. Note that each vertex in the K-representation associated to HM (resp. HN) is a direct sum of copies of M_0 (resp. N_0), and for any morphism $f = (f_0) \in \operatorname{Hom}_{K\langle a_1, a_2\rangle}(M, N)$, the linear map $(Hf)_y$ is a block diagonal matrix with diagonal entries f_0 for all $y \in Q_0$. Thus, the proof of the exactness of H is identical to the proof of the exactness of the functor F in Lemma 4.1.6. $\square$

Lemma 4.1.12. *The functor $H = GF : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$ maps indecomposable $K\langle a_1, a_2\rangle$-modules to indecomposable A-modules.*

Proof. We will prove that G maps indecomposable KQ'-modules to indecomposable A-modules. It is sufficient to prove the contrapositive statement. Namely, that if $M' = (M'_i, \varphi_{\zeta'})_{i \in Q'_0, \zeta' \in Q'_1}$ is a representation associated to a KQ'-module such that GM' is decomposable, then M' is decomposable.

So suppose $GM' = ((GM')_y, \phi_\zeta)_{y \in Q_0, \zeta \in Q_1}$ is decomposable. Then there exists an isomorphism of representations $\Phi = (\Phi_y)_{y \in Q_0} : GM' \to N \oplus L$ for some non-zero K-representations $N = (N_y, \theta_\zeta)_{y \in Q_0, \zeta \in Q_1}$ and $L = (L_y, \psi_\zeta)_{y \in Q_0, \zeta \in Q_1}$ of Q such that for any arrow $\zeta : y \to z$ in Q, the square

$$\begin{tikzcd}
(GM')_y \arrow[r, "\phi_\zeta"] \arrow[d, "\Phi_y"'] & (GM')_z \arrow[d, "\Phi_z"] \\
N_y \oplus L_y \arrow[r, "\begin{pmatrix} \theta_\zeta & 0 \\ 0 & \psi_\zeta \end{pmatrix}"] & N_z \oplus L_z
\end{tikzcd}$$

commutes. For each $y \in Q_0$, write

$$(GM')_y = \bigoplus_{\pi_0(i)=y} M'_i \qquad \text{and} \qquad (GM')_z = \bigoplus_{\pi_0(k)=z} M'_k$$

and consider each map ϕ_ζ as a block matrix. Since, Φ_y and Φ_z are bijective, we may then write

$$N_y = \bigoplus_{\pi_0(i)=y} U_i, \qquad\qquad N_z = \bigoplus_{\pi_0(k)=z} U_k,$$
$$L_y = \bigoplus_{\pi_0(i)=y} V_i, \qquad\qquad L_z = \bigoplus_{\pi_0(k)=z} V_k,$$

where U_i and V_i are vector subspaces such that $U_i \oplus V_i = \Phi_y(M'_i) \cong M'_i$ for all i such that $\pi_0(i) = y$, and U_k and V_k are vector subspaces such that $U_k \oplus V_k = \Phi_z(M'_k) \cong M'_k$ for all k such that $\pi_0(k) = z$.

For each arrow $\zeta : y \to z$ in Q and for each i such that $\pi_0(i) = y$, define K-linear maps

$$\theta'_i = \Phi_z \phi_\zeta \Phi_y^{-1}|_{U_i} : U_i \to N_z, \text{ and}$$

$$\psi'_i = \Phi_z \phi_\zeta \Phi_y^{-1}|_{V_i} : V_i \to L_z.$$

Note that $\operatorname{Im} \Phi_y^{-1}|_{U_i}, \operatorname{Im} \Phi_y^{-1}|_{V_i} \subseteq M'_i$. Thus, we have linear maps

$$\theta'_{k,i} = (\Phi_z \phi_\zeta)_{k,i} \Phi_y^{-1}|_{U_i} = \left(\sum_{\pi_0(j)=z} (\Phi_z)_{k,j} (\phi_\zeta)_{j,i} \right) \Phi_y^{-1}|_{U_i}, \text{ and}$$
$$\psi'_{k,i} = (\Phi_z \phi_\zeta)_{k,i} \Phi_y^{-1}|_{V_i} = \left(\sum_{\pi_0(j)=z} (\Phi_z)_{k,j} (\phi_\zeta)_{j,i} \right) \Phi_y^{-1}|_{V_i},$$

where $(\Phi_z)_{k,j} : M'_j \to U_k \oplus V_k$ and $(\phi_\zeta)_{j,i} : M'_i \to M'_j$ are linear maps, as defined. But by Remark 4.1.8(a), (b) and (c), each row and column of ϕ_ζ contains at most one non-zero entry. In particular, an entry $(\phi_\zeta)_{j,i}$ is non-zero only if $(\phi_\zeta)_{j,i} = \varphi_{\zeta'}$ for some arrow $\zeta' : i \to j$ in Q' such that $\pi_1(\zeta') = \zeta$ (this follows from the definition of G). So

$$\theta'_{k,i} = (\Phi_z)_{k,j} \varphi_{\zeta'} \Phi_y^{-1}|_{U_i}, \text{ and}$$
$$\psi'_{k,i} = (\Phi_z)_{k,j} \varphi_{\zeta'} \Phi_y^{-1}|_{V_i}$$

for some arrow $\zeta' : i \to j$ in Q' such that $\pi_1(\zeta') = \zeta$. But $\operatorname{Im} \varphi_{\zeta'} \Phi_y^{-1}|_{U_i}, \operatorname{Im} \varphi_{\zeta'} \Phi_y^{-1}|_{V_i} \subseteq M'_j$ and $M'_j \cong \Phi_z(M'_j) = U_j \oplus V_j$. So $(\Phi_z)_{k,j} = 0$ whenever $k \neq j$. Thus, $\theta'_{k,i} = 0$ if there exists no arrow $\zeta' : i \to k$ in Q'. So for each i such that $\pi_0(i) = y$ and each j such that $\pi_0(j) = z$, define vector spaces $N'_i = U_i$, $L'_i = V_i$, $N'_j = U_j$ and $L'_j = V_j$. Then for each arrow $\zeta' : i \to j$ in Q', there exist K-linear maps $\theta''_i = \theta'_{j,i} : N'_i \to N'_j$ and $\psi''_i = \psi'_{j,i} : L'_i \to L'_j$ such that the square

$$\begin{tikzcd} M'_i \arrow[r, "\varphi_{\zeta'}"] \arrow[d, "\Xi_i"'] & M'_j \arrow[d, "\Xi_j"] \\ N'_i \oplus L'_i \arrow[r, "\begin{pmatrix} \theta''_i & 0 \\ 0 & \psi''_i \end{pmatrix}"] & N'_j \oplus L'_j \end{tikzcd}$$

commutes, where $\Xi_i : M'_i \to N'_i \oplus L'_i$ is defined such that $\Xi_i = \Phi_y|_{M'_i}$ for all i such that $\pi_0(i) = y$. Since the maps Ξ_i are bijections for all i, we have shown that $M' \cong N' \oplus L'$ for some KQ'-modules N' and L'. Thus M' is decomposable, as required.

Since $F : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} KQ'$ is a representation embedding, it maps indecomposable $\operatorname{fin} K\langle a_1, a_2\rangle$-modules to indecomposable KQ'-modules. Since G maps indecomposable KQ'-modules to indecomposable A-modules, the composite functor $H = GF$ maps indecomposable $\operatorname{fin} K\langle a_1, a_2\rangle$-modules to indecomposable A-modules, as required. $\square$

Lemma 4.1.13. *The functor $H = GF : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$ respects isomorphism classes. That is, if $HM \cong HN$ then $M \cong N$ for all $M, N \in \operatorname{fin} K\langle a_1, a_2\rangle$.*

Proof. Let M and N be finite-dimesnional $K\langle a_1, a_2\rangle$-modules such that $GFM \cong GFN$. Let

$$FM = ((FM)_i, \varphi_{\zeta'})_{i\in Q_0', \zeta'\in Q_1'}, \qquad FN = ((FN)_i, \varphi'_{\zeta'})_{i\in Q_0', \zeta'\in Q_1'}$$
$$GFM = ((GFM)_y, \phi_\zeta)_{y\in Q_0, \zeta\in Q_1}, \qquad GFN = ((GFN)_y, \phi'_\zeta)_{y\in Q_0, \zeta\in Q_1}.$$

Then there exists a bijective K-linear map $(\Phi_y)_{y\in Q_0}$ such that the squares

$$\begin{tikzcd}
(GFM)_y \arrow[r, "\phi_\zeta"] \arrow[d, "\Phi_y"'] & (GFM)_z \arrow[d, "\Phi_z"] \\
(GFN)_y \arrow[r, "\phi'_\zeta"] & (GFN)_z
\end{tikzcd}$$

commute. Write

$$(GFM)_y = \bigoplus_{\pi_0(i)=y} (FM)_i, \qquad (GFM)_z = \bigoplus_{\pi_0(k)=z} (FM)_k,$$
$$(GFN)_y = \bigoplus_{\pi_0(i)=y} (FN)_i, \qquad (GFN)_z = \bigoplus_{\pi_0(k)=z} (FN)_k.$$

Then each Φ_y can be viewed as a block matrix, where each entry $(\Phi_y)_{j,i} : (FM)_i \to (FN)_j$ is a K-linear map such that $\pi_0(i) = \pi_0(j) = y$. Recall further that for any arrow $\zeta : y \to z$ in Q, the maps ϕ_ζ and ϕ'_ζ can be viewed as a block matrices such that the entries $(\phi_\zeta)_{k,i} : (FM)_i \to (FM)_k$ and $(\phi'_\zeta)_{k,i} : (FN)_i \to (FN)_k$ are K-linear maps such that $\pi_0(i) = y$ and $\pi_0(k) = z$. If $(\phi_\zeta)_{k,i}$ (resp. $(\phi'_\zeta)_{k,i}$) is a non-zero entry of ϕ_ζ (resp. ϕ'_ζ), then $(\phi_\zeta)_{k,i} = \varphi_{\zeta'}$ and $(\phi'_\zeta)_{k,i} = \varphi'_{\zeta'}$ for some arrow

$\zeta' : i \to k$ in Q' such that $\pi_1(\zeta') = \zeta$. By Remark 4.1.8(a), (b) and (c), each row and column of ϕ_ζ contains at most one non-zero entry. Thus for each arrow $\zeta' : i \to k$ in Q', we have commutativity relations of the form

$$(\phi'_{\pi_1(\zeta')}\Phi_{\pi_0(i)})_{k,i} = \varphi'_{\zeta'}(\Phi_{\pi_0(i)})_{i,i} = (\Phi_{\pi_0(k)})_{k,k}\varphi_{\zeta'} = (\Phi_{\pi_0(k)}\phi_{\pi_1(\zeta')})_{k,i}. \qquad (*)$$

Note that the relations $(*)$ are precisely the commutativity relations that determine the space $\mathrm{Hom}_{KQ'}(FM, FN)$. So $\Phi' = (\Phi'_i)_{i \in Q'_0} \in \mathrm{Hom}_{KQ'}(FM, FN)$, where $\Phi'_i = (\Phi_{\pi_0(i)})_{i,i}$. Recall that the space $\mathrm{Hom}_{KQ'}(FM, FN)$ was calculated in the proof of Lemma 4.1.6. Specifically, each K-linear map $\Phi'_i : (FM)_i \to (FN)_i$ is a block diagonal matrix with diagonal entries f_0, where $(f_0) = f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$.

Recall that Q' is constructed using a string. It follows from Remark 4.1.7(a) that

$$(GFM)_{\pi_0(n+3)} = \begin{cases} (FM)_{n+3} & \text{if } \mathrm{val}(v_2) > 2 \\ (FM)_{-3} \oplus (FM)_{n+3} & \text{if } \mathrm{val}(v_2) = 2. \end{cases}$$

$$(GFN)_{\pi_0(n+3)} = \begin{cases} (FN)_{n+3} & \text{if } \mathrm{val}(v_2) > 2 \\ (FN)_{-3} \oplus (FN)_{n+3} & \text{if } \mathrm{val}(v_2) = 2. \end{cases}$$

In the case where $\mathrm{val}(v_2) > 2$, we have $\Phi_{\pi_0(n+3)} = \Phi'_{n+3}$, which by assumption is bijective. But since Φ'_{n+3} is a block diagonal matrix with diagonal entries $f_0 = f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M, N)$ it follows that Φ'_{n+3} is bijective only if f_0 is bijective. Thus in this case, there exists an isomorphism $f : M \to N$.

In the case where $\mathrm{val}(v_2) = 2$, we have

$$\Phi_{\pi_0(n+3)} = \begin{pmatrix} (\Phi_{\pi_0(n+3)})_{-3,-3} & (\Phi_{\pi_0(n+3)})_{-3,n+3} \\ (\Phi_{\pi_0(n+3)})_{n+3,-3} & (\Phi_{\pi_0(n+3)})_{n+3,n+3} \end{pmatrix}.$$

Firstly, we will make the observation that $\dim_K M = \dim_K N$, since $GFM \cong GFN$, which implies

$$\dim_K GFM = (3n + 15) \dim_K M_0 = (3n + 15) \dim_K N_0 = \dim_K GFN,$$

where n is the length of the string used in the construction of Q'. Secondly, note from the definition of F and the above observation that

$$\dim_K(FM)_{-3} = \dim_K(FM)_{n+3} = \dim_K(FN)_{-3} = \dim_K(FN)_{n+3},$$

so each block entry of $\Phi_{\pi_0(n+3)}$ is a square matrix. Thirdly, note from the definition of F that the map

$$\varphi_{\beta'_2} = \varphi'_{\beta'_2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

is injective (where $\beta'_2 : n+3 \to n \in Q'_1$, as defined earlier in this section). Fourthly, by Remark 4.1.8(a), the map $\phi_{\pi_1(\beta'_2)}$ (resp. $\phi'_{\pi_1(\beta'_2)}$) has at most one non-zero entry, which is $(\phi_{\pi_1(\beta'_2)})_{n,n+3} = \varphi_{\beta'_2}$ (resp. $(\phi'_{\pi_1(\beta'_2)})_{n+3,n} = \varphi'_{\beta'_2}$). Thus, we have a commutative square of the form

$$\begin{tikzcd} (FM)_{-3} \oplus (FM)_{n+3} \arrow[r, "\left(\begin{smallmatrix} 0 & \varphi_{\beta'_2} \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[d, "\Phi_{\pi_0(n+3)}"'] & (FM)_n \oplus X \arrow[d, "\Phi_{\pi_0(n)}"] \\ (FN)_{-3} \oplus (FN)_{n+3} \arrow[r, "\left(\begin{smallmatrix} 0 & \varphi'_{\beta'_2} \\ 0 & 0 \end{smallmatrix}\right)"] & (FN)_n \oplus X' \end{tikzcd}$$

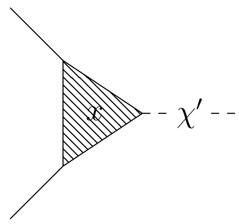
where $X = (GFM)_{\pi_0(n)}/(FM)_n$ and $X' = (GFN)_{\pi_0(n)}/(FN)_n$. From this, we obtain the relation $\varphi'_{\beta'_2}(\Phi_{\pi_0(n+3)})_{n+3,-3} = 0$. Since $\varphi'_{\beta'_2}$ is injective, it has a left inverse. This implies that $(\Phi_{\pi_0(n+3)})_{n+3,-3} = 0$, so $\Phi_{\pi_0(n+3)}$ is a block triangular matrix. Thus,

$$\det \Phi_{\pi_0(n+3)} = \det(\Phi_{\pi_0(n+3)})_{-3,-3} \det(\Phi_{\pi_0(n+3)})_{n+3,n+3}.$$

But $\Phi_{\pi_0(n+3)}$ is bijective by assumption and hence has non-zero determinant. Thus, both $(\Phi_{\pi_0(n+3)})_{-3,-3}$ and $(\Phi_{\pi_0(n+3)})_{n+3,n+3}$ have non-zero determinant, and hence both are bijections. But both $(\Phi_{\pi_0(n+3)})_{-3,-3}$ and $(\Phi_{\pi_0(n+3)})_{n+3,n+3}$ are diagonal matrices with diagonal entries f_0, where $(f_0) = f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M,N)$. Thus, there must exist an isomorphism $f \in \mathrm{Hom}_{K\langle a_1,a_2\rangle}(M,N)$, as required. $\square$

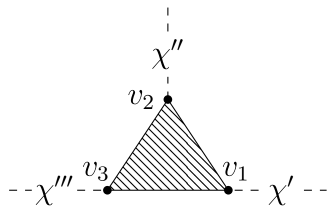
We can now prove Proposition 4.1.1, which we restate here for convenience.

Proposition. *Let A be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ contains a 3-gon x. Suppose further that χ contains a cycle or a vertex v such that $\mathfrak{e}_v > 1$. If χ is not of the form*



where χ' is a subconfiguration of χ and all vertices except those in χ' have multiplicity one, then A is wild.

Proof. Suppose χ is not of the form above. If x is self-folded then A is wild by Proposition 3.2.1, so assume that this is not the case. Suppose then that χ is locally of the form



where χ', χ'' and χ''' are subconfigurations of χ. If any two of the subconfigurations χ', χ'' and χ''' contain a polygons belonging to a cycle or vertices of multiplicity strictly greater than one, then A is also wild by Proposition 3.2.1, so assume that this is not the case either. Thus, χ', χ'' and χ''' are pairwaise disjoint and all cycles and vertices of χ of higher multiplicity must belong to precisely one of χ', χ'' and χ'''. Let χ' be this subconfiguration, which we may assume without loss of generality. Then χ'' and χ''' are multiplicity-free trees. Since χ is not of the form given in the Proposition statement, there necessarily exists at least two polygons in χ'' or χ'''. Suppose (without loss of generality) that χ''' contains at least two polygons. Then χ satisfies Assumption 4.1.5. Thus, there exists a K-linear functor $H : \operatorname{fin} K\langle a_1, a_2\rangle \to \operatorname{mod} A$ (defined above) that is exact (Lemma 4.1.11), maps indecomposable $K\langle a_1, a_2\rangle$-modules to indecomposable A-modules (Lemma 4.1.12), and respects isomorphism classes (Lemma 4.1.13), and hence, is a representation embedding. Thus, A is wild. □

4.2 Brauer Configurations with at Least Two 3-gons

We will address the final case of wild symmetric special triserial algebras, which is where the Brauer configuration contains multiple 3-gons. We begin with the following lemma.

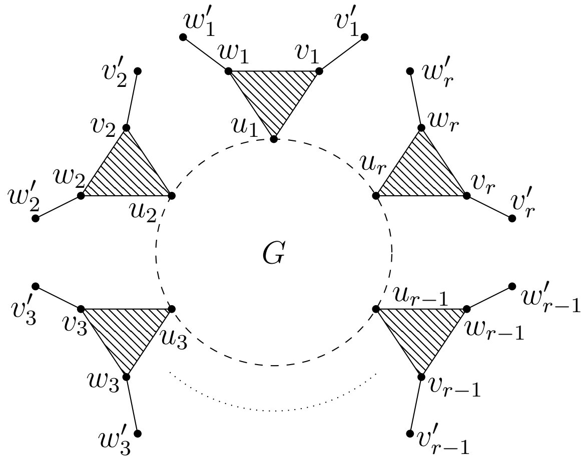
Lemma 4.2.1. *Let $A = KQ/I$ be a Brauer configuration algebra. Suppose there exists a connected acyclic subquiver $Q' \subset Q$ such that KQ' is a wild hereditary algebra and every (directed) path $\alpha_1 \dots \alpha_n$ in Q' is not in I. Then A is wild.*

Proof. Define a functor $F : \operatorname{mod} KQ' \to \operatorname{mod} A$ in the following way. For any KQ'-module M defined by a quiver representation $(M_x, \varphi_\alpha)_{x \in Q'_0, \alpha \in Q'_1}$, we define FM to be the A-module given by the quiver representation $((FM)_x, \phi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ such that

$$(FM)_x = \begin{cases} M_x, & \text{if } x \in Q'_0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \phi_\alpha = \begin{cases} \varphi_\alpha, & \text{if } \alpha \in Q'_1 \\ 0, & \text{otherwise.} \end{cases}$$

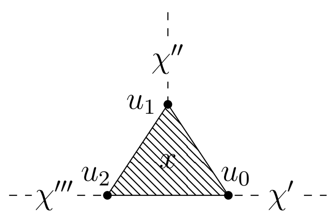
For any KQ'-modules M and N and any morphism $\Phi = (\Phi_x)_{x \in Q'_0} : M \to N$, we define a morphism $F\Phi = ((F\Phi)_x)_{x \in Q_0} : FM \to FN$ by $(F\Phi)_x = \Phi_x$ if $x \in Q'_0$ and $(F\Phi)_x = 0$ otherwise. It is easy to see that this functor is exact and fully faithful, and hence, is a (strict) representation embedding. Since, KQ' is a wild algebra, this implies A is also a wild algebra. □

Proposition 4.2.2. *Let $A = KQ/I$ be a Brauer configuration algebra associated to a Brauer configuration χ. Suppose χ contains at least two 3-gons. Suppose further that χ is not of the form*



where G is a Brauer graph connecting the (not necessarily distinct) vertices $u_1, \ldots, u_r$ and $\mathfrak{e}_{v_i} = \mathfrak{e}_{v'_i} = \mathfrak{e}_{w_i} = \mathfrak{e}_{w'_i} = 1$ for all i. Then A is wild.

Proof. If χ contains a 3-gon and an n-gon with $n > 3$, then A is wild by Theorem 3.2.3. If χ contains a cycle or a vertex with multiplicity strictly greater than one, then the result follows from Proposition 4.1.1. So assume that χ contains no n-gon with $n > 3$, no cycles and no vertices with multiplicity strictly greater than one. (Note that this implies that no 3-gon of χ is self-folded.) Under this assumption, if χ is not of the form given in the proposition statement, then χ contains a 3-gon x that is locally of the form



where χ', χ'' and χ''' are pairwise disjoint subconfigurations of χ such that at least two of χ', χ'' and χ''' contain more than one polygon distinct from x.

Let y be some 3-gon in χ distinct from x. Since Q is connected, there exists a string $w = \beta_1 \ldots \beta_n$ such that $s(w) = x$ and $e(w) = y$. Recall that since x is a 3-gon in χ, x is the source of 3 distinct arrows and the target of 3 distinct arrows in Q. Since at least two of χ', χ'' and χ''' contain more than one polygon distinct from x, there exist pairwise distinct symbols $\alpha_1, \alpha_2, \alpha_3 \in Q_1 \cup Q_1^{-1}$ such that α_1, α_2 and α_3 are not symbols of w and $\alpha_1 w$ and $\alpha_2\alpha_3 w$ are strings. Since y is the source of 3 distinct arrows and the target of 3 distinct arrows in Q, there exist pairwise distinct symbols $\gamma_1, \gamma_2, \in Q_1 \cup Q_1^{-1}$ such that γ_1 and γ_2 are not symbols of w and $w\gamma_1$ and $w\gamma_2$ are strings. Thus, there exists a wild subquiver Q' of Q with underlying graph

$$X: \quad \begin{tikzcd} & & \bullet \arrow[d, dash, "\alpha_1"] & & & & \bullet \arrow[d, dash, "\gamma_1"] & \\ \bullet \arrow[r, dash, "\alpha_2"] & \bullet \arrow[r, dash, "\alpha_3"] & x \arrow[r, dash, "\beta_1"] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash, "\beta_n"] & y \arrow[r, dash, "\gamma_2"] & \bullet \end{tikzcd}$$

The orientation of Q' is determined by the string w and the symbols α_1, α_2, α_3, γ_1 and γ_2. Since Q' is constructed from strings, every directed path of Q' avoids the relations in I. Thus KQ' is a wild hereditary algebra and by Lemma 4.2.1, A is wild. $\square$

4.3 The Proof of the Main Theorem

We can now bring together all of the results in Sections 3.2 and 3.3 to prove the Main Theorem in the Introduction. Let A be a Brauer configuration algebra associated to a Brauer configuration χ. Cases (a) and (b) of the Main Theorem follow from Theorem 3.2.3. For Case (c) of the main theorem, if χ contains at least two 3-gons, then χ cannot be of the form in (c)(ii) of the main theorem. By Proposition 4.2.2, A is wild if it is also not of the form in (c)(i). Now suppose χ contains precisely one 3-gon. If χ contains a cycle or a vertex of multiplicity strictly greater than one, then χ is necessarily not of the form in (c)(ii). By Proposition 4.1.1, if χ is also not of the form in (c)(i) of the main theorem (with $r = 1$), then A is wild.

The only remaining case is where χ contains precisely one 3-gon and χ is a multiplicity-free tree. If χ is of the form in (c)(i) with $r = 1$, then A is tame (in fact, of finite representation type), as by Proposition 3.3.12, it is derived equivalent to the trivial extension of some orientation of $\mathbb{D}_n$. So suppose this is not the case. Then at least two of the distinct subtrees connected to the unique 3-gon in χ contain at least two polygons each. By Theorem 3.3.1, if χ is not of the form in (c)(ii) of the main theorem, then A is wild. This completes the proof.

Glossary of Notation

$\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_p$	Dynkin diagrams of respective types.
$\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p$	Euclidean diagrams of respective types.
$\mathrm{add}(T)$	The full subcategory of $K^b(\mathrm{proj}\, A)$ consisting of direct summands of direct sums of copies of T.
$\mathfrak{C}_v$	The cycle of arrows in a quiver generated by a vertex v in a Brauer tree/graph/configuration.
$\mathrm{Coker}\, \phi$	The cokernel of a morphism ϕ.
D	The standard K-linear dual $D = \mathrm{Hom}_K(-, K)$.
det	The determinant of a matrix.
dim	The dimension of a vector space or module.
$e(\alpha)$	The target of an arrow α.
$\widehat{e}(\alpha)$	The half-edge or germ of a polygon corresponding to the target of an arrow α.
$\mathfrak{e}_v$	The multiplicity of a vertex v in a Brauer tree/graph/configuration.
$\mathrm{End}_A(M)$	The algebra of endomorphisms $M \to M \in \mathrm{mod}\, A$.
$\mathrm{End}_{K^b(\mathrm{proj}\, A)}(T)$	The algebra of endomorphisms $T \to T \in K^b(\mathrm{proj}\, A)$.
$\mathrm{fin}\, A$	The category of finite dimensional right A-modules.
G	A Brauer graph.
$\mathrm{Hom}_A(M, N)$	The vector space of morphisms $M \to N \in \mathrm{mod}\, A$.
$\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(T_1, T_2)$	The vector space of morphisms $T_1 \to T_2 \in K^b(\mathrm{proj}\, A)$.
I	An admissible ideal of a path algebra.
$I(M)$	The injective envelope of a module M.

$I(x)$	The indecomposable (right) injective module corresponding to a vertex x in a quiver/Brauer graph/Brauer configuration.
id_X	An identity morphism $X \to X$.
$\operatorname{Im} \phi$	The image of a morphism ϕ.
K	An algebraically closed field.
KQ	The path algebra of Q over the field K.
$K^b(\operatorname{proj} A)$	The bounded homotopy category of chain complexes over $\operatorname{proj} A$.
$\operatorname{Ker} \phi$	The kernel of a morphism ϕ.
$\operatorname{mod} A$	The category of finitely generated right A-modules.
$\overline{\operatorname{mod}}\, A$	The injectively stable category of finitely generated right A-modules.
$\underline{\operatorname{mod}}\, A$	The projectively stable category of finitely generated right A-modules.
$\mathfrak{o}_v$	The cyclic ordering of a vertex v in a Brauer tree/graph/configuration.
$\mathcal{P}(S)$	The power set of S.
$P(M)$	The projective cover of a module M.
$P(x)$	The indecomposable (right) projective module corresponding to a vertex x in a quiver/Brauer graph/Brauer configuration.
$\operatorname{proj} A$	The full subcategory of $\operatorname{mod} A$ consisting of projective A-modules.
Q	A quiver, unless otherwise stated.
Q_0	The vertex set of a quiver Q.
Q_1	The arrow set of a quiver Q.
$\operatorname{rad} M$	The radical of a module M.
$s(\alpha)$	The source of an arrow α.
$\widehat{s}(\alpha)$	The half-edge or germ of a polygon corresponding to the source of an arrow α.

$S(x)$	The simple (right) module corresponding to a vertex x in a quiver/Brauer graph/Brauer configuration.
$\operatorname{soc} M$	The socle of a module M.
$\operatorname{top} M$	The top of a module M. That is, the module $M/\operatorname{rad} M$.
Tr	The Auslander-Reiten transpose.
val	The valency of a vertex.
ε_x	The stationary path at a vertex x in a quiver.
χ	A configuration or Brauer configuration.
χ_0	The vertex set of a configuration or Brauer configuration.
χ_1	The polygon set of a configuration or Brauer configuration.
$\Omega(M)$	The syzygy of a module M.
τ	The Auslander-Reiten translate $\tau = DTr$.

Bibliography

[1] T. Adachi, T. Aihara, A. Chan, *Classification of two-term tilting complexes over Brauer graph algebras*, arXiv:1504.04827.

[2] T. Aihara, *Mutating Brauer trees*, Math. J. Okayama Univ. **56** (2014), 1–16.

[3] T. Aihara, *Derived equivalences between symmetric special biserial algebras*, J. Pure Appl. Algebra **219** (2015), no. 5, 1800–1825.

[4] L. A. Hügel, D. Happel, H. Krause, *Handbook of tilting theory*, London Mathematical Society Lecture Note Series, **332**. Cambridge University Press, Cambridge, 2007. viii+472 pp.

[5] H. Asashiba, *The derived equivalence classification of representation-finite self-injective algebras*, J. Algebra **214** (1999), no. 1, 182–221.

[6] I. Assem, J. Nehring, A. Skowroński, *Domestic trivial extensions of simply connected algebras*, Tsukuba J. Math. **13** (1989), no. 1, 31–72.

[7] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Volume 1, Techniques of Representation Theory*, London Mathematical Society Student Texts, **65**, Cambridge University Press, Cambridge, 2006.

[8] M. Auslander, *Representation theory of Artin algebras II*, Comm. Algebra, **1** (1974), 269–310.

[9] M. Auslander, I. Reiten, *Representation theory of Artin algebras III*, Comm. Algebra, **3** (1975), 269–310.

[10] M. Auslander, I. Reiten, *Representation theory of Artin algebras IV: Invariants given by almost split sequences*, Comm. Algebra, **5** (1977), 443–518.

[11] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1997.

[12] D. J. Benson, *Representations and Cohomology, I. Basic representation theory of finite groups and associative algebras*, second edition, Cambridge Studies in Advanced Mathematics, **30**, Cambridge University Press, Cambridge, 1998.

[13] D. J. Benson, *Resolutions over symmetric algebras with radical cube zero*, J. Algebra **320** (2008), no. 1, 48–56.

[14] R. Brauer, *Investigations on group characters*, Ann. of Math., **42**, (1941), 936-958.

[15] F. M. Bleher, T. Chinburg, *Locations of modules for Brauer tree algebras*, J. Pure Appl. Algebra **169** (2002), no. 2-3, 109–135.

[16] R. Bocian, A. Skowroński, *Symmetric special biserial algebras of Euclidean type*, Colloq. Math. **96** (2003), no. 1, 121–148.

[17] M. C. R. Butler, C. M. Ringel, *Auslander-Reiten Sequences with Few Middle Terms*, Comm. in Algebra **15** (1987), 145–179.

[18] W. W. Crawley-Boevey, *Tameness of biserial algebras*, Arch. Math. (Basel) **65** (1995), no. 5, 399–407.

[19] W. W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56** (1988), no. 3, 451–483.

[20] W. W. Crawley-Boevey, *Maps between representations of zero-relation algebras*, J. Algebra **126** (1989), no. 2, 259–63.

[21] Y. A. Drozd, *Tame and wild matrix problems*, In: Representations and Quadratic Forms, Akad. Nauk Ukr. S.S.R., Inst. Matem., Kiev 1979, 39–74.

[22] D. Duffield, *Auslander-Reiten components of Brauer graph algebras*, arXiv:1509.02478.

[23] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics **1428**, Springer-Verlag, New York/Berlin (1990).

[24] K. Erdmann, A. Skowroński, *On Auslander-Reiten components of blocks and self-injective biserial algebras*, Trans. Amer. Math. Soc., **330** (1992), no. 1, 165–189.

[25] E. A. Fernáandez, *Extensiones triviales y algebras inclinadas iteradas*, PhD thesis, Universidad Nacional del Sur, Argentina, 1999.

[26] E. A. Fernáandez, M. I. Platzeck, *Isomorphic trivial extensions of finite dimensional algebras*, J. Pure Appl. Algebra, **204** (2006), no. 1, 9–20.

[27] K. R. Fuller, *Biserial rings*, Ring theory, (Proc. Conf., Univ. Waterloo, Waterloo, 1978), pp. 64–90, Lecture Notes in Math., 734, Springer, Berlin, 1979.

[28] P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103.

[29] P. Gabriel, Ch. Riedtmann, *Group Representations without Groups*, Comment. Math. Helv. **54** (1979), no. 2, 240–287.

[30] C. Geiss, *On components of type* $\mathbb{Z}A_\infty^\infty$ *for string algebras*, Comm. Algebra **26** (1998), no. 3, 749–758.

[31] I. M. Gel'fand, V. A. Ponomarev, *Indecomposable representations of the Lorentz group*. Russian Mathematical Surveys 23, pp.1–58, 1968.

[32] E. L. Green, S. Schroll, N. Snashall, *Group Actions and Coverings of Brauer Graph Algebras*, Glasg. Math. J. **56** (2014), no. 2, 439–464.

[33] E. L. Green, S. Schroll, N. Snashall, R. Taillefer, *The Ext algebra of a Brauer graph algebra*, J. Noncommut. Geom. **11** (2017), no. 2, 537–579.

[34] E. L. Green, S. Schroll, *Almost gentle algebras and their trivial extensions*, arXiv:1603.03587.

[35] E. L. Green, S. Schroll, *Brauer configuration algebras: A generalization of Brauer graph algebras*, arXiv:1508.03617.

[36] E. L. Green, S. Schroll, *Multiserial and special multiserial algebras and their representations*, Adv. Math. **302** (2016), 1111–1136.

[37] J. A. Green, *Walking around the Brauer Tree*, Collection of articles dedicated to the memory of Hanna Neumann, VI. J. Austral. Math. Soc. **17** (1974), 197–213.

[38] Y. Han, *Controlled wild algebras*, Proc. London Math. Soc. (3) **83** (2001), no. 2, 279–298.

[39] D. Happel, U. Preiser, C. M. Ringel, *Vinbergs characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules*, Proc. ICRA II (Ottawa 1979), **832**, Lecture Notes in Math., 280294, Springer, 1980.

[40] T. Holm, A. Skowroński, *Derived equivalence classification of symmetric algebras of domestic type*, J. Math. Soc. Japan **58** (2006), no. 4, 1133–1149.

[41] H. Krause, *Maps between tree and band modules*, J. Algebra **137** (1991), no. 1, 186–194.

[42] H. Krause, *Stable equivalence preserves representation type*, Comment. Math. Helv. **72** (1997), no. 2, 266–284.

[43] R. Marsh, S. Schroll, *The geometry of Brauer graph algebras and cluster mutations*, J. Algebra **419** (2014), 141–166.

[44] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint, 1998.

[45] Z. Pogorzaly, A. Skowroński, *Self-injective biserial standard algebras*, J. Algebra, **138** (1991), no. 2, 491–504.

[46] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[47] C. M. Ringel, *The repetitive algebra of a gentle algebra*, Bol. Soc. Mat. Mexicana (3) **3** (1997), no. 2, 235–253.

[48] K. W. Roggenkamp, *Biserial algebras and graphs*, Algebras and Modules, II, 481-496, CMS Conf. Proc., **24**, Amer. Math. Soc., Providence, RI, 1998.

[49] J. Schröer, *On the quiver with relations of a repetitive algebra*, Arch. Math. (Basel) **72** (1999), no. 6, 426–432.

[50] S. Schroll, *Trivial Extensions of Gentle Algebras and Brauer Graph Algebras*, J. Algebra, **444** (2015), 183–200.

[51] D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Vol. 3, Representation-Infinite Tilted Algebras*, London Mathematical Society Student Texts, **72**, Cambridge University Press, Cambridge, 2007.

[52] A. Skowroski, *Selfinjective algebras: finite and tame type*, Trends in representation theory of algebras and related topics, 169–238, Contemp. Math. **406**, Amer. Math. Soc., Providence, RI, 2006.

[53] A. Skowroński, J. Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew Math., **345** (1983), 172–181.

[54] H. Tachikawa, *On algebras of which every indecomposable representation has an irreducible one as the top or the bottom Loewy constituent.*, Math. Z. **75** (1961), 215–227.

[55] H. Tachikawa, *Representations of trivial extensions of hereditary algebras.* Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 579–599, Lecture Notes in Math., **832**, Springer, Berlin, 1980.

[56] H. Von Höhne, J. Waschbüsch, *Die struktur n-reihiger algebren*, Comm. Algebra **12** (1984), no. 9-10, 1187–1206.

[57] B. Wald, J. Waschbüsch, *Tame Biserial Algebras*, J. Algebra **95** (1985), no. 2, 480–500.