Modern Mathematical Methods for Actuarial Sciences

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Abstract

In the ruin theory, premium income and outgoing claims play an important role. We introduce several ruin type mathematical models and apply various mathematical methods to find optimal premium price for the insurance companies. Quantum theory is one of the significant novel approaches to compute the finite time non-ruin probability. More exactly, we apply the discrete space Quantum mechanics formalism

$$\langle x|\exp(-tH)|x'\rangle = \sum_i \langle x|\exp(-tH)|i\rangle\langle i|x'\rangle$$

and continuous space Quantum mechanics formalism

$$\langle x|\exp(-\tau H)|x'\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\langle x|\exp(-\tau H)|p\rangle\langle p|x'\rangle$$

with the appropriately chosen Hamiltonians.

Several particular examples are treated via the traditional basis and quantum mechanics formalism with the different eigenvector basis. The numerical results are also obtained using the path calculation method and compared with the stochastic modeling results.

In addition, we also construct various models with interest rate. For these models, optimal premium prices are stochastically calculated for independent and dependent claims with different dependence levels by using the Frank copula method.

Acknowledgements

I would like to express my immeasurable appreciation to my supervisor, Prof. Sergey Utev, for his perfect guidance, continuous support, immense knowledge, patience, motivation, enthusiasm and providing me excellent atmosphere during my research.

I sincerely thank the Ministry of National Education in Republic of Turkey for providing me a funding to carry out my PhD study in the UK.

I also reserve my deepest thanks to my dearest wife, Merve, who was always with me through my PhD journey. She had continuously encouraged me with her sincere optimism and enthusiasm for my work.

To my beloved wife Merve Tunçarslan Kaya..

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Chapter 1

Introduction

1.1 Classical Ruin Theory

Financial companies' main aim of risk-hedging is to gain high profits. Thus, they must avoid negative displays of their financial situation because this is unfavorable for them. An important model, called **ruin theory**, operates in the actuarial world to determine the companies' situation against ruin.

Ruin theory has always been the most significant theory in actuarial mathematics [59] and this necessary tool was developed by Lundberg [4] [51]. The theory gives a useful information about how much premium must be charged continuously to customers by insurance companies to protect their financial situation from bankruptcy.

There are two main assumptions in the classical ruin theory:

1. Independent claim occurrences
2. No interest rate

Define a classical surplus process by [3] [13] [33] [34] [45] [51]

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i$$

where u is initial capital, c is premium amount, X_i are iid claims and independent of a Poisson process N_t (claim occurrences process). The goal is to compute the **infinite time ruin probability** $P_u(T < \infty) = \psi(u)$ where $T = inf\{t > 0 : R_t \leq 0\}$ [45] [51]. In our definition it is more useful to use $\leq$ instead of $<$. The main objective is to find optimal premium that is under which the ruin probability smaller than or equal a fixed small barrier, chosen as 5%.

1.2 General Ruin Model

To make ruin model realistic, we construct a variety of models both with interest rate and dependent claim occurrences [3] [52]. We also consider finite time non-ruin probability.

Firstly, we concentrate on Markovian structures and treat several examples of ruin type models. Models considered are both with and without interest rate. Also they may incorporate claim occurrences dependence.

Secondly, we apply the discrete space Quantum mechanics formalism [6] [16] [62]

$$\langle x| \exp(-tH)|x'\rangle = \sum_i \langle x| \exp(-tH)|i\rangle\langle i|x'\rangle$$

and continuous space Quantum mechanics formalism [6] [16] [62]

$$\langle x| \exp(-\tau H)|x'\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x| \exp(-\tau H)|p\rangle\langle p|x'\rangle$$

with the appropriately chosen Hamiltonians, to compute the finite time non-ruin probability and thus to find the optimal premium.

Thirdly, by using the Quantum mechanics approach and via stochastic method we define and compute the non-ruin operator

$$A_t f(x) = E[f(\xi_t)I(T > t)|\xi_0 = x]$$

where ξ_t is a generalize surplus process, I is an indicator function and T is the ruin time.

Fourthly, we apply stochastic modeling and the Frank copula [20] [29] [55] [74]

$$C_\eta(u) = -\frac{1}{log\eta}\left(1 + \frac{\prod_{i=1}^{n}(\exp(\eta_i^u - 1))}{(\eta - 1)^{n-1}}\right)$$

to find the optimal premium price where u is the uniform variable $U(0, 1)$ and η is the dependence parameter.

1.3 Results

- Results have been presented and well received with the oral presentation entitled "Quantum mechanism approach for ruin probability" at ICCMSA 2016: 18th International conference on computational modeling, simulation and analysis in Amsterdam. I have been awarded with the best presentation certificate.
- An abstract entitled "Hamiltonian approach and path integral method in ruin theory" has been accepted for oral presentation at EAJ 2016 conference and IA summer school in Lyon.
- The joint paper with S. Utev is under preparation. The paper includes Hamiltonian technique for three-state system with change basis and tensor

product of three-state system with change basis. Also, we use path calculation and regression for Quantum Binomial market. We treat ruin probability as a Binomial market with Maxwell-Boltzmann statistics.

1.3.1 Theoretical Results

- Finite time non-ruin operator for Gambler's ruin is calculated in the 4.1.1 in page 50.
- Finite time non-ruin operator for claim $X_j = 1$ and infinite claim is computed in the *Lemma* 4.1.2 and for claims $X_j = 0$, 1 and infinite claim is calculated in the *Lemma* 4.1.3.
- Hamiltonian technique is used to compute semigroup for two-state system in *Lemma* 6.2.1 with change and without change basis.
- Hamiltonian technique is applied to compute semigroup for three-state system in *Lemma* 6.3.1.
- Hamiltonian technique is used to compute tensor product of two-state system in *Lemma* 6.4.4 with change basis.
- Several models with interest rate are constructed.

1.3.2 Numerical Results

- Path calculation for non-ruin probability is implemented for several examples.
- Optimal premium price for various models of interest rate are found by using the Copula claim occurrences via stochastic modeling.

1.4 Structure of the Dissertation

- Chapter 2 contains terminology which comes from various fields such as Quantum physics in section 2.1, Hamiltonian system in section 2.2, path integral for Quantum theory in section 2.3, tensor product in section 2.4, stochastic process in section 2.5, ruin theory in section 2.6, Markovian structure and modeling construction in sections 2.7 and 2.8, reflection in section 2.9 and copula in section 2.10.

- Chapter 3 contains methods and techniques for transform of transition operator in discrete and continuous time in sections 3.1 and 3.2, for Hamiltonians and probability calculations in section 3.3. Also, path calculation and preservation under linear transform is introduced in sections 3.4 and 3.5.

- Chapter 4 contains several examples of non-ruin operator in discrete time.

- Chapter 5 contains several examples of non-ruin operator in continuous time.

- Chapter 6 contains advanced examples via Hamiltonian technique such as two-state system examples in section 6.1, Hamiltonian method with eigenvector basis for two-state and three-state system examples in sections 6.2 and 6.3, example of tensor product of 2×2 Hamiltonian in section 6.4, examples of surplus process different claim sizes in sections 6.5, 6.6 and 6.7, example of Brownian motion in section 6.8.

- Chapter 7 contains various models of interest rate with using Copula method.

- Chapter 8 contains ruin probability via comparison in section 8.1, operator method to compute finite and infinite time non-ruin probability for exponential claims with infinite jumps in section 8.2 and Riemann-Liouville integral operator in section 8.3.

- Chapter 9 contains conclusion and future research.

- I will be a lecturer after my PhD. So, a couple of obvious examples are treated in this thesis to show my ability to teach.

1.5 Overview

The classical collective risk theory has introduced by Lundberg in 1903. The surplus process with ruin probability has been studied in several literature such as Buhlmann [13], Gerber [32], Bowers, et al. [12], Rolski, et al. [59] and Asmussen [4].

The usual assumption in risk theory is that the claim sizes are independent, when claims occur [74]. The dependence structure known as Sklar's theorem and copula has firstly appeared in Sklar [65] [74]. The dependence between claim occurrences and effect of dependence in the individual risk process has been initiated by Denuit, et al [19] [74]. They also introduced two type of dependence: global dependence and local dependence.

Various types of dependence in the individual risk theory has been studied by Marceau, et al. [48] [74]. Also, Cossette and Marceau have studied on ruin probability in discrete time with dependence on the claim number process [74]. The joint distribution of the claims as a copula and its Archimedean form is introduced by this paper and later work Genest, et al. [31] [74]. Marshall and Olkin [49] has constructed the algorithm of Frank copula from a frailty framework [74]. In this thesis, we use this algorithm to find optimal premium.

Furthermore, effect of dependence of consecutive claims on the probability of ruin according to a Markov-type risk process has been simulated by Albrecher and Kantor [1] [74].

Chapter 2

Terminology

One can see that ruin probability analysis requires broad knowledge of mathematics. In this chapter, terminology of various fields are implemented.

2.1 Quantum Physics

Quantum mechanics is one of the most significant theory in science and gives more accurate experimentally results as explained by Baaquie [6]. Calculus and linear algebra are commonly used in mathematical formalism of Quantum mechanics [6]. To define a necessary notation in physics, variety of mathematical tools are required. For the quantum mechanics, characteristics such as **inner product** and **vector space** are necessary and **Hilbert space** play a special role [6] [16] [42] [62]

A Hilbert space, denoted by H, is a complete, separable, vector space equipped with the norm generated by the inner product $(x, y) \ : \ H \times H \to \mathbb{C}$ [16]

$$\|u\| := \sqrt{(u,u)}\ , \quad u \in H$$

An inner product is a map $(x, y) : H \times H \to \mathbb{C}$ satisfying [6] [16] [62]

1. $(\alpha, \beta) = \overline{(\beta, \alpha)} \quad for\ \alpha, \beta \in H,$

2. $(\alpha + \beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma) \quad for\ \alpha, \beta\ and\ \gamma \in H,$

3. $(c\alpha, \beta) = c(\alpha, \beta),\ for\ \alpha, \beta \in H\ and\ c \in \mathbb{C}$

4. $(\alpha, \alpha) \geq 0,\ where\ equality\ holds\ if\ and\ only\ if\ \alpha = 0.$

A normed vector space is **complete** if every Cauchy sequence converges. In other words, if $(\alpha_n)_{n\in\mathbb{N}}$ is such that [16] [62]

$$\|\alpha_n - \alpha_m\| \to 0\ as\ m, n \to \infty$$

there exist an $\alpha \in H$ such that $\alpha_n \to \alpha\ ,\ n \to \infty$.
A space H is **separable** if there exists a countable dense subset $\{\alpha_1, \alpha_2, \ldots\} \subset H$. In other words for any $\beta \in H$ there exists a subsequence $(\alpha_{i_j})_{j\in\mathbb{N}}$ such that [16] [62]

$$\lim_{j\to\infty} \left\|\alpha_{i_j} - \beta\right\| = 0$$

A vector $|\alpha\rangle$ in Hilbert space $\mathcal{H}$ is called **ket vector**. Similarly, a vector $\langle\beta|$ in the adjoint conjugate dual space $\mathcal{H}^*$ is called **bra vector** [16] [27] [62]. Dirac has introduced these two vectors to deal with the difficulty resulting from infinite dimensional systems. To construct the inner product in Dirac notation, we put adjoint conjugate of $\langle\alpha|$ as bra vector in front of the $|\beta\rangle$, written as [16] [62]

$$\langle\alpha||\beta\rangle \equiv \langle\alpha|\beta\rangle$$

In addition, the sum of two ket (or bra) vectors is another ket (or bra) vector [62]

$$|\alpha\rangle + |\beta\rangle = |\delta\rangle$$

The multiplication of a ket (or bra) vector and a constant number $c \neq 0$ is given by [62]

$$c|\alpha\rangle = |\alpha\rangle c$$

There is no difference where the constant c stands on the right or left of the ket vector. Particularly, the result is **null ket (bra)** where $c = 0$.

Vectors $|\alpha\rangle$ and $|\beta\rangle$ are **orthogonal** if $\langle\alpha|\beta\rangle = 0$. It also implies that $\langle\beta|\alpha\rangle = 0$ [62].

A **normalized ket** $|\widetilde{\alpha}\rangle$ is denoted by [62]

$$|\widetilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}\right)|\alpha\rangle$$

where $|\alpha\rangle$ is not a null ket and $\langle\widetilde{\alpha}|\widetilde{\alpha}\rangle = 1$.

A **linear operator** A transforms a ket vector to another one, i.e [6] [16] [62]

$$A|\alpha\rangle = |\alpha'\rangle$$

In addition, operator A is a **null operator** if [62]

$$A|\alpha\rangle = 0$$

for any $|\alpha\rangle$.

Let A, B and C be three operators. Multiplication operation is said to be noncommutative but associative [62]

$$AB \neq BA$$
$$A(BC) = (AB)C = ABC$$

Then, we have [62]

$$A(B|\alpha\rangle) = (AB)|\alpha\rangle = AB|\alpha\rangle \quad , \quad (\langle\beta|X)Y = \langle\beta|(XY) = \langle\beta|XY$$

Furthermore, the inner product of vector $|\alpha\rangle$ and an acted operator $A|\beta\rangle = |\beta'\rangle$ is written as [6] [16] [62]

$$\langle\alpha|\beta'\rangle = \langle\alpha|(A|\beta\rangle) = \langle\alpha|A|\beta\rangle$$

where the vector $|\beta\rangle$ is transformed by an operator A. On the other hand, if the operator is acting on the ket vector, the inner product will be as follows [16] [62]

$$\langle\alpha'|\beta\rangle = (\langle\alpha|A^{+})|\beta\rangle = \langle\alpha|A^{+}|\beta\rangle$$

where the vector $|\alpha\rangle$ is converted by an operator A^{+} which is adjoint of the linear operator A.

The set of eigenvector $\mu = \{|a_1\rangle, |a_2\rangle, \cdots, |a_i\rangle, \cdots\}$ forms the orthonormal basis of Hilbert space $\mathcal{H}$. Any vector $|\alpha\rangle$ is a linear combination of its basis as follows [6] [16] [62]

$$|\alpha\rangle = \sum_j \alpha_j |a_j\rangle$$

where the coefficient α_j is obtained by the inner product is written as follows [16]

[62]

$$\langle a_i|\alpha\rangle = \langle a_i|\sum_j \alpha_j|a_j\rangle = \sum_j \alpha_j \delta_{ji} = \alpha_i$$

and corresponding decomposition of identity in discrete basis is [7]

$$I = \sum_j |a_j\rangle\langle a_j|$$

Decomposition of identity operator in continuous basis is denoted by [7]

$$I = \int_{-\infty}^{\infty} |\alpha\rangle\langle\alpha| \; d\alpha$$

Additionally, the projection operator onto the subspace $|\alpha\rangle$ is formed and called dyad projector, i.e [6] [16] [62]

$$P_\alpha = |\alpha\rangle\langle\alpha|$$
$$P_\mu = \sum_i |\alpha_i\rangle\langle\alpha_i|$$

where $\mu = \{|a_1\rangle, |a_2\rangle, \cdots, |a_i\rangle, \cdots\}$ is the set of eigenvectors.

2.2 Hamiltonian System

A general dynamical theory was established by Newton as laws of motion. Then significant developments were made by Lagrange and Hamilton. In the Hamiltonian method [7] [26] [27], a pair, a dynamical coordinate and time is transformed to a new dynamical state and time. A system of ordinary differential equations, is called Hamiltonian system, which forms [7] [26] [50]

$$\dot{q} = H_p \quad , \quad \dot{p} = -H_q$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}(t,q,p) \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(t,q,p)$$

for $i = 1, \ldots, n$, where Hamiltonian $H = H(t,q,p)$ depends on the position vectors $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ and time t. These vector p and q are conjugate vectors such as p is conjugate to q.

2.3 Path Integrals in Quantum Physics

Consider a moving from a position q at time $t = 0$ to another position q' at a later time $t = T$ with some probability by using the form of Hamiltonian such as [7] [16] [62]

$$H = \frac{p^2}{2m} + V(q)$$

where p is the momentum and V(q) is the potential.
Let the initial state $|\psi(0)\rangle = |q\rangle$. We get the amplitude [7] [16] [62]

$$A = K(q,0;q',t) = \langle q| \exp(-Ht) |q'\rangle$$

and also it is known as propagator which is independent of the origin of time [5] [7] [16] [62]

$$K(q,t;q',T+t) = K(q,0;q',T)$$

The amplitude can be derived in the form of integral summation over all possible paths from beginning to end point so path integral is derived. In the first studies

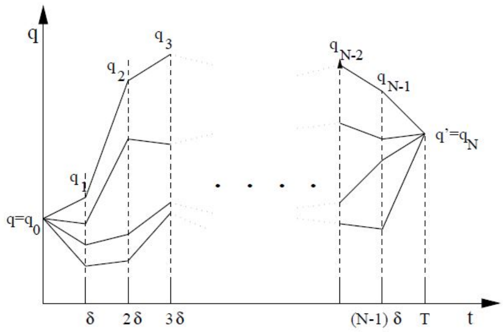


Figure 2.1: Amplitude as a sum over all N-legged paths [47].

in path integral, Feynman has explained equivalence of path integral on the formulation of quantum theory. Then, if we split up the time evolution as two small parts, the amplitude is derived by [5] [7] [16] [57] [62]

$$A = \langle q| \exp(-Ht_1) \exp(-H(T - t_1))|q'\rangle$$

Inserting factor $\int dq_1 |q_1\rangle\langle q_1| = 1$ to the amplitude, it can be reached that

$$\begin{aligned} A =& \langle q| \exp(-Ht_1) \int dq_1 |q_1\rangle\langle q_1| \exp(-H(T - t_1))|q'\rangle \\ =& \int dq_1 K(q, 0; q_1, t_1)K(q_1, t_1; q', T) \end{aligned}$$

Now, if we divide the time evolution with a large number N, the time interval between the each paths are $\delta = \frac{T}{N}$. Afterwards amplitude becomes [7] [16] [46] [62]

$$K(q', T|q, t) := \lim_{N\mapsto\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(- \frac{1}{2\sigma^2 N\delta} \sum_{i=0}^{N-1} (q' - q)^2 \right) \frac{dq_1}{\sqrt{2\pi\sigma^2\delta}} \cdots \frac{dq_{N-1}}{\sqrt{2\pi\sigma^2\delta}}$$

where $V(q) = 0$.

2.4 Tensor Product

Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces with different degrees of freedom. Tensor product space is denoted by $\mathcal{V} \otimes \mathcal{W}$ [6] [26]. For ket vector $|x\rangle \in \mathcal{V}$ and $|y\rangle \in \mathcal{W}$, tensor product is defined by [6] [26]

$$|x\rangle \otimes |y\rangle = |x\rangle|y\rangle$$

If ket vector $|x\rangle \in \mathcal{V}_N$ is N-dimensional and and $|y\rangle \in \mathcal{W}_M$ is M-dimensional, tensor product space $\mathcal{V}_N \otimes \mathcal{W}_M$ is MN-dimensional vector.[6] [26]
General information and several examples are given in the Chapter 6.

2.5 Stochastic Processes

Let $T \in [0, \infty)$ be an index set and t be a parameter running over this set. A family of random variables X_t or $X(t)$ is called stochastic process [61] [70]. If index t corresponds to discrete unit time, and the index set $T = N$, it is called a discrete-time process, and if index t corresponds to continuous unit time, and index set $T = [0, \infty)$, it is said to be a continuous-time process which is particularly important in applications [61] [70].
Stochastic processes are split up by their state space, or the range of possible values for the random variables X_t, by their index set T, and by the dependence relations among the random variables X_t.

2.5.1 Compound Poisson Processes

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) positive random variables. Then [51] [61]

$$N(t) = \#\{i \geq 1 : T_i \leq t\} \quad , \quad t \geq 0$$

is said to be corresponding renewal (counting) process where T_i are inter-arrival times and also the random walk is called a renewal sequence [32] [51] [61]

$$S_t = X_1 + X_2 + \cdots + X_{N(t)} \quad , \quad N(t) \geq 1$$

with $S_t = 0$ if $N(t) = 0$.
If $N(t)$ is a Poisson process with parameter $\lambda > 0$, S_t is called a compound Poisson process with this specific parameter λ [32] [45] [51] [61] [70].

2.5.2 Brownian Motion Stochastic Processes

Brownian motion process is a continuous time continuous state space stochastic process [36]. Modern mathematical method and mathematical aspect of Brownian motion is constructed by Norbert Wiener [36]. A **standard Wiener Process** is a random stochastic process $W = W_t : t \in [0, \infty)$ with state space R satisfies the following properties: [36] [70]

- $W_0 = 0$ with probability 1.
- For $s < t \in [0, \infty)$, the distribution of $W_t - W_s$ is equal to distribution of W_{t-s} that is W has stationary increments.
- W_t is normally distributed with mean zero and variance $t - s$ for every

increment $W_t - W_s$.

$$W_t - W_s \sim N(0, t-s)$$

- For every pair of disjoint intervals $(t_1, t_2], (t_3, t_4]$ with $0 \leq t_1 < t_2 \geq t_3 < t_4$, the increments $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent random variables. Similarly, the increments are independent for any n disjoint time intervals, $0 \leq t_0 < t_1 < \cdots < t_n$, where n is an arbitrary positive integer.
- W_t is continuous as a function of t (continuous sample path).

The definition says that if we know $W_0 = 0$ and $W_s = x_0$, knowledge of values of W_τ for past times $\tau < s$ don't have any effect on the future movement $W_t - W_s$. The statement of the Markov property of the Wiener Process is defined by [70]

$$P[W_t \geq x | W(t_0) = x_0, W(t_1) = x_1, \ldots, W(t_n) = x_n] = P[W_t \geq x | W(t_n) = x_n]$$

if $0 \leq t_0 < t_1 < \ldots < t_n < t$.

2.5.3 Levy Processes

Levy processes are stochastic processes with stationary and independent increments [36]. The Levy processes include Poisson processes and Brownian motion [36]. They used for modeling in various area such as physics, biology, option pricing and finance [36].
A process $L = \{L_t : 0 \leq t \leq T\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a Levy process if it has the following properties: [36] [43] [58]

- **Independent increments:** $L_t - L_s$ is independent of F_s for any $0 \leq s \leq t \leq T$.

- **Stationary increments:** The distribution of $L_{t+s} - L_t$ does not depend on t for any $0 \leq s \leq t \leq T$.
- **Stochastically continuous:** $\lim_{s \mapsto t} P(|L_t - L_s| > \varepsilon) = 0$ for every $0 \leq s, t \leq T$ and $\varepsilon > 0$.

2.6 Ruin Theory

The insurance companies like to take a certain amount of risk in a branch of insurance [59]. If the claim surplus exceeds the fixed beginning value, the company will have to take a rigid response (attitude). Thus, the company begins to make an investment with specified amount as a level $R_0 = u$ which is called *initial capital.* The vital criterion to optimize the insurance policy is minimizing the probability that the claim surplus exceeds the initial capital u [59]. At this point, actuaries have to determine the amount of premium C in a certain period and which type of insurance will be taken by policyholders [59]. This determination is reached depending on company policy and tariffs of rivals. In addition to this, Ct is total premium amount in the interval $(0, t)$ [32]. S_t is aggregate claims in this time interval and policyholders receive claims according to the Poisson Process with rate λ [32].
Total claim amount process [13] [15] [22] [45] [51]

$$S_t = \sum_{i=1}^{N(t)} X_i, t \geq 0$$

in the renewal model and the claim sizes sequence (X_i) is independent and identically distributed (iid) with distribution function F [51]. Claim arrival sequence (T_n) is defined as follows: [15] [45] [51]

$$T_0 = 0 \ , \ T_n = W_1 + \cdots + W_n \ , \ n \geq 1$$

where W_i are iid interarrival times. In that case, T_n is referred to as a renewal sequence. The arrival process is Poisson process if and only if W_1 has a negative exponential distribution.

One of the most important cases at this point is that renewal process and claim arrival sequence (T_n) are individually independent of (X_i).

In a discrete time, the surplus process is defined by [3] [13] [15] [33] [34] [45] [51]

$$\begin{aligned} R_t =& R_0 + Ct - \sum_{i=1}^{t} X_i \\ =& R_0 + \sum_{i=1}^{t} (X_i - C) \end{aligned}$$

Assume a continuous premium income $p(t) = Ct$ in the homogenous portfolio where p is a deterministic function and linear [51]. The risk process (or surplus) of the portfolio is described by [4] [51]

$$R_t = R_0 + p(t) - S_t, t \geq 0$$

At the time t, if R_t is positive, the insurance companies make profit and they gain capital; if it is not (negative) companies have lost their capital [51].

Define a random variable $\tau = inf\{t \geq 0 : R_t < 0\}$. Ruin time of the portfolio is given instant time τ and is the main characteristic of ruin event [32] [34] [45] [51] [59]. The target of ruin theory is to define the probability of ruin where τ refers to specific time period (It can be unbounded for ultimate ruin). Additionally, there is possibility that ruin time τ is dependent on all stochastic elements in the risk process R_t.

Idealized path of the process R_t is seen in the Figure 2.2 [51]. The ruin process R_t begins with initial capital $R_0 = u$ at the beginning of the contract and then when the first claim occurs, an increase is observed in the path with premium C until a first event occurs at $T_1 = W_1$. Besides, the process moves downward by the

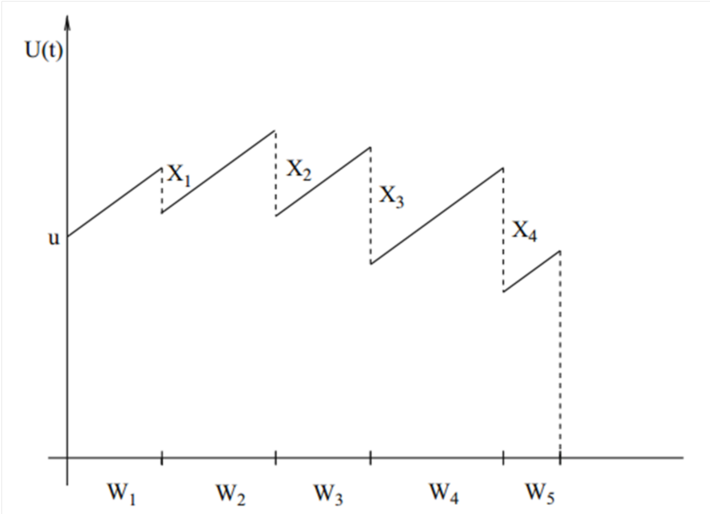


Figure 2.2: Idealized path of the process of capital R_t.

claim size of X_1. In the next step, the process is again increasing with premium C in the interval $[T_1, T_2]$ and also it decreases by the amount of X_2. Then, the process keeps its behaviour as the above structure [51]. As a result, if the claim sizes X_i are large, capital R_t may take negative values. When R_t values fall under the limit zero, this situation is called *ruin* [51].

2.6.1 Infinite Time Ruin Probability

A claim surplus process $\{K_t\}_{t\geq 0}$ is defined by

$$K_t = u - R_t = \sum_{t=1}^{N_t} X_t - ct$$

where the time to ruin is $\tau(u) = inf\{t \geq 0 \ : \ R_t \leq 0\} = inf\{t \geq 0 \ : \ K_t > u\}$, u is the initial capital, c is the premium amount and R_t is the capital [15].

Let $L = sup_{0<t<\infty}\{K_t\}$ and $L_T = sup_{0<t<T}\{K_t\}$. The ruin probability in infinite time is written as [15] [24] [25]

$$\psi(u) = P(\tau(u) < \infty) = P(L > u)$$

Also, the ruin probability in finite time T is defined by [15] [24]

$$\psi(u,T) = P(\tau(u) < T) = P(L_T > u)$$

Hence, it can be said that $\psi(u,T) < \psi(u)$ and infinite time ruin probability may be relevant to finite time ruin probability. [15]

2.7 Markovian Structure

2.7.1 Introduction to Markov Chains

Simple structure of Markov Chains gives much knowledge about their behaviour. They are the most significant type of random processes since their richness and practicability are sufficient to serve a diverse range of applications [37] [56] [61] [67].

Memory-less property refers to a Markov Process for stochastic processes. Numerous phenomena can be modeled and identified with a most effective probability law that the future state of this stochastic process depends only on its current state, without its previous state. In other words, no subsequent state is affected by probability distribution of earlier states. Its past states only contribute to reach its current state and play no role in the probability of its future states. This characteristic is called the *Markov Property* [36] [56] [61] [70].

The Markov Processes are classified thus: characteristic of state space being measured and observation of discrete and continuous intervals [40].

General definition: Mathematically, the Markov Property is written as [35] [61] [70]

$$P[X_t \in \mathcal{A} | X_{S_1} = x_1, X_{S_2} = x_2, \cdots, X_{S_n} = x_n, X_S = x] = P[X_t \in \mathcal{A} | X_S = x]$$

for all $s_1 < s_2 < \cdots < s_n < s < t$ in the time set; all states $x_1, x_2, \cdots, x_n, x$ in the state space S and all subsets $\mathcal{A}$ of S.

2.7.2 Disrete Time Markov Chains

Consider a sequence of random variable $(X_n)_{n\geq 0}$ that takes on a finite number of possible values on a countable state space S, the process is in state i at time n if $X_n = i$ [60] [61]. Then, the state is changed from state i to the next state j with a fixed probability P_{ij}. This stochastic process is called a *Markov Chain*, if it has a Markov Property [14] [36] [60] [61] [66] [70]

$$P[X_{n+1} = j | X_n = i; X_{n-1} = i_{n-1}, \cdots, X_1 = i, X_0 = i_0] = P(X_{n+1} = j | X_n = i) = P_{ij}$$

for all states $i_0, i_1, \cdots, i_{n-1}, i, j$ and all $n \geq 0$.

2.7.3 Chapman-Kolmogorov Equation for DTMCs

Based on Markov Property, one-step transition probabilities P_{ij} have already been defined and n-step transition probabilities $P_{ij}^{(n)}$ can easily be proved. After n additional transitions, process will pass to state j from state i. That is, [14] [61] [66]

$$P_{ij}^{n} = P[X_{n+m} = j | X_m = i] n \geq 0, i, j \geq 0$$

A method is satisfied by the *Chapman-Kolmogorov Equation* with regards to n-step transition probabilities for $P'_{ij} = P_{ij}$, i.e, [61]

$$P_{ij}{}^{n+m} = \sum_{k=0}^{\infty} P_{ik}{}^{n} P_{kj}{}^{m}$$

for all $n, m \geq 0$ and all i, j. If $P^{(n)}$ is defined as a matrix for n-step transition probabilities, the following equation is derived (e.g. [14] [61] [66])

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

$$P^{(n)} = \quad P^{(n-1)} \cdot P = P^{(n-2)} \underbrace{\cdot P \cdot P}_{P^2} = P^{(1)} \underbrace{\cdot P \cdot P \cdots P}_{P^{n-1}} = \cdots = P^n$$

Hence, n times matrix multiplication of matrix P by itself gives n-step transition matrix [14] [61].

2.7.4 Continuous Time Markov Chains

Continuous Time Markov Chains (CTMCs), which are applied in different applications and fields in the real world, behave with the same characteristics as Discrete Time Markov Chains (DTMCs) in terms of the Markov Property that future state is affected by current state and independent of past states [41] [61].
In other words, CTMCs are analogous to DTMCs, but there is such an important difference between them that the times between transitions of states are exponentially distributed, not deterministic [71] [72].
CTMCs, which are stochastic processes, draw a model that enters a state i at time $t \geq 0$ and remains in this state for a while; that is, it keeps its situation for a random period of time, after which it jumps into the next state j $(j \neq i)$ with a known rule. [14]

Formally, a continuous time stochastic process $[x_{(t)}, t \geq 0]$ with a discrete state space S is CTMCs if [14] [36] [61] [69] [71] [72]

$$P[X(t+s) = j | X(s) = i, X(u) = x(u), 0 \leq u < s] = P_{ij}(t)$$

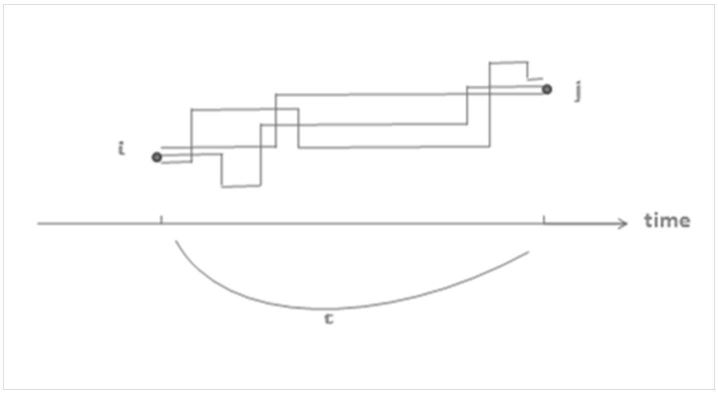


Figure 2.3: The total probability of all paths leading from i to j in time t.

for all $s, t \geq 0$ and all states i and j. $P_{ij}(t)$ is the probability of being in state j from current state i in t times unit. Figure 2.3 shows that total probability of all paths leading from i to j in time t in terms of trajectory [69].

The conditional distribution of future depends only on the present due to the fact that CTMCs have the Markov Property [61].

The conditional probabilities of process $P[X(t+s) = j|X(s) = i]$ are called the transition probabilities. The CTMCs have stationary transition probabilities (sometimes called homogenous transition probabilities), when [14]

$$P[X(t+s) = j|X(s) = i] = P[X(t) = j|X(0) = i] = P_{ij}(t)$$

for all $s, t \geq 0$ and all states i and j.

2.7.5 Chapman-Kolmogorov Equation for CTMCs

Chapman-Kolmogorov Equation (C-K Equation) describes change of transition probabilities in a similar way in discrete time, although the form is different in continuous time [14] .

For all $s \geq 0$ and $t \geq 0$,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s)$$

is called *Chapman-Kolmogorov Equation for CTMCs* [14] [66].

A square matrix $P(t)$ is called a transition function and is written for transition probabilities $(P_{ij}(t))$ with using matrix notation. The following matrix multiplication is obtained by C-K Equation [14] [66]:

$$P(t+s) = P(t) \cdot P(s)$$

2.8 Stochastic Modeling Construction

2.8.1 Modeling of Discrete Time Markov Chains

Let $X_k, k = 0, 1, 2, \ldots$ be a discrete time Markov chain with transition matrix $P = P_{ij}$ for any $i, j \in Z$. To model it, we create a path between states. Our subprogram is defined by

$$\Delta_{xy} = \sum_{j=1}^{y} P_{xj}$$

- Fix $X_0 = x$
- $\Delta_{xy} = P_{i1} + \ldots + P_{ij}$
- Generate $U \sim U[0,1]$

If $\Delta_{xj-1} < U \leq \Delta_{xj}$ take $X_{n+1} = y$. Then loop it.

Example: Let transition matrix $P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} \end{array}.$

$$\text{For } X_0 = 1 \qquad \longmapsto \qquad \Delta_{xy} = \Delta_{1y} = \sum_{j=1}^{y} P_{1j}$$
$$\longmapsto \qquad \Delta_{10} = 0\,,\ \Delta_{11} = P_{11} = 0.3\,,\ \Delta_{12} = P_{11} + P_{12} = 1$$

If generated random number U_1 is in the interval of $[0, \Delta_{11}]$, we jump the another state. If not, it stays at the same state. We assume the generated random number $U_1 = 0.53$. So, it jumps to the other state $X_1 = 2$.
Similarly, for $X_1 = 2$, we assume generated random number is $U_2 = 0.37$ which is in the interval $(0, \Delta_{21} = 0.4)$. So, it jumps to the other state as well.

2.8.2 Modeling of Continuous Time Markov Chains

Let $X_t, t < T$ be a continuous time Markov chain with

$$X_t = i \mapsto \begin{cases} i+1 & , \quad \lambda_{ij}\Delta_t + o(\Delta_t) \\ i & , \quad 1 - \lambda_{ii}\Delta_t + o(\Delta_t) \end{cases}$$

where $\lambda_{ii} = \sum_{j \neq i} \lambda_{ij}$. From state i to j, probability is defined by

$$P_{ij}(\Delta_t) = P(X_{t+\Delta_t} = j \,|\, X_t = i) = \lambda_{ij}\Delta_t + o(\Delta_t)$$

- Fix initial state $x = X_0 = i$
- Take random exponential number $NE(\lambda_{ii}) = T$
- Move time $0 \Rightarrow 0 + T$
- Jump from i to $j \neq i$ with probability $\frac{\lambda_{ij}}{\lambda_{ii}} = P_{ij}$
- To do it, generate a random number $U \sim U[0, 1]$

If $\Delta_{xj-1} < U \leq \Delta_{xj}$ take $X_{n+1} = y$. Then loop it.

2.9 Combinatorial Approach and Reflection

Consider there is a path from (t_0, k_0) to (t_n, k_n) for $t_n > t_0$ where k_0 is the position at time t_0 and k_n is the position at time t_n. Then, we say that (t_n, k_n) is reachable from (t_0, k_0) [11]. A possible path from (t_0, k_0) to (t_n, k_n) is the set $(X_{t_0} \cdots , X_{t_n})$ such that [11] [73]

$$X_{t_0} = k_0 \quad , \quad X_{t_n} = k_n$$

where $X_{t+1} - X_t = 1$ or -1 in $\mathbb{Z}$.

Reachability

Let m be the total number of moves up and n be the total number of moves down. In order to reach (t_n, k_n) from origin, we say that [11] [73]

$$m + n = t_n \quad \text{and} \quad m - n = k_n$$

and so

$$m = \frac{t_n + k_n}{2} \quad \text{and} \quad n = \frac{t_n - k_n}{2}$$

Here $t_n + k_n$ and $t_n - k_n$ must be even and they must have the same parity because m and n must be integer numbers. [11] [73]

Definition: The number of possible path to reach the reachable point (t, k) is denoted by [11] [73]

$$N_{t,k}$$

If (t, k) is not reachable, then $N_{t,k} = 0$.

Also, the number of possible path from $(0, k_0)$ to (t_n, k_n) is defined by [11] [73]

$$N_{t_n}(k_0, k_n)$$

Proposition: Number of path to reach (t,k)

If (t_n, k_n) is a reachable point, then

$$\begin{aligned} N_{t_n}(k_0, k_n) &= \binom{t_n}{m} = \binom{t_n}{n} \\ &= \binom{t_n}{(t_n + k_n - k_0)/2} = \binom{t_n}{(t_n - k_n + k_0)/2} \end{aligned}$$

where $(t_n + k_n - k_0)$ is even number [11] [73].

The Reflection Principal

Let $N_{t_n}(k_0, k_n)$ be the number of all possible path from (t_0, k_0) to (t_n, k_n). If the path touch x-axis in any point $0 < t < t_n$, then [11] [73]

$$N_{t_n}(k_0, k_n) = N_{t_n}(-k_0, k_n)$$

The Ballot Theorem

Let $N_{t_n}(0, k_n)$ be the number of all possible paths from $(0, 0)$ to (t_n, k_n). The number of paths from $(1, 1)$ to (t_n, k_n) which do not cross the x-axis is identified by [11]

$$N_{t_n-1}(0, k_n - 1) - N_{t_n-1}(-1, k_n)$$

Let m be the integer number of plus ones and n be the integer number of minus ones. As we defined before $m + n = t_n$, $m - n = k_n$ and $t_n + k_n = 2m$. By doing the algebraic calculations [11]

$$\begin{aligned} N_{t_n-1}(0,k_n-1) - N_{t_n-1}(-1,k_n) =& \binom{t_n-1}{(t_n+k_n-2)/2} - \binom{t_n-1}{(t_n+k_n)/2} \\ =& \binom{m+n-1}{m-1} - \binom{m+n-1}{m} \\ =& \frac{k_n}{t_n} N_{t_n}(0,k_n) \end{aligned}$$

Combinatorial Approach and Probability

Assume that

$$k \mapsto \begin{cases} k+1 & \text{with probability } p \\ k-1 & \text{with probability } q \end{cases}$$

for each integer k. Each path in set $(X_0, \ldots, X_n)$ has an equal probability

$$p^{(t_n+k_n-k_0)/2} q^{(t_n-k_n+k_0)/2}$$

where $X_0 = k_0$ and $X_n = k_n$ [11] [73].

For symmetric random walk (same probability $p = q = 1/2$), each path of length t_n has probability 2^{-t_n} [11] [73].

2.10 Copulas

Model dependence beyond multivariate normality has become more important in recent years [20]. The importance of dependencies among risks is well define in actuarial theory [19]. Kaas illustrated disastrous effect of dependencies on stop-loss premiums [19] [38]. Then, Dhaene and Goovaerts have increased the attention for dependence among risks [19] [23]. Albrecher et al. [3] have used the Archimedean survival copulas to compute ruin probabilities [17]. Then Albrecher and Boxma

[2] considered inter-arrival time depend on the previous claim size [17]. Also, broad class of examples are also presented by Marceau et al. [48] and Genest at al. [31], separately [19]. Effect of dependence in the individual risk model has been analysed and separated two different types: global dependence and local dependence [19]. Another study shows that the probability of ruin is increased and adjustment coefficient is decreased under the dependence structure in claim number process [48].

Assume that an N-dimensional random vector $X = (X_1, X_2, \ldots, X_N)$ has the cumulative distribution function as below

$$F(x_1, x_2, \ldots, x_N) = Pr[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_N \leq x_N]$$

The copula method is based on the decomposition of the cumulative distribution function F via the univariate marginals of X_k for $k = 1, 2, \ldots, N$ and another distribution function called a copula as stated below in $(*)$ [20] [74].

Copula, which provides a connection amongst the several univariate marginal distributions to their multivariate distribution, is a function [20] [74]. It constructs a structure amongst the marginals for dependence. In other words, a copula is a multivariate distribution function with specified uniform marginals [20] [74].

Conversely, for any multivariate distribution, there exists a copula function to link it with its marginals. Furthermore, Sklar illustrates that the copula is called **unique** if that marginal distribution is continuous.

Suppose a portfolio weights for the N assets whose returns are univariate marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_N(x_N)$. The copula function C gives the joint density as follows [20] [55]

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1, x_2, \ldots, x_N) \qquad (*)$$

Also, pseudo inverse function is defined [20] [55]

$$x = F^{-1}(u) \equiv sup\{x|F(x) \leq u\}$$

Consider the distribution [20] [55]

$$F(x,y) = \exp(-(\exp(-x) + \exp(-y) - (\exp(-\delta x) + \exp(-\delta y))^{-\frac{1}{\delta}}))$$

for $\delta > 0, -\infty < x$ and $y < \infty$. Its univariate marginals are $F_1(x) = \exp(-\exp(-x))$ and $F_1(y) = \exp(-\exp(-y))$ where $x \mapsto \infty$ and $y \mapsto \infty$. By substituting $u = F_1(x)$ and $v = F_1(y)$, the copula is found

$$C(u,v) = uv\exp(((-logu)^{-\delta} + (-logv)^{-\delta})^{-\frac{1}{\delta}})$$

Joint distribution of the claim occurrence has stated as copula in the individual risk model as a form of Archimedean copula as follows [2] [3] [20] [30] [31] [55] [74]

$$C(F_{I_1}(i_1), \ldots, F_{I_n}(i_n)) = \psi^{-1}[\psi(F_{I_1}(i_1)) + \cdots + \psi(F_{I_n}(i_n))]$$

where $F_{I_1 \cdots I_n}$ is the multivariate distribution function and Archimedean generator ψ is decreasing, convex and $\psi(1) = 0$.

Also, the multivariate Frank copula is defined by [15] [20] [29] [55] [74]

$$C_\theta(u) = -\frac{1}{\theta}\left(1 + \frac{\prod_{i=1}^{n}(\exp(-\theta u_i - 1))}{(\exp(-\theta u_i) - 1)^{n-1}}\right)$$

where $\theta > 0$. For $\eta = \exp(-\theta)$, this notation is written as [20] [29] [55] [74]

$$C_\eta(u) = -\frac{1}{log\eta}\left(1 + \frac{\prod_{i=1}^{n}(\exp(\eta_i^u - 1))}{(\eta - 1)^{n-1}}\right)$$

Chapter 3

Methods and Techniques

In this chapter, methods and techniques for transform of transition operator in discrete and continuous time process, probability calculation and path calculation are implemented. Also, preservation under linear transform is introduced.

3.1 Transform of Transition Operator in Discrete Time

Let ξ_t be an abstract surplus process $\xi_t = \xi_0 + Ct - \sum_{i=1}^{N_t} X_i$ where C is a premium rate, X_i are claim sizes and T is a ruin time defined by $T = inf\{t > 0 \; s.t \; \xi_t \leq 0\}$. We emphasize that it is more convenient to us to include 0 to a ruin region. The non-ruin operator in discrete time for $t = \{0, 1, 2, \cdots\}$ is defined by

$$A_t f(x) = E[f(\xi_t)I(T > t)|\xi_0 = x]$$

where A_t is a semi-group. Then, finite time non-ruin probability is found by using this definition.

Lemma 3.1.1. *Fix a function $H : R \to R$. Define a family of operators $\{A_t; t \geq 0\}$ by*

$$A_t f(x) = E[H(\xi_1, \ldots, \xi_t) f(\xi_t) | \xi_0 = x]$$

where function $H(\xi_1, \ldots, \xi_t) = H(\xi_1) \cdots H(\xi_t)$.

Then, $\{A_t; t \geq 0\}$ is a semigroup and $A_t = A^t$ where $A = A_1$.

Corollary:

$$A_t f(x) = E[I(T > t) f(\xi_t) | \xi_0 = x] = A_1^t f(x)$$

where $A_1 f(x) = E[I(T > 1) f(\xi_1) | \xi_0 = x]$.

This corollary establishes an important semigroup property which allows to compute the ruin operator in a simpler way.

Proof: When we fix H, it admits

$$\begin{aligned} A_t f(\xi_0) =& E(H(\xi_1) \ldots H(\xi_t) f(\xi_t) | \xi_0)(\xi_0 = x) \\ =& E[E\{[H(\xi_1) \ldots H(\xi_{t-1})](H(\xi_t) f(\xi_t) | F_{t-1})\} | F_0] \end{aligned}$$

Using the tower lemma, we get

$$A_t f(\xi_0) = E(H(\xi_1) \cdots H(\xi_{t-1}) E(H(\xi_t) f(\xi_t) | F_{t-1}) | F_0)$$

Applying the Markov property, we find

$$A_t f(\xi_0) = E(H(\xi_1) \cdots H(\xi_{t-1}) E(H(\xi_t) f(\xi_t) | f(\xi_{t-1})) | \xi_0)$$

By definition of the operator

$$\begin{aligned} A_1 f(\xi_{t-1}) =& E(H(\xi_t) f(\xi_t) | \xi_{t-1}) \\ =& g(\xi_{t-1}) \end{aligned}$$

and so

$$\begin{aligned} A_t f(x) =& E[H(\xi_1)\dots H(\xi_{t-1})g(\xi_{t-1})|\xi_0 = x] \\ =& A_{t-1}g(x) \end{aligned}$$

Afterwards,if we substitute $f(x)$ for $g(x)$

$$A_t f(x) = (A_{t-1}A_1 f)(x)$$

Hence, the proof is completed as follows

$$A_t f(x) = A^t f(x)$$

3.2 Transform of Transition Operator in Continuous Time

Lemma 3.2.1. ***(Continuous time general transformation lemma)***
The generator of the non-ruin semi-group A^t is the following matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -\lambda_{11} & \lambda_{12} & \cdots & \cdots & \lambda_{1i-1} & \lambda_{1i} & \lambda_{0i+1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ 0 & \lambda_{i1} & \lambda_{i2} & \cdots & \cdots & \lambda_{ii-1} & -\lambda_{ii} & \lambda_{ii+1} & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where λ_{ij} is a transition rate from state i to j.

Proof: Now, consider the time interval is divided by $N+1$ steps, then the non-ruin probability is found via the discretization technique. The discrete transition matrix is defined by

$$P_{ij}^{(N+1)} = P(X_{N+1} = m | X_0 = x)$$

By conditioning on the one step, the non-ruin operator is obtained by

$$\begin{aligned} A^{N+1}f(x) =& \sum_k E[f(X_{N+1})I(T > N+1)I(X_j = k)|X_0 = x] \\ =& \sum_k E[f(X_{N+1})I(T > N+1)|X_1 = k]P(X_1 = k|X_0 = x) \end{aligned}$$

From the definition of the operator, we have

$$A^{N+1}f(x) = \sum_k A^N f(k) P(X_1 = k|X_0 = x)$$

On the step $N+1$, the value at time $t + \Delta_t$ is obtained as follows

$$A^{t+\Delta_t} f(x) = \sum_k A^t f(k) P(\xi_{\Delta_t} = k | \xi_0 = x)$$

By using SDL (subtract-divide-limit) as $\Delta_t \mapsto 0$

$$(A^t f(x))' = -A^t f(x)\lambda_{ii} + \sum_{k \neq i, k \neq 0} A^t f(k)\lambda_{ik}$$

Then the pseudo $Q-$matrix is referred to the non-ruin generator as

$$Q = \begin{cases} -\lambda ii & , \quad \text{if } i = j > 0 \\ \\ \lambda ij & , \quad \text{if } j \neq i \ , \quad i, j > 0 \end{cases}$$

where λ_{ij} is a transition rate from state i to j.

Here, Q is not a q-matrix in general, that is, it does not generate a semigroup of transition matrices.

3.3 Hamiltonians and Probability Calculations

Let (x, y) be an inner product in coordinate if basis are given as

$$x = (x_1, \cdots, x_k) \text{ and } y = (y_1, \cdots, y_k)$$

and identity operator is defined by [16] [62]

$$I = \sum_{i=1}^{k} |i\rangle\langle i|$$

If identity operator is acting a ket vector, it can be obtained as follows [16] [62]

$$\begin{aligned} \sum_i |i\rangle\langle i|x\rangle &= \sum_i \langle i|x\rangle|i\rangle \\ &= \sum_i x_i e_i \end{aligned}$$

In particular, an arbitrary matrix operator A is decomposed in the following way

$$\begin{aligned} \langle x|A|x'\rangle =& \langle x|AI|x'\rangle \qquad \text{(applying } A = AI\text{)} \\ =& \langle x|A\sum_i^k |i\rangle\langle i||x'\rangle \qquad \text{(since } I = \sum |i\rangle\langle i|\text{)} \end{aligned}$$

By linearity and Dirac convention $\langle i||x'\rangle = \langle i|x'\rangle$

$$\langle x|A|x'\rangle = \sum_i \langle x|A|i\rangle\langle i|x'\rangle$$

for any bra vector $\langle x|$ and ket vector $|x'\rangle$.

Let $H : \mathbb{H} \mapsto \mathbb{H}$ be a Hamiltonian matrix operator and $A_t := \exp(-tH)$ be a semigroup for the previous statement, then

$$\langle x|\exp(-tH)|x'\rangle = \sum_i \langle x|\exp(-tH)|i\rangle\langle i|x'\rangle$$

where the notation on the left side, the "pinching" with the operator between bra and ket vectors, is called standard Dirac notation [16] [62].

Assume that the basis $|i\rangle$ is the set of eigenvectors of H. Then

$$H|i\rangle = \delta_i|i\rangle$$

where δ_i is a corresponding eigenvalue.

Semigroup A_t is applied to a ket vector $|i\rangle$ as follows

$$\begin{aligned} A_t|i\rangle &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} H^k|i\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \delta_i^k|i\rangle \\ &= \exp(-t\delta_i)|i\rangle \end{aligned}$$

where δ_i are eigenvalues of H. Hence, it is concluded as follows for this unique operator

$$\langle x|\exp(-tH)|x'\rangle = \sum_i \langle x|i\rangle\langle i|x'\rangle \exp(-t\delta_i)$$

We start by treating obvious example.

Example: Let $H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ be a real matrix and $A_t = \exp(-tH)$ is a semigroup.

Identity operator will be $\sum|i\rangle\langle i| = |1\rangle\langle 1| + |2\rangle\langle 2|$ for this matrix.

Then, applying H to a ket vector $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that

$$H|1\rangle = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a|1\rangle$$
$$H|2\rangle = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b|2\rangle$$

Afterwards, applying semigroup A_t to a ket vectors $|1\rangle$ and $|2\rangle$, it can be obtained as follows

$$\exp(-\tau H)|1\rangle = \exp(-\tau a) \quad \text{and} \quad \exp(-\tau H)|2\rangle = \exp(-\tau b)$$

Using this statements for standard Dirac notation

$$\begin{aligned} \langle x| \exp(-\tau H)|x'\rangle =& \langle x|i\rangle \exp(-\tau \delta_i)\langle i|x'\rangle \\ =& \langle x|1\rangle\langle 1|x'\rangle \exp(-a\tau) + \langle x|2\rangle\langle 2|x'\rangle \exp(-b\tau) \end{aligned}$$

Hence, transition probabilities can be obtained as

For $1 \to 1$

$$P_{11}(\tau) = \langle 1|A_\tau|1\rangle = \langle 1|1\rangle\langle 1|1\rangle \exp(-a\tau) + \langle 1|2\rangle\langle 2|1\rangle \exp(-b\tau) = exp(-a\tau)$$

For $1 \to 2$

$$P_{12}(\tau) = \langle 1|A_\tau|2\rangle = \langle 1|1\rangle\langle 1|2\rangle \exp(-a\tau) + \langle 1|2\rangle\langle 2|2\rangle \exp(-b\tau) = 0$$

For $2 \to 1$

$$P_{21}(\tau) = \langle 2|A_\tau|1\rangle = \langle 2|1\rangle\langle 1|1\rangle \exp(-a\tau) + \langle 2|2\rangle\langle 2|1\rangle \exp(-b\tau) = 0$$

For $2 \to 2$

$$P_{22}(\tau) = \langle 2|A_\tau|2\rangle = \langle 2|1\rangle\langle 1|2\rangle \exp(-a\tau) + \langle 2|2\rangle\langle 2|2\rangle \exp(-b\tau) = exp(-b\tau)$$

Notice that $p_{11}(\tau)+p_{12}(\tau) = \exp(-a\tau) < 1$ in general and so A_τ is not a stochastic semigroup of transition probabilities.

3.4 Path Calculation

Let $P = \begin{bmatrix} P_{11} & P_{12} & \cdots & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & \cdots & P_{2N} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ P_{N1} & P_{12} & \cdots & \cdots & P_{1N} \end{bmatrix}$ is a real matrix for states $x_1\ ,\ \ldots\ ,\ x_N \in \mathbb{Z}$.

The non-ruin probability from state x_1 to x_N with k steps is defined by

$$\begin{aligned} P^{(k)}_{x_1 x_N} &= P(x_1 \mapsto x_N \text{ in } k \text{ steps}) \\ &= \sum_{x_1=i_0,\ldots,i_k=x_N} P_{i_0 i_1} \cdots P_{i_{k-1}k} \end{aligned}$$

for arbitrary integer $i_j \in \mathbb{Z}$.

Obvious Example: Let $P = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$ is a probability matrix for states x and $y \in \{1,2\}$. For $k = 2$ steps, the non-ruin probability from any state to another one is calculated as follows:

For $1 \mapsto 1$ in 2 steps

$$\begin{aligned} P^{(2)}_{11} &= P_{11}P_{11} + P_{12}P_{21} \\ &= 0.46 \end{aligned}$$

Similarly, from any point to another one can be calculated.

3.5 Preservation Under Linear Transform and Convolution

Stochastic ordering was not applied explicitly in our study. It motivated construction of comparison in future research in Chapter 8 and application of copula in Chapter 7.

3.5.1 Usual Stochastic Order (Stochastic Dominance)

Suppose given two random variable X and Y such that [64]

$$P(X > k) \leq P(Y > k) \ for \ all \ k \in \mathbb{R}$$

X is said to be smaller than Y. This is called **usual stochastic order** and denoted by $X \leq_{st} Y$.
We say that if all the above are valid [64]

$$P(X \leq k) \geq P(Y \leq k) \ for \ all \ k \in \mathbb{R}$$

then again $X \leq_{st} Y$.
The equivalent common definition is $X \leq_{st} Y$ if and only if $E[f(X)] \leq E[f(Y)]$ for all increasing function f with finite expectations [54] [64].

Remark 1. If for two random variables X and Y we have $E[f(X)] \leq E[f(Y)]$ for ALL functions f, then $X =_d Y$, i.e. equal in distribution, then $E[f(X)] = E[f(Y)]$.
Proof of remark 1: For $\tilde{f} = -f$,

$$E[\tilde{f}(X)] = E[-f(X)] = -E[f(X)] \leq E[\tilde{f}(Y)] = -E[f(Y)]$$

$$\Rightarrow E[f(Y)] \leq E[f(X)]$$

and also we know that $E[f(X)] \leq E[f(Y)]$. Combining them, we prove $E[f(X)] = E[f(Y)]$ for any function f.

Remark 2. In addition, it can be shown that if the expected values of two stochastically ordered random variables are equal, then they must have the same distribution, more exactly if $X \leq_{st} Y$ and $E[X] = E[Y]$ then $X =^d Y$.

3.5.2 Stochastic Ordering of Convex/Concave Function

Stochastic ordering are used in several different areas of probability, statistic, etc. Variety of discrete stochastic orderings are defined to compare random variables. Large number of applications are made by Shantikumar, Fishburn, etc [18] [28] [63].

Integral Stochastic Ordering: We follow [18] See also Muller [53] for a general study for integral stochastic orderings. Let X and Y be two random variables which take on values in $\mathbb{R}$ and $\mathcal{F}$ be a class of measurable functions u. For some specific sense, X is smaller than Y if

$$E[u(x)] \leq E[u(y)] \qquad \text{for all function } u \in \mathcal{F}$$

where expectations exist and F is associated classes of convex/concave type continuous functions [18].

Suppose a function u is s-convex for some $s \in \mathbb{N}$, u is s-increasing convex if

$$u^{(k)} \geq 0 \qquad \text{for all } k = 1, \ldots, s$$

where $y^{(s)}$ is the s-th derivative of u [18].

For $s = 2$, u is non-decreasing function if

$$det \begin{bmatrix} 1 & 1 \\ u(x) & u(y) \end{bmatrix} \geq 0$$
$$= u(y) - u(x) \geq 0$$

for all $x < y$.

Also u is convex if [18]

$$det \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ u(x) & u(y) & u(z) \end{bmatrix} \geq 0$$
$$= \frac{y-x}{z-x} u(z) + \frac{z-y}{z-x} u(x) \geq u(y)$$

If we substitute $u(x)$ for x and $u(z)$ for z, we get $u(y) = y$.

Moreover, u is increasing convex if u'' exists and $u'' \geq 0$.

Preliminaries: Let X and Y be two random variables. Shantikumar and Stoyan refer that X is smaller than Y if [44] [63] [68]

$$\frac{E(X-u)_+}{EX} \leq \frac{E(Y-u)_+}{EY} \qquad \text{for all } u \geq 0$$

where expectations exist.

Assume $F_k \leq G_k$ for $k = 1, \ldots, m$. The order $\leq$ satisfies the mixture property [44]

$$\sum_{k=1}^{m} F_k p_k \leq \sum_{k=1}^{m} G_k p_k$$

where $p_1 + \cdots + p_m = 1$ for non-negative $p_i, \ldots, p_m$.
The order $\leq$ also satisfies convolution property if [44]

$$F_1 * \cdots * F_m \leq G_1 * \cdots * G_m$$

Additionally, scaling property is satisfied by $\leq$ [44]

$$aX \leq aY \qquad if \quad X \leq Y$$

for any $a \geq 0$.

3.5.3 Preservation Under Convolution

Let X be any random variable and Y_1 and Y_2 be independent random variables with $Y_1 \leq_{st} Y_2$. Convolution property is satisfied by stochastic order $\leq$ if and only if [44]

$$X + Y_1 \leq X + Y_2$$

Linear Perturbation

Perturbation theory combined with the stochastic comparison is a powerful mathematical method. It makes comparison amongst the mathematical methods to find an approximate solution for a problem [18] [19].
We say that X approximately satisfies property P, if perturbed X, say X_δ, satisfies P.
We say that a random variable A is linearly perturbed by a random variable B if

$$A_\varepsilon \;=\; (1 - V_\delta)A + V_\delta B$$

where V_δ is independent of A and B and $P(V_\delta = 1) = \delta$, $P(V_\delta = 0) = 1-\delta$. Then,

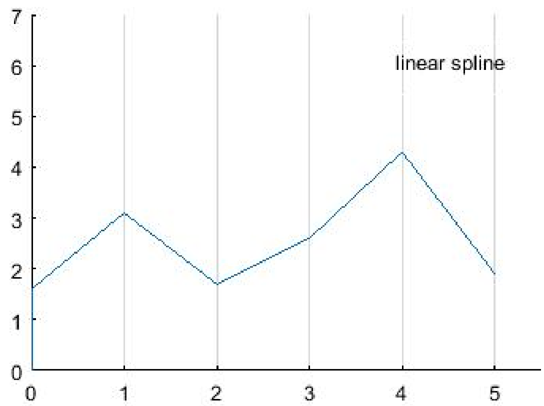


Figure 3.1: Sample path for linear perturbation.

the distribution function of A_δ is

$$P(A_\delta > t) = P(A \geq t)(1 - \delta) + \delta$$

where $P(V_\delta = 1) = \delta$ and $P(B) = 1$.

Example 1: Let $X = U_a$ and $Y = U_b$ be two uniform random variables such on intervals [0,a] and [0,b], respectively and from the definition of stochastic ordering $P(X \geq t) \ \leq \ P(Y \geq t))$ if $X \leq_{st} Y$. Now, there are two significant properties for perturbation

1. $a \geq b \ \rightarrow \ X \geq_{st} Y$ and $a < b \ \rightarrow \ X <_{st} Y$

2. $a < b \ \rightarrow \ X_\delta \geq Y$

The probability density function of the continuous uniform distribution on interval $[a, b]$ is [44]

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{for } x \leq a \text{ or } x \geq b \end{cases}$$

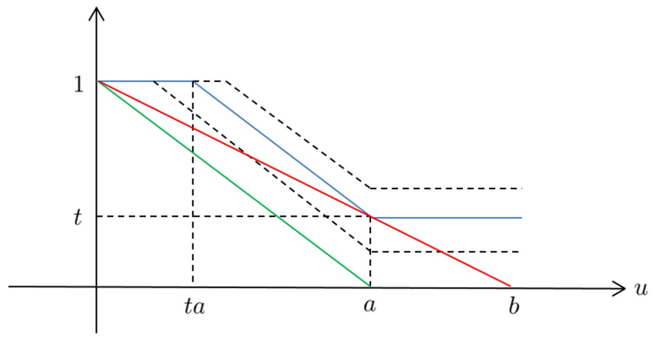


Figure 3.2: Perturbation for uniform distribution.

on the interval $[a, b]$. Assume perturbed X is

$$X_\delta = (1 - V_\delta)X + V_\delta Z$$

where $P(V_\delta = 1) = \delta = 1 - P(V_\delta = 0)$ and $Z \mapsto \infty$. The probability function is defined as [44]

$$P(X \geq t) = \frac{(a-t)_+}{a} \quad = \begin{cases} 0, & t \geq a \\ \frac{a-t}{a}, & 0 < a \leq t \end{cases}$$

and after applying it for random variables X and Y with stochastic ordering, we get

$$\frac{(a-t)_+}{a} \leq \frac{(b-t)_+}{b} \longrightarrow 1 - \frac{t}{a} \leq 1 - \frac{t}{b}$$

if $a < b$. Afterwards, the probability for perturbation function has to be found by using the definition of probability function as follows

$$\begin{aligned} P(X_\delta \geq t) =& P(X \geq t)(1-\delta) + P(Z \geq t)\delta \\ =& P(X \geq t)(1-\delta) + \delta \end{aligned}$$

where $Z \mapsto \infty$.

After applying probability function for random variables X_δ and Y with stochastic ordering, we find that

$$\frac{(a-t)_+}{a}(1-\delta)+\delta \geq \frac{(b-t)_+}{b} \rightarrow (1-\frac{t}{a})_+(1-\delta)+\delta \geq (1-\frac{t}{b})$$

where $P(X_\delta \geq t) \geq P(Y \geq t)$. Then, it is concluded as

$$\delta \geq 1-\frac{a}{b}$$

and so

$$min\{\delta > 0 : X_\delta \geq Y\} = 1-\frac{a}{b}$$

Example 2: Let $X = U_a$ and $Y = U_b$ be two uniform random variables such on intervals [0,a] and [0,b], respectively and from the definition of stochastic ordering $P(X \geq t) \leq P(Y \geq t))$ if $X \leq_{st} Y$. Now, perturbation is defined for a new tale defined by $P(T \geq u) = min\{P(X \geq u)+t, 1)\}$ and we need to find the optimal δ. We want to find t and delta such that

$$P(T_\delta > u) \geq P(Y \geq u)$$

Notice that

$$P(T_\delta \geq u) = P(T \geq u)(1-\delta)+\delta = min\{(P(X \geq u)+t, 1)\}(1-\delta)+\delta$$

Clearly if $P(T > u) = 1$ then $P(T_\delta > u) = 1 > P(Y > u)$. By using the probability function in previous example, we get

$$\begin{aligned} P(Y \geq u) =& \frac{(b-u)_+}{b} \\ =&(1-\frac{u}{b})_+ \end{aligned}$$

and

$$
\begin{aligned}
P(X \geq u) + t =& \frac{(a-u)_+}{a} + t \\
=& (1 - \frac{u}{a})_+ + t
\end{aligned}
$$

where $a > u$ and $b > u$. For probability to be less than 1, we need $ta < u$. So by making simple calculation, we obtain that $ta \leq u$ if $a > u$. If $P(X \geq t) \leq P(Y \geq t))$, the intersection is found for case of equation as follows

$$
1 - \frac{u}{a} + t = 1 - \frac{u}{b}
$$

so $t = \frac{b-a}{b}$ if $u = a$.

Then we get

$$
\begin{aligned}
P(X_\delta \geq u) = (\frac{(a-u)_+}{a} + t)(1-\delta) + \delta \ \geq_{st} \ P(Y \geq u) = \frac{(b-u)_+}{b} \\
(1 - \frac{u}{a} + t)(1-\delta) + \delta \geq 1 - \frac{u}{b}
\end{aligned}
$$

Hence, for $t = \frac{u}{a} - 1$, δ is found as $\delta \geq 1 - \frac{a(t+1)}{b}$.

Example 3: Let X and Y be exponential variables denoted by $EX(a)$ and $EX(b)$ with parameters a and b, respectively, that is $P(X \geq u) = e^{-au}$ and $P(Y \geq u) = e^{-bu}$. Now, perturbation is defined for a new tale defined by $P(T \geq u) = min\{P(X \geq u) + t, 1)\}$ and we need to find the optimal δ. We want to find t and delta such that

$$
P(T_\delta > u) \geq P(Y \geq u)
$$

Notice that

$$
P(T_\delta \geq u) = P(T \geq u)(1-\delta) + \delta = min\{(P(X \geq u) + t, 1)\}(1-\delta) + \delta
$$

Clearly if $P(T \geq u) = 1$ then $P(T_\delta \geq u) = 1 > P(Y > u)$.

By making simple calculation, we find that

$$\frac{ln(t-1)}{a} \leq u$$

If $P(X \geq t) \;\leq\; P(Y \geq t))$, we get

$$e^{-au} + t = e^{-bu} \Rightarrow t = e^{-bu} - e^{-au}$$

Then t is obtained as follows

$$t = e^{-ba} - e^{-a^2} \Rightarrow t = e^{-a}(e^b - e^a)$$

So

$$P(X_\delta \geq u) = (e^{-au} + t)(1-\delta) + \delta \;\geq_{st}\; P(Y > u) = e^{-bu}$$

Hence, δ is found as

$$\delta \geq \frac{e^{-bu} - e^{-au} - t}{1 - e^{-au} - t}$$

Example 4: Using notation in Examples 2 and 3, let $X = X_a = U_a + EX(\lambda)$ and $Y = X_b = U_b + EX(\mu)$ be two random variables where U_a,U_b, $EX(\lambda)$ and $EX(\mu)$ are all independent. From the definition of stochastic ordering $P(X \geq t) \;\leq\; P(Y \geq t))$ if $X \leq_{st} Y$. By using the probability, we get

$$P(Y \geq u) = (1-\varepsilon)\frac{(b-t)_+}{b} + \varepsilon e^{-\mu t}$$

and

$$P(X \geq u) + t = (1-\varepsilon)\frac{(a-t)_+}{a} + \varepsilon e^{-\lambda t}$$

where $a > u$ and $b > u$. Then, assume perturbation is defined by

$$(1-\delta)((1-\varepsilon)P(U_a \geq t) + \varepsilon P(E(\lambda) \geq t)) + \delta \geq (1-\varepsilon)P(U_b \geq 0) + \varepsilon P(E(\mu) \geq t)$$
$$(1-\delta)((1-\varepsilon)\frac{(a-t)_+}{a} + \varepsilon e^{-\lambda t}) + \delta \geq (1-\varepsilon)\frac{(b-t)_+}{b} + \varepsilon e^{-\mu t}$$

where $P(Z) \mapsto 1$. For $\varepsilon = 0$, we get

$$\delta \geq \frac{b-a}{b} = 1 - \frac{a}{b}$$

where $a \geq u$ and $b \geq u$. For $\varepsilon = 1$, the similar result of exponential variable example is found as follows

$$e^{-\lambda u} + \delta(1 - e^{-\lambda u}) \geq e^{-\mu u}$$

Hence, after making some simple calculations, δ is found as $\delta \geq \frac{(e^{\lambda t - \mu t} - 1)}{e^{\lambda t} - 1}$.

Chapter 4

Examples for Non-Ruin Operator in Discrete Time

In this chapter, several examples of non-ruin operator in discrete time are treated and most common technique path integral and combinatorics are used to treat them.

4.1 General Non-Ruin Operator in Discrete Time

Let $\{\xi_k\ ,\ k = 0, 1, 2, \ldots\}$ be a discrete time Markov chain and transition probabilities be $P = (P_{ij})$ where $i, j \in \mathbb{Z}$. We apply the discrete time general transformation *Lemma* 3.1.1. We emphasize that jumps to non-positive integer values occur (otherwise there is no ruin). Then, the non-ruin operator matrix is defined by

$$a_{ij} \;=\; \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ P_{ij}, & \text{if } i, j \geq 1 \end{cases}$$

In the matrix form

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & \cdots & \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
0 & P_{11} & P_{12} & P_{13} & P_{14} & \cdots & \cdots \\
0 & P_{21} & P_{22} & P_{23} & P_{24} & \cdots & \cdots \\
0 & P_{31} & P_{32} & P_{33} & P_{34} & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{pmatrix}
\end{array}
$$

In general, A is not a transition matrix. Roughly, in this case jumps to non-positive values are not allowed.

Path Calculation: Assume the probability of being in state y from state x in k steps is defined by

$$
\begin{aligned}
P_{xy}^{(k)} &= P(x \mapsto y \text{ in } k \text{ steps}) \\
&= \sum_{x=i_0,\ldots,i_k=y} P_{i_0 i_1} \cdots P_{i_{k-1} k}
\end{aligned}
$$

for arbitrary integer $i_j \in \mathbb{Z}$.

Then, non-ruin probability from state x to y in k steps is also defined by

$$
\begin{aligned}
A_{xy}^{(k)} &= P(x \mapsto y \text{ in } k \text{ steps without ruin}) \\
&= \sum_{x=i_0>0, i_1>0, \ldots, i_k=y>0} P_{i_0 i_1} \cdots P_{i_{k-1} k}
\end{aligned}
$$

for arbitrary integer $i_j \in \mathbb{Z}^+$ and $i_j > 0$.

Now, variety of examples are calculated for ruin operator in discrete time.

4.1.1 Examples of Non-Ruin Operator in Discrete Time without Interest Rate

Example 1: In this example, we make a new proof for a well-known Gambler's ruin model [51].
Let the surplus process be

$$R_k = u + \sum_{j=1}^{k}(C - X_j)$$

where u is the initial capital, C is the premium price, X_j are the claims defined by

$$C - X_j \ = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } q \end{cases}$$

($i.e$ $P(X_j = C - 1) = p$, $P(X_j = C + 1) = q$). This process is called Gambler's ruin. The transform of transition operator is defined by

$$A_t f(x) = E[f(R_t)I(T > t)|R_{t-1} = u]$$

where $A_1 = A$, T is a ruin time defined by $T = inf\{t > 0 \ s.t \ R_t \leq 0\}$.

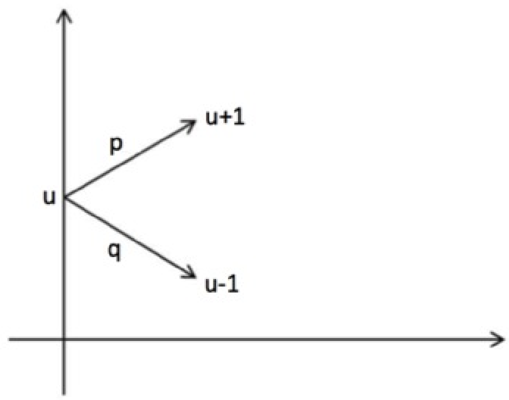


Figure: Sample path for Gambler's ruin

For the initial capital $u = 0$, it is convenient to assume that the zero income is ruin, that is $Af(0) = 0$. Notice that we also assume that $f(x) = 0$ for $x \leq 0$.

For the initial capital $u = 1$, there is just one way to avoid ruin

$$\begin{aligned} Af(u) =& E[f(R_1)I(T > 1)|R_0 = 1] \\ =& pf(2) \end{aligned}$$

For the initial capital $u \geq 2$, there is no ruin and the operator is calculated by

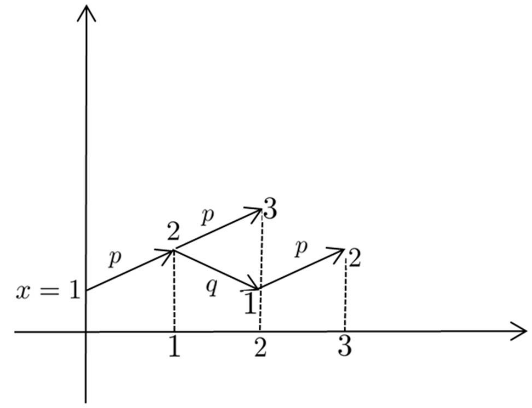
$$\begin{aligned} Af(u) =& E[f(R_1)I(T > 1)|R_0 = 2] \\ =& pf(u+1) - qf(u-1) \end{aligned}$$

Then, the operator matrix is defined as follows

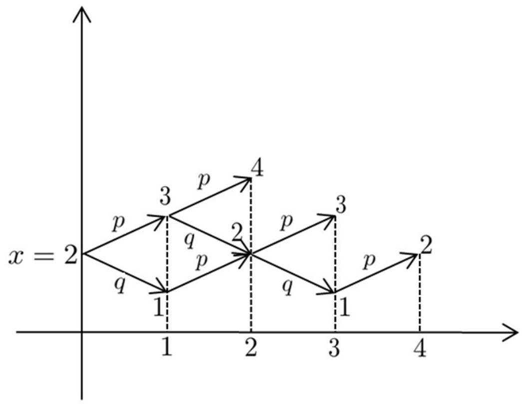
$$Af = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & \cdots \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & p & 0 & \cdots & \cdots & \cdots \\ 0 & q & 0 & p & 0 & \cdots & \cdots \\ 0 & 0 & q & 0 & p & 0 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \end{array} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \end{pmatrix}$$

The finite time non-ruin probability $P_u(T > 1) = P(T > 1|u)$ is then computed by applying operator A to the unit function $\mathbb{1}(u)$

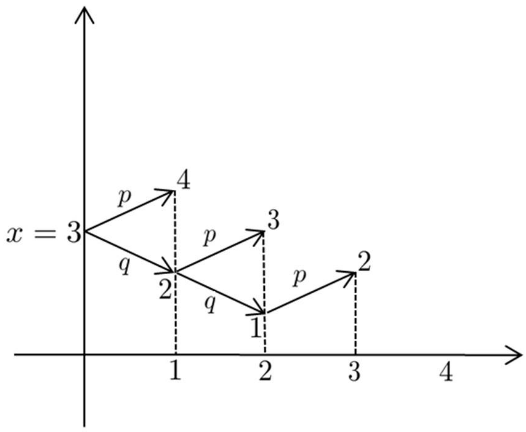
$$A\mathbb{1}(u) = \begin{cases} 0, & u = 0 \\ p, & u = 1 \\ 1, & u \geq 2 \end{cases}$$



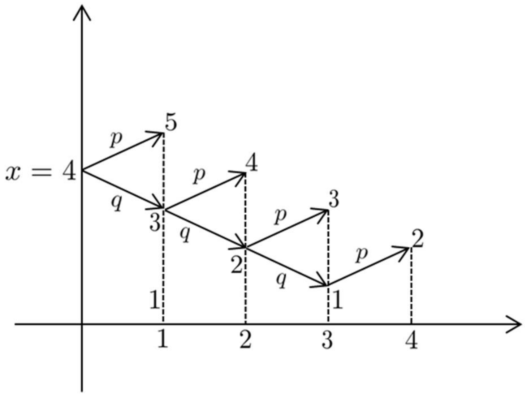
(a) The case of u=1.



(b) The case of u=2.



(c) The case of u=3.



(d) The case of u=4.

Figure 4.1: Sample path for initial capital u=1,2,3 and 4 (Example 1).

Example 1: Proof by reflection approach

Consider that there is a path from $(0, u)$ to (t, j) where u is the initial capital at time $t = 0$ and j is the final value at time t. The number of possible paths are denoted by

$$\begin{aligned} N_t(u, j) =& N_t(0, j - u) \\ =& \binom{t}{\frac{t+(j-u)}{2}} \end{aligned}$$

The probability is also defined by the reflection approach as follows

$$\begin{aligned} P(N_t(0, j - u)) =& p^{\#ups} q^{\#downs} \\ =& p^{\frac{t+(j-u)}{2}} q^{\frac{t-(j-u)}{2}} \end{aligned}$$

Notice that the set of paths without crossing x-axis is equal to the set of all paths − the set of paths with crossing x-axis. In particular, the number of paths from $(0, u)$ to (t, j) is equal to

$$N_t(u, j) - N_t(-u, j)$$

We calculate this subtraction because crossing x-axis means ruin.

Theorem 4.1.1. *By using the definition of non-ruin operator, we derive*

$$\begin{aligned} A_t f(u) =& E_u[f(R_t) I(T > t)] \\ =& \sum_j f(j) P_u(R_t = j | \textit{no ruin up to time } t) \end{aligned}$$

Hence, we derive

$$Af(u) = \sum_j f(j)\{N_t(u, j) - N_t(-u, j)\} p^{\frac{t+(j-u)}{2}} q^{\frac{t-(j-u)}{2}}$$

In the example, there is just up and down cases. The capital at time $t = 1$ is defined by

$$j = R_1 = \begin{cases} u + 1 & , \quad \text{with } p \\ u - 1 & , \quad \text{with } q \end{cases}$$

By making simple algebraic calculation, we find

$$\begin{aligned} \frac{t + (j - u)}{2} &= 1 \ \vee \ 0 \\ \frac{t - (j - u)}{2} &= 1 \ \vee \ 0 \end{aligned}$$

Hence, we conclude that

$$
\begin{aligned}
A_1 f(u) = Af(u) = & \sum_j f(j)\{N_1(u,j) - N_1(-u,j)\} p^{\frac{1+(j-u)}{2}} q^{\frac{1-(j-u)}{2}} \\
= & f(u-1)\left[\binom{1}{0} - \binom{1}{u}\right] q + f(u+1)\left[\binom{1}{1} - \binom{1}{u+1}\right] p
\end{aligned}
$$

So, we get

$$
Af(u) = \begin{cases} 0 \quad , & \text{if } u = 0 \\ f(2)p \quad , & \text{if } u = 1 \\ f(u-1)q + f(u+1)p \quad , & \text{if } u \geq 2 \end{cases}
$$

where $\binom{1}{u} = 0$ for any $u \geq 2$.

Example 2: In this example, we construct a new case of the well-known model. The result is included in [39]. Let the surplus process be

$$
R_k = u + \sum_{j=1}^{k} (C - X_j)
$$

where u is the initial capital, C is the premium price, X_j are defined by

$$
C - X_j = \begin{cases} 1, & \text{with probability } p \\ -\infty, & \text{with probability } q \end{cases}
$$

($i.e$ $P(X_j = C - 1) = p$, $P(X_j = \infty) = q$). In a similar way as example 1, the operator matrix is now defined by

$$Af(x) = \begin{array}{c|c} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & p & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & p & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & p & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & p & 0 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \end{array} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ \vdots \end{pmatrix}$$

The finite time non-ruin probability $P_x(T > 1) = P(T > 1|R_0 = u)$ is now calculated by applying operator A to the unit function

$$Af(u) \;=\; \begin{cases} 0, & u = 0 \\ p, & u \geq 1 \end{cases}$$

where T is a ruin time defined by $T = inf\{t > 0 \; s.t \; R_t \leq 0\}$.

Lemma 4.1.2. $A^{k+1} = (a_{ij}^{(k+1)})$ *with*

$$\begin{cases} a_{i,j}^{(k+1)} = p^{k+1}, & for \; j = i + k + 1 \\ a_{i,j}^{(k+1)} = 0, & else \end{cases}$$

Proof: *From definition, we know that* $a_{i,m}^{(k)} = p^k$ *where* $m = i + k$ *and* $a_{m,j} = p$ *where* $j = m + 1$*. By induction, operator matrix is defined as*

$$A^{(k+1)} = (A^k A)_{i,j} = a_{i,j}^{(k+1)} = \sum_m (a_{i,m}^{(k)}) a_{m,j}$$

where $j = i + k + 1$*.*

In particular, the non-ruin probability is found by

$$
\begin{aligned}
P(T > k|x) &= A^k \mathbb{1}(x) \\
&= \begin{cases} 0, & for\ x = 0 \\ p^k, & for\ x \geq 1 \end{cases}
\end{aligned}
$$

Example 3: In this example, we construct a new case of the well-known model. The result is included in [39]. Let the surplus process be

$$R_k = u + \sum_{j=1}^{k} (C - X_j)$$

where u is initial capital, C is the premium price, X_j are defined

$$
C - X_j \ = \begin{cases} 1, & \text{with probability } p_1 \\ 0, & \text{with probability } p_0 \\ -\infty, & \text{with probability } p_{-\infty} \end{cases}
$$

($i.e$ $P(X_j = C - 1) = p_1$, $P(X_j = C) = p_0$, $P(X_j = \infty) = p_{-\infty}$).

The operator matrix is defined by

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & p_0 & p_1 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & p_0 & p_1 & 0 & \cdots & \cdots \\
0 & 0 & 0 & p_0 & p_1 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{pmatrix}
\end{array}
$$

Lemma 4.1.3. *The general formula for A^k is found as follows*

$$\begin{cases} a_{ij}^{(k)} = p_0^k, & for\ i = j \geq 1 \\ a_{ij}^{(k)} = \binom{k}{j-i} p_1^{j-i} p_0^{k-(j-i)}, & for\ j > i \geq 1 \\ a_{i,j}^{(k)} = 0, & else \end{cases}$$

Proof: From definition, we know that

$$(A)_{sj} = \begin{cases} p_0 & for\ s = j \geq 1 \\ p_1 & for\ j = s+1 \geq 2 \\ 0 & else \end{cases}$$

By induction, operator matrix is defined as

$$(A^{(k+1)})_{ij} = (A^k)_{i,s}(A)_{s,j}$$

For $s = j$, it is equal to $(A^{(k+1)})_{ij} = (A^k)_{ij}(A)_{jj}$ and for $s = j - 1$, it equals $(A^{(k+1)})_{ij-1} = (A^k)_{ij-1}(A)_{j-1j}$. So

$$(A^{(k+1)})_{ij} = (A^k)_{ij} p_0 I(j \geq 1) + (A^k)_{ij-1} p_1 I(j \geq 2)$$

Hence, for case $i = j \geq 1$,

$$(A^{(k+1)})_{jj} = (A^k)_{j,j} p_0 I(j \geq 1) + (A^k)_{jj-1} p_1 I(j \geq 2)$$

By induction assumption,

$$\begin{aligned} (A^{(k+1)})_{jj} =& p_0^k p_0 I(j \geq 1) \\ =& p_0^{k+1} \end{aligned}$$

Similarly, for case $j > i \geq 1$,

$$(A^{(k+1)})_{ij} = (A^k)_{i,j} p_0 I(j \geq 1) + (A^k)_{ij-1} p_1 I(j \geq 2)$$

By induction assumption,

$$\begin{aligned}(A^{(k+1)})_{ij} =& \binom{k}{j-i} p_1^{j-i} p_0^{k-(j-i)} p_0 I(j \geq 1) \\ =& \binom{k+1}{j-i} p_1^{j-i} p_0^{k+1-(j-i)} I(j \geq 1)\end{aligned}$$

In particular, the non-ruin probability is defined by

$$P(T > k|u) = A^k \mathbb{1}(u) = \begin{cases} 0, & for\ u = 0 \\ (p_0 + p_1)^k, & for\ u \geq 1 \end{cases}$$

where T is a ruin time defined by $T = inf\{t > 0\ s.t\ R_t \leq 0\}$.

Example 4: In this example, we construct a new case of the well-known model. The result is included in [39]. Let the surplus process be

$$R_k = u + \sum_{j=1}^{k} (C - X_j)$$

where u is initial capital, C is the premium price, X_j are defined by

$$C - X_j \ = \begin{cases} 1, & \text{with probability } p_1 \\ 0, & \text{with probability } p_0 \\ -1, & \text{with probability } p_{-1} \\ -\infty, & \text{with probability } p_{-\infty} \end{cases}$$

($i.e\ P(X_j = C - 1) = p_1$, $P(X_j = C) = p_0$, $P(X_j = C + 1) = p_{-1}$, $P(X_j =$

$\infty) = p_{-\infty})$.

Similar as in the previous examples, the operator matrix for $t = 1$ is defined by

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & p_0 & p_1 & 0 & \cdots & \cdots & \cdots \\ 0 & p_{-1} & p_0 & p_1 & 0 & \cdots & \cdots \\ 0 & 0 & p_{-1} & p_0 & p_1 & 0 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \end{array}$$

The finite time non-ruin probability $P_u(T > 1) = P(T > 1|R_0 = u)$ is then calculated by applying operator A to the unit function

$$Af(u) = \begin{cases} 0, & u = 0 \\ p_0 + p_1, & u = 1 \\ p_{-1} + p_0 + p_1, & u \geq 2 \end{cases}$$

where T is a ruin time defined by $T = inf\{t > 0 \; s.t \; R_t \leq 0\}$.

Another Approach:

Consider there is no infinite claim before the specific time t. The non-probability is defined by

$$\begin{aligned} P(T > t|u) =& P(u_1 < \infty, \ldots, u_t < \infty; T > t|u) \\ =& P(T > t|u, x_1 < \infty, \ldots, u_t < \infty) P(u_1 < \infty, \ldots, u_t < \infty) \end{aligned}$$

Hence, we get

$$P(T > t|u) = P(\widetilde{T} > t|u) P(x_1 < \infty)^t$$

By using the operator notation and this result, we find that

$$\begin{aligned} E[f(R_t)I(T > t)|R_0 = u] =& E[f(R_t)I(u_1 < \infty, \ldots, u_t < \infty)I(T > t)|R_0 = u] \\ =& P(u_1 < \infty, \ldots, u_t < \infty)E[f(\widetilde{R}_t)I(\widetilde{T} > t)|R_0 = u] \end{aligned}$$

So

$$E[f(R_t)I(T > t)|R_0 = u] = p^t \widetilde{A}^t f(u)$$

where $P(u_1 < \infty) = p$.

Another Approach for Example 2: In this example, we construct new case of the well-known model.

Let claims X_j be

$$C - X_j = \begin{cases} 1 & \text{with probability } p \\ \infty & \text{with probability } q \end{cases}$$

The new operator matrix is defined by

$$\widetilde{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \end{array}$$

The finite time non-ruin probability is calculated as

$$\widetilde{A}\mathbb{1}(x) = P(\widetilde{T} > t|x) = 1$$

Then, we get the same result

$$\begin{aligned} P(T > t|x) =& p^t P(\widetilde{T} > t|x) \\ =& p^t \end{aligned}$$

4.1.2 Examples of Non-Ruin Operator in Discrete Time with Interest Rate

Example 1: Here, we construct a new example in discrete time with interest rate. The result is included in [39].

Let the surplus process be

$$R_{k+1} = 2R_k + C - X_k$$

where R_0 is initial capital, C is the premium price, X_j are the claims on condition with

$$X_k = \begin{cases} K+1, & \text{with probability } q \\ 0, & \text{with probability } p \end{cases}$$

Then using the non-ruin operator notation, we get

$$\begin{aligned} Af(n) =& E[f(R_1)I(T > 1)|R_0 = u] \\ =& f(2u+1)P(T > 1, R_1 = 2u+1|R_0 = u) + f(2u-K)P(T > 1, R_1 = 2u-K|R_0 = u) \end{aligned}$$

Hence, for the initial capital $u = 0$, it is not possible to gain some income, so $Af(0) = 0$. For the initial capital $2u \leq K$, there is just one way to avoid ruin

$$Af(u) = pf(2u + 1)$$

For the initial capital $2u > K$, there is no ruin and the operator is calculated by

$$Af(u) = pf(2u + 1) + qf(2u - K)$$

where T is a ruin time defined by $T = inf\{t > 0 \ s.t \ R_t \leq 0\}$.

Let $C = 1$ be the premium price and $K = 2$ be the constant value for the claim. Then, the matrix form of operator is represented by

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & p & 0 & 0 \end{pmatrix} \end{array}$$

Example 2: Here, we construct a new example in discrete time with interest rate. The result is included in [39].

Let the surplus process be

$$R_{k+1} = Y_j R_k + C - X_k$$

where R_0 is initial capital, C is the premium price,

$$Y_j = \begin{cases} 2, & \text{with probability } p_2 \\ 1, & \text{with probability } p_1 \\ 0, & \text{with probability } p_0 \end{cases}$$

, X_j are the claims on condition with

$$X_k = \begin{cases} K+1, & \text{with probability } q \\ 0, & \text{with probability } p \end{cases}$$

So the surplus process R_{k+1} will be as follows

$$u \longrightarrow \begin{cases} 2u+1\ , & p_2 p \quad \text{(no claim with gain)} \\ u+1\ , & p_1 p \quad \text{(no claim and no gain)} \\ 1\ , & p_0 p \quad \text{(no claim but ruin)} \\ 2u-K\ , & p_2 q \quad \text{(claim and gain)} \\ u-K\ , & p_1 q \quad \text{(claim and no gain)} \\ -K\ , & p_0 q \quad \text{(claim but ruin)} \end{cases}$$

where $R_k = u$. Then using the non-ruin operator notation, we have

$$\begin{aligned} Af(u) =& E[f(R_1)I(T>1)|R_0=u] \\ =& f(2u+1)P(T>1\ ,\ R_1=2u+1|R_0=u) + f(u+1)P(T>1\ ,\ R_1=u+1|R_0=u)) \\ &+ f(1)P(T>1\ ,\ R_1=1|R_0=u) + f(2u-K)P(T>1\ ,\ R_1=2u-K|R_0=u) \\ &+ f(u-K)P(T>1\ ,\ R_1=u-K|R_0=u) \end{aligned}$$

Hence

$$Af(u) = f(2u+1)p_2p + f(u+1)p_1p + f(1)p_0p + f(2u-K)p_2q + f(u-K)p_1q$$

where T is a ruin time defined by $T = inf\{t > 0 \ s.t \ R_t \leq 0\}$.

Let $C = 1$ be the premium price and $K = 2$ be the constant value for the claim.

Then, the matrix form of operator is represented by

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_0p & p_1p & p_2p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_0p+p_2q & 0 & p_1p & 0 & p_2p & 0 & 0 & 0 & 0 \\ 0 & p_0p & 0 & p_2q & p_1p & 0 & 0 & p_2p & 0 & 0 \\ p_0p+p_1q & 0 & 0 & 0 & p_1p+p_1q & 0 & 0 & 0 & 0 & p_2p \end{pmatrix} \end{array}$$

If the capitals are same, we add probabilities. (e.g $2u - K = u + 1$)

For each example, non-ruin probability from state i_1 to i_{k+1} with k steps are calculated by using the path calculation method. Table 4.1 shows the path calculation results for $i_1 = 2$ and $i_{k+1} = 3$ in $k = 3$ steps where $p_{-\infty} = 0.1$, $p_{-1} = 0.3$, $p_0 = 0.2$, $p_1 = 0.4$.

No Interest	i_1	i_{k+1}	Probability ($P^{(k)}_{i_1 i_k}$)
Example 1	2	3	$P^{(3)}_{23} = 0.1440$
Example 2	2	3	$P^{(3)}_{23} = 0$
Example 3	2	3	$P^{(3)}_{23} = 0.0480$
Example 4	2	3	$P^{(3)}_{23} = 0.1920$

Table 4.1: Table shows the path calculation results from $i_1 = 2$ to $i_{k+1} = 3$ in $k = 3$ steps for each examples without interest.

Table 4.2 shows the path calculation results for $i_1 = 2$ and $i_{k+1} = 3$ in $k = 3$ steps where $q = 0.4$, $p = 0.6$, $p_0 = 0.1$, $p_1 = 0.5$, $p_1 = 0.4$.

With Interest	i_1	i_{k+1}	Probability ($P^{(k)}_{i_1 i_k}$)
Example 1	2	3	$P^{(3)}_{23} = 0.0960$
Example 2	2	3	$P^{(3)}_{23} = 0.0434$

Table 4.2: Table shows the path calculation results from $i_1 = 2$ to $i_{k+1} = 3$ in $k = 3$ steps for each examples with interest.

Chapter 5

Examples for Non-Ruin Operator in Continuous Time

In this chapter, several examples of non-ruin operator in continuous time are treated.

5.1 General Non-Ruin Operator in Continuous Time

Let $\{\xi_t\ ,\ t \geq 0\}$ be a continuous time discrete space Markov chain with state space $\mathbb{Z}$ and transition rate λ_{ij} for $i \neq j$ and $i, j \in \mathbb{Z}$. We apply the continuous time general transformation *Lemma* 3.2.1, the non-ruin generator is defined by

$$Q = (q_{ij}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -\lambda_{11} & \lambda_{12} & \cdots & \cdots & \lambda_{1i-1} & \lambda_{1i} & \lambda_{0i+1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ 0 & \lambda_{i1} & \lambda_{i2} & \cdots & \cdots & \lambda_{ii-1} & -\lambda_{ii} & \lambda_{ii+1} & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Now, variety of examples are calculated for non-ruin operator in discrete time.

5.2 Non-Ruin Operator in Continuous Time for Two State Case

Example 1: Using particular matrix $H = \begin{pmatrix} 0 & 0 \\ 0 & -\mu \end{pmatrix}$, it can be showed that $\lim\limits_{h \mapsto 0} \frac{A^h - A^0}{h} = H$ where $A^t = \exp(-tH)$. From this equation, operator matrix is defined

$$A^t = \begin{pmatrix} 0 & 0 \\ 0 & \exp(-\mu t) \end{pmatrix} \quad \text{and} \quad A^{t+\Delta_t} = \begin{pmatrix} 0 & 0 \\ 0 & \exp(-\mu(t+\Delta_t)) \end{pmatrix}$$

From expansion of exponential function for a small time period Δ_t,

$$A^{t+\Delta t} = \begin{pmatrix} 0 & 0 \\ 0 & \exp(-\mu t) - \mu \exp(-\mu t)\Delta_t + o(\Delta_t) \end{pmatrix}$$

and after splitting the matrix, we get

$$A^{t+\Delta t} = A^t - \mu \Delta_t A^t + o(\Delta_t)$$

Example 2: In this example, obvious example is done for well-known Markov chains process [4].

Consider the transition operator $P = P_{ij}(t)$ with parameters $\lambda_{01} = \lambda$ and $\lambda_{10} =$

μ. Then, assume $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$. The eigenvalues are $k_1 = 0$ and $k_2 = -(\lambda+\mu)$. So diagonal form of the matrix is $D = \begin{pmatrix} -(\lambda+\mu) & 0 \\ 0 & 0 \end{pmatrix}$ and $f(D) = \begin{pmatrix} f(-(\lambda+\mu)) & 0 \\ 0 & f(0) \end{pmatrix}$. Then eigenvectors are calculated as $\begin{pmatrix} \lambda \\ -\mu \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So the matrix T and its inverse are defined as

$$T = \begin{pmatrix} \lambda & 1 \\ -\mu & 1 \end{pmatrix} \quad , \quad T^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 & -1 \\ \mu & \lambda \end{pmatrix}$$

where $\Delta = \lambda + \mu$. By using the formula $Q = TDT^{-1}$, we derive the probability transition matrix as follows

$$P(t) = Te^{tD}T^{-1} = \begin{pmatrix} \lambda & 1 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} \exp(-(\lambda+\mu)t) & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\lambda+\mu} \begin{pmatrix} 1 & -1 \\ \mu & \lambda \end{pmatrix}$$

$$= \frac{1}{\Delta}\begin{pmatrix} \lambda\exp(-\Delta t) + \mu & -\lambda\exp(-\Delta t) + \lambda \\ -\mu\exp(-\Delta t) + \mu & \mu\exp(-\Delta t) + \lambda \end{pmatrix}$$

where $\Delta = \lambda + \mu$. On the other hand, by using the formula $P(t) = e^{tQ}$, the same result is found by

$$\begin{aligned} \exp(tQ) &= \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} \\ &= I + \sum_{k=1}^{\infty} \frac{t^k(-(\Delta))^{k-1}Q}{k!} \\ &= I + \frac{Q}{-\Delta}(\exp(-t\Delta) - 1) \end{aligned}$$

where $\Delta = \lambda + \mu$.

5.3 Non-Ruin Operator in Continuous Time With Interest Rate

Let the surplus process be

$$R_{k+1} = Y\ R_k + C - X_k$$

where Y is a random gain, R_0 is initial capital, C is the premium price and X_j are the claims

$$X_k \;=\; \begin{cases} K, & \text{with probability } q \\ 0, & \text{with probability } p \end{cases}$$

for any $K \in \mathbb{Z}$.

The forward equation and pseudo Q-matrix are found with the transition probabilities.

Example 1: Here, we construct a new example and the result is included in [39]. Let X_t be the continuous time Markov chain on the state space $Z_t = \{0, 1, 2, \ldots\}$ where premium rate is 1, claim size is $K + 1$ and random gain is $u \mapsto 2u + 1$ or $u \mapsto 1$ from time t to $t + \Delta_t$ as seen in the transition scheme

$$u \longmapsto \begin{cases} 2u+1 & \lambda_2\Delta_t \quad \text{(investment in)} \\ 1 & \lambda_0\Delta_t \quad \text{(ruin, no gain)} \quad (\text{ if } u \neq 1) \\ u-(K+1) & \lambda_q\Delta_t \quad \text{(claim)} \\ u+1 & \lambda\Delta_t \quad \text{(premium)} \\ u & 1-(\lambda_2+\lambda_0+\lambda_q+\lambda)\Delta_t \quad \text{(nothing)} \end{cases}$$

The transition matrix $P(t)$ is defined by

$$P_{uv}(t+\Delta_t) = P(X_{t+\Delta_t} = v | X_0 = u)$$

Lemma 5.3.1. *The limit of non-ruin probability is found by*

$$(P_{uv}(t))' = \lambda_2 P_{2u+1,v} + \lambda_0 P_{1,v} + \lambda_q P_{u-(K+1),v} + \lambda P_{u+1,v} - \lambda_{uu} P_{uv}(t)$$

where $\lambda_{uu} = \lambda_2 + \lambda_0 + \lambda_q + \lambda$.

Proof: Using the diagram and the backward argument (Chapman-Kolmogoroff equation), we derive

$$\begin{aligned} P_{ij}(t+\Delta_t) &= \sum_k P(X_{t+\Delta_t} = v, X_{\Delta_t} = k | X_0 = u) \\ &= \sum_k P(X_{t+\Delta_t} = v | X_{\Delta_t} = k) P(X_{\Delta_t} = k | X_0 = u) \\ &= \sum_k P_{kv}(t) P_{uk}(\Delta_t) \end{aligned}$$

If we separate the cases for jumps and stay

$$\begin{aligned} P_{uv}(t+\Delta_t) = &\sum_k P_{kv}(t)\lambda_{uk}\Delta_t + P_{uv}(t)(1-\lambda_{uu})\Delta_t \\ =&\lambda_2\Delta_t P_{2u+1,v} + \lambda_0\Delta_t P_{1,v} + \lambda_q\Delta_t P_{u-(K+1),v} + \lambda\Delta_t P_{u+1,v} \\ &+ (1-(\lambda_2+\lambda_0+\lambda_q+\lambda))\Delta_t P_{u,v=u} + o(\Delta_t) \end{aligned}$$

In a limit

$$\begin{aligned}(P_{uv}(t))' &= \lim_{\Delta_t \mapsto 0} \frac{P_{uv}(t+\Delta_t) - P_{uv}(t)}{\Delta_t} \\ &= \lim_{\Delta_t \mapsto 0} \frac{\lambda_2 \Delta_t P_{2u+1,v} + \lambda_0 \Delta_t P_{1,v} + \lambda_q \Delta_t P_{u-(K+1),v} + \lambda \Delta_t P_{u+1,v} - \lambda_{uu} P_{uv}(t)\Delta_t + o(\Delta_t)}{\Delta_t} \\ &= \lambda_2 P_{2u+1,v} + \lambda_0 P_{1,v} + \lambda_q P_{u-(K+1),v} + \lambda P_{u+1,v} - \lambda_{uu} P_{uv}(t)\end{aligned}$$

where $\lambda_{uu} = \lambda_2 + \lambda_0 + \lambda_q + \lambda$.

Continuing Example 1: The pseudo $Q-$matrix, which is referred to the non-ruin generator, is derived by

$$Q = \begin{cases} \lambda_2 & , \text{ if } v = 2u+1 \quad , \ u > 0 \\ \lambda & , \text{ if } v = u+1 \quad , \ u > 0 \\ \lambda_0 & , \text{ if } v = 1 \quad , \ v > 0 \\ \lambda_q & , \text{ if } v = u-(K+1) \quad , \ v > 0 \\ 1-\lambda_{uu} & , \text{ if } u = v > 0 \end{cases}$$

where $\lambda_{uu} = \lambda_2 + \lambda + \lambda_0 + \lambda_q$.

For example, in the matrix form for $K = 2$

$$Q = \begin{array}{c c} & \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -\lambda_{11} & \lambda & \lambda_2 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \lambda_0 & -\lambda_{22} & \lambda & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \lambda_0 & 0 & -\lambda_{33} & \lambda & \lambda_2 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \lambda_0+\lambda_q & 0 & 0 & -\lambda_{44} & \lambda & \lambda_2 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{array}$$

If numbers are same, rates usually sum up.

Notice that non-ruin generators are not generators of transition matrices, in general. For this reason, we refer to them as pseudo Q-matrices.

Example 2: Here, we construct a new example and the result is included in [39]. Now, let the transition probability scheme be as follows

$$u \longrightarrow \begin{cases} 2u \ , & \lambda_2 \Delta_t \quad \text{(no claim with gain)} \\ u+1 \ , & \lambda \Delta_t \quad \text{(no claim and no gain)} \\ u-K \ , & \lambda_q \Delta_t \quad \text{(claim and no gain)} \\ 0 \ , & \lambda_0 \Delta_t \quad \text{(ruin)} \\ u \ , & 1-(\lambda_2+\lambda+\lambda_q+\lambda_0)\Delta_t \quad \text{(ruin)} \quad \text{(nothing)} \end{cases}$$

where premium rate is 1, claim size is K and random gain is $u \mapsto 2u$ (twice) or $u \mapsto 0$ (nothing). As explained in the theory, the pseudo $Q-$matrix is referred to the non-ruin generator as

$$Q = \begin{cases} \lambda_2 & , \quad \text{if } v = 2u \quad , \ u > 0 \\ \lambda & , \quad \text{if } v = u+1 \quad , \ u > 0 \\ \lambda_q & , \quad \text{if } v = u-K \quad , \ v > 0 \\ \lambda_0 & , \quad \text{if } v = 0 \\ 1-\lambda_{uu} & , \quad \text{if } u = v > 0 \end{cases}$$

where $\lambda_{uu} = \lambda_2 + \lambda + \lambda_0 + \lambda_q$.

For example, in the matrix form for $K = 2$

$$
Q = \begin{array}{c|cccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
1 & 0 & -\lambda_{11} & \lambda+\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
2 & 0 & 0 & -\lambda_{22} & \lambda & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
3 & 0 & \lambda_q & 0 & -\lambda_{33} & \lambda & 0 & \lambda_2 & 0 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Example 3: Consider the transition probabilities scheme

$$
u \longrightarrow \begin{cases} u+1 \ , & \text{with } \lambda_1 \Delta_t \\ 0 \ , & \text{with } \lambda_0 \Delta_t \\ u \ , & \text{with } 1-(\lambda_1+\lambda_0)\Delta_t \end{cases}
$$

After making similar calculations, the non-ruin generator is derived as follows

$$
Q = \begin{cases} \lambda uu = -(\lambda_0+\lambda_1) & , \ \text{if } u = v > 0 \\ \lambda uv = \lambda_{uu+1} = \lambda_1 & , \ \text{if } v \neq u \ , \ v \neq 0 \quad , \ u, v > 0 \\ 0 & , \ else \end{cases}
$$

For example, in the matrix form for $K = 2$

$$
Q = \begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
1 & 0 & -\lambda_{11} & \lambda_1 & 0 & 0 & \cdots & \cdots \\
2 & 0 & 0 & -\lambda_{22} & \lambda_1 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Chapter 6

Advanced Examples via Hamiltonian Technique

In this chapter, lengthy algebraic arguments for Hamiltonian technique are calculated and several advanced examples are treated.

6.1 Two State Examples

6.1.1 Hamiltonian Method with Traditional Basis

Example: Here, we construct Hamiltonian method with traditional basis and the result is included in [39].

Let $H = \begin{pmatrix} \lambda & -\lambda \\ -\mu & \mu \end{pmatrix}$ be a real matrix, $A_\tau = \exp(-\tau H)$ be a semigroup. Assume that the basis is represented by vectors $|i\rangle$.

We start with the decomposition

$$\langle x|A_t|x'\rangle = \sum_i \langle x|A_t|i\rangle\langle i|x'\rangle$$

Notice that

$$H|1\rangle = \begin{pmatrix} \lambda \\ -\mu \end{pmatrix} = |a\rangle \quad and \quad H|2\rangle = -|a\rangle$$

Hence

$$H^2|1\rangle = H|a\rangle = (\lambda+\mu)|a\rangle$$

By computing kth power, we get

$$H^k|1\rangle = H^{k-1}|a\rangle = (\lambda+\mu)^{k-1}|a\rangle$$

So

$$A_\tau = \exp(-\tau H) = \textstyle\sum_{k=0}^{\infty}(-\tau)^k \frac{H^k}{k!} = I + \sum_{k=1}^{\infty}(-\tau)^k \frac{H^k}{k!}$$

In particular,

$$\begin{aligned} A_\tau|1\rangle &= |1\rangle + (\textstyle\sum_{k=1}^{\infty}(-\tau)^k \frac{H^k}{k!})|1\rangle \\ &= |1\rangle + (\textstyle\sum_{k=1}^{\infty}(-\tau)^k \frac{(\lambda+\mu)^{k-1}}{k!})|a\rangle \\ &= |1\rangle + \frac{1}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1)|a\rangle \end{aligned}$$

Similarly,

$$\begin{aligned} A_\tau|2\rangle =& |2\rangle - \frac{1}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1)|a\rangle \\ A_\tau|1\rangle =& |1\rangle + K|a\rangle \quad \text{and} \quad A_\tau|2\rangle = |2\rangle - K|a\rangle \end{aligned}$$

where $K = \frac{1}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1)$.

Then, the operator defined by

$$\langle x|\exp(-\tau H)|x'\rangle = [\langle x|1\rangle + K\langle x|a\rangle]\langle x'|1\rangle + [\langle x|2\rangle - K\langle x|a\rangle]\langle x'|2\rangle$$

Overall, for $1 \to 1$

$$P_{11}(\tau) = \langle 1|A_\tau|1\rangle = 1 + K\lambda = 1 + \frac{\lambda}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1)$$

For $1 \to 2$

$$P_{12}(\tau) = \langle 1|A_\tau|2\rangle = -K\lambda = -\frac{\lambda}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1) = \frac{\lambda}{\lambda+\mu}(1 - \exp(-\tau(\lambda+\mu)))$$

For $2 \to 1$

$$P_{21}(\tau) = \langle 2|A_\tau|1\rangle = -K\mu = -\frac{\mu}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1) = \frac{\mu}{\lambda+\mu}(1 - \exp(-\tau(\lambda+\mu)))$$

For $2 \to 2$

$$P_{22}(\tau) = \langle 2|A_\tau|2\rangle = 1 + K\mu = 1 + \frac{\mu}{\lambda+\mu}(\exp(-\tau(\lambda+\mu)) - 1)$$

6.2 Hamiltonian Method with Eigenvector Basis

Changing the Coordinates: In this part, we work with the different bases. Let $|i\rangle$ and $|i^*\rangle$ be two different bases.

Assume that $\langle i|j\rangle = \langle i^*|\Sigma j^*\rangle$ where $i*$ and $j*$ are new coordinates. If we also assume $|1^*\rangle = \begin{pmatrix} m_{11} \\ m_{21} \\ \vdots \\ m_{N1} \end{pmatrix}$, $|2^*\rangle = \begin{pmatrix} m_{12} \\ m_{22} \\ \vdots \\ m_{N2} \end{pmatrix}$ and similarly $|N^*\rangle = \begin{pmatrix} m_{1N} \\ m_{2N} \\ \vdots \\ m_{NN} \end{pmatrix}$. Then the matrix $\Sigma = M^T M$ where $M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{NN} \end{pmatrix}$ and M^T is the transpose of the matrix M.

Example 1: We construct an obvious example for change of basis.
Let $|1^*\rangle = \begin{pmatrix} A \\ 0 \end{pmatrix}$, $|2^*\rangle = \begin{pmatrix} 0 \\ B \end{pmatrix}$ be the ket vectors. So,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle = \frac{1}{A}|1^*\rangle = \begin{pmatrix} \frac{1}{A} \\ 0 \end{pmatrix}_* \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |2\rangle = \frac{1}{B}|2^*\rangle = \begin{pmatrix} 0 \\ \frac{1}{B} \end{pmatrix}_*$$

To simplify notation we skip $*$ whenever it is possible.

By changing the coordinates

$$\langle 1^*|\Sigma 1^*\rangle = \begin{pmatrix} \frac{1}{A} & 0 \end{pmatrix} \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \begin{pmatrix} \frac{1}{A} \\ 0 \end{pmatrix}$$

and

$$\langle 2^*|\Sigma 2^*\rangle = \begin{pmatrix} 0 & \frac{1}{B} \end{pmatrix} \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{B} \end{pmatrix}$$

Then, the same result is found by making simple calculations

$$\langle 1|1\rangle = \langle 1^*|\Sigma 1^*\rangle = 1 \quad \text{and} \quad \langle 2|2\rangle = \langle 2^*|\Sigma 2^*\rangle = 1$$

Example 2: Let $|1^*\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|2^*\rangle = \begin{pmatrix} -\lambda \\ \mu \end{pmatrix}$ be the ket vectors. By definition, vector $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is the representation of $|1\rangle$ in new coordinates. Then, we get

$$|1\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T = x_1|1^*\rangle + x_2|2^*\rangle = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -\lambda \\ \mu \end{pmatrix}$$

By algebraic calculation,

$$\begin{aligned} |1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}^T &= \frac{\mu}{\Delta}|1^*\rangle - \frac{1}{\Delta}|2^*\rangle \quad , \\ |2\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}^T &= \frac{\lambda}{\Delta}|1^*\rangle + \frac{1}{\Delta}|2^*\rangle \end{aligned}$$

where $\Delta = \lambda + \mu$. To change the coordinates of vectors, we take matrix Σ

$$\Sigma = M^T M = \begin{pmatrix} 1 & 1 \\ -\lambda & \mu \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} 2 & \mu - \lambda \\ \mu - \lambda & \lambda^2 + \mu^2 \end{pmatrix}$$

Note that

$$\langle 1^* | \Sigma 1^* \rangle = \begin{pmatrix} \frac{\mu}{\Delta} & -1 \end{pmatrix} \begin{pmatrix} 2 & \mu - \lambda \\ \mu - \lambda & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} \frac{\mu}{\Delta} \\ -1 \end{pmatrix}$$

Hence, it is proved that

$$\langle 1|1\rangle = \langle 1^*|\Sigma 1^*\rangle = 1$$

and by making similar calculation, we get

$$\langle 2|2\rangle = \langle 2^*|\Sigma 2^*\rangle = 1$$

Lemma 6.2.1. *Let* $H = \begin{pmatrix} \lambda & -\lambda \\ -\mu & \mu \end{pmatrix}$ *be a real matrix and* $A_\tau = \exp(-\tau H)$ *be generated semigroup. Then*

$$A_\tau = \begin{pmatrix} P_{11}(\tau) & P_{12}(\tau) \\ P_{21}(\tau) & P_{22}(\tau) \end{pmatrix} = \begin{pmatrix} 1 + \frac{\lambda}{\Delta}(\exp(-\tau\Delta) - 1) & \frac{\lambda}{\Delta}(1 - \exp(-\tau\Delta)) \\ \frac{\mu}{\Delta}(1 - \exp(-\tau\Delta)) & 1 + \frac{\mu}{\Delta}(\exp(-\tau\Delta) - 1) \end{pmatrix}$$

Here, we construct a new lemma and the result is included in [39].

Proof: Let $|1^*\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|2^*\rangle = \begin{pmatrix} -\lambda \\ \mu \end{pmatrix}$ be ket vectors. Statements are defined similar to example 2.

$$|1\rangle = \frac{\mu}{\Delta}|1^*\rangle - \frac{1}{\Delta}|2^*\rangle \quad \text{and} \quad |2\rangle = \frac{\lambda}{\Delta}|1^*\rangle + \frac{1}{\Delta}|2^*\rangle$$

where $\Delta = \lambda + \mu$. To change the coordinates of vectors, we take matrix Σ

$$\Sigma = M^T M = \begin{pmatrix} 1 & 1 \\ -\lambda & \mu \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} 2 & \mu - \lambda \\ \mu - \lambda & \lambda^2 + \mu^2 \end{pmatrix}$$

Hence

$$\begin{aligned} \Sigma 1 &= \frac{1}{\Delta} \begin{pmatrix} 2 & \mu - \lambda \\ \mu - \lambda & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} \mu \\ -1 \end{pmatrix} \\ &= |1^*\rangle - \lambda|2^*\rangle \end{aligned}$$

and similarly

$$\begin{aligned} \Sigma 1 &= \frac{1}{\Delta} \begin{pmatrix} 2 & \mu - \lambda \\ \mu - \lambda & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \\ &= |1^*\rangle + \mu|2^*\rangle \end{aligned}$$

Notice that a vector can have different representations in different bases. However, the different bases generate different inner products, in general.

In particular, the starting Dirac formalism can be written in two different forms as follows

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle &= \langle x|\exp(-\tau H)|\Sigma x'\rangle_* \\ = \langle \Sigma x|\exp(-\tau H)|1^*\rangle_* \langle 1^*|x'\rangle_* &+ \langle \Sigma x|\exp(-\tau H)|2^*\rangle_* \langle 2^*|x'\rangle_*\end{aligned}$$

Here, the second representation is in eigen-bases of operator H.

By applying ket vectors $|1\rangle$ and $|2\rangle$ to the matrix H, we find that

$$H|1^*\rangle = 0|1^*\rangle \qquad \text{and} \qquad H|2^*\rangle = \Delta|2^*\rangle$$

where $\Delta = \lambda + \mu$. The operator is derived by

$$\langle x|\exp(-\tau H)|x'\rangle = \langle \Sigma x|1^*\rangle_* \langle 1^*|x'\rangle_* + \exp(-\tau\Delta)\langle \Sigma x|2^*\rangle_* \langle 2^*|x'\rangle_*$$

So the probabilities are calculated as follows:

For $1 \to 1$

$$\begin{aligned}P_{11}(\tau) = \langle 1|A_\tau|1\rangle[\langle 1^*|1^*\rangle - \lambda\langle 2^*|1^*\rangle][\frac{\mu}{\Delta}\langle 1^*|1^*\rangle - \frac{1}{\Delta}\langle 1^*|2^*\rangle] \\ + [\langle 1^*|2^*\rangle - \lambda\langle 2^*|2^*\rangle][\frac{\mu}{\Delta}\langle 2^*|1^*\rangle - \frac{1}{\Delta}\langle 2^*|2^*\rangle] \\ =1 + \frac{\lambda}{\Delta}(\exp(-\tau\Delta) - 1)\end{aligned}$$

For $1 \to 2$

$$\begin{aligned}P_{12}(\tau) = \langle 1|A_\tau|2\rangle[\langle 1^*|1^*\rangle - \lambda\langle 2^*|1^*\rangle][\frac{\lambda}{\Delta}\langle 1^*|1^*\rangle + \frac{1}{\Delta}\langle 1^*|2^*\rangle] \\ + [\langle 1^*|2^*\rangle - \lambda\langle 2^*|2^*\rangle][\frac{\lambda}{\Delta}\langle 2^*|1^*\rangle + \frac{1}{\Delta}\langle 2^*|2^*\rangle] \\ =\frac{\lambda}{\Delta}(1 - \exp(-\tau\Delta))\end{aligned}$$

For $2 \to 1$

$$\begin{aligned}P_{21}(\tau) = \langle 2|A_\tau|1\rangle[\langle 1^*|1^*\rangle + \mu\langle 2^*|1^*\rangle][\frac{\mu}{\Delta}\langle 1^*|1^*\rangle - \frac{1}{\Delta}\langle 1^*|2^*\rangle] \\ + [\langle 1^*|2^*\rangle + \mu\langle 2^*|2^*\rangle][\frac{\mu}{\Delta}\langle 2^*|1^*\rangle - \frac{1}{\Delta}\langle 2^*|2^*\rangle] \\ =\frac{\mu}{\Delta}(1 - \exp(-\tau\Delta))\end{aligned}$$

Finally, for $2 \to 2$

$$\begin{aligned}P_{22}(\tau) = \langle 2|A_\tau|2\rangle[\langle 1^*|1^*\rangle + \mu\langle 2^*|1^*\rangle][\frac{\lambda}{\Delta}\langle 1^*|1^*\rangle + \frac{1}{\Delta}\langle 1^*|2^*\rangle] \\ + [\langle 1^*|2^*\rangle + \mu\langle 2^*|2^*\rangle][\frac{\lambda}{\Delta}\langle 2^*|1^*\rangle + \frac{1}{\Delta}\langle 2^*|2^*\rangle] \\ =1 + \frac{\mu}{\Delta}(\exp(-\tau\Delta) - 1)\end{aligned}$$

Notice that the answers of course match the answers in the previous calculations. However, the arguments here are different. The argument is then applied in the three-state case.

6.3 Hamiltonian Method with the Eigenvector Basis for Three-State Case

Lemma 6.3.1. *Let* $H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\lambda+\mu) & \lambda \\ 0 & \mu & -2\mu \end{pmatrix}$ *be a real matrix and* $A_\tau = \exp(-\tau H)$ *be generated semigroup. Then*

$$A_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\exp(\tau\mu)}{\Delta}(\mu + \lambda\exp(\tau\Delta)) & \frac{\lambda\exp(\tau\mu)}{\Delta}(1 - \exp(\tau\Delta)) \\ 0 & \frac{\mu\exp(\tau\mu)}{\Delta}(1 - \exp(\tau\Delta)) & \frac{\exp(\tau\mu)}{\Delta}(\lambda + \mu\exp(\tau\Delta)) \end{pmatrix}$$

where $\Delta = \lambda + \mu$.

Here, we construct a new lemma and the result is included in [39].

Proof: Let $|1^*\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2^*\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $|3^*\rangle = \begin{pmatrix} 0 \\ \lambda \\ -\mu \end{pmatrix}$ be ket vectors. Statements are defined similar to the Lemma 1.

$$
\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T &= |1\rangle = |1^*\rangle \quad , \\
\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T &= |2\rangle = \frac{\mu}{\Delta}|2^*\rangle + \frac{1}{\Delta}|3^*\rangle \quad , \\
\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T &= |3\rangle = \frac{\lambda}{\Delta}|2^*\rangle + \frac{-1}{\Delta}|3^*\rangle
\end{aligned}
$$

where $\Delta = \lambda + \mu$. To change the coordinates of vectors, we take matrix Σ

$$
\Sigma = M^T M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \lambda & -\mu \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 1 & -\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda - \mu \\ 0 & \lambda - \mu & \lambda^2 + \mu^2 \end{pmatrix}
$$

Hence

$$
\begin{aligned}
\Sigma 1 &= \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda - \mu \\ 0 & \lambda - \mu & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1^*\rangle \quad , \\
\Sigma 2 &= \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda - \mu \\ 0 & \lambda - \mu & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} 0 \\ \mu \\ 1 \end{pmatrix} = |2^*\rangle + \lambda|3^*\rangle \quad , \\
\Sigma 3 &= \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda - \mu \\ 0 & \lambda - \mu & \lambda^2 + \mu^2 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda \\ -1 \end{pmatrix} = |2^*\rangle - \mu|3^*\rangle
\end{aligned}
$$

Then, the operator in the new coordinates is defined by

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =&\langle x|\exp(-\tau H)|\Sigma x'\rangle_* \\ =&\langle \Sigma x|\exp(-\tau H)|1^*\rangle_*\langle 1^*|x'\rangle_* + \langle \Sigma x|\exp(-\tau H)|2^*\rangle_*\langle 2^*|x'\rangle_* \\ &+ \langle \Sigma x|\exp(-\tau H)|3^*\rangle_*\langle 3^*|x'\rangle_*\end{aligned}$$

By applying ket vectors $|1\rangle$, $|2\rangle$ and $|3\rangle$ to the matrix H, we find that

$$\begin{aligned}H|1^*\rangle = 0|1^*\rangle \quad &\Rightarrow \quad \exp(-\tau H)|1^*\rangle = |1^*\rangle \\ H|2^*\rangle = -\mu|2^*\rangle \quad &\Rightarrow \quad \exp(-\tau H)|2^*\rangle = \exp(-\tau\mu)|2^*\rangle \\ H|3^*\rangle = -(\lambda+2\mu)|3^*\rangle \quad &\Rightarrow \quad \exp(-\tau H)|3^*\rangle = \exp(-\tau(\lambda+2\mu))|3^*\rangle\end{aligned}$$

where $\Delta = \lambda + \mu$. The operator now derived by

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =&\langle \Sigma x|1^*\rangle_*\langle 1^*|x'\rangle_* + \exp(\tau\mu)\langle \Sigma x|2^*\rangle_*\langle 2^*|x'\rangle_* \\ &+ \exp(\tau(2\mu+\lambda))\langle \Sigma x|3^*\rangle_*\langle 3^*|x'\rangle_*\end{aligned}$$

So the probabilities are calculated as follows:

For $1 \to 1$

$$P_{11}(\tau) = \langle 1|A_\tau|1\rangle =[\langle 1^*|1^*\rangle\langle 1^*|1^*\rangle] + [\langle 1^*|2^*\rangle\langle 2^*|1^*\rangle] + [\langle 1^*|3^*\rangle\langle 3^*|1^*\rangle] = 1$$

For $1 \to 2$

$$P_{12}(\tau) = \langle 1|A_\tau|2\rangle =[\langle 1^*|1^*\rangle\langle 1^*|2^*\rangle] + [\langle 1^*|2^*\rangle\langle 2^*|2^*\rangle] + [\langle 1^*|3^*\rangle\langle 3^*|2^*\rangle] = 0$$

For $1 \to 3$

$$P_{13}(\tau) = \langle 1|A_\tau|3\rangle =[\langle 1^*|1^*\rangle\langle 1^*|3^*\rangle] + [\langle 1^*|2^*\rangle\langle 2^*|3^*\rangle] + [\langle 1^*|3^*\rangle\langle 3^*|3^*\rangle] = 0$$

For $2 \to 1$

$$\begin{aligned}P_{21}(\tau) = \langle 2|A_\tau|1\rangle =& [\langle 2^*|1^*\rangle + \lambda\langle 3^*|1^*\rangle]\langle 1^*|1^*\rangle + [\langle 2^*|2^*\rangle + \lambda\langle 3^*|2^*\rangle]\langle 2^*|1^*\rangle \\ &+ [\langle 2^*|3^*\rangle + \lambda\langle 3^*|3^*\rangle]\langle 3^*|1^*\rangle = 0\end{aligned}$$

For $2 \to 2$

$$\begin{aligned}P_{22}(\tau) = \langle 2|A_\tau|2\rangle =& \frac{1}{\Delta}[\langle 2^*|1^*\rangle + \lambda\langle 3^*|1^*\rangle][\mu\langle 1^*|2^*\rangle + \langle 1^*|3^*\rangle] \\ &+ \frac{1}{\Delta}[\langle 2^*|2^*\rangle + \lambda\langle 3^*|2^*\rangle][\mu\langle 2^*|2^*\rangle + \langle 2^*|3^*\rangle] \\ &+ \frac{1}{\Delta}[\langle 2^*|3^*\rangle + \lambda\langle 3^*|3^*\rangle][\mu\langle 3^*|2^*\rangle + \langle 3^*|3^*\rangle] \\ =& \frac{1}{\Delta}\mu\exp(\tau\mu) + \frac{1}{\Delta}\lambda\exp(\tau(2\mu+\lambda)) = \frac{\exp(\tau\mu)}{\Delta}(\mu + \lambda\exp(\tau\Delta))\end{aligned}$$

For $2 \to 3$

$$\begin{aligned}P_{23}(\tau) = \langle 2|A_\tau|3\rangle =& \frac{1}{\Delta}[\langle 2^*|1^*\rangle + \lambda\langle 3^*|1^*\rangle][\lambda\langle 1^*|2^*\rangle - \langle 1^*|3^*\rangle] \\ &+ \frac{1}{\Delta}[\langle 2^*|2^*\rangle + \lambda\langle 3^*|2^*\rangle][\lambda\langle 2^*|2^*\rangle - \langle 2^*|3^*\rangle] \\ &+ \frac{1}{\Delta}[\langle 2^*|3^*\rangle + \lambda\langle 3^*|3^*\rangle][\lambda\langle 3^*|2^*\rangle - \langle 3^*|3^*\rangle] \\ =& \frac{1}{\Delta}\lambda\exp(\tau\mu) - \frac{1}{\Delta}\lambda\exp(\tau(2\mu+\lambda)) = \frac{\lambda\exp(\tau\mu)}{\Delta}(1 - \exp(\tau\Delta))\end{aligned}$$

For $3 \to 1$

$$\begin{aligned}P_{31}(\tau) = \langle 3|A_\tau|1\rangle =& [\langle 2^*|1^*\rangle - \mu\langle 3^*|1^*\rangle]\langle 1^*|1^*\rangle + [\langle 2^*|2^*\rangle - \mu\langle 3^*|2^*\rangle]\langle 2^*|1^*\rangle \\ &+ [\langle 2^*|3^*\rangle - \mu\langle 3^*|3^*\rangle]\langle 3^*|1^*\rangle \\ =& 0\end{aligned}$$

For $3 \to 2$

$$\begin{aligned}
P_{32}(\tau) = \langle 3|A_\tau|2\rangle =& \frac{1}{\Delta}[\langle 2^*|1^*\rangle - \mu\langle 3^*|1^*\rangle][\mu\langle 1^*|2^*\rangle + \langle 1^*|3^*\rangle] \\
&+ \frac{1}{\Delta}[\langle 2^*|2^*\rangle - \mu\langle 3^*|2^*\rangle][\mu\langle 2^*|2^*\rangle + \langle 2^*|3^*\rangle] \\
&+ \frac{1}{\Delta}[\langle 2^*|3^*\rangle - \mu\langle 3^*|3^*\rangle][\mu\langle 3^*|2^*\rangle + \langle 3^*|3^*\rangle] \\
=& \frac{1}{\Delta}\mu\exp(\tau\mu) - \frac{1}{\Delta}\mu\exp(\tau(2\mu+\lambda)) = \frac{\mu\exp(\tau\mu)}{\Delta}(1 - \exp(\tau\Delta))
\end{aligned}$$

For $3 \to 3$

$$\begin{aligned}
P_{33}(\tau) = \langle 3|A_\tau|3\rangle =& \frac{1}{\Delta}[\langle 2^*|1^*\rangle - \mu\langle 3^*|1^*\rangle][\lambda\langle 1^*|2^*\rangle - \langle 1^*|3^*\rangle] \\
&+ \frac{1}{\Delta}[\langle 2^*|2^*\rangle - \mu\langle 3^*|2^*\rangle][\lambda\langle 2^*|2^*\rangle - \langle 2^*|3^*\rangle] \\
&+ \frac{1}{\Delta}[\langle 2^*|3^*\rangle - \mu\langle 3^*|3^*\rangle][\lambda\langle 3^*|2^*\rangle - \langle 3^*|3^*\rangle] \\
=& \frac{1}{\Delta}\lambda\exp(\tau\mu) + \frac{1}{\Delta}\mu\exp(\tau(2\mu+\lambda)) = \frac{\exp(\tau\mu)}{\Delta}(\lambda + \mu\exp(\tau\Delta))
\end{aligned}$$

6.4 Tensor Product

Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces with different degrees of freedom. Tensor product space is denoted by $\mathcal{V} \otimes \mathcal{W}$ [6] [26]. For ket vector $|x\rangle \in \mathcal{V}$ and $|y\rangle \in \mathcal{W}$, tensor product is defined by [6] [26]

$$|x\rangle \otimes |y\rangle = |x\rangle|y\rangle$$

If ket vector $|x\rangle \in \mathcal{V}_N$ is N-dimensional and and $|y\rangle \in \mathcal{W}_M$ is M-dimensional, tensor product space $\mathcal{V}_N \otimes \mathcal{W}_M$ is MN-dimensional vector.[6] [26]

Let

$$|x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad , \quad |y\rangle = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

be two ket vectors. Tensor product of ket vectors $|x\rangle$ and $|y\rangle$ is defined by multiplying of each elements in ket vector $|x\rangle$ into all elements in ket vector $|y\rangle$. Tensor product vector is denoted by [6] [26]

$$|x\rangle \otimes |y\rangle = |x\rangle|y\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}_N \otimes \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}_M = \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ \vdots \\ x_1y_M \\ \vdots \\ x_Ny_1 \\ x_Ny_2 \\ \vdots \\ x_Ny_M \end{pmatrix}_{NxM}$$

6.4.1 Tensor Product of a Matrix

Let

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NN} \end{bmatrix}_{NxN} \quad \text{and} \quad B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{M1} & x_{M2} & \cdots & x_{MM} \end{bmatrix}_{MxM}$$

be two matrices on spaces $\mathcal{V}$ and $\mathcal{W}$, respectively. The tensor product of the matrix A and B is denoted by [6] [9] [26]

$$A \otimes B = \langle x'|\langle y'|A \otimes B|y\rangle|x\rangle = \langle x'|A|x\rangle \otimes \langle y'|B|y\rangle$$

The matrix elements of tensor product matrix $A \otimes B$ are defined as follows: [6] [9] [26]

$$A \otimes B = \begin{bmatrix} x_{11}\begin{bmatrix} y_{11} & \cdots & y_{1M} \\ \vdots & \ddots & \vdots \\ y_{M1} & \cdots & y_{MM} \end{bmatrix} & \cdots & \cdots & x_{1N}\begin{bmatrix} y_{11} & \cdots & y_{1M} \\ \vdots & \ddots & \vdots \\ y_{M1} & \cdots & y_{MM} \end{bmatrix} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{N1}\begin{bmatrix} y_{11} & \cdots & y_{1M} \\ \vdots & \ddots & \vdots \\ y_{M1} & \cdots & y_{MM} \end{bmatrix} & \cdots & \cdots & x_{NN}\begin{bmatrix} y_{11} & \cdots & y_{1M} \\ \vdots & \ddots & \vdots \\ y_{M1} & \cdots & y_{MM} \end{bmatrix} \end{bmatrix}_{(NxM)x(NxM)}$$

$$= \begin{bmatrix} x_{11}y_{11} & \cdots & x_{11}y_{1M} & \cdots & \cdots & x_{1N}y_{11} & \cdots & x_{1N}y_{1M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ x_{11}y_{M1} & \cdots & x_{11}y_{MM} & \cdots & \cdots & x_{1N}y_{M1} & \cdots & x_{1N}y_{MM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{N1}y_{11} & \cdots & x_{N1}y_{1M} & \cdots & \cdots & x_{NN}y_{11} & \cdots & x_{NN}y_{1M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ x_{N1}y_{M1} & \cdots & x_{N1}y_{MM} & \cdots & \cdots & x_{NN}y_{M1} & \cdots & x_{NN}y_{MM} \end{bmatrix}_{(NxM)x(NxM)}$$

6.4.2 Tensor Product of Operators

Let x and y be two initial state and x' and y' be the final points respectively. The operator is denoted by [6]

$$\begin{aligned}\langle x \otimes y|A \otimes B|x' \otimes y'\rangle =&(x \otimes y)^T(A \otimes B)(x' \otimes y')\\ =&(x^T \otimes y^T)(A \otimes B)(x' \otimes y')\end{aligned}$$

By the property of tensor product [6]

$$\begin{aligned}\langle x \otimes y|A \otimes B|x' \otimes y'\rangle =&[(x^T A \otimes y^T B)](x' \otimes y')\\ =&(x^T Ax' \otimes y^T By')\end{aligned}$$

Then, we get [6]

$$\langle x \otimes y|A \otimes B|x' \otimes y'\rangle = \langle x|A|x'\rangle \otimes \langle y|B|y'\rangle$$

We know that $\langle x|A|x'\rangle$ and $\langle y|B|y'\rangle$ are numbers.

Hence, the operator is derived by [6]

$$\langle x \otimes y|A \otimes B|x' \otimes y'\rangle = \langle x|A|x'\rangle\langle y|B|y'\rangle$$

Fact 1: Let A, B, C and D be matrix. Multiplication of tensor products $A \otimes B$ and $C \otimes D$ is denoted by

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

Proof of fact 1: From the definition of tensor product, we know that

$$(A_{ij}) \otimes (B_{km}) = (A \otimes B)_{(ik)(jm)} = A_{ij}B_{km}$$

In the new labels (ik) and (km), the first index is taken from matrix A and the second one is taken from matrix B. By definition of matrix multiplication, we get

$$\begin{aligned}((A \otimes B)(C \otimes D)) &= \sum (A \otimes B)_{(ik)(jm)}(C \otimes D)_{(jm)(st)} \\ &= \sum_{jm} A_{ij} B_{km} C_{js} D_{mt}\end{aligned}$$

Then, we separate them as follows

$$\begin{aligned}((A \otimes B)(C \otimes D)) =& (\sum_{j} A_{ij} C_{js})(\sum_{m} B_{km} D_{mt}) \\ =& (AC)_{is}(BD)_{kt} = (AC \otimes BD)_{(ik)(st)}\end{aligned}$$

Fact 2: If the notation $Ax = A|x\rangle$, then

$$(A \otimes B)|C \otimes D\rangle = (A|C\rangle) \otimes (B|D\rangle)$$

Now, if we generalize our system for initial states $\{x_1, \ldots, x_n\}$, final points $\{x_1{}', \ldots, x_n{}'\}$ and set of matrix $\{A_1, \ldots, A_n\}$, the operator is defined by

$$\begin{aligned}\langle x_1 \otimes \cdots \otimes x_n | A_1 \otimes \cdots \otimes A_n | x_1{}' \otimes \cdots \otimes x_n{}' \rangle =& \langle x_1 | A | x_1{}' \rangle \otimes \cdots \otimes \langle x_n | A | x_n{}' \rangle \\ =& \prod_{i=1}^{n} \langle x_i | A | x_i{}' \rangle\end{aligned}$$

Example 1: Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be two matrix, vectors $x = y = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the initial states and vectors

$x' = y' = |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the final points. To calculate the tensor product argument of operator, we apply the tensor product for the matrix and vectors.

$$(x \otimes y)^T = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad (x' \otimes y')^T = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$A \otimes B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{22} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Then, by simple algebraic calculations

$$\langle x \otimes y | A \otimes B | x' \otimes y' \rangle = (x \otimes y)^T (A \otimes B)(x' \otimes y') = a_{12}b_{12}$$

and by making matrix calculations, we get

$$\langle x|A|x' \rangle = a_{12} \quad \text{and} \quad \langle y|B|y' \rangle = b_{12}$$

So

$$\langle x|A|x' \rangle \langle y|B|y' \rangle = a_{12}b_{12}$$

Hence, the notations are equal

$$\langle x \otimes y | A \otimes B | x' \otimes y' \rangle = \langle x|A|x' \rangle \langle y|B|y' \rangle$$

Probability Argument: Let x_i be initial states, $x_i{}'$ be the final states and A_i is a semi-group. Probability argument is denoted by

$$\langle x_i|A_i|x_i{}'\rangle = P(Z_i{}' = x_i{}'|Z_i = x_i)$$

where Z_i are independent.

$$\begin{aligned}\langle x_i \otimes \cdots \otimes x_n|A_i \otimes \cdots \otimes A_n|x_i{}' \otimes \cdots \otimes x_n{}'\rangle =& P(Z_1{}' = x_1{}', \ldots, Z_n{}' = x_n{}'|Z_1 = x_1, \ldots, Z_n = x_n)\\ =& \frac{P(\cap\{Z_i{}' = x_i{}'\}\ ,\ \cap\{Z_i = x_i\})}{P(\cap\{Z_i = x_i\})}\end{aligned}$$

From independence

$$\begin{aligned}\langle x_i \otimes \cdots \otimes x_n|A_i \otimes \cdots \otimes A_n|x_i{}' \otimes \cdots \otimes x_n{}'\rangle &= \prod_{i=1}^{n} P(Z_i{}' = x_i{}', Z_i = x_i)\\ &= \prod_{i=1}^{n} \langle x_i|A_i|x_i{}'\rangle\end{aligned}$$

Lemma 6.4.1. *Let K_i^A and K_i^B be the eigenvalues for matrix A and B, respectively. If $H_{A\otimes B} = I_A \otimes H_B + H_A \otimes I_B$*

$$H_{A\otimes B}|i \otimes j\rangle = K_\sigma|\sigma\rangle$$

where $K_\sigma = K_j^B + K_i^A$ and $\sigma = |i \otimes j\rangle = |i\rangle \otimes |j\rangle$.

Proof: If K_i^A and K_j^B are eigenvalues, we say that

$$H_A|i\rangle = K_i^A|i\rangle \quad \text{and} \quad H_B|j\rangle = K_j^B|j\rangle$$

Using the notation $H_{A\otimes B} = I_A \otimes H_B + H_A \otimes I_B$, we get

$$H_{A\otimes B}|i\otimes j\rangle = I_A \otimes H_B|i\otimes j\rangle + H_A \otimes I_B|i\otimes j\rangle$$

From the properties of tensor product

$$\begin{aligned} H_{A\otimes B}|i\otimes j\rangle =& I_A|i\rangle \otimes H_B|j\rangle + H_A|i\rangle \otimes I_B|j\rangle \\ =& |i\rangle \otimes K_j^B|j\rangle + K_i^A|i\rangle \otimes |j\rangle \end{aligned}$$

Hence

$$H_{A\otimes B}|i\otimes j\rangle = (K_j^B + K_i^A)|i\otimes j\rangle = K_\sigma|\sigma\rangle$$

Lemma 6.4.2. *Let K_i^A, K_j^B and K_k^C be the eigenvalues for matrix A, B and C, respectively. If $H_{A\otimes B\otimes C} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C$*

$$H_{A\otimes B\otimes C}|i\otimes j\otimes k\rangle = K_\sigma|\sigma\rangle$$

where $K_\sigma = K_i^A + K_j^B + K_k^C$ and $\sigma = |i\otimes j\otimes k\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle$.

This lemma is similar to *Lemma* 6.4.1. We prove it here for better understanding.
Proof: If K_i^A, K_j^B and K_k^C are eigenvalues, we say that

$$H_A|i\rangle = K_i^A|i\rangle \quad , \quad H_B|j\rangle = K_j^B|j\rangle \quad , \quad H_C|k\rangle = K_k^C|k\rangle$$

Using the notation $H_{A\otimes B\otimes C} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C$, we get

$$\begin{aligned} H_{A\otimes B\otimes C}|i\otimes j\otimes k\rangle =& H_A \otimes I_B \otimes I_C|i\otimes j\otimes k\rangle + I_A \otimes H_B \otimes I_C|i\otimes j\otimes k\rangle \\ &+ I_A \otimes I_B \otimes H_C|i\otimes j\otimes k\rangle \end{aligned}$$

From the properties of tensor product

$$\begin{aligned}H_{A\otimes B\otimes C}|i\otimes j\otimes k\rangle =& H_A|i\rangle\otimes I_B|j\rangle\otimes I_C|k\rangle + I_A|i\rangle\otimes H_B|j\rangle\otimes I_C|k\rangle + I_A|i\rangle\otimes I_B|j\rangle\otimes H_C|k\rangle\\ =& H_A|i\rangle\otimes|j\rangle\otimes|k\rangle + |i\rangle\otimes H_B|j\rangle\otimes|k\rangle + |i\rangle\otimes|j\rangle\otimes H_C|k\rangle\end{aligned}$$

Hence

$$H_{A\otimes B\otimes C}|i\otimes j\otimes k\rangle = (K_i^A + K_j^B + K_k^C)|i\otimes j\otimes k\rangle = K_\sigma|\sigma\rangle$$

Lemma 6.4.3. *By induction, we generalize it as follows*

$$\begin{aligned}H_A|i_1\otimes\cdots\otimes i_n\rangle =& (\sum_{j=1}^n K_{i_j}^{A_j})|i_1\otimes\cdots\otimes i_n\rangle\\ =& K_\sigma|\sigma\rangle\end{aligned}$$

where $H_{A_j}|i_j\rangle = K_{i_j}^{A_j}|i_j\rangle$ *and*

$$H_A = H_{(A_1\otimes\cdots\otimes A_n)} = \sum_{j=1}^n I_{A_1}\otimes\cdots\otimes I_{A_{j-1}}H_{A_j}I_{A_{j+1}}\otimes\cdots\otimes I_{A_n}$$

Then, the operator notation is derived by

$$\langle x|\exp(-\tau H)|x'\rangle = \sum_\sigma\langle x|\exp(-\tau H)|\sigma\rangle\langle\sigma|x'\rangle$$

where $x = x_1\otimes\cdots\otimes x_n$ are initial states, $x = x_1{}'\otimes\cdots\otimes x_n{}'$ are final points and $H|\sigma\rangle = K|\sigma\rangle$. So

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =& \sum_\sigma\exp(-\tau K_\sigma)\langle x|\sigma\rangle\langle\sigma|x'\rangle\\ =& \sum\prod_{j=1}^n\exp(-\tau K_{i_j}^{A_j})\langle x_j|i_j\rangle\langle i_j|x_j{}'\rangle\end{aligned}$$

From properties of sum and product function, we derive

$$\langle x|\exp(-\tau H)|x'\rangle = \prod_{j=1}^{n}\sum_{j}\exp(-\tau K_{i_j}^{A_j})\langle x_j|i_j\rangle\langle i_j|x_j{}'\rangle$$
$$= \prod_{j=1}^{n}\langle x_j|\exp(-\tau H_j)|x_j{}'\rangle$$

Lemma 6.4.4. *Let* $H_1 = \begin{pmatrix} \lambda_1 & -\lambda_1 \\ -\mu_1 & \mu_1 \end{pmatrix}$ *and* $H_2 = \begin{pmatrix} \lambda_2 & -\lambda_2 \\ -\mu_2 & \mu_2 \end{pmatrix}$ *be two real matrices and*

$$A_t = \exp(-tH) = \exp(-tH_1) \otimes \exp(-tH_2)$$
$$= \exp(-t(H_1 \otimes I_2 + I_1 \otimes H_2))$$

be generated semigroup. Then the elements of non-ruin probability matrix are computed by algebraic calculation as seen in the proof.

Here, we construct a new lemma for tensor product approach of Hamiltonian method and the result is included in [39].

Proof: By using the notation $H = H_1 \otimes I_2 + I_1 \otimes H_2$, the matrix H is found by

$$H = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_2 & -\lambda_1 & 0 \\ -\mu_2 & \lambda_1+\mu_2 & 0 & -\lambda_1 \\ -\mu_1 & 0 & \mu_1+\lambda_2 & -\lambda_2 \\ 0 & -\mu_1 & -\mu_2 & \mu_1+\mu_2 \end{bmatrix}$$

Let $|1^*\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $|2^*\rangle = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ -\mu_1 \\ -\mu_1 \end{pmatrix}$, $|3^*\rangle = \begin{pmatrix} \lambda_2 \\ -\mu_2 \\ \lambda_2 \\ -\mu_2 \end{pmatrix}$ and $|4^*\rangle = \begin{pmatrix} -\lambda_1\lambda_2 \\ \lambda_1\mu_2 \\ \lambda_2\mu_1 \\ -\mu_1\mu_2 \end{pmatrix}$ be eigenvectors of matrix H for corresponding eigenvalues $K_1 = 0,\ K_2 = \lambda_1 + \mu_1,\ K_3 =$

$\lambda_2+\mu_2$ and $K_4 = \lambda_1+\mu_1+\lambda_2+\mu_2$, respectively. For simplicity, we use $\Delta_1 = \lambda_1+\mu_1$ and $\Delta_2 = \lambda_2 + \mu_2$ in our calculations.

Ket vectors $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$ are calculated in the new coordinates as follows

$$
\begin{aligned}
|1\rangle = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T &= \frac{\mu_1\mu_2}{\Delta_1\Delta_2}|1^*\rangle + \frac{\mu_2}{\Delta_1\Delta_2}|2^*\rangle + \frac{\mu_1}{\Delta_1\Delta_2}|3^*\rangle - \frac{1}{\Delta_1\Delta_2}|4^*\rangle \\
|2\rangle = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T &= \frac{\mu_1\lambda_2}{\Delta_1\Delta_2}|1^*\rangle + \frac{\lambda_2}{\Delta_1\Delta_2}|2^*\rangle + -\frac{\mu_1}{\Delta_1\Delta_2}|3^*\rangle + \frac{1}{\Delta_1\Delta_2}|4^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
|3\rangle = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^T &= \frac{\lambda_1\mu_2}{\Delta_1\Delta_2}|1^*\rangle - \frac{\mu_2}{\Delta_1\Delta_2}|2^*\rangle + \frac{\lambda_1}{\Delta_1\Delta_2}|3^*\rangle + \frac{1}{\Delta_1\Delta_2}|4^*\rangle \\
|4\rangle = \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^T &= \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}|1^*\rangle - \frac{\lambda_2}{\Delta_1\Delta_2}|2^*\rangle - -\frac{\lambda_1}{\Delta_1\Delta_2}|3^*\rangle - \frac{1}{\Delta_1\Delta_2}|4^*\rangle
\end{aligned}
$$

To change the coordinates of vectors, we take matrix

$$
\Sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_1 & -\mu_1 & -\mu_1 \\ \lambda_2 & -\mu_2 & \lambda_2 & -\mu_2 \\ -\lambda_1\lambda_2 & \lambda_1\mu_2 & \lambda_2\mu_1 & -\mu_1\mu_2 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 & -\mu_2 & \lambda_1\mu_2 \\ 1 & -\mu_1 & \lambda_2 & \lambda_2\mu_1 \\ 1 & -\mu_1 & -\mu_2 & -\mu_1\mu_2 \end{pmatrix}
$$

Then, if we apply the matrix to our ket vector $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$, we get

$$
\begin{aligned}
\Sigma 1 &= |1^*\rangle + \lambda_1|2^*\rangle + \lambda_2|3^*\rangle - \lambda_1\lambda_2|4^*\rangle \\
\Sigma 2 &= |1^*\rangle + \lambda_1|2^*\rangle + \lambda_2|3^*\rangle - \lambda_1\lambda_2|4^*\rangle \\
\Sigma 3 &= |1^*\rangle - \mu_1|2^*\rangle + \lambda_2|3^*\rangle\lambda_2\mu_1|4^*\rangle \\
\Sigma 4 &= |1^*\rangle - \mu_1|2^*\rangle - \mu_2|3^*\rangle - \mu_1\mu_2|4^*\rangle
\end{aligned}
$$

Then, the operator in the new coordinates is defined by

$$
\langle x|\exp(-\tau H)|x'\rangle = \langle x|\exp(-\tau H)|\Sigma x'\rangle_*
$$

So

$$\langle x|\exp(-\tau H)|x'\rangle = \langle \Sigma x|\exp(-\tau H)|1^*\rangle_* \langle 1^*|x'\rangle_* + \langle \Sigma x|\exp(-\tau H)|2^*\rangle_* \langle 2^*|x'\rangle_* + \langle \Sigma x|\exp(-\tau H)|3^*\rangle_* \langle 3^*|x'\rangle_* + \langle \Sigma x|\exp(-\tau H)|4^*\rangle_* \langle 4^*|x'\rangle_*$$

By applying eigenvectors to the matrix H, we find

$$H|1^*\rangle = 0|1^*\rangle \ , \ H|2^*\rangle = \Delta_1|2^*\rangle \ , \ H|3^*\rangle = \Delta_2|3^*\rangle \ , \ H|4^*\rangle = (\Delta_1+\Delta_2)|4^*\rangle$$

then, the operator is derived by

$$\langle x|\exp(-\tau H)|x'\rangle = \langle \Sigma x|1^*\rangle_* \langle 1^*|x'\rangle_* + \exp(-\tau\Delta_1)\langle \Sigma x|2^*\rangle_* \langle 2^*|x'\rangle_* + \exp(-\tau\Delta_2)\langle \Sigma x|3^*\rangle_* \langle 3^*|x'\rangle_* + \exp(-\tau(\Delta_1+\Delta_2))\langle \Sigma x|4^*\rangle_* \langle 4^*|x'\rangle_*$$

Hence, the probabilities are calculated as follows:

For $1 \mapsto 1$

$$P_{11}(\tau) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $1 \mapsto 2$

$$P_{12}(\tau) = \frac{\mu_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $1 \mapsto 3$

$$P_{13}(\tau) = \frac{\lambda_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $1 \mapsto 4$

$$P_{14}(\tau) = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $2 \mapsto 1$

$$P_{21}(\tau) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $2 \mapsto 2$

$$P_{22}(\tau) = \frac{\mu_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $2 \mapsto 3$

$$P_{23}(\tau) = \frac{\lambda_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $2 \mapsto 4$

$$P_{24}(\tau) = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2}$$

For $3 \mapsto 1$

$$P_{31}(\tau) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$

For $3 \mapsto 2$

$$P_{32}(\tau) = \frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$

For $3 \mapsto 3$

$$P_{33}(\tau) = \frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$

For $3 \mapsto 4$

$$P_{34}(\tau) = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\lambda_2\mu_1}{\Delta_1\Delta_2}$$

For $4 \mapsto 1$

$$P_{41}(\tau) = \frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$

For $4 \mapsto 2$

$$P_{42}(\tau) = \frac{\lambda_2\mu_1}{\Delta_1\Delta_2} - \exp(-\tau\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$

For $4 \mapsto 3$

$$P_{43}(\tau) = \frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\mu_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} - \exp(-\tau(\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$

For $4 \mapsto 4$

$$P_{44}(\tau) = \frac{\lambda_1\lambda_2}{\Delta_1\Delta_2} + \exp(-\tau\Delta_1)\frac{\lambda_2\mu_1}{\Delta_1\Delta_2} + \exp(-\tau\Delta_2)\frac{\lambda_1\mu_2}{\Delta_1\Delta_2} + \exp(-\tau(\Delta_1+\Delta_2))\frac{\mu_1\mu_2}{\Delta_1\Delta_2}$$

The sum of rows are 1.

6.5 $R_t = u + bN_t$

Let $R_t = u + bN_t$ be a surplus process with initial capital u and premium rate $C = 0$ where $N_0 = 0$. Consider the semigroup $A_t = \exp(-tH)$ which satisfies the semigroup property $\lim\limits_{t\mapsto 0} \frac{A_0 - A_t}{t} = H$ where H is a Hamiltonian matrix. By using the operator statement $A_t f(u) = E[f(R_t)|R_0 = u]$, Hf is found by

$$\lim_{t\mapsto 0}\frac{f(u)-Ef(u+bN_t)}{t} =$$
$$\lim_{t\mapsto 0}\left[\frac{f(u)-[f(u)P(N_t=0)+f(u+b)P(N_t=1)]}{t}+\frac{o(P(N_t\geq 2))}{t}+\frac{\mathcal{O}(t^2)}{t}\right]$$
$$=\lim_{t\mapsto 0}\frac{f(u)-[f(u)\exp(-\lambda t)-f(u+b)\exp(-\lambda t)\lambda t]}{t}$$
$$=\lambda f(u)-\lambda f(u+b)=Hf(u)$$

Hence, the acted operator is defined by $Hf = \lambda\Delta_b f$ where Δ is a backward equation.

Now, assume $f(k) = |p\rangle = \exp(ipk)$ where variable k is an integer. Then, for $b = -1$, we follow the four steps:

Step 1: As calculated in the previous statement

$$Hf(y) = \lambda\Delta_{y-1}f = \lambda[f(y)-f(y-1)]$$

and by using the assumption $f(k) = \exp(ipk)$, we get

$$Hf(y) = \lambda(\exp(ipy)-\exp(ip(y-1))) = \lambda\exp(ipy)(1-\exp(-ip)) = K_p|p\rangle$$

where $K_p = \lambda(1-\exp(-ip))$.

Step 2: Applying ket vector $|p\rangle$ to the matrix H, we find that

$$H|p\rangle = K|p\rangle \;\Rightarrow\; H^j|p\rangle = K^j|p\rangle$$

and so

$$\exp(-tH)|p\rangle = \exp(-tK_p)|p\rangle$$

Step 3: Let $x = u$ be the initial capital and $x' = m$ be the capital value at maturity time and assume that $\langle u|p\rangle = \exp(ipu)$ and $\langle p|x'\rangle = \langle m|p\rangle = \exp(-ipm)$. The transition probabilities are

$$\begin{aligned}
\langle x|\exp(-tH)|x'\rangle =& \langle u|\exp(-tH)|m\rangle \\
=& \int_0^{2\pi} \frac{dp}{2\pi} \langle x|\exp(-tH)|p\rangle\langle p|x'\rangle \\
=& \int_0^{2\pi} \frac{dp}{2\pi} \exp(-tK_p + ip(u-m)) \\
=& P_{um}^{(t)}
\end{aligned}$$

Step 4: For a specific $K_p = \lambda(1 - \exp(-ip))$, the statement is found by

$$\begin{aligned}
&\int_0^{2\pi} \frac{dp}{2\pi} \exp(-t\lambda(1 - \exp(-ip)) + ip(u-m)) \\
&= \frac{\exp(-t\lambda)}{2\pi i} \int_\Gamma \frac{dp}{2\pi} \exp(-t\lambda(1 - \exp(-ip)) + ip(u-m))
\end{aligned}$$

where we integrate over terms $\Gamma = \big(z : z = \exp(-ip)\big)$ and by using the Cauchy-integral in complex analysis, the result is found by

$$= \exp(-t\lambda)\frac{(t\lambda)^{u-m}}{(u-m)!}$$

For $t = 1$, we get

$$P(N_t = u - m) = P(R_t = m)$$

6.6 $R_t = u + at + bN_t$

Let $R_t = u + at + bN_t$ be a surplus process with initial capital u and premium rate $C = 0$ where $N_0 = 0$. Consider the semigroup $A_t = \exp(-tH)$ which satisfies the semigroup property $\lim\limits_{t \mapsto 0} \frac{A_0 - A_t}{t} = H$ where H is a Hamiltonian matrix. By using the operator statement $A_t f(u) = E[f(R_t)|R_0 = u]$, Hf is found by

$$\lim_{t\mapsto 0} \frac{f(x) - Ef(x + at + bN_t)}{t} =$$
$$\lim_{t\mapsto 0} \left[\frac{f(x) - [f(x+at)P(N_t = 0) + f(x+at+b)P(N_t = 1)]}{t} + \frac{o(P(N_t \geq 2))}{t} + \frac{\mathcal{O}(t^2)}{t} \right]$$
$$= \lim_{t\mapsto 0} \frac{f(x) - [f(x+at)\exp(-\lambda t) - f(x+at+b)\exp(-\lambda t)\lambda t]}{t}$$
$$= \lim_{t\mapsto 0} \frac{f(x) - (atf'(x) + f(x))\exp(-\lambda t) - (atf'(x+b) + f(x+b))\exp(-\lambda t)\lambda t}{t}$$
$$= \lambda f(x) - af'(x) - \lambda f(x+b) = Hf(x)$$

Hence, our operator can be obtained as $Hf = -af' - \lambda \Delta_b f$ where Δ is a backward equation. For $a = 1$ and $b = -2$, we get

$$H = -[\lambda \Delta_{-2} f + f']$$

6.7 Up and Down Example

Consider pseudo $Q-$matrix is defined by

$$\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & \cdots \end{pmatrix}
\end{array}$$

as seen in figure. Then, acted operator is defined by $Hf = \lambda\Delta_y f - \mu\Delta_{y-1} f$ where Δ is a backward equation.

Now, assume $f(k) = |p\rangle = \exp(ipk)$ where variable $k \in \mathcal{Z}$ and operator $H|p\rangle = K_p|p\rangle$. Then, we follow next four steps:

Step 1: By using the previous statement for backward equation, we derived

$$\begin{aligned} Hf(y) =& \lambda\Delta_y f - \mu\Delta_{y-1} f = \lambda[f(y) - f(y-1)] \\ =& \lambda f(y+1) + \mu f(y-1) - (\lambda+\mu) f(y) \end{aligned}$$

and using the assumption $f(k) = \exp(ipk)$, we found that

$$\begin{aligned} Hf(y) =& \lambda(\exp(ip(y+1)) - \exp(ip(y))) - \mu(\exp(ip(y)) - \exp(ip(y-1))) \\ =& \lambda \exp(ipy)(1 - \exp(-ip)) = K_p|p\rangle \end{aligned}$$

where $K_p = \lambda(\exp(ip) - 1) - \mu(1 - \exp(-ip))$.

Step 2: Applying ket vector $|p\rangle$ to the matrix H, we get

$$H|p\rangle = K|p\rangle \;\Rightarrow\; H^j|p\rangle = K^j|p\rangle$$

and so

$$\exp(-\tau H)|p\rangle = \exp(-\tau K_p)|p\rangle$$

Step 3: Let $x = u$ be the initial capital and $x' = m$ be the capital value at maturity time and assume that $\langle u|p\rangle = \exp(ipu)$ and $\langle p|x'\rangle = \langle m|p\rangle = \exp(-ipm)$. Then, transition probabilities are

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =& \langle u|\exp(-\tau H)|m\rangle \\ =& \int_0^{2\pi} \frac{dp}{2\pi} \langle x|\exp(-\tau H)|p\rangle\langle p|x'\rangle \\ =& \int_0^{2\pi} \frac{dp}{2\pi} \exp(-\tau K_p + ip(u-m))\end{aligned}$$

<u>Step 4:</u> For a specific $K_p = \lambda(\exp(ip)-1) - \mu(1-\exp(-ip))$, the statement is derived by

$$P = \int_0^{2\pi} \frac{dp}{2\pi} \lambda(\exp(ip)-1) - \mu(1-\exp(-ip)) + ip(u-m))$$

If we substitute $\exp(-ip)$ by z, we get

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =& \langle x|\exp(-\tau H)|x'\rangle \\ =& -\frac{\exp(\tau(\lambda+\mu))}{2\pi i} \int_0^{2\pi} \frac{dp}{2\pi} \frac{g(z)}{z^{u-m+1}} dz = P_{um}^{(\tau)}\end{aligned}$$

where $g(z) = \exp(-\tau\lambda Z^{-1} - \tau\mu Z)$. So, there is no ready simple formula to calculate the final statement.

6.8 Brownian Motions

Gaussian

(I) In this part, we compute the so-called solvency probability $P_u(Rt < 0)$ where $R_t = u + \sigma B_t$ is the capital.

$$\begin{aligned}
P_u(R_t < 0) =& P(u + \sigma B_t < 0)\\
=& P\big(B_t < -\frac{u}{\sigma}\big)\\
=& P\big(\sqrt{\tau}N(0,1) < -\frac{u}{\sigma}\big)\\
=& P\big(N(0,1) < -\frac{u}{\sigma\sqrt{\tau}}\big)\\
=& \psi\big(-\frac{u}{\sigma\sqrt{\tau}}\big)
\end{aligned}$$

$(II) - Computation$: Let H be an operator on a Hilbert space $L_2(\mathbb{R})$, $x \in \mathbb{R}$ and $|p\rangle = exp(ipx)$.

$$P^{(\tau)}_{xx'} = \langle x|\exp(-\tau H)|x'\rangle H|p\rangle = K_p|p\rangle \;\mapsto\; \exp(-\tau K_p)|p\rangle$$

Then

$$\begin{aligned}
\langle x|\exp(-\tau H)|x'\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\langle x|\exp(-\tau H)|p\rangle\langle p|x'\rangle\\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\langle x|p\rangle\langle p|x'\rangle\exp(-\tau K_p)\\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\exp(-\tau K_p + ip(x - x'))
\end{aligned}$$

where $\langle x|p\rangle = \exp(ipx)$ and $\langle p|x'\rangle = \langle x'|p\rangle = \exp(-ipx')$.

(III) Let $A_t = \exp(-tH)$ be a semigroup. Then

$$\begin{aligned}\frac{I - A_t}{t} f(x) =& \frac{1}{t}[f(x) - E[f(R_t)|R_0 = x]] \\ =& \frac{1}{t}[f(x) - E[f(x + \sigma B_t)]]\end{aligned}$$

If we assume $f(x + \sigma B_t) = G(B_t)$ and using Ito Calculus

$$\begin{aligned}dG(B_t) =& G_B dB_t + \frac{1}{2} G''_{BB} dt \\ G(B_t) =& G(0) + \int_0^t G'_B(B_u) dB_u + \int_0^t \frac{1}{2} G''_{BB}(B_u) du \\ E[G(B_t)] =& G(0) + \underbrace{E(\int_0^t G'_B(B_u) dB_u)}_{E(Ito)=0} + E(\int_0^t \frac{1}{2} G''_{BB}(B_u) du) \\ E[f(x + \sigma B_t)] =& f(x) + \frac{1}{2} \int_0^t E(G''_{BB}(B_u)) du\end{aligned}$$

Afterwards we return the previous equation and put this result there, we get

$$\begin{aligned}&\frac{1}{t}[f(x) - f(x) - \frac{1}{2} \int_0^t E(G''_{BB}(B_u)) du] \\ =& -\frac{1}{2} \cdot \frac{1}{t} \int_0^t E(G''_{BB}(B_u)) du \\ =& -\frac{1}{2} G''(0) = -\frac{1}{2} \sigma^2 f''(x)\end{aligned}$$

where $t \mapsto 0$, $B_u \mapsto 0$ and $G''(y) = \sigma^2 f''(x + \sigma y)$. Hence,

$$\begin{aligned}Hf(x) &= -\frac{\sigma^2}{2} f''(x) \\ H|p\rangle &= -\frac{\sigma^2}{2} (\exp(ipx))'' \\ &= \frac{\sigma^2 p^2}{2} |p\rangle = K_p\end{aligned}$$

where $K_p = \sigma^2 p^2/2$.

(IV) For this specific K_p, we get

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle &= P^{\tau}_{x,x'}\\ I_1 &= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\exp(-\tau(\frac{\sigma^2 p^2}{2})) + ip(x-x'))\end{aligned}$$

If we substitute $y = p\sigma\sqrt{\tau}$, we found

$$\begin{aligned}I_1 = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\exp(-\frac{y^2}{2} - iyu)dy\\ = \exp(\frac{(iu)^2}{2})\frac{1}{2\pi}\frac{1}{\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\exp(-\frac{1}{2}(y+iu)^2)dy\end{aligned}$$

By substitution $z = y + iu$, the next term is derived by

$$\begin{aligned}I_1 =& \exp\left(-\frac{(x'-x)^2}{2(\sigma\sqrt{\tau})^2}\right)\frac{1}{2\pi}\frac{1}{\sigma\sqrt{\tau}}\int\exp(-\frac{z^2}{2})dz\\ =& \exp\left(-\frac{(x'-x)^2}{2(\sigma\sqrt{\tau})}\right)\frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{\tau}}\\ =& \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(-\frac{(x'-x)^2}{\sqrt{2\sigma^2\tau}}\right)\end{aligned}$$

from the integral $\int_{-\infty}^{\infty}\exp(\frac{-x^2}{2})dx = \sqrt{2\pi}$.

Chapter 7

Models

In this chapter, our aim is to find optimal premium which guarantees that the ruin probability is smaller than or equal to 5%. In simulations, this means that the frequency of sample paths with ruin does not surpass 5%.

We get the data from the real situation of four happened accidents in 2012 with the claim arrival times $T_k = [3/12\ ,\ 5/12\ ,\ 8/12\ ,\ 11/12]$ in one year time period. Because of that we study the finite non-ruin probability for one year time period in this chapter. By using these data, the time of occurrence is modeled as a Poisson process with parameter $\lambda = 4$ and claims are modeled as exponential with parameter μ which is estimated by maximum likelihood estimation. Then, the ruin probability is calculated by the surplus process $R_{k+1} = R_k + C - X_k$ where C is the premium price, X_t are claims and $\Delta_t = T_{k+1} - T_k$ is the iid inter-arrival time. Our objectives is to minimize the premium C. To find optimal premium, we assume $C = 1$ is a fixed premium price at the beginning and initial capital is $R_0 = u = 5$. The surplus process is computed a hundred times to find how many of R_k are in ruin at the stopping time. When the frequency of ruin is over 5%, premium C will be increased by 0.1. This process is done until frequency of ruin is smaller than or equal to 5%. When the premium C is 1.3, the frequency of ruin does not surpass 5% in the simulation.

All the insurance models constructed in this chapter are relatively new up to our knowledge.
See the introduction for the related literature.

7.1 Ruin Probability with Dependent Claims

Now, our main objective is finding the ruin probability with using the various level of dependence under the condition that frequency of ruin is smaller than or equal 5%. In this chapter, modeling of dependence structure is constructed and copulas are used to make a dependence amongst the claims.

7.1.1 Numerical Simulations

To generate multivariate outcomes for Frank copula, frailty model for discrete logarithmic random variables with parameter $1-\eta$ was constructed by Marshall and Olkin [49] [74]. Also, an algorithm is introduced by Devroye [21] to generate r from the frailty distribution as follows [74]

- For Frank copula, take the parameter as $1-\eta$.
- Generate $x=(x_1,x_2,\ldots,x_n)^T$ where $x_i \in U[0,1]$ are iid for $i=1,\ldots,n$
- The new x^* is defined by [74]

$$x^* = M_Z(r^{-1}logx)$$

where $logx=(logx_1,\ldots,logx_n)^T$ and

$$M_Z(t) = \frac{log(1-(1-\eta)\exp(t))}{log(\eta)}$$

is the moment generating function.

- Then $z = (F_1^{-1}(x_1^*), \ldots, F_n^{-1}(x_n^*))$ where F_i^{-1} are the inverses of the marginal distribution function.

As a starting point, initial surplus and premium price are introduced as u and C, respectively. Claims X_i are generated as exponential random variables. To construct the model, we specifically used Frank copula method with regards to chosen parameter $\eta = 0.3,\ 0.5,\ 0.8$. To find optimal premium, simulation processes repeated 100 times by using the Matlab program.
In our study, we show the effect of copula on the ruin probability and time of ruin. Firstly, we illustrate the effect of different initial capitals for ruin probability with various dependence level as presented in the Figure 7.1.
Figure 7.1 shows that being at ruin decreases and reaches to almost zero at some

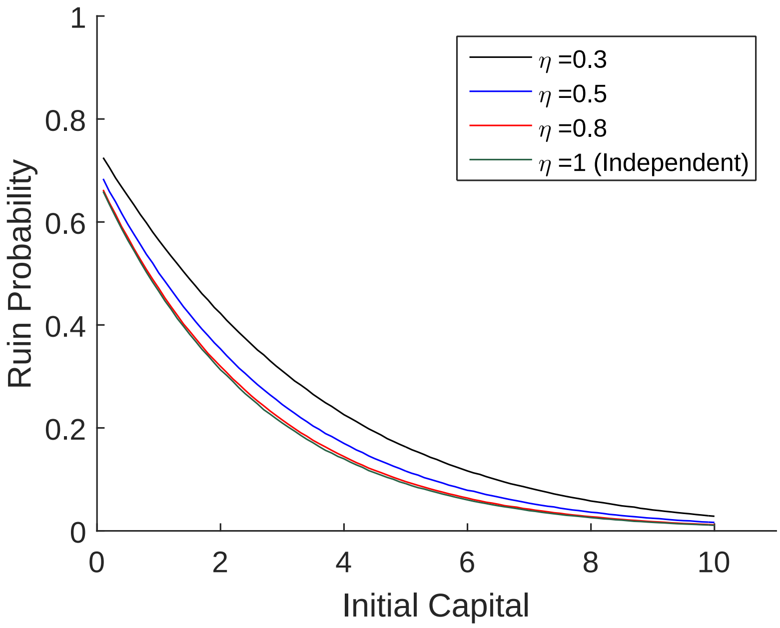


Figure 7.1: Plot of ruin probability against initial capital for some different dependence levels of claims with Frank copula.

point when we increase the initial capital. In other words, being at ruin is inverse proportional to how large initial capital gained from customer [74]. In addition, dependence structure also effects the being at ruin for different size of initial capitals. As illustrated in the Figure 7.1, it can be said that when a dependence structure placed on the claim occurrence, the probability of ruin increases. Hence, it can be reached that, the probability of ruin and level of dependence increases proportionally [74].

Secondly, effect of dependence structure on the time to ruin is investigated over different sets of initial capital. Time of ruin tends to decrease as the initial capital decreases and it tends to increases when dependency level decreases. Also, time to ruin is gradually growing where initial capital is increasing as presented in the Figure 7.2. The time of ruin limited to 1 in the this figure. Because we study the finite non-ruin probability for one year time period in this chapter.

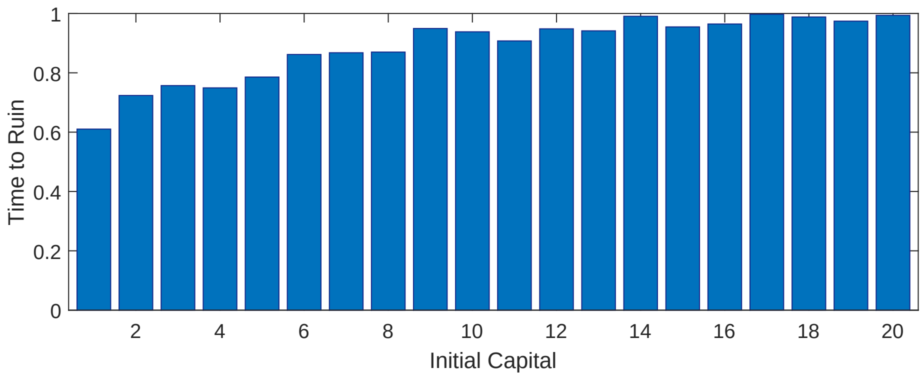


Figure 7.2: Plot of time to ruin against initial capital for independent claims in one year time period.

7.2 Modeling with Interest Rate

We start by plotting a typical sample path behaviour of the modified surplus process where the capital is increasing with the interest rate during the time between jumps (claim payments).

Assume money in at time t is $C\delta t$ and money out at time T_k is claim X_k where δ is an interest rate. The profit between starting point $t = 0$ and any point $t = m$

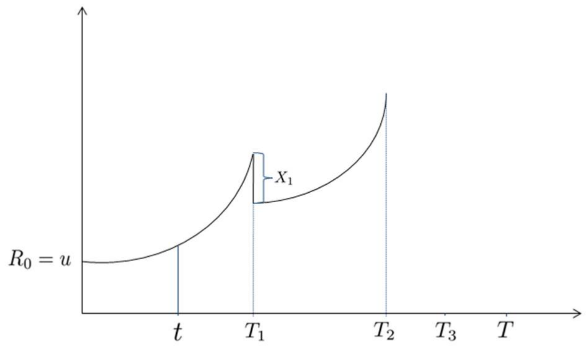


Figure 7.3: Surplus process with interest rate.

is integrated by

$$\int_{t=0}^{t=m} C\exp(\rho(m-t))dt \;=\; C\frac{\exp(\rho(m-t))}{-\rho}\int_{t=0}^{t=m} \;=\; \frac{C(\exp(\rho m)-1)}{\rho}$$

Let $R_0 = u$ be the initial capital. When the first claim occurs, our capital is defined by

$$R_1 = u\exp(\rho(T_1 - T_0)) + C\frac{(\exp(\rho(T_1 - T_0)) - 1)}{\rho} - X_1$$

In general, the capital is calculated by

$$R_k = R_{k-1}\exp(\rho(T_k - T_{k-1})) + C\frac{(\exp(\rho(T_k - T_{k-1})) - 1)}{\rho} - X_k$$

Now, we introduce four different models for interest rate.

7.2.1 Invest once at the beginning

Here, we construct a well-known model to find optimal premium and the result is included in [39].

In this model, we divide our capital into a two different amount: invested (u_1) and not invested(u_0). We assume that there is no possibility to access invested money at any time till its maturity date. Then we define the surplus process for accessible and non-accessible parts as follows:

Accessible amount (u_0):

$$R_k^{(1)} = (R_{k-1}^{(1)} + C(T_k - T_{k-1})) - X_k$$

where $u_0 = R_0$ is the not invested part of initial capital, C is the premium price, X_t are the claims and T_k are the inter-arrival times.

NON-Accessible amount:

$$R(t) = u_1 \exp(\rho t)$$

where u_1 is the invested part of initial capital and ρ is the interest rate. At the end of first year ($t = 1$), invested part is found

$$R(1) = u_1 \exp(\rho)$$

Oveerall, the capital is found by

$$R_k = R_k^{(1)} + R(1)$$

7.2.2 Accessible invested money with penalty

Here, we construct a new model to find optimal premium and the result is included in [39].

Let $R_0 = u$ be the initial capital, C be the premium price, X_k be claims and T_k be the inter-arrival time. We invest all our money with interest rate ρ but now money can be out by investor by applying the penalty. The surplus process is calculated by

$$R_k = R_{k-1} + C(T_k - T_{k-1}) - X_k$$

Then, we apply the interest rate to the smallest amount of capital as a penalty at the end of the maturity time. Hence, the capital is found by

$$R_k + min(R_j)_+(\exp(\rho) - 1) \qquad 1 \leq j \leq k$$

7.2.3 Accessible invested money with paying interest penalty

Here, we construct a new model to find optimal premium and the result is included in [39].

Let $R_0 = u$ be the initial capital, C be the premium rate, X_t be the claims and T_k be the inter-arrival times. We invest all the capital money with the interest rate ρ. If we withdraw any amount from our invested capital when the claim occurs at any time T_k, we pay interest penalty with parameter ϵ. Consider first claim happens at time T_1. Before payment, the surplus process is computed by

$$R_1^{(1)} = R_0(\exp(\rho(T_1 - 0))) + C\left(\frac{\exp(\rho(T_1 - 0))}{\rho}\right)$$

Then, after the payment for claim, we get

$$R_1 = (R_1^{(1)} - X_1) - \epsilon(R_1^{(1)} - X_1)_+(e^{\rho} - e^{\rho T_1})$$

So, by making similar calculations until time T_k, the capital is found by

$$R_k^{(1)} = R_{k-1}(\exp(\rho(T_k - T_{k-1}))) + C\left(\frac{\exp(\rho(T_k - T_{k-1}))}{\rho}\right)$$

7.2.4 Accessible invested money with paying interest penalty for splitting amount

Here, we construct a new model to find optimal premium and the result is included in [39].

Now, at time $T_0 = t$, we again split up initial capital to two different part: invested in u_1 and not invested in u_0. Overall capital is found by

$$R_0 = u_0 + u_1$$

When the first claim occurs at time T_1, we subtract claim from the not invested part $R_2^{(1)} = u_0$. If the claim is bigger than u_0, we get extra amount from the invested part u_1 with interest penalty. So

$$\widetilde{X}_1 = (X_1 \wedge u_0) + (X_1 - (x_1 \wedge u_0)) + \varepsilon(x_1 - (x_1 \wedge u_0))(\exp(\rho) - \exp(\rho T_1))$$

where X_1 is the first claim and $\wedge$ is meaning the minimum of a pair. Also, we get interest for the invested part of the capital as follows

$$R_1^{(1)} = u_1(\exp(\rho(T_1 - T_0))) + C(\frac{\exp(T_1 - T_0)}{\rho})$$

Then, the overall claim is found by

$$R_1 = R_1^{(1)} + R_1^{(2)} - \widetilde{X}_1$$

The similar process is done until time $T_k = 1$. The not invested part is calculated by

$$R_k^{(2)} = R_{k-1}^{(2)} - (X_k \wedge R_{k-1}^{(1)})$$

and invested part is computed by

$$R_k^{(1)} = \big(R_{k-1}^{(1)} - Y_{k-1}) - \varepsilon Y_{k-1}(\exp(\rho(T_k - T_{k-1})))\big)(\exp(\rho(T_k - T_{k-1})))$$
$$+ C\left(\frac{\exp(\rho(T_k - T_{k-1}))}{\rho}\right)$$

where $Y_{k-1} = X_{k-1} - (X_{k-1} \wedge R_{k-1}^{(2)})$. So, overall capital is found by

$$R_k = R_k^{(1)} + R_k^{(2)}$$

We apply the Frank copula to incorporate the dependence between time occurrences. This is done as in Section 7.1.1.

Then, optimal premium amounts are found for different dependence levels η in the interval $(0, 1)$. When η approaches 0, dependence increases; and it goes to 1, dependence decreases. Also, $\eta = 1$ shows the independent claims.

Results are stated in Table 7.1 for four specific dependence levels $\eta = 0.3,\ 0.5,\ 0.8, 1$.

	$\eta = 1$ $(Independent)$	$\eta = 0.8$	$\eta = 0.5$	$\eta = 0.3$
Case 0	1.3112	1.3347	1.3614	1.4327
Case 1	1.2644	1.2941	1.3347	1.3921
Case 2	1.2921	1.3228	1.3564	1.4139
Case 3	1.1278	1.1422	1.1687	1.2108
Case 4	1.1348	1.1456	1.1802	1.2208

Table 7.1: Table shows the optimal premiums for dependence levels $\eta = 0.3\ ,\ 0.5\ ,\ 0.8$ and independent claims for each models.

Chapter 8

Miscellaneous

In this chapter, we present overall preliminary results on the finite time non-ruin probabilities.

- Section 8.1 is dealing with comparison technique.
- In Section 8.2, we apply integral operator technique to the modified surplus process.
- Section 8.3 is dealing with the Riemann Liouville operator.

8.1 Ruin Probability via Comparison

The insurance example in this part is modelled by a continuous time Markov chains (CTMC) $\{X_t : t \geq 0\}$ on the state space $S = \{0, 1, 2, \ldots\}$.
We assume that X_t is defined by the following rules

- Premium is coming in with rate λ.
- Claim of size is coming out with rate μ_{ij}.

Then, we construct the matrix as

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & \cdots & \cdots \\ \mu_{10} & -(\mu_{10}+\lambda) & \lambda & 0 & \cdots & \cdots & \cdots \\ \mu_{20} & \mu_{21} & -(\mu_{20}+\mu_{21}+\lambda) & \lambda & 0 & \cdots & \cdots \\ \mu_{30} & \mu_{31} & \mu_{32} & -(\mu_{30}+\mu_{31}+\mu_{32}+\lambda) & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

Example: Claim size $k-j$ occurs with intensity. As a realistic example we can consider the case

$$\underset{k}{sup} \sum_{j=0}^{k-1} \exp(-\theta(k-j)\delta) = \mu$$

The further comparison is motivated by the Schur-convexity.

For two vectors $x, y \in \mathbb{R}^n$, the vector x majorizes vector y (denoted $x >_{sc} y$) if [8] [10]

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad for\ k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{k} y_i$$

Then, a function $f : \mathbb{R}^n \to \mathbb{R}$ is called Schur-convex if [8] [10]

$$x >_{sc} y \Rightarrow f(x) \geq f(y)$$

If we compare Q_- and Q_+ by the general comparison arguments, we get

$$Q_- = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & \cdots & \cdots \\ \mu & -\Delta & \lambda & 0 & \cdots & \cdots \\ 0 & \mu & -\Delta & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -\Delta & \lambda & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \leq Q \leq \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & \cdots & \cdots \\ \mu & -\Delta & \lambda & 0 & \cdots & \\ \mu & 0 & -\Delta & \lambda & 0 & \cdots \\ \mu & 0 & 0 & -\Delta & \lambda & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = Q_+$$

The comparison is also intuitive since we have smaller claims in Q_- and we have larger claim in Q_+ with the same intensity. Notice that $Q_- = -H_-$ and $Q_+ = -H_+$ and further we work with the operators H_- and H_+.

$$\begin{array}{ccccc} Q_- & \leq & Q & \leq & Q_+ \\ -H_- & \leq & Q & \leq & -H_+ \end{array}$$

Then, we get

$$-H_+ = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & \cdots & \cdots \\ \mu & -\Delta & \lambda & 0 & \cdots & \\ \mu & 0 & -\Delta & \lambda & 0 & \cdots \\ \mu & 0 & 0 & -\Delta & \lambda & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

So, $H_+|p\rangle = K_p|p\rangle$ where $K_p = (-\Delta + \lambda \exp(ip))$ and $|p\rangle = \exp(ipx)$. If we apply $f(x) = |p\rangle$ to our statement, we find that

$$H_+ f(x) = f(x)(-\Delta) + f(x+1)\lambda$$

By applying the Hamiltonian to $|p\rangle$, we obtain that

$$H|p\rangle = K|p\rangle \;\Rightarrow\; H^2|p\rangle = K^2|p\rangle \;\Rightarrow\; H^j|p\rangle = K^j|p\rangle$$
$$\exp(-\tau H)|p\rangle = \sum \tfrac{(-\tau)^j H^j}{j!} = \exp(-\tau K_p)|p\rangle$$

Let $x = u$ be the initial capital and $x' = m$ be the last capital value. It can be said that $\langle u|p\rangle = \exp(ipu)$ and $\langle p|x'\rangle = \langle m|p\rangle = \exp(-ipm)$. Then, transition probabilities are

$$\begin{aligned}\langle x|\exp(-\tau H)|x'\rangle =& \langle u|\exp(-\tau H)|m\rangle \\ =& \int_0^{2\pi} \frac{dp}{2\pi} \langle x|\exp(-\tau H)|p\rangle\langle p|x'\rangle \\ =& \int_0^{2\pi} \frac{dp}{2\pi} \exp(-\tau K_p + ip(u-m))\end{aligned}$$

For a specific $K_p = (-\Delta + \lambda\exp(ip))$, the statement is derived by

$$I = \int_0^{2\pi} \frac{dp}{2\pi} \exp(-\tau(-\Delta + \lambda\exp(ip))) + ip(u-m)) = I$$

If we substitute z by $\exp(ip)$, we get

$$I = \frac{\exp(\tau\Delta)}{2\pi i} \int_0^{2\pi} \frac{\exp(-\tau\lambda z)}{z^{m-u}} dz$$

From Cauchy-integral in complex analysis, we find that

$$I = \exp(\tau\lambda)\frac{(-\tau\lambda z)^{m-u}}{(m-u)!}$$

If we do the same for H_-

$$\begin{aligned} H_-|p\rangle &= |p\rangle(-\Delta + \lambda \exp(ip) + \mu \exp(-ip)) - I(x=1)\exp(ip(-\Delta + \lambda \exp(ip))) \\ &= K_p|p\rangle + C_p I(x=1) \end{aligned}$$

where $K_p = (-\Delta + \lambda \exp(ip) + \mu \exp(-ip))$ and $C_p = -\exp(ip(-\Delta + \lambda \exp(ip)))$

$$Ae_i = \lambda_i e_i + c_i e_0 \quad \Rightarrow \quad Ae_i - \lambda_i e_i = c_i e_0$$

Hence

$$H_- I(x=1) = -\Delta I(x=1) + \mu I(x=2)$$

Unfortunately, we were unable to do the same calculation for H_- as for H_+.

It is interesting to continue the research.

8.2 Finite and Infinite Time Non-Ruin Probability for Exponential Claims with Infinite Jumps

In this part, we consider a modified version of the classical surplus process

$$R_t = R_0 + Ct - S_t$$

in a continuous time where R_0 is initial capital, C is premium and S_t are compound Poisson process.

We assume that the claim size distribution is the mixture of the exponential distribution and infinite claim. More exactly, the probability of having any finite claim be $P[X = NE(\lambda)] = p$ and the probability of having infinite claim be

$P[X = \infty] = q$. Then, obviously

$$\begin{aligned}
P(T = \infty) \geq & P_u(\text{at least one infinite claim occurs over infinite time period })\\
= & 1 - P_u(\text{no claim on } (0, \infty))\\
= & 1 - \lim_{M \mapsto \infty} P_u(\text{no claim on } (0, M)
\end{aligned}$$

when q is positive. Notice that infinite claim occurs according to Poisson process with rate λq.

$$\begin{aligned}
P(T = \infty) \geq & 1 - \lim_{M \mapsto \infty} P(N_M^\infty = 0)\\
= & \lim_{M \mapsto \infty} P_u(NE(\lambda q) \geq M)\\
= & 1 - \lim \exp(-\lambda q M) = 1
\end{aligned}$$

So

$$P(T = \infty) = 1.$$

Finite time non-ruin probability

By conditioning on the first claim, ruin probability is obtained as follows (we use the abbreviation $e^x = \exp(x)$)

$$\begin{aligned}
P_u(T > t) = & P_u(T_1 \geq t) + \int_0^t P_u(T > t | T_1 = y) dF_{T_1}(y)\\
= & e^{-\lambda t} + \int_0^t \lambda e^{-\lambda y} P_u(T > t | T_1 = y) dy\\
= & e^{-\lambda t} + \int_{y=0}^{y=t} \lambda e^{-\lambda y} \left(\int_{x=0}^{x=\infty} P_u(T > t | T_1 = y, \xi = x) dF_\xi \right) dy
\end{aligned}$$

$$= e^{-\lambda t} + \int_{y=0}^{y=t} \lambda e^{-\lambda y} \Big(\int_{x=0}^{x=u+cy} P\mu e^{-\lambda x} P_{u+cy-x}(T > t-y) dx \Big) dy$$

Then, we find that

$$P_u(T > t) = \varphi(u,t) = e^{-\lambda t} + \lambda\mu P \int_{y=0}^{y=t} e^{-\lambda y} \int_{x=0}^{x=u+cy} e^{-\lambda x} P_{u+cy-x}(T > t-y) dx dy$$

By substitution $\theta = u + cy - x$ and $z = t - y$

$$\begin{aligned}
\varphi(u,t) =& e^{-\lambda t} + \lambda\mu P \int_{z=0}^{z=t} \int_{\theta=0}^{\theta=u+c(t-z)} \varphi(\theta,z) e^{-\lambda(t-z)} e^{\mu(\theta-u-c(t-z))} d\theta dz \\
=& e^{-\lambda t} + e^{-\lambda t} e^{-\mu(ct+u)} \lambda\mu P \int_{z=0}^{z=t} \int_{\theta=0}^{\theta=u+c(t-z)} \varphi(\theta,z) e^{\lambda z} e^{\mu(\theta+cz)} d\theta dz
\end{aligned}$$

If we simply the equation as the region

$$\varphi(u,t) = e^{-\lambda t} + e^{-\lambda t} e^{-\mu(ct+u)} D \int_{K(t,u)} \varphi(\theta,z) d\theta dz$$

multiply by $g(u,t) = e^{-\lambda t} e^{-\mu(ct+u)}$ on both sides of the equation,

$$\begin{aligned}
\varphi(u,t) e^{-\lambda t} e^{-\mu(ct+u)} =& \varphi(u,t) g(u,t) \\
=& H(u,t) + D \int_{K(t,u)} W(\theta,z) d\theta dz
\end{aligned}$$

where $W(\theta,z) = \varphi(\theta,z) e^{\mu\theta+\mu cz}$ and $H(u,t) = e^{\mu ct+\mu u}$. Taking the derivation by u with using Leibniz integration formula

$$\begin{aligned}
\frac{\partial \varphi g}{\partial u} =& \frac{\partial H}{\partial u} + D \int_{z=0}^{z=t} W(u + c(t-z), z) dz \\
=& \frac{\partial H}{\partial u} + D \int_{z=0}^{z=t} W(r,z) dz
\end{aligned}$$

where $r = u + c(t - z)$. Then, if we take the partial derivation by t with using the Leibniz integration formula

$$\frac{\partial^2 \varphi g}{\partial t \partial u} = \frac{\partial^2 H}{\partial t \partial u} + DEW(u + ct, z) \tag{8.2.1}$$

Unfortunately, we were unable to find the theoretical solution to this equation.

Numerical operator approach

In this part, we find the Taylor type expansion for the solution of 8.2.1.

$$\begin{aligned}\varphi(u,t) =& e^{-\lambda t} + e^{-\lambda t} e^{-\mu(ct+u)} \lambda \mu P \int\limits_{z=0}^{z=t} \int\limits_{\theta=0}^{\theta=u+c(t-z)} \varphi(\theta, z) e^{\lambda z} e^{\mu(\theta + cz)} d\theta dz \\ =& H(x) + G(x) D \int\limits_{\Delta} I(v \in K_x) \varphi(v) e^{<a,v>} dv\end{aligned}$$

where $x = (u,t) \in R^2$, $v = (\theta, z) \in R^2$, $H(x) = e^{-\lambda t}$, $G(x) = e^{-\lambda t} e^{-\mu(ct+u)}$ and $D = \lambda \mu P$. If we divide by $G(x)$ through on both side of the equation

$$\begin{aligned}Q(x) = \frac{\varphi(x)}{G(x)} =& \frac{H(x)}{G(x)} + D \int\limits_{\Delta} I(v \in K_x) \frac{\varphi(v)}{G(v)} e^{<a,v>} dv \\ =& q(x) + D \int\limits_{\Delta} I(v \in K_x) Q(v) dv\end{aligned}$$

where $q(x) = \frac{H(x)}{G(x)}$, $G(v) = e^{-\lambda z - \mu c z - \mu \theta}$ and $< a, v >= e^{\lambda z + \mu c z + \mu \theta}$. Then using the operator

$$Af(x) = \int\limits_{\Delta} I(v \in K_x) D Q(v) dv$$

and then the solution found from the integral equation (operator equation)

$$\begin{aligned} Q = q + AQ \quad &\Rightarrow \quad Q(I - AD) = q \\ &\Rightarrow \quad Q = \sum_{j=0}^{\infty} A^j q \\ \varphi = QG = G \sum_{j=0}^{\infty} A^j q \end{aligned}$$

Observe that

$$\begin{aligned} A1 =& \lambda\mu P \int_{\Delta} I(y \in K_x) dy = \lambda\mu P(K_x) \\ =& \lambda\mu P \frac{(u + u + ct)}{2} \end{aligned}$$

Hence, one $A1 < 1$ we can write the expansion on t

$$\begin{aligned} \varphi(u,t) =& e^{-\lambda t} + e^{-\lambda t} e^{-\mu(ct+u)} \lambda\mu P \int_{z=0}^{z=t} \int_{\theta=0}^{\theta=u+c(t-z)} \varphi(\theta, z) e^{\lambda z} e^{\mu(\theta + cz)} d\theta dz \\ =& \varphi(u, 0) + \frac{\partial \varphi}{\partial t}(u, 0) t + \cdots \\ =& 1 + \frac{\partial \varphi}{\partial t}(u, 0) t + \cdots \end{aligned}$$

where $t \mapsto 0$.

It is an interesting problem to solve partial differential equations and relative integral equations in this chapter.

8.3 Riemann-Liouville Integral Operator

Riemann-Liouville is the most frequently used fractional integration method although integral operator can be formed by many different type of methods. Riemann's form of integral operator is defined as follows

$$Af(x) = \int_{t=0}^{x} f(t)dt$$

By applying the operator again

$$\begin{aligned} A_2f(x) &= \int_{t=0}^{x} (\int_{y=0}^{t} f(y)dy)dt \\ &= \int_{t=0}^{x} \int_{y=0}^{x} [f(y)I(y \leq t)]dydt \\ &= \int_{y=0}^{x} f(y)(x-y)dy \end{aligned}$$

By induction, Cauchy formula is generalized by Riemann's form of integral operator

$$\begin{aligned} A_nf(x) &= \int_{0}^{x} dx_1 \int_{0}^{x_1} dx_2 \cdots \int_{0}^{x_{n-1}} f(x_n)dx_n \\ &= \frac{1}{(n-1)!} \int_{0}^{x} f(y)(x-y)^{n-1}dy \\ &= \frac{1}{\Gamma(n)} \int_{0}^{x} f(y)(x-y)^{n-1}dy \end{aligned}$$

where $(n-1)! = \Gamma(n)$. If we apply another operator, we derived that

$$\begin{aligned} A_u(A_vf)(x) &= \frac{1}{\Gamma(u)} \int_{y=0}^{x} (x-y)^{u-1} \frac{1}{\Gamma(v)} [\int_{t=0}^{y} f(t)(y-t)^{v-1}dt]dy \\ &= \frac{1}{\Gamma(u+v)} \int_{t=0}^{x} f(t)(x-t)^{v+u-1}dt \\ &= A_{u+v}f(x) \end{aligned}$$

This proves the semigroup $A_t = A^t$ property for the discrete time family , for $t = 1, 2, 3....$

Now, we consider continuous time.

Let $f(x) = x^k$ be a function. Riemann integral is defined by

$$A_t x^k = \frac{1}{\Gamma(t)} \int\limits_{y=0}^{x} y^k (x-y)^{t-1} dy$$

By substitution $u = \frac{y}{x}$,

$$\begin{aligned} A_t x^k =& \frac{x^{k+t}}{\Gamma(t)} B(k+1, t) \\ =& x^k + k(x^k lnx + x^k C_k) + \cdots \end{aligned}$$

where Beta function $B(u,v) = \frac{\Gamma(v)\Gamma(u)}{\Gamma(u+v)}$. Furthermore,

$$\frac{[I - A_t]f(x)}{t} = \frac{1}{t}[f(x) - \frac{1}{\Gamma(t)} \int\limits_{0}^{x} f(y)(x-y)^{t-1} dy]$$

By substitution $u = \frac{x-y}{x}$,

$$\begin{aligned} \frac{[I - A_t]f(x)}{t} =& x^t \int\limits_{0}^{1} f(x - ux) u^{t-1} du \\ =& \frac{f(x)}{t}(1 - \frac{x^t}{\Gamma(t+1)}) + \frac{x^t}{\Gamma(t+1)} \int\limits_{0}^{1} (f(x) - f(x-ux)) u^{t-1} du \end{aligned}$$

By expanding x^t and Gamma function $\Gamma(t+1)$ at $t = 0$; we derive

$$\begin{aligned} \frac{[I - A_t]f(x)}{t} \rightarrow& f(x)(lnx - \Gamma'(1)) + \int\limits_{0}^{1} [\frac{(f(x) - f(x-ux))}{ux}] du \\ Hf =& f(x)(lnx - \Gamma'(1)) + \int\limits_{0}^{x} [\frac{(f(x) - f(x-z))}{z}] dz \end{aligned}$$

where $z = ux$.

It is interesting to continue the study on the Riemann-Liouville integral operator.

Chapter 9

Conclusion and Future Research

Premium income and outgoing claims play an important role to determine the companies' profit or loss. In this matter, finite time non-ruin probability provides to determine the companies' situation against ruin.

We discovered Quantum mechanics is one of the significant approaches to compute the finite time non-ruin probability.

To make the ruin model realistic, we construct variety of models both with interest rate and dependent claim occurrences. There are many ways to extend this research for example dependent renewal processes i.e claim occurrences in a stationary sequence.

We treated several particular examples. We also applied the path calculation method to find the numerical results. More research should be done in this area such as example in Section 6.7.

Optimal premium price for various models of interest rate are found by using the Copula claim occurrences via stochastic modeling. It is a challenging question to find a theoretical result for the model with dependent claim occurrences.

It is also interesting to continue the study on the Riemann-Lioville integral operator and to solve partial differential equations and relative integral equations in Chapter 8.

Appendix

The main body of Matlab code for ruin probability and finding the optimal premium without interest rate is shown as follows.

```
% Given claim amounts to generate new ones
X=[2.5 1.1 1.7 2.1];
%u is initial capital, C is premium and \mu is found by maximum
    likelihood estimation
u=5;lambda=4;C=1;mu=lambda/sum(X);
% Generating uniform variable for inter-arrival times
while TA(i) < 1
        T(i+1)=random('Exponential',1/lambda);
        TA(i+1)=T(i+1)+TA(i);
        i=i+1;
end
Y=zeros(1,i-1);  R=zeros(1,i);
% Generating claim amounts with parameter \mu
for k=2:(i-1)
    Y(k)=random('Exponential',1/mu);
    while Y(k)>u
        Y(k)=random('Exponential',1/mu);
    end
end
% Ruin probability calculation
R(1)=u;
for k=2:i-1
    R(k)=R(k-1)+C-Y(k);
end
R(i)=R(i-1);
```

Also, the ruin probability calculation part of Matlab code for our four different models with interest rate are shown as:

Model 1

```
rho=0.05; % Interest rate
% Ruin probability calculation
R(1)=u/2;
for k=2:i-1
    R(k)=R(k-1)+C-Y(k);
end
P=(u-R(1))*exp(rho);
R(i)=R(i-1)+P;
```

Model 2

```
rho=0.05; % Interest rate
% Ruin probability calculation
R(1)=u;
for k=2:i-1
    R(k)=R(k-1)+C-Y(k);
end
R(i)=R(i-1);
R(i)=R(i)+(min(R)*exp(rho)-min(R));
```

Model 3

```
% Interest rate rate rho and interest penalty rate F
rho=0.05; F=0.005;
% Ruin probability calculation
R(1)=u;
for k=2:i-1
    R(k)=R(k-1)*exp(rho*(PA(k)-PA(k-1)))+C*(exp(rho)-1)/rho;
    R(k)=(R(k)-Y(k))-F*(R(k)-Y(k))*(exp(rho)-exp(rho*(PA(k))));
end
R(i)=R(i-1)*exp(rho*(PA(i)-PA(i-1)))+C*(exp(rho)-1)/rho;
if R(i)<0
```

Model 4

```
% Introducing interest rate rho and interest penalty rate F
rho=0.05; F=0.005;
% Ruin probability calculation by splitting up the initial capital (R
    (1)=u/2 and S(1)=u/2)
R(1)=u/2; S(1)=(u-R(1)); P(1)=0;
RSUM(1)=R(1)+S(1)-P(1);
for k=2:i-1
    R(k)=( R(k-1)-( Y(k-1)  - min(Y(k-1),S(k-1))  ) - F*( Y(k-1)  -
        min(Y(k-1),S(k-1))  )* (exp(rho)-exp(rho*PA(k-1)))  )*(exp(rho
        *(PA(k)-PA(k-1))))+C*(exp(rho)-1)/rho;
    S(k)=S(k-1)- min(Y(k-1),S(k-1));
    P(k)=min(Y(k-1),S(k-1))+(Y(k-1)-min(Y(k-1),S(k-1)))+F*(Y(k-1)-min
        (Y(k-1),S(k-1)))*(exp(rho)-exp(rho*PA(2)));
    RSUM(k)=R(k-1)+S(k-1)-P(k-1);
end
    R(i)=( R(i-1)-( Y(i-1)  - min(Y(i-1),S(i-1))  ) - F*( Y(i-1)  -
```

```
        min(Y(i-1),S(i-1))  )*  (exp(rho)-exp(rho*PA(i-1)))  )*(exp(rho
        *(PA(i)-PA(i-1))))+C*(exp(rho)-1)/rho;
    S(i)=S(i-1)- min(Y(i-1),S(i-1));
    RSUM(i)=R(i-1)+S(i-1)-P(i-1);
```

The following Matlab code has written to make path calculation for any matrix P. The initial state x_1 , final state x_n and step size are given by the user.

```
function pathcalculation(P,steps,x1,xn)
l=length(P); n=steps+1;
t=0; k=0;sum=0;
while t<n
   k=k+(l*(10^t));
   t=t+1;
end
display(k);
A=ones(k);
% Paths from any state x_1 to x_n in k steps is written as vector
    elements (e.g from 1 to 1 in 2 steps A(111)=P(11)*P(11) , A(121)=P
    (12)*P(21) )
for i=1:(k+1)
    for j=1:l
        if rem (i,10) < l+1
            if rem (i,10) > 0
            A(i*10+j)=A(i)*P(rem (i,10),j);
            end
        else
            A(i*10+j)=1;
        end
    end
end
% The sum of all paths for initial state $x_1$ and final state $x_n$
```

```
for i=(10^(n-1)):(10^n)
    if floor(i/(10^(n-1))) == x1
        if rem(i,10) == xn
            sum=sum+A(i);
        end
    end
end
display(sum-(10^(steps-1))+1^(steps-1));
end
```

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