The Two-Faced God Janus

or

What Does n-Hausdorfness Have to Do With Dynamics and Topology?

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Abstract

This thesis is centered around the study of topological dynamics and analytic topology, as well as an unexpected intersection between the two, which revolves around the notion of an n-Hausdorff space.

In the Dynamics part of this thesis, we discuss the author's two main results in topological dynamics. The first is about the Ellis semigroup of substitution systems, which extends previous results in this area. It states that the Ellis semigroup of a certain type of constant-length substitution dynamical systems has two minimal ideals, and further calculates the number of idempotents in these ideals. This requires a novel approach towards considering the factor maps to the maximal equicontinuous factor of these dynamical systems - a reworking of an old theorem which takes up a chapter in the thesis. The second result is about the Furstenberg topology of a point-distal dynamical system. Since the constant-length substitution systems we had considered in the previous sections are also point-distal, it can be considered a rather general result. It shows that if a point-distal system is an almost k-to-1 extension of its maximal equicontinuous factor, the Furstenberg topology restricted to a (in some sense canonical) subspace is at most $k+1$-Hausdorff.

In the Analytic Topology part of the thesis, we discuss the n-Hausdorff property in its original context, as a natural part of a series of combinatorial generalisations of separation axioms. These combinatorial generalisations were introduced by several authors throughout the past 20 years. However, n-Hausdorffness in particular is interesting in light of a couple of still-open questions of Arhangelskii. The more easily stateable of the two is whether the cardinality of a T_1 first countable Lindelöf space exceeds continuum. The main work of the author in this part involves the many examples of spaces which satisfy a combinatorial separation axiom and also have (or lack) various other properties, such as being Lindelöf, first countable, compact, or being T_1. The author has contributed towards the proofs of the theorems given in this part.

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Contents

Part I

Introduction

Present-day Pure mathematics has become so specialised that oftentimes one can find few areas of intersection between subjects. In particular, general topology has been criticized by some for having developed concepts which are too abstract to be ‘naturally’ found in other areas of mathematics. For example, some non-topologists assume that the spaces they deal with are Hausdorff, and that weakenings of this separation axiom are mostly an intellectual pursuit with slight relevance outside of general topology. Thus, it is of great surprise and interest when one can find relations between general topology and other fields, such as dynamical systems. While studying the Ellis semigroup of a dynamical system, the present author discovered some natural topologies which are associated with certain dynamical properties (those of proximality), and which are often not Hausdorff. These topologies were introduced in the 1960’s by H Furstenberg, and bear his name. The two main trains of thought followed in this thesis naturally converge upon this class of examples: that of exploring the Ellis semigroup of a substitution dynamical system, and that of considering the properties of n-Hausdorffness, which is the natural weakening of the Hausdorff property found in the Furstenberg topologies.

The Ellis (originally called the ‘enveloping’) semigroup was introduced by Ellis in 1960 [Ell60], and has since shown to be a useful tool in topological dynamics. In [Ell69], R Ellis develops the theory of this semigroup, showing how the asymptotic, or eventual, properties of a topological dynamical system can be captured by the topological and algebraic properties of the Ellis semigroup. Its machinery has helped provide much shorter and cleaner proofs of several key theorems in topological dynamics, such as the the Auslander-Ellis Theorem [EE14]. More recently, E Glasner has given alternative and shorter proofs of the theorems of Maliutin and Margulis [Gla17] via the Ellis semigroup.

In spite of its usefulness and academic interest, specific calculations and description of the Ellis semigroup remain rare. The few examples include those given by Namioka [Nam84], Milnes [Mil86] and [Mil91], Glasner [Gla76] and [Gla93], Berg, Gove and Haddad [BGH98], Haddad and Johnson [HJ97], Budak, Isik, Milnes and Pym [BIsMP01], and Glasner and Megrelishvili [GM06]. One recent example is Marcy Barge’s calculation of the Ellis semigroup of the Thue-Morse system [Bar], which involves various auxiliary codings and Brattelli diagrams.

In [Sta18], the present author calculates the Ellis semigroup of a certain class of constant length substitutions over arbitrary alphabets. This generalises both Barge's result and an earlier result by Haddad and Johnson [HJ97]. It also fixes an error in the latter's proof. In their paper [HJ97], Haddad and Johnson prove that the Ellis semigroup of any generalised Morse sequence has two minimal ideals with two idempotents each. Their main technique uses the theory around IP cluster points and IP sets, which are certain 'combinatorially rich' subsets of the integers. They first compute the idempotents in the case of $\mathbb{N}$-cascades and use the fact that the closure of the set of idempotents is precisely the set of IP cluster points, so when this set is finite, every IP cluster point is an idempotent. Then they use a technical proposition, which they give without proof, to extend this computation of the IP cluster points of the $\mathbb{N}$-cascade to the $\mathbb{Z}$-cascade case. In Section 3.3, we show that any binary Morse-like substitution is a counterexample to their proposition. Our generalisation of their main theorem does not use IP cluster points. Instead, we combine some ideas from [HJ97] with properties of the Ellis semigroup given in [EE14] and a new approach to the construction of a certain AI extension using notions from [Mar71]. Combining this with the result of Coven and Keane [CK71], we give a complete characterization of the minimal ideals and idempotents in the Ellis semigroup of a constant-length binary substitution system. These results are presented in greater depth in Chapter 3.

Our construction of an AI extension to the maximal equicontinuous factor of a substitution system is interesting in its own right, as it gives an explicit intermediate substitution system and a sliding block code from the main space to the intermediate space. In this way, it provides a clear generalisation of an earlier construction by Coven and Keane [CK71], which was done only for the binary case. The most general result in this study is that of Dekking [Dek78] about the maximal equicontinuous factor of a constant length substitution system. When seen solely as a final result, the existence of an AI extension to the maximal equicontinuous factor which we consider is an easy corollary. However, the approach to the construction itself is of interest. This is explored in more depth in Chapter 2.

In Chapter 4, we consider an unlikely collection of examples of n-Hausdorff spaces, which arises naturally in topological dynamics. These are the so-called Furstenberg topologies, which were initially introduced by Hillel Furstenberg to study distality in dynamical systems.

0.1 Definitions, Notation, Standard Results

We now give some standard definitions, notation, and theorems which will be used throughout this thesis. For notation particular to certain parts of the thesis, the reader is referred to Section IV. These notions are very commonly used, and can be found, for example, in [Eng89] or [Fog02].

0.1.1 Notation. We will use the following notation:

- $\mathbb{N}$ for the non-negative integers (so, $0 \in \mathbb{N}$)
- $\mathbb{N}^+$ for the strictly positive integers
- $\mathbb{Z}$ for the integers
- $\mathbb{R}$ for the real numbers

0.1.2 Definition (topology, topological space, open sets, closed sets, [Eng89])**.** Let X be a set. A collection $\tau \subset \mathcal{P}(X)$ is called a *topology on the set* X if and only if τ satisfies the following conditions:

1. $\emptyset \in \tau$, $X \in \tau$;
2. $U, V \in \tau \;\Rightarrow\; U \cap V \in \tau$;
3. $\{U_i\}_{i \in I} \subset \tau \;\Rightarrow\; \bigcup_{i \in I} U_i \in \tau$.

In this case, the pair (X, τ) is called a *topological space*, and is often just denoted by X when the topology is understood. The elements of τ are called *open sets*, and their complements are *closed sets.*

0.1.3 Definition (basis, [Eng89])**.** Let $\mathcal{B} \subset \mathcal{P}(X)$ be a nonempty collection of sets such that

- $\bigcup\{B : B \in \mathcal{B}\} = X$, and
- for all $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 \neq \emptyset$, there is a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Then $\mathcal{B}$ is a *basis* for a unique topology τ on X, namely

$$\tau = \{\cup A : A \subseteq \mathcal{B}\}.$$

0.1.4 Definition (Hausdorff space, [Eng89])**.** We call a topological space (X, τ) *Hausdorff* or T_2 if and only if for any distinct points $x, y \in X$ there exist disjoint open sets U_x, U_y with $x \in U_x$ and $y \in U_y$.

A weaker though still useful notion is:

0.1.5 Definition (T_1 space, [Eng89])**.** We call a topological space (X, τ) T_1 if and only if for every two distinct points $x, y \in X$, there exist open neighbourhoods $U_x, U_y \in \tau$ such that $x \notin U_y \ni y$ and $y \notin U_x \ni x$.

0.1.6 Definition (Tychonoff space, [Eng89])**.** A space X is *Tychonoff* if and only if it is Hausdorff and for any closed set $C \subset X$ and any point $x \notin C$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) \subset \{1\}$.

0.1.7 Definition (compact space, [Eng89])**.** A space X is *compact* if and only if any open cover of X has a finite subcover.

Compactness is closely related to the following notion:

0.1.8 Definition (finite intersection property (FIP), [Eng89])**.** Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{A}$ has the *finite intersection property* if and only if whenever $\{A_i\}_{i \in I}$ is a finite collection of elements of $\mathcal{A}$, then $\bigcap_{i \in I} A_i \neq \emptyset$ - in other words, any finite subcollection of $\mathcal{A}$ has a non-empty intersection.

A natural generalisation of compactness (which will be of use in Part III), is that of a Lindelöf space.

0.1.9 Definition (Lindelöf, [Eng89])**.** A topological space (X, τ) is called *Lindelöf* if and only if every open cover of X has a countable subcover.

0.1.10 Definition (continuous map, [Eng89]). A map $f : X \to Y$ between topological spaces is called *continuous* if and only if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

0.1.11 Definition (dense set, [Eng89]). A subset A of a topological space X is called *dense in* X if and only if its closure is the whole space. Using the alternative characterisation of closed sets, we can also say that A is dense in X if and only if for any open set U, we have $U \cap A \neq \emptyset$.

0.1.12 Proposition ([Eng89]). *Let X be a topological space, Y be Hausdorff, D a dense subset of X and $f : D \to Y$ be continuous. Then there is a unique continuous function $F : X \to Y$ such that $F|_D = f$.*

0.1.13 Note. Any map f from a discrete space X to a topological space Y is continuous. Indeed, since all subsets of X are open, then for any open $U \subset Y$, the pre-image $f^{-1}(U)$ is a subset (possibly empty) of X, and thus open.

0.1.14 Theorem (Tychonoff's Theorem, [Eng89]). *The product of arbitrarily many compact spaces is also compact, assuming the Axiom of Choice.*

0.1.15 Theorem (Zorn's Lemma, [Eng89]). *If $(S, \leqslant)$ is a partially ordered set such that any increasing chain $s_1 \leqslant s_2 \leqslant \dots$ has a supremum in S, then S has a maximal element, that is, an element $m \in S$ such that $s \leqslant m$ for all $s \in S$.*

0.1.16 Definition (regular, [Eng89]). A topological space (X, τ) is called *regular* or T_3 if and only if it is T_1 and for any closed set C and any point $x \notin C$, there exist disjoint open sets U_x, U_C with $x \in U_x$ and $C \subset U_C$.

0.1.17 Lemma (Alternative characterisation of closed sets, [Eng89]). *A subset A of a topological space (X, τ) is* closed *if and only if for any point $x \in A$ and any open set $U \ni x$, $A \cap U \neq \{x\}$.*

0.1.18 Definition (nowhere dense, [Eng89]). A subset $A \subset X$ is called *nowhere dense* if and only if the interior of its closure is empty.

0.1.19 Definition (residual set, [Eng89]). A subset $A \subset X$ of a complete metric space X is called *residual* if and only if $X \setminus A$ is a countable union of nowhere dense sets.

0.1.20 Definition (uniformly continuous, [Eng89])**.** We call a map $f : X \to X$ on a metric space X *uniformly continuous* if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$ for all points $x, y \in X$.

0.1.21 Theorem (Heine-Cantor Theorem, [Eng89])**.** *Any continuous function on a compact set is uniformly continuous.*

0.1.22 Definition (group, [Eng89])**.** A *group* is a set G with a binary operation $+ : G \times G \to G$ satisfying the following axioms:

- Associativity: for all $a, b, c \in G$, $(a + b) + c = a(b + c)$,
- Identity element: there exists a special element $e \in G$ such that for every $a \in G$, $e + a = a + e = a$. We call this element e the *identity* of G,
- Inverses: for each $a \in G$ there is a $b \in G$ such that $a + b = b + a = e$.

If moreover the binary operation satisfies commutativity (for all $a, b \in G$, $a+b = b + a$), we call the group *Abelian*.

0.1.23 Example. The integers $\mathbb{Z}$ together with addition give an example of a group; the identity element is 0. Note that the integers are not a group under multiplication, as we do not have multiplicative inverses.

0.1.24 Definition (semigroup, [HS98])**.** A *semigroup* is a set S together with an associative binary operation.

0.1.25 Example ([HS98])**.** An example of a semigroup is the natural numbers $\mathbb{N}$ together with addition. Another example is the integers $\mathbb{Z}$, together with multiplication.

Thus, a group can be defined as a semigroup with identity and inverses.

Part II

Dynamical Systems

Chapter 1

Introduction to Topological Dynamics

In this chapter, we introduce the general theory of dynamical systems, as well as some results important in our particular setting. Almost all of the definitions and results are standard; the ones from topological dynamics can be found in [dV93], [EE14], [Fog02], or [BG13]; the analytic topology parts can be found in [Eng89] or [HS98].

In general, a dynamical system is a space X with a semigroup action on it. But what does that mean?

1.0.26 Definition (semigroup action, [EE14])**.** Let X be a set and T be a semigroup. Then an *action of* T *on* X is a function $\pi : X \times T \to X$, $(x, t) \mapsto tx$, such that $(ts)x = t(sx)$ for all $x \in X$ and $s, t \in T$. In other words, $\pi((x, ts)) = \pi((\pi(x, s), t))$ for all $x \in X$ and $s, t \in T$. If T also has an identity, we require that $ex = x$ for all $x \in X$.

Similarly, we can define a group action on a space X.

1.0.27 Definition (dynamical system, [EE14])**.** A *dynamical system* is a triple (X, T, π) where X is a compact Hausdorff topological space, T is a topological semigroup (in other words, T is a topological space with (semi-) group structure where the (semi-) group operation $\cdot$ is continuous) and π is an action of T on X such that the map π is continuous. Sometimes, a dynamical system with acting (semi-) group $\mathbb{N}$ or $\mathbb{Z}$ is called an $\mathbb{N}$- *(respectively,* $\mathbb{Z}$*-) cascade* for short.

Also, sometimes we will shorten the phrase 'dynamical system' to just 'system'.

When we talk about a dynamical system (X, T, π), we will implicitly take T to be discrete, unless otherwise specified. Thus, for π to be continuous (in light of Note 0.1.13), it suffices that the maps $\pi^t : X \to X$, $x \mapsto tx$ be continuous for all $t \in T$.

1.0.28 Example. Let X be any compact Hausdorff topological space and f be any continuous map $f : X \to X$. Then we may define an action of the semigroup $\mathbb{N}^+$ on X via the maps $\pi(x, n) = f^n(x)$. Moreover, if f is a homeomorphism of X, then the maps $\pi(x, n) = f^n(x)$ give an action of the group $\mathbb{Z}$ on X.

We would like to have a way to 'compare' two dynamical systems over the same group, or to be able to find a relationship between their properties. For this, we need a special type of map:

1.0.29 Definition (homomorphism of dynamical systems, factor, extension, [EE14])**.** Let (X, T) and (Y, T) be dynamical systems and $f : X \to Y$. Then f is a *homomorphism* if f is continuous and $f(tx) = tf(x)$ for all $x \in X$ and $t \in T$. If f is surjective, then we call (Y, T) a *factor* of the system (X, T), and we call the system (X, T) an *extension* of (Y, T). Note that in some more recent texts, the term 'conjugacy' is used for a homomorphism of dynamical systems, and the systems in question are called 'conjugate'.

In studying the dynamics of a certain system, it is often useful to find a factor which has simpler properties. Usually, one considers the relatively 'tame' notion of equicontinuity:

1.0.30 Definition (equicontinuous dynamical system, [EE14])**.** A dynamical system (X, T) is called *equicontinuous* if and only if it is a metric system (with metric d), and for all $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(tx, ty) < \epsilon$ for all $t \in T$.

In fact, to each compact Hausdorff dynamical system, we may associate an equicontinuous factor which is in some strict sense 'maximal':

1.0.31 Definition (maximal equicontinuous factor, [EE14])**.** A dynamical system (Y, T) is the *maximal equicontinuous factor* of a system (X, T) if and only if (Y, T) is an equicontinuous factor of (X, T) and whenever (Z, T) is an equicontinuous factor of (X, T), then (Z, T) is also a factor of (Y, T).

By an application of Zorn's Lemma, one can show that the maximal equicontinuous factor always exists [EE14].

We are interested in the way points in the dynamical system behave 'eventually' under the (semi-)group action. To make this more specific, we introduce the following notions.

1.0.32 Definition (positively/negatively asymptotic points, [BG13])**.** Let X be a dynamical system over $\mathbb{Z}$ via the homeomorphism $f : X \to X$, and let $x, y \in X$. We say x and y are *positively (resp, negatively) asymptotic* if and only if $\lim_{n\to+\infty} f^n x = \lim_{n\to+\infty} f^n(y)$ (respectively, $\lim_{n\to-\infty} f^n x = \lim_{n\to-\infty} f^n(y)$).

Asymptoticity is a fairly restrictive notion. It is often useful to consider points which are not quite asymptotic, but satisfy the milder condition of proximality, defined below.

1.0.33 Definition (proximal/distal points, distal system, [EE14])**.** Let (X, T) be a dynamical system. We call two points $x, y \in X$ *proximal* if and only if there is a point $z \in X$, and a net $\{t_\alpha\}_{\alpha\in A} \subset X^X$ such that $\lim_{\alpha\in A} t_\alpha x = \lim_{\alpha\in A} t_\alpha y = z$. A point $x \in X$ is called *distal* if and only if whenever $y \in X$ is proximal to x, then $y = x$.

1.0.34 Definition (distal system, [EE14])**.** A dynamical system (X, T) is called *distal* if and only if every point in X is distal.

It is easy to see that every asymptotic pair of points is proximal. However, there are examples of proximal pairs which are not asymptotic, as we will see in the Thue-Morse system introduced in the following section [BG13]. Distal systems will be considered in more detail in Section **??**. A generalisation of distal systems will be studied in Chapter 4, where we will link properties of the type of extension to the maximal equicontinuous factor with properties of the Furstenberg topologies which are not always Hausdorff.

1.1 Symbolic Systems

Most of the dynamical systems we consider from now on come from certain symbolic systems. We will now formally introduce these systems. For these notions and results, we use [dV93] and [BG13]. A few notions can be found in [Fog02].

1.1.1 Definition (alphabet, letters, finite, infinite, bi-infinite words, concatenation, [dV93])**.** We call a finite set $\mathcal{A}$ an *alphabet*, and its elements $a \in A$, *letters*. The set $\mathcal{A}^{<\mathbb{N}}$ of *finite words*, also called *blocks*, over the alphabet $\mathcal{A}$, consists of elements formed by concatenation of finitely many letters, in other words $w \in \mathcal{A}^{<\mathbb{N}}$ has the form $w = a_0 \dots a_n$ where $a_i \in \mathcal{A}$ for $i = 0, \dots, n$. Given two finite words $w = w_0 \dots w_n$ and $v = v_0 \dots v_m$ we may *concatenate* the two by writing $vw = v_0 \dots v_m w_0 \dots w_n$; the operation of concatenation makes the set $\mathcal{A}^{<\mathbb{N}}$ into a semigroup. Similarly, we form the set $\mathcal{A}^{\mathbb{N}}$ of *(right-) infinite words* and the set $\mathcal{A}^{\mathbb{Z}}$ of *bi-infinite words*, with elements $w = a_0 \dots a_n \dots \in \mathcal{A}^{\mathbb{N}}$ and $w = \dots a_{-1} \cdot a_0 a_1 \dots \in \mathcal{A}^{\mathbb{Z}}$, respectively. The dot gives us some sense of 'center'.

1.1.2 Example. Taking the binary alphabet $\mathcal{A} = \{0, 1\}$, we give the following examples of finite, infinite, and bi-infinite words:

$$\begin{aligned} 101010 &\in \mathcal{A}^{<\mathbb{N}} & 100 &\in \mathcal{A}^{<\mathbb{N}} \\ 101010\dots &\in \mathcal{A}^{\mathbb{N}} & 100000\dots &\in \mathcal{A}^{\mathbb{N}} \\ \dots 0101010 \cdot 10101 \dots &\in \mathcal{A}^{\mathbb{Z}} & \dots 0000 \cdot 1000 \dots &\in \mathcal{A}^{\mathbb{Z}}. \end{aligned}$$

1.1.3 Definition (length of a word, [dV93])**.** We define the *length* of a finite word $w = w_0 \dots w_n$ by $|w| = n + 1$. For a letter $a \in \mathcal{A}$, we will denote by $|w|_a$ the number of occurrences of the letter a in the word w.

1.1.4 Example. Using the finite words from the previous example (1.1.2), we have $|101010| = 6$, and $|100| = 3$, and also $|100|_1 = 1$, and $|100|_0 = 2$.

1.1.5 Definition (metric on $\mathcal{A}^{\mathbb{N}}$, resp $\mathcal{A}^{\mathbb{Z}}$, [dV93])**.** We define a metric d on $\mathcal{A}^{\mathbb{N}}$ by $d(v, w) = 0$ if and only if $v = w$, and else $d(v, w) = 1/2^k$, where $k = \min\{n \in \mathbb{N} : v_n \neq w_n\}$. Similarly, we define a metric on $\mathcal{A}^{\mathbb{Z}}$ by $d(v, w) = 1/2^k$, if $0 \neq k = \min\{n \in \mathbb{N} : v_n \neq w_n \text{ or } v_{-n-1} \neq w_{-n-1}\}$, and again set $d(v, w) = 0$ if and only if $v = w$.

It can be shown that with the metric d, $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ are compact spaces [dV93].

1.1.6 Example. Returning to the words from Example 1.1.2, we have:

$$\begin{aligned} d(101010\ldots, 1000\ldots) &= \frac{1}{4} \\ d(\ldots 0101010 \cdot 101010\ldots, \ldots 0000 \cdot 1000\ldots) &= \frac{1}{2}. \end{aligned}$$

1.1.7 Definition (bar operation, dual word, [dV93])**.** When $\mathcal{A}$ is the binary alphabet $\{0, 1\}$, we define the *bar operation* first by $\overline{0} = 1$, $\overline{1} = 0$, and extend this to a (finite, infinite, bi-infinite) word w by $(\overline{w})_i := \overline{w_i}$. In other words, $\overline{w}$ swaps all the 0's and 1's in w. The word $\overline{w}$ will be called the *dual* of w.

1.1.8 Example. Noting that the finite blocks from Example 1.1.2 are binary, we have that $\overline{101010} = 010101$ and $\overline{100} = 011$.

This dual operation will play an important part in expressing the fixed points of some key words later on, such as the Thue-Morse one, as well as expressing some important concepts (called Morse-like substitutions).

1.1.9 Definition (shift, [dV93])**.** We define the *shift* $s : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by $s(w)_i = w_{i+1}$, i.e. $s(w) = w_1 w_2 w_3 \ldots$. Similarly, we may define the *shift* on $\mathcal{A}^{\mathbb{Z}}$ by $s : \mathcal{A}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}$, $s(w) = \ldots w_{-2} w_{-1} w_0 \cdot w_1 w_2 \ldots$.

Note that the shift is not invertible on the space $\mathcal{A}^{\mathbb{N}}$, but is invertible on $\mathcal{A}^{\mathbb{Z}}$. It is easy to check that s is continuous, and that $s : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is 1-1 and with continuous inverse.

The shift operation is illustrated below:

$$\begin{tikzcd}[column sep=small]
w \arrow[d, "s"'] & = & \ldots & w_{-3} & w_{-2} \arrow[dl] & w_{-1} \arrow[dl] & \cdot & w_0 \arrow[dll] & w_1 \arrow[dl] & w_2 \arrow[dl] & w_3 \arrow[dl] & \ldots \\
u = s(w) & = & \ldots & w_{-2} & w_{-1} & w_0 & \cdot & w_1 & w_2 & w_3 & w_4 & \ldots
\end{tikzcd}$$

So, how do we obtain a dynamical system from these symbolic notions?

1.1.10 Definition (shift-orbit, shift-orbit closure, [dV93])**.** For a word x in $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$, we define the *shift orbit* of x by $O_x := \{s^n(x) : n \in \mathbb{N}\} \subset \mathcal{A}^{\mathbb{N}}$, respectively $O_x := \{s^n(x) : n \in \mathbb{Z}\} \subset \mathcal{A}^{\mathbb{Z}}$. For a word x in $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$, we define the *shift-orbit closure* as $\overline{O_x} \subset \mathcal{A}^{\mathbb{N}}$, respectively $\overline{O_x} \subset \mathcal{A}^{\mathbb{Z}}$.

Note that the shift orbit closure is a closed, hence compact, subset of $\mathcal{A}^{\mathbb{Z}}$ (or $\mathcal{A}^{\mathbb{N}}$), which is invariant under the shift operator, i.e. $s(\overline{O_x}) \subseteq \overline{O_x}$. Hence, $(\overline{O_x}, s)$ is a dynamical system (with acting (semi-) group $\mathbb{Z}$ or $\mathbb{N}$, respectively). Sometimes, we will call such a system a *shift space*.

We will mostly be concerned with symbolic systems which are associated with a certain type of 'rule', made more rigorous in the following definition:

1.1.11 Definition (substitution, [dV93])**.** A *substitution* is a map $\sigma : \mathcal{A} \to \mathcal{A}^{<\mathbb{N}}$ from the alphabet to the set of finite words over the alphabet. This map can be extended to a map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by concatenation, so for $w \in \mathcal{A}^{\mathbb{N}}$, $\sigma(w) = \sigma(w_0)\sigma(w_1)\ldots$. Similarly, the map can be extended to the space $\mathcal{A}^{\mathbb{Z}}$ by setting $\sigma(w) = \ldots\sigma(w_{-2})\sigma(w_{-1}) \cdot \sigma(w_0)\sigma(w_1)\ldots$.

1.1.12 Definition (primitive substitution, [dV93])**.** A substitution σ is called *primitive* if and only if there exists a positive integer k such that, for every $a, b \in \mathcal{A}$, the letter a occurs in $\sigma^k(b)$.

1.1.13 Definition ([dV93])**.** For a primitive substitution θ, there is at least one periodic point, i.e. a word $w \in \mathcal{A}^{\mathbb{N}}$ such that for some $n \in \mathbb{N}^+$, $\theta^n(w) = w$. Without loss of generality in what follows, we may consider θ^n instead of θ, so instead of 'periodic', we will call such a w a *fixed point* of θ.

1.1.14 Definition (constant length substitution, [dV93])**.** If there is a number $n \in \mathbb{N}^+$ such that for all letters a, $|\theta(a)| = n$, we say θ is of *constant length* and call the number n its *length*.

1.1.15 Definition (coincidence free or simple substitution, [dV93])**.** A constant-length substitution σ of length r is called *coincidence-free* or *simple* if and only if for each $a, b \in \mathcal{A}$, and for all indexes $i \in \{0, \ldots, r-1\}$, we have that $\sigma(a)_i \neq \sigma(b)_i$.

As we have noted above, one can show that every primitive substitution has a power n such that σ^n has a fixed point in $\mathcal{A}^{\mathbb{N}}$ (resp, $\mathcal{A}^{\mathbb{Z}}$). In Proposition 1.4.7, we will see that the shift-orbit closures of any two such fixed points coincide (as sets) and give rise to a unique dynamical system (X_σ, s) which is associated with the substitution.

1.1.16 Definition (admissible substitution, [dV93])**.** If $\theta(a) \neq \theta(b)$ for all letters $a \neq b$, and its fixed point is not a periodic word, we say θ is *admissible*.

We shall be interested in a special type of constant length substitutions:

1.1.17 Definition (continuous substitution, [CK71])**.** Following the terminology of Coven and Keane [CK71], if $\mathcal{A} = \{0, 1\}$, θ is admissible and of constant length, and $\theta(0) = \overline{\theta(1)}$, we say θ is a *continuous substitution*.

A particularly interesting continuous length substitution which will be a running example throughout Part II is the Prohuet-Siegel-Thue-Morse substitution. For short, we will follow [dV93] and call this the Thue-Morse substitution.

1.1.18 Example (The Thue-Morse substitution, [dV93])**.** Let us take the alphabet $A = \{0, 1\}$ and the substitution σ where $\sigma(0) = 01$ and $\sigma(1) = 10$. Then σ is a primitive substitution with $k = 1$. Also, σ^2 has four fixed points.

1.1.19 Definition (Thue-Morse word, [dV93])**.** The *Thue-Morse word* is defined as the right-infinite fixed point of σ^2 starting from the letter 0:

$$\lim_{n\to\infty} \sigma^{2n}(0) = 0110100110010110\ldots$$

As noted before, we will usually denote the nth letter of a bi-infinite word w by w_n. This is useful in many cases, for example to give a rule by which the Thue-Morse word can be described:

$$w_n = \mathbf{s}_2(n) \mod 2,$$

where $\mathbf{s}_2(n)$ is the sum of the non-zero coefficients in the binary expansion of the number n.

However, there are cases when such notation is cumbersome, for example, when the indexes are long and involved. For the Thue-Morse word w as above, we have that:

$$w_{2^n+2^{n+1}+2^{n+2}} = 1 \ \forall n \in \mathbb{N}.$$

In such cases, it would be more useful to denote the nth letter of a word w by $w[n]$. This notation carries another advantage, namely that instead of writing $w_n \dots w_k$, we can write $w[n;k]$ to denote the subword of w of length $k-n$ starting from the letter $w[n]$ (this notation will make sense only when $k > n$). Thus, the above expression can be written more clearly as:

$$w[2^n + 2^{n+1} + 2^{n+2}] = 1 \ \forall n \in \mathbb{N}.$$

However, at other times we may consider maps of functions of words, so to avoid piling up many parentheses, the subscript notation would be more legible and preferable. Therefore, from now on we will use the two notations, $w_n = w[n]$, interchangeably, to improve readability.

It would sometimes be useful to consider not all n-letter words over an alphabet $\mathcal{A}$, but only those which can be in some sense 'obtained' from a given substitution.

1.1.20 Definition (legal words, P_θ, [dV93], [Mar71])**.** A finite word $A \in \mathcal{A}^{<\mathbb{N}}$ is called θ-*legal* if and only if there is a word $y \in X_\theta$ such that A appears in y. We denote by P_θ the set of all θ-legal two-letter words, and by $\mathcal{L}(m)$ the set of all θ-legal m-letter words.

Given two symbolic dynamical systems (X, s) and (Y, s), one might wonder what type of homomorphisms might be induced. An important type are the so-called sliding block codes, given below.

1.1.21 Definition (sliding block code, [LM95])**.** Let $w = \dots w_{-1} \cdot w_0 w_1 \dots$ be a bi-infinite word in a shift space X over $\mathcal{A}$. We can transform w into a new sequence $v = \dots v_{-1} \cdot v_0 v_1 \dots$ over another alphabet $\mathcal{B}$ as follows. Fix integers m and n with $-m \leqslant n$. To compute the ith coordinate v_i of the transformed sequence, we use a function Φ that depends on the "window" of coordinates of

w from $i-m$ to $i+n$. Here $\Phi : \mathcal{L}(m+n+1) \to \mathcal{B}$ is a fixed *block map*, called an $(m+n+1)$-*block map* from allowed $(m+n+1)$-blocks in X to symbols in $\mathcal{A}$, and so

$$v_i = \Phi(w_{i-m}w_{i-m+1}\dots w_{i+n}) = \Phi(w[i-m, i+n]). \tag{1.1}$$

Now, let X be a shift space over $\mathcal{A}$, and $\Phi : \mathcal{B}(n+m+1) \to \mathcal{B}$ be a block map. Then the map $\phi : X \to \mathcal{B}^{\mathbb{Z}}$ defined by $v = \phi(w)$ with v_i given by (1.1) is called the *sliding block code* with *memory* m and *anticipation* n *induced by* Φ. We will denote the formation of ϕ from Φ by $\phi = \Phi_\infty^{[-m,n]}$, or more simply by $\phi = \Phi_\infty$ if the memory and anticipation of ϕ are understood. If Y is a shift space contained in $\mathcal{B}^{\mathbb{Z}}$ and $\phi(X) \subseteq Y$, we write $\phi : X \to Y$.

To check that a given homomorphism between symbolic dynamical systems is indeed a sliding block code, we have the following theorem:

1.1.22 Theorem ([LM95]). *Let X and Y be shift spaces. A map $\phi : X \to Y$ is a sliding block code if and only if $\phi \circ s_X = s_Y \circ \phi$ and there exists $N \geqslant 0$ such that $\phi(w)_0$ is a function of $w[-N, N]$.*

1.2 Minimal Dynamical Systems

The majority of the dynamical systems studied in this thesis are in some sense 'minimal'. For background and standard results about minimal dynamical systems, we will use [dV93] or [EE14]. Minimal systems can be thought of as the building blocks of 'larger' dynamical systems, hence the interest in understanding their properties. To rigorously introduce the notion of minimality, we first need the following definition.

1.2.1 Definition (invariant set, [EE14]). Let (X, T, π) be a dynamical system. We say that $A \subset X$ is *invariant* if $TA := \{ta : a \in A, t \in T\} \subset A$. If A is also closed, the resulting system (A, T, π) is called a *subsystem* of (X, T, π); note that the restriction of π to $A \times T$ defines an action of T on A.

1.2.2 Example. Take $\mathcal{A} = \{0, 1\}$, and consider the set of all bi-infinite binary sequences $\mathcal{A}^{\mathbb{Z}}$. The set $Y \subset \mathcal{A}^{\mathbb{Z}}$ given by $x \in Y$ if and only if x has finitely many

0's, is an invariant subset of $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \pi)$ - a left shift does not change the number of 0's in a sequence. Note, however, that $(Y, \mathbb{Z}, \pi)$ is not a subsystem, as Y is not closed (to see this, note that the zero sequence can be approximated arbitrarily well by sequences with finitely many 0's, but it itself is not an element of Y). However, $(\overline{Y}, \mathbb{Z}, \pi)$ is a subsystem of $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \pi)$ - in fact, $(\overline{Y}, \mathbb{Z}, \pi) = (\mathcal{A}^{\mathcal{A}}, \mathbb{Z}, \pi)$!

1.2.3 Definition (minimal set, minimal dynamical system, [EE14])**.** A subset M of the dynamical system (X, T, π) is called *minimal* if M is nonempty, closed, invariant, and minimal with respect to these properties. In other words, if $N \subseteq M$ and N is nonempty closed and invariant, then $N = M$. The dynamical system (X, T, π) is called *minimal* if and only if the set X is minimal.

Recall Example 1.0.28, where X is any compact Hausdorff topological space and f is any continuous map $f : X \to X$. We define an action of the semigroup $\mathbb{N}^+$ on X via the maps $\pi(x, n) = f^n(x)$. Moreover, if f is a homeomorphism of X, then the maps $\pi(x, n) = f^n(x)$ give an action of the group $\mathbb{Z}$ on X.

1.2.4 Example. In Example 1.0.28, assume further that f has a fixed point $x \in X$. Then $\{x\}$ is an invariant subset of the dynamical system (X, T, π). Since we assumed X to be Hausdorff, this is also a closed subset of X, so $(\{x\}, T, \pi)$ is in fact minimal. Similarly, the orbit of any periodic point is an invariant subset of a dynamical system, and is in fact a minimal system.

1.2.5 Proposition ([EE14])**.** *For a dynamical system (X, T, π) and a closed invariant set $M \subset X$, the following are equivalent:*

1. *M is minimal,*
2. *$\overline{Tx} = M$ for all $x \in M$,*
3. *if $U \subset X$ is open with $M \cap U \neq \emptyset$, then $M = T(U \cap M)$.*

We have the following closely related notion:

1.2.6 Definition (point transitive system, [EE14])**.** We call a system (X, T) *point transitive* if and only if there is a point $x_0 \in X$ with a dense orbit.

1.2.7 Remark. Note that it is not enough for a subsystem to be point transitive for it to be minimal. For example, $(\{0,1\}^{\mathbb{Z}}, s)$ has a point with dense orbit, namely the point given by concatenating first all the words of one letter, then all finite two-letter words, etc: $\omega = 0\,1\,00\,01\,10\,11\,000\,001\,\ldots$. However, this space is not minimal, as it has many closed invariant subsets. Thus, we need the orbit of *every* point to be dense for minimality to occur.

But do minimal sets always exist? An application of Zorn's Lemma (Theorem 0.1.15) can be used to show that:

1.2.8 Proposition ([EE14]). *Every dynamical system (X, T, π) has a minimal subset.*

It is natural to ask how minimal sets behave under homomorphisms of dynamical systems.

1.2.9 Proposition ([EE14]). *Let $\phi : (X, T) \to (Y, T)$ be a homomorphism of dynamical systems.*

1. *If M is a minimal subset of X, then $\phi(M)$ is a minimal subset of Y.*
2. *If N is a minimal subset of $\phi(X)$, then there exists a minimal subset M of X with $\phi(M) = N$.*

We can translate minimality in terms of sequences in the following way:

1.2.10 Definition (minimal sequence, [Fog02]). Let $x \in A^{\mathbb{Z}}$, in other words, let x be a bi-infinite sequence, and consider its shift-orbit $O(x) = \{s^n(x) : n \in \mathbb{Z}\}$. Then we say that x is *a minimal sequence* if and only if the dynamical system $(\overline{O(x)}, s)$ is minimal.

1.2.11 Definition (syndetic set, [EE14]). A subset P of $\mathbb{N}$ or $\mathbb{Z}$ is called *syndetic* if and only if there is a positive integer N such that all of the gaps between two consecutive elements of P are bounded by N, i.e. smaller than N.

1.2.12 Proposition ([EE14]). *A sequence $x \in A^{\mathbb{Z}}$ is minimal if and only if for every finite subword u of the word x, the following set is syndetic:*

$$x|_u := \{n \in \mathbb{Z} : x_n x_{n+1} \ldots x_{n+|u|-1} = u\}.$$

In other words, every word occurring in x, occurs in an infinite number of positions with bounded gaps. This motivates an alternative name for minimal sequences - *uniformly recurrent sequences.*

The link between minimality and primitivity (see Definition 1.1.12) can be seen via the following proposition:

1.2.13 Proposition ([EE14]). *If a substitution σ is primitive, then any of its periodic points is a minimal sequence.*

In particular, we have that:

1.2.14 Corollary ([Fog02]). *The Thue-Morse substitution is primitive, hence any of the four fixed points of its square is a minimal sequence.*

1.3 Required Background from other fields

1.3.1 Inverse Limits and Adding Machines

1.3.1 Definition (inverse limit of a sequence of groups, [dV93]). The *inverse limit of a sequence of groups* comprises of a sequence of groups $G_0, G_1, G_2, \ldots$ and homomorphisms $\phi_n : G_n \to G_{n-1}$, $n \geqslant 1$. The *inverse limit group* G_∞ of such a sequence is the collection of all sequences $(g_0, g_1, g_2, \ldots)$ with g_i in G_i and such that $g_i = \phi_{i+1}(g_{i+1})$ for all i. The product of two such elements, say $(f_0, f_1, \ldots)$ and $(g_0, g_1, \ldots)$ is given by the formula

$$\{f_i\} \cdot \{g_i\} = \{f_i \cdot g_i\},$$

where the dot on the right indicates the group operation in G_i. One may show that G_∞ is indeed a group. Note that G_∞ always contains at least one element, namely $(e_0, e_1, \ldots)$, where e_i denotes the identity element of G_i.

A similar definition can be made for inverse limits of dynamical systems.

The following well-known results can be found in [dV93].

1.3.2 Proposition ([dV93]). *Every factor of an inverse limit of equicontinuous (similarly, distal) compact dynamical systems is again an equicontinuous (respectively, distal) compact system.*

1.3.3 Proposition ([dV93]). *Let (Z, T) be a compact dynamical system. An inverse limit of compact proximal (respectively - distal) extensions of (Z, T) is also a proximal (respectively - distal) extension.*

1.3.4 Definition ($\mathbb{Z}(r)$, [dV93]). Let $\mathbb{Z}(r)$ be the *r-adic adding machine*, defined as follows. We consider this as the set of all sequences $z_0z_1z_2\ldots$, where $z_i \in \{0, \ldots, r-1\}$ for $i \geqslant 0$. Such a sequence will be viewed as a formal r-adic expansion $z_0+z_1r+z_2r^2+\ldots$, and the group addition is defined accordingly with carry. We define a metric ρ on $\mathbb{Z}(r)$ as follows: $\rho(\{a_i\}, \{b_i\}) = 1/(k+1)$, where $k := \max\{j : a_i = b_i \text{ for } i = 0, \ldots, j-1\}$, if $a_0 = b_0$, and as $\rho(\{a_i\}, \{b_i\}) = 1$ otherwise. The map $T : \mathbb{Z}(r) \to \mathbb{Z}(r)$ is the homeomorphism of $\mathbb{Z}(r)$ onto itself corresponding to addition of the group element $100\ldots$. We denote by $\mathcal{Z}(r)$ the dynamical system $(\mathbb{Z}(r), T)$. By an *integer* in $\mathbb{Z}(r)$, we mean an element of the form $T^n(000\ldots)$, for $n \in \mathbb{Z}$. Note that a positive integer will have infinitely many 0s in its tail, while a negative integer will have infinitely many 1s in its tail. Correspondingly, a *non-integer* is any element not of this form (note that it will have infinitely many 0s and infinitely many 1s in its tail).

1.3.5 Definition ($\mathbb{Z}_m$, [dV93]). We denote by $\mathbb{Z}_m$ the cyclic group of order m, where $\mathbb{Z}_m$ acts on itself via addition modulo m.

1.3.2 The Stone-Čech Compactification

The running example for this section will be the natural numbers, $\mathbb{N}$. We note that most of the subsequent theorems and constructions do not rely on the fact we chose $\mathbb{N}$ as the underlying set - the arguments will carry through just as easily if we were using any other discrete topological space X. The main source for this section is [HS98], though we complement the exposition with that from other sources, which we will cite where appropriate.

1.3.6 Definition (filter, principal filter [HS98]). Given a space X, a *filter $\mathcal{F}$ on X* is a nonempty set of subsets of X, $\emptyset \neq \mathcal{F} \subset \mathcal{P}(X)$, such that:

1. $\emptyset \notin \mathcal{F}$,

2. If $U \in \mathcal{F}$ and $V \supset U$, then $V \in \mathcal{F}$ (so $\mathcal{F}$ is closed with respect to supersets), and

3. If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$ (so F is closed with respect to finite intersections).

We call a filter $\mathcal{F}$ *principal* if and only if there is a set $A \subset X$ such that $\mathcal{F} = \{U \subset X : A \subset U\}$, in other words, $\mathcal{F}$ is the collection of all supersets of the set $\mathcal{A}$.

Note that if $\mathcal{F}$ is a filter on X, then it is a collection of sets which satisfy the finite intersection property. In other words, every finite collection of nonempty subsets of a filter has a nonempty intersection.

1.3.7 Definition (basis for a filter, [HS98])**.** Let X be a set, $\mathcal{F}$ a filter on X. Then $\mathcal{B} \subset \mathcal{P}(X)$ is a *basis for the filter* $\mathcal{F}$ if and only if

$$\mathcal{F} = \{F \subset X : \exists B \in \mathcal{B} : B \subset F\}.$$

Let X be a set, and let $\emptyset \neq \mathcal{B} \subset \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a unique filter on X if and only if for all $B_1, B_2 \in \mathcal{B}$, there is a $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. Also, if $\mathcal{B}$ satisfies FIP (recall Definition 0.1.8), then $\mathcal{B}$ can be extended to a basis for a unique filter on X. Moreover, if $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ is a chain of filters on X, i.e. for each λ, F_λ is a filter on X and for all $\lambda, \mu \in \Lambda$, $\mathcal{F}_\lambda \leqslant \mathcal{F}_\mu$ or $\mathcal{F}_\mu \leqslant \mathcal{F}_\lambda$, then $\bigcup_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a filter on X.

1.3.8 Note ([HS98])*.* By Zorn's Lemma, there exist maximal elements in the space of all filters on a set X, if we consider it as partially ordered by $\subseteq$. We call these maximal elements *ultrafilters.*

Ultrafilters were first defined and shown to exist on $\mathbb{N}$ by F. Riesz in [R.09] and Ulam [Ula29], respectively. One can see that the principal filter at any point $x \in X$ is maximal, i.e. an ultrafilter.

1.3.9 Lemma ([HS98])**.** *Suppose X, Y are sets, $\mathcal{F}$ is a filter on X and $f : X \to Y$ is continuous. Then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is not a filter in general, but is always a filter base. If $\mathcal{F}$ is an ultrafilter on X then the filter generated by $f(\mathcal{F})$ is an ultrafilter on Y.*

A compactification is roughly a compact space which, in some sense, preserves "most" of the properties of the original space. To put this more precisely,

1.3.10 Definition (compactification, [HS98])**.** Let X be a (Hausdorff, Tychonoff) topological space. A topological space Y is called a *compactification* of X if and only if Y is compact, and there is a homeomorphic embedding $h : X \to Y$ such that $h(X)$ is dense in Y. We will sometimes call the pair (Y, h) a compactification of X, and moreover, will sometimes identify X with its image under h in Y.

Besides the Ellis semigroup, we will sometimes consider another compactification of the acting group, which is in some sense a 'maximal' element of the partially ordered space of compactifications of a given space X. This is known as the Stone-Čech compactification. So, what is a Stone-Čech compactification? It is a compactification of a space which has the following 'maximality' property:

1.3.11 Definition (Stone-Čech compactification, [HS98])**.** Let X be a Tychonoff (Hausdorff) space. A pair, consisting of a compact space and a continuous map from X into the space, is called a *Stone-Čech compactification* of X, and denoted by $(\beta X, \beta)$, if it is a Hausdorff compactification of X satisfying the following universal property:

(SC) For each compact Hausdorff space Y and each continuous mapping $f : X \to Y$, there is a uniquely determined continuous mapping $\beta f : \beta X \to Y$ such that

$$\beta f \circ \beta = f.$$

When a space satisfies the condition in Definition 1.3.11 we say that it has the *Stone-Čech property.*

We now introduce the topology on $\beta\mathbb{N}$:

1.3.12 Definition (the topological space $\beta\mathbb{N}$, [HS98])**.** We define the space $\beta\mathbb{N}$ as the set of all ultrafilters on $\mathbb{N}$:

$$\beta\mathbb{N} := \{p \subseteq \mathcal{P}(\mathbb{N}) : p \text{ is an ultrafilter on } \mathbb{N}\}.$$

We define a topology on $\beta\mathbb{N}$ through basic open sets $\hat{A}$ as follows. For a set $A \subseteq \mathbb{N}$, define $\hat{A}$ as the collection of all ultrafilters on $\mathbb{N}$ which contain the set A:

$$\hat{A} := \{p \in \beta\mathbb{N} : \ A \in p\}.$$

This definition immediately yields the following equivalence:

$$p \in \hat{A} \Leftrightarrow \ A \in p. \tag{1.2}$$

The base for the topology on $\beta\mathbb{N}$ will be the collection of all $\hat{A}$, i.e.

$$\mathcal{B} := \{\hat{A} : A \subseteq \mathbb{N}\} = \{\hat{A} : A \in \mathcal{P}(\mathbb{N})\},$$

which is known as the *Stone topology* on $\beta\mathbb{N}$.

Let us now consider some properties of the basic open sets. The collection $\mathcal{B} = \{\hat{A} : A \subseteq \mathbb{N}\}$ is a basis of clopen sets for $\beta\mathbb{N}$. The topology generated by this basis makes $\beta\mathbb{N}$ into a compact Hausdorff space. Also, the topological space $\beta\mathbb{N}$ contains a dense subset which is homeomorphic to $\mathbb{N}$. Furthermore, the topological space $\beta\mathbb{N}$ satisfies the Stone-Čech property. Thus, we have that:

1.3.13 Theorem ([HS98])**.** *The space $\beta\mathbb{N}$ with the Stone topology is the Stone-Čech compactification of $\mathbb{N}$ as a discrete space.*

The space $(\beta\mathbb{N}, \tau)$ can also be endowed with an algebraic structure which in some natural sense 'extends' the semigroup structure of $(\mathbb{N}, +)$. We give the rigorous definition of addition in $\beta\mathbb{N}$ here, but what we need for the rest of this thesis will be the algebraic and topological properties of $\beta\mathbb{N}$, rather than make use of the specific way in which the semigroup operation of $\beta\mathbb{N}$.

1.3.14 Definition ($\mathcal{F} \oplus \mathcal{G}$, [HS98])**.** For two filters $\mathcal{F}, \mathcal{G}$ on a discrete semigroup X, we define $\mathcal{F} \oplus \mathcal{G}$ as:

$$\mathcal{F} \oplus \mathcal{G} := \{A \in \mathcal{P}(X) : \big\{x \in X : \{y \in X : xy \in A\} \in \mathcal{G}\big\} \in \mathcal{F}\}.$$

In our case, we often take $(\mathbb{N}, +)$ to be the discrete semigroup in question. Then, for a subset $A \subset \mathbb{N}$ a natural number $n \in \mathbb{N}$, and filters $\mathcal{F}, \ \mathcal{G}$ on $\mathbb{N}$, we have

$$\mathcal{F} \oplus \mathcal{G} = \{A \in \mathcal{P}(\mathbb{N}) : \{n \in \mathbb{N} : \{m \in \mathbb{N} : n + m \in A\} \in \mathcal{G}\} \in \mathcal{F}\}.$$

We list some facts from [HS98] about this operation. If $\mathcal{F}, \mathcal{G}$ are ultrafilters on $\mathbb{N}$, then so is $\mathcal{F} \oplus \mathcal{G}$. If moreover $\mathcal{F}$ and $\mathcal{G}$ are principal, then $\mathcal{F} \oplus \mathcal{G}$ is also a principal ultrafilter on $\mathbb{N}$. In addition, $\mathcal{F}_n \oplus \mathcal{F}_m = \mathcal{F}_{n+m}$, where $\mathcal{F}_n, \mathcal{F}_m$ are the principal ultrafilters at $\{n\}$, resp $\{m\}$. We have that the binary operation of addition, $\oplus : \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$, is continuous in the left argument. In other words, for each $\mathcal{G} \in \beta\mathbb{N}$, we have that the function $\rho_{\mathcal{G}} : \beta\mathbb{N} \to \beta\mathbb{N}$, given by $\rho_{\mathcal{G}}(\mathcal{F}) = \mathcal{F} \oplus \mathcal{G}$, is continuous.

To finish our exploration of the algebraic structure of $\beta\mathbb{N}$ that we have just introduced, we will define one of the most important (for our purposes) special elements in it:

1.3.15 Definition (idempotent, [HS98])**.** An element p of a semigroup G is called an *idempotent* if and only if $p + p = p$.

So, are there any such elements of $\beta\mathbb{N}$? Note that the existence of an idempotent ultrafilter in $\beta\mathbb{N}$ would be trivial if we let $0 \in \mathbb{N}$ (for then the principal ultrafilter at $\{0\}$ would be an idempotent). However, if we take $0 \notin \mathbb{N}$, we need to show the existence of a non-principal idempotent ultrafilter - something that is prima facie very difficult, as we have no concrete way of representing non-principal ultrafilters. The result we now turn to answers this question in the positive. It is originally due to Ellis, and proven in a more general setting of a compact right-topological semigroup [Ell58].

1.3.16 Theorem ([Ell58])**.** *Idempotent ultrafilters exist in $\beta\mathbb{N}$.*

In Chapter 3, we will consider a special type of subsets of the integers and natural numbers, called IP sets:

1.3.17 Definition (IP set, generating sequence, [HS98])**.** An IP set P in $\mathbb{N}$ (respectively, in $\mathbb{Z}$), is a subset of $\mathbb{N}$ (resp $\mathbb{Z}$) which coincides with the set of finite sums $p_{n_1} + \ldots + p_{n_k}$, for distinct indices $n_1 < n_2 < \ldots < n_k$, taken from a sequence $(p_n)_{n=1}^{\infty}$ of distinct elements in $\mathbb{N}$ (resp in $\mathbb{Z}$). The sequence $(p_n)_{n=1}^{\infty}$ is called the *generating sequence* of P.

We will construct certain generating sequences to some IP sets in order to provide a counterexample to [HJ97, Proposition 3.4], which is central to the main theorem of that paper. However, the main theorem - that a binary continuous substitution has an Ellis semigroup with two minimal ideals - still holds. Moreover, in Chapter 3 we generalise it for a certain class of substitutions over arbitrary finite alphabets.

1.4 Substitution Dynamics

We now return to the symbolic spaces introduced in Section 1.1. In this section, we will list some facts from substitution dynamical systems. Since our counterexample to Haddad and Johnson's proposition, which we give in Section 3.3, will require some specific properties of binary words, we will intersperse comments which are specific to the binary case whenever needed in the below discussion. We will also use the Thue-Morse substitution, $0 \mapsto 01$, $1 \mapsto 10$, as a running example.

1 Hypothesis. From now on, let θ be an admissible substitution of constant length r over the alphabet $\mathcal{A}$.

Let $w \in \mathcal{A}^{\mathbb{Z}}$ be any fixed point of θ. Let X_θ be the orbit closure of w in $\mathcal{A}^{\mathbb{Z}}$; it is well-known that X_θ does not depend on the choice of fixed point $w \in \mathcal{A}^{\mathbb{Z}}$. Then (X_θ, s) is the unique substitution dynamical system associated with θ.

1.4.1 Notation ([dV93]). We note that if a fixed point $x \in \{0,1\}^{\mathbb{N}}$ of a continuous binary uniformly recurrent sequence is not periodic, its set of legal subwords must be $\{01, 10, 00, 11\}$. Thus, it can be extended to a sequence in $\{0,1\}^{\mathbb{Z}}$ in precisely two ways, with the left extensions being dual to each other [HJ97]. In other words, if $x \in \{0,1\}^{\mathbb{N}}$ is recurrent and $y \in \{0,1\}^{\mathbb{Z}}$ is such that there are $\mu, \nu \in X_y \subset \{0,1\}^{\mathbb{Z}}$ with $\mu_i = \nu_i = x_i$ for $i \geqslant 0$, then either $\mu_j = \nu_j$ for all $j < 0$ or $\mu_j = \overline{\nu_j}$ for all $j < 0$. Moreover, if $y' \in \{0,1\}^{\mathbb{Z}}$ has the same property, then $X_y = X_{y'}$, in other words, the orbit-closure does not depend on the choice of fixed point. (In Proposition 1.4.7 below we will see this observation holds more generally.) From now on, whenever x is a right-hand fixed point of a continuous substitution, we will write ω and ν for its two left extensions.

1.4.2 Definition (basic r^k-block, [Mar71])**.** For $k \in \mathbb{N}^+$, we call a word B of length r^k a *basic r^k-block* if and only if there is a letter $a \in \mathcal{A}$ such that $\theta^k(a) = B$.

Recall Definition 1.1.20 of a θ-legal word - a finite word which appears as a sub-block of a word $y \in X_\theta$, and note the difference between a legal word and a basic block. Every basic block is legal, but not every legal word is a basic block.

1.4.3 Definition (coincidence-free, [dV93])**.** We call a substitution θ *coincidence-free* if and only if for all letters $a \neq b$, for all $n \in \{0, \ldots, r-1\}$, we have $\theta(a)_n \neq \theta(b)_n$.

Recall that a continuous substitution is a length binary substitution θ where $\theta(0) = \overline{\theta(1)}$ (definition 1.1.17), and note that any continuous substitution is coincidence-free.

2 Hypothesis. Note that for each constant length coincidence-free substitution, there is a power $n \in \mathbb{N}^+$ such that for any letter a, $\theta^n(a)_0 = \theta^n(a)_{r-1} = a$. From now on, without loss of generality, assume any coincidence-free θ is already in this standard form.

1.4.1 The Period Doubling System and Thue-Morse Systems

1.4.4 Definition (Period Doubling substitution, [BG13])**.** The *Period Doubling* substitution is the binary constant length substitution defined by the rule:

$$0 \mapsto 01$$
$$1 \mapsto 00.$$

We will denote by (X_{PD}, s) the resulting dynamical system.

Let us revisit the Thue-Morse substitution given in Example 1.1.18.

1.4.5 Definition (Thue-Morse substitution, [BG13])**.** The *Thue-Morse substitution* is defined over the binary alphabet $\{0, 1\}$ as:

$$0 \mapsto 01$$
$$1 \mapsto 10$$

Obviously, this is a constant length substitution. In light of Hypothesis 2, we may instead consider the square of this substitution, $0 \mapsto 0110,\ 1 \mapsto 1001$, so that the Thue-Morse has the following four fixed points:

$$\dots 0110 \cdot 0110 \dots$$
$$\dots 0110 \cdot 1001 \dots$$
$$\dots 1001 \cdot 0110 \dots$$
$$\dots 1001 \cdot 1001 \dots$$

1.4.6 Notation. In light of notation 1.4.1, we shall use the following notation in the context of the Thue-Morse sequence: $v := \dots 1001 \cdot 1001 \dots$ and $w := \dots 0110 \cdot 1001 \dots$. Then the four fixed points can be represented as the set $\{w, \overline{w}, v, \overline{v}\}$.

Recall Corollary 1.2.14, that any of the four fixed points forms a minimal sequence. Now, all we have to do to define the set is to show that the four minimal sets, defined as the orbit closures of the fixed points, coincide.

1.4.7 Proposition ([BG13])**.** *The orbit closures of each of the four fixed points under the two-sided shift map coincide as sets, i.e.*

$$\overline{Orb(v)} = \overline{Orb(\overline{v})} = \overline{Orb(w)} = \overline{Orb(\overline{w})}.$$

Now we are in a position to make the following definition:

1.4.8 Definition (Thue-Morse dynamical system, [BG13])**.** We define the *Thue-Morse dynamical system* as the orbit closure of any of the four fixed points of the Thue-Morse substitution. This set, denoted X_{TM}, together with the two-sided shift map, form a $\mathbb{Z}$-cascade.

1.4.9 Proposition ([BG13]). *The system (X_{TM}, s) factors onto the period doubling system (X_{PD}, s) via a 2-to-1 map.*

1.4.10 Remark ([BG13]). This map is a sliding block code with memory 1, anticipation 0, induced by the block map from the two-letter words P_{TM} given by $\Phi : P_{TM} \to \{0, 1\}$, given by $\{00, 11\} \mapsto 1$ and $\{10, 01\} \mapsto 0$.

Thus, we have the following diagram for the Thue-Morse system, where $(\mathbb{Z}(2), +)$ is the binary adding machine:

$$\begin{tikzcd}
(X_{TM}, s) \arrow[r, "\pi_1", "{\pi_1(w)=y_1,\ \pi_1(v)=y_2}"'] \arrow[rr, bend left, "\pi:=\pi_2\circ\pi_1"] & (X_{PD}, s) \arrow[r, "\pi_2", "{\pi_2(y_1)=\pi_2(y_2)=1}"'] & (\mathbb{Z}(2), +)
\end{tikzcd}$$

This diagram will be of great use in Chapter 3.2, where we will calculate the Ellis semigroup of this and more general systems.

1.5 The Ellis Semigroup

The Ellis semigroup is often useful in studying the 'eventual', or asymptotic, properties of a dynamical system. If we consider X^X to be the set of all (not necessarily continuous) functions from X to itself, then the Ellis semigroup of a dynamical system (X, T) is a closed subset of X^X. Moreover, if we endow X^X with the semigroup operation of composition of functions, then the Ellis semigroup is indeed a sub-semigroup with respect to the operation. Thus, let us start by familiarising ourselves with the space X^X.

1.5.1 Definition ([Eng89]). Let X be a topological space. Then X^X is the set of all functions from X to itself, with the topology of pointwise convergence.

The topology of pointwise convergence is also known as the Tychonoff or product topology [Eng89]. In it, a net of functions $\{f_n\}_{n\in\alpha} \subset X^X$ converges to a limit $f \in X^X$ if and only if for every point $x \in X$, the net $\{f_n(x)\}_{n\in\alpha}$ converges to $f(x)$ in the topological space X, [Eng89].

To give two very basic examples of the product space:

1.5.2 Example. Let $X = \{0, 1\}$, a space of two points, with any (fixed) topology. Then X^X consists of four functions, which, by setting $f = (f(0), f(1))$, can be visualised as points in the plane:

- the identity map: $f_1 = (0, 1)$
- a contraction map: $f_2 = (0, 0)$
- another contraction map: $f_3 = (1, 1)$
- the swap map: $f_4 = (1, 0)$.

If we endow X with the Sierpinski topology, making $\{0\}$ the only non-trivial open set, then the swap map is the only non-continuous map, as $f_4^{-1}(\{0\}) = \{1\}$, which is not open.

More generally, if X is any compact Hausdorff space, Tychonoff's Theorem (0.1.14) gives us that X^X is also compact (and Hausdorff). Thus, X^X has the desirable property that every infinite sequence has an accumulation point. But the space X^X can also be viewed as a semigroup. To be more precise about its structure, we have the following:

1.5.3 Proposition ([EE14]). *Let X be a compact Hausdorff space. Then:*

1. *composition of functions provides X^X with a semigroup structure;*
2. *composition is continuous on the right for all continuous $f \in X^X$, i.e. the map $\rho^f : X^X \to X^X$ given by $\rho^f(g) = f(g)$ is continuous for all continuous $f \in X^X$;*
3. *composition is always continuous on the left, i.e. the map $\rho_f : X^X \to X^X$ given by $\rho_f(g) = g(f)$ is continuous for all $f \in X^X$;*
4. *composition of functions defines an action of the semigroup X^X on the set X^X.*

If we have a subset $A \subset X^X$, it is sensible to consider its closure in X^X. This brings us to the definition of the Ellis semigroup:

1.5.4 Definition (Ellis semigroup, [EE14])**.** For a dynamical system (X, T), we define the *Ellis semigroup* (also know as the enveloping semigroup) as

$$E(X,T) := \overline{\{f \in X^X : \exists t \in T \text{ such that } \pi(x,t) = f(x)\ \forall x \in X\}} \subset X^X.$$

In other words, it is the 'closure of the (semi-)group T' when viewed as a subspace of X^X with the Tychonoff topology. When there is no ambiguity, we will write just $E(X)$. One can easily see that the Ellis semigroup is a compactification of $\mathbb{Z}$ (or of whatever (semi-)group is acting on the space).

The set $E(X, T)$ is a semigroup with respect to composition of functions.

Alternatively, we can define the Ellis semigroup via the Stone-Čech compactification of the acting group T. For the dynamical system (X, T), we consider the map from T to X^X which associates to each element $t \in T$ the respective homeomorphism in X^X of X, which we will also denote by t. By the properties of the Stone-Čech compactification, this map has a continuous extension $\Phi_X : \beta T \to X^X$. Thus, $\Phi_X(\beta T) = E(X, T)$.

1.5.5 Notation. In what follows, we will mostly consider dynamical systems over a certain fixed semigroup (or group) T. When the (semi-)group is understood, we can shorten the notation for the Ellis semigroup of the dynamical system to $E(X)$. In several sections, we will consider the Ellis semigroup of substitution dynamical systems. Since the group will always be $\mathbb{Z}$ and the space is uniquely determined by the substitution θ, we will write E_θ for even shorter notation for the Ellis semigroup of the substitution system.

In general, it is very difficult to calculate the Ellis semigroup of a dynamical system. However, we begin with a few easy examples.

1.5.6 Example (Identity)**.** Let X be any compact Hausdorff space, and i be the identity on it. Then $E(X, \{i^n\}_{n\in\mathbb{Z}}) = \{i\}$, since limit points are unique in this case.

1.5.7 Example (periodic examples)**.**

1. Let $\omega \in \mathcal{A}^{\mathbb{Z}}$ be periodic, and consider $X = \overline{O_s}(\omega)$. Then X has finitely many points, and is a discrete space in the subspace topology. Thus, $E(X, \{s^n\}_{n\in\mathbb{Z}}) = \{s^n : n \in \mathbb{Z}\}$, since any finite subset of a compact Hausdorff space is closed. If ω is period 2, and we

replace 0 by ω and 1 by $s(\omega)$ in Example 1.5.2 (and consider the discrete topology instead of the Sierpinski one), then $E(X, \{s^n\}_{n\in\mathbb{Z}}) = \{f_1, f_4\}$, and in general, $|E(X, \{s^n\}_{n\in\mathbb{Z}})| = k$, where k is the period of ω.

2. Let X be any compact Hausdorff space, and $f : X \to X$ be a homeomorphism such that $f^{-n} = f = f^n$. Then $E(X, \{f^n\}_{n\in\mathbb{Z}})$ will consist of finitely many functions (or points), and thus will be a closed subset of X^X. Thus, $E(X, \{f^n\}_{n\in\mathbb{Z}}) = \{f^n : n \in \mathbb{Z}\}$, as this set is already closed.

1.5.8 Example (Ellis semigroup of shift of one-point sequence)**.** Let $\omega = \ldots\overline{0}00 \cdot 10\overline{0}\ldots$, and let $X = O_{\mathbb{Z}}(\omega)$ - the shift-orbit closure of ω. Then $X = \{s^n(\omega) : n \in \mathbb{Z}\} \cup \{\ldots\overline{0} \cdot \overline{0}\ldots\}$. Then $E(X, \{s^n\}_{n\in\mathbb{Z}}) = \{s^n : n \in \mathbb{Z}\} \cup \{f_0\}$, where $f_0(u) = \ldots\overline{0} \cdot \overline{0}\ldots$ for any $u \in X$.

Though difficult to calculate, there are many ways in which the Ellis semigroup is a very natural object to associate to a dynamical system.

For example, how do the Ellis semigroups behave under homomorphisms? In fact, any surjective dynamical system homomorphism can be extended to a homomorphism between the respective Ellis semigroups:

1.5.9 Proposition ([EE14])**.** *Let $\pi : (X, T) \to (Y, T)$ be a surjective homomorphism of dynamical systems. There exists a unique map $\pi^* : E(X) \to E(Y)$ such that π^* is surjective and continuous, $\pi^*(pq) = \pi^*(p)\pi^*(q)$ for all $p, q \in E(X)$, and such that the following diagram is commutative:*

$$\begin{tikzcd} E(X) \arrow[r, "\pi^*", "p\mapsto q"'] \arrow[d, "p\mapsto px_0"'] & E(Y) \arrow[d, "q\mapsto q\pi(x_0)"] \\ X \arrow[r, "\pi", "x_0\mapsto\pi(x_0)"'] & Y \end{tikzcd}$$

The following proposition clarifies how the properties of the Stone-Čech compactification βT relate to the properties of the Ellis semigroup.

1.5.10 Proposition ([EE14])**.** *Let (X, T) be a dynamical system. Then:*

1. *The action $(p, t) \mapsto p\pi^t : E(X) \times T \to E(X)$, where $\pi^t = R_t : E(X) \to E(X)$ by $R_t(q) = qt$, makes $E(X, T)$ a point transitive dynamical system (i.e. there is a point with dense orbit).*

2. *The canonical map $\Phi_X : \beta T \to E(X)$ is both a dynamical system and a semigroup homomorphism.*

3. *The map $p \mapsto xp \ : E(X) \to X$ is a dynamical system homomorphism for all $x \in X$.*

4. *The map $\Phi_{\beta T} : \beta T \to E(\beta T)$ is an isomorphism.*

5. *Let $f : (X,T) \mapsto (Y,T)$ be a homomorphism of dynamical system s. Then $f(xp) = f(x)p$ for all $x \in X$ and $p \in \beta T$.*

1.5.1 Some Theorems and Propositions Around the Ellis Semigroup, or, 'Why is the Ellis Semigroup Useful?'

Unless otherwise stated, all notions and statements here can be found in [EE14].

1.5.11 Proposition ([EE14])**.** *Let (X,T) be a dynamical system. Then:*

1. *The set $E(X,T)$ is a dynamical system (with a point with dense orbit) under the action $E(X) \times T \to E(X)$ given by $(p,t) \mapsto \pi^t p$, where $\pi^t = L_t : E(X) \to E(X)$ by $\pi^t q = tq$.*

2. *The canonical map $\Phi_X : \beta T \to E(X)$ is both a dynamical system and a semigroup homomorphism.*

3. *For $x \in X$ arbitrary but fixed, the map $e_x : E(X) \to X$ given by $e_x(p) = p(x)$ is a homomorphism of dynamical systems.*

4. *The map $\Phi_{\beta T} : \beta T \to E(\beta T, T)$ is an isomorphism.*

5. *For a homomorphism of dynamical systems $f : (X,T) \to (Y,T)$, $f(px) = pf(x)$ for all $x \in X$ and all $p \in \beta T$.*

For what follows, recall Definition 1.2.3 of a minimal set - this is a nonempty closed, T-invariant subset of X which is minimal in this respect. Similarly, (X,T) is a minimal system whenever the space X is itself minimal. We do not necessarily have that the Ellis semigroup $E(X)$ of a minimal dynamical system is itself minimal - in fact, this is far from the case. To further consider this, let us introduce a more algebraic notion - that of a minimal ideal.

1.5.12 Definition ([EE14])**.** Let (X, T) be a dynamical system with Ellis semigroup $E(X)$. A nonempty subset I of $E(X)$ is a (left) *ideal* if and only if $EI \subset I$. The ideal is called *minimal* if and only if it contains no ideals as proper subsets.

Note that by the definition of an ideal $I \subset E(X)$, $TI \subset I$. So, if I is closed, then (I, T) is a dynamical subsystem of $E(X, T)$ (and so in fact $E(X, T)$ is not itself minimal if it has any nontrivial minimal ideals). In fact, we have that (I, T) is a minimal dynamical system:

1.5.13 Proposition ([EE14])**.** *Let (X, T) be a dynamical system with Ellis semigroup $E(X)$, and let $I \subset E(X)$ be an ideal. Then I is a minimal ideal if and only if I is closed and the dynamical system (I, T) is minimal.*

1.5.14 Corollary ([EE14])**.** *Let (X,T) be a dynamical system with Ellis semigroup $E(X)$. Then every ideal $I \subset E(X)$ contains a minimal ideal.*

This Corollary can alternatively be shown through a lemma of Ellis and Numakura [Ell58] and an application of Zorn's lemma.

1.5.15 Definition (idempotent, minimal idempotent, [EE14])**.** We call an element $u \in E(X)$ an *idempotent* if and only if $u^2 = u$. If $u \in I$ for some minimal ideal $I \in E(X)$, then we call u *minimal*.

1.5.16 Corollary (Ellis-Numakura Lemma, [Ell58, Num52])**.** *Let X be a compact Hausdorff semigroup such that the maps $R_x : X \to X$ (defined similarly as in Proposition 1.5.10) are continuous for all $x \in X$. Then there exists an idempotent $u \in X$.*

1.5.17 Corollary ([EE14])**.** *Let X be a compact T_1 group such that left multiplication is continuous, and let S be a closed sub-semigroup of X. Then S is a subgroup of X.*

1.5.18 Theorem ([EE14])**.** *Let (X, T) be a dynamical system and let $I \subset E(X)$ be a minimal ideal in its Ellis semigroup. Then:*

1. *The set J of idempotents of I is non-empty.*
2. $pv = p$ *for all* $v \in J$ *and* $p \in I$.
3. vI *is a group with identity* v*, for all* $v \in J$.
4. $\{vI : v \in J\}$ *is a partition of* I.
5. *If we set* $G := uI$ *for some* $u \in J$*, then* $I = \bigsqcup\{vG : v \in J\}$*, where* $\bigsqcup$ *denotes the disjoint union of the groups* vG.

1.5.19 Lemma ([EE14])**.** *Let* $\phi : E(X,T) \to E(Y,T)$ *be a homomorphism of dynamical systems, and let* $u \in E(Y,T)$ *be an idempotent. Then there is an idempotent* $v \in E(X,T)$ *such that* $u = \phi(v)$.

Here we provide a proof to ease the reader into the subject.

Proof. We have that $\phi^{-1}(u)$ is a closed subset of $E(X,T)$, since $E(Y,T)$ is Hausdorff and $E(X,T)$ is compact. Moreover, it is a semigroup: if $p, q \in \phi^{-1}(u)$, then $\phi(pq) = \phi(p)\phi(q) = uu = u$, so $pq \in \phi^{-1}(u)$. Thus, by the Ellis-Numakura Lemma 1.5.16, we have that there is an idempotent $v^2 = v \in \phi^{-1}(u)$. $\square$

1.5.20 Definition (equivalent idempotents, [EE14])**.** Let (X,T) be a dynamical system. If $u, v \in E(X,T)$ are idempotents such that $uv = v$ and $vu = u$, then we will call u and v *equivalent idempotents* and write this as $u \sim v$.

We expand on a note in [EE14] to show the relation between equivalent idempotents and minimal ideals:

1.5.21 Proposition ([EE14])**.** *The relation* $\sim$ *is an equivalence relation on the set of idempotents in* $E(X,T)$*. Moreover, the equivalence class* $[u]$ *of any minimal idempotent* $u \in E(X,T)$ *contains only minimal idempotents and intersects each minimal ideal at most once.*

Proof. The relation $\sim$ is obviously reflexive and symmetric; all we have to show is transitivity. Let $u, v, w \in E(X,T)$ be idempotents such that $u \sim v$ and $v \sim w$.

Then

$$\begin{aligned} uw &= u(vw) && \text{since } w \sim v \\ &= (uv)w = vw && \text{since } u \sim v \\ &= w && \text{since } w \sim v. \end{aligned}$$

Similarly, $wu = w(vu) = (wv)u = (v)u = u$, so indeed $u \sim w$ as required.

Now, let $I \subset E(X,T)$ be a minimal ideal, $u^2 = u \in I$, and consider its equivalence class $[u] := \{v \in E(X,T) : v \sim u\}$. Let $v \in [u]$; then the map $\rho_v : g \to gv$ from $E(X,T)$ to itself is continuous by Proposition 1.5.3. The system $(\rho_v[I],T)$ is minimal in $E(X,T)$ by Proposition 1.2.9 and contains v (since $uv = v$), hence $\rho_v[I]$ is a closed set. Hence, by Proposition 1.5.13, $\rho_v[I]$ is a minimal ideal in $E(X,T)$, so $v \in \rho_v[I]$ is a minimal idempotent, as required. □

1.5.22 Proposition ([EE14])**.** *Let I, K be minimal ideals in the Ellis semigroup $E(X)$ of the dynamical system (X,T), and let $u \in I$ be an idempotent. Then there exists a unique idempotent $v \in K$ which is equivalent to u.*

What is the relation between idempotents in different minimal ideals? We have the following proposition:

1.5.23 Proposition ([EE14])**.** *Let (X,T) be a dynamical system with Ellis semigroup $E(X)$, let I, K be minimal ideals in $E(X)$ and let $u^2 = u$ be an idempotent in I. Then there exists a unique idempotent $v \in K$ such that $u \sim v$. Moreover, if $u^2 = u \in E(X)$ is minimal, and $v \sim u$, then v is a minimal idempotent, as well.*

1.5.24 Proposition ([EE14])**.** *Let I, K be minimal ideals in the Ellis semigroup $E(X)$ of the dynamical system (X,T), let u be an idempotent in I and let v be the unique idempotent in K which is equivalent to u.*

1.5.25 Theorem ([EE14])**.** *Let M be a minimal subset of βT and (X,T) be a minimal dynamical system. Then there exists a surjective homomorphism of dynamical systems $f : M \to X$.*

We will mostly concern ourselves with minimal dynamical systems. However, we will still make use of non-minimal systems; in particular, the Ellis semigroup, when viewed as a system $(E(X,T),T)$, is usually not a minimal system. Thus, following [EE14], we introduce the following notion:

1.5.26 Definition (almost periodic point, [EE14])**.** Let (X,T) be a dynamical system and $x \in X$. Then x is an *almost periodic point* if and only if the orbit closure of x is minimal. We say that the dynamical system (X,T) is *pointwise almost periodic* if every point is almost periodic.

By Proposition 1.2.5, every minimal dynamical system is pointwise almost periodic.

1.5.27 Definition (set of return times, [EE14])**.** For any $x \in X$ and neighbourhood U of x we define the set of *return times to* U by $A(U) := \{t \in T : tx \in U\}$.

1.5.28 Proposition ([EE14])**.** *Let (X,T) be a dynamical system, x a point in X, and let $A(U)$ denote the set of return times to an open set U. Then x is an almost periodic point if and only if for every open neighbourhood U of x, there exists a finite set $F \subset T$ such that $FA(U) = T$. In other words, $A(U)$ is a* syndetic *subset of T.*

As the following proposition shows, the almost periodic points of (X,T) (and respectively, the notion of minimality) can be characterized in terms of the minimal idempotents in $E(X)$.

1.5.29 Proposition ([EE14])**.** *Let (X,T) be a dynamical system with Ellis semigroup $E(X)$, let $I \subset E(X)$ be a minimal left ideal and let $x \in X$. Then $\overline{\{t(x) : t \in T\}}$ is minimal if and only if there exists an idempotent $u^2 = u \in I$ with $ux = x$.*

There are many ways to characterize the dynamical notion of proximality via the Ellis semigroup $E(X)$ and the Stone-Čech compactification βT. We now give several equivalent definitions of proximality, found also in [EE14], where the first one will be the most useful throughout this thesis:

1.5.30 Proposition ([EE14])**.** *For two points x and y in the dynamical system (X,T), the following are equivalent:*

- *there is $p \in E(X)$ such that $px = py$,*

- *there exists a net* $\{t_i\} \subset T$ *with* $\lim t_i(x) = \lim t_i(y)$,
- $\overline{\{(tx, ty) : t \in T\}} \cap \Delta \neq \emptyset$, *where* Δ *is the diagonal in* $X \times X$,
- *there exists* $p \in \beta T$ *with* $p(x) = p(y)$,
- *there exists* $r \in E(X)$ *with* $r(x) = r(y)$,
- *there exists a minimal left ideal* $I \subset E(X)$ *with* $r(x) = r(y)$ *for all* $r \in I$,
- *there exists a minimal left ideal* $K \subset \beta T$ *with* $q(x) = q(y)$ *for all* $q \in K$.

In Chapter 2, Chapter 3 and Chapter 4 we will deal with distal and almost automorphic dynamical systems. Thus, it is useful to acquaint ourselves with the terminology and how these notions are reflected in the Ellis semigroup.

1.5.31 Definition (almost periodic point, [EE14])**.** Let (X, T) be a dynamical system. We say that $x \in X$ is *almost periodic* if and only if its orbit closure $\overline{xT}$ is minimal as a subset of (X, T).

1.5.32 Proposition ([EE14])**.** *Let* (X, T) *be a dynamical system with Ellis semigroup* E*, and let* $x \in X$*. Then* x *is almost periodic if and only if for any minimal ideal* $I \subseteq E$ *there is an idempotent* $u \in I$ *with* $ux = x$*.*

Proof. Let $x \in X$ be almost periodic and let $I \subseteq E$ be an arbitrary but fixed minimal ideal. Then $\phi_x : E \to X$ given by $p \mapsto px$ is a homomorphism of E onto $\overline{Tx}$, and $Ix = \phi_x(I)$ is closed and invariant subset of $\overline{Tx}$ as a continuous projection of I into X. Since x is almost periodic, this means that $Ix = \overline{Tx}$. Then the set $S = \{p \in I : px = x\}$ is a closed nonempty subsemigroup of I, so by the Ellis-Numakura Lemma 1.5.16 there exists an idempotent $u \in S(\subseteq I)$ with $ux = x$, as required.

Now, let $x \in X$ be such that for every minimal ideal I there is an idempotent $u \in I$ with $ux = x$. Then $Tx = T(ux) = (Tu)x \subseteq Ix$, and Ix is closed (recall $Tu \subseteq I$ and $TI \subset I$ since I is an ideal), so $\overline{Tx} \subseteq Ix$ and since I is minimal, $\overline{Tx} = Ix$. So $\overline{Tx}$ is minimal, so x is an almost periodic point, as required. $\square$

Recall Proposition 1.5.30 which gave alternative definitions of proximality. Also, we recall Definition 1.0.34 of a distal dynamical system - a system which has no proximal pairs.

1.5.33 Proposition ([EE14]). *Let (X, T) be a minimal dynamical system, and $x \in X$. Then x is a distal point if and only if for all idempotents $u \in E$, we have $u(x) = x$.*

This proposition differs slightly from the one given in more generality in [EE14]. Here, we add the assumption of minimality to simplify the proof, since all the spaces we are interested in are minimal.

Proof. Let $x \in X$ be distal, and let $u \in E$ be an idempotent. Consider (ux, x). We have that $u(ux, x) = u(ux, ux) \in \Delta$, so $(ux, ux) \in \overline{(x, ux)T} \cap \Delta \neq \emptyset$, so by Proposition 1.5.30, we have that x is proximal to ux. Since x is distal, $x = ux$.

Conversely, assume that for all idempotents $u \in E$, $ux = x$, and let y be proximal to x. Then there exists a minimal right ideal $K \subseteq E$ with $px = py$ for all $p \in K$. Since y is almost periodic, by Proposition 1.5.32, we have that in every minimal ideal there exists an idempotent v such that $vy = y$. In particular, $\exists v \in K$ with $vy = y$. So,

$$\begin{aligned} x &= vx && \text{by assumption} \\ &= vy && \text{since } v \in K \\ &= y && \text{since } y \text{ is an almost periodic point.} \end{aligned}$$

Thus, $x = y$ so x is distal, as required. $\square$

The above Proposition 1.5.33 does not hold without the additional assumption of minimality, as illustrated in the example below.

1.5.34 Example. There is a nonminimal dynamical system (X, T) with a point $x \in X$ such that $ux = x$ for all idempotents $u^2 = u \in E(X)$, but x is not distal.

Construction. Consider the orbit closure of the bi-infinite word $\omega = \ldots 0 \cdot 10 \ldots$, $X := \overline{Orb(\omega)}$. Then in fact

$$X = \{s^n(\omega) : n \in \mathbb{Z}\} \cup \{\ldots 0 \cdot 0 \ldots\}.$$

Also, $E(X) = \{s^n : n \in \mathbb{Z}\} \cup \{\mathbf{0}\}$, where $\mathbf{0}(x) = \ldots 0 \cdot 0 \ldots$ for any $x \in X$.

Then the idempotents in $E(X)$ are the identity Id_X and $\mathbf{0}$. Note that the point $\ldots 0 \cdot 0 \ldots$ is fixed by both idempotents. However, every point $x \in X$ is proximal with $\ldots 0 \cdot 0 \ldots$, so it is very far from distal!

The next two propositions and lemma give examples of how the Ellis semigroup captures the asymptotic properties of a dynamical system.

1.5.35 Proposition ([EE14])**.** *Let (X, T) be a dynamical system with Ellis semigroup $E(X)$. Then (X, T) is distal if and only if $E(X)$ is a group, if and only if $Id_X \in T$ is the only idempotent in $E(X)$.*

We show the first equivalence by expanding on the proof given in [EE14], as this proposition will be important later on.

Proof. Assume X is distal, let u, v be two idempotents in E, and let $x \in X$ be arbitrary but fixed. By Proposition 1.5.33, we have that $ux = x = vx$. Since x was arbitrary, we have that for all $x \in X$, $ux = vx$, so $u \equiv v$. Thus, there is just one idempotent (call this idempotent e) in E. Thus, E is a group.

Now, assume that E is a group, and let $x, y \in X$ be proximal. Then by Proposition 1.5.30 there is a minimal ideal $K \subseteq E(X)$ with $px = py\ \forall p \in K$. This ideal contains an idempotent, so $e \in K$, since e is the only idempotent in the group E. Then

$$\begin{aligned} x &= ex && \text{since } e \text{ is the identity} \\ &= ey && \text{since } e \in K \\ &= y && \text{since } e \text{ is the identity,} \end{aligned}$$

so indeed $x = y$ as required. Thus, x is distal, and since it was arbitrary, we have that X is a distal dynamical system. □

1.5.36 Proposition ([EE14])**.** *A dynamical system (X, T) is equicontinuous if and only if its Ellis semigroup $E(X, T)$ is a group of homeomorphisms of X.*

Using the propositions above, we give a much shorter proof of [HJ97, Lemma 3.3]

1.5.37 Lemma ([HJ97])**.** *For a minimal system X over $\mathbb{N}$ or $\mathbb{Z}$ (or even more generally, any (semi-) group T), if X is not distal, then every minimal left ideal of $E(X)$ contains more than one idempotent.*

Proof. Assume X is minimal not distal, and suppose $I \subset E(X)$ is a minimal ideal with only one idempotent $u^2 = u \in I$. Since X is minimal, Proposition 1.5.29 yields that for each $x \in X$, there is an idempotent $v \in I$ such that $vx = x$. Since I has only one idempotent, $vx = x$ for all $x \in X$. So $ux = x$ for all $x \in X$, so $u = Id_X$, so $E(X)$ is a group. Then by Proposition 1.5.35, X is distal - a contradiction to the assumption that it is not. □

We continue with a generalisation of an analogue of a Lemma in [HJ97]. This recasts their Lemma, which concerns IPCPs in dynamical systems over $\mathbb{N}$, in terms of idempotents in arbitrary dynamical systems over the same group:

1.5.38 Lemma. *Given an extension (X,T) of (Y,T), the idempotents of $E(X)$ project to idempotents of $E(Y)$.*

Proof. Given an extension $\pi : (X,T) \to (Y,T)$, by Proposition 1.5.9, we have an induced homomorphism π^* between Ellis semigroups, such that the diagram is commutative for all $x_0 \in X$:

$$\begin{tikzcd} E(X) \arrow[r, "\pi^*", "p\mapsto q"'] \arrow[d, "p\mapsto px_0"'] & E(Y) \arrow[d, "q\mapsto q\pi(x_0)"] \\ X \arrow[r, "\pi", "x_0\mapsto\pi(x_0)"'] & Y \end{tikzcd}$$

Note that $\pi^*(pp') = \pi^*(p)\pi^*(p')$. Thus, if $u \in E(X)$ is an idempotent and $v = \pi^*(u)$, then $v \in E(Y)$ is also an idempotent:

$$\begin{aligned} vv &= \pi^*(u)\pi^*(u) && \text{by definition} \\ &= \pi^*(uu) && \text{since } \pi^* \text{ is a homomorphism} \\ &= \pi^*(u) && \text{since } u \text{ is an idempotent} \\ &= v && \text{by definition.} \end{aligned}$$

This proves the required result. □

Finally, we recall the following Proposition from [dV93] which will be useful throughout the rest of this thesis. We emphasize its importance by also expanding on the exposition of the proof given in [dV93].

1.5.39 Proposition. *Let* (X, T) *be a dynamical system, and* $u \in E(X)$ *be a minimal idempotent. The subspace* $u[X] \subset X$ *does not contain any proximal pairs.*

Proof. Assume that $x, y \in u[X]$ are proximal points; by Proposition 1.5.30 there exists $p \in E(X)$ such that $px = py$. Since $u \in I$, and I is a minimal ideal, for all $\gamma \in I$, $E\gamma$ is a minimal left ideal which is a subset of I, thus $E\gamma = I$. Applying this for $\gamma = u$, we get that $pu \in I$. Applying this again for $\gamma = pu$, we get that $Epu = I \ni u$, and thus there is $q \in E$ such that $qpu = u$. From this, we obtain:

$$\begin{aligned}
x &= ux && (x \in u[X]) \\
&= qpux && (qpu = u) \\
&= qpx && (x \in u[X]) \\
&= qpy && (px = py) \\
&= qpuy && (y \in u[X]) \\
&= uy && (qpu = u) \\
&= y && (y \in u[X]),
\end{aligned}$$

as required. Thus, there are no proximal pairs in $u[X]$. $\square$

1.5.40 Proposition ([EE14])**.** *Proximality is a transitive relation on the dynamical system* (X, T) *if and only if the Ellis semigroup* $E(X, T)$ *has only one minimal ideal.*

1.5.41 Remark. Note that since proximality is already reflexive and symmetric, the requirement that it be transitive is equivalent to making proximality an equivalence relation.

In Chapter 2, Chapter 3 and Chapter 4, we will be considering systems which satisfy a more relaxed condition than distality. This condition will be central to some of the arguments given especially in Chapter 4.

1.5.42 Definition (point-distal dynamical system, [EE14])**.** A dynamical system (X, T) is called *point-distal* if and only if it has a distal point with a dense orbit.

1.5.43 Proposition ([BG13])**.** *Point-distal dynamical systems are minimal.*

Proof. Recall Proposition 1.5.29 that $\overline{Tx}$ is minimal if and only if in every minimal ideal in $E(X)$ there is an idempotent v such that $vx = x$. Also, recall Proposition 1.5.33 that $x \in X$ is distal if and only if $ux = x$ for all idempotents $u \in E$. Combining these two statements with the fact that there is a distal point $x \in X$ with a dense orbit, we get that X is minimal. □

1.6 Almost Automorphic Extensions and Almost Automorphic dynamical systems

We have the following two closely related notions:

1.6.1 Definition (almost one-to-one extension, [dV93])**.** An extension (X, T) of a system (Y, T) via the homomorphism $\pi : X \to Y$ is called *almost one-to-one* if and only if the restriction of π to a residual set is one-to-one. We will sometimes call almost one-to-one extensions *almost automorphic extensions*.

1.6.2 Definition (almost automorphic, [dV93])**.** We say that a dynamical system (X, T) is *almost automorphic* if and only if there is a point $x_0 \in X$ with a dense orbit, such that whenever $\{t_i\} \subset T$ is a net and $x' \in X$ are such that $\lim t_i x_0 = x'$, we also have that $\lim t_i^{-1} x' = x_0$. Such points are called *almost automorphic*.

1.6.3 Proposition ([dV93])**.** *Let (X, T) be an almost automorphic system. Then (X, T) is minimal.*

Proof. Let $x_0 \in X$ be an almost automorphic point and take an arbitrary but fixed idempotent $u \in E(X)$. Then there is a net $\{t_i\} \subset T$ with $\lim_i t_i = u$. Also, $\{t_i^{-1}\}$ is a net in $E(X)$, let it converge to $v \in E(X)$. Define $x^* := ux_0$ $(= \lim t_i x_0)$.

Recall Proposition 1.5.29 that $\overline{Tx}$ is minimal if and only if in every minimal ideal in $E(X)$, there is an idempotent w such that $wx = x$. Note that $ux^* = uux_0 = ux_0 = x^*$, and since u was arbitrary, we have that $X^* := \overline{Tx^*}(= \{px : p \in E(X)\})$ is minimal. Now, note that $vx^* = vux_0 = \lim t_i^{-1}(\lim t_i x_0) = x_0$ (since x_0 is almost automorphic). Thus, $x_0 \in X^*$ (since $v \in E(X)$ is such that $vx^* = x_0$). □

1.6.4 Proposition ([dV93]). *Every minimal equicontinuous dynamical system is almost automorphic.*

For completeness we give a paraphrased exposition of the proof.

Proof. Let (X, T) be equicontinuous with metric d and let $x_0 \in X$ be arbitrary. Assume $\{t_i\}_{i\in I} \subseteq T$ and $x' \in X$ are such that $\lim_{i\in I} t_i x_0 = x'$. Then for all $\epsilon > 0$ there is $i \in I$ such that for all $j \geqslant i$ we have $d(t_j x_0, x') < \epsilon$. But then,

$$\begin{aligned} d(t_j^{-1} t_j x_0, t_j^{-1} x') &< \epsilon \text{ since } (X,T) \text{ is equicontinuous} \\ d(x_0, t_j^{-1} x') &< \epsilon \text{ by rewriting,} \end{aligned}$$

so in fact we have $\lim_{i\in I} t_i^{-1} x' = x_0$, as required. □

Note that we have in fact proven something stronger, that in equicontinuous dynamical systems all points are almost automorphic.

1.6.5 Proposition ([dV93]). *Let the minimal system (X, T) be an almost automorphic extension of its maximal equicontinuous factor (Y, T). Then any two points in a fiber of X are proximal.*

Proof. Let $\phi : (X, T) \to (Y, T)$. Note that since (X, T) is minimal and ϕ is surjective (since (X, T) is a factor of (Y, T)), Proposition 1.2.9 gives us that (Y, T) is minimal, as well. Since (X, T) is an almost automorphic extension of (Y, T), let $y_0 \in Y$ be such that $\phi^{-1}(y_0)$ is a singleton, say $\phi^{-1}(y_0) = \{x_0\}$. Let $x_1, x_2 \in X$, $y \in Y$ be such that $\phi(x_1) = \phi(x_2) = y$. Since (Y, T) is minimal, there is a net $\{t_n\}_{n\in\alpha} \subset T$ such that $t_n y \to y_0$.

Consider the sequence $\{t_n x_1\}_{n\in\alpha}$. Since X is compact metric, there is a $\beta \subseteq \alpha$ so that the subsequence $\{t_n x_1\}_{n\in\beta}$ is convergent, say $\lim_{n\in\beta} t_n x_1 = x^*$.

Then $x^* = x_0$: for assume that $x^* \neq x_0$; then $\phi(x^*) \neq y_0$, so $\lim_{n\in\beta} \phi(t_n x_1) = \phi(x^*) \neq y_0$, a contradiction since $\lim_{n\in\alpha} \phi(t_n x_1) = y_0$.

Now, consider the sequence $\{t_n x_2\}_{n\in\beta}$. Again, since X is compact metric, there is a $\gamma \subseteq \beta$ such that the subsequence $\{t_n x_2\}_{n\in\gamma}$ is convergent. By the same argument as above, we have that $\lim_{n\in\gamma} t_n x_2 = x_0$.

Thus, $\lim_{n\in\gamma} t_n x_1 = \lim_{n\in\gamma} x_2 = x_0$, so x_1 and x_2 are proximal, as required.

$\square$

We also have the following Corollary to Proposition 1.6.5:

1.6.6 Corollary ([dV93]). *Let (X, T) be an almost automorphic extension of its maximal equicontinuous factor. Then the proximal relation is transitive.*

Proof. Let $x_1, x_2, x_3 \in X$ be three points such that x_1, x_2 are proximal, and x_2, x_3 are proximal, and let $\pi : X \to Y$ be the map of X onto its maximal equicontinuous factor Y. Then $\pi(x_1) = \pi(x_2)$ and $\pi(x_2) = \pi(x_3)$, thus $\pi(x_1) = \pi(x_3)$. But this means that x_1 and x_3 are in the same fiber, so by Proposition 1.6.5, we have that x_1 is proximal to x_3, as required. $\square$

1.6.7 Example. Note that the Period Doubling sequence (Definition 1.4.4) is an almost automorphic extension of its maximal equicontinuous factor. Thus the proximal relation is transitive in the Period Doubling dynamical system.

One of the classical theorems in this area is Furstenberg's Structure Theorem, which classifies minimal distal systems in terms of towers of isometric extensions [Fur63]. Furthermore, in [Vee70], W Veech proves that every minimal point-distal system with a residual set of distal points has an almost automorphic extension which is an AI dynamical system (see Definition 2.2.4 for a description of AI systems).

Chapter 2

Factorizations of Substitution Dynamical Systems

2.1 Introduction

In [Kea68] and [CK71], Coven and Keane gave an explicit construction of a two-step factor $(X_\theta, s) \to (X_\phi, s) \to \mathbb{Z}(r)$ for continuous substitutions θ of constant length r. There, the map from X_θ to X_ϕ is isometric, and the map from X_ϕ to the r-adic adding machine $\mathbb{Z}(r)$ is almost automorphic. This result was generalized by Martin in [Mar71], where he shows that a similar two-step factor map exists for a certain class of substitutions over an arbitrary finite alphabet. In the same paper, he also shows that the maximal equicontinuous factor of any admissible substitution θ is $\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r)$, where r is the length of θ and $m(\theta)$ is a constant related to the substitution. Soon after, the question about the maximal equicontinuous factor of any constant length substitution was completely settled by Dekking [Dek78]. Similar, though more complicated and abstract, constructions have been used by Veech in [Vee70], where he proves that every point-distal dynamical system with a residual set of distal points has an almost automorphic extension which is an AI dynamical system. A generalisation of a similar flavor is obtained by Eli Glasner in [Gla75], where he proves that a metric minimal dynamical system whose Ellis semigroup has finitely many minimal ideals, is a PI system (for the notion of a PI system, see

Remark 2.2.5). In a subsequent paper [GG18] he expands upon an example which shows the reverse does not hold: that there exists a PI system whose Ellis semigroup has uncountably many minimal ideals.

Here, we will use notions introduced by Martin to construct a two-step factor as above for our substitution space (X_θ, s). However, our construction differs from Martin's through a closer investigation of the intermediate space X_ϕ. Unlike Martin, we do not consider X_ϕ as a quotient of X_θ, but instead we show X_ϕ is a substitution space over a potentially smaller alphabet $\mathcal{B}$. Moreover, we prove that the map $\Psi : (X_\phi, s) \to (\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r), +)$ is one to one everywhere outside of the orbits of the fixed points of ϕ. Hence, we give a novel presentation of these results.

2.2 The AI Factor of a Generalised Morse System

Let us now introduce the notions and results which will be called upon in the following discussion. All non-standard definitions and results can be found in [Mar71]. For brevity, we will sometimes write $\mathcal{X}$ (respectively - $\mathcal{Y}, \mathcal{Z}$) for the dynamical system (X, T) (respectively, $(Y, T), (Z, T)$), when the underlying space and action on it are understood.

2.2.1 Definition (proximal extension)**.** Let $\Phi : X \to Y$ be a homomorphism of dynamical systems (X, T) and (Y, T). We say that (X, T) *is a proximal extension of* (Y, T) if and only if whenever $\Phi(x_1) = \Phi(x_2)$, we have that the points $x_1, x_2 \in X$ are proximal. In other words, the homomorphism Φ has proximal fibers.

2.2.2 Definition (isometric extension)**.** Let $\Phi : X \to Y$ be a homomorphism of dynamical systems (X, T) and (Y, T), and let $K := \{(x, y) \in X \times X : \Phi(x) = \Phi(y)\}$. We say that (X, T) *is an isometric extension of* (Y, T) if and only if there is a continuous function $F : K \to \mathbb{R}$ such that:

1. For each $y \in Y$, $F : \Phi^{-1}(y) \times \Phi^{-1}(y) \to \mathbb{R}$ defines a metric on $\Phi^{-1}(y)$, and
2. $F(tx, ty) = F(x, y)$ for all $t \in T$.

Moreover, we assume that for each $y \in Y$, the fiber $\Phi^{-1}(y)$ contains at least two points.

2.2.3 Definition (AI extension)**.** Let (X,T), (Y,T), and (Z,T) be dynamical systems with homomorphisms $\Phi : X \to Y$ and $\Psi : Y \to Z$. We say that (X,T) is an *AI extension of* (Z,T) if and only if (Y,T) is an almost automorphic extension of (Z,T) and (X,T) is an isometric extension of (Y,T).

2.2.4 Definition (AI dynamical system)**.** We call a dynamical system $\mathcal{X} = (X,T)$ an *AI system* if and only if there exists an ordinal α and an inverse system $\{\mathcal{X}_\beta; \Phi_{\beta_\gamma}(\gamma \leqslant \beta)\}_{\beta \leqslant \alpha}$ such that

1. $\mathcal{X}_\alpha = \mathcal{X}$,
2. $\mathcal{X}_0$ is the one-point dynamical system,
3. If $\beta + 1 < \alpha$, then $\mathcal{X}_{\beta+1}$ is an AI extension of $\mathcal{X}_\beta$; if $\beta + 1 = \alpha$, then $\mathcal{X}_{\beta+1}$ is an AI extension of $\mathcal{X}_\beta$, where we do not require the final isomorphic extension to have fibers of at least two points, and
4. If $\beta \leqslant \alpha$ is a limit ordinal, then $\mathcal{X}_\beta = \lim^{-1}_{\gamma<\beta} \mathcal{X}_\gamma$.

2.2.5 Remark. If in Definition 2.2.4 we replaced 'almost automorphic' with 'proximal', we would obtain the definition of a PI dynamical system. Since in this thesis we focus on AI extensions, we do not formally introduce the definition.

Recall Hypotheses 1 and 2 from earlier, namely that we assume θ is an admissible substitution of constant length r over the alphabet $\mathcal{A}$, and that if θ is coincidence-free, then it is in the standard form where $\theta(a)_0 = \theta(a)_{r-1} = a$ for any letter $a \in \mathcal{A}$. Also recall Definition 1.3.4 of an adding machine and Definition 1.3.5 of a finite group.

2.2.6 Lemma ([Mar71])**.** *Let θ be an admissible substitution of length r. There is a dynamical system homomorphism $f : (X_\theta, s) \to (\mathbb{Z}(r), +)$.*

Recalling Definition 1.4.2 of basic r^k-blocks, we introduce the following notation from [Mar71] and note the following Lemma:

2.2.7 Notation. For $x \in X_\theta$, $z = z_0 z_1 \ldots \in \mathbb{Z}(r)$, and $k \in \mathbb{N}^+$, we denote by $x[(z); k+1]$ the r^{k+1}-block

$$x\big[-\sum_{i=0}^{k} z_i r^i, -\sum_{i=0}^{k} z_i r^i + r^{k+1} - 1\big].$$

2.2.8 Lemma ([Mar71])**.** *Let $x \in X_\theta$, $z = z_0 z_1 \ldots \in \mathbb{Z}(r)$. For the function f as in Lemma 2.2.6, we have $f(x) = z$ if and only if for all $k \in \mathbb{N}^+$, $x[(z); k+1]$ is a basic r^{k+1}-block.*

2.2.9 Notation (special point of X_θ). From now on, for a constant-length substitution θ, let the *special point x_θ of θ* be any bi-infinite fixed point of θ such that $x_\theta[0] = 0$, i.e. such that $x_\theta = \ldots \cdot 0 \ldots$.

2.2.10 Definition ([Mar71], [Dek78], height of a substitution)**.** For $n \in \mathbb{N}^+$ with prime factorization $n = p_1 \ldots p_k$ (potentially with repetition of factors p_i), we denote by n^* the product of all factors p_i which do not divide r, the length of θ. We define $M := \{n \in \mathbb{N}^+ : x_n = 0\}$, i.e. M is the set of indexes of all positive occurrences of 0 in the special point x_θ. Denote by d_θ the greatest common divisor of elements of M. Finally, we define $m(\theta) := d_\theta^*$ to be the *height* of the substitution θ.

We follow [Mar71] and define an equivalence relation on the alphabet $\mathcal{A}$ via the following sets:

2.2.11 Definition (S_p)**.** For $i \in \mathcal{A}$, let $z(i) = \min\{n \geqslant 0 : \theta(i)_n = 0\} \mod m(\theta)$. For $p \in \{0, \ldots, m(\theta)-1\}$, we define $S_p := \{i \in \mathcal{A} : z(i) \cong -p \mod m(\theta)\}$.

2.2.12 Theorem ([Mar71])**.** *Let θ be an admissible substitution of constant length r. Then its maximal equicontinuous factor is $\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r)$.*

Martin shows that the map to the maximal equicontinuous factor is $x \mapsto (\alpha(x), f(x))$, where $f(x)$ is as in Lemma 2.2.6, and $\alpha(x) = -p(x) \mod m(\theta)$, where $p(x) := \min\{i \geqslant 1 : x_i = 0\}$.

Moreover, Martin links a type of partial coincidence within an equivalence class S_i with the property of being an almost automorphic extension of its maximal equicontinuous factor. More precisely:

2.2.13 Theorem. *The dynamical system* (X_θ, s) *is an almost automorphic extension of its maximal equicontinuous factor if and only if for some* $i \in \{0, \ldots, m(\theta) - 1\}$*, there are integers* $k \in \mathbb{N}^+$*,* $m \in \{0 \ldots, r^k - 1\}$*, such that if* $p, q \in S_i$*, then* $\theta^k(p)_m = \theta^k(q)_m$*.*

2.2.14 Lemma. *If* θ *is coincidence-free, then all* S_i *are equicardinal.*

2.2.15 Definition ($P(i, j, k)$, $[ab]$)**.** For $i \in \{0, \ldots, m(\theta) - 1\}$, k a positive integer, and $j \in \{0, \ldots, r^k - 2\}$, define $P(i, j, k) := \{\theta^k(p)[j, j+1] : p \in S_i\}$. When the set $\{P(i, j, k) : i \in \{0, \ldots, h(\theta - 1)\}, k \in \mathbb{N}^+, j \in \{0, \ldots, r^k - 2\}\}$ partitions the set of legal 2-letter words P_θ, we will write $[ab]$ for the equivalence class of $ab \in P_\theta$.

2.2.16 Example. For the Thue-Morse substitution

$$0 \mapsto 0110$$
$$1 \mapsto 1001$$

we have that $m(\theta) = 1$, so $i = 0$ and $P(0, 0, 1) = \{01, 10\}$, $P(0, 1, 1) = \{00, 11\}$; all other $P(0, j, k)$ coincide with one of these two classes. Thus, the $P(i, j, k)$ partition the set of legal words $P_\theta = \{00, 01, 10, 11\}$.

2.2.17 Theorem ([Mar71])**.** *The dynamical system* (X_θ, s) *is an AI extension of its maximal equicontinuous factor* $\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r)$ *if and only if the following condition holds:*

(A) The collection $\{P(i, j, k) : i \in \{0, \ldots, m(\theta) - 1\}, k \in \mathbb{N}^+, j \in \{0, \ldots, r^k - 2\}\}$ *is a partition of* P_θ*.*

2.2.18 Remark. From now on let θ be a fixed substitution which satisfies condition (A) from Theorem 2.2.17, let (X_θ, s) be the associated shift space, and let X_ϕ be a compact Hausdorff space such that there is an action of $\mathbb{Z}$ on X_ϕ such that $(X_\phi, \mathbb{Z})$ is the intermediate space postulated in Theorem 2.2.17.

We now proceed to develop the new presentation of the construction of the AI factor. For this, we will need to prove some additional results.

2.2.19 Proposition. *Let θ be of constant length r, primitive and in canonical (as in Hypothesis 2) form. If $ab \in P_\theta$ and $a \in S_i$ for some i, then $b \in S_{i+1 \mod m(\theta)}$.*

Proof. By definition of $m(\theta)$, whenever $w = w_0 \dots w_n$ is a finite θ-legal word with $w_0 = w_n = 0$, then the indexes $0 \equiv n \mod m(\theta)$, and so $|w| = n+1 \equiv 1 \mod m(\theta)$. In particular, $r \equiv 1 \mod m(\theta)$ (*).

If $ab \in P_\theta$, then $\theta(ab)$ is a θ-legal word of length $2r$. Let $a \in S_i$, $b \in S_j$, so if $\theta(a) = \alpha_0 \dots \alpha_{r-1}$, then the index i' of the first letter where 0 occurs is congruent to $-i \mod m(\theta)$. In other words, $i' \cong -i \mod m(\theta)$ (**). Similarly, if $\theta(b) = \beta_0 \dots \beta_{r-1}$, then the first j' such that $\beta_{j'} = 0$ satisfies $j' \cong -j \mod m(\theta)$ (***). (By definition of S_i, S_j, respectively.) Let w be the subword of $\theta(ab)$ defined as $w = \alpha_{i'}\alpha_{i'+1} \dots \alpha_{r-1}\beta_0 \dots \beta_{j'}$. Since $\alpha_{i'} = \beta_{j'} = 0$, by the remark above we have that $|w| \cong 1 \mod m(\theta)$. Also, by direct calculation, $|w| = |\alpha_{i'} \dots \alpha_{r-1}| + |\beta_0 \dots \beta_{j'}| = (r-1-i'+1)+(j'+1) = r-i'+j'+1$. So we have

$$\begin{aligned}
&1 \cong r - i' + j' + 1 \mod m(\theta) & \\
&0 \cong r + i - j \mod m(\theta) & \text{by (**) and (***)} \\
&j \cong r + i \mod m(\theta) & \text{by modular arithmetic} \\
&j \cong i + 1 \mod m(\theta) & \text{by (*).}
\end{aligned}$$

Since all indexes of S_i are elements of $\{0, \dots, m(\theta)-1\}$, this means that $j = i+1 \mod m(\theta)$, as required. ☐

2.2.20 Corollary. *If in addition to the conditions of Proposition 2.2.19, θ is simple, for each $P(i,j,k)$ there exists a unique S_i such that*

$ab \in P(i,j,k)$ implies that $a \in S_i$, and $b \in S_{i+1 \mod m(\theta)}$.
Moreover, for all $a \in S_i$, there exists a letter $b \in S_{i+1 \mod m(\theta)}$ such that $ab \in P(i,j,k)$.

Proof. By definition, $P(i,j,k) := \{\theta^k(p)[j, j+1] : p \in S_i\}$, so $|P(i,j,k)| \leqslant |S_i|$. Since θ is simple, $|P(i,j,k)| = |S_i|$(*). By Proposition 2.2.19, $\theta^k(p)(j) \in S_{i+j \mod m(\theta)}$ and so indeed there exists a unique $S_{i+j \mod m(\theta)}$ such that $ab \in$

$P(i,j,k) \to a \in S_{i+j \mod m(\theta)}$, $b \in S_{i+j+1 \mod m(\theta)}$. Also by (*) and since θ is simple, we conclude that for all $a \in S_{i+j \mod m(\theta)}$ there exists a $b \in S_{i+j+1 \mod m(\theta)}$ such that $ab \in P(i,j,k)$. $\square$

Now we move onto one of our main theorems - that the intermediate space $\mathcal{X}_\phi = (X_\phi, s)$ (from Remark 2.2.18) is in fact a substitution system, with the homomorphism between the spaces being a sliding block code.

2.2.21 Theorem. *Let θ be a simple substitution in canonical form of length r over $\mathcal{A}$ and let $P(i,j,k)$ partition P_θ into n equivalence classes. Then there exists a substitution ϕ on $\mathcal{B} = \{0,\dots,n-1\}$ and a sliding block code $\Psi : \mathcal{X}_\theta \to \mathcal{X}_\phi$. In fact, we also show that this is a $|P(i,j,k)|$-to-1 extension.*

Informally, we construct a map $\Psi : \mathcal{X}_\theta \to \mathcal{X}_\phi$ which is defined via an 'encoding' of the set P_θ of admissible two-letter words. This encoding sends all two-letter words in the same partition P_j of the set P_θ to a given letter in $\mathcal{B}$. Explicitly defining this Ψ so that it maps fixed points of θ to fixed points of ϕ requires careful combinatorial choices, detailed in the proof which we now give.

Proof. Let us label the partitions of P_θ as $P_0,\dots,P_{n-1}$ with the rule that the *last* letters of P_0 belong to S_0 (so in particular, for some $a \in \mathcal{A}$, $a0 \in P_0$). For $ab \in P_\theta$, define $[ab] := k$, where $ab \in P_k$ (since the $P(i,j,k)$ partition P_θ, this k is uniquely defined for any $ab \in P_\theta$). For $b \in \{0,\dots n-1\}$, let a_b be any last letter of a word in P_b. Define $\phi(b) = b_0 \dots b_{r-1}$ by $b_0 = b$ and $b_h := [\theta(a_b)(h-1,h)]$. Note that by Corollary 2.2.20, $\phi(b)$ does not depend on the particular choice of a_b - if $c, d \in S_i$, then $[\theta(c)(h-1,h)] = [\theta(d)(h-1,h)]$ for all $h \in \{1,\dots,r-1\}$, by definition of $P(i,j,k)$. Now let $\Psi : \mathcal{X}_\theta \to \mathcal{X}_\phi$ be the sliding block code defined by $\Psi(x)_i = [x_{i-1}x_i] = [x(i-1,i)]$. We use Theorem 1.1.22 to confirm that Ψ is indeed a sliding block code by checking $\Psi \circ s_\theta = s_\phi \circ \Psi$. For $x \in X_\theta$, $\Psi(s(x))_i = [s(x)(i-1,i)] = [x(i,i+1)] = \Psi(x)_{i+1} = s(\Psi(x))$, as required. $\square$

To be able to explore the properties of the Ellis semigroups of the shift spaces X_θ and X_ϕ, we will need to further determine the structure of X_ϕ and the homomorphism from it to the maximal equicontinuous factor. We begin with the following lemmas.

2.2.22 Lemma. $\Psi(\theta(x)) = \phi(\Psi(x))$.

Proof. We want to show $\Psi(\theta(x))_i = \phi(\Psi(x))_i$ for any $i \in \mathbb{Z}$. Let $i = mr + n$. We have two possibilities.

Case 1 - $n = 0$. Recalling Definition 2.2.15 of the equivalence class $[ab]$ of a word $ab \in P_\theta$, we have that

$$\begin{aligned}
\Psi(\theta(x))_i &= [\theta(x)_{i-1}\theta(x)_i] && \text{by definition of } \Psi \\
&= [\theta(x_{m-1})_{r-1}\theta(x_m)_0] && \text{since } i = mr + n \\
&= [x_{m-1}x_m] && \text{since } \theta \text{ is in canonical form.}
\end{aligned}$$

Also, $\phi(\Psi(x))_i = \phi(\psi(x)_m)_0 = \phi([x_{m-1}x_m])_0 = [x_{m-1}x_m]$, by definition of ϕ. Hence, when $i = mr$, $\Psi(\theta(x)) = \phi(\Psi(x))$, as required.

Case 2 - $n \in \{1, \ldots, r-1\}$. Arguing as in the previous case, $\Psi(\theta(x))_i = [\theta(x)_{i-1}\theta(x)_i] = [\theta(x_m)_{n-1}\theta(x_m)_n] = [\theta(x_m)(n-1, n)]$. Similarly, $\phi(\Psi(x))_i = \phi(\Psi(x)_m)_n = \phi([x_{m-1}x_m])_n = [\theta(x_m)(n-1, n)]$, where the final equality holds by definition of ϕ and Corollary 2.2.20. Thus again, $\Psi(\theta(x)) = \phi(\Psi(x))$. □

2.2.23 Lemma. $m(\theta) = m(\phi)$.

Proof. Let w be the right-hand infinite fixed point of ϕ starting from the letter 0, and let u be the right-infinite fixed point of θ starting with the letter 0. Note that $\lim_{n\to\infty} \Psi(\theta^n(0)) = \lim_{n\to\infty} \phi^n(0)$, since $\phi(b)_0 = b$ for all letters $b \in \mathcal{B}$ and since by definition, $a0 \in P_0$ for some $a \in \mathcal{A}$. Thus w is the image under Ψ of u. Since only 2-letter blocks in P_0 are mapped to $0 \in \mathcal{B}$ by Ψ and since $a0$ is the only word in P_0 ending in '0' (by Corollary 2.2.20), we have that $w_i = 0$ implies $u_i = 0$. Thus,

$$M_\phi := \{n \in \mathbb{N} : w_n = 0\} \subset \{n \in \mathbb{N} : u_n = 0\} =: M_\theta,$$

and so $gcdM_\theta$ divides $gcdM_\phi$, and so $m(\theta)$ divides $m(\phi)$, as required. Hence, $m(\theta) \leqslant m(\phi)$.

It is not too difficult, using a similar line of argument, to show that in fact $m(\theta) = m(\phi)$. □

Thus, we now provide a different proof to the following theorem given by Martin, that the intermediate space is an almost automorphic extension of its maximal equicontinuous factor. We should note that this proof differs significantly from Martin's, as he did not represent the intermediate space as a substitution system.

2.2.24 Theorem. *$\mathcal{X}_\phi$ is an almost automorphic extension of its maximal equicontinuous factor.*

Proof. We will use Theorem 2.2.13, by showing a stronger condition than the one needed for the Theorem holds, namely that for *any* equivalence class $S_i' \subset \mathcal{B}$, for just the first iteration of ϕ and *for any* integer $m \in \{1, \ldots, r-1\}$, the m-th column of $\phi(p)$ coincides for all $p \in S_i'$, i.e. $\phi(p)(1, r-1) = \phi(q)(1, r-1)$ for all $p, q \in S_i'$.

Let $S_i' := \{b \in \mathcal{B} : z(b) \cong -i \mod m(\phi)\}$ be an equivalence class of $\mathcal{B}$, and let $c, d \in S_i'$. By definition of ϕ, $\phi(c) = c_0 \ldots c_{r-1}$, $\phi(d) = d_0 \ldots d_{r-1}$, where $c_0 = c$, $d_0 = d$, and there exist $a, b \in \mathcal{A}$ such that for all $h \in \{1, \ldots, r-1\}$, $c_h = [\theta(a)(h-1, h)]$ and $d_h = [\theta(b)(h-1, h)]$. By Lemma 2.2.23, $m(\theta)$ divides $m(\phi)$, so $z(c) \cong -i \mod m(\phi)$ implies that $z(c) \cong -(i \mod m(\phi)) \mod m(\theta)$. Writing $j := i \mod m(\phi)$ for short, we have $z(c) \cong -j \mod m(\theta)$; similarly, $z(d) \cong -j \mod m(\theta)$. As we remarked before in the proof of Lemma 2.2.23, we can choose $a, b \in \mathcal{A}$ so that $c_h = 0$ implies that $a_h = 0$, and similarly, $d_h = 0$ implies that $b_h = 0$. Combining this with the above congruences and the fact that all zeroes in X_θ are spaced at least $m(\theta)$ apart force us to conclude that both $z(a) \cong -j \mod m(\theta)$ and $z(b) \cong -j \mod m(\theta)$. Thus, $a, b \in S_j$, so $c_h = [\theta(a)(h-1, h)] = [\theta(b)(h-1, h)] = d_h$.

Hence by Theorem 2.2.13, $\mathcal{X}_\phi$ is an almost automorphic extension of its maximal equicontinuous factor. □

2.2.25 Note. Note that we have in fact shown something stronger than what was needed for Theorem 2.2.13 - that *whenever* $c, d \in S_i' \subset \mathcal{B}$, $c_h = d_h$ for all $h = 1, \ldots, r-1$.

We use this fact to prove the following theorem.

2.2.26 Theorem. *The map $\Psi : \mathcal{X}_\phi \to \mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r)$ as previously defined is one to one outside of the orbits of the fixed points of ϕ.*

Proof. By Martin's proof of Lemma 2.2.6, $f^{-1}(00\ldots)$ contains only and all of the fixed points of the substitution ϕ, and Ψ maps orbits to orbits. Moreover, since Ψ is surjective, we have that for all $z \in \mathbb{Z}$, there exist $x_1, \ldots x_{m(\theta)}$ such that $\Psi(x_j) = (z, j) \in \mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r)$. Thus, $|f^{-1}(z)| \geqslant m(\theta)$. Therefore, recalling that $(\alpha, f) : \mathcal{X}_\theta \to \mathcal{Z}(m(\theta), r)$, to show Ψ is one to one, we need to show that $|f^{-1}(z)| = m(\theta)$ for all non-integer $z \in \mathbb{Z}(r)$.

So, let $z = z_0z_1 \ldots \in \mathbb{Z}(r)$ be a non-integer, and let z_i be a nonzero term of z. Since z is not an integer, it does not have a tail of zeroes, so there is $j > i$ such that $z_j \neq 0$. Then by Lemma 2.2.8, $x[(z); j + 1]$ is a basic r^{j+1}-block. Note that since $i < j$, the word $x[(z); i + 1]$ is a basic sub-block of $x[(z); j + 1]$. Moreover, since $z_j \neq 0$, $x[(z); i + 1]$ is not a prefix of $x[(z); j + 1]$; informally we can say it is a basic "tail-end" r^{i+1}-block of $x[(z); j + 1]$.

Note that by the way ϕ is defined, if $a, b \in S_l$, then for all $k \in \mathbb{N}^+$, for all $n \in \{1, \ldots, r^k - 1\}$, we have $\phi^k(a)_n = \phi^k(b)_n$. In other words, we have only $m(\theta)$-many options for $x[(z); i+1]$. By the same type of argument, we also have only $m(\theta)$-many choices for $x[(z); j + 1]$. Let us label them as $w_1^{(j)}, \ldots, w_{m(\theta)}^{(j)}$. Moreover, again by definition of ϕ, if $a \in S_l,\ b \in S_m$, and $l \neq m$, then $\phi^k(a)_n \neq \phi^k(b)_n$, for all $k \in \mathbb{N}^+$ and $n \in \{0, \ldots, r^k - 1\}$. Hence, out of the $m(\theta)$-many choices for $x[(z); i+1]$, there is precisely one which is a subword of a given choice of the $m(\theta)$ possibilities for $x[(z); i + 1]$. Hence, for each $w_n^{(j)}$, for each $i < j$, there is precisely one $w_m^{(i)}$ which is a subword of $w_n^{(j)}$ starting at the appropriate index. Without loss of generality, let us relabel the words for each j so that $w_n^{(i)}$ is a subword at the appropriate place of $w_n^{(j)}$ for all $n \in \{1, \ldots, m(\theta)\}$. Therefore, for $z \in \mathbb{Z}(r)$, we have only $m(\theta)$-many $x_i \in X_\phi$ such that $f(x) = z$, namely $x_n = \lim_{i\to\infty} w_n^{(i)}$ for $n \in \{1, \ldots, m(\theta)\}$. Therefore, $|f^{-1}(z)| = m(\theta)$, as required. $\square$

Hence we have the following diagram

$$\begin{tikzcd}(X_\theta, s) \arrow[r, "\Psi", "\Psi(w)=y_1,\ \Psi(v)=y_2"'] \arrow[rr, bend left, "\Phi\circ\Psi"] & (X_\phi, s) \arrow[r, "\Phi", "\Phi(y_1)=\Phi(y_2)=\mathbf{0}"'] & (\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r), +)\end{tikzcd} \tag{2.1}$$

We note that since the extension $\Psi : \mathcal{X}_\theta \to \mathcal{X}_\phi$ is distal (and isometric), all points outside the orbits of the fixed points of θ are distal, as well.

Chapter 3

The Ellis Semigroup of Certain Substitution Systems

3.1 Introduction

As we have mentioned before, concrete calculations of the Ellis semigroup are very few. The few examples include those given by Namioka [Nam84], Milnes [Mil86] and [Mil91], Glasner [Gla76] and [Gla93], Berg, Gove and Haddad [BGH98], Haddad and Johnson [HJ97], Budak, Isik, Milnes and Pym [BIsMP01], and Glasner and Megrelishvili [GM06], as well as a more recent one by Barge [Bar]. In Section 3.2 we add to this list a wide range of constant length substitution systems over arbitrary finite alphabets whose Ellis semigroups have one or two minimal ideals. The two minimal ideals have q idempotents each, where q can be any natural number greater than 1. Furthermore, in Section 3.3, we provide a counterexample to a key proposition to [HJ97], a paper which aims to calculate the Ellis semigroup of some binary constant length 'Morse-like' substitution systems. However, their main theorem still holds, and is shown to be a corollary of our theorem for substitutions over arbitrary alphabets.

3.2 Calculating the Ellis Semigroup of Certain Constant Length Substitution Systems

We proceed by recalling Proposition 1.5.9 about the map between respective Ellis semigroups which is induced by homomorphisms of dynamical systems and considering Diagram 2.1 from the end of Chapter 2:

$$\begin{tikzcd}
(X_\theta, s) \arrow[r, "\Psi", "\Psi(w)=y_1,\ \Psi(v)=y_2"'] \arrow[rr, bend left, "\Phi\circ\Psi"] & (X_\phi, s) \arrow[r, "\Phi", "\Phi(y_1)=\Phi(y_2)=\mathbf{0}"'] & (\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r), +)
\end{tikzcd}$$

We move "backwards" (i.e. from factors to extensions) through this diagram, going from the simpler semigroup $E(\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r))$ to the more complicated ones for the other two spaces. All spaces are the same as introduced in Chapter 2; we sometimes write $\mathcal{X}_\theta$ for the dynamical system (X_θ, s) and similarly $\mathcal{X}_\phi$ for (X_ϕ, s).

Since $\mathcal{Z} := \mathcal{Z}(m(\theta), r) = (\mathbb{Z}_{m(\theta)} \times \mathbb{Z}(r), +)$ is equicontinuous, it is distal and so by Proposition 1.5.35, its Ellis semigroup is a group (in fact, $E(\mathcal{Z}) \cong \mathcal{Z}$). Thus, the only idempotent in $E(\mathcal{Z})$ is the identity map, $I_{\mathcal{Z}}$.

3.2.1 Definition (q)**.** For the substitution ϕ on the alphabet $\mathcal{B}$ defined as in the proof of Theorem 2.2.21 , we define sets of letters $C_1, \dots, C_{r-1} \subset \mathcal{B}$ by $C_i := \{\phi(b)_i : b \in \mathcal{B}\}$. Define $q := |\{C_i : i = 1, \dots, r-1\}|$.

In other words, the set C_i is the set of all letters in the ith 'column' of the substitution ϕ, where we only consider the 'tail-ends' $\phi(b)[1; r-1]$, for a letter b. Then, q is the number of distinct sets of letters in the same column. Note that $|C_i| = m(\theta)$ for any $i = 1, \dots, r-1$, since by definition of ϕ, we have only $m(\theta)$-many possibilities for "tail-ends", i.e. blocks $\phi(b)[1; r-1]$.

3.2.2 Notation. By $\lim_{n\to\infty} \phi^n(a \cdot b)$ we mean that we keep the 'center dot' fixed, so $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$, etc.

We now can state one of our auxiliary theorems of this section.

3.2.3 Theorem. *The Ellis semigroup of the space $\mathcal{X}_\phi$ has one minimal ideal with q idempotents.*

Proof. If f is an idempotent in E_ϕ, then f (as in Lemma 2.2.6) should project to $I_{\mathcal{Z}}$, i.e. $\Phi \circ f = I_{\mathcal{Z}} \circ \Phi$. Recall Theorem 2.2.26, that Φ is 1-1 on the set $X'_\phi := X_\phi \setminus \cup\{O(w) : w \text{ is a fixed point of } \phi\}$. Then proximality is trivially seen to be a transitive relation, hence by Proposition 1.5.40, we have that $E(X_\phi)$ has one minimal ideal. Moreover, if $x \in X'_\phi$, then $\Phi(f(x)) = I(\Phi(x)) = \Phi(x)$, so we have $f(x) = x$, i.e. x is a fixed point of f. Noting that all maps in E_ϕ commute with powers of the shift, we only need to determine the values of f on the preimage of $\mathbf{0} \in \mathcal{Z}$, i.e. on the fixed points of ϕ, namely $w_1, \ldots, w_d$.

We make a couple of observations about the fixed points of ϕ. All such fixed points are images under Ψ of fixed points of θ. Since Ψ identifies the fixed points $\ldots a \cdot b \ldots$ with $\ldots c \cdot d \ldots$ if and only if $ab \sim_\theta cd$, then the number of fixed points of ϕ is equal to the number of distinct equivalence classes $P(i,j,k)$ of θ, which is also equal to $|\mathcal{B}|$. Moreover, from the way in which ϕ was defined, we have only $m(\theta)$-many possibilities for "tail-ends" $\phi(b)[1, r-1]$ for $b \in \mathcal{B}$. Hence, if w' and w'' are two-sided fixed points of ϕ such that $w'_0 = w''_0$, then $w'_{-n} = w''_{-n}$ for all $n \in \mathbb{N}^+$.

We now claim that if w', w'' are distinct and negatively asymptotic, then in fact they only differ in the 0th letter and thus are also positively asymptotic. Indeed, if $w'_{-n} = w''_{-n}$ for all $n \in \mathbb{N}^+$, and if $u' \in \Psi^{-1}(w')$ and $u'' \in \Psi^{-1}(w'')$, then $u'[-(n+1), -n] \sim_\theta u''[-(n+1), -n]$ for all $n \in \mathbb{N}^+$. Let $S_a \subset \mathcal{A}$ be the equivalence class of last letters of $P_{w'_{-1}}$. Then the set of all first letters of $P_{w'_0}$ is the same as the set of all first letters of $P_{w''_0}$, i.e. is the set $S_{a+1 \mod m(\theta)}$. So, $u'_0 \sim_\theta u''_0$, so for all $n \in \mathbb{N}^+$, $u'[n, n+1] \sim_\theta u''[n, n+1]$. Hence for all $n \in \mathbb{N}^+$, $w'_n = w''_n$. Hence if w', w'' are distinct and negatively asymptotic, then they are also positively asymptotic and differ only in the 0th letter. By the same argument, if w, w' are positively asymptotic, then they are also negatively asymptotic, and again might differ only in the 0th letter.

We will prove our theorem through the following steps:

1. We define a set of special sequences $s^{k_i(n)}$ of shifts, such that the limit of each such sequence is idempotent on the set of fixed points of ϕ.

2. We next show these limits not only exist on all of X_ϕ, but are also idempotent. Thus, these maps belong to the Ellis semigroup $E(\mathcal{X}_\phi)$.

3. Finally, we show that these are both minimal idempotents, and the *only* possible minimal idempotents.

Let $i \in \{1, \ldots, r-1\}$ and consider the sequences $s^{k_i(n)}$, where $k_i(n) = ir^n$, for $n \in \mathbb{N}$. Then note $s^{k_i(n)}(x)[-1,0] = x[ir^n - 1, ir^n]$ for all $n \in \mathbb{N}$, by the definition of the shift. Consider the set C_{r-1} of final letters of images $\phi(a)$ for $a \in \mathcal{B}$. We make the following observation: (A) For each $a \in C_i$, there is a unique $b \in C_{r-1}$ such that for all $c \in \mathcal{B}$, $s^{ir^n}(c)[-1,0] = ba$ for all $n \geqslant 1$. In other words, each $a \in C_i$ has a unique predecessor in the limit.

For $a \in S_i$, let $pred_i(a)$ be any letter in $\mathcal{B}$ such that $\phi(pred_i(a))_{r-1} = b$. Then for any $c \in \mathcal{B}$, $\lim_{n\to\infty} s^{ir^n}(c) = \lim_{n\to\infty} \phi^n(pred_i(a) \cdot a) = \ldots b \cdot a \ldots$, where $a = \phi(c)_i$.

Let F be the set of fixed points of ϕ, and define $f_i|_F := \lim_{n\to\infty} s^{ir^n}|_F$. Then $f_i|_F$ is indeed an idempotent on F. Let $x = \ldots c \cdot d \ldots \in F$ be a fixed point of ϕ, and let $a := \phi(d)_i$. Then $f_i(x) = \lim_{n\to\infty} s^{ir^n}(\ldots c \cdot d \ldots) = \lim_{n\to\infty} \phi^n(pred_i(a) \cdot a) = \ldots b \cdot a \ldots = \ldots \phi(pred_i(a)) \cdot \phi(a) \ldots$, where b is the unique predecessor of $a \in S_i$. Also,

$$
\begin{aligned}
f_i(f_i(x)) &= f_i(\ldots b \cdot a \ldots) \\
&= f_i(\ldots \phi(pred_i(a)) \cdot \phi(a) \ldots) \\
&= \ldots \phi(pred_i(a)) \cdot \phi(a) \ldots && \text{by definition of } pred_i(a) \\
&= \ldots b \cdot a \ldots = f_i(x).
\end{aligned}
$$

Hence $f_i|_F$ is an idempotent.

Moreover, f_i identifies all points which are proximal to the right, as f_i is a limit of positive powers of the shift s. In other words, if $f_i(\ldots a_{-1} \cdot a \ldots) = \ldots b_{-1} \cdot b_0 \ldots$, and $c_0 \in \mathcal{B}$ has the same tail-end as a_0, then $f_i(\ldots c_{-1} \cdot c_0 \ldots) = f(\ldots a_{-1} \cdot a_0 \ldots) = \ldots b_{-1} \cdot b_0 \ldots$, since $\phi^\infty(a_0)$ and $\phi^\infty(c_0)$ coincide on the right.

Now, since $|C_i| = q$, we have only q-many distinct f_i. In other words, $C_i = C_j$ if and only if $f_i = f_j$. This is obvious from the definition of the C_i and f_i.

Now, we show the maps f_i can be extended to all of $x \in X_\phi$. In other words, we show that $f_i := \lim_{n\to\infty} s^{ir^n}$ converges for all $x \in X_\phi$ and is an idempotent, for all $i = 1, \ldots, r-1$. Recall that the following diagram is commutative:

$$\begin{array}{ccc} X_\phi & \xrightarrow{\quad s^k \quad} & X_\phi \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ \mathbb{Z}_r \times \mathbb{Z}(m(\theta)) & \xrightarrow{+(0,k)} & \mathbb{Z}_r \times \mathbb{Z}(m(\theta)) \end{array}$$

Moreover, Φ is one to one outside the orbits of the fixed points of ϕ. We have, for w not an integer:

$$\begin{aligned} \psi(\lim_{n\to\infty} s^{r^n}(w)) &= \lim_{n\to\infty} \Phi(s^{r^n}(w)) && \text{since } \Phi \text{ is continuous} \\ &= \lim_{n\to\infty} [\Phi(w) + (0, r^n)] && \text{note that } r^n \in \mathbb{Z}(m(\theta)) \\ &= \Phi(w) + \lim_{n\to\infty} (0, r^n) \\ &= \Phi(w) + (0,0) = \Phi(w). \end{aligned}$$

Therefore, $\{s^{r^n}\}_{n\in\mathbb{N}}$ converges to an idempotent, as Φ is one to one outside the orbits of the fixed points. Therefore, $f_i \in E(X_\phi)$, for all $i = 1, \ldots, r-1$.

We now show that the f_i are minimal idempotents in $E(X_\phi)$. First recall that we have enumerated all possible values an idempotent $f \in E(X_\phi)$ can take, since it has to commute with the shift and commute with the map Φ, which is one to one on X'_ϕ. Note that all our f_i act as identity on the right of the other f_i, and since we know that $E(X_\phi)$ has only one minimal (by Proposition 1.5.40) ideal with at least two idempotents in it (by Proposition 1.5.37), we conclude that in fact all f_i are minimal idempotents in the same minimal ideal $I \subset E(X_\phi)$. □

Also note this - that f_i are limits of sequences - is consistent with Eli Glasner's result that in cases such as our X_ϕ, the Ellis semigroup is Fréchet.

To state our main theorem, we recall Definition 2.2.15 of the equivalence classes of two-letter words $P(i,j,k)$, and Definition 3.2.1 of the special number q associated to a substitution.

3.2.4 Theorem. *If θ is a simple substitution with n fixed points, such that $\{P(i,j,k)\}$ partition the set of legal two-letter words P_θ, its Ellis semigroup has*

2 minimal ideals with q idempotents each.

Proof. In Theorem 3.2.3, we have shown that E_ϕ has one minimal ideal with q idempotents.

Moving to the extension $\mathcal{X}_\theta$ of $\mathcal{X}_\phi$, any minimal idempotent in E_θ is mapped to a minimal idempotent in E_ϕ. We note again that idempotents commute with powers of the shift, so are fully determined by their value on a point per orbit. Since points in $X'_\theta := X_\theta \setminus \cup\{O(w) : w \text{ is a fixed point of } \phi\}$ get mapped to points in X'_ϕ, fibers of Ψ are distal, points in X'_θ are distal, and idempotents map distal points to themselves, we have that an idempotent $f \in E_\phi$ will be the identity on X'_θ. Thus, we only need to determine the value f takes on the fixed points of θ. Since it gets mapped to an idempotent in E_ϕ, we have $\Psi \circ f = g \circ \Psi$, for one of the q-many idempotents g in E_ϕ.

Let us consider what an idempotent $f \in E_\theta$ 'does' to the fixed points of θ. Recall from Proposition 1.5.30 that for any minimal idempotent u, the points ux and x are proximal. Note that since $\theta(a)_0 = \theta(a)_{r-1}$, every legal word in P_θ is a fixed point of θ, so θ has $|P_\theta|$ many fixed points. Also note that for two such fixed points x and y, either $x_n = y_n$ for all $n \in \mathbb{N}$, or $x_n \neq y_n$ for all $n \in \mathbb{N}$; similarly x_n and y_n are either all the same or all different for all negative integers n. Thus, if x and y are proximal, they either coincide in all their non-negative or all their negative indexes.

Fix a minimal idempotent g in E_ϕ, and let the minimal idempotent $f \in E_\theta$ be such that $\Psi \circ f = g \circ \Psi$. Let $a \in X_\theta$ be a fixed point of θ. Since $ux = uy$ implies that x is proximal to y (for a minimal idempotent u), each one of the $m(\theta)$-many points b in the fiber of $\Psi^{-1}(a)$ can only get mapped to two potential points in the fiber of $\Psi^{-1}(g(a))$ - call them b', which is proximal with b on the right, and b'', which is proximal with b on the left. Note that the choice of b' or b'' also uniquely determines the choice of $f(c)$ for any other point c in the same fiber $\Psi^{-1}(a)$, since θ is coincidence-free (and so would the tails of its fixed points be coincidence-free). Hence, for each idempotent $g \in E_\phi$, we have exactly two choices of $f \in E_\theta$ of idempotents such that $\Psi^*(f) = g$. By almost the same argument as that in the proof of Theorem 3.2.3, we can show that both f are limits of shift maps, hence are indeed in the Ellis semigroup of $\mathcal{X}_\theta$. Recalling that equivalent idempotents get mapped to equivalent idempotents (so in this case, equivalent idempotents in E_θ get mapped to the same idempotent in E_ϕ), we have only two equivalent idempotents in E_θ. Hence, we have only two minimal ideals in E_θ, with q many idempotents each. □

Thus, as corollary, we have the following restatement of the Theorem of Haddad and Johnson:

3.2.5 Theorem ([HJ97]). *The Ellis semigroup E_θ of a continuous constant-length binary substitution has two minimal ideals with two idempotents each.*

3.2.6 Remark. Furthermore, let $v, \overline{v}, w, \overline{w}$ be the four fixed points of the substitution θ, where $v[-1,0] = 11$ and $w[-1,0] = 01$. Then in light of the argument in the proof of Theorem 3.2.4, we may express the four minimal idempotents g_1, g_2, g_3, and g_4 in shorthand as:

	v	$\overline{v}$	w	$\overline{w}$
g_1	w	$\overline{w}$	w	$\overline{w}$
g_2	$\overline{w}$	w	w	$\overline{w}$
g_3	v	$\overline{v}$	v	$\overline{v}$
g_4	v	$\overline{v}$	$\overline{v}$	v

3.3 The Counterexample

We now give a counterexample to Haddad and Johnson's proposition 3.4 [HJ97], which is essential to their proof that the Ellis semigroup of a binary continuous substitution system contain two minimal ideals with two idempotents each. We emphasize that their theorem still holds, and was generalised in the previous section.

Recall Definition 1.3.17 of an IP set. Certain idempotents in the Ellis semigroup can be thought of as cluster points 'along an IP set'. In [Had96], Kamel Haddad introduces this notion as:

3.3.1 Definition. For a dynamical system over $\mathbb{N}$ (or $\mathbb{Z}$), a cluster point f of the Ellis semigroup $E(X)$ is called an *IP cluster point along an IP subset P of*

$\mathbb{N}$ *(or $\mathbb{Z}$)* if and only if for every neighbourhood U of f in X^X, there is a IP subset Q_U of P, such that $Q_U \subseteq \{n \in P : T^n \in U\}$.

3.3.2 Remark. Note that if f is an IP cluster point (written IPCP for short) along the set P, and if $Q \supset P$, then f is also an IPCP along Q.

The main aim of this section is to give a counterexample to Proposition 3.4 from Haddad and Johnson's paper, which states:

3.3.3 Proposition ([HJ97], Proposition 3.4)**.** *Let P be an IP subset of $\mathbb{Z}$, generated by $\{p_n\}_{n=1}^{\infty}$. If p_n is positive for an infinite number of n, we denote by P^+ the IP set generated by the positive p_n's. If p_n is negative for an infinite number of n, we denote by P^- the IP set generated by the negative p_n's. Then f is an IPCP for a $\mathbb{Z}$-cascade along an IP set P if and only if f is an IPCP for at least one of the corresponding $\mathbb{Z}^+$ or $\mathbb{Z}^-$ actions, along P^+ or P^- respectively.*

Recall Definition 1.1.17 of a continuous substitution as a constant length binary one where $\theta(0) = \overline{\theta(1)}$, and the fixed points of θ are not periodic.

From now on, let θ be a continuous binary substitution of length r. We provide an alternative way of defining continuous substitutions in the following Proposition. To make the proof of this proposition clearer, we need the notion of 'disjoint support'.

3.3.4 Definition (disjoint support)**.** Let m, k be two natural numbers with binary expansions $m = \sum_{i \in \mathbb{N}} 2^{m_i}$, $k = \sum_{i \in \mathbb{N}} 2^{k_i}$, where $m_i, k_i \in \{0, 1\}$. We say that the binary expansions of m and k have *disjoint support* for the 1's (for the 0's) if and only if for every $i \in \mathbb{N}$, $m_i = 1$ implies $k_i = 0$ (respectively, $m_i = 0$ implies $k_i = 1$).

3.3.5 Proposition. *Let θ be a continuous binary substitution of length r, so $\theta(0) = a$, $\theta(1) = \overline{a}$, where $a = a_0 a_1 \dots a_{r-1}$, and $a_0 = 0$. Define the function $P : \mathbb{N} \to \{0, \dots, r-1\}^{<\mathbb{N}}$ by $\mathcal{P}(k)_m = b_m$, where k has base r expansion $k = b_0 r^l + b_1 r^{l-1} + \dots b_{l-1} r + b_l$. Then the one-sided fixed point of θ defined as $\omega := \lim_{n \to \infty} \theta^n(0)$ can be represented as $\omega_k = (\sum_{i \in I} |\mathcal{P}(k)|_i) \mod 2$, where $I := \{m \in \{0, \dots, r-1\} : a_m = 1\}$.*

Proof. We proceed by induction on θ^k, noting that $\theta^k(0)$ is a prefix of length r^k of ω.

Base Case: It is immediate that for $\theta^1(0) = a = a_0 \dots a_{r-1}$, $a_m = 1$ if and only if $m \in I$, and since $0 \leqslant m < r$, $\mathcal{P}(m) = m$, so indeed $a_m = (\sum_{i \in I} |\mathcal{P}(m)|_i) \mod 2$.

Inductive Step: Assume that for some $k \in \mathbb{N}^+$, $\theta^k(0)$ is such that $\theta^k(0)_m = (\sum_{i\in I} |\mathcal{P}(m)|_i) \mod 2$ for $0 \leqslant m < r^k = |(\theta^k(0))|$, and consider $\theta^{k+1}(0)$.

We observe that $\theta^{k+1}(0) = \theta^k(a_0)\theta^k(a_1) \dots \theta^k(a_{r-1})$ and $\theta^k(a_i) = \theta^k(1)$ if and only if $a_i \in I$.

Let $\alpha \in \{0, \dots, r^{k+1} - 1\}$ be arbitrary but fixed. Then α can be uniquely written as $\alpha = jr^k + m$ where $j \in \{0, \dots, r-1\}$ and $0 \leqslant m < r^k$. Note that $\mathcal{P}(jr^k + m) = j\mathcal{P}(m)$ (*) and that $\theta^{k+1}(0)_\alpha = \theta^k(a_j)_m$. We want to show that $\theta^{k+1}(0)_\alpha = (\sum_{i\in I} |\mathcal{P}(\alpha)|_i) \mod 2$.

Now, $j \in I$ if and only if $a_j = 1$ if and only if $\theta^k(a_j) = \theta^k(1)$. So,

$$\sum_{i\in I} |\mathcal{P}(jr^k + m)|_i = \sum_{i\in I} |\mathcal{P}(m)|_i + \sum_{i\in I} |\mathcal{P}(j)|_i = \begin{cases} \sum_{i\in I} |\mathcal{P}(m)|_i + 1 & \text{iff } j \in I \\ \\ \sum_{i\in I} |\mathcal{P}(m)|_i & \text{iff } j \notin I. \end{cases}$$

Thus,

$$\left(\sum_{i\in I} |\mathcal{P}(jr^k + m)|_i\right) \mod 2 = \begin{cases} \left(1 + \sum_{i\in I} |\mathcal{P}(m)|_i\right) \mod 2 = \overline{\theta^k(0)}_m & \text{if } j \in I \\ \\ \left(\sum_{i\in I} |\mathcal{P}(m)|_i\right) \mod 2 = \theta^k(0)_m & \text{if } j \notin I \end{cases}$$

Recall that $\theta^{k+1}(0)_\alpha = \theta^k(0)_m$ if and only if $j \in I$. Combining these two facts gives

$$\left(\sum_{i\in I} |\mathcal{P}(jr^k + m)|_i\right) \mod 2 = \theta^{k+1}(0)_\alpha,$$

as required $\square$

3.3.6 Remark. Recall that $w_{-n} = \overline{v}_{-n}$ for any $n \in \mathbb{N}$, so $s^{-n}(w)$ and $s^{-n}(v)$ are always distance 1 apart. Thus, g_1 cannot be an IPCP along any $P^- \subset \mathbb{Z}^-$.

Now the following Lemma is all we need to finish our construction of the counterexample:

3.3.7 Lemma. *Further to the conditions of Proposition 3.3.5, let* $j := \min I$ *and* $p := jn + j$*, so* $\mathcal{P}(p) = jj$*. Then the idempotent* g_1 *defined in Remark 3.2.6 is not an IPCP along the IP set generated by* $Q^+ := \{pr^{2m} : m \in \mathbb{N}\}$*.*

Proof. Recall that $g_1(w) = w = g_1(v)$, where $v = \ldots 1 \cdot 0 \ldots$ and $w = \ldots 0 \cdot 0 \ldots$. So, if g_1 is an IPCP along the IP set P^+, we will need for $s^q(w)$ to get arbitrarily close to w for $q \in P^+$. Note that since $v_n = w_n$ for $n \in \mathbb{N}$, this also means $s^q(v)$ will get arbitrarily close to w.

Note that for all $\rho \in Q^+$, $(\sum_{i\in I} |\mathcal{P}(\rho)|_i) \mod 2 = 0$, so $w_\rho = 0$ for all $\rho \in Q^+$. Also, since all $\rho \in Q^+$ have disjoint support, we have that for $\rho_1, \ldots, \rho_m \in Q^+$,

$$\left(\sum_{i\in I} |\mathcal{P}(\sum_{k=1}^{m} \rho_k)|_i\right) \mod 2 = \left(\sum_{k=1}^{m}\sum_{i\in I} |\mathcal{P}(\rho_k)|_i\right) \mod 2 = 0.$$

Also, ρ will have a 'tail' of $2m$ zeroes, so $\mathcal{P}(\rho-1)$ will have an odd number of j's, an even number of $(n-1)$'s (in the tail), and one $j-1$ (which, since $j = \min I$, is not an element of I hence not counted). Thus, $\left(\sum_{i\in I} |\mathcal{P}((\sum_{k=1}^{m} \rho_k) - 1)|_i\right) \mod 2 = 1$. So, if ρ is in P^+, $s^\rho(w) = \ldots w_{\rho-1} \cdot w_\rho \ldots = \ldots 1 \cdot 0 \ldots$, which is distance 1 from w. So, g_1 cannot be an IPCP along P^+, as required. □

3.3.8 Note. In fact, it is not hard to amend the proof above to show that the idempotent g_3 (as in Remark 3.2.6) is an IPCP along P^+.

3.3.9 Counterexample. Let θ be a continuous binary substitution of length r. Then by Theorem 3.2.5, we have that the Ellis semigroup of (X_θ, s) has two minimal ideals with two idempotents each. Following the notation of Remark 3.2.6, we denote the four minimal idempotents as g_1, g_2, g_3, g_4, where $g_1 \sim g_3$, $g_2 \sim g_4$, and g_1 and g_2 are in the same minimal ideal, as are g_3 and g_4.

Since g_1 is an idempotent in $E(X_\theta)$, by [Had96], g_1 is an IP cluster point in $E(X_\theta)$. Then by Remark 3.3.2, g_1 is also an IPCP along $\mathbb{Z}$ (since any IP sequence is contained in $\mathbb{Z}$).

We now construct a generating set for $\mathbb{Z}$. Since θ is continuous, we may write $\theta(0) = a$, $\theta(1) = \overline{a}$, where $a = a_0a_1 \dots a_{r-1}$, and $a_0 = 0$. As in Proposition 3.3.5, define $I := \{m \in \{0, \dots, r-1\} : a_m = 1\}$. Also, define the function $\mathcal{P} : \mathbb{N} \to \{0, \dots, r-1\}^{<\mathbb{N}}$ by $\mathcal{P}(k)_m = b_m$, where k has base r expansion $k = b_0r^l + b_1r^{l-1} + \dots b_{l-1}r + b_l$. Furthermore, let $j := \min I$ and $p := jr + j$, so $\mathcal{P}(p) = jj$. We take as generating set for $\mathbb{Z}$ the sequence given by $P := \{m \in \mathbb{Z} : m < 0\} \cup \{pr^{2m} : m \in \mathbb{N}\}$.

Then by Remark 3.3.6, g_1 cannot be an IPCP along P^-. Moreover, by Lemma 3.3.7, g_1 also cannot be an IPCP along P^+. Therefore, the idempotent g_1 combined with the IP set $\mathbb{Z}$ generated by the sequence P provide the necessary counterexample to Proposition 3.3.3.

Chapter 4

The Furstenberg Topology

4.1 Introduction

Let (X, T) be a dynamical system, let u be a minimal idempotent in its Ellis semigroup, and let $M := uE$ be the corresponding minimal left ideal in $E(X, T)$. In general the set $u[X]$ is not closed in X, and so cannot be compact in the subspace topology. However, sets of the form $u[X]$ are of fundamental importance for phenomena related with distality. For example, recall Proposition 1.5.39, which has so far been used in several places throughout this thesis:

1.5.39 Proposition. *Let (X, T) be a dynamical system, and $u \in E(X)$ be a minimal idempotent. The subspace $u[X] \subset X$ does not contain any proximal pairs.*

This proposition can be used to give an alternative proof of Proposition 1.5.40 that the Ellis semigroup has only one minimal ideal if and only if proximality is an equivalence relation. Moreover, it is used in many theorems about extensions of various dynamical systems. Because of this importance, it is useful to weaken the topology of $u[X]$ so as to get a compact space. This weakening is the so-called Furstenberg Topology. It was introduced in [Fur63] by Hillel Furstenberg, where he investigated the structure of distal dynamical systems. It was later used by William Veech [Vee70] to give a necessary and sufficient condition for a point-distal dynamical system to be an almost automorphic extension of its maximal equicontinuous factor. This topology has subsequently proven useful for investigation of distality within a dynamical system.

4.2 Definition and Properties of the Furstenberg Topology

Let us begin by considering the Furstenberg topology. This topology is defined on a special subset of the phase space X, using a minimal idempotent. So, let I be a minimal ideal in $E(X)$, so $EI \subseteq I$, and let $u^2 = u \in I$ be an idempotent. We define

$$X_u := \{x \in X : ux = x\},$$

in other words, X_u is the subset of X consisting of all points which are fixed by u. Note that:

4.2.1 Proposition ([dV93])**.** *$X_u = u[X]$, the image of X under the mapping u.*

Proof. $X_u \subseteq u[X]$: let $x \in X_u$. Then $ux = x$, so $x \in u[X]$.

$X_u \supseteq u[X]$: let $x \in u[X]$. Then there exists $y \in X$ with $uy = x$. But then $ux = uuy = uy = x$, so in fact $x \in X_u$. □

Let $x, y \in X$. We define a function $F(x, y) : X \times X \to \mathbb{R}_0^+$ by

$$F(x, y) := \inf\{d(tx, ty) : t \in T\}.$$

We note that this function is upper semicontinuous as the greatest lower bound of continuous functions, namely $d(tx, ty) : X \times X \to \mathbb{R}_0^+$. Moreover, $F(x, y) = 0$ if and only if x, y are proximal. However, $F(x, y)$ does not satisfy the triangle inequality. Indeed, in the Thue-Morse system (X_{TM}, s), we have that $F(v, w) = 0$ and $F(w, \overline{v}) = 0$, but $F(v, \overline{v}) = 1$. Hence, $F(x, y)$ does not induce a metric on X.

This function will be used to define a topology on X, and later on a special subset of X, via sets

$$U_a(x) := \{y \in X : F(x,y) < a\}.$$

We begin with a Lemma, which will then help us show the sets $\{U_a(x) : x \in X_u, a \in \mathbb{R}^+\}$ form a basis for a topology (the Furstenberg topology) on X_u.

4.2.2 Lemma ([Vee70])**.** *Let $x, y \in X_u$ and suppose $y \in U_a(x)$ for some $a > 0$. There exists $\epsilon > 0$ such that $U_\epsilon(y) \subseteq U_a(x)$.*

Here, we give an expanded and clarified proof of this lemma.

Proof. Let W be the set

$$W := I(x,y) = \{(ix, iy) : i \in I\},$$

for some I - a minimal ideal in $E(X,T)$. This is a minimal subset, and thus (I,T) is a minimal dynamical system. Now, the mapping $\phi : I \to X \times X$ given by $i \mapsto (ix, iy)$ is a homomorphism, so $W = \phi(I)$ is a minimal set in $(X \times X, T)$. Since $u(x,y) = (x,y) \in I(x,y)$, the set $I(x,y)$ is the orbit closure of (x,y) in $X \times X$.

Let $b > 0$ be such that $F(x,y) < b < a$, and consider the set

$$V = \{(x', y') \in W : d(x', y') < b\}.$$

Note that $F(x,y) < b$, so $\inf_{t \in T} d(tx, ty) < b$, so there exists $t_0 \in T$ with $d(t_0x, t_0y) < b$. Further note that $I(x,y)$ is the orbit closure of (x,y), so $(t_0x, t_0y) \in I(x,y)$, and $d(t_0x, t_0y) < b$. Thus, $(t_0x, t_0y) \in V \neq \emptyset$. Moreover, this set V is open in W.

Note that since $V \ni (t_0x, t_0y)$ and since $t \in T$ is continuous for all $t \in T$, and since W is the orbit closure of (x,y), we have that $\{t^{-1}(V) : t \in T\}$ forms an open cover of W. Since W is minimal, it is compact, so there exist $t_1, \ldots, t_n \in T$ such that

$$\bigcup_{j=1}^{n} t_j^{-1}(V) = W.$$

Let $\epsilon > 0$ be arbitrary but fixed. Recall that by the Heine-Cantor Theorem 0.1.21, any continuous function on a compact set is uniformly continuous. Now, t_j is uniformly continuous on the compact set W, so for $\epsilon > 0$ we have that there exists $\delta_j > 0$ such that $(w_1, w_2) \in W$ with $d(w_1, w_2) < \delta_j$ implies that $d(t_j w_1, t_j w_2) < \epsilon$. We take $\delta := \min\{\delta_j : j = 1, \dots, n\}$ and get that $(w_1, w_2) \in W$ with $d(w_1, w_2) < \delta$ implies that $d(t_j w_1, t_j w_2) < \epsilon$, i.e. these maps are equicontinuous on W.

Now take $\epsilon = a - b$, so there is $\delta > 0$ such that if $d(x', y') < \delta$ then $d(t_j x', t_j y') < a - b$ for $j = 1, \dots, n$. Now let $z \in U_\delta(y)$. There exists $t \in T$ such that $d(ty, tz) < \delta$, and therefore also $d(t_j ty, t_j tz) < a - b$, $1 \leqslant j \leqslant n$. By the triangle inequality for d, we get

$$d(t_j tx, t_j tz) < d(t_j tx, t_j ty) + d(t_j ty, t_j tz) < b + a - b = a.$$

Thus $F(x, z) < a$, so $z \in U_a(x)$, as required. $\square$

The above proof can be easily modified to show the sets $U_a(x)$ form a basis for a topology on all of X. The reader is referred to [Fur63] for more details.

4.2.3 Proposition ([Vee70]). *The Furstenberg topology is a compact T_1 topology and the sets defined before form a basis.*

Proof. Consider $U_a(x_1) \cap U_b(x_2) \neq \emptyset$ for some $x_1, x_2 \in X$, and some $a, b > 0$. Without loss of generality $x' \in U_a(x_1) \cap U_b(x_2)$. In particular, $x' \in U_a(x_1)$, so by Lemma 4.2.2, there exists $\epsilon_1 > 0$ with $U_{\epsilon_1}(x') \subseteq U_a(x_1)$. Similarly, $x' \in U_b(x_2)$, so by Lemma 4.2.2, there exists $\epsilon_2 > 0$ with $U_{\epsilon_2}(x') \subseteq U_b(x_2)$. Taking $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$, we have that $U_\epsilon(x') \subseteq U_a(x_1) \cap U_b(x_2)$, as required. Thus, the collection $\{U_a(x) : a \in \mathbb{R}^+, x \in X_u\}$ forms a basis for a topology on X_u.

To show that the Furstenberg topology is T_1, we use Proposition 4.3.7. If $x, y \in X_u$ with $x \neq y$, we have that x, y are not proximal, thus $F(x, y) > 0$. If $a \leqslant F(x, y)$, then $x \notin U_a(y)$ and $y \notin U_a(x)$, as required.

To see that this topology is compact, we let $\mathcal{U}$ be an F-open cover of X_u by basic open sets, so $\mathcal{U} := \{X_u \cap U_{a_\lambda}(x_\lambda)\}_{\lambda \in \Lambda}$. Let $w \in X$. Then there is a λ such that $uw \in X_u \cap U_{a_\lambda}(x_\lambda)$. Then $F(uw, w) = 0$ so $w \in U_\epsilon(uw)$ for any $\epsilon > 0$. Then $w \in U_{a_\lambda}(x_\lambda)$ by Lemma 4.2.2. Thus, $\mathcal{U}' := \{U_{a_\lambda}(x_\lambda)\}_{\lambda \in \Lambda}$ is an open cover of X, which is assumed compact, thus there is a finite subcover. This would also be a finite subcover of $X_u \subset X$. $\square$

4.3 Veech's Theorem

Our main motivation for considering the Furstenberg Topology is the following theorem by Veech [Vee70]:

4.3.1 Theorem (Theorem 6.5 of [Vee70])**.** *A dynamical system (X, T) is almost automorphic if and only if X is point-distal and (X_u, F_u) is Hausdorff.*

In this section, we will significantly rewrite the proof of this theorem, filling in gaps, improving the overall structure. We also present a new line of reasoning for the 'only if' part of the proof, which uses properties of the Ellis semigroup of the respective spaces. We recall the Heine-Cantor Theorem 0.1.21, that any continuous function on a compact set is uniformly continuous, which will be of use in this section.

We will first prove the latter direction of Veech's Theorem (4.3.1) - i.e. we consider a point-distal dynamical system such that (X_u, τ_F) is Hausdorff, and show that it is an almost 1-1 extension of its maximal equicontinuous factor.

We begin with a point-distal dynamical system (X, T), where X is compact with metric d and T is a group, and $x_0 \in X$ is a point such that Tx_0 is dense in X. For the sake of clarity, we denote by τ the usual topology on X.

Also, we denote by $P(X)$ the set of proximal pairs in X, and by E_X the Ellis semigroup of (X, T).

We denote the topology generated by the basis $\{U_a(x) : a \in \mathbb{R}, x \in u[X]\}$ by τ_F, and call it the Furstenberg topology. We note that $\tau_F \subseteq \tau$.

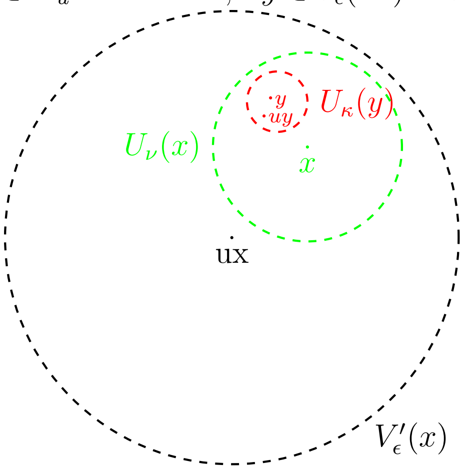
Note that:

4.3.2 Proposition. *The map $u : (X, \tau) \to (X_u, \tau_F)$ is continuous.*

Proof. Let $x \in X$ and recall that u is continuous at x if and only if for any neighbourhood V of ux there is a neighbourhood U of x such that $u[U] \subset V$. So, let $\epsilon > 0$ be arbitrary, and consider $V_\epsilon(ux) = V'_\epsilon(ux) \cap X_u$, where $V'_\epsilon(ux)$ is a τ_F-open in X neighbourhood of ux (in black in the image below) and V_ϵ is its trace in X_u.

Now, $F(ux, x) = 0$, so $x \in V'_\epsilon(ux) := \{x' \in X : F(x', ux) < \epsilon\}$, and the latter is open, so there is a $\nu > 0$ such that $U_\nu(x) \subseteq V'_\epsilon(ux)$ ($U_\nu(x)$ is portrayed in green in the diagram below).

We wish to show that $u[U_\nu(x)] \subset V_\epsilon(ux)$. So, let $y \in U_\nu(x)$. Then there is $\kappa > 0$ such that $U_\kappa(y) \subset U_\nu(x)$ ($U_\kappa(y)$ is denoted in red below). Also, $F(uy, y) = 0$, so $uy \in U_\kappa(y)$. But then $uy \in U_\kappa(y) \subset U_\nu(x) \subset V'_\epsilon(ux)$, and $uy \in X_u$. Therefore, $uy \in V_\epsilon(ux) = V'_\epsilon(x) \cap X_u$, as required.



□

We will use the following statement from [Vee70]:

4.3.3 Proposition ([Vee70])**.** *If $x, y \in X_u$, then $F(x, y) = F(px, py)$ for any $p \in E_X$.*

From now on, assume τ_F is Hausdorff.

4.3.4 Proposition. *(X_u, T, τ_F) is a dynamical system.*

Proof. We already know that X_u is a compact Hausdorff space.

Let us show X_u is invariant under the action of T. Let $t \in T$, $x \in X_u$. Then there exists $y \in X$ such that $uy = x$. But then $tx = tuy = uty \in X_u$ (since T is the centralizer (in the semigroup-theoretic sense, i.e. the set which commutes with all elements of the bigger space) of $E_X = E(X, \tau, T)$ - the 'bigger' space [EE14]).

Next, we need that $T : X_u \to X_u$ acts continuously with respect to the Furstenberg topology. We will show a bit more than that. Let $p \in E_X$. By Proposition 4.3.3, if $x, y \in X_u$, then $F(x, y) = F(px, py)$, so if $x_n \to x$ in the Furstenberg topology, then also

$$\lim_{n\to\infty} F(px_n, px) = \lim_{n\to\infty} F(x_n, x) = 0,$$

so $px_n \to px$ in the Furstenberg topology. Thus, for every $p \in E_X$, $p : (X_u, \tau_F) \to (X_u, \tau_F)$ is continuous. In particular, T acts continuously on X_u. □

Denote by E_u the Ellis semigroup of (X_u, T, τ_F).

4.3.5 Lemma. *The map $\phi : E_X \to E_u$ defined by $\phi(g) := g(u(x))$ is continuous from E_X to E_u.*

Proof. This follows directly from Proposition 4.3.2, and the facts that $X_u \subset X$ (as sets), and that precomposition by a function is a continuous map from X^X to itself.

Recall (Proposition 4.3.2) that $u : (X, \tau) \to (X_u, \tau_F)$ is continuous. We wish to show continuity of ϕ by the fact that the image of a convergent in E_X sequence under ϕ converges in E_u. So, let $g_\alpha \to^\tau g$, so $g_\alpha(y) \to^\tau g(y)$ for all $y \in X$. In particular, setting $y = ux$, we have $g_\alpha(u(x)) \to^\tau g(ux)$ for all $x \in X$. Thus, for all $x \in X_u$,

$$(g_\alpha \circ u)(x) = g_\alpha(u(x)) \to^\tau g(ux) = (g \circ u)(x).$$

Since $\tau_F \subset \tau$, we have that any τ-convergent sequence is also τ_F-convergent, so we conclude that

$$g_\alpha(u(y)) \to^{\tau_F} g(u(y)),$$

as required. So, $\phi(g_\alpha) \to^{\tau_F} \phi(g)$.

□

4.3.6 Lemma. *The induced map $\theta : E_X \to E_u$ coincides with the map $\phi(g) = g \circ u$.*

Proof. From Proposition 2.10 page 23 of [EE14], we have that for any surjective homomorphism between dynamical systems (in particular, for $u : (X, \tau, T) \to (X_u, \tau_F, T)$), there exists an induced map $\theta : E_X \to E_u$ such that the following diagram commutes:

$$\begin{tikzcd} E_X \arrow[r, "\theta"] \arrow[d, "p \mapsto px"'] & E_u \arrow[d, "p \mapsto pux"] \\ X \arrow[r, "u"] & X_u \end{tikzcd}$$

Since both θ and ϕ are continuous maps between Hausdorff spaces, we just need to show they agree on a dense set of their domains, in particular, it is sufficient to show they agree on $T \subset E_X$.

We obtain that

$$\begin{aligned} (\theta(t))(ux) &= utx && \text{by the diagram above} \\ &= tux && \text{since } T \text{ is the centralizer of } E_X \\ &= (\phi(t))(ux) && \text{by defn of } \phi, \end{aligned}$$

as required.

Thus the continuous functions θ and ϕ agree on T, hence they are equal. □

Recall Proposition 1.5.39, that the system (X_u, T, τ) is distal. However, this does not imply that the system with the Furstenberg topology is still distal (since proximality in the finer τ topology does not imply proximality in the coarser τ_F topology). Thus, we need to show the latter.

4.3.7 Lemma. *The system (X_u, T, τ_F) is distal (equivalently, E_u is a group).*

Proof. The equivalence between distality of X_u and E_u being a group follows from Proposition 1.5.35.

Note that since (as sets), $X_u \subset X$, Lemma 4.3.6 gives us that $E_u \subset E_X$ (again, only as sets - not as topological spaces). So, $E_u = \theta[E_X]$ (as a set). Let us consider what θ 'does' to E_X.

First, denote I the minimal left ideal such that $u \in I$ (so $E_X I \subset I$), and note that $E_X u \subset E_X I \subset I$ and moreover $E_X u$ is an ideal (since $E_X(E_X u) \subset E_X u$), so $E_X u = I$. So $pu \in I$ for all $p \in E_X$.

So in fact, θ given by $\theta(g) = gu$ maps E_X to I. Now, let $x \in X$, then $gux \in X_u$, so $ugux = gux$. Since $x \in X$ was arbitrary, we have that $ugu = gu$ for all $g \in E_X$. So $\theta[E_X] \subset uI$, where the latter is a group with identity u 1.5.35 (note the semigroup structure of the Ellis semigroup is independent of the topology since it rests solely on composition of functions). Thus, $\theta[E_X] = uI$, i.e. E_u is a group with identity $u = id$, so in fact (X_u, T, τ_F) is distal, as required. $\square$

4.3.8 Lemma. *The system (X_u, T, τ_F) is equicontinuous.*

Proof. By Proposition 1.5.36, X is equicontinuous if and only if $E(X)$ is a group of homeomorphisms of X. So, we need to show the latter.

By the argument in the proof of Proposition 4.3.4, we have that all $p \in E_X$ (respectively, $p \in E_u$) are continuous on (X_u, τ_F). They are also 1-1: if $x, y \in X_u$ with $px = py$, then $(x, y) \in P(X_u)$, and since X_u is distal (Lemma 4.3.7), we conclude $x = y$. Since again by Lemma 4.3.7, E_u is a group, we have that for each $p \in E_u$, there is $p^{-1} \in E_u$ such that $pp^{-1} = p^{-1}p = id$, so we conclude that each $p \in E_u$ is both a surjection and has a continuous inverse.

Therefore, each $p \in E_u$ is a homomorphism of X_u, as required. $\square$

4.3.9 Proposition (Veech pg 227)**.** *Let $\pi : X \to Y$ be a homomorphism of minimal dynamical systems, with induced homomorphism between the Ellis semigroups $\pi_0 : E_X \to E_Y$. Let u be a minimal idempotent in E_X and $u_0 := \pi_0(u)$ be its image (also an idempotent) in E_Y. Then $\pi[X_u] = Y_{u_0}$.*

Proof. Everything is based on the commutativity of the following diagram:

$$\begin{tikzcd} E_X \arrow[r, "\pi_0", "u\mapsto u_0"'] \arrow[d, "u\mapsto ux"'] & E_Y \arrow[d, "u_0\mapsto u_0\pi(x)"] \\ X \arrow[r, "\pi", "x\mapsto\pi(x)"'] & Y \end{tikzcd}$$

'$\subseteq$' Let $y \in \pi[X_u]$. Then there is an $x \in X_u$ with $y = \pi x$. Note that by the diagram above, $u_0 y = \pi_0 u \pi x = \pi ux$, and since $x \in X_u$, $\pi ux = \pi x = y$, so $u_0 y = y$, so $y \in Y_{u_0}$, as required.

'$\supseteq$' Let $y \in Y_{u_0}$, so $u_0 y = y$, and let $y = \pi x$ for some $x \in X$. Then also

$$\begin{aligned} \pi u x &= (\pi_0 u)\pi x && \text{by the diagram above} \\ &= u_0 y && \text{because } u_0 = \pi_0 u \\ &= y && \text{because } y \in Y_{u_0}, \end{aligned}$$

so $y \in \pi X_u$ (since $ux \in X_u$ as $u(ux) = (ux)$), as required. $\square$

4.3.10 Proposition. *The map $\pi : X_u \to Y_{u_0}$ is τ_F-continuous, where Y_{u_0} is as in Proposition 4.3.9.*

Proof. We prove this via convergent sequences. Let $\{x_n\}$ be a sequence in X_u which is $\tau_{F,X}$-convergent to some $x \in X_u$. This means that $\lim_{n\to\infty} F_X(x, x_n) = 0$, so

$$\lim_{n\to\infty} \inf_{t\in T} d(tx, tx_n) = \lim_{n\to\infty} F_X(x, x_n) = 0.$$

WLOG, $F_X(x, x_{n-1}) > F_X(x, x_n)$ (as otherwise, we can re-order the sequence and throw away some members to satisfy this inequality). So for each $n \in \mathbb{N}$ there exists $t_n \in T$ such that $F_X(x, x_{n-1}) > d_X(t_n x, t_n x_n)$. Since $\lim_{n\to\infty} F_X(x, x_n) = 0$, we get that there is a sequence $\{t_n\}$ in T such that $\lim_{n\to\infty} d_X(t_n x, t_n x_n) = 0$. Since π is uniformly continuous by the Heine-Cantor Theorem 0.1.21, we have

$$\lim_{n\to\infty} d_Y(\pi t_n x_n, \pi t_n x) = \lim_{n\to\infty} d_Y(t_n \pi x_n, t_n \pi x) = 0.$$

Therefore, $\lim_{n\to\infty} F_Y(\pi x_n, \pi x) = 0$, so $\lim_{n\to\infty} F_Y(\pi x_n, \pi x) = 0$, and so $\{\pi x_n\}$ is $\tau_{F,Y}$-convergent to πx, as required.

$\square$

Thus, we have practically shown that:

4.3.11 Theorem. *For a point-distal dynamical system (X, T) with minimal idempotent $u \in E(X, T)$, if (X_u, τ_F) is Hausdorff, then (X_u, T, τ_F) is the maximal equicontinuous factor of (X, T, τ).*

Proof. Let (Y, T) be an equicontinuous factor of (X, T), witnessed by the dynamical system homomorphism $\pi : X \to Y$. Note $\pi(ux) = \pi(x)$ for all $x \in X$.

Thus, π is a map from (X_u, T, τ_F) to (Y, T). Moreover, π is onto by Proposition 4.3.9 and the fact that (Y, T) is equicontinuous hence distal hence $Y_{\pi(u)} = Y$. Finally, π is τ_F-continuous by Proposition 4.3.10 and the fact that since Y is equicontinuous, $\tau_F \equiv \tau_Y$. Thus, π is a dynamical system homomorphism of (X_u, T, τ_F) onto (Y, T), as required. □

4.3.12 Lemma. *If (X, T) is point-distal, then $u : (X, T, \tau) \to (X_u, T, \tau_F)$ is almost 1-1.*

Proof. If (X, T) is point-distal, then there is a distal point $x_0 \in X$ with dense orbit. Then $x_0 \in X_u$, since $(ux_0, x_0) \in P(X)$ (for any point), so $x_0 = ux_0 \in X_u$. Moreover, $u^{-1}(x_0) = x_0$, since if $uy = ux_0$ then y is proximal to x_0 - contradiction to x_0 being distal. □

In conclusion, we have shown that when τ_F is Hausdorff, the point-distal system (X, T, τ) is an almost 1-1 extension of its maximal equicontinuous factor (X_u, T, τ_F) with the respective continuous homomorphism being $u : X \to X_u$.

We now prove the other direction of Veech's Theorem.

4.3.13 Theorem. *If a dynamical system (X, T) is an almost 1-1 extension of its maximal equicontinuous factor, then X is point-distal and (X_u, F_u) is Hausdorff.*

We loosely follow Veech's original train of thought, filling in the details and gaps in the original proof.

Proof. Suppose that (X, T) is an almost 1-1 extension $\pi : (X, T) \to (Y, T)$ of its maximal equicontinuous factor (Y, T), and let $u \in J(I)$. If $y \in Y$, any two points of $X_y := \pi^{-1}(y)$ are proximal by Proposition 1.6.5. Since by Proposition 4.3.7, there are no proximal pairs in X_u, we have that $|X_y \cap X_u| \leqslant 1$.

Let $\pi_0 : E_X \to E_Y$ be the induced homomorphism between the Ellis semigroups. Since Y is equicontinuous, E_Y is a group, so $\pi_0(u)$ is the identity in E_Y. Thus, by Proposition 4.3.9, we have that $\pi(X_u) = Y$, so $|X_y \cap X_u| \geqslant 1$. Therefore, $\pi : X_u \to Y$ is one-to-one and onto.

Since Y is equicontinuous, the Furstenberg topology agrees with the usual topology on Y (Y has an invariant metric). Thus Proposition 4.3.10 gives us that π is continuous from the Furstenberg topology on X_u to the (unique metric) topology on Y. Since moreover π is a one-to-one map of a compact space onto a Hausdorff space, it is a homeomorphism, and in particular, (X_u, F_u) is T_2. $\square$

There are several places in which the Hausdorff property is used. The first one is in the 'if' part of the statement; the main point being the following:

4.3.14 Proposition. *Let $f : X \to Y$ be a continuous 1-1 map between topological spaces X and Y. If Y is Hausdorff, then X is also Hausdorff.*

The proof is simple:

Proof. Let $x_1, x_2 \in X$ be two distinct points in X. Since f is one-to-one, we have $f(x_1) \neq f(x_2)$, and since Y is Hausdorff, there exist open in Y sets $U_1 \ni f(x_1)$, $U_2 \ni f(x_2)$ which are disjoint. Then their pre-images $f^{-1}(U_1) \ni x_1$, $f^{-1}(U_2) \ni x_2$ are disjoint, open (as f is continuous) neighbourhoods of the original points in X. Therefore, X is Hausdorff. $\square$

As a side note of curiosity, we combine Proposition 1.6.5 with the observation that any two proximal points $x_1, x_2 \in X$ get mapped to the same point $y \in Y$, where Y is the maximal equicontinuous factor, to obtain:

4.3.15 Proposition. *In a minimal point-distal dynamical system (X, T), the proximal relation is transitive if and only if there is an idempotent $u \in E(X)$ such that (X_u, τ_F) is a Hausdorff space.*

Proof. If there is a minimal idempotent $u \in E(X)$ such that (X_u, τ_F) is Hausdorff, then by Veech's Theorem 4.3.1, we have that (X, T) is an almost automorphic extension of its maximal equicontinuous factor. Conversely, if proximality is a transitive relation $\sim$, then one can see that the space $(X/\sim, T)$ is a dynamical system which is in fact the maximal equicontinuous factor of (X, T). Moreover, since (X, T) is point-distal, it is an almost automorphic extension of $(X/\sim, T)$. Hence, again by an application of Theorem 4.3.1, we have that there is a minimal idempotent $u \in E(X, T)$ such that (X_u, τ_F) is Hausdorff. $\square$

4.4 A Generalisation Using n-Hausdorffness

In [Bon13], M Bonanzinga introduces the following generalization of the Hausdorff separation axiom:

4.4.1 Definition (n-Hausdorff)**.** Let $n \in \mathbb{N} \setminus \{0, 1\}$. We say that a topological space X is *n-Hausdorff* if and only if for any n distinct points $x_1, \dots, x_n$, we have open neighbourhoods $U_1, \dots, U_n$ with $x_i \in U_i$, such that $\bigcap_{i=1}^{n} U_n = \emptyset$.

Note that every Hausdorff space is 2-Hausdorff, but also n-Hausdorff, for any $n > 2$. Similarly, any k-Hausdorff space is n-Hausdorff for $n > k$.

To link n-Hausdorffness with a natural generalisation of an almost 1-1 extension, we use the following notion, which is also studied elsewhere is topological dynamics:

4.4.2 Definition (almost n-to-1 extension)**.** Let $\pi : X \to Y$ be a homomorphism of minimal dynamical systems. We say that Y is *an almost n-to-1 extension* of X if and only if

$$\min\{|\pi^{-1}(y)| : y \in Y\} = n.$$

Now, let $n \in \mathbb{N}$ be an integer greater than 2. Similarly to Proposition 4.3.14, we can prove the following:

4.4.3 Proposition. *Let $f : X \to Y$ be a continuous $(n-1)$-to-one map between topological spaces X and Y. If Y is Hausdorff, then X is k-Hausdorff, where $2 \leqslant k \leqslant n$.*

Proof. Let $x_1, \dots, x_n$ be n distinct points in X. Since f is $(n-1)$-to-one, we have wlog that $f(x_1) \neq f(x_2)$. So there are disjoint open sets $V_1 \ni f(x_1)$ and $V_2 \ni f(x_2)$; let $U_1 := f^{-1}(V_1)$ and $U_2 := f^{-1}(V_2)$; then $U_1 \ni x_1$ and $U_2 \ni x_2$ are disjoint open neighbourhoods in X. Take any neighbourhoods $U_i \ni x_i$ for $i = 3, \dots, n$; we have found a family of open neighbourhoods $U_i \ni x_i$ for $i = 1, \dots, n$ with empty intersection. Thus, X is at most n-Hausdorff. $\square$

We prove an analogue of Proposition 1.6.5, which we will need in the proof of the n-Hausdorff analogue of Theorem 4.3.1. First, we prove the case when $n = 3$:

4.4.4 Proposition. *Let $\phi : (X,T) \to (Y,T)$ be a surjective homomorphism of dynamical systems, where Y is minimal and there is a $y_0 \in Y$ such that $\phi^{-1}(y_0) = \{x', x''\}$. Then whenever $x_1, x_2, x_3 \in \phi^{-1}(y)$ for some $y \in Y$, we have that there is at least one proximal pair in $\{x_1, x_2, x_3\}$.*

Proof. Let $y \in Y$, $x_1, x_2, x_3 \in \phi^{-1}(y)$. Since Y is minimal, there is a net $\{t_n\}_{n\in\alpha} \subseteq T$ such that $\lim_{n\in\alpha} t_n y = y_0$. This is equivalent to $\lim_{n\in\alpha} \phi(t_n x_i) = y_0$, for any $i = 1, 2, 3$.

Consider the net $\{t_n x_1\}_{n\in\alpha} \subseteq X$. Since X is compact and metric, there exists a subset $\beta \subseteq \alpha$ such that the associated subnet $\{t_n x_1\}_{n\in\beta}$ converges, say $\lim_{n\in\beta} t_n x_1 = x^*$. Then $x^* \in \{x', x''\}$: for otherwise, $\lim_{n\in\beta} \phi(t_n x_1) = \phi(x^*) \neq y_0 = \phi(x') = \phi(x'')$, which is a contradiction to the assumption that $\lim_{n\in\alpha} \phi(t_n x_1) = y_0$.

Applying the same argument as above to the net $\{t_n x_2\}_{n\in\beta}$, we obtain a subset $\gamma \subseteq \beta$ such that $\lim_{n\in\gamma} t_n x_2 = x^{**} \in \{x', x''\}$. In the same way we obtain a subset $\delta \subseteq \gamma$ with $\lim_{n\in\delta} t_n x_3 = x^{***} \in \{x', x''\}$.

Thus, we have that $\{x^*, x^{**}, x^{***}\} \subseteq \{x', x''\}$, so in fact at least two are equal. Thus, at least two of the points x_1, x_2, x_3 are proximal, as required. □

4.4.5 Remark. Ultimately $\phi^{-1}(y_0)$ gives the set of "possible limits" for x_1, x_2, x_3, and since there are three limits but only two possible values for them, the pigeonhole principle tells us that two of the three limits must co-incide.

4.4.6 Corollary (Corollary to Proposition 4.4.4)**.** *If (X,T) is an almost 2-to-1 extension of its maximal equicontinuous factor (Y,T), we have that there is a proximal pair amongst any three points in the same fiber.*

4.4.7 Proposition. *Let $\phi : (X,T) \to (Y,T)$ be a surjective homomorphism of dynamical systems, where Y is minimal and there is a $y_0 \in Y$ such that $|\phi^{-1}(y_0)| = n$. Then whenever $x_1, \ldots, x_{n+1} \in \phi^{-1}(y)$ for some $y \in Y$, we have that there is at least one proximal pair in $\{x_1, \ldots, x_{n+1}\}$.*

Proof. Let $y \in Y$, $x_1, \ldots, x_{n+1} \in \phi^{-1}(y)$. Since Y is minimal, there is a net $\{t_n\}_{n\in\alpha_0} \subseteq T$ such that $\lim_{n\in\alpha_0} t_n y = y_0$. This is equivalent to $\lim_{n\in\alpha_0} \phi(t_n x_i) = y_0$, for any $i = 1, \ldots, n+1$.

Consider the net $\{t_n x_1\}_{n \in \alpha_0} \subseteq X$. Since X is compact and metric, there exists a subset $\alpha_1 \subseteq \alpha_0$ such that the associated subnet $\{t_n x_1\}_{n \in \alpha_1}$ converges, say $\lim_{n \in \alpha_1} t_n x_1 = x_1^*$. Then $x_1^* \in \phi^{-1}(y_0)$: for otherwise, $\lim_{n \in \alpha_1} \phi(t_n x_1) = \phi(x_1^*) \neq y_0$, which is a contradiction to the assumption that $\lim_{n \in \alpha_0} \phi(t_n x_1) = y_0$.

We inductively apply the same argument as above to the nets $\{t_n x_i\}_{n \in \alpha_{i-1}}$ to obtain subsets $\alpha_i \subseteq \alpha_{i-1}$ such that $\lim_{n \in \alpha_i} t_n x_i = x_i^* \in \phi^{-1}(y_0)$, for $i = 2, \ldots, n+1$.

Thus, we have that $\{x_1^*, \ldots, x_{n+1}^*\} \subseteq \phi^{-1}(y_0)$, and since $|\phi^{-1}(y_0)| = n$, we have that at least two x_i^* are equal. Thus, at least two of the points $x_1, \ldots, x_{n+1}$ are proximal, as required. $\square$

Finally, we prove an analogue of Theorem 4.3.13 for n-Hausdorff spaces:

4.4.8 Theorem. *If a dynamical system (X, T) is an almost n-to-1 extension of its maximal equicontinuous factor, then (X_u, F_u) is k-Hausdorff, where $2 \leqslant k \leqslant n+1$.*

Proof. Let u be a minimal idempotent and let $\pi : X \to Y$ be a homomorphism of X onto its maximal equicontinuous factor Y. As in the proof of Theorem 4.3.1, we have that $\pi : X_u \to Y$ is surjective and continuous in the Furstenberg topology (and that the Furstenberg topology on Y coincides with the usual one, since Y is equicontinuous). Thus, $|X_y \cap X_u| \geqslant 1$.

Moreover, we have that no two points of X_u are proximal by Proposition 4.3.7, so by Proposition 4.4.7, we have that $|X_u \cap X_y| \leqslant n$. Thus, the map $\pi : X_u \to Y$ is a surjection which is at most n-to-1, so by Proposition 4.4.3, we have that X_u is k-Hausdorff, where $2 \leqslant k \leqslant n+1$. $\square$

4.5 Calculations of the Furstenberg Topology and its link with n-Hausdorff spaces

Let (M, σ) be the Thue-Morse dynamical system, and recall the following four idempotents in $E(M)$ (written in shorthand, using Notation 1.4.6 for the fixed points of the Thue-Morse substitution):

	v	$\overline{v}$	w	$\overline{w}$
g_1	w	$\overline{w}$	w	$\overline{w}$
g_2	$\overline{w}$	w	w	$\overline{w}$
g_3	v	$\overline{v}$	v	$\overline{v}$
g_4	v	$\overline{v}$	$\overline{v}$	v

Now, consider $X_3 := g_3[M] = dist(X) \cup Orb(v) \cup Orb(\overline{v}) = g_4[M]$ (since $g_3 \sim g_4$), where $dist(X)$ is the set of distal points in X. We will show that $\overline{v}$ and v do not have any disjoint open neighbourhoods. Hence, we will have that (X_3, τ_F) is at least 3-Hausdorff (and since it is an almost 2-to-1 extension of its maximal equicontinuous factor, we will have that it is precisely 3-Hausdorff). Let $U \ni \overline{v}$, $V \ni v$ be τ_F-open neighbourhoods. Since $g_3, g_4 : (M, \tau) \to (X_3, \tau_F)$ are continuous, $W_1 := g_3^{-1}[V]$ is a τ-open set containing v and w. Similarly, $W_2 := g_4^{-1}[U]$ is a τ-open set containing $\overline{v}$ and w. Thus, $W := W_1 \cap W_2$ is a τ-open neighbourhood of w. Since M is point-distal, there is $x \in W$ such that x is distal. Then $g_3(x) = x = g_4(x)$, and so

$$
\begin{aligned}
x =& g_3(x) \in g_3[W] \subset g_3[g_3^{-1}[V]] = V \\
x =& g_4(x) \in g_4[W] \subset g_4[g_4^{-1}[U]] = U,
\end{aligned}
$$

so $x \in U \cap W \neq \emptyset$, as required.

One can use Theorem 4.4.8 together with Theorem 4.3.1 to conclude that (X_3, τ_F) is precisely 3-Hausdorff. However, in this case it is easy to see 3-Hausdorffness directly. We have that $\pi : (M, \sigma) \to (G, p)$ (where (G, p) is the maximal equicontinuous factor of the Thue-Morse system), when restricted to X_3, is precisely 2-to-1. Thus, amongst any three points $x_1, x_2, x_3 \in X_3$, at least two are in different fibers, so without loss of generality $\pi(x_1) \neq \pi(x_2)$. As (G, p, τ_G) is Hausdorff, there are disjoint open neighbourhoods $U_1 \ni \pi(x_1)$, $U_2 \ni \pi(x_2)$. Since, as in the argument in Theorem 4.4.8, $\pi : (X_3, \tau_F) \to (G, p)$ is continuous, we have that $\pi^{-1}[U_1]$, $\pi^{-1}[U_2]$ are disjoint open neighbourhoods of x_1 and x_2, respectively. Thus, there exist neighbourhoods $V_i \ni x_i$, $i = 1, 2, 3$ such that $\bigcap_{i=1,2,3} V_i = \emptyset$, as required. Thus, (X_3, τ_F) is precisely 3-Hausdorff.

Part III

Analytic Topology

Chapter 5

Introduction

Separation axioms play a fundamental role in topology. There exist numerous generalisations and weakenings of those notions, considered in various fields even outside of general topology, such as domain theory [GL13], computer science [MS11], and quantum physics [HPS11]. Separation axioms also play an essential role in the theory of cardinal invariants, and some problems related to weakening the separation axioms in certain central cardinal inequalities have remained open for years and only got solved by considering additional set-theoretic axioms. Most of the generalisations studied until recently have been based on various weakenings of the notion of open sets (σ-Hausdorff in symmetrizable or o-metrizable spaces [As63], [Ned71]) or considering separation through special classes of open sets ('regularly open', for example). But if we consider separation axioms from the point of view of cardinal invariant inequalities, it seems more natural to consider some combinatorial-type generalisations. It is interesting that even outside the setting of cardinal invariants, combinatorial generalisations of normality have been investigated by Reed, Dolecky, Nogura and Peirome as early as 2001 [DNPR03], while the notion of 3-normal spaces has been introduced and considered in the setting of topological manifolds by Nykos in [Nyi92]. Dolecky and co-authors were mainly interested in the notion itself and constructed many examples for such combinatorial non-normal spaces - with finite and infinite non-normality number.

For the lower separation axioms, Bonanzinga, Cammaroto and Matveev

[BCM11] considered first combinatorial generalisations of non-Urysohn spaces, and later Bonanzinga [Bon13] introduced the notion of a Hausdorff number for spaces which might not necessarily have the Hausdorff separation axiom. Non-Urysohn spaces and anti-Urysohn spaces have also recently been studied by Juhász, Soukup, and Szentmiklóssy in [JSS16]. Those ways of investigations were aimed mainly at cardinal invariant settings.

It is worth pointing out that Hausdorffness, Urysohn-ness, and normality could be considered in essence homogeneous axioms of separation - in other words, they deal with "separation" of two "similar" topological objects. This is not the case with regularity, where a point and a closed set which does not contain it are separated. Here, we give a couple of possible combinatorial generalisations of regular spaces. We investigate some basic and classical results in the setting of all the above combinatorial separation axioms, and prove several theorems restricting cardinality of topological spaces, replacing standard separation axioms with their combinatorial analogues.

As a rule, so far, combinatorial separation axioms have been considered only in relation with other problems. The internal structure of a space possessing some of these axioms, as well as their relation to other basic and important topological properties, have not been investigated. Here, we give several such results and interrelations. We also pose some open questions for further investigation.

5.1 Some More Definitions and Notations

Let X be a topological space. All notions not defined here can be found in [Juh80] and [Eng89]. We deviate from standard notation, using $\langle a, b\rangle$ for an ordered pair (in other words, an element of the Cartesian product $A \times B$, of two sets A and B), reserving (a, b) for the open interval on the real line. We use $\mathbb{N}^+$ for the set of positive integers. In most cases, without ambiguity, we will consider the defined cardinal invariants to be infinite. For completeness, we recall some basic notions.

The following notion captures the 'size' of the local basis at a point x in a topological space X.

5.1.1 Definition (character [Juh80])**.** The *character* of X at the point $x \in X$, denoted by $\chi(x, X)$ is the following cardinal:

$$\chi(x, X) = min\{\tau : \text{there is a local base at } x \text{ with cardinality } \tau\ \}.$$

Then the character of X, $\chi(X)$ will be:

$$\chi(X) = sup\{\chi(x, X) : x \in X\}$$

A similar notion, related to the local pseudobase, is that of pseudocharacter.

5.1.2 Definition (pseudocharacter [Juh80])**.** The *pseudocharacter of* X *at the point* $x \in X$, denoted by $\psi(x, X)$ is the following cardinal:

$$\psi(x, X) = min\{\tau : \text{there is a local open neighbourhood system } \mathcal{B}(x) \text{ at } x \text{ with cardinality } \tau \text{ such that } x = \cap\{U : U \in \mathcal{B}(x)\}\ \}.$$

Then the *pseudocharacter of* X, $\psi(X)$ will be:

$$\psi(X) = sup\{\psi(x, X) : x \in X\}$$

It is interesting to note that in compact Hausdorff spaces, $\psi(x, X) = \chi(x, X)$ for all x in X. Such an equality rarely happens in non-compact spaces.

Compact spaces played a fundamental role in the initial development of analytic topology. The first more general compactness-like property was introduced by Lindelöf, and captures the ability to find a countable, instead of finite, subcover of every open cover of a topological space. If we consider a similar notion for higher cardinal numbers, we come to the following:

5.1.3 Definition (Lindelöf number, [Juh80])**.** The *Lindelöf number of* X, $L(X)$, is defined in the following way:

$$L(X) = \omega.\min\{\tau : \forall \text{ open cover } \gamma \text{ of } X \text{ there exists } \gamma' \in [\gamma]^{\leqslant\tau} \text{ such that } X \subseteq \cup\{U : U \in \gamma'\}\,\}.$$

So, if $L(X)$ is countable then X is called Lindelöf.

Recall that a subset D of a topological space X is dense in X if $\overline{D} = X$.

5.1.4 Definition (density character, [Juh80])**.** We define the *density character* $d(X)$ *of* X as

$$d(X) := \min\{|D| : D \subset X, \overline{D} = X\}.$$

We can also recall that a topological space X is said to have the countable chain condition (CCC) if the maximal family of mutually disjoint open sets in it is at most countable. This gives rise to the notion of a cellularity number:

5.1.5 Definition (cellularity number, [Juh80])**.** We define the *cellularity number* $c(X)$ *of* X as

$$c(X) := \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a family of mutually disjoint open subsets of } X\}.$$

Sometimes we will use the term 'Souslin number' instead of 'cellularity number'.

The spread of X reflects the possible size of discrete subspaces of X:

5.1.6 Definition (spread, [Juh80])**.** The *spread* $s(X)$ *of* X is defined as:

$$s(X) := \sup\{|A| : A \text{ is a discrete subspace of } X\}.$$

5.1.7 Notation. In general, if $\phi(X)$ is a cardinal invariant of the topological space X, then by $h\phi(X)$ we will denote the property of being hereditarily $\phi(X)$.

Some other more specific cardinal invariants will be defined later on. We now introduce the main combinatorial versions of the standard separation axioms, which will be the focus of the following sections.

5.1.8 Definition (Hausdorff number, [Bon13])**.** The *Hausdorff number* of X, written $H(X)$, is defined as:

$$H(X) = \min\Big\{\tau : \text{ whenever } \{x_\alpha : \alpha \in \tau\} \text{ is a subset of different points in } X, \text{ then } \\ \forall \alpha \in \tau \text{ there is an open } U_\alpha \subset X \text{ such that } x_\alpha \in U_\alpha \text{ and } \bigcap_{\alpha \in \tau} U_\alpha = \emptyset\Big\}.$$

5.1.9 Definition (Urysohn number, [BCM11]). The *Urysohn number of* X, denoted $U(X)$, is defined as:

$$U(X) = \min\big\{\tau : \text{ whenever } \{x_\alpha : \alpha \in \tau\} \text{ is a subset of different points in } X, \text{ then } \forall \alpha \in \tau \text{ there is an open } U_\alpha \subset X \text{ such that } x_\alpha \in U_\alpha \text{ and } \bigcap_{\alpha\in\tau} \overline{U}_\alpha = \emptyset\big\}$$

5.1.10 Definition (normality number, [DNPR03]). The *normality number* of X, denoted $N(X)$, is defined as:

$$N(X) = \min\big\{\tau : \text{whenever } \{F_\alpha : \alpha \in \tau\} \text{ is a disjoint family of closed nonempty different subsets of } X \text{ then } \exists \text{ open sets } U_\alpha \supset F_\alpha,\ \forall \alpha \in \tau, \text{ such that } \bigcap_{\alpha\in\tau} U_\alpha = \emptyset\big\}.$$

As we have already mentioned, due to the non-homogeneous nature of regularity, we can have various combinatorial approaches to defining the regularity number. We start with one of them.

5.1.11 Definition (weak regularity number, [BSS16]). The *weak regularity number of* X, denoted $R_0(X)$, is defined as:

$$R_0(X) = \min\big\{\tau : \text{ whenever } \emptyset \neq F \subset X \text{ is closed and } \{x_\alpha : \alpha \in \tau\} \text{ is a set of different points in } X \text{ such that } F \cap \{x_\alpha : \alpha \in \tau\} = \emptyset \text{ then } \exists \text{ open } U \supset F,\ U_\alpha \subset X,\ \mathbf{x}_\alpha \in U_\alpha\ \forall \alpha \in \tau, \text{ such that } U \cap \bigcap_{\alpha\in\tau} U_\alpha = \emptyset\big\}.$$

We note that τ-Hausdorfness implies weak τ-regularity. We also define the following stronger combinatorial regularity notion, that, in some intuitive sense, is "closer" to the notion of normality number.

5.1.12 Definition (regularity number, [BSS16]). The regularity number of X, denoted $R(X)$, is defined as:

$$R(X) = \min\big\{\tau :\ \text{whenever } \{F_\alpha : \alpha \in \tau\} \text{ is a disjoint family of closed nonempty different subsets of } X \text{ and a point } a \notin \bigcup\{F_\alpha : \alpha \in \tau\},\ \text{then there are open sets } U \ni a, U_\alpha \supset F_\alpha,\ \forall \alpha \in \tau,\ \text{such that } U \cap \bigcap_{\alpha \in \tau} U_\alpha = \emptyset\big\}.$$

Again, in T_1 spaces we have $H(X) \leqslant R_0(X) \leqslant R(X) \leqslant N(X)$, and if X is τ-regular, then X is α-regular for all $\alpha < \tau$.

If $H(X) \leqslant \tau$, we call X τ-Hausdorff. Similarly, if $U(X) \leqslant \tau$, $R_0(X) \leqslant \tau, R(X) \leqslant \tau$, $N(X) \leqslant \tau$ we call X respectively τ-Urysohn, τ-weakly regular, τ-regular or τ-normal. Let us point out that when X is τ-Hausdorff (τ-Urysohn, τ-weakly regular, τ-normal), then it is also β-Hausdorff (β-Urysohn, β-weakly regular, or β-normal) for every $\beta \geqslant \tau$. We also easily have in T_1 spaces that $H(X) \leqslant R_0(X) \leqslant N(X)$, and always $H(X) \leqslant U(X)$. It is tempting to claim that, as in the case of traditional separation axioms, in T_1 spaces we have $U(X) \leqslant R_0(X)$, but as we shall show through examples afterwards, this is unfortunately not the case.

In [Bon13], examples of T_1 n-Hausdorff not Hausdorff spaces are given for any $n \geqslant 3$, as well as an example of a T_1 ω-Hausdorff not n-Hausdorff (for any $n \in \mathbb{N}$) space. In [Sta13], such examples with additional topological properties are given. In both of the above papers, examples of not T_1 n-Hausdorff spaces were given, thus showing that τ-Hausdorff property is independent of T_1. Most of the examples in [Bon13] are countable and those in [Sta13] are first countable with cardinality continuum and compact if $H(X)$ is finite, or Lindelöf if $H(X) = \omega$.

In [DNPR03], for every cardinal κ, examples of completely regular spaces with $N(X) = \kappa$ are constructed. For some cardinals κ, we will provide examples of such spaces with additional properties.

Chapter 6

Investigating Combinatorial Separation Axioms

6.1 Examples

It is a basic result that Hausdorff compact spaces are normal. The following example shows that even the strongest "non-normality" property, i.e. T_1 3-normality, does not imply even Hausdorffness in compact spaces.

6.1.1 Example. There is a T_1, first countable, compact not Hausdorff (hence not normal and not regular) 3-normal space X (with cardinality 2^ω, by construction).

Construction. Let $X = ([0,1] \times \{0\}) \cup \left\{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle\right\} \subset \mathbb{R}^2$, where $[0,1]$ is the standard unit interval in $\mathbb{R}$. Topologize X as follows:

- all points on $[0,1] \times \{0\}$ have the Euclidean neighborhoods;
- the neighborhoods of $\{\langle \frac{1}{2}, \frac{1}{2} \rangle\}$ consist of

$$U_p\left(\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle\right) = \left\{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle\right\} \cup \left(\left(\left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} + \frac{1}{p}\right) \setminus \left\{\frac{1}{2}\right\}\right) \times \{0\}\right),\ p \in \mathbb{N}^+.$$

Let us point out that the subspace topology on $[0,1] \times \{0\}$ coincides with the Euclidean topology.

This space is not T_2 as the points $\langle 1/2, 0\rangle$ and $\langle 1/2, 1/2\rangle$ cannot be separated via disjoint open neighbourhoods.

Let us prove that X is 3-normal. Let F_1, F_2, F_3 be mutually disjoint closed subsets of X. The point $\langle 1/2, 1/2\rangle$ is the only point in X which is not in $[0,1]\times\{0\}$. Thus, at least two of the F_i's are subsets of $[0,1]\times\{0\}$, and so they have disjoint open neighbourhoods.

By not excluding $\langle 1/2, 0\rangle$ from the neighbourhoods of $\langle 1/2, 1/2\rangle$, we similarly obtain

6.1.2 Example. There is a first countable, compact, not T_1, (not regular nor Hausdorff), 3-Hausdorff, 3-normal (and weakly 3-regular) space X with cardinality 2^ω.

Using the same idea as in Example 6.1.1, for any $k \in \mathbb{N}$ we can construct first countable compact T_1 space that is $(k+1)$-normal not k-normal. We can also have a not T_1 example of such a space, thus showing that T_1 is independent of k-normality. In addition, the not T_1 example can be constructed in such a way that it has only one point at which the T_1 axiom is violated (as in Example 6.1.2). Hence, we have:

6.1.3 Example. There is a T_1, first countable, compact space X not k-normal but $(k+1)$-normal for any $k \in \mathbb{N}$ with $k \geqslant 2$, with cardinality 2^ω by construction.

Construction. Let $X = ([0,1]\times\{0\}) \cup \left\{\left\langle \frac{1}{2}, \frac{1}{m}\right\rangle : 0 < m < k\right\}$, and topologize X as follows:

- all points of $[0,1]\times\{0\}$ have Euclidean neighbourhoods;
- the neighbourhoods of $\langle 1/2, 1/m\rangle$ are

$$U_p\left(\left\langle \frac{1}{2}, \frac{1}{m}\right\rangle\right) = \left\{\left\langle \frac{1}{2}, \frac{1}{m}\right\rangle\right\} \cup \left(\left(\left(\frac{1}{2}-\frac{1}{p}, \frac{1}{2}+\frac{1}{p}\right) \setminus \left\{\frac{1}{2}\right\}\right) \times \{0\}\right),\ p \in \mathbb{N}^+$$

6.1.4 Example. There is a not T_1 at only one point, first countable, compact space X which is not k-normal but is $(k+1)$-normal for any $k \in \mathbb{N}^+$, $k \geqslant 2$. X is also not k-weakly regular.

Construction. Let the underlying set of X be as in Example 6.1.3 and let us topologize X as follows:

- points on $[0,1]\times\{0\}$ have Euclidean neighbourhoods;
- the neighbourhoods of $\langle 1/2, 1/m\rangle$, $m<k$, are

$$U_p\left(\left\langle\frac{1}{2},\frac{1}{m}\right\rangle\right)=\left\{\left\langle\frac{1}{2},\frac{1}{m}\right\rangle\right\}\cup\left(\left(\frac{1}{2}-\frac{1}{p},\frac{1}{2}+\frac{1}{p}\right)\times\{0\}\right),\ p\in\mathbb{N}^+.$$

Let us point out again that the subspace topology on $[0,1]\times\{0\}$ coincides with the Euclidean topology; and the only point in X which is not closed is, as in Example 6.1.2, the point $\langle 1/2, 0\rangle$.

We also have

6.1.5 Example. There is an ω-normal, T_1 space X which is first countable, Lindelöf, not k-normal (k-weakly regular, k-Hausdorff), for all $k\in\mathbb{N}$.

Construction. Let $X=([0,1]\times\{0\})\cup\{\langle 1/2,1/m\rangle : m\in\mathbb{N}^+\}$. Topologize X as follows:

- All points on $[0,1]\times\{0\}$ have Euclidean neighbourhoods;
- The neighbourhoods of $\langle 1/2, 1/m\rangle$, $m\in\mathbb{N}^+$, are

$$U_p\left(\left\langle\frac{1}{2},\frac{1}{m}\right\rangle\right)=\left\{\left\langle\frac{1}{2},\frac{1}{m}\right\rangle\right\}\cup\left(\left(\frac{1}{2}-\frac{1}{p},\frac{1}{2}+\frac{1}{p}\right)\setminus\left\{\frac{1}{2}\right\}\right)\times\{0\},\ p\in\mathbb{N}^+.$$

Let us show directly that X is ω-Hausdorff. The only essential case we have to consider is how to separate the points in $\{\langle 1/2,1/m\rangle : m\in\mathbb{N}^+\}\cup\{\langle 1/2,0\rangle\}$. For each $m\in\mathbb{N}^+$, consider $U_m(\langle 1/2,1/m\rangle)$ and any neighbourhood U of $\langle 1/2,0\rangle$ on $[0,1]\times\{0\}$. Then $U\cap\left(\bigcap_{m\in\omega}U_m\right)=\emptyset$.

The space is not k-normal, as we can take the closed points $\{\langle 1/2,1/m\rangle\}$ for $m=1,\ldots,k$, and they do not have open neighbourhoods with empty intersection.

Let us now show that X is also ω-normal. Let $\{F_n : n\in\mathbb{N}^+\}$ be a family of mutually disjoint closed subsets of X. We have the following cases:

Case 1: If two of $\{F_i : i \in \mathbb{N}^+\}$ are subsets of $[0,1] \times \{0\}$, we are done, as they can be separated in the Euclidean topology.

Case 2: Suppose that at most one of $\{F_i : i \in \mathbb{N}^+\}$ is a subset of $[0,1] \times \{0\}$. Write $F_i = F_i' \cup K_i$, where $F_i' = F_i \cap ([0,1] \times \{0\})$, and $K_i = F_i \setminus F_i'$. Then, $\{F_i' : i \in \mathbb{N}^+\}$ is a family of disjoint, Euclidean-closed subsets of $[0,1] \times \{0\}$, at most one of which contains the point $\langle 1/2, 0\rangle$. Without loss of generality, after re-numbering if necessary, we can write our family as $F_0 \cup \{F_i' : i \in \mathbb{N}^+\}$, where $\langle 1/2, 0\rangle \in F_0$. For all $n \in \mathbb{N}^+$, let

$$O_n = \left(\frac{1}{2} - \frac{1}{n+1}, \frac{1}{2} + \frac{1}{n+1}\right) \setminus \left\{\left\langle \frac{1}{2}, 0\right\rangle\right\}.$$

We shall inductively define open $W_n \supset F_n$ such that $\bigcap_{n\in\mathbb{N}} W_n = \emptyset$.

First, let us separate F_1' and F_2' by disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $U_1 \supset F_1'$, $U_2' \supset F_2'$. Let us separate F_1' from $\langle 1/2, 0\rangle$ by disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $V_1 \supset F_1'$ and $Q_1 \ni \langle 1/2, 0\rangle$. Let

$$W_1 = (U_1 \cap V_1) \cup (O_1 \cap Q_1) \cup K_1,$$

and note that W_1 is an open neighbourhood of F_1.

Now let us separate F_2' and F_3' by disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $U_2 \supset F_2'$ and $U_3' \supset F_3'$. Let us separate F_2' from $\langle 1/2, 0\rangle$ by disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $V_2 \supset F_2'$ and $Q_2 \ni \langle 1/2, 0\rangle$. Let

$$W_2 = (U_2 \cap V_2 \cap U_2') \cup (O_2 \cap Q_2) \cup K_2,$$

and note that W_2 is an open neighbourhood of F_2.

Now assume that for $k = 1, \ldots, n-1$, we have defined disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $U_k \supset F_k'$, $U_{k+1} \supset F_{k+1}'$, and disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $V_k \supset F_k'$ and $Q_k \ni \langle 1/2, 0\rangle$. Consider F_n' and F_{n+1}'. Let us separate them via disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $U_n \supset F_n'$, $U_{n+1}' \supset F_{n+1}'$. Let us separate F_n' from $\langle 1/2, 0\rangle$ by disjoint Euclidean-open subsets of $[0,1] \times \{0\}$, $V_n \supset F_n'$ and $Q_n \ni \langle 1/2, 0\rangle$. Define

$$W_n = (U_n \cap U_n' \cap V_n) \cup (O_n \cap Q_n) \cup K_n,$$

and note that W_n is an open neighbourhood of F_n. Then we have $\bigcap_{n\in\mathbb{N}^+} W_n = \emptyset$. Let W_0 be an arbitrary neighbourhood of F_0, then we still have $\bigcap_{n\in\mathbb{N}} W_n = \emptyset$, which proves ω-normality.

The fact that X is not k-Hausdorff for any $k \in \omega$ follows in the same way as in Example 6.1.1 and by construction. Since X is T_1, from here it also follows that X is not k-normal (weakly k-regular) for any $k \in \mathbb{N}^+$.

The facts that X is first countable and Lindelöf follow directly by the construction: $[0,1] \times \{0\}$ is compact in the Euclidean topology, and outside it we have only countably many points.

We can modify the previous example in order to obtain a compact space with the same properties.

6.1.6 Example. There is a T_1, ω-normal, first countable compact space X which is not k-Hausdorff (k-normal, weakly k-regular) for any $k \in \mathbb{N}^+$.

Construction. Let X be the set as in Example 6.1.5, and topologize X as follows:

- all points on $[0,1] \times \{0\}$ again have Euclidean neighbourhoods;
- the neighbourhoods of $\langle 1/2, 1/m\rangle$, $m \in \mathbb{N}^+$, are the same as in Example 6.1.5;
- the neighbourhoods of $\langle 1/2, 0\rangle$ are

$$U_p\left(\left\langle \frac{1}{2}, 0\right\rangle\right) = \left(\left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} + \frac{1}{p}\right) \times \{0\}\right) \cup \left\{\left\langle \frac{1}{2}, \frac{1}{m}\right\rangle : p \leqslant m \in \omega\right\},\ p \in \mathbb{N}^+.$$

The arguments given in Example 6.1.5 can be modified to show that X is ω-normal. X is also T_1 and with the usual arguments as in all of the above examples, it follows that X is not k-Hausdorff for any $k \in \mathbb{N}^+$. By construction, X is first countable and also by construction, X is compact, as union of its compact subset $[0,1]\times\{0\}$ and the convergent sequence $\{\langle 1/2, 0\rangle\} \cup \{\langle 1/2, 1/m\rangle : m \in \mathbb{N}^+\}$.

6.1.7 Note. If on the real line, we take only the points from $\mathbb{Q}$, we obtain countable versions of Examples 6.1.1-6.1.6.

We have already seen that first countability and compactness cannot "convert" even the strongest "non-normal" property into normality. One might speculate that, by adding more "nice" properties, one could achieve this. But even Example 6.1.1 shows that another classic result does not extend even to the 3-normal case - namely, the Urysohn metrization theorem. If we point out that the spaces in all of the above examples are even second countable, we have:

6.1.8 Example. There is a T_1, 3-normal compact second countable space which is not normal (and hence not metrizable).

We might ask, what is the regularity number of some classic Hausdorff not regular spaces. We concentrate on the following one:

6.1.9 Example. There is a Hausdorff, second countable, not regular, weakly 2-regular space.

Construction. Take the real line $\mathbb{R}$ with the following topology τ: the basis of τ consists of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$, where $K = \{1/n : n \in \omega\}$. In this topology, K is closed and cannot be separated from $\{0\}$ by disjoint open sets; hence it is not regular.

It is Hausdorff and second countable by construction. It is easily seen that X is weakly 2-regular.

We note that, alongside K, every subsequence of K is also closed in this topology. Indeed, let K_1 be a proper subsequence of K. Then suppose $1/p \notin K_1$, for some $p \in \omega$. Then there are disjoint Euclidean-open U_1, V_1 such that $U_1 \ni 1/p$, $V_1 \ni 1/(p+1)$. Similarly, there are disjoint Euclidean-open $U_2 \ni 1/p$, $V_2 \ni 1/(p-1)$. Let $U_p = U_1 \cap U_2$, and let $W_1 = \cup\{U_p : 1/p \notin K_1\}$. Then $K_1 \cap W_1 = \emptyset$ by construction, and hence $\mathbb{R} \setminus W_1$ is a closed superset of K_1. Since $K_1 = K \cap (\mathbb{R} \setminus W_1)$, it follows that K_1 is also closed.

Let us only further point out that trivially, the above example is Lindelöf.

Then on the basis of Example 6.1.9, we obtain the following

6.1.10 Example. There is a Hausdorff second countable space X which is not ω-regular (hence not k-regular for any finite $k \in \omega$) but is ω_1-normal (and hence ω_1-regular).

Construction. Let $X = \mathbb{R}$, and let Q^* be the rationals in $(0,1)$. Decompose Q^* into countably many disjoint Euclidean dense sets $\{Q^*_n : n \in \mathbb{N}\}$. Topologize X as follows: the pseudobasis of X consists of all open intervals (a,b) and sets of the form $(a,b) \setminus Q^*_n$ for $n \in \mathbb{N}$. To show that X is not ω-regular, let us note that Q^*n , for $n \in \mathbb{N}$, are closed and disjoint in X. Also, $0 \notin Q^* = \cup\{Q^*_n : n \in \mathbb{N}\}$, and 0 and $\{Q^*_n : n \in \mathbb{N}\}$ cannot be separated in the sense of w-regularity.

To show that X is ω_1-normal, let us point out that if we have a family of disjoint closed sets $\{F_\alpha : \alpha \in \omega_1\}$ in X, then at least three are disjoint from Q^* and hence can be Euclidean-separated by normality of the Euclidean topology.

6.2 Combinatorial Separation Axioms in the Setting of Some Classic Topological Constructions and Properties

We can also pose questions about inheritance of the Hausdorff number:

Open Question. Given an uncountable τ-Hausdorff (τ-normal) space X, do we have that, for each $\kappa < \tau$, there is an uncountable κ-Hausdorff (respectively, κ-normal) subspace of X?

As we have pointed out, in [DNPR03] there is a construction of a T_1 completely regular space which are not τ-normal for every cardinal τ. All of those spaces are trivially weakly τ-regular, and hence we readily have T_1 τ-regular spaces which are not τ-normal.

Dolecky and al [DNPR03] proved that the non-normality number of every separable regular topology with a closed discrete subset of cardinality 2^ω is at least 2^ω. The following example shows that it can be greater than 2^ω, but in addition it shows that n-normality, is not productive even in the case of Hausdorff normal spaces.

6.2.1 Example. There is a Hausdorff regular not 2^ω-normal space X which is a product of two Hausdorff normal spaces.

The space in question is the Sorgenfrey plane. The only thing we have to show is that it is not 2^ω-normal. To do this, let L be the line $y = -x$. Decompose L into 2^ω Euclidean dense disjoint subsets $\{L_\alpha : \alpha \in 2^\omega\}$. Note that all L_α's are closed discrete in X and cannot be separated in the sense of 2^ω-normality.

The above example leads to the first open problem that we pose:

Open Question. Is the product of a 3-normal space with the unit interval $[0, 1]$ also 3-normal? More generally, is the product of a k-normal space with $[0, 1]$ also k-normal? Or, in the terminology adopted in [TT11], is $[0, 1]$ productively 3-normal?

Open Question. Is any metric space productively 3-normal?

We know that the Hausdorff property is productive. That is why it is interesting to investigate the finite Hausdorff number in product spaces.

We have the following straightforward result:

6.2.2 Proposition. *The product of a 3-Hausdorff space X with a Hausdorff space Y is 3-Hausdorff.*

However, the same conclusion is not true when both spaces are 3-Hausdorff, as the following example shows.

6.2.3 Example. There exists a compact, first countable, 3-Hausdorff T_1 space such that X^2 is not 3-Hausdorff, but 5-Hausdorff.

Construction. Let $X = [0, 1] \cup \{2\}$. All points from $[0, 1]$ have Euclidean neighbourhoods. The neighbourhoods of 2 are:

$$U_n(2) = \{2\} \cup \left(\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n} \right) \setminus \left\{ \frac{1}{2} \right\} \right), \ n \in \mathbb{N}^+, n > 1.$$

It can be seen that X is compact, 3-Hausdorff, first countable, T_1 not Hausdorff. The product $X \times X$, though, is not 3-Hausdorff. It is exactly 5-Hausdorff.

One possible hypothesis about the Hausdorff number that the above example disproves, is that the product of two 3-Hausdorff spaces is exactly 6-Hausdorff. Here we prove the most general case.

6.2.4 Theorem. *The product of m many n-Hausdorff spaces is $((n-1)^m+1)$-Hausdorff.*

Proof. Let X_k, $k = 1, \dots, m$ be n-Hausdorff spaces, and consider $\prod_{k=1}^m X_k$.

First, we will show that the Hausdorff number of $\prod_{k=1}^m X_k$ is at least $(n-1)^m+1$. For each X_k, there exists a set of points $P_k = \{x_k^{(1)}, \dots, x_k^{(n-1)}\}$ such that for all neighbourhoods $U_k^{(1)}, \dots, U_k^{(n-1)}$ of the points, we have that $\bigcap_{i=1}^{n-1} U_k^{(i)} \neq \emptyset$. Then the points $\{(y_1, \dots, y_m) : y_k \in P_k\}$ are $(n-1)^m$-many, and we will show that all their neighbourhoods have a nonempty intersection. Let $W_i \subset \prod_{k=1}^m X_k$, $i = 1, \dots, (n-1)^m$ be basic open neighbourhoods of the points, so $W_i = V_1^{(i)} \times \dots \times V_m^{(i)}$, where $V_k^{(i)} \subseteq X_k$. For $x_k^{(j)} \in P_k$, let

$$U_k^j := \bigcap \left\{ V_k^{(i)} : x_k^{(j)} \in V_k^{(i)} \right\}.$$

Then $U_k^{(j)}$ are neighbourhoods of $x_k^{(j)}$ in X_k, so by n-Hausdorffness, there exists a point $y \in \bigcap_{j=1}^{n-1} U_k^{(j)}$. Then the point

$$(y_1, \dots, y_m) \in \bigcap_{i=1}^{(n-1)^m} W_i,$$

showing that the intersection is indeed non-empty, as required.

Next, we will show that $\prod_{k=1}^m X_k$ is at most $((n-1)^m+1)$-Hausdorff. Let $x^{(i)}$, $i = 1, \dots, (n-1)^m+1$ be distinct points in $\prod_{k=1}^m X_k$, with coordinates $x^{(i)} = (x_1^{(i)}, \dots, x_m^{(i)})$. Consider the sets

$$P_k := \{x_k^{(i)} : i = 1, \dots, (n-1)^m + 1\} \subset X_k,$$

the set of k-th coordinates of the points $x^{(i)}$. Note that if $i \neq j$, $x_k^{(i)}$ is not necessarily distinct from $x_k^{(j)}$, but this does not create any difficulties in the subsequent proof. Since $(n-1)^m + 1 > (n-1)^m$, it is easy to see that for some P_k, we have $|P_k| > n-1$. Since X_k is n-Hausdorff, the at-least-n-many points in P_k have open neighbourhoods $U_k^{(i)} \ni x_k^{(i)}$, $i = 1, \dots, (n-1)^m+1$, with

$\bigcap_{i=1}^{(n-1)^m+1} U_k^{(i)} = \emptyset$. Consider the open sets

$$V_j := \prod_{l=1}^{k-1} X_l \times U_k^{(j)} \times \prod_{l=k+1}^{m} X_l.$$

By construction, each point $x^{(i)}$ lies in an open set V_j. Moreover, by construction, we have that $\bigcap_{j=1}^{(n-1)^m+1} V_j = \emptyset$, as required.

Therefore, $\prod_{k=1}^m X_k$ is precisely $((n-1)^m+1)$-Hausdorff, as required. □

Now, we can prove an analogue of Proposition 6.2.2 in the most general setting.

6.2.5 Theorem. *If X is n-Hausdorff and Y is m-Hausdorff, then $X \times Y$ is $((n-1)(m-1)+1)$-Hausdorff. Here, we allow n or m to be equal to 2.*

Proof. Let X be an n-Hausdorff space, Y be m-Hausdorff, and let $k := (n-1)(m-1)+1$.

We first show that $X \times Y$ is at least k-Hausdorff. Since X is n-Hausdorff, there exist $x_1, \ldots, x_{n-1}$ such that any neighbourhoods $U_i \ni x_i$ will have $\cap_{i=1}^{n-1} U_i \neq \emptyset$. Similarly, there are $y_1, \ldots, y_{m-1} \in Y$ such that for any neighbourhoods $V_i \ni y_i$, we have $\cap_{i=1}^{m-1} V_i \neq \emptyset$. So, consider the set of $(k-1)$-many points $\{(x_i, y_j) : i = 1, \ldots, n-1,\ j = 1, \ldots, m-1\} \subset X \times Y$. Whatever set of basic open neigbourhoods $U_i \times V_j$ we choose for the points (x_i, y_j), we will have that there is $x^* \in \cap_{i=1}^{n-1} U_i \neq \emptyset$ and $y^* \in \cap_{i=1}^{m-1} V_i \neq \emptyset$, so that $(x^*, y^*) \in \cap\{U_i \times V_j : 1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant m-1\}$, so the latter would be nonempty.

We now show that $X \times Y$ is at most k-Hausdorff. Let $P := \{(x_i, y_i) : i = 1, \ldots, k\}$ be a set of k-many distinct points in $X \times Y$. Consider $P_X := \{x \in X : \exists y \in Y,\ (x, y) \in P\}$ and $P_Y := \{y \in Y : \exists x \in X,\ (x, y) \in P\}$ - the sets of x- (or respectively, y-) coordinates of points in P. Then either $|P_X| \geqslant n$ or $|P_Y| \geqslant m$. Without loss of generality, assume $|P_X| = n' \geqslant n$. Express $P_X = \{x_i' : 1 \leqslant i \leqslant n'\}$. Since X is n-Hausdorff, one can take neighbourhoods $U_i \ni x_i'$, $1 \leqslant i \leqslant n'$, such that $\bigcap_{i=1}^{n'} U_i = \emptyset$, and set $W_i := U_i \times Y$. Then, for $1 \leqslant j \leqslant k$, every $(x_i, y_i) \in W_i$ for some i, $1 \leqslant i \leqslant n'$, and $\bigcap_{i=1}^{n'} W_i = \emptyset$, as required. This shows that $X \times Y$ is k-Hausdorff, and thus completes the proof. □

It is well known that the situation with the productivity of normality is not straightforward. The classical example of E. Michael [Mic63] is that the product of a normal space and a metric space need not be normal. Let us just recall Michael's example. The space is $Z = \mathbb{I} \times \mathbb{M}$, where $\mathbb{I}$ is the irrationals and $\mathbb{M}$ is the real line with the stronger topology obtained by isolating the points of $\mathbb{I}$. We note that Michael's example is not 3-normal, which motivates the following questions:

1 Question. Is Michael's example n-normal for some $n \geqslant 4$?

2 Question. Is there a normal space X such that X^2 is not normal, but 3-normal?

Similarly as in the case of Hausdorff number, we can ask:

3 Question. What is the normality number of the product of m-many n-normal spaces?

4 Question. What is the cardinality of a T_1, 3-normal, first countable, weakly Lindelöf space?

In fact, weak 2-regularity in examples 6.2.1 and 6.1.9 trivially follows from the following straightforward Proposition:

6.2.6 Proposition. *Every Hausdorff space X is weakly 2-regular.*

More generally, we have the following:

6.2.7 Proposition. *Every τ-Hausdorff space is weakly τ-regular for every cardinal τ.*

The above facts have in fact initially motivated us to define the stronger form of regularity number.

For compact spaces, we have the following analogue of an already discussed classical result:

6.2.8 Theorem. *If X is compact, 3-Hausdorff, then X is 3-normal.*

Proof. Let A, B, C be three mutually disjoint closed subsets of X. Note that they are also compact.

Fix a point $b \in B$ and a point $c \in C$. Then for any $a \in A$, there are open neighbourhoods $U_a \ni a$, $V_a \ni b$ and $W_a \ni c$ such that $U_a \cap V_a \cap W_a = \emptyset$. From the cover $\{U_a : a \in A\}$, choose a finite subcover $\{U_{a_i} : i = 1, \ldots, n\}$, and set

$$\begin{aligned} U_A &= \bigcup\{U_{a_i} : i = 1, \ldots, n\} \supset A \\ V_b &= \bigcap\{V_{a_i} : i = 1, \ldots, n\} \ni b \\ W_c &= \bigcap\{W_{a_i} : i = 1, \ldots, n\} \ni c, \end{aligned}$$

and note that $U_A \cap V_b \cap W_c = \emptyset$.

Hence for all $b \in B$ and for each fixed $c \in C$, we can find open sets $U_A^{(b,c)} \supset A$, $V_b^{(b,c)} \ni b$, $W_c^{(b,c)} \ni c$ with empty intersection. From the cover $\{V_b^{(b,c)} : b \in B\}$ of B, we choose a finite subcover $\{V_{b_i}^{(b_i,c)} : i = 1, \ldots, k\}$, and let

$$\begin{aligned} V_B^c &= \bigcup\{V_{b_i}^{(b_i,c)} : i = 1, \ldots k\} \supset B \\ U_A^c &= \bigcap\{U_A^{(b_i,c)} : i = 1, \ldots k\} \supset A \\ W_c &= \bigcap\{W_c^{(b_i,c)} : i = 1, \ldots k\} \ni c. \end{aligned}$$

Then, $U_A^c \cap W_c \cap V_B^c = \emptyset$.

From $\{W_c : c \in C\}$, choose a finite subcover $\{W_{c_i} : i = 1, \ldots, m\}$, and let

$$\begin{aligned} W_C &= \bigcup\{W_{c_i} : i = 1, \ldots, m\} \\ U_A &= \bigcap\{U_A^{c_i} : i = 1, \ldots, m\} \\ V_B &= \bigcap\{V_B^{c_i} : i = 1, \ldots, m\}. \end{aligned}$$

Then, we have $W_C \cap U_A \cap V_B = \emptyset$, as required.

□

With a similar argument, we have:

6.2.9 Theorem. *If X is a compact k-Hausdorff space, then X is k-normal.*

As for the construction of the quotient space, one might that collapsing all points of a local Hausdorff width in one might give us a Hausdorff space. So, it is of interest to ask:

5 Question. Is it true that every T_1 n-Hausdorff space has an equicardinal Hausdorff quotient?

In general, the study of quotient of n-Hausdorff spaces might help provide a converse to the author's Theorem 4.4.8 about the Furstenberg topologies.

Chapter 7

Cardinality Restrictions

7.1 History, Motivation, and First Results

In the previous section, we defined some of the basic cardinal invariants, such as Lindelöf number, Souslin number, density and spread. Another basic cardinal invariant is the weight of the topological space, which is the smallest cardinality of a basis $\mathcal{B}$ of X. Fundamental theorems and problems in topology immediately show the importance of these basic notions. For example, a regular space of countable weight is metrizable (Urysohn-Tychonoff metrization theorem, 1925); a compact Hausdorff space is metrizable if and only if its weight is countable; the Lindelöf number of every space of countable weight is itself countable. Hence, the question of comparison of cardinal invariants might lead to significant conclusions about the structure of the space. This question is central in the theory of cardinal invariants.

Much research in the theory of cardinal invariants was stimulated by the problem of estimating the cardinality of compact Hausdorff spaces satisfying the first axiom of countability.

As early as 1915 [Arh00], Luzin asked the following question:

6 Question (Luzin, 1915)**.** Is the cardinality of every uncountable Borel subset of the real line precisely 2^ω?

The answer was given by Alexandroff [Arh00], who proved

7.1.1 Theorem (Alexandroff, 1915)**.** *If a Borel subset A of a complete, separable, metrizable space is uncountable, then it contains a topological copy of the Cantor set.*

During 1921-23, Alexandroff and Urysohn made an extensive study of compact topological spaces and published their famous 'Memoir on Compact Topological Spaces' [AU29], which provides many beautiful theorems and fascinating examples. Amongst them is one of the first theorems that gives a restriction of the cardinality of a topological space which is implied by its properties, namely:

7.1.2 Theorem (Alexandroff, Urysohn, 1921-1923)**.** *The cardinality of any perfectly normal compact space does not exceed 2^ω.*

At that time (1922), Alexandroff posed his famous question:

7 Question (Alexandroff, 1922)**.** Does the cardinality of a Hausdorff, first countable, compact space not exceed 2^ω?

This gave rise to the theory of cardinal invariants. It took some time before the first systematic results in the field were obtained. The computation of cardinal invariants took place in all parts of general topology, because of its set-theoretic nature. Cardinal invariants appear both as a principal tool in the investigation of the structure of spaces, and as a means of classification and selection of new classes of topological spaces. Some of the most classical results are the following.

7.1.3 Theorem (de Groot, [dG65])**.** *If X is Hausdorff, then $|X| \leqslant 2^{hL(X)}$.*

7.1.4 Theorem (Hajnal and Juhasz, [HJ67])**.** *If X is Hausdorff, then $|X| \leqslant 2^{c(X)\chi(X)}$. If X is T_1, then $|X| \leqslant 2^{s(X)\psi(X)}$.*

Hajnal and Juhasz used mainly combinatorial ways of proving their results, in particular, the partition theorem of Erdös-Rado.

7.1.5 Theorem (Čech-Pospišil, [ČP38])**.** *If X is a Hausdorff space, then $|X| \leqslant d(X)^{\chi(X)}$.*

Alexandroff's problem has been answered only 50 years later by Arhangelskii [Arh69], who proved:

7.1.6 Theorem. *If X is a Hausdorff, compact first countable space, then $|X| \leqslant 2^{\omega}$.*

Since he worked in compact spaces, he developed a special method using the so-called 'free sequences'. He was also able to extend this result to the more general one [Arh69]:

7.1.7 Theorem. *If X is Hausdorff, then $|X| \leqslant 2^{L(X)\chi(X)}$.*

Shortly after that, independently of Arhangelskii's work, Gryzlov [Gry80], Pol [Pol74] and Shapirovski [Sha72b], [Sha72a] developed the so-called 'closure method', which they used to give another proof of the theorems of Hajnal and Juhasz and Arhangelskii. They also used this method to extend Arhangelskii's result to the following:

7.1.8 Theorem. *If X is Hausdorff, then $|X| \leqslant 2^{L(X)\psi(X)t(X)}$.*

At that time, there arose an interest in whether Arhangelskii's result could be further extended by omitting one or another of the conditions. We formulate the countable case, which is still of interest:

Open Question (Arhangelskii)**.** Does the cardinality of a T_1 first countable Lindelöf space exceed 2^{ω}?

Here, the condition of Hausdorffness is relaxed to that of the T_1 axiom, which is much weaker. Some partial answers of this question will be provided here.

Open Question (Arhangelskii)**.** Does the cardinality of Hausdorff Lindelöf spaces with countable pseudocharacter exceed 2^{ω}?

Here, the condition of first countability (i.e. the requirement that the space has a countable character) is relaxed to that of having a countable pseudocharacter.

A very strong but partial answer to both questions was given by Gryzlov in 1981 [Gry80], who proved that:

7.1.9 Theorem (Gryzlov,[Gry80])**.** *If X is a T_1 compact space, then $|X| \leqslant 2^{\psi(X)}$.*

In the early 1990's Gorelic [Gor93] was able to construct, using the forcing method, a T_1 Lindelöf space with countable pseudocharacter and with cardinality exceeding 2^ω. The first question, though, still remains open, and very few partial results in that direction have been obtained.

Analyzing the above results leads us to the following observation. All of them could be expressed in the following general setting: we are given a separation property, a covering property, and a local cardinal invariant, and this leads to a restriction of the cardinality of the space.

We can also observe that various authors used four distinct approaches in proving cardinality restrictions: combinatorial (Hajnal and Juhasz [HJ67, HJ80], Hodel [Hod91]), the closure method (Arhangelskii [Arh69], Gryzlov [Gry77], Shapirovski [Sha72b, Sha72a], Pol [Pol74], Hodel [Hod76]), elementary submodels (Dow [Dow88], Spadaro [Spa11]), and games (Tall and Scheepers [ST10], Cammaroto, Santoro, Bella and Spadaro [BS15], Aurichi). It is also worth mentioning that for various theorems, some of these methods are more appropriate than the others, but it also interesting that all those methods are interchangeable, in the sense that (with various levels of difficulty) any of them can be used for proving the results proved by using another method. In our humble opinion, the closure method is the most natural one from the topological point of view, and uses only purely topological means.

The results of Archangelskii gave rise to considering cardinal invariants similar to the Lindelöf number. One such was introduced in 1984 by Dissanayake and Willard [WD84]

7.1.10 Definition. For a topological space X and $Y \subset X$, let:

$$\begin{aligned} aL(Y,X) =& \omega \min\{\tau : \text{ for each open in } X \text{ cover } \gamma \text{ of } Y\ \exists \gamma' \in [\gamma]^{\leqslant\tau} \\ & \text{such that } Y \subseteq \cup\{\overline{U} : U \in \gamma'\}\}, \\ aL_c(X) =& \omega \min\{aL(Y,X) : Y \text{ closed } \subset X\}, \\ aL(X) =& aL(X,X) \quad \text{(almost Lindelöf number)}. \end{aligned}$$

We know that every closed subspace of a Lindelöf space is also Lindelöf, but that is not the case with almost Lindelöf spaces. This necessitates the introduction of $aL_c(X)$, which assures inheritance of the almost Lindelöf property by closed subspaces. We obviously have $aL_c(X) \leqslant L(X)$. Dissanayake and Willard proved that:

7.1.11 Theorem ([WD84])**.** *If X is Hausdorff then* $|X| \leqslant 2^{aL_c(X)\chi(X)}$.

One of the few improvements of de Groot's result (Theorem 7.1.3) is

7.1.12 Theorem (Stavrova, [Sta92])**.** *If X is Hausdorff then* $|X| \leqslant \psi_c(X)^{haL(X)}$.

In this result, $\psi_c(X)$ is the closed pseudocharacter of X, defined as follows:

7.1.13 Definition (closed pseudocharacter)**.** The *closed pseudocharacter of a Hausdorff space X at the point $x \in X$*, denoted by $\psi_c(x, X)$ is the following cardinal:

$$\psi_c(x, X) = min\{\tau : \text{there is a local open neighbourhood system } \mathcal{B}(x) \text{ at } x \text{ with cardinality } \tau \text{ such that } x = \cap\{\overline{U} : U \in \mathcal{B}(x)\}\ \}.$$

Then the *closed pseudocharacter of X*, $\psi_c(X)$ will be:

$$\psi_c(X) = sup\{\psi_c(x, X) : x \in X\}$$

The presence of $\psi_c(X)$ is essential, as the following example shows:

7.1.14 Example (Hajnal, Szentmiklossy 1981)**.** For every $\tau \geqslant \omega$, there is a Hausdorff space with cardinality τ and $haL(X) \leqslant \omega$.

7.2 Combinatorial Separation Axioms and Restrictions of Cardinality

A possible way of approaching Arhangelskii's problem about the cardinality of T_1 first countable Lindelöf spaces is to investigate cardinality restriction in T_1 spaces with separation-type axioms, which are analogous to the classical ones (in terms of intersections of open sets). It is worth considering somehow 'step-by-step approximation' of Hausdorffness in T_1 spaces.

The "worst" such spaces should be 'nowhere Hausdorff', i.e. no two distinct points can be separated by disjoint neighbourhoods. Such a notion has already been introduced in the literature:

7.2.1 Definition. A topological space is called hyperconnected if every two non-empty open subsets in it intersect.

That is why it seems to be easier to first look at the following question:

8 Question. Does the cardinality of a T_1 first countable hyperconnected Lindelöf space not exceed 2^ω?

Later, we shall give some examples in that direction.

The second "level" of "approximating" Hausdorffness is to consider not Hausdorff spaces in which any three different points x, y, z have open neighbourhoods U_x, U_y, U_z such that $U_x \cap U_y \cap U_z = \emptyset$, in other words, 3-Hausdorff spaces. We have a positive result:

7.2.2 Theorem ([Bon13])**.** *If X is T_1 3-Hausdorff then $|X| \leqslant 2^{L(X)\psi(X)t(X)}$.*

Bonanzinga also introduced the notion of the weak Hausdorff number:

7.2.3 Definition (weak Hausdorff number, [Bon13])**.** The weak Hausdorff number of X is:

$$H^*(X) = \min\{\tau : \forall A \subset X \text{ with } |A| \geqslant \tau \exists B \subset A,\ |B| < \tau \text{ and open} \\ U_b \ni b, \forall b \in B \text{ such that } \bigcap_{b\in B} U_b = \emptyset\}.$$

It is easily observed that $H^*(X) \geqslant H(X)$.

For all values of $H^*(X)$ up to the countable one, Bonanzinga also proved:

7.2.4 Theorem ([Bon13])**.** *Let X be a T_1 space, $H^*(X) \leqslant \omega$. Then $|X| \leqslant 2^{L(X)\chi(X)}$.*

But, if we replace $H^*(X) = \omega$ with $H(X) = \omega$, we still have an open problem:

9 Question ([Bon13])**.** Is it true that if X is T_1, ω-Hausdorff, Lindelöf, first countable, then $|X| \leqslant 2^\omega$?

Also, for all values of $H^*(X)$ up to the countable, we have another open problem:

10 Question ([Bon13])**.** Is it true that if X is a T_1 space with $H^*(X) \leqslant \omega$ then $|X| \leqslant 2^{L(X)t(X)\psi(X)}$?

She also obtained the following generalisation of Pospišil's Theorem:

7.2.5 Proposition ([Bon13])**.** *If X is T_1 n-Hausdorff, then $|X| \leqslant d(X)^{\chi(X)}$.*

One of the stepping stones in the proofs of cardinal inequalities involving the Hausdorff separation axiom is that in Hausdorff spaces, $\{x\} = \cap\{\overline{U} : U \text{ open } \ni x\}$, for all $x \in X$. That is not true in T_1 n-Hausdorff spaces.

This motivates us to consider the following notion:

7.2.6 Definition (Hausdorff width, [BSS17])**.** Let X be a T_1 topological space and for all x in X, let

$$Hw(x) = \cap\{\overline{U} : U \in \mathcal{U}_x,\ |\mathcal{U}_x| \leqslant \psi(X),\ \mathcal{U}_x \text{ is an open neighborhood system for } x\}.$$

The *Hausdorff width of the space* X, denoted $HW(X)$, is:

$$HW(X) = \sup\{|Hw(x)| : x \in X\}.$$

Let us point out that in T_1 n-Hausdorff spaces, $HW(X)$ can be strictly smaller than $|X|$ [Sta13].

We also note that $HW(X)$ is a hereditary invariant i.e. $hHW(X) = HW(X)$ because we have:

7.2.7 Lemma. *If $A \subseteq X$ then $HW(A) \leqslant HW(X)$.*

This allows us to obtain the following generalisation of the above theorem of Dissanayake and Willard:

7.2.8 Theorem ([BSS17])**.** *If X is a T_1 n-Hausdorff space, then $|X| \leqslant HW(X)2^{aL_c(X)\chi(X)}$.*

Proof. Let $HW(X) \leqslant \kappa$, $aL_c(X)\chi(X) \leqslant \tau$. For all x in X, let $\mathcal{U}_x$ be a local base and $|\mathcal{U}_x| \leqslant \tau$. Note that for all $x \in X$, $Hw(x) = \cap\{\overline{U} : U \in \mathcal{U}_x\}$. Construct $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1. $H_\alpha \subset H_{\alpha'} \subset X$, $\forall \alpha \in \alpha' \in \tau^+$
2. H_α is closed $\forall \alpha \in \tau^+$
3. $|H_\alpha| \leqslant 2^\tau$ $\forall \alpha \in \tau^+$
4. If $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \cup\{U_x : x \in \cup\{H_\beta : \beta \in \alpha\}\}$
5. If $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leqslant\tau}$ is such that $X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$ then $H_\alpha \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $\mathcal{W}$ as in (5), choose a point $x(\mathcal{W}) \in X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\})$ and let C_α be the set of these points. Let $H_\alpha = \overline{\cup\{H_\beta : \beta \in \alpha\} \cup C_\alpha}$. In order to conclude that $|H_\alpha| \leqslant 2^\tau$ we use the Bonanzinga's generalisation of Pospišil's Theorem 7.2.5.

Let $H = \cup\{H_\beta : \beta \in \tau^+\}$. Since $t(X) \leqslant \chi(X) \leqslant \tau$, τ^+ is regular, and $\{H_\alpha : \alpha \in \tau^+\}$ is τ^+-inductive, we have that H is closed. Also, $|H| \leqslant 2^\tau$. Let $H^* = \cup\{Hw(x) : x \in H\} \supseteq H$. Then $|H^*| \leqslant \kappa 2^\tau$.

Suppose $q \in X \setminus H^* \subset X \setminus H$. Then for all $x \in H$ there is $U(x) \in \mathcal{U}_x$ such that $q \notin \overline{U(x)}$. From $aL_c(H) \leqslant \tau$ choose $H' \in [H]^{\leqslant\tau}$ such that $H \subseteq \cup\{\overline{U(x)} : x \in H'\}$. Then $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and hence $\mathcal{W} = \{\overline{U(x)} : x \in H'\} \in [\mathcal{B}_{\alpha+1}]^{\leqslant\tau}$ and $q \in X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in H_{\alpha+1} \cap (H \setminus \cup\{\overline{U(x)} : x \in H'\}) \subseteq H \cap (X \setminus H)$ - a contradiction. Hence $X = H^*$ and $|X| \leqslant \kappa 2^\tau$.

□

The Hausdorff width allows us to generalise de Groot's result as follows:

7.2.9 Theorem ([BSS17]). *Let X be a T_1 space. Then $|X| \leqslant HW(X)2^{haL(X)\psi(X)}$.*

Proof. Let $haL(X)\psi(X) \leqslant \tau$ and $HW(X) \leqslant \kappa$. Let $\forall x \in X$, $\mathcal{U}_x$ be a local open neighbourhood system such that $|\mathcal{U}_x| \leqslant \tau$ and $\{x\} = \cap\{U : U \in \mathcal{U}_x\}$. By transfinite induction, define two families $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1. $\{H_\alpha : \alpha \in \tau^+\}$ is τ^+-inductive
2. $|H_\alpha| \leqslant 2^\tau \ \forall \alpha \in \tau^+$
3. If $\{H_\beta : \beta \in \alpha\}$ are already defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \cup\{\mathcal{U}_x : x \in \cup\{H_\beta : \beta \in \alpha\}\}$
4. If $\alpha \in \tau^+$, $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leqslant\tau}$ and $H \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$, then $H_\alpha \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$.

Suppose $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined. If $\mathcal{W}$ is as in (3), choose $x(\mathcal{W}) \in X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\})$ and let C_α be the set of these points. Let $H_\alpha = \cup\{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Then $|H_\alpha| \leqslant 2^\tau$.

Let $H = \cup\{H_\alpha : \alpha \in \tau^+\}$ and $H^* = \cup\{Hw(x) : x \in H\}$. Then $|H^*| \leqslant \kappa 2^\tau$.

Suppose $q \in X \setminus H^*$. Then $q \notin Hw(x) \ \forall x \in H$. Hence for all $x \in H$ there exists $U(x) \in \mathcal{U}_x$ such that $q \notin \overline{U(x)}$. By $haL(X) \leqslant \tau$ we can choose $H' \in [H]^{\leqslant\tau}$ such that $H \subset \cup\{\overline{U(x)} : x \in H'\}$. Let $\mathcal{W} = \{U(x) : x \in H'\}$. We have that $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_{\alpha+1}]^{\leqslant\tau}$ and $X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in X \setminus (\cup\{\overline{U} : U \in \mathcal{W}\}) \subseteq X \setminus H$ and $x(\mathcal{W}) \in H$ - a contradiction.

□

7.2.10 Corollary ([Sta93])**.** *If X is Hausdorff then $|X| \leqslant \psi_c(X)^{haL(X)}$.*

So far, we have considered conditions which are weaker than Hausdorff and other separation axioms. However, we can also make use of one 'far-fetched' generalisation of the pseudocharacter, arising from a work of Stavrova [Sta01], and defined by Bonanzinga in [Bon13]:

7.2.11 Definition (n-Hausdorff pseudocharacter)**.**

$$
\begin{aligned}
n - H\psi(X) = \min\{\tau : {} & \forall x \in X \text{ there is a collection } \{V(\alpha, x) : \alpha \in \tau\} \text{ of open} \\
& \text{neighbourhoods of } x \text{ such that for any } n \text{ distinct } x_1, \ldots, x_n \in X \\
& \text{there are } \alpha_1, \ldots, \alpha_n \in \tau \text{ such that } \textstyle\bigcap_{i=1}^n V(\alpha_i, x_i) = \emptyset\}.
\end{aligned}
$$

Reflecting the famous Gryzlov's theorem [Gry80] that every T_1 compact space with countable pseudocharacter has cardinality at most 2^ω, and our Example 6.1.2, we can ask the following

Open Question. Let X be 3-Hausdorff compact space with countable 3-Hausdorff pseudocharacter. Is it true that $|X| \leqslant 2^\omega$?

Recall that for a Hausdorff compact space X, $\psi(X) = \chi(X)$. A similar result need not be true if one replaces "Hausdorff" with "3-Hausdorff" and "3-$H\psi(X)$" with "$\psi(X)$", as the following example shows.

7.2.12 Example. There is a 3-Hausdorff compact space with countable 3-Hausdorff pseudocharacter having uncountable character.

Construction. Let $Y = \omega \cup \{p\}$, where p is an ultrafilter on ω. Y is a n-Hausdorff not first countable space such that n-$H\psi(X) = \omega$. Let $X = Y \cup \{*\}$ the space topologized as follows: Y is open in X and a basic neighbourhood of $*$ takes the form $U_* = \{*\} \cup F$, where $F \subset Y$ is such that $|X \setminus F| < \aleph_0$.

Let $n \in \omega$ with $n \geqslant 3$. X is n-Hausdorff. Indeed Y is n-Hausdorff; for every $x_1, ..., x_{n-2} \in \omega$, where $x_1 < ... < x_{n-2}$, $U_i = \{x_i\}$, $i = 1, ..., n-2$, $U_p = Y \setminus \{0, ..., x_{n-2}+1\}$, $U_* = X \setminus \{0, ..., x_{n-2}+1\}$ are disjoint neighbourhoods of $x_1, ..., x_{n-2}$, p and $*$ respectively; further, if $x_1, ..., x_{n-1} \in \omega$, where $x_1 < ... < x_{n-1}$, $U_i = \{x_i\}$, $i = 1, ..., n-2$, $U_* = X \setminus \{0, ..., x_{n-2}+1\}$ are disjoint neighbourhoods of $x_1, ..., x_{n-1}$ and $*$ respectively.

n-$H\psi(X) = \omega$ for $n \geqslant 3$. Indeed, $\mathcal{U}_m = \{\{m\}\}$, $m \in \omega$, $\mathcal{U}_p = \{Y \setminus \{0, 1, ..., n\} : n \in \omega\}$, $\mathcal{U}_* = \{X \setminus \{n\}, X \setminus \{p\} : n \in \omega\}$ are countable families of open neighbourhoods of m, p and $*$. Then for every distinct points of $x, y, z \in X$ there exist $U_x \in \mathcal{U}_x$, $U_y \in \mathcal{U}_y$ and $U_z \in \mathcal{U}_z$ such that $U_x \cap U_y \cap U_z = \emptyset$.

It is a classical result that the weight of a compact Hausdorff space $\omega(X)$ is less than its cardinality. However, this is no longer true if we replace "Hausdorff" by the strongest combinatorial Hausdorff generalisation, namely "T_1 3-Hausdorff":

7.2.13 Example. There is a T_1 3-Hausdorff compact space X such that $\omega(X) > |X|$.

This is in fact Example 7.2.12, since it is a countable space with uncountable character.

7.2.14 Example. There is a T_1 compact and countable 3-regular space X which is not 2-regular. Also, X is a 4-normal space which is not 3-normal.

Construction. Let $X = 4 \cup \{\omega \times 4\}$, where $4 = \{0, 1, 2, 3\}$, be the space topologized as follows: all points from $\omega \times \{i\}$, where $i \in 4$ are isolated; a basic neighbourhood of $i \in 4$ takes the form $U(i, N) = \{i\} \cup \{(m, j) : j \neq i, m \geqslant N\}$, where $N \in \omega$. Note that X is countable T_1. In [Bon13], it is proved that X is 4-Hausdorff not 3-Hausdorff. By Theorem 6.2.9, X is 4-normal and hence 3-regular: indeed, for the closed set $F = \{3, 4\}$ and the points 0 and 1, we cannot find open sets containing them and with empty intersection. Also, since X it is not 2-regular, it is not 3-normal; thus, X is also a 4-normal space which is not 3-normal.

A similar idea can be used to construct

7.2.15 Example. There is a T_1 compact and countable k-regular space X which is not weakly $(k-1)$-regular. In addition X is also $(k+1)$-normal not k-normal.

The following question is related to the Jones' Lemma that every closed and discrete subspace of a normal separable space has cardinality at most 2^ω:

Open Question. Let X be a T_1 3-normal separable space. Is it true that every closed and discrete subspace of X has cardinality at most 2^ω?

The next example gives a partial negative answer to the above questions:

7.2.16 Example. There is a T_1 ω-normal space X with a closed discrete subspace of cardinality greater than 2^ω.

Construction. Consider the space $X = \omega \cup A$, where $|A| > 2^\omega$ topologized as follows: the points of ω are isolated and basic neighborhoods of $a \in A$ take the form $a \cup (\omega \setminus H)$, where H is a subset of ω such that $|H| < \omega$. X is T_1, $d(X) = \omega$. It was noted in [Bon13] that $H(X) = \psi(X, a) = \omega$, where a is any point in A and ψ is the pseudocharacter; then X is ω-Hausdorff. It is easy to see that X is ω-normal.

It is known that if X is a T_1 regular space then $\omega(X) \leqslant 2^{d(X)}$. Hence we can naturally ask

Open Question. If X is a T_1 2-regular space, is it true that $\omega(X) \leqslant 2^{d(X)}$?

Example 7.2.16 above gives a partial negative answer:

7.2.17 Example. There is an ω-normal (hence ω-regular) space X such that $\omega(X) > 2^{d(X)}$.

Construction. It is enough to point out that the space X in Example 7.2.16 is separable and $\omega(X) = |A| > 2^{\omega}$.

With respect to the inheritance of those separation properties, we can point out that:

7.2.18 Proposition. *Every closed subspace of an k-normal space is k-normal, for all $k \in \omega$.*

But 3-normality is not hereditary with respect to open sets as the following example shows.

7.2.19 Example. There is a T_1 3-normal space X with an open subspace which is not 3-normal.

Construction. Let L be the Niemitzky plane and let $X = L \cup \{*\}$ topologized as follows: N is open in X and a basic neighbourhood of $*$ takes the form: $U_* = \{*\} \cup F$, where $F \subset L$ and $|X \setminus F| < \aleph_0$. X is compact. Also X is 3-Hausdorff. Indeed L is 3-Hausdorff; also, considering $x, y \in L$ and the point $*$, there exists open sets $U_x \ni x$, $U_y \ni y$ and $U_* = \{*\} \cup \{z\}$, where $z \notin U_x \cup U_y$, such that $U_x \cap U_y \cap U_* = \emptyset$. Then, X is compact and 3-Hausdorff, hence X is 3-normal. However, L is not 2^{ω}-normal hence it is not 3-normal.

Recall Theorem 7.2.5 which generalises the Czech-Pospišil inequality. A natural question related to this result is:

11 Question. Let X be a T_1 space such that $H(X) = \omega$. Is it true that $|X| \leqslant d(X)^{\chi(X)}$?

The following example shows that the answer is negative:

7.2.20 Example. There is a T_1 ω-Hausdorff first countable, locally compact, separable, not hyperconnected, not Hausdorff space X with arbitrarily large cardinality.

Construction. Let B be any set of cardinality κ which is disjoint from $\mathbb{N}^+$. Let $X = \mathbb{N}^+ \cup B$. All points of $\mathbb{N}^+$ are isolated. If $b \in B$ then a neighbourhood of b takes the form $U_b = \{b\} \cup (\mathbb{N}^+ \setminus F)$, where F is a finite subset of $\mathbb{N}^+$.

$\mathbb{N}^+$ is dense in X; X is first countable, X is not Hausdorff (points from B cannot be separated), X is not hyperconnected (as it has isolated points); moreover, X is not n-Hausdorff for any $n \in \mathbb{N}$ (any neighbourhoods of any n-many points from B have a nonempty intersection). However, X is ω-Hausdorff, because if $\{y_n : n \in \mathbb{N}^+\} \subseteq B$, then for all $n \in \mathbb{N}^+$ we may take $U_{y_n} = \{y_n\} \cup (\mathbb{N}^+ \setminus \{1, \ldots, n\})$. Then $\bigcap_{n\in\mathbb{N}^+} U_{y_n} = \emptyset$.

We aim at hyperconnected spaces not just out of idle curiosity. As we pointed out, this notion is closely related to the attempt to find a proof or a ZFC counterexample of Arhangel'skii's initial question about the cardinality of T_1 first countable Lindelöf spaces. In trying to do so it is natural to look at the spaces as much non-Hausdorff as possible. We have the following result:

7.2.21 Theorem ([BSS17])**.** *If the cardinality of every T_1, first countable, Lindelöf, separable, hyperconnected space is at most $2^{\aleph_0}$, then the cardinality of every T_1, first countable, Lindelöf space is at most $2^{\aleph_0}$.*

Proof. Let Y be T_1 first countable, Lindelöf and $Y = k$ where k is any infinite cardinal. Let $X = Y \cup D$, where D is countable and $Y \cap D = \emptyset$. Let the topology τ on X have the following local basis: all $d \in D$, have neighborhoods $V \subseteq D$ such that $d \in V$ and $D \setminus V$ is finite; $\emptyset$ is open and $U \cup (D \setminus F)$ is open where U is open in the initial topology in Y and F is a finite subset of D. Then $|X| = |Y|$, X is T_1, first-countable, Lindelöf , hyperconnected and D is dense in X. $\square$

Examples of hyperconnected spaces are hard to find, moreover such which are separable, T_1, Lindelöf and first countable. So, this gives us some hope of a positive solution of Arhangel'skii's initial question. Let us note that if in the above example we suppose in addition that Y is ω-Hausdorff then we get:

7.2.22 Theorem ([BSS17]). *If the cardinality of every T_1, first countable, Lindelöf, separable, ω-Hausdorff, hyperconnected space is at most $2^{\aleph_0}$, then the cardinality of every T_1, first countable, ω-Hausdorff, Lindelöf space is at most $2^{\aleph_0}$.*

Examples of T_1, first countable, separable, Lindelöf, ω-Hausdorff, hyperconnected spaces are even harder to find.

Here we give two such examples - both with cardinality at most $2^{\aleph_0}$.

7.2.23 Example. There is a T_1 hyperconnected, ω-Hausdorff, first countable, Lindelöf, separable space with topology that is not the co-finite topology.

Construction. Let $X = \mathbb{Z} \times \mathbb{Z}$ and let $\mathcal{B} = \{\mathcal{V}_{n,m}, \mathcal{U}_{n,m} : n, m \in \mathbb{Z}\}$ be the subbase for a topology of X where

$$\begin{aligned}\mathcal{V}_{n,m} :=& \{(x, y) \in \mathbb{Z}^2 : x < n \text{ or } y < m\}\\ \mathcal{U}_{n,m} :=& \{(x, y) \in \mathbb{Z}^2 : x > n \text{ or } y > m\}.\end{aligned}$$

The following one is an uncountable such example:

7.2.24 Example. There is a T_1, Lindelöf, first countable, ω-Hausdorff, separable, hyperconnected space with cardinality continuum.

Construction. Let $X = \mathbb{R} \cup (\mathbb{N}^+ \times \{1\})$. Points on $\mathbb{N}^+ \times \{1\}$ have the cofinite topology in $\mathbb{N}^+ \times \{1\}$. Points $x \in \mathbb{R}$, have neighbourhoods $U \cup (\mathbb{N}^+ \times \{1\} \setminus F)$, where U is open in $\mathbb{R}$, $x \in U$ and F is a finite subset of $\mathbb{N}^+ \times \{1\}$. The empty set is open by definition.

Then $|X| = 2^\omega$, X is hyperconnected, Lindelöf, first countable, ω-Hausdorff and separable.

In trying to solve the second of the two main problems of Arhangelskii, A. Charlesworth proved the following:

7.2.25 Theorem ([Cha77]). *If X is T_1 then $|X| \leqslant psw(X)^{L(X)\psi(X)}$.*

In the above $psw(X)$ is the *point separating weight of* X which is defined as

$$psw(X) := \omega + \min\{\tau : X \text{ has a separating open cover } \gamma \text{ such that}$$
$$\text{for all } x \in X,\ x \text{ is in at most } \tau \text{ members of } \gamma\}.$$

Charlesworth's result inspires the following questions:

12 Question. If X is T_1, hyperconnected, separable, Lindelöf and first countable, how big is $psw(X)$?

13 Question. If X is T_1, ω-Hausdorff, hyperconnected, separable, Lindelöf and first countable, how big is $psw(X)$?

We suspect that examples for which $psw(X) > 2^\omega$ exist.

As we have pointed out, one of the stepping stones of the proofs of cardinality restrictions involving the Hausdorff axiom is equivalent to, $\{x\} = \bigcap\{\overline{U} : U \text{ open}, x \in U\}$ for all $x \in X$. One might expect that in n-Hausdorff spaces $|\bigcap\{\overline{U} : U \text{ open}, x \in U\}| = n$. This is not true as the following example shows:

7.2.26 Example. There is a T_1 compact 3-Hausdorff not Hausdorff, not separable space X for which there is a point x such that $\bigcap\{\overline{U} : U \text{ open}, x \in U\}$ has big cardinality.

Construction. Let $A \neq \emptyset$ be any set such that $|A| > \omega$ and $X = (\mathbb{N}^+ \times A) \cup \{0\}$, with the basis

$$\mathcal{B} := \{\{(n,a)\} : a \in A, n \in \mathbb{N}^+ \setminus \{1\}\} \cup \{U : 0 \in U, U \text{ is a cofinite subset of } X\}$$
$$\cup \{V : \text{ for some } a \in A, (1,a) \in V, V \text{ is a cofinite subset of } \mathbb{N}^+ \times \{a\}\}$$

Let us point out that neighbourhoods of 0 are cofinite subsets of X; neighbourhoods of $(1,a)$ are cofinite subsets of $\mathbb{N}^+ \times \{a\}$, and $(n,a), a \in A$, $n \geqslant 2$ are isolated.

Then X is T_1, compact, not first countable, 3-Hausdorff, not Hausdorff,

$|X| = |A|$, not separable, and

$$\bigcap\{\overline{U} : U \ni 0, U \text{ open}\} = \{0\} \cup (\{1\} \times A)$$
$$\bigcap\{\overline{U} : U \ni (1,a)\} = \{0\} \cup \{(1,a)\}$$
$$\bigcap\{\overline{U} : U \ni (n,a) : n \geqslant 2\} = \{(n,a)\}.$$

The above example can be easily generalised for the n-Hausdorff case.

Those two examples give us another justification for our defining the Hausdorff width earlier. This notion also gives rise to the following generalisation of pseudocharacter

7.2.27 Definition ([Juh80])**.** For every $x \in X$, let us also define

$$\psi w(x) = min\{|\mathcal{U}_x| : \bigcap\{\overline{U} : U \in \mathcal{U}_x\} = Hw(x), \mathcal{U}_x \text{ is a family of open neighbourhoods of } x\}$$

and

$$\psi w(X) = sup\{\psi w(x) : x \in X\}$$

7.2.28 Note. Let us point out that if $\mathcal{U}_x$ is a local base at x then $Hw(x) = \bigcap\{\overline{U} : U \in \mathcal{U}_x\}$. Hence $\psi w(X) \leqslant \chi(X)$ in T_1 spaces and in addition $\psi w(X) = \psi_c(X)$ in Hausdorff spaces.

We now use both $HW(X)$ and $\psi w(X)$ to also generalise de Groot's result:

7.2.29 Theorem ([BSS17])**.** *Let X be a T_1 space. Then $|X| \leqslant HW(X)\psi w(X)^{haL(X)}$.*

Proof. Let $haL(X) \leqslant \tau$, $\psi w(X) \leqslant \lambda$ and $HW(X) \leqslant \kappa$. Let $\forall x \in X$, $\mathcal{U}_x$ be a family of open neighbourhoods of x such that $|\mathcal{U}_x| \leqslant \lambda$ and $Hw(x) = \bigcap\{\overline{U} : U \in \mathcal{U}_x\}$. By transfinite induction, define two families $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1. $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing sequence of subsets of X
2. $|H_\alpha| \leqslant \kappa\lambda^\tau \ \forall \alpha \in \tau^+$
3. If $\{H_\beta : \beta \in \alpha\}$ are already defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \bigcup\{\mathcal{U}_x : x \in \bigcup\{Hw(y) : y \in \bigcup\{H_\beta : \beta \in \alpha\}\}\}$

4. If $\alpha \in \tau^+$, $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leqslant\tau}$ and $X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$, then $H_\alpha \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$.

Suppose $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined. If $\mathcal{W}$ is as in (4), choose $x(\mathcal{W}) \in X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\})$ and let C_α be the set of these points. Let $H_\alpha = \bigcup\{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Then $|H_\alpha| \leqslant \kappa\lambda^\tau$.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$ and $H^* = \bigcup\{Hw(x) : x \in H\}$. Then $|H^*| \leqslant \kappa\lambda^\tau$.

Suppose $q \in X \setminus H^*$. Then $q \notin Hw(x)$, $\forall x \in H$. Hence for all $x \in H$ there exists $U(x) \in \mathcal{U}_x$ such that $q \notin \overline{U(x)}$. By $haL(X) \leqslant \tau$ we can choose $H' \in [H]^{\leqslant\tau}$ such that $H \subset \bigcup\{\overline{U(x)} : x \in H'\}$. Let $\mathcal{W} = \{U(x) : x \in H'\}$. We have that $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_{\alpha+1}]^{\leqslant\tau}$ and $X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\}) \subseteq X \setminus H$ and $x(\mathcal{W}) \in H$ - a contradiction. $\square$

7.2.30 Theorem (Arhangelskii, [Arh79])**.** *If X is regular, then $|X| \leqslant 2^{wL_c(X)\chi(X)}$.*

Here:

7.2.31 Definition (quasi-Lindelöf number, [Juh80])**.**

$$wL_c(X) := \omega \min\{\tau : \forall Y \text{ closed } \subset X \text{ and for all open in } X \text{ cover } \gamma \text{ of } Y\ \exists \gamma' \in [\gamma]^{\leqslant\tau} \text{ such that } Y \subseteq \overline{\bigcup \gamma'}\}.$$

14 Question (Arhangelskii)**.** If X is a Hausdorff space, do we have that $|X| \leqslant 2^{wL_c(X)\chi(X)}$?

Here, we provide an improvement of a theorem of [Ala93], which also gives a partial answer to the above question:

7.2.32 Theorem. *If X is T_1 space such that $H^*(X) \leqslant \omega$ and has a dense set of isolated points then*

$$|X| \leqslant 2^{wL_c(X)\chi(X)}$$

Proof. Let $wL_c(X)\chi(X) \leqslant \tau$ and let $\forall x \in X$, $\mathcal{U}_x$ be a local base at x with $|\mathcal{U}_x| \leqslant \tau$. Construct $\{A_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$, where $\forall \alpha \in \tau^+$, $\mathcal{B}_\alpha$ is a family of open sets such that $|\mathcal{B}_\alpha| \leqslant 2^\tau$, so that the following conditions are satisfied:

1. $A_\alpha \subset A_{\alpha'}$ if $\alpha \in \alpha' \in \tau^+$,

2. A_α are closed and $|A_\alpha| \leqslant 2^\tau$ for all $\alpha \in \tau^+$,

3. If for some $\beta \in \tau^+$, $\{A_\alpha : \alpha \in \beta\}$ are constructed, then $\mathcal{B}_\beta = \bigcup\{U_x : x \in \bigcup\{A_\alpha : \alpha \in \beta\}\}$,

4. If $\beta \in \tau^+$ and $\gamma \in [\mathcal{B}_\beta]^{\leqslant\tau}$ is such that $X \setminus \overline{\bigcup\gamma} \neq \emptyset$, then $A_\beta \setminus \overline{\bigcup\gamma} \neq \emptyset$.

Let $\{A_0\} = \{x_0\}$ and let for $\alpha \in \beta \in \tau^+$, $\{A_\alpha : \alpha \in \beta\}$ be defined. Let $\mathcal{E}_\beta = \{\gamma : \gamma \in [\mathcal{B}_\beta]^{\leqslant\tau}, X \setminus \overline{\bigcup\gamma} \neq \emptyset\}$. Choose $\phi(\gamma) \in X \setminus \overline{\bigcup\gamma}$ for every $\gamma \in \mathcal{E}_\beta$ and let $E_\beta = \{\phi(\gamma) : \gamma \in \mathcal{E}_\beta\}$. Then $A_\beta = \overline{E_\beta \cup (\bigcup\{A_\alpha : \alpha \in \beta\})}$. Because [Bon13, Proposition 28] we have constructed $\{A_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ as required.

Let $A = \bigcup\{A_\alpha : \alpha \in \tau^+\}$. Since $t(X) \leqslant \chi(X) \leqslant \tau$ we have that A is closed and $|A| \leqslant 2^\tau$. Suppose $X \setminus A \neq \emptyset$. Then there exists $a^* \in X \setminus A$, a^* isolated. By T_1 for every $a \in A$, there exists $W_a \ni a$, $W_a \in \mathcal{U}_x$, $a^* \notin W_a$. Then $A \subset \bigcup\{W_a : a \in A\}$. Since $wL_c(X) \leqslant \tau$ we can find $A' \in [A]^{\leqslant\tau}$ such that $A \subseteq \overline{\bigcup\{W_a : a \in A'\}}$ and since a^* is isolated, $a^* \notin \overline{\bigcup\{W_a : a \in A'\}}$. Since τ^+ is regular, there exists $\alpha_0 \in \tau^+$ such that $A' \subseteq A_{\alpha_0}$. We have that $\gamma = \{W_a : a \in A'\} \in [\mathcal{B}_{\alpha_0}]^{\leqslant\tau}$ and $X \setminus \overline{\bigcup\gamma} \neq \emptyset$. Then we have already chosen $\phi(\gamma) \in A_{\alpha_0+1} \setminus \overline{\bigcup\gamma} \subset A$, i.e. $\phi(\gamma) \in A \subset \overline{\bigcup\gamma}$ and $\phi(\gamma) \notin \overline{\bigcup\gamma}$ - a contradiction. $\square$

Part IV

Indexes and References

Chapter 8

List of Notation

To observe general practice in the two fields of dynamics and topology, we sometimes use the same notation for different notions, assuming that the meaning should be clear from the context. Below are some pieces of notation which are used throughout larger portions of this thesis, but not universally.

- $\mathcal{A}$, $\mathcal{B}$ - alphabets in topological dynamics or families (often of open sets) in analytic topology
- $\mathcal{A}^{<\mathbb{N}}$ - set of finite words over the alphabet $\mathcal{A}$
- $\mathcal{A}^{\mathbb{N}}$, $\mathcal{A}^{\mathbb{Z}}$ - set of finite (or bi-infinite) words over the alphabet $\mathcal{A}$
- $w_n = w[n]$ - the nth letter of the word w
- $w[n;k] = w_n \dots w_k$, the subword of w of length $k-n$ starting from the letter $w[n]$
- $|w|$ - the length of a finite word
- σ - a substitution
- s - the shift map (on the appropriate symbolic space)
- r - the length of the substitution
- ϕ, ψ - substitutions (on the appropriate alphabets)

- Φ, Ψ, π - homomorphisms between dynamical systems
- τ - the topology on a space, which may be indexed to indicate one of several topologies on the same space; alternatively, it is a cardinal in the analytic topology section
- U, V - open sets

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