

Dynamic Cooperative Investment

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by

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"I can calculate the movement of the stars, but not the madness of men"

Sir Isaac Newton after losing a fortune in the South Sea Bubble

Abstract

In this thesis we develop dynamic cooperative investment schemes in discrete and continuous time. Instead of investing individually, several agents may invest joint capital into a commonly agreed trading strategy, and then split the uncertain outcome of the investment according to the pre-agreed scheme, based on their individual risk-reward preferences. As a result of cooperation, each investor is able to get a share, which cannot be replicated with the available market instruments, and because of this, cooperative investment is usually strictly profitable for all participants, when compared with an optimal individual strategy. We describe cooperative investment strategies which are Pareto optimal, and then propose a method to choose the most ‘fair’ Pareto optimal strategy based on equilibrium theory. In some cases, uniqueness and stability for the equilibrium are justified.

We study a cooperative investment problem, for investors with different risk preferences, coming from expected utility theory, mean-variance theory, mean-deviation theory, prospect theory, etc. The developed strategies are time-consistent; that is the group of investors have no reasons to change their mind in the middle of the investment process. This is ensured by either using a dynamic programming approach, by applying the utility model based on the compound independence axiom.

For numerical experiments, we use a scenario generation algorithm and stochastic programming model for generating appropriate scenario tree components of the *S&P* 100 index. The algorithm uses historical data simulation as well as a GARCH model.

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Abbreviations

CI = Cooperative investment

DCI = Dynamic cooperative investment

IV = Individual investment

CDF = Cumulative distribution function

VaR = Value at risk

CVaR = Conditional value at risk

EUT = Expected utility theory

MP = Multi-period

SP = Stochastic programming

DP = Dynamic programming

DCI = Dynamic cooperative investment

GARCH = General autoregressive conditional heteroskedastic

ARCH = Autoregressive conditional heteroskedastic

COGARCH = Continuous GARCH

EWMA = Exponential weighting moving average

ACF = Autocorrelation function

PACF = Partial autocorrelation function

r.v. = Random variable

WN = Weight noise

PT = Prospect theory

C = Certainty equivalent

To my parents

Chapter 1

Introduction

1.1 Criteria for comparing portfolios

Portfolio optimisation is the process of selecting a trading strategy for a financial market, such that the resulting outcome, which is the rate of return of the final portfolio, often modeled as a random variable (r.v.), is “optimal” for the investor, with respect to some criterion. The choice of a reasonable criterion is of central importance. The theory of choice under uncertainty suggest a number of possible criteria based on preference modeling axioms. In the thesis, we mainly focus on the investors whose preferences are described by modern portfolio theory, expected utility theory or prospect theory. *We show that in all these theories, the investors having different preferences can use cooperation to achieve strictly better outcomes, compared to optimal individual investment.*

Modern portfolio theory, originated by Markowitz [106], states that an investor aims to minimise portfolio risk, subject to achieving a pre-specified level of expected return. In Markowitz [106], risk is measured by the standard deviation of the portfolio’s rate of return. Later, a number of alternative ways for measuring risk have been suggested. For example, Artzner et al. [13], presented and justified a set of four properties for measure of risk, which is named coherent risk measure. A similar approach was suggested by Rockafellar et al. [122] who suggested replacing standard deviation by a general deviation measure, that is, functionally satisfying four axioms: non-negativity, sub-additivity, positive homogeneity, and lower semi continuity. For background literature of risk measures and many of their applications we refer to [54, 51, 59].

Another popular theory for comparing uncertain outcomes comes from von Neumann and Morgenstern (1953) who proved that any investor whose risk preferences satisfy four basic assumptions, or axioms, should maximise the expected utility of his/her uncertain outcome for some utility functions, which are normally assumed to be convex and increasing. This theory is called expected utility theory (EUT), and is still very popular nowadays, along with Markowitz's mean-variance theory. The relation between these two theories was discussed where mean-variance approach does not conform with the expected utility theory unless uncertain outcome is normal distributed or a utility function is quadratic, for more discussion about the relation see, [94] and [99].

However, Allais [3] demonstrated that the main axiom of EUT is often violated, which motivated the development of alternative theories for comparing uncertain outcome. Maybe the most successful one is prospect theory, proposed by Kahneman and Tversky [83], or, more precisely, its cumulative version, see Tversky and Kahneman [138]. Prospect theory describes the way of how the person (investors) measures his/her gain or loss relative to the natural reference level. While in the case of the normal distribution there is no significant difference between the mean variance analysis and prospect theory, in general they differ significantly. Prospect theory is able to describe both risk-averse and risk-seeking investors, while mean-variance analysis assumes that investors are always risk-averse. Prospect theory suggests that the investor or agents make their decision based on the change of the wealth as gain or loss relative to the natural reference point, using some value function. Moreover, the possible choices for value functions are discussed, for example in Tversky and Kahneman [138] and Prelec [113]. Koszegi and Rabin [90, 91, 92] make an important attempt to explain how people think about either gain and loss. In addition, Tversky and Kahneman [138] provide brief guidance on how to choose an appropriate reference point. But, many researchers are working in this direction to determine the natural reference level, as well as clarifying other relevant aspects. For example, Schmidt [129] suggested that the reference point may be stochastic, while Benartzi and Thaler [22] said that the reference point is not static and actually it will adapt with the market and may change from the initial price with a relevant change in the security price. Gneezy [64] chooses reference point as a historical peak point. Oden [111] focuses his study on real estate and claims that the reference point is the buying price. Moreover, initial price was considered as a reference point by Shefrin and Statman [134]. Barberis et al.[14] claim that the reference point is scaled up by the risk-free rate. We will use the

latter idea in this thesis.

1.2 The idea of cooperation

Once the method for comparing the outcomes from trading strategies is chosen by the investor, the optimal strategy can be found from a straightforward optimisation problem. However, it is known (see e.g. Grechuk and Zabaranin [68]) that, for any group of investors with different risk preferences, it is better to join their capital, develop a common cooperative investment strategy, and then divide the outcome of this strategy optimally among participants. To illustrate this we use the following simple example from Grechuk and Zabaranin [68] and modify it slightly, see Almualim [5].

Example 1.1 (cash-or-nothing binary options). *A cash-or-nothing binary option O pays some fixed amount of cash $C(O)$ if the option expires in-the-money but nothing otherwise. Suppose a market offers two cash-or-nothing binary options A and B for the same price p with $C(A) = 2p$ and $C(B) = 3p$. Then investing capital x into option A and capital y into option B , one earns: $-x - y$ if none of the options pays, $x - y$ if only option A pays, $-x + 2y$ if only option B pays, or $x + 2y$ if both options pay. For simplicity, we assume that these four outcomes are equally probable. Thus, investment profits are random variables on a four-element probability space and will be denoted as four-component vectors $X = (x_1, x_2, x_3, x_4)$; where each element x_i represents the amount of money which we get in the corresponding scenario. Suppose there are two agents with initial capital of \$1 each. The first one is risk neutral with utility function $U_1(X)$ being the expected profit: $U_1(X) = \frac{1}{4} \sum_{i=1}^4 x_i$, whereas the second one is risk averse with utility function $U_2(X)$ being the second worst outcome of X , i.e. if the components of X are ordered as $x_1 \leq x_2 \leq x_3 \leq x_4$, then $U_2(X) = x_2$. Both agents maximise their corresponding utility functions.*

Each agent can invest capital x into option A and capital $y = 1 - x$ into option B , which would result in profit $X = (-x - y, x - y, -x + 2y, x + 2y)$. For the first agent, the optimal investment is, obviously, $x = 0$ and $y = 1$ with the profit $X_1^ = (-1, -1, 2, 2)$ for which $U_1(X_1^*) = E[X_1^*] = 0.5$. For the second agent, $U_2(X) = \min\{x - y, -x + 2y\}$ attains a maximum when $x - y = -x + 2y$, so the optimal investment is $x = 0.6$ and $y = 0.4$ with the profit $X_2^* = (-1, 0.2, 0.2, 1.4)$ for which $U_1(X_2^*) = 0.2$. Now suppose that the agents agree to invest two units of*

their combined capital into a joint portfolio with $x = 0.7$ and $y = 2 - x = 1.3$, and agree to divide the resulting joint profit $X^* = (-x - y, x - y, -x + 2y, x + 2y) = (-2, -0.6, 1.9, 3.3)$ into two shares: $Y_1 = (-1, -1, 1.5, 2.9)$ for the first agent and $Y_2 = X^* - Y_1 = (-1, 0.4, 0.4, 0.4)$ for the second agent. Then $U_1(Y_1) = E[Y_1] = 0.6 > 0.5 = U_1(X_1^*)$ and $U_2(Y_2) = 0.4 > 0.2 = U_2(X_2^*)$, so each agent has the value of his/her utility function being strictly greater than that from the optimal individual investment.

The study of cooperative investment divides naturally into the following parts;

- (i) Finding an investment strategy, optimal for the group of investors. In the example above, was the joint portfolio with $x = 0.7$ and $y = 1.3$ optimal, or does a better one exist?
- (ii) For the resulting investment outcome, find the way to divide it optimally among investors. In the example above, was the suggested way to divide $X^* = (-2, -0.6, 1.9, 3.3)$ optimal, or does a better one exist?

In (i)-(ii), the notion of ‘optimal’ needs clarification. It is easy to divide X^* into shares Y'_1 and Y'_2 such that $U_1(Y'_1) > U_1(Y_1)$ but $U_2(Y'_2) < U_2(Y_2)$, that is, the first investor is better at the cost of the second one (see below). The investment strategy, together with the method of outcome division, is called *Pareto optimal* if there are no other investment strategies and outcome divisions, which is strictly better for at least one investor, and at the same time not worse for all other investors.

- (iii) After solving (i)-(ii), we end up with a set of Pareto optimal strategies, some of which are better for one investor, and some of which are better for others. So, which one would it be fair to choose? In the example above we could divide the same $X^* = (-2, -0.6, 1.9, 3.3)$ as $X^* = Y'_1 + Y'_2$, where $Y'_1 = (-0.9, -0.9, 1.6, 3)$ and $Y'_2 = X^* - Y'_1 = (-1.1, 0.3, 0.3, 0.3)$. Then $U_1(Y'_1) = 0.7 > 0.5 = U_1(X_1^*)$ and $U_2(Y'_2) = 0.3 > 0.2 = U_2(X_2^*)$. So, once again, each agent utility function is strictly greater than that from the optimal individual investment. So, would it be fair to choose the division (Y_1, Y_2) , (Y'_1, Y'_2) , or some third one?

In some cases, see for example Grechuk et al. [67], it is possible to prove that the group of investors behave as a single investor with some ‘group’ utility function

U^* , and then (i) reduces to a standard portfolio optimisation problem with U^* . In turn, problem (ii) for optimal division of *fixed* outcome is known as a problem of optimal risk sharing, which is a well-studied problem that has a long history in insurance as well as in mathematical economics, starting from the 60s of the last century. Borch [32] studied Pareto optimal risk exchanges for investors using EUT, and showed that in many cases it will lead to familiar linear quota-sharing of the total losses known as stop-loss contracts. Additionally, the problem has been studied in various contexts in hundreds of papers, including Dana [47] and related papers [109, 105, 103, 30, 67]. For more details see Grechuk et al. [67] and references therein. The problem (iii) of fair division is also well-studied and a standard approach to it uses equilibrium theory, see the end of this introduction for a detailed discussion.

However, in most cases sub-problems (i)-(iii) cannot be solved separately; it is unclear what trading strategy would be optimal for coalition on stage (i) before the method of fair division is agreed on stage (iii). While there is plenty of literature studying portfolio optimisation, risk sharing, and equilibrium theory separately, the theme of cooperative investment is much less studied. Grechuk et al. [67] solved the cooperative investment problem for investors using general deviation measures in a one-period model. In the dynamic settings, the problem was studied in Parkes [113] for agents using expected utility theory, where he suggested that the simple communication mechanism of explicit hint exchange yields an increased performance. He also studied multi-agents in multi-period and showed that a system of independent agents will outperform a single agent. Furthermore, this system can improve their performance by sharing a short-term portfolio strategy. Additionally, he showed that the communication through hint exchange is considered redundant in a stochastic market, and at the same time it satisfied the Capital Asset Pricing Model (CAPM). Later, Xia [143] studied a cooperative investment problem in the expected utility framework, and he gave a characterisation of Pareto optimal strategies. Also, the problem was studied in Grechuk and Zabaranin [68] for investors using a cash invariant utility function, with the special focus on so-called drawdown risk measure. The theme of this thesis is a systematic study of the dynamic cooperative investment problem in various utility models, including EUT, mean-variance model, mean-deviation model, and prospect theory, illustrating it with case studies with real market data. We will show that cooperation is almost always strictly profitable for all participating investors, and hence should be extensively used in practice.

1.3 Time consistency of dynamic strategies

One of the main issues in dynamic portfolio selection is the issue of time inconsistency. To illustrate this, imagine you have won the possibility to participate in one of the following two games of your choice. A fair coin is tossed twice, and the first game pays you 5 million in case of HH and nothing otherwise, while the second game pays you 1 million in cases HH and HT and nothing otherwise. Assume that your risk preferences are such that you would prefer a 25% chance to get 5 million to a 50% chance to get 1 million, because of the much higher expected profit. However, you would prefer to get 1 million for sure rather than a 50% chance to get 5 millions, because 1 million is a huge amount of money for you, and, having a chance to get this much for sure, you would prefer to avoid any risk. Given that, which game would you select, the first one or the second one? It seems that, given your risk preferences, you would initially select the first one, but then, after the first H happened, you would regret your risky decision, and ask if it is possible to switch to the second game. A similar situation can happen in portfolio optimisation: you may initially select some trading strategy only because it promises you good profit as a result of risky decisions in the last period, but then, when it is time to implement that risky decision, you would rather prefer a safer one with lower profit.

The situation becomes much worse in the context of dynamic cooperative investment, as studied in this thesis. While investing individually you could change your mind, but participation in a cooperative investment scheme assumes that all investors sign a contract and are obliged to follow a pre-specified trading strategy until the very end. It may happen that *all* investors in the group would prefer to change their mind, and then it seems illogical to follow the strategy they have agreed.

Can we develop a time-consistent trading strategy, that is, one that we will be happy to follow at any moment during its implementation? Bellman [20] suggested using the dynamic programming technique, that is, start from the last period and find the optimal strategy for it, then, assuming we know how we will behave during the last period, find the optimal strategy for the pre-last one, and so forth, until we arrive at the first period. In the example above, we first analyse the last period and decide that game 2 should be selected, and then, based on this, we select game 2 at the beginning as well. There is a lot of literature investigating optimal *individual* investments, in both pre-commitment and time-consistent settings.

Dynamic mean-variance portfolio optimisation was studied in [100, 95, 24], and by many other authors. The problem of finding the best time-consistent trading strategy was studied in Bjork and Murgoci [25], who study time-inconsistence in the sense that they do not admit the Bellman optimality principle, see [25]. In my thesis we study time-inconsistence for mean-variance portfolio optimization where the term $(E[X])^2$ in variance causes time-inconsistence. Actually, we studied this problem by viewing them within a game theoretic framework, and we looked for the trading strategy as a function instead of being constant. Bodnar et al. [27] derived the close form solution for exponential utility function in multiple periods which admits a Bellman equation of the dynamic portfolio choice problem, both with and without a risk-less asset under rather weak assumptions. Furthermore, along this line there are many works in either discrete or continuous time; see e.g. [26, 137, 144, 57, 44].

An alternative way to theoretically justify time consistent trading strategies is based on compound independence axioms, together with refuting the reduction axiom, see Segal [132, 133] and [53]. In the example above, reduction axiom states that receiving 5 million in case of HH (and nothing otherwise) is equivalent to a one-round lottery of receiving 5 million with a probability of 25% (and nothing otherwise). That is, we look only at the final outcome, how much will be got with what probability, ignoring all the dynamics. Segal argues that people do not think like this; dynamics matter, and this provides a solution to the described paradox. With this theory, an investor can analyse the problem from the end, and replace any optimal portfolio received during the last period by its certainty equivalent. This simplified the problem for the pre-last period, and so on. This procedure resembles dynamic programming, but there is a crucial philosophical difference: at time t , we care about the certainty equivalent of the portfolio we receive at time $t + 1$, not about the final wealth. Deep theoretical work of Segal [132, 133] and [53], confirmed by significant empirical investigation, argues that this method is a better approximation to the way people *actually* think.

We have applied this idea to the cooperative investment problem, and developed a method for finding the approximation of certainty equivalent observed for each investor and for the coalition. Then we complete the process in the recursive manner until we arrive at the first period. We notice that this procedure has a significant reduction in computational process compared to other methods. We solve the dynamic cooperative investment (DCI) problem in various utility models. In some cases, we find both an optimal pre-commitment trading strategy and

an optimal time-consistent trading strategy based on the dynamic programming technique. While the first one provides better Pareto optimal solutions, the second one has the advantage of avoiding the break-down of the contract between the investors in the middle of the investment period. In Chapter 4 we will explain more about how can we use dynamic programming in cases of discrete time and continuous time.

1.4 Modelling the market with GARCH model

While working with historical data, one of the main problems that we will face is how to generate the scenario trees for future tiers of returns in multiple periods. We study several ways to generate the scenario tree. The first one, a method uses simulation and a randomised clustering approach, see Nalan Glupiner [70] and Chen and Xu [41] and the second method by using binary tree. In our experiments we generate scenario trees by using all the described methods, and use clustering in such a way that we get the same number N of scenarios at each node in each period, avoiding the exponential growth. To make the model realistic, we should check that it is arbitrage-free, and for this we use an algorithm developed by Klaassen [88].

There are many economic models, such as *ARCH*, *ARMA*, *GARCH*, *NGARCH*, *IGARCH*, are used to characterise and model observed time series. We will apply a $GARCH(p, q)$ (Generalised AutoRegressive Conditional Heteroscedasticity of lag p and q) model, which was introduced by Bollerslev in 1986 [29]. The $GARCH(p, q)$ model is designed in order to describe and capture the ‘volatility clustering’ effect in returns. Moreover, in continuous time the volatility is highly heteroskedastic. In most cases, we will use the $GARCH(1, 1)$ time series model which is becoming widely used in econometrics and finance, and is considered as one of the most common and simplest ways to produce estimates of current and future levels of volatilities by using historical data. The main idea is describing a volatile σ_t as a random variable, e.g. an asset price on day t , as estimated at the end of the previous day $t - 1$, see [50].

In continuous time it is natural to model the logarithm of the asset price, that is $G_t = \ln P_t$. The continuous version of $GARCH(1, 1)$ has abbreviation $COGARCH(1, 1)$. In financial econometrics, $COGARCH(1, 1)$ is used in modelling irregularly spaced data. $COGARCH(1, 1)$ was recently constructed and

studied by Nelson [110] and Kluppelberg [89], who show that the *COGARCH*(1, 1) model is an analogue of the discrete time *GARCH*(1, 1) model, based on a single background driving Levy process. Since the wealth of the investor, $X(t)$ satisfies the stochastic differential equation (SDE); we will solve the SDE by using backward numerical approximation; then we complete our solution by using dynamic programming the same way as in the discrete time. Furthermore, we will solve the stochastic differential equation by numerical methods. One of these models we use is due to Euler-Maruyams (1955), which is considered as an analogue of the Euler method for ordinary differential equations. Alternatively, the other method which could be applied to solve SDE is called the Milstein Method, a method which has order one. Moreover, the Euler-Maruyams and Milstein methods are identical in cases where there is no X term in the diffusion part of $b(X, t)$ of the SDE equation. Parallel to this, and in the same line of numerical methods, there are many literature reviews which have other numerical methods to solve stochastic differential equations with higher orders, which give more accurate results, such as Runge-Kutta methods of order one, as well as Taylor methods of order 1.5. The higher order methods can develop the numerical solution for SDE, but become a much more complicated solution corresponding to the degree at which the order grows, so that in my thesis we solve SDE of portfolio wealth by using Euler-Maruyams. In this work, firstly we will use the *GARCH*(1, 1) diffusion approximation of Nelson [110]. Then we solve the corresponding SDE by using the Euler-Maruyams method, since it allows one to solve the cooperative investment in simplistic and realistic ways.

1.5 Choosing a fair allocation using equilibrium theory

Chapter 5 of this thesis studies the equilibrium-based method for selecting the fair allocation point among Pareto optimal ones, which means selecting the fair division of uncertain outcome at the end of the investment period among participants. Intuitively, the group of investors can earn some extra profit as a result of collaboration, and we need to find a method for how this extra profit should be divided among all participating investors (agents) in a fair way. For this, equilibrium theory has been applied to find a fair allocation point, which is also called fair equilibrium allocation. The idea is that we allow the agent to ‘trade’, that is,

selling their parts of risk, and buying other parts of risk from different participants. Because investors value risks differently, such trading may benefit all of them. After some time, this ‘market’ would converge to the equilibrium state, where the supply and demand for any risk equalise, and the price at which this happens is called the equilibrium price. This price can be used to calculate how extra profit from cooperation should be divided among participating investors. This method will be called equilibrium allocation, and will be considered as a ‘fair’ method.

In case of cooperative investment portfolio optimization in multiple periods, we need to find the equilibrium allocation and equilibrium price corresponding to it, which are found by solving maximized individual objective function in order to get the equilibrium allocation in terms of equilibrium price. Thus, the formalization of the notion of efficiency and competitive equilibrium can find the pair of equilibrium allocation and equilibrium price, and this method was developed by Bergson [23]. A few years later, two fundamental theorems of welfare economics were developed by Arrow and Debreu [9], which give us the basic concept of equilibrium and show that every equilibrium allocation is Pareto optimal among all the Pareto optimal sets, which is considered as a key word. For the study of equilibrium in (multi-period) dynamic settings, see Hu et al. [78], who defined the equilibrium via open loop controls, while Debreu [49] and Henriksen and Spear [74] studied multi-period equilibrium as a sequence of equilibrium allocations and a sequence of equilibrium prices, which is one that applied in this thesis. In addition, an important part of this research direction is questions of the existence, uniqueness and stability of the equilibrium. Some of the literature review which shows a way to prove the existence of the equilibrium comes from global analysis, by using Sard’s lemma and the Baire category theorem, see [49, 135], in other words solving maximized individual objective function. Moreover, in this thesis we use that fact from Levin [81] and Quah [117], where Levin [81] shows that the number of the equilibriums should be finite and odd according to the regular economy ‘risk sharing’, while Quah [117] shows the sufficient condition that guarantees that the unique equilibrium is held by satisfying some conditions of local weak axioms.

Thus, in this thesis we derive an explicit formula to find a ‘fair’ equilibrium allocation as a function of equilibrium price. A version of the first fundamental theorem of welfare economics in our setting guarantees that equilibrium allocations are always Pareto optimal. Moreover, we study the conditions under which the equilibrium is unique in our settings. Also, we study the stability level of the fair equilibrium point, which is related to the uniqueness. In addition, the

uniqueness of the equilibrium allocations and equilibrium price is shown under some conditions, see [117]. Furthermore, the local stability is also discussed.

1.6 Thesis contribution and outline

In Chapter 2, we document some definitions, and we formulate the problem of individual and cooperative investment in a single period. Also, we describe our framework for numerical experiments. We solve a cooperative investment in single period for investors whose preferences are described by expected utility theory and Markowitz's mean-variance model, as well as prospect theory. Then, we replace standard deviation in Markowitz's model by other deviation/risk measures such as standard lower and semi-deviation or coherent risk measure such as conditional value at risk and mix-conditional value at risk, as well as expected utility function and value function from prospect theory. In this chapter we show that the risk from cooperative investment for each investor is less than the risk from individual investment for each participant. On the other hand, the expected return from the cooperative investment for each investor is greater than the return from individual investment for each participant.

Chapter 3 focuses on cooperative investment in a multi-period setting and solves it using the dynamic programming approach. In addition, we solve it again in an alternative way by assuming utility model based on compound independence axiom by force in back technique, then finding the certainty equivalent for the corresponding one-period in each step. The numerical experiments based on historical data simulation are observed, and we concentrate on the mean-variance model. We calculate the trading strategy by solving the problem in a recursive manner; furthermore, stability of the results for the whole investment period is discussed, we compare between the efficient frontier from dynamic programming with a time-inconsistent and efficient frontier from a global solution for cooperative investment with a pre-commitment trading strategy. In this chapter we show the contribution of this thesis which is solving cooperative investment in multi-period by using dynamic programming since the mean-variance and mean-semi-variance problem in multi-period face time in-consistency, so we treat this problem in different ways: firstly, by using dynamic programming technique according to Bjork and Murgoci, see [25] and secondly, by finding the certainty equivalent according to Segal [132, 133] and [53].

While Chapters 2 and 3 use naive historical simulations for numerical experiments, Chapter 4 solves the cooperative investment problem in discrete and continuous time assuming that the underlying rates of return follow the $GARCH(p, q)$ model. Actually, we applied this model in discrete time and $COGARCH(p, q)$ in continuous time. We show how can we derive the rate of return according to $GARCH(p, q)$ and $COGARCH(p, q)$ models. Then, we solve cooperative investment and dynamic cooperative investment for the mean-variance model in either discrete time or continuous time, as well as the numerical experiments supporting our result. In this chapter according to numerical experiments we show that applying $GARCH(1, 1)$ in discrete and continuous time in order to forecast the future return in a more realistic way can preserve the main result in our thesis which is the risk from cooperative investment for each investor being less than the risk from individual investment for each investor, and conversely the expected return from cooperative for each investor is greater than the return from individual investment for each investor. Graphically, it represents as an efficient frontier curve. In addition, the difference between the efficient frontier comes from solving cooperative investment without dynamic programming (global solution) and with dynamic programming is not being very significant.

In chapter 5 we derive an explicit formula to get the fair allocation based on equilibrium allocation and equilibrium price according to risk-reward preferences, expected utility model and certainty equivalent. Moreover, the uniqueness and stability of equilibrium allocation is also investigated for some cases, and relevant numerical experiments are performed. We prove that forming a coalition, investing together, and then dividing the outcome based on equilibrium allocation is (with minor additional assumptions) strictly preferable to an individual investment. In this chapter we determine the fair allocation point for each investor at the end of the investment period, which is called the equilibrium allocation point and is represented in an explicit formula.

The conclusion and summary of the thesis, as well as the future works, are shown in Chapter 6. Thus, in the thesis, we mainly focus on the investors whose preferences are described by modern portfolio theory, expected utility theory or prospect theory. *We show that in all these theories, the investors having different preferences can use cooperation to achieve strictly better outcomes, compared to optimal individual investment.*

Chapter 2

Cooperative Investment in single period

The material in this chapter is also the basis for papers by Almulaim [5, 4].

2.1 Problem formulation for individual investment

We start with a one-period model where the portfolio is formed from n risky instruments A_1, \dots, A_n . Suppose the current price of the instrument i is P'_i , and the price at the end of investment period is P_i . Then $r_i = \frac{P_i - P'_i}{P'_i}$ is the return of the asset i . A risk-free instrument A_0 with constant rate of return r_0 may also be included.

In the one-period model, the rates of returns of financial instruments are modeled as random variables (r.v.s), which are the measurable functions from some probability space $\Omega = (\Omega, \mathcal{M}, \mathbb{P})$ to the real line \mathbb{R} , where Ω is the set of future states ω , \mathcal{M} is a σ -algebra of sets in Ω , and \mathbb{P} is a probability measure on (Ω, \mathcal{M}) . Let $\mathcal{L}^p(\Omega)$ be the space of r.v.s X with finite norm $\|X\|_p = (E|X|^p)^{1/p}$. A cumulative distribution function (cdf) of r.v. X is a function $F_X(x) := \mathbb{P}[X \leq x]$.

Let \mathcal{F} is the set of all feasible investment opportunities. If risk-free asset is available on the market, then \mathcal{F} includes it. However, an investor may decide to invest fixed amount of capital into risky assets only, and then feasible set \mathcal{F} do not contain risk-free assets. In most (but not all) of our examples and applications, we assume

that risk-free asset is available, and then (in single period). If there are one risk-free r_0 and n risky instruments A_1, \dots, A_n with the returns r_1, \dots, r_n and initial capital W_0 , then the set of feasible portfolios is

$$\mathcal{F} = \{X = (W_0 - \sum_{i=1}^n w_i)r_0 + \sum_{i=1}^n w_i r_i, \text{ s.t. } \sum_{i=1}^n w_i = 1\}$$

where $w_i \in \mathbb{R}$ are the proportions of capital invested in the instrument A_i .

Investor i introduces preference relation \succeq_i on $\mathcal{L}^p(\Omega)$, that is, $X \succeq_i Y$ if he/she weakly prefers the portfolio with rate of return X to the one with rate of return Y . We will write $X \succ_i Y$ if X is strictly preferable over Y , and $X \sim_i Y$ if the investor is indifferent between X and Y . We assume that \succeq_i is complete, reflexive and transitive, and has a numerical representation U_i , that is $X \succeq_i Y$ if and only if $U_i(X) \geq U_i(Y)$. Functional $U_i : \mathcal{L}^p(\omega) \rightarrow [-\infty, \infty)$ will be called utility functional of investor i . For any portfolio X , its *certainty equivalent* for investor i is a real number C such that $U_i(X) = U_i(C)$.

Traditionally, portfolio optimisation means that a single investor acts alone and wants to achieve his/her investment goals. The individual portfolio optimisation problem for investor i to find maximum of utility function can be formulated as

$$\max_{X \in \mathcal{F}} U_i(X) \tag{2.1}$$

Next we discuss several forms of utility functional that an investor may choose.

2.1.1 Expected utility

Maybe the most popular form for utility functional U is the expected utility model: $U(X) = E[u(X)]$ for some utility function $u : \mathbb{R} \rightarrow \mathbb{R}$.

The basic axioms underlying the expected utility model are formulated in terms of lotteries. A p -lottery between r.v.s X and Y is an r.v. $Z = pX \oplus (1-p)Y$ corresponding to getting X with probability p and Y with probability $1-p$. In other words, Z is an r.v. with cdf $F_Z(x) = pF_X(x) + (1-p)F_Y(x)$.

If the complete transitive preference relation \succeq satisfies in addition

- i- Continuity: If $X \succeq Z \succeq Y$, then there exists a $p \in [0, 1]$ such that $pX \oplus (1 - p)Y \sim Z$; and
- ii- Independence: If $X \succeq Y$, then for any Z and $p \in (0, 1)$,

$$pX \oplus (1 - p)Z \succeq pY \oplus (1 - p)Z.$$

Independence axioms means that preference ordering of two r.v.s is not changed if each of them is mixed with a third r.v. in the same way.

Then the corresponding utility functional U can be represented in the expected utility form

$$U(X) = E[u(X)]$$

for some utility function u . Sometimes additional properties of u are assumed, such as continuity, monotonicity or concavity, see e.g. Follmer and Schied [59].

The portfolio optimisation problem for expected utility maximiser can be written as

$$\max_{X \in \mathcal{F}} E[u(X)]. \quad (2.2)$$

Problem (2.2) can also be rewritten as follows:

$$\begin{aligned} & \max_w E[u(X)] \\ & \text{s.t.} \\ & \sum_{i=1}^n (w_i) = 1, \\ & X = w^T r \end{aligned} \quad (2.3)$$

where $r = (r_1, \dots, r_n)$ is the return vector of risky instruments, $w = (w_1, \dots, w_n)$ is the vector of weights, and T is the symbol for matrix transposition. We can also add the constraint $w \geq 0$ if there is no short selling allowed.

Definition 2.1. [59] Let $S \subseteq R$, where R is the real line and $u : S \rightarrow R$ be a twice continuously differentiable and strictly increasing function on S . Then

$$\alpha(x) = \frac{-u''(x)}{u'(x)}, \quad x \in S$$

is called the Arrow-Pratt coefficient of absolute risk aversion of u at level x .

The following classes of utility functions u and their corresponding coefficients of risk aversion are standard examples, see [59]:

- Constant absolute risk aversion (CARA): $\alpha(x)$ equals some constant α and utility function $u(x) = a - b \exp(-\alpha x)$, where $a \in R$, and $b, \alpha > 0$. Note that u can be normalised to $u(x) = 1 - \exp(-\alpha x)$.
- Hyperbolic absolute risk aversion (HARA): $\alpha(x) = (1 - \gamma)/x$ on $S = (0, \infty)$ and for some $\gamma < 1$. Up to affine transformations, we have

$$u(x) = \log(x), \text{ for } \gamma = 0$$

$$u(x) = \frac{1}{\gamma} x^\gamma \text{ for } \gamma \neq 0$$

Note that sometimes these functions are also called CRRA utility functions, because their relative risk aversion $x\alpha(x)$ is constant. In addition, these utility functions can be shifted to any interval $S = (a, \infty)$. The risk neutral limiting case $\gamma = 1$ would correspond to an affine function $u(x) = x$.

- Quadratic utility function which represents it as $u(x) = x - \alpha x^2$, where $\alpha > 0$ is the risk aversion, $S = R$ and $\alpha(x) = \frac{2\alpha}{1-2\alpha x}$.

2.1.2 Mean-deviation approach

As an alternative method, Markowitz in 1952 [106] suggested that investors can minimise the risk of their portfolios subject to the constraint that the expected return should be at least at the specified level π . He used the variance of the final wealth, or equivalently standard deviation, as the measure of risk, and formulated the portfolio optimisation problem as follows:

$$\begin{aligned} & \min_{X \in \mathcal{F}} \sigma^2(X) \\ & \text{s.t.} \\ & E[X] \geq \pi \end{aligned} \tag{2.4}$$

If the portfolio optimal in (2.4) also achieves the maximum expected level of return among the portfolios which have the same variance, it is said to be efficient. Then we call the pair of the minimum variance and the maximum expected level of return the efficient frontier. For a portfolio consisting of n risky assets we have

$\sigma^2(X) = w'Vw$ where V is the variance-covariance matrix with entries $Cov(r_i, r_j)$, $i, j = 1, \dots, n$, $w = (w_1, \dots, w_n)$, $X = (1 - \sum_{i=1}^n (w_i))r_0 + w^T r$ and r_0 is the return of the risk-free instrument. The exact solution of problem (2.4) is

$$w = \frac{(\pi - r_0)V^{-1}(E[r] - r_0e)}{(E[r] - r_0)^T V^{-1}(E[r] - r_0e)}$$

where $e = (1, 1, \dots, 1)$ is the n -dimensional unit vector, see Markowitz [106].

Standard deviation σ is not an ideal objective function to minimise in (2.4), because it is, for example, symmetric, and penalises losses and profits equally. Rockafellar et al. [122] suggested to replace it by a general deviation measure. firstly, we will define a feasible set as follows:

Let \mathcal{F} be a feasible set which is non-empty subset of locally convex topological vector space \mathcal{X} . Typical settings $\mathcal{X} = \mathcal{L}^p(\Omega) = \mathcal{L}^p(\Omega, \mathcal{M}, \mathbb{P})$, where $\omega \in \Omega$ and Ω denoting the designated space of future states ω , \mathcal{M} is a field of set in Ω and \mathbb{P} is probability measure on (Ω, \mathcal{M}) . In my thesis we use $\mathcal{L}^2(\Omega)$ to ensure the $\sigma(y_i)$ and $\sigma_-(y_i)$ exist and finite.

Definition 2.2. [66] General deviation measure is a functional $\mathcal{D} : \mathcal{L}^2(\omega) \rightarrow [0, \infty]$ satisfying axioms

- (1) $\mathcal{D}(X) = 0$ for constant X , but $\mathcal{D}(X) > 0$ otherwise (nonnegative);
- (2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and $\lambda > 0$ (positive homogeneity);
- (3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y (subadditivity); and
- (4) set $\{X \in \mathcal{L}^2(\omega) | \mathcal{D}(X) \leq C\}$ is closed for all $C < \infty$ (lower semicontinuity).

Then portfolio optimisation problem (2.4) for individual investment can be extended to the form:

$$\begin{aligned} & \min_{X \in \mathcal{F}} \mathcal{D}(X) \\ & \text{s.t.} \\ & E[X] \geq \pi \end{aligned} \tag{2.5}$$

Examples of deviation measures are standard deviation

$$\sigma(X) = \sqrt{E[X - E[X]]^2},$$

and standard lower semi-deviation

$$\sigma_-(X) = \sqrt{E[[X - E[X]]_-]^2},$$

where the minus subscript means taking the negative part, $g_- = \max\{-g, 0\}$, and mean absolute deviation

$$MAD(X) = \|X - EX\|_1,$$

see [122] for more examples.

General deviation measure D has a close correspondence with coherent risk measure R by four axioms that introduced by Artzner et al.[13] in the following definition.

Definition 2.3. [13] A function R is called a coherent risk measure if it satisfies the following axioms

- (1) $R(X_1) \geq R(X_2)$ if $X_1 \leq X_2$ (monotonicity);
- (2) $R(tX_1 + (1-t)X_2) \leq tR(X_1) + (1-t)R(X_2)$ for all X_1, X_2 and all $t \in (0, 1)$ (convexity);
- (3) $R(\lambda X) = \lambda R(X)$ for all $\lambda > 0$ (positive homogeneity); and
- (4) $R(c + X) \leq R(X) - c$ for any X and any positive c (translation equivariance).

A popular example of coherent risk measure, which is often used to reduce the probability that a portfolio will incur large losses, is the conditional value at risk, defined as

$$CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_\beta(X) d\beta,$$

where

$$VaR_\alpha(X) = -\inf\{x | \mathbb{P}(X \leq x) \geq \alpha\},$$

is called the value at risk of X at confidence level $\alpha \in (0, 1)$.

Another example is Mixed Conditional Value at Risk $MCVaR_{\alpha, \lambda}(X)$, defined as

$$MCVaR_{\alpha, \lambda} = \sum_{k=1}^L \lambda_k CVaR_{\alpha_k}(X),$$

where $\alpha = (\alpha_1, \dots, \alpha_L)$ is such that $\alpha_k \in (0, 1)$, $k = 1, \dots, L$, and $\lambda = (\lambda_1, \dots, \lambda_L)$, where $\lambda_k \geq 0$, $k = 1, \dots, L$ and $\sum_{k=1}^L \lambda_k = 1$.

If the risk preferences of an investor are represented by a coherent risk measure R , the portfolio optimisation problem for individual investment becomes

$$\begin{aligned} & \min_{X \in \mathcal{F}} R(X) \\ & \text{s.t.} \\ & E[X] \geq \pi \end{aligned} \tag{2.6}$$

Note that the relation between the problem (2.5) and (2.6) just replaced the deviation measure D by coherent risk measure R .

2.1.3 Prospect theory

Prospect theory (**PT**) is the one of the behaviour economic theories in recent years that people use to make decisions based on the potential value of losses and gains rather than the final outcome. Hence, it considers preferences as a function decision weight and we assume that the weight is not necessary to match with probability. That means there is no conception of different domains of gains and losses and the decision weight add up to one. If all outcomes (excess rate of return) were in the domain of gains then the summation of decision weight is equal to one and similarly if all the outcome were in the domain of losses then the summation of decision weight is equal to one. Specially, **PT** suggests that decision weight tends to overweight small probabilities and underweight moderate and high probabilities. Moreover, in order to describe how investors perceive risk and with appropriate modelling it can be made consistent with rational decision making, so this theory describes it very well.

Prospect theory was created in 1979 and developed in 1992 (under the name of cumulative Prospect theory **CPT**) by Daniel Kahneman and Amos Tversky [83], [138] as a psychologically more accurate description of decision making compared with the expected utility theory. Moreover, investors tend to evaluate prospects or discrete (r.v.s) in terms of gains and losses relative to some natural reference point rather than the final state of wealth. The reference point could be initial wealth, see Shefrin and Statman [134], status quo wealth, average wealth and rational expectations of future wealth (Koszegi and Rabin [90, 91, 92]), historical peak (Gneezy [64]), the purchase price in the real state market (Odean[111]), or uncertain stochastic point (Schmidt [129]).

In this thesis, we assume that in the context of portfolio optimisation, the natural choice for the reference point is the risk-free rate of return r_0 , see [14]. For simplicity, we will describe **CPT** for the case when risks of returns are modelled as discrete r.v.s. For any portfolio A , let $X = r_A - r_0$ be the difference between rate of return of portfolio A and the risk-free rate r_0 , where X is also called excess return. We assume that X can take M negative and N positive outcomes, sort all outcomes in the increasing order, and denote them $x_{-M} < x_{-M+1} < \dots < x_{-1} < x_0 = 0 < x_1 < \dots x_N$. Let outcome x_t , $t = -M, \dots, N$, happen with probability p_t . For such an r.v., we will use the following notation

$$X = (x_{-M}, p_{-M}; x_{-M+1}, p_{-M+1}; \dots; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; x_2, p_2; \dots; x_N, p_N)$$

Example 2.1. Suppose that the risk free rate is 5%, and the rate of return of risky asset can be 35% or -5% with 0.5 probabilities for each. Then the investor can gain 30% or lose 10%, compared to the risk-free rate. Hence, in this case,

$$X = (-0.1, 0.5; +0.3, 0.5)$$

.

Definition 2.4. [138] According to the **CPT**, the value of the discrete r.v.s X (called ‘prospect’ in [138]) is given by

$$U(X) = \sum_{t=-M}^N \pi_t \nu(x_t), \quad (2.7)$$

where $\nu : \mathbb{R} \rightarrow \mathbb{R}$ is the value function satisfying

- $\nu(0) = 0$ (reference dependence);
- $\nu(x)$ is convex for $x \leq 0$ declining sensitivity for losses;
- $\nu(x)$ is concave for $x \geq 0$ declining sensitivity for gains; and
- $-\nu(-x) > \nu(x)$ for $x > 0$ loss aversion.

and π_t are decision weights which are defined differently for the domain of gain and loss.

Definition 2.5. [2]

Domain of gain

$$\pi_N = w^+(p_N)$$

$$\pi_{N-1} = w^+(p_{N-1} + p_n) - w^+(p_N)$$

$$\pi_i = w^+(\sum_{j=i}^N p_j) - w^+(\sum_{j=i+1}^N p_j)$$

$$\pi_1 = w^+(\sum_{j=1}^N p_j) - w^+(\sum_{j=2}^N p_j)$$

Domain of Loss

$$\pi_{-M} = w^-(p_{-M})$$

$$\pi_{-M+1} = w^-(p_{-M+1} + p_{-m}) - w^-(p_{-M})$$

$$\pi_j = w^-(\sum_{i=-M}^j p_i) - w^-(\sum_{i=-M}^{j-1} p_i)$$

$$\pi_{-1} = w^-(\sum_{i=-M}^{-1} p_i) - w^-(\sum_{i=-M}^{-2} p_i)$$

where $w^+(\cdot)$ and $w^-(\cdot)$ are weighing functions for the domains of gains and losses are defined from [138] or [116], respectively. Also, we need $w^+(\cdot)$ and $w^-(\cdot)$ to calculate a value function as shown below.

Examples of value function

- Tversky and Kahnman [138] suggested to use the following value function

$$\nu(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -\lambda(-x)^\beta & \text{if } x < 0 \end{cases}$$

with parameters $\alpha = \beta = 0.88$ and coefficient of loss aversion $\lambda = 2.25$. They also suggested [138] that weighing functions $w^+(p)$ and $w^-(p)$ [138] and the definition (2.5) can be chosen as

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^\gamma}$$

while

$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^\delta}$$

where $\gamma = 0.61$ and $\delta = 0.69$.

- Prelec [116] suggested the power value function ν is given by

$$\nu(x) = \begin{cases} x^{\phi_+} & \text{if } x \geq 0 \\ -(-x)^{\phi_-} & \text{if } x < 0 \end{cases}$$

where ϕ_+ and ϕ_- are equal to 0.88 as in Tversky and Kahneman [138], and weighting functions are defined as

$$w^+(p) = \exp(-\beta^+(-\ln p)^\alpha)$$

$$w^-(p) = \exp(-\beta^-(-\ln p)^\alpha)$$

where $\alpha = 0.65$, $\beta^+ = 1$, $\beta^- = 0.99$. The Prelec value function $\nu(x)$ has S-shape. It is then strictly convex for low probabilities (loss case) $w^-(p)$, while strictly concave for high probabilities (gain case) $w^+(p)$, see Prelec [116].

Problem formulation for the individual investor is given by (2.1) with U given by (2.7).

Example 2.2. *This example shows how to apply the prospect theory for individual investors. Suppose that the first investor follows the value function see [138] and the second investor follows the value function by Prelec in [116], where risk-free $r_{f,t} = 3\%$. Hence, the return of risky asset, excess return and value functions are shown in the following table the value function for each investor are shown in*

probability	return of r_i	x_t excess return	value function 1 st	value function 2 nd
0.1770	-21 %	-24 %	-32.7895%	-14.573 %
0.1917	3%	0	0	0
0.1990	33%	30%	19.9465 %	19.9465 %
0.2015	34%	31%	20.5304 %	20.5304 %
0.2309	46%	43%	27.3811%	27.3811%

TABLE 2.1: Example of value function

Table 2.1 and the decision weights for the first investor and second investor are shown in Table 2.2.

π	weight decision for 1 st	weight decision for 2 nd
π_{-1}	0.2572	0.2763
π_0	0.136018	0.16139
π_1	0.10536	0.13695
π_2	0.1067	0.1325
π_3	0.2796	0.2774

TABLE 2.2: Example of decision weight

Note that $\pi_{-1} = w^-(p_{-1})$, $\pi_3 = w^+(p_3)$, $\pi_2 = w^+(p_3 + p_2) - w^+(p_3)$, $\pi_1 = w^+(p_3 + p_2 + p_1) - w^+(p_3 + p_2)$, and $\pi_0 = w^+(p_3 + p_2 + p_1 + p_0) - w^+(p_3 + p_2 + p_1)$

for both investors according to gain and loss of returns. However, $w^+(p)$, $w^-(p)$ for the first and second investors are defined by Tversky and Kahneman [138] and Prelec [116], respectively.

Now, we need to compute the value function for both investors; according to definition (2.7) the value function for the first investor = 3.514581 while the value function for the second investor = 0.090209.

In addition, in order to find the certainty equivalent for the first investor and second investor by using the formula $U(X) = U(C)$ where $U(X)$ from (2.7) which implies $C = U^{-1}U(X)$ then we get

for the first investor the certainty equivalent computed as

$$\begin{aligned} \pi_{-1}(-(2.25)(-C)^\beta + \pi_0(C)^\beta + \pi_1(C)^\beta + \pi_2(C)^\beta + \pi_2(C)^\beta) &= \sum_{t=-1}^3 \pi_t \nu(x_t) \\ &= \\ 0.2572(-(2.25)(-C)^\beta + 0.136018(C)^\beta + 0.10536(C)^\beta + 0.1067(C)^\beta + 0.796(C)^\beta) &= 3.514581 \end{aligned}$$

hence $C = (2.25)(\frac{3.514581}{0.048978})^{1/0.88} = 128.51\% \approx 1.2851$.

Similarly for the second investor we can find the certainty equivalent as follows:

$$\begin{aligned} \pi_{-1}(-C)^\phi + \pi_0(C)^\phi + \pi_1(C)^\phi + \pi_2(C)^\phi + \pi_3(C)^\phi &= \sum_{t=-1}^3 \pi_t \nu(x_t) \\ &= \\ 0.2763(-C)^\phi + 0.16139(C)^\phi + 0.13695(C)^\phi + 0.1325(C)^\phi + 0.2774(C)^\phi &= 0.090209 \end{aligned}$$

Hence $C = 0.0661$

2.2 Cooperative investment in single period

We argue that it would be beneficial for the investors to cooperate with each other for many reasons, such as: (i) the risk may be too high for a single investor to undertake, (ii) the capital of a single investor may not be sufficient for single period trade without borrowing.

Cooperative investment suggests that m agents/investors with utility functional U_1, \dots, U_m collect all their initial capitals S_i $i = 1, \dots, m$ and invest the total capital $S = \sum_{i=1}^m S_i$ in a common portfolio, and finally, divide the uncertain outcome $X \in \mathcal{F}$ optimally among all the participants at the end of the investment period such that $\sum_{i=1}^m y_i = X$, y_i be shares of investor i . Then, the cooperative investment problem is formulated as follows:

$$\max_{X \in \mathcal{F}} (U_1(y_1), U_2(y_2), \dots, U_m(y_m)) \text{ s.t. } \sum_{i=1}^m y_i = X, X \in \mathcal{F} \quad (2.8)$$

Problem (2.8) is a multi-objective portfolio optimisation problem and each investor ‘agent’ participate in ‘cooperative investment’, and therefore the solution should be understood in the sense of Pareto optimality. Pareto optimality for cooperative investment in a single period can be described as a state of affairs in which resources are distributed in such a way that it is impossible to improve a single individual without causing at least one other person to become worse off than before the change, see Pareto [112]. In an other words, for any vector (y_1, y_2, \dots, y_m) is called a pareto optimal allocation, if there is no allocation $(y'_1, y'_2, \dots, y'_m)$, such that $U_i(y'_i) \geq U_i(y_i)$, with at least one inequality to be strict. Moreover, any vector $Y = (y_1, y_2, \dots, y_m)$, where y_i , $i = 1, \dots, m$ are r.v.s will be called *allocation*. The allocation $Y = (y_1, y_2, \dots, y_m)$ is called *feasible* if $\sum_{i=1}^m y_i = X \in \mathcal{F}$. We say that allocation $Z = (z_1, z_2, \dots, z_m)$ dominates $Y = (y_1, y_2, \dots, y_m)$ if $U_i(z_i) \geq U_i(y_i)$ for $i = 1, \dots, m$, with at least one inequality being strict. Allocation $Y = (y_1, y_2, \dots, y_m)$ is called *Pareto optimal* if there are no feasible allocations dominating it.

Remark 2.6. Cooperative investment problems with agents using expected utility theory is (2.8) with $U_i = E[u_i(X)]$ for $i = 1, \dots, m$. For agents using, for example, prospect theory we have

$$U_i(X) = \sum_{t=-M}^N \pi_t \nu_i(x_t),$$

where π_t is independent of investor $i = 1, \dots, m$, see Tversky and Kahnman[138]. Agents using coherent risk measures can solve (2.8) with $U_i(X) = -R_i(X)$ for some coherent risk measures R_1, \dots, R_m , or $U_i(X) = -D_i(X)$, D_i is deviation measure.

The key benefit from cooperative is that investors use different utility functions or different risk measures, and therefore can act as insurers for each other. Moreover, Assume that individual preferences of investors $i = 1, \dots, m$ are given by (2.5) for

some deviation measures D_1, \dots, D_m and some desired levels of expected returns π_1, \dots, π_m , such that $\pi_i > r_0, i = 1, \dots, m$. In this case, Grechuk et al. [66] show that investors can form a “cooperative portfolio” that solves problem (2.5) for a single investor with certain deviation measure D_s given by the following definition.

Definition 2.7. [66]

Let risk preferences of investor i be expressed by a law-invariant deviation measure D_i , and let m investors form a collation s . Then, a functional defined by

$$D_s(X) = \inf_{y \in A(X)} \max\{D_1(y_1), D_2(y_2), \dots, D_m(y_m)\}$$

is called a deviation measure of the collation s , where $A(X)$ be the set of divisions of the uncertain outcome X among investors $i = 1, \dots, m$.

Let $\mathcal{H} = \{y = (y_1, y_2, \dots, y_m), y_i \in \mathcal{L}^2(\omega)\}$ be a Hilbert space with the inner product $\langle y, z \rangle = \sum_{i=1}^n E[y_i z_i]$, and let $A(X) = \{y \in \mathcal{H} : X = \sum_{i=1}^n y_i\}$ be the set of divisions of the uncertain pay off X among investor $i = 1, \dots, m$, see Grechuk et al. [66]. In this case, the cooperative investment problem will be of the form:

$$\begin{aligned} & \min_{X \in \mathcal{F}} \max(D_1(y_1), D_2(y_2), \dots, D_m(y_m)) \\ & \text{s.t.} \\ & E[X] \geq \pi \\ & \sum_{i=1}^m y_i = X \\ & X \in \mathcal{F} \end{aligned} \tag{2.9}$$

Example 2.3. Assume that there are $m = 2$ investors whose preferences are given by (2.5). Then the cooperative investment problem which is quadratic for $D_1(y_1) = \sigma(y_1)$ and $D_2(y_2) = \sigma_-(y_2)$, and it can be written as follows:

$$\begin{aligned} & \min_{y_1, y_2} D_1(y_1) \\ & \text{s.t.} \\ & D_2(y_2) \leq \beta \\ & E[y_1] \geq \pi_1 \\ & E[y_2] \geq \pi_2 \\ & y_1 + y_2 = X, X \in \mathcal{F} \end{aligned} \tag{2.10}$$

Here, β is the maximum level of deviation that the second investor is willing to take.

Example 2.4. Similarly, if preferences of $m = 2$ investors are described by coherent risk measures R_1 and R_2 , the cooperative investment problem which is linear programme can be written as follows:

$$\begin{aligned}
 & \min_{y_1, y_2} R_1(y_1) \\
 & s.t. \\
 & R_2(y_2) \leq \beta \\
 & E[y_1] \geq \pi_1 \\
 & E[y_2] \geq \pi_2 \\
 & y_1 + y_2 = X, \quad X \in \mathcal{F}
 \end{aligned} \tag{2.11}$$

where π_1 and π_2 is the expected return for each investor and β is the maximum level of risk that the second investor is willing to take; which are chosen after solving individual investments for each investor to have more return and less risk, where $R_1(y_1) = CVaR_{\alpha_1}(y_1)$, and $R_2(y_2) = CVaR_{\alpha_2}(y_2)$. Moreover, we can solve (2.10) in case of utility function $u(\cdot)$ where $U(\cdot) = E[u(\cdot)]$ as follows:

$$\begin{aligned}
 & \max_{y_1, y_2} U_1(y_1) \\
 & s.t. \\
 & U_2(y_2) \geq \beta \\
 & y_1 + y_2 = X, \quad X \in \mathcal{F}
 \end{aligned} \tag{2.12}$$

Note that, the case of 2 investors is the simplest non-trivial case of cooperation, which already contains many ideas which can be applied to the more general cases. It also has a practical importance, because it may be easier for an investor to find one partner whom he/she would trust, than to form a big coalition. We have demonstrated that even this simple coalition of 2 investors already can achieve strictly better results than optimal individual investment. The choice of variance and semi-variance in (2.10) is because these are the most popular risk measures used in portfolio optimization, see [54, 51, 59].

For instance, the investors have different risk measures, with deviation measure such as standard deviation σ for the first investor and standard lower semi-deviation σ_- for the second investor. Hence, $D_1(y_1) = \sigma(y_1)$ and $D_2(y_2) = \sigma_-(y_2)$.

In the numerical experiment we will focus on problem (2.10), (2.11) and (2.12) with $U_i(X) = E[u_i(X)]$.

Instruction for Solving Cooperative Investment Numerically

In the above examples, we will illustrate how to solve cooperative investment in a single period with $m = 2$ investors; let us have a portfolio consisting of n risky assets and one risk-free asset r_0 and the uncertainty outcome X which represents the return of the portfolio at the end of this period, where $X = (X_1, X_2, \dots, X_n)$ and the division of this uncertainty between two investors are $y_1 = (y_1^1, y_1^2, \dots, y_1^n)$ and $y_2 = (y_2^1, y_2^2, \dots, y_2^n)$ respectively. Note that feasible set

$$\mathcal{F} = \{X = (W_0 - \sum_{i=1}^n w_i)r_0 + \sum_{i=1}^n w_i r_i, \text{ s.t. } \sum_{i=1}^n w_i = 1\}$$

where W_0 is initial joint capital between the investor and the rate of return for risky assets r_i and proportion of capital ω_i for asset i . Now we need to find the return r_i that computes as follows: let us have historical price matrix $P = (p_{i,j})$ for time $i = 1, \dots, n$ and asset $j = 1, \dots, T$ weekly, daily, etc. Thus, price matrix P of size $t \times n$ then we will compute the return matrix $R = (r_{i,j})$ where $r_{i,j} = \frac{p_{i,j+1} - p_{i,j}}{p_{i,j}}$. Thus return matrix R of size $n \times (T - 1)$. Then we need to compute the expected return of matrix R we get $r = E[R]$ which is vector of size $n \times 1$ and $r = (r_1, \dots, r_n)$ and $\omega = (\omega_1, \dots, \omega_n)$. Hence, we can define $X = (\omega_1 r_1, \dots, \omega_n r_n)$ which is vector of size $n \times 1$ and it is called uncertain outcome and $E[X] = w.r'$. Also, $D_1(y_1) = \sqrt{E[(y_1 - E[y_1])^2]}$ and $D_2(y_2) = \sqrt{E[(y_2 - E[y_2])^2]}$. Then, plug in the value of $D_1(y_1)$, $D_2(y_2)$ and X in problem (2.10) and solve it by CVX in MATLAB over y_1, y_2 to get the optimal values of $D_1(y_1)$ for first investor and $D_2(y_2)$ for second investor. Note that, we can replace deviation measure $D_i(\cdot)$ by coherent risk measure $R_i(\cdot)$ and solve problem (2.11). Also, we can replace $D_i(\cdot)$ by utility function $u(\cdot)$ and solve problem (2.12) by following the same Instruction, where how to write coherent risk measure $R_i(\cdot)$ in the code in CVX in MATLAB are shown in the Remark (2.8), see the code in Appendix A.

2.3 Numerical experiments

We will solve the problems (2.10) as a cooperative investment.

1) Our assumption is that we have a portfolio containing one hundred risky assets which are selected from *S&P 100*. Also, weekly closing prices of these stocks

from 1/January/2010 to 1/June/2012, are downloaded from Yahoo Finance. Consequently, historical price provides us with 95 total returns for each stock after that we need to compute the expected return of matrix R we get vector r of size 1×100 . Then, we will solve the problem by minimising the risk subject to portfolio expected return exceeding some desired level π_i $i = 1, 2$ greater than level r_0 (risk free), see Remark (2.8) below. The problems (2.4) and (2.5) are for individual investment (**IV**) and (2.10) for cooperative investment(**CI**) where β is chosen as the risk level for second investor which should be less than the risk that we have it from solving (IV) for second investor with the same preferences, where $\pi = 0.0002$, $r_0 = 0.0001$ and $m = 2$, which is two investors.

Remark 2.8. All the codes for these experiments are shown in Appendix A, and also we can check about the result by using linear programme and quadratic programme in the optimisation toolbox, MATLAB. Moreover, we checked the result for the problem (2.4) by using the exact solution by Markowitz [106]. Hence, the Tables below illustrate the optimal value for cooperative and individual investments in a single period where the Tables show the risk of portfolio return that the investor faced during the investment period as well as the profit in case of expected utility and prospect theory just we need to follow instruction to solve cooperative investment in single period numerically.

- **1)** In case of standard deviation $\sigma(X)$ which is defined as $D(X) = \sqrt{E[(X - E[X])^2]}$ and in case we defined it for first investor we will rewrite it in the code as follow: $D(y_1) = \sqrt{E[(y_1 - E[y_1])^2]}$, where share division y_1 for first investor is a vector of size 100×1 in my experiment.
- **2)** In case of standard lower semi-deviation $D(X) = \sqrt{E[(X - E[X])_-^2]}$. Note that using the writing code by using the CVX-file for convex optimisation problem in MATLAB, we can write the minus subscript that means taking the negative part, $g_- = \max\{-g, 0\}$ as *square - pos* $(-(X - EX))$, where $-$ between square and pos is underscore symbol. Then take the square root of it to get standard lower semi-deviation. Note that this is a convex optimisation in quadratic form as well as standard deviation.
- **3)** In case of risk measure defined as $R_1(X) = E[X] - \inf(X)$ or $R_2(X) = \sup(X) - \inf(X)$ will be a linear programming. In this case we need to find $\sup(X), \inf(X)$ or $E[X]$ in order to construct the deviation risk measure.
- **4)** In case of $R(X) = MAD(X)$, where the $MAD(X) = E[|X - EX|]$ which is mean absolute deviation measure and is also a linear programme.

- **5)** In case of $R(X) = CVaR_\alpha(X)$. Firstly, we need to define *CDF* of r.v. X where in my experiment X has an uniform distribution. Then use it to find the quantile function $q_\alpha(X) = \inf\{X | F_X(x) > \alpha\}$ and $VaR_\alpha(X) = -q_\alpha(X)$, see the code in Appendix A. Taking in account that α is the confidence level, we already get VaR_α which is the value at risk with confidence level α , where usually α is very close to 1. Than, when an extreme loss occurs, VaR_α is exceeded. Consequently, the actual loss can be much higher than value at risk. So that, we can employ a coherent risk measure to better quantify and note that VaR_α is not a coherent risk measure because of a lack sub-additive axioms. So that we deal with expected shortfall which is called conditional value at risk. We can consider that the $CVaR_\alpha$ is called conditional value at risk or expected short full, too. Also, this will be a linear programme.

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$\sigma(X)$	+0.000179576	+0.00100099	-0.082 %
$\sigma_-(X)$	+0.0002	+0.00043462	-0.023 %

TABLE 2.3: Optimal value for cooperative investment (2.10) in single period with $\sigma(x)$ for first investor and $\sigma_-(x)$ for second investor

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$E(X) - \inf(X)$	+0.000370372	+0.02	-1.9 %
$\sup(X) - \inf(X)$	+0.01	+0.667155	-65.7 %

TABLE 2.4: Optimal value for cooperative investment (2.10) in single period with $E(X) - \inf(X)$ for first investor and $\sup(X) - \inf(X)$ for second investor

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$MAD(X)$	$+1.68867 \times 10^{-10}$	+0.0490376	- 4.9%
$\sigma(X)$	+0.0001	+0.00100099	-0.09%

TABLE 2.5: Optimal value for cooperative investment (2.10) in single period with $MAD(X)$ for first investor and $\sigma(x)$ for second investor

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$MCVaR_\alpha(X)$	+0.00987789	+0.0197558	-0.98%
$CVaR_{\alpha=99\%}(X)$	+0.01	+0.0237576	-1.3%

TABLE 2.6: Optimal value for cooperative investment (2.11) in single period with $MCVaR_\alpha(X) = \lambda_1 CVaR_{\alpha=95\%}(X) + \lambda_2 CVaR_{\alpha=97\%}(X)$, where $\lambda_1 = 0.25$ and $\lambda_2 = 0.75$ for first investor and $CVaR_{\alpha=99\%}(X)$ for second investor

Risk measure	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$CVaR_{\alpha=95\%}(X)$	+0.00790429	+0.012508	-0.4603%
$CVaR_{\alpha=99\%}(X)$	+0.01	+0.0237576	-1.3%

TABLE 2.7: Optimal value for cooperative investment (2.11) in single period with $CVaR_{\alpha=95\%}(X)$ for first investor and $CVaR_{\alpha=99\%}(X)$ for second investor

- 6) Solving quadratic utility function see Table 2.8.

utility function	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$u_1(X) = X - \alpha_1 X^2$	+0.7336	+0.0511391	68.2%
$u_1(X) = X - \alpha_2 X^2$	+0.500	+0.0100631	48.9%

TABLE 2.8: Optimal value for cooperative investment(2.8) in single period with quadratic utility function, where $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$

- 7) Solving exponential utility function see Table 2.9.

utility function	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$u_1(X) = 1 - \exp(-\alpha_1 X)$	+1.7433	+0.800372	94.2%
$u_1(X) = 1 - \exp(-\alpha_2 X)$	+0.4	+0.0553202	34.4 %

TABLE 2.9: Optimal value for cooperative investment(2.8) in single period with exponential utility function, where $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$

- 8) Solving cooperative investment in multi-period with prospect theory, see Table 2.10.

utility function	optimal value for CI	optimal value for IV	(CI-IV) \times 100
$\nu_1(X)$	+3.843	+3.073	77%
$\nu_2(X)$	+6.3287	+3.019	33.09%

TABLE 2.10: Optimal value for cooperative investment in prospect theory in single period with solving the problem (2.8) where $\nu_1(X)$ see [138] and $\nu_2(X)$ see [116]

Note that, according to all the Tables in these experiments we notice that when we solve cooperative investment with risk measure we find that the risk from cooperative investment for each investor is less than the risk from individual investment for each investor which cause sign $-$. On the other hand, when we solve cooperative investment with utility function we find that the expected return from cooperative for each investor is greater than the expected return by solving individual investment for each investor.

2.4 Discussion and concluding remarks

The experiments in this chapter showed that the risk from cooperative investment for each investor $R_i(y_i)$ will be less than the risk from individual investment for each investor $R_i(x_i)$ $i = 1, 2$ and R could be coherent risk measure or deviation measure $R = D$. However, the expected return from cooperative investment for each investor it will be greater than the expected return from individual investment for each investor i , where $i = 1, 2$. Generally, optimal cooperative investment has two advantages for the investor: agent sharing creates instruments that, firstly, satisfy individual risk preferences; and, secondly, they are not available on the market unless we have cooperative investment.

Chapter 3

Cooperative Investment in Multi-Period with Dynamic Programming Approach

The material in this chapter is also the basis for papers by Almualim [5, 4].

In this chapter we formulate the dynamic cooperative investment problem in multi-period setting, and solve this problem for the case of two investors, who wants to minimise the deviation of their final outcome subject to the constraint on the expected return. The first investor minimises standard deviation, while the second one minimises standard lower semi-deviation.

First, we find the optimal pre-commitment strategy for them, and show how to construct the set of Pareto optimal allocations. Then we show that this strategy is unstable, meaning that the investors would want to change their minds in the middle of the investment period. We then formulate the problem of finding an optimal strategy amongst those, which are stable, or time consistent, see Bjork and Murgoci [25]. We first use a dynamic programming approach, which breaks down the problem into sub-problems and solve it recursively, so that it transforms a complex problem into a sequence of simpler problems; its essential characteristic is the multi-stage nature of the optimisation procedure, see Bjork and Murgoci [25] and Cui. et al.[44].

Alternatively, we suggest an approach based on a compound independence axiom, see Segal [132, 133]. In this approach, each investor replaces an uncertain future

outcome from investment by its certainty equivalent, which simplifies the computation significantly. This approach also results in the trading strategy which is stable for the investment period.

3.1 Problem formulation

In this section we generalise the method to solve cooperative investment (CI) in a single period to the multi-period setting, see Xia [143], and Follmer [59].

Let (Ω, \mathcal{M}, P) be a complete probability space. We assume that the rate of return $r_{i,t}$ of asset i at time t is a random variable on Ω , and that vector $r_t = (r_{1,t}, r_{2,t}, \dots, r_{n,t})$ is measurable with respect to σ -algebra $\mathcal{M}_t \subset \mathcal{M}$. Furthermore, a family $(\mathcal{M}_t)_{0 \leq t \leq T}$ of σ -algebras satisfying $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M}_T$ is called a filtration. In case $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t=0, \dots, T}, P)$ is called a filtered probability space, and to simplify we assume, $\mathcal{M}_0 = \{\emptyset, \Omega\}$, $\mathcal{M}_T = \mathcal{M}$ see Follmer [59] chapter 6. The positive number T is a fixed and finite time horizon. A stochastic process $R = (r_t)_{t=0, \dots, T}$ is called adapted with respect to the filtration $(\mathcal{M}_t)_{t=0, \dots, T}$ if each r_t is (\mathcal{M}_t) measurable.

A stochastic process $Z = (z_t)_{t=0, \dots, T}$ is called predictable with respect to the filtration $(\mathcal{M}_t)_{t=0, \dots, T}$ if each z_t is (\mathcal{M}_{t-1}) measurable. A predictable \mathbb{R}^n -valued process $z_t = (z_{1,t}, z_{2,t}, \dots, z_{n,t})$ is called a *trading strategy*. $z_{i,t}$ is the proportion of total capital an agent invests into asset i at time t . A trading strategy is called *self-financing* if all the money we get after time period $t - 1$ is invested at time period t . Let $X(z_t)$ be a random variable written as a linear combination of rate of return of the final outcome and a self-financing trading strategy z_t see example of uncertain outcome X in Algorithms 3.2 and 3.3. The set of all possible r.v.s $X(z_t)$ will be denoted \mathcal{F} , and called the *feasible set*.

Then individual and cooperative investment problems in multi-period can be written as follows:

- Individual investment

$$\max_{X \in \mathcal{F}} U(X), \quad (3.1)$$

- cooperative investment

$$\max_{X \in \mathcal{F}} (U_1(y_1), U_2(y_2), \dots, U_m(y_m)) \text{ s.t. } \sum_{i=1}^m y_i = X, \quad X \in \mathcal{F}, \quad (3.2)$$

where $U_i(\cdot)$ is the utility functional representing the preferences of agent i . Note that y_i $i = 1, \dots, m$ is r.v so it is measurable and moreover $y_i \in \mathcal{L}^2$.

- in case of 2 agents and $U(\cdot) = -R(\cdot)$ the problem is written as follows:

$$\min_{X \in \mathcal{F}} R_1(y_1) \text{ s.t } R_2(y_2) \leq \beta, E[y_1] \geq \pi_1, E[y_2] \geq \pi_2, y_1 + y_2 = X, X \in \mathcal{F}, \quad (3.3)$$

Note that β represents the level of risk for a second investor which can be chosen to be less than the risk for the same investor in an individual investment. Also, the optimal trading strategy in (3.1), (3.3) and (3.2) may be time inconsistent, that is, unstable during investment period. We propose the dynamic programming method for finding the best time-consistent trading strategy in both individual and cooperative investment cases.

Definition 3.1: Bellman [20]

Dynamic programming means breaking down the problem like (3.1), (3.3) or (3.2) into sub-problems and solving it into a recursive algorithm by shifting the time recursively until the initial period.

First of all we explain the recursive algorithm procedure for the individual investor using the following illustrating example, in which we derive the optimal stable trading strategy for one risky asset and two periods. After that, we develop a general framework for n risky assets and T periods in case of cooperative investment which is our goal in this thesis.

3.1.1 Dynamic programming for individual investment: An example

We consider two investment periods, during which an agent may invest either into a risk-free asset with $r_0 = 0$, or into the risky asset whose rate of return after each period is an r.v. assuming values 2 and -1 with equal probabilities. That is, if we invest x to the risky asset, it either returns $3x$, or nothing.

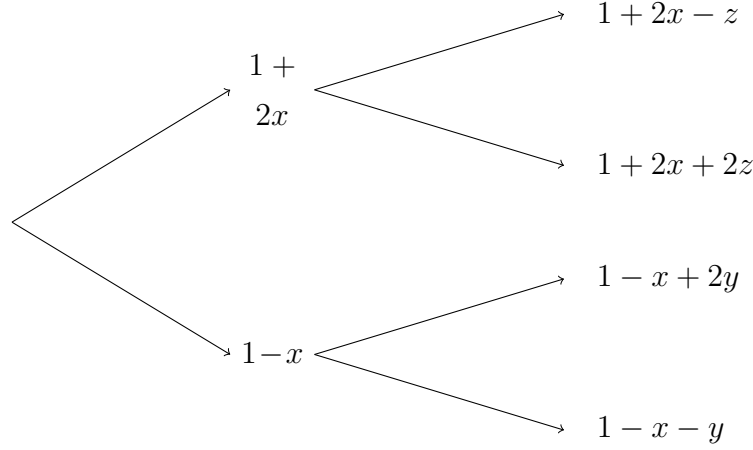
In this simple case, all possible investment strategies are completely described by 3 numbers, (x, y, z) , where

x is the proportion of money invested into the risky asset in the first; period

y is the proportion of money invested into the risky asset in the second period, provided that its price went down during the first period; and

z is the proportion of money invested into the risky asset in the second period, provided that its price went up during the first period.

Then all possible outcomes can be visualised as follows:



Let $X = X(x, y, z)$ be an uncertain final outcome from the trading strategy (x, y, z) . Then X is an r.v. assuming values of each outcome with equal probabilities.

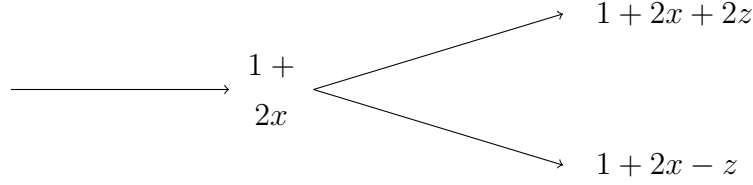
In this example, the mean-variance portfolio optimisation problem can be formulated as follows:

$$\begin{aligned}
 & \min_{x, y, z} \quad \sigma^2(X) \\
 & \text{s.t.} \\
 & E[X] \geq \pi \\
 & -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1,
 \end{aligned} \tag{3.4}$$

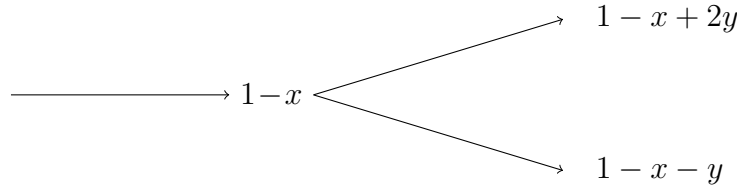
The portfolio positions are always between 0 and 1 if the short sales are not allowed. We decided to relax this assumption to the positions between -1 and 1 to allow reasonable short sales. However, we think that allowing unbounded short sales is unrealistic. Let $\pi = 1.7$. The optimal value for the individual investment in a two-period problem (3.4) is $\sigma^2 = 2.0889$, and the optimal trading strategy is $x = 0.736838$, $y = 0.884193$, and $z = 0.44213$. However, this solution

is time-inconsistent: after investing $x = 0.736838$ in the first period, and getting $1 + 2x$ back, an investor with mean-variance risk preferences will *not* invest $y = 0.884193$ into a risky asset during the second period since this problem faced time-inconsistency.

Our aim is to find the best strategy that an investor would actually follow. For this, we use a recursive algorithm that breaks down the problem into two sub-problems (A and B) as follows: **A**



B



- Firstly, we will solve the mean-variance problem (3.4) for (**A**) in order to find the trading z as a function of x and by using Lagrange-multiplier we get $z(x) = 2(\pi - 1) - 4x$ directly from constraint $E[X] \geq \pi$. Note that the observation uncertain outcome X has two random variables as shown in **A**.
- Similarly, we will solve the mean variance problem (3.4) for (**B**) in order to find the trading y as a function of x and by using Lagrange-multiplier we get $y(x) = 2(\pi - 1) + 2x$ directly from constraint $E[X] \geq \pi$. Finally, we plug the $z(x)$ and $y(x)$ into problem (3.4). Note that in this case the problem will be with one variable x only, where the X will be a random variable which depends only on x . Thus, we can solve the one-parameter optimisation problem to get $x = 0.14$ and the optimal value $+3.969$. So that we get the optimal value of trading strategy x then plug the value of x into functions $z(x)$ and $y(x)$ to get the trading strategy for z and y corresponding to the value of x .

Remark 3.1. We will note that the optimal value V_1 for the problem (3.4) by solving the individual investment directly in two periods over the variables x, y and z will be less than the the optimal value V_2 for the problem (3.4) by solving the problem by using a recursive algorithm (force in back technique). In addition, if we fix the value of x that comes from optimisation over the variables x, y and z and plugging the value of $x = 0.736838$ into the function $z(x)$ and $y(x)$. Then resolve the optimisation problem over one variable x we get the optimal value V_3 and we will notice that $V_1 < V_2 < V_3$ which means that the optimal value V_1 is the best one since it is a global solution. Furthermore, we will explain the meaning of that in the next section and we will illustrate it in the efficient frontier for the problem (3.4) in order to find the best trading strategy that is stable during whole investment periods.

3.1.2 Certainty equivalent for individual investment

An alternative method to solve time-inconsistent trading strategy by using utility model based on the compound independence axiom of Segal [132, 133] implies that an uncertain outcome during each period can be replaced by its certainty equivalent which has less computational than dynamic programming shown in the previous section, and also has deep theoretical work confirmed by empirical investigation for decision-makers. In order to understand what is the certainty equivalent in the mean-variance model, we need to formulate the portfolio optimisation problem using utility functional U as a function of 2 variables, $U = U(\mu, \sigma)$.

It is easy to see that the linear combination $U(\mu, \sigma) = \mu - \frac{\sigma}{2\rho}$, where ρ is the risk aversion, is positive homogeneous and cannot be used to identify optimal portfolio. Hence, we need to find a non-linear combination between σ and μ , that is non-linear utility function $U(\mu, \sigma)$. Once U is found, we can easily find the certainty equivalent C of any portfolio X by equation $U(X) = U(C)$ according to Segal [132, 133].

Sargent [125] suggests to use the following non-linear function

$$U(\mu, \sigma) = \mu - \frac{\sigma^2}{2\rho}. \quad (3.5)$$

Because $U(C) = U(C, 0) = C - \frac{0^2}{2\rho} = C$ for any constant C , this leads to the same certainty equivalent: $C = \mu - \frac{\sigma^2}{2\rho}$.

We justify this choice of U using analogy with expected utility theory. In general, there are three types of risk attitudes for decision maker; risk-averse, risk-neutral and risk-seeking. Moreover, the risk-averse is the most common one. Thus, the risk-attitude for a rational decision maker assumed it to be proper risk attitude. Within the framework of expected utility theory, utility functions representing the risk-averse attitude are usually assumed to be differentiable, increasing and concave. An important example is exponential utility function $u(X) = -\exp(-\lambda X)$, where λ indicates less risk-averse of a rational decision maker. Let $C(X)$ be the corresponding certainty equivalent found from the equation $E[u(X)] = E[u(C)]$.

Proposition 3.2. *If X is normally distributed with mean μ and variance σ^2 , then $C(X) = \mu - \frac{\sigma^2}{2\rho}$.*

Proof. The details of how to derive this formula see Sargent [125], page.155.

□

Proposition 3.2 justifies the choice of utility function (3.5): the corresponding certainty equivalents will coincide at least for exponential utility and normal distributions.

Similarly, for an investor using standard lower semi-deviation σ_- as deviation measure, we suggest to use the following certainty equivalent

$$U(\mu, \sigma_-) = \mu - \frac{\sigma_-^2}{\rho}, \quad (3.6)$$

which is motivated by the following Proposition.

Proposition 3.3. *If X is normally distributed with mean μ and lower semi-variance σ_-^2 , then $C(X) = \mu - \frac{\sigma_-^2(X)}{\rho}$, where $C(X)$ is the certainty equivalent for exponential expected utility function defined before Proposition 3.2 and ρ is risk aversion.*

Proof. Since semi-variance $= \frac{1}{2}$ variance for normal distribution. This also comes from the fact that $(X - \mu)_- = (X - \mu)_+$, where $\sigma_-^2 = E[(X - \mu)_-]^2$, thus $\sigma_-^2 = \frac{1}{2}\sigma^2$. Hence, from the proposition (3.2) we have the utility $U(\mu, \sigma_-) = \mu - \frac{2\sigma_-^2}{2\rho}$, then $U(\mu, \sigma_-) = \mu - \frac{\sigma_-^2}{\rho}$. In other words, we can find this relation directly from moment-generating function for normal distribution. Moreover, the $E(C) = C$ and $\sigma_-^2(C) = 0$ so that the approximation of $U(X) = U(C)$ implies $C = \mu(X) - \frac{\sigma_-^2(X)}{\rho}$. □

3.1.3 Two-period Example: First investor

In this section we explain time consistent mean-variance model based on compound independent axiom using this illustrate example. Let the first investor wants to maximise $C = \mu - \frac{\sigma^2}{2\rho_1}$. We suppose that the risk free rate is 0 and single risky assets has statistically independent rate of return $a, -b$ with probability p and $1 - p$ respectively. Where $(a \geq 0)$, $(b \geq 0)$, $(0 \leq p \leq 1)$. Then, after the first period the investment outcome will be

$$X_1 = X_1(x) = \begin{cases} 1 + ax & \text{with prob. } p \\ 1 - bx & \text{with prob. } 1 - p \end{cases}$$

Note that, the x is a fraction of the initial unit capital invested in the risky asset, so that after the second period

$$X_2 = X_2(x) = \begin{cases} 1 + ax + ay & \text{with prob. } p^2 \\ 1 + ax - by & \text{with prob. } p(1 - p) \\ 1 - bx + az & \text{with prob. } (1 - p)p \\ 1 - bx - az & \text{with prob. } (1 - p)^2 \end{cases}$$

It follows directly from compound independent axiom, that every outcome of any two-stage lottery can be replaced by its certainty equivalent, thus reducing a two-stage lottery to a one-stage one. This suggests the following dynamic programming approach to any multi-period portfolio optimisation. Solve-one period portfolio optimisation at the last stage. We will find the certainty equivalent of the optimal portfolio, that we will use compound independence axiom to reduce T period portfolio optimisation problem to $T - 1$ period one. Suppose the first investor (individual investment) wants to

$$\max_y E[X_2] - \frac{\sigma(X_2)^2}{2\rho_1}$$

where ρ_1 is the risk aversion for the first investor. Now, we will find the certainty equivalent at the second period for first scenario, since we start from the last period to find y and find C as constant and does not depend on x . The scenario will be as following

$$X_2 = X_2(y) = \begin{cases} 1 + ax + ay & \text{with prob. } p \\ 1 + ax - by & \text{with prob. } 1 - p \end{cases}$$

since the $\sigma^2(X_2)$ in this scenario will be $\sigma^2 = (a+b)^2 y^2 p(1-p)$ and the expectation is $E[X_2] = 1 + ax + (pa - b(1-p))y$ since the objective function $E[X_2] - \frac{\sigma(X_2)^2}{2\rho_1}$ to find the optimal value of y . By plunging the value of the variance and expectation on the objective function and solve it by maximize the function U , where $U = E[X_2] - \frac{\sigma(X_2)^2}{2\rho_1}$. We get

$$\frac{\partial U}{\partial y} = pa - b(1-p) - \frac{2}{2\rho_1}(a+b)^2 y p(1-p) = 0$$

$$\text{Hence, } y^* = \frac{[(-b(1-p) + ap)]\rho_1}{(a+b)^2 p(1-p)}$$

with certainty equivalent of the optimal value portfolio given by

$$EX'_2(y^*) - \frac{\sigma(X_2)^2}{2\rho_1} = 1 + ax + \frac{(ap - b(1-p))^2 \rho_1}{(a+b)^2 p(1-p)} - \frac{1}{2\rho_1} \frac{(a+b)^2 p(1-p)(ap - b(1-p))^2 \rho_1^2}{(a+b)^4 (p(1-p))^2}$$

by simplifying to get the optimal

$$\begin{aligned} & 1 + ax + \frac{(ap - b(1-p))^2 \rho_1}{(a+b)^2 p(1-p)} - \frac{\rho_1}{2} \frac{(ap - b(1-p))^2}{(a+b)^2 (p(1-p))} \\ & = 1 + ax + \frac{1}{2} \frac{(ap - b(1-p))^2 \rho_1}{(a+b)^2 (p(1-p))} \end{aligned}$$

Hence, the optimal

$$X_1 = X_1(x) = \begin{cases} 1 + ax + C & \text{with prob. } p \\ 1 - bx + C & \text{with prob. } 1 - p \end{cases}$$

where $C = \frac{\rho_1(ap - b(1-p))^2}{2(a+b)^2 p(1-p)}$ which is constant function and does not depend on x as well as we can show that $y^* = z^*$ by similar argument. Finally we need to find the optimal value of x^* . Then, the optimal objective value

$$EX'_1(x^*) - \frac{\sigma(X_1)^2}{2\rho_1} \text{ where}$$

$$EX'_1(x^*) = 1 + (pa - (1-p)b)x + C, \text{ and}$$

$$\sigma(X_1)^2 = (a+b)^2 x^2 p(1-p)$$

plug them in the objective function we get:

$$\begin{aligned} EX'_1(x^*) - \frac{\sigma(X_1)^2}{2\rho_1} &= 1 + (pa - b(1-p)) \frac{[(ap - b(1-p))\rho_1]}{(a+b)^2 p(1-p)} + \frac{\rho_1 [ap - b(1-p)]^2}{2(a+b)^2 p(1-p)} - \frac{1}{2\rho_1} (a + \\ & b)^2 \frac{[ap - b(1-p)]^2 \rho_1^2}{[(a+b)^2 p(1-p)]^2} p(1-p) \\ &= \end{aligned}$$

Thus, the optimal trading strategy is

$$x^* = y^* = z^* = \frac{[(-b(1-p) + ap)]\rho_1}{(a+b)^2 p(1-p)},$$

Note that, $x^* = z^* = y^*$ is due to simplicity of this example. In general, these

equalities do not hold. Moreover, the $E(C) = C$ and $\sigma^2(C) = 0$ so that the approximation of $U(X) = U(C)$ implies $C = \mu(X) - \frac{\sigma^2(X)}{\rho}$.

The optimal objective value will be

$$1 + \frac{[pa - b(1-p)]^2 \rho_1}{(a+b)^2 p(1-p)}$$

3.1.4 Two-period Example: Second investor

Similar to the example in section 3.1.3 in 2-periods with binomial tree. Now, we suppose the second investor has a rational decision maker that can be measured by C so he/she wants to maximise $\mu - \sigma_-^2/\rho_2$. We need to prove that the certainty is constant and does not depend on x . Similar to example 1 we start by the last period $X_2(y)$ in order to find the optimal value of y^* and we need to compute $\sigma_-^2(X_2(y))$ and $E[X_2(y)]$. Where

$\sigma_-^2(X_2(y)) = p[\max(b(p-1)y, 0)]^2 + (1-p)[\max((a+pb)y, 0)]^2$ and $E[X_2(y)] = 1 + ax + pay - b(1-p)y$ by plugging the value of semi-variance and the expected return in the objective function, then solve the problem by maximize the function U , where $U = \mu - \sigma_-^2/\rho_2$. Then, plug in the value of $E[X_2(y)]$ and $\sigma_-^2(X_2(y))$ we get

$$U = 1 + ax + pay - b(1-p)y - \frac{1}{\rho_2} [p[\max(b(p-1)y, 0)]^2 + (1-p)[\max((a+pb)y, 0)]^2]$$

$$\frac{\partial U}{\partial y} = p + b(1-p) - \frac{1}{\rho_2} [2p \max(b(p-1)y, 0)(b(p-1)) + 2(1-p) \max((a+pb)y, 0)(a+pb)]$$

and we have two cases

$$\frac{\partial U}{\partial y} = \begin{cases} p + b(1-p) - \frac{1}{\rho_2} [2(1-p)(a+pb)^2 y] & \text{with } y \geq 0 \\ p + b(1-p) - \frac{1}{\rho_2} [2p(b(p-1))^2 y] & \text{with } y \leq 0 \end{cases}$$

- **First case if $y \geq 0$** so that to complete and find the optimal value of y^* by solving $\frac{\partial U}{\partial y} = 0$

$$[p + b(1-p) - \frac{1}{\rho_2} (2(1-p)(a+pb)^2)y] = 0$$

implies that

$$\frac{1}{\rho_2} [2(1-p)(a+pb)^2 y] = p + b(1-p)$$

solve last equation in y

Hence,

=

$$y^* = \frac{\rho_2(p + b(1 - p))}{2(1 - p)(a + pb)^2}$$

where $x^* = z^* = y^*$

Now, to find the certainty equivalent in this case by plugging the value of y^* in $E(X_2) - \frac{\sigma_-^2(X_2)}{\rho_2}$

$$1 + ax + pa \left[\frac{\rho_2(p + b(1 - p))}{2(1 - p)(a + pb)^2} \right] - b(1 - p) \left[\frac{\rho_2(p + b(1 - p))}{2(1 - p)(a + pb)^2} \right] - \frac{1}{\rho_2} \left[\frac{\rho_2^2(p + b(1 - p))^2}{4(1 - p)(a + pb)^2} \right]$$

simplifying

$$1 + x + \rho_2 \frac{(2pa + 2b(1 - p))(p + b(1 - p)) + (p + b(1 - p))^2}{4(1 - p)(a + pb)^2}$$

Thus

$$C = \rho_2 \frac{(2pa + 2b(1 - p))(p + b(1 - p)) + (p + b(1 - p))^2}{4(1 - p)(a + pb)^2}$$

- **Second case if $y \leq 0$** hence the optimal value of y^* by solving $\frac{\partial L}{\partial y} = 0$ then

$$p + b(1 - p) - \frac{1}{\rho_2} [2p(b(p - 1)y(b(p - 1)))]$$

solving last equation in y which implies

$$y^* = \frac{\rho_2(p + b(1 - p))}{2p(b(p - 1))^2}$$

where $x^* = z^* = y^*$

Now, to find the certainty equivalent in this case by plugging the value of y^* in $E(X_2) - \frac{\sigma_-^2(X_2)}{\rho_2}$, then

$$1 + ax + pay - b(1 - p)y - \frac{1}{\rho_2} [b^2(p - 1)^2 y^2]$$

=

$$1 + ax + pa \left[\frac{2\rho_2(p + b(1 - p))}{4(p(b(p - 1)))^2} \right] - b(1 - p) \left[\frac{2\rho_2(p + b(1 - p))}{4(p(b(p - 1)))^2} \right] - \frac{1}{\rho_2} \left[\frac{b^2(p - 1)^2 \rho_2}{4(p(b(p - 1)))^2} \right]$$

simplifying

$$1 + ax + \rho_2 \frac{b^2(p-1)^2 + (2pa - 2b(1-p))(p + b(1-p))}{4(p(b(p-1)))^2}$$

hence,

$$C = \rho_2 \frac{b^2(p-1)^2 + (2pa - 2b(1-p))(p + b(1-p))}{4(p(b(p-1)))^2}$$

Thus,

$$\frac{\partial L}{\partial y} = \begin{cases} p + b(1-p) - \frac{1}{\rho_2} [2(1-p)(a+bp)^2 y] & \text{with } y \geq 0 \\ p + b(1-p) - \frac{1}{\rho_2} [2p(b(b(p-1)))^2 y] & \text{with } y \leq 0 \end{cases}$$

implies that the solution ‘optimal trading strategy’ y^*

$$y^* = \begin{cases} \frac{\rho_2(p+b(1-p))}{2(1-p)(a+pb)^2} & \text{when } y \geq 0 \\ \frac{\rho_2(p+b(1-p))}{2p(b(p-1))^2} & \text{when } y \leq 0 \end{cases}$$

where $x^* = y^* = z^*$.

- Now, to find the objective value for $X_1(x)$, where

$$X_1 = X_1(x) = \begin{cases} 1 + ax + C & \text{with prob. } p \\ 1 - bx + C & \text{with prob. } 1 - p \end{cases}$$

and

$$\mu_1 = E[X_1(x^*)] = 1 + (pa - (1-p)b)x + C$$

$$\sigma_-^2(X_1) = p[\max((p-1)(a+b)x, 0)]^2 + (1-p)[\max(p(a+b)x, 0)]^2$$

Hence, the objective function will be as follows:

$$\mu_1 - \frac{\sigma_-^2(X_1)}{\rho_1} = \begin{cases} 1 + (pa - (1-p)b)x + C - \frac{(1-p)[p(a+b)x]^2}{\rho_1} & \text{with } x \geq 0 \\ 1 + (pa - (1-p)b)x + C - \frac{p[(p-1)(a+b)x]^2}{\rho_1} & \text{with } x \leq 0 \end{cases}$$

plug in the value of C and the value of x^* for the objective function to get the optimal objective value **O.V** in each case and simplifying we get

$$O.V = \begin{cases} 1 + pa + \rho_2 \frac{[2 + (2pa + 2b(1-p))(p+b(1-p)) + [1-p^2(a+b)^2](p+b(1-p))^2]}{4(1-p)(a+pb)^2} & \text{with } x \geq 0 \\ 1 + \rho_2 \frac{[(a+b)^2(p+b(1-p))^2 + 2p(pa - (1-p)b)(p+b(1-p))(1+b^2(p-1)^2)]}{4(p(b(p-1)))^2} & \text{with } x \leq 0 \end{cases}$$

According to requirement to maximise the objective value $\mu_1 = E[X_1(x^*)]$, the optimal trading strategy will be

$$x^* = z^* = y^* = \frac{\rho_2(p + b(1 - p))}{2(1 - p)(a + pb)^2}$$

Note that, Note that, $x^* = z^* = y^*$ is due to simplicity of this example. In general, these equalities do not hold. Moreover, the $E(C) = C$ and $\sigma_-^2(C) = 0$ so that the approximation of $U(X) = U(C)$ implies $C = \mu(X) - \frac{\sigma_-^2(X)}{\rho}$. The optimal objective value will be

$$1 + \rho_2 \frac{[(a + b)^2(p + b(1 - p))^2 + 2p(pa - (1 - p)b)(p + b(1 - p))(1 + b^2(p - 1)^2)]}{4(pb(p - 1))^2}$$

3.1.5 Two-period Example: Cooperative Investment

We continue to discuss the two-periods example introduced in section 3.1.3 with binomial tree. Now, suppose the two agents (investors) agree to invest their joint capital into the risky instrument. Then, divide the random variable X by the amount of money investors (agents) get at the end of the investment period, where y_1 and y_2 are the optimal allocation of the first and second agents, respectively, such that $X = y_1 + y_2$. Furthermore, the investors (agents) have different risk measures. For instance, variance for first investor and semi-variance for second investor or equivalently standard deviation and standard lower semi-deviation, respectively. Now, the portfolio optimisation for the individual investment for first and second investors is formulated as follows:

$$\begin{aligned} & \min_{x,y,z} \quad \sigma^2[X] \\ & \text{s.t.} \\ & E[X] \geq \pi_1 \\ & X \in \mathcal{F} \\ & -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1, \end{aligned} \tag{3.7}$$

Individual investment for the second investor

$$\begin{aligned}
& \min_{x,y,z} \quad \sigma_-^2[X] \\
& \text{s.t.} \\
& E[X] \geq \pi_2 \\
& X \in \mathcal{F} \\
& -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1,
\end{aligned} \tag{3.8}$$

Corresponding to the cooperative investment of the form

$$\begin{aligned}
& \min_{x,y,z} \quad \sigma^2[y_1] \\
& \text{s.t.} \\
& \sigma_-^2[y_2] \leq \beta \\
& E[y_1] \geq \pi_1 \\
& E[y_2] \geq \pi_2 \\
& y_1 + y_2 = X \\
& X \in \mathcal{F} \\
& -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1
\end{aligned} \tag{3.9}$$

Hence, we will solve the problems (3.9) as a cooperative investment in multi-period over the variables $(x, y$ and $z)$, and N is the number of nodes in the scenario tree at period t . In our example, we have binomial tree and two periods, then we will solve the problem again by using the recursive algorithm and break down the problem (3.9) into sub-problems at each node. Note that, if we have N nodes at time t , then we will have N sub-problems.

Now, to solve this problem with dynamic programming we will complete, which means a recursive manner. Furthermore, the other types of the problems (3.7), (3.8) and (3.9) can be written with constant absolute risk aversion during the investment period as follows:

1) Individual for first investor

$$\begin{aligned}
& \min_{x,y,z} \quad \sigma^2[X] - \alpha_1 E[X] \\
& \text{s.t.} \\
& X \in \mathcal{F} \\
& -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1
\end{aligned} \tag{3.10}$$

2) Individual for second investor

$$\begin{aligned}
& \min_{x,y,z} \quad \sigma_-^2[X] - \alpha_2 E[X] \\
& \text{s.t.} \\
& X \in \mathcal{F} \\
& -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1
\end{aligned} \tag{3.11}$$

3) Cooperative between two investors (agents)

$$\begin{aligned}
& \min_{x,y,z} \quad \sigma^2[y_1] - \alpha_1 E[y_1] \\
& \text{s.t.} \\
& \sigma_-^2[y_2] - \alpha_2 E[y_2] \leq \beta \\
& y_1 + y_2 = X \\
& X \in \mathcal{F} \\
& -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1
\end{aligned} \tag{3.12}$$

Note that, the portfolio positions are always between 0 and 1 if the short sales are not allowed. We decided to relax this assumption to the positions between -1 and 1 to allow reasonable short sales. However, we think that allowing unbounded short sales is unrealistic.

Algorithm 3.1:

Suppose we have two-periods with a binomial tree, which an agent may invest either into a risk-free asset with $r_0 = 0$, or into risky assets, where the rate of return after each period is an r.v. assuming values 2 and -1 with equal probabilities. That is, if we invest x in the risky asset, it either returns $3x$, or nothing. Hence, at the end of two-investment periods with a binomial tree, the possible investment strategies are completely described by three numbers, (x, y, z) where x is the proportion of money invested into the risky asset in the first period and y is the proportion of money invested into the risky asset in the second period, provided that its price went down during the first period, and z is the proportion of money invested into the risky asset in the second period, provided that its price went up during the first period. Also $\mathcal{F} = \{X = (X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)) \mid -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\}$, where we have four scenarios at the end of the investment represented as follows:

$$X(\omega_1) = 1 + 2x + 2z;$$

$$X(\omega_2) = 1 + 2x - z;$$

$X(\omega_3) = 1 - x + 2y$; and

$X(\omega_4) = 1 - x - y$.

a- In order to solve the problems (3.7) and (3.8), or for the other type (3.10) and (3.11), note that each of these problems has four scenarios solved by CVX in MATLAB over three variables x, y and z , where $\sigma^2(X) = \frac{1}{4} \sum_{i=1}^4 (X_i)^2 - (\frac{1}{4} \sum_{i=1}^4 X_i)^2$ or another individual investment problem defined with

$\sigma_-^2(X) = \frac{1}{4} \sum_{i=1}^4 (\max(-(X_i - (\frac{1}{4} \sum_{i=1}^4 X_i)), 0))^2$, $E[X] = \frac{1}{4} \sum_{i=1}^4 X_i$, and the result is that we get the optimal value for the investor individually.

b- In order to solve the problems (3.9), or for the other type (3.12) as a cooperative investment over the variables x, z and y and as a cooperative investment in multi-periods, note that we need to write the share division for each investor y_1 and y_2 as a vector of the same size of $X(\omega)$ in my case $y_1 = (y_1^1, y_1^2, y_1^3, y_1^4)$ and $y_2 = (y_2^1, y_2^2, y_2^3, y_2^4)$. Hence, in this cooperative investment we need to write $\sigma^2(y_1) = \frac{1}{4} \sum_{i=1}^4 (y_1^i)^2 - (\frac{1}{4} \sum_{i=1}^4 y_1^i)^2$ and $\sigma_-^2(y_2) = \frac{1}{4} \sum_{i=1}^4 (\max(-(y_2^i - (\frac{1}{4} \sum_{i=1}^4 y_2^i)), 0))^2$, $E[y_1] = \frac{1}{4} \sum_{i=1}^4 y_1^i$, and $E[y_2] = \frac{1}{4} \sum_{i=1}^4 y_2^i$. Then, plugging the value of $\sigma^2(y_1)$, $\sigma_-^2(y_2)$, $E[y_1]$ and $E[y_2]$ into problems (3.9) or (3.12) in order to solve portfolio optimization over the variables x, y and z with four scenarios $y_1^i + y_2^i = X(\omega_i)$ for $i = 1; 2; 3; 4$. Note that we need to choose β to represent the level of risk for a second investor, which is chosen to be less than the optimal value that we got from step **a**. Consequently, we get the optimal value for first investor which denote by $\sigma(\beta)$. Then, repeating the last step with a different value of β we get the whole efficient frontier which represents the optimal value for the first investor in the x-axis and optimal value for the second investor in the y-axis, which approximately represents the value of β .

c- In order to solve the problems again by using the recursive algorithm and break down the problems (3.9) or for the second type (3.12) into sub-problems at each node.

- **i-** We have a two-periods model with a binary tree. Then, we will have $N = 2$ sub-problems. In our case, since we have a binomial tree then we have two sub-problems as shown in **A** and **B**, shown below. Hence, we need to solve three portfolio optimization problems. Firstly, according to **A** we solve (3.12) or (3.9) as cooperative investments in a single period by CVX in MATLAB over variables x and z where we have just two scenarios $X(\omega_1) = 1 + 2x + 2z$ and $X(\omega_2) = 1 + 2x - z$. Note that $y_1 = (y_1^1, y_1^2)$ as well as $y_2 = (y_2^1, y_2^2)$, where $y_1^i + y_2^i = X(\omega_i)$ for $i = 1, 2$. Also, $\sigma^2(y_1) = \frac{1}{2} \sum_{i=1}^2 (y_1^i)^2 - (\frac{1}{2} \sum_{i=1}^2 y_1^i)^2$ and $\sigma_-^2(y_2) = \frac{1}{2} \sum_{i=1}^2 (\max(-(y_2^i - (\frac{1}{2} \sum_{i=1}^2 y_2^i)), 0))^2$, $E[y_1] = \frac{1}{2} \sum_{i=1}^2 y_1^i$, and

$E[y_2] = \frac{1}{2} \sum_{i=1}^2 y_2^i$. Then, plug in the value of $\sigma^2(y_1)$, $\sigma_-^2(y_2)$, $E[y_1]$ and $E[y_2]$ in problems (3.9) or (3.12) in order to solve portfolio optimization over the variables x and z in this stage we will get the value of trading strategy $z(X)$ as a function of x . Also, choose the value of β as exactly the same value of β in step **b**.

- **ii**– Similarly, according to **B** we will solve (3.12) or (3.9) as a cooperative investment in a single period by Lagrange multiplier over variables x and y where we have two r.v. in this case, which is represented as follows: $X(\omega_1) = 1 - x + 2y$ and $X(\omega_2) = 1 - x - y$ and similarly to the previous step $y_1^i + y_2^i = X(\omega_i)$ for $i = 1, 2$ in order to solve portfolio optimization over the variables x and y . In this stage we will get the value of the trading strategy $y(x)$ as a function of x . Also, choose the value of β as exactly the same value of β in step **b**.
- **iii**– Plug in the value of $z(x)$ and $y(x)$ in problems (3.9) or (3.12) and solve them again over one variable x . Note that, in this case we have four r.v. $x(\omega_i)$ for $i = 1, 2, 3, 4$, where $X(\omega_1) = 1 + 2x + 2z(x)$; $X(\omega_2) = 1 + 2x - z(x)$; $X(\omega_3) = 1 - x + 2y(x)$; and $X(\omega_4) = 1 - x - y(x)$. Then compute $\sigma^2(y_1) = \frac{1}{4} \sum_{i=1}^4 (y_1^i)^2 - (\frac{1}{4} \sum_{i=1}^4 y_1^i)^2$ and $\sigma_-^2(y_2) = \frac{1}{4} \sum_{i=1}^4 (\max(-(y_2^i - (\frac{1}{4} \sum_{i=1}^4 y_2^i)), 0))^2$, $E[y_1] = \frac{1}{4} \sum_{i=1}^4 y_1^i$, and $E[y_2] = \frac{1}{4} \sum_{i=1}^4 y_2^i$. Then, plug in the value of $\sigma^2(y_1)$, $\sigma_-^2(y_2)$, $E[y_1]$ and $E[y_2]$ in problems (3.9) or (3.12) in order to solve portfolio optimization over the variables x , and $y_1^i + y_2^i = X(\omega_i)$. Also, choose the value of β as exactly the same value of β in step **b**. Consequently, we get the optimal value for the first investor which is denoted by $\sigma(\beta)$. Then, repeating the last step with a different value of β we get the whole efficient frontier which represents the optimal value for the first investor in the x-axis and the optimal value for the second investor in the y-axis, which approximately represents the value of β .

d- Finally, we will compare between plotting the efficient frontier that comes from step (b) for solving CI and step(c) part(iii) for DCI, respectively.

Example 3.1

First of all, we will solve the problems (3.10) and (3.11) as individual investments and we will choose the $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$ as a risk aversion for first and second investor shown in the previous section. Then we will solve the problem (3.12) as a cooperative investment in two periods over the variables x, y and z . We will plot the efficient frontier by changing the value of β as shown in Figure

3.1. Since this efficient frontier is unstable, which means the mean-variance prob-

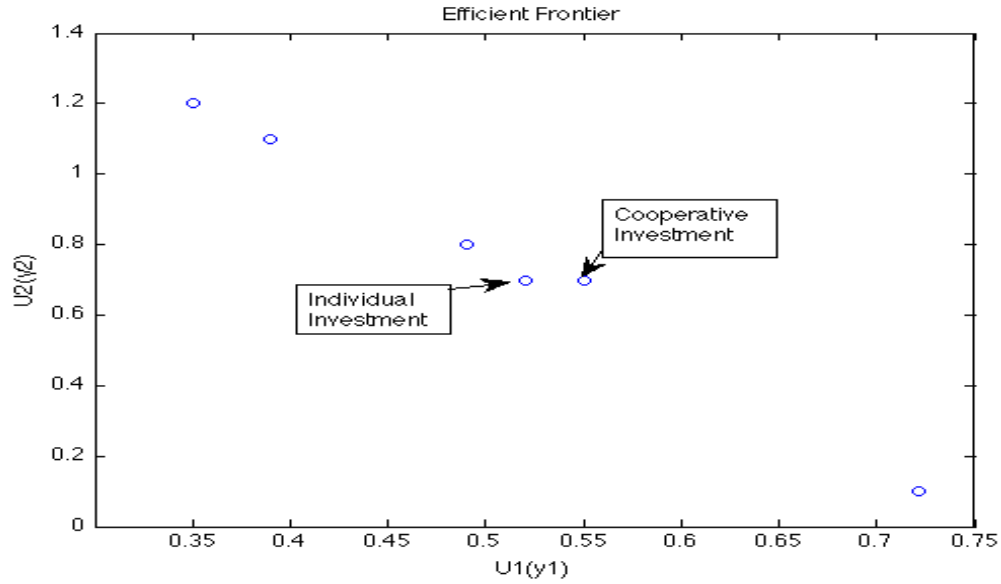
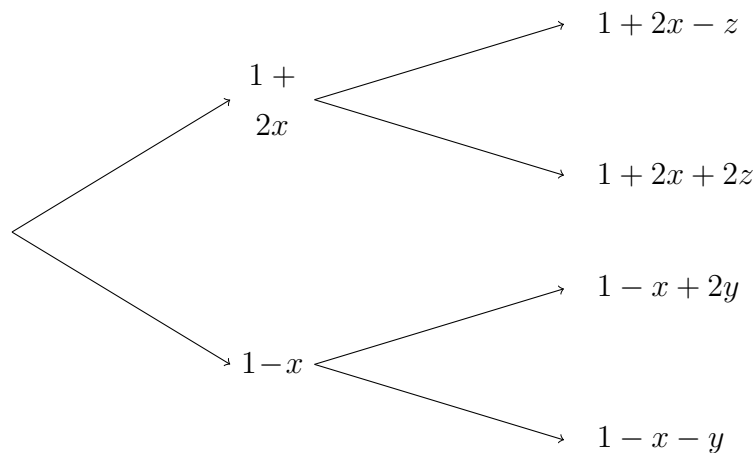


FIGURE 3.1: Example showing comparison between the optimal individual and cooperative investment solving problem (3.12) with $U(.) = -R$, and $R = D_i^2 - \alpha_i E[y_i]$ where $D_1 = \sigma$ and $\alpha_1 = 0.5$, and $D_2 = \sigma_-$ and $\alpha_2 = 0.25$

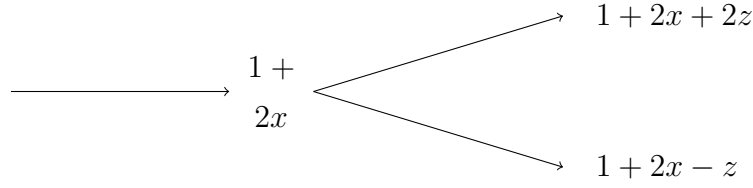
lem faced time inconsistency where the term of $E[X^2]$ from the definition of $\sigma(X)$ cause time inconsistent. Actually, the part $E[y_i^2]$, $i = 1, 2$ in the definition of $\sigma^2(y_1)$ and $\sigma_-^2(y_2)$ cause time-inconsistency. Thus, we resolved it by solving the problem (3.12) as in force in back technique as follows:

1) We will break down the scenario tree



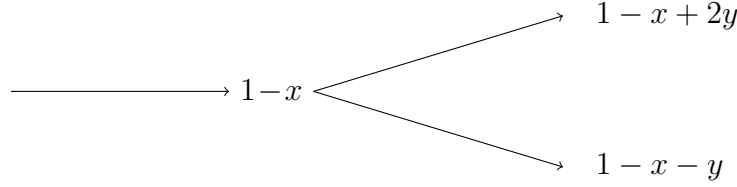
to solve the problem (3.12) into two sub-problems as shown in (A) and (B).

Note that, (A) and (B) as CI in single period and over the variables z and y respectively, Firstly, we take the problem (A).

A

Furthermore, we will solve the problem (3.12) as cooperative investment in single period as shown in **A** above, which means in this part we have only two scenarios. Also, we will solve the problem **A** over the variables x and z and fix $\beta = 0.7$ to get the value of z and x will be arbitrary. Then, we will resolve the same problem after fixing the value of z at $\beta = 0.7$ to get the efficient value of x at $\beta = 0.7$.

2) Similarly, we will break down the problem (3.12) into sub-problems as shown in **(B)**.

B

Note that in this part we have only two scenarios. Also, we will solve the problem **B** over the variables x and y and fix the $\beta = 0.7$ we got y and x will be arbitrary. Then, we will resolve the same problem after fix the value of y at $\beta = 0.7$ to get the efficient value of x at $\beta = 0.7$.

From the Algorithm 3.1 step c we have the value of $z = 0.5548$ and $y = 0.7807$ for arbitrary value at x . Then we plug the values of z and y into the problem (3.12). Hence, this problem will be over one variable x and we solve it again in order to get the optimal trading strategy x . Now, change the value of β in order to get the whole efficient frontier which is the optimal and stable efficient frontier for the whole problem as shown in Figure 3.2.

Definition 3.4. The efficient frontier is the set of points $(R_1, R_2) \in F$, where F is convex set that offers the best optimal value for investor i , $i = 1, 2$. Moreover, there exists a strategy such that the risk of investor i is R_i $i = 1, 2$ but cannot be improved upon, which means there is no point $(R'_1, R'_2) \in F$ such that $R'_1 \leq R_1$ and $R'_2 \leq R_2$, and at least one of inequality being strict.

Theorem 3.5. The set F for portfolio optimisation problem (3.3) in multi-period is convex, provided that R_1 and R_2 are convex.

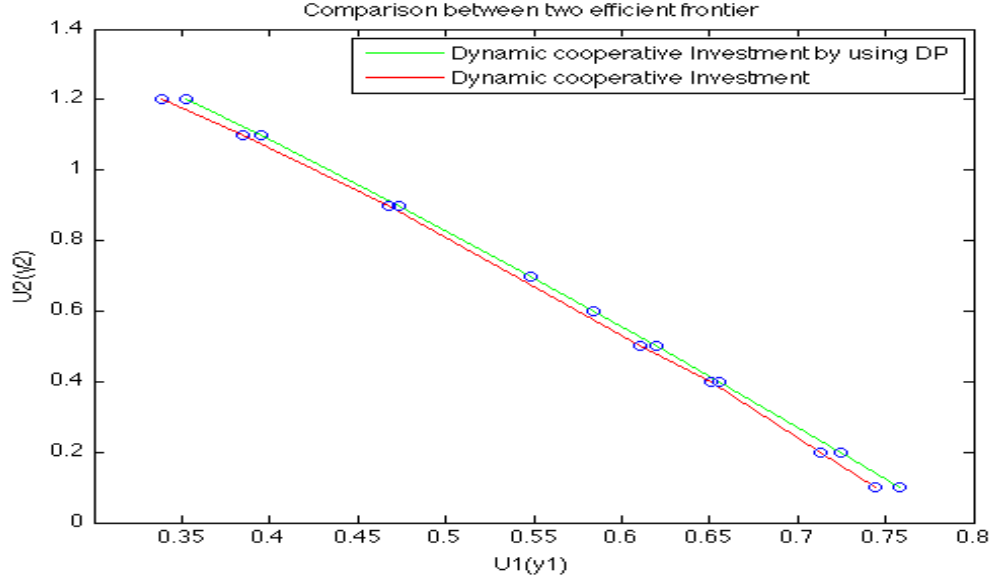


FIGURE 3.2: Efficient frontier curve for problem (3.12) with $U(.) = -R$, and $R = D_i^2 - \alpha_i E[y_i]$ where $D_1 = \sigma$ and $\alpha_1 = 0.5$, and $D_2 = \sigma_-$ and $\alpha_2 = 0.25$ where the curve of dynamic cooperative investment comes from solve cooperative investment in step **b** in Algorithm 3.1 and the curve dynamic cooperative investment by using DP comes from solve cooperative investment in recursive manner as in step **c** part **iii** in Algorithm 3.1. In addition, the difference between efficient frontier is not very significant. Thus, the investor follows the trading strategies come from solving problem (3.12) by using dynamic programming.

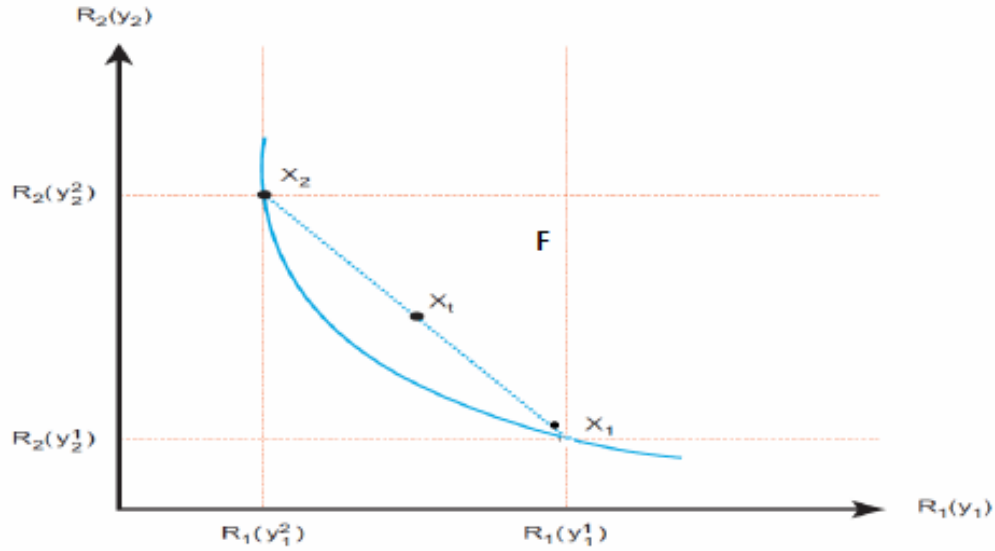
Proof. The proof for this theorem in single period is represented in De la Fuente, theorem 1.13 [48] and the proof for multi-period is similar.

□

3.1.6 Dynamic cooperative investment in real data

In this section we develop and present a novel technique for solving large-scale multi-period cooperative investment problems with real data, obtained as a result of historical simulation. We assume the scenario tree comes from the historical simulation. We will address several aspects of this problem.

- First, we need to check that the scenario tree obtained from the data is arbitrage free, otherwise the model would be unrealistic.

FIGURE 3.3: The efficient frontier of DCI with $T = 2$

- Second, we will find an optimal time-consistent trading strategy, so that the group of investors would not change their minds in the middle of the investment period. For this, we will use the dynamic programming approach.
- Third, we introduce an approximation procedure which allow us to avoid the exponential growth of scenario trees.

In this section we will study only the case of two investors; one uses the mean-variance approach to portfolio optimisation, and the second uses the mean-semi-variance since this problem faced time-inconsistent and we need to show how to solve it again by DP to get the time-consistent trading strategy. In this case, cooperative investment in multi-period can be written as the following optimisation

problem

$$\begin{aligned}
& \min_{x, z_{i,t}} \quad \sigma^2[y_1] \\
& \text{s.t.} \\
& \sigma_-^2[y_2] \leq \beta \\
& E[y_1] \geq \pi_1 \\
& E[y_2] \geq \pi_2 \\
& y_1 + y_2 = X \\
& X \in \mathcal{F} \\
& \sum_{i=1}^n z_{i,t} = 1 \\
& -1 \leq z_{i,t} \leq 1 \text{ for } t = 1, \dots, T
\end{aligned} \tag{3.13}$$

where π_1 is the fixed return level for the first investor, and π_2 is the fixed return level for the second investor and β is the fixed σ_- level for the second investor. The trading strategy z_t is a vector of function $z_t = \{z_{1,t}(\cdot), \dots, z_{n,t}(\cdot)\}$, T is the number of scenarios, and period t . The choice of variance and semi-variance in (3.13) is because these are the most popular risk measures used in portfolio optimization. The next algorithm defines the solution of problem (3.13).

Algorithm 3.2:

a- Let $r_{i,t}$, $i = 1, \dots, n$, $t = 1, \dots, T$ be the historical rate of return of instrument i at time t . We use them to generate the two-period scenario tree as follows. We assume that there are T scenarios at the first period, and then from each node T scenarios at the second period, totalling in T^2 scenarios at the second period. We denote these scenarios as (t_1, t_2) , $t_1 = 1, \dots, T$, $t_2 = 1, \dots, T$. Let $a_{(t_1, t_2), i}$ be the (predicted) rate of return of asset i under scenario (t_1, t_2) . Numbers $a_{(t_1, t_2), i}$ form a matrix A of size $T^2 \times n$. In this section, we use a historical simulation method, and assume that

$$a_{(t_1, t_2), i} = (1 + r_{t_1, i})(1 + r_{t_2, i}) - 1, \quad \forall t_1, t_2, i.$$

In the next section we will use the GARCH model for much better prediction for $a_{(t_1, t_2), i}$.

Next we do the following trick to reduce the number of scenarios. We sort the rows of matrix A such that

$$\frac{1}{n} \sum_{i=1}^n a_{h_1,i} \leq \frac{1}{n} \sum_{i=1}^n a_{h_2,i}$$

for every row $1 \leq h_1 \leq h_2 \leq T^2$. Then, to reduce the size of the scenario tree, we group ‘good’ scenarios with ‘good’ ones, and ‘bad’ scenarios with ‘bad’ ones, that is, take

$$b_{h,i} = \frac{1}{T} \sum_{t=(h-1)*T+1}^{hT} a_{t,i}, \quad h = 1, \dots, T, \quad i = 1, \dots, n.$$

where $b_{h,i}$ is the average of the matrix A after sort it. This reduces $T^2 \times n$ matrix from two-period simulation to just $T \times n$ matrix. Note that, we can do the same process for many time periods T if we want more periods by just repeating the same process for each time period. This step will let us to avoid exponential growth for the number of scenarios.

We have no formal proof that this approximation is acceptable, but it works surprisingly well in numerical computations. Namely, we solve the problems of moderate size exactly (without this averaging procedure), and then with it, and the resulting optimal portfolio agreed and coincide (in my experiment they are agreed up to 5-digit).

b- Now to solve cooperative investment after we set up the scenario tree with two-periods, where (r_i) is the rate of return for a risky asset i , $i = 1, \dots, n$ in first period, $b_{h,i}$ is the expected rate of return for asset i , $i = 1, \dots, n$ node h in the scenario tree, $h = 1, \dots, T$ and $x = (x_1, x_2, \dots, x_n)$ is the trading strategy at first period $t = 1$ and $z_t = (z_{1,t}, z_{2,t}, \dots, z_{n,t})$, $t = 1, \dots, T$ where each $z_{i,t}$ is a number of the trading strategy at each node for each risky asset i , $i = 1, \dots, n$, note that the number of nodes are equal to the number of scenarios $h = t = 1, \dots, T$ in my scenario tree and $-1 \leq z_{i,t} \leq 1$ and the initial capital is W_0 with risk-free r_0 .

- **i-** In order to solve the problems (3.13) as cooperative investment over the variables x and $z_t = (z_{1,t}, z_{2,t}, \dots, z_{n,t})$, $t = 1, \dots, T$ and where z_t is the trading strategy at node t , where $-1 \leq z_{i,t} \leq 1$ for all risky assets i , also $\sum_{i=1}^n z_i = 1$. Hence, we have T scenarios $X = (X(\omega_1), \dots, X(\omega_T))$ where $X(\omega_t) = (W_0 - \sum_{i=1}^n x)r_0 + xr' - \sum_{i=1}^n z_{i,t}r_0 + z_t b'_{h,i}$, for $t = 1, \dots, T$ and similarly as in Algorithm 3.1 we have $y_1 = (y_1^1, \dots, y_1^n)$ as well as $y_2 = (y_2^1, \dots, y_2^n)$, where $y_1^t + y_2^t = X(\omega_t)$ for $t = 1, 2, \dots, T$. Also, $\sigma^2(y_1) =$

$\frac{1}{n} \sum_{i=1}^n (y_1^i)^2 - (\frac{1}{n} \sum_{i=1}^n y_1^i)^2]$ and $\sigma_-^2(y_2) = E[(\max(-(y_2^i - (\frac{1}{n} \sum_{i=1}^n y_2^i)), 0))^2]$, $E[y_1] = \frac{1}{n} \sum_{i=1}^n y_1^i$, and $E[y_2] = \frac{1}{n} \sum_{i=1}^n y_2^i$. Then, plug in the value of $\sigma^2(y_1), \sigma_-^2(y_2), E[y_1]$ and $E[y_2]$ in problems (3.13) and solve it by CVX over the variables x and z_t , where the value of β is chosen to be less than the optimal value for the second investor from individual investment. Then, after we get the value of x and $z_t, t = 1, \dots, T$ and the optimal value for the first investor $\sigma(\beta)$. Then, repeat the last step with a different value of β to get the whole efficient frontier.

- **ii-** In order to solve the problems again by using the recursive way and break down the problems (3.13) into sub-problems in our case we have T sub-problems. Note that in this case we have the random variables $X(\omega_t) = (W_0 - \sum_{i=1}^n x)r_0 + xr' - \sum_{i=1}^n z_{i,t}r_0 + z_t a'_{(t_1, t_2), i}$, for $t = 1, \dots, T$ and similar to Algorithm 3.1. Fix the same value of β as in step **i**. Then, we will solve it as a cooperative investment (3.13) in a single period at each node individually. Note that, for these problems, we will solve them over the variable x and z_t where z_t is the trading strategy at node and has a size $1 \times n$ as a function of x which is formulated of the form $z_t = \{z_{1,t}(x), \dots, z_{i,t}(x) \dots z_{n,t}(x)\}$. Then repeat this process for each node to get all of trading strategy z_t as a function of x at each node in each period. Note that the number of nodes is equivalent to number of sub-problems.
- **iii-** Plug the $z_t(x)$ for $t = 1, \dots, T$ into the problem (3.13) in order to get the value of x which is the trading strategy at the first period. Similar to Algorithm 3.1, we have T scenarios of $X = (X(\omega_1), X(\omega_2), \dots, X(\omega_T))$, where $X(\omega_t) = (W_0 - \sum_{i=1}^n x_i)r_0 + xr' - \sum_{i=1}^n z_{i,t}(x)r_0 + z_t(x)b'_{h,i}$, for $t = 1, \dots, T$. In this stage we will solve it as a cooperative investment in multi-periods over one variable x . Fix the same value of β as in step **i**. Then, after we get the value of x and the optimal value for the first investor $\sigma(\beta)$ we can now repeat the last process with a different value of β to get the whole efficient frontier.

e- Finally, we will compare between the efficient frontier coming from steps (i) and (iii) to show the difference between the curves and how are they significant.

3.1.7 Dynamic cooperative investment with certainty equivalent

For cooperative investment portfolio optimisation our problem can be written as follows:

$$\begin{aligned}
 & \max_{X \in \mathcal{F}} \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1} \\
 & \text{s.t.} \\
 & \mu_2 - \frac{\sigma(y_2)^2}{\rho_2} \geq \pi \\
 & y_1 + y_2 = X \\
 & X \in \mathcal{F}
 \end{aligned} \tag{3.14}$$

In this problem we will solve the time inconsistent mean-variance model based on compound independence axioms. We will start with the last period and find the certainty equivalent for each scenario at each node for both investors $C_1 = \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1}$ and $C_2 = \mu_2 - \frac{\sigma(y_2)^2}{\rho_2}$ and actually $C_2 \approx \pi$ since we fixed it. Then Carry on the process for each scenario at each period. Note that we start with the last and use force in back technique until we arrive at the first period. We will solve the problem in two ways as illustrated in the following algorithm. Moreover, the reason to solve problem (3.14) by certainty equivalent is shown in section (3.1.2).

Algorithm 3.3:

First way

Solve the problem (3.14) globally which means writing the uncertain outcome for each scenario $X(\omega)$ exactly as in Algorithm 3.2 part (i) and $X(\omega) = (W_0 - \sum_{i=1}^n x_i)r_0 + x'r_i) - \sum_{i=1}^n z_1^i)r_0 + z_1'.r_{1,t_1}) - \sum_{i=1}^n z_2^i)r_0 + z_2'.r_{2,t_2})$ this is in the case of three periods $T = 3$ and complete the same for more period. Then, solve it over the trading strategy (x, z_1, z_2) ; in this alternative method of dynamic programming we are looking for the optimal value for each investor, where $C_1 = \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1}$ and $C_2 = \mu_2 - \frac{\sigma(y_2)^2}{\rho_2}$ and actually $C_2 \approx \pi$ according to problem (3.14), π is the fix level for the second investor and it is chosen to be greater than the optimal value for the individual investment for the second investor. Thus, we get the optimal value $U_1(\pi)$ for the first investor and repeat the same process for a different value of π to get the whole efficient frontier.

Second way

- **a)** We will solve the problem (3.14) by setting up a time consistent mean variance model based on compound independence axioms, so that we will solve the problem by the force in back technique and we will break down the problem (3.14) into T sub-problems at each period see Algorithm 3.2 the same as part (ii). But, in this case we are looking for the optimal value for each investor instead of the trading strategy (x, z_1, z_2) and will solve problem (3.14) for each node separately as cooperative investment in a single period. Thus, for the first scenario up to $T = 3$ we have $X_{t_3}(\omega) = (((W_0 - \sum_{i=1}^n x_i)r_0 + x'r_i) - \sum_{i=1}^n z_1^i)r_0 + z_1'.r_{1,t_1}) - \sum_{i=1}^n z_2^i)r_0 + z_2'.r_{2,t_2}$ and solve problem (3.14) over x, z_1, z_2 , where in this stage we get the trading strategy z_2 as a function of x and z_1 , but we are looking to find the optimal amount of cash for each investor, which is the optimal value for each investor $C_{1,t_3} = \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1}$ and $C_{2,t_3} = \mu_2 - \frac{\sigma(y_2)^2}{\rho_2}$ so that the whole amount of money that the investors have in this period at this node is $C_{t_3} = C_{1,t_3} + C_{2,t_3}$ and similarly we will complete recursively for the previous node; we have $X_{t_2}(\omega) = ((W_0 - \sum_{i=1}^n x_i)r_0 + x'r_i) - \sum_{i=1}^n z_1^i)r_0 + z_1'.r_{1,t_1})$ and again we will solve problem (3.14) over x, z_1 , where in this case the trading strategy z_1 will be as a function of x , but the same as previously in this alternative method we are looking at the optimal amount of cash for each investor $C_{1,t_2} = \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1}$ and $C_{2,t_2} = \mu_2 - \frac{\sigma(y_2)^2}{\rho_2}$ so that the whole amount of money that the investors have in this period at this node is $C_{t_2} = C_{1,t_2} + C_{2,t_2}$.
- **b)** Plug the value of C_{t_i} $i = 2, 3$ into the scenario tree from step **a** until reaching to the first period.
- **c)** Solve the problem (3.14) at the first period over one variable x which is the trading strategy at the first period; exactly the same procedure as in Algorithm 3.2 part (iii) where the uncertainty outcome $X(\omega)$ can be written in this case as follows: $X(\omega) = (W_0 - \sum_{i=1}^n x_i)r_0 + x'r_i + \sum_{i=2}^{T=3} C_{t_i}$ where W_0 is the initial capital and r_0 is risk-free. In this stage we get the optimal value for each investor C_1 and C_2 , where $C_1 = E(y_1) - \frac{\sigma(y_1)^2}{2\rho_1}$ which is represented on x-axis and $C_2 = E(y_2) - \frac{\sigma(y_2)^2}{\rho_2}$ which is represented on y-axis. Repeat the same process in this step for a different value of π to get the whole efficient frontier. More explanation for the Algorithm is shown in Figure 3.4.
- **d)** Comparing between the efficient frontier comes from the first way and the second way at step **c**.

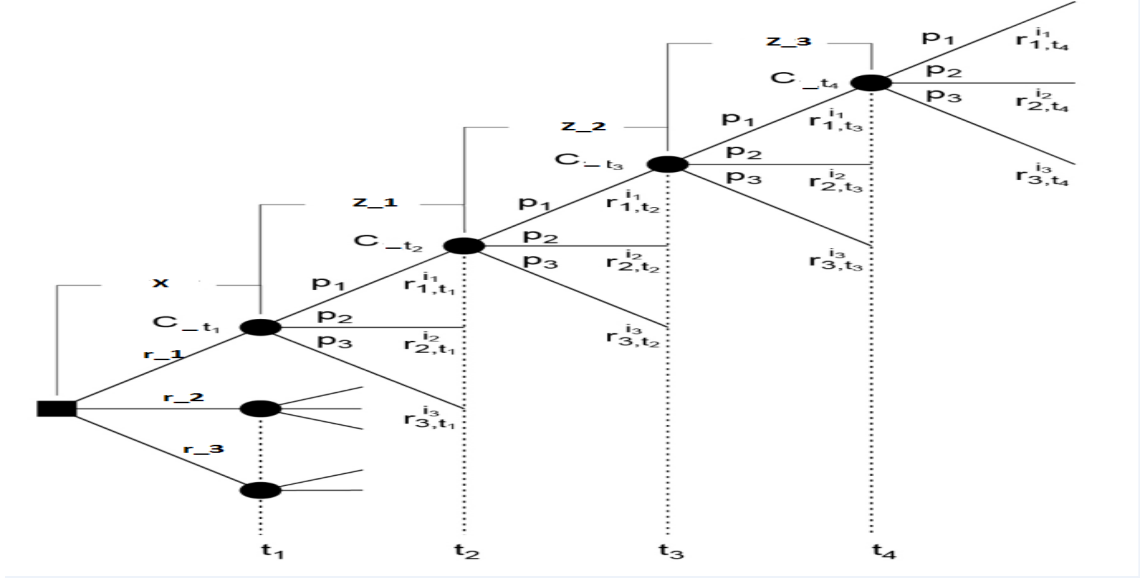


FIGURE 3.4: Scenario tree with certainty equivalent where P_i and r_{i,t_i} represent the probability and return at the asset i at period t_i and we assume that all scenarios has equivalent probability

3.2 Numerical experiments

In my following experiments we show that the difference between the curves comes from cooperative investment in multi-period with pre-commitment trading strategy and cooperative investment by dynamic programming with time-consistent trading strategy is not very significant.

- **First Experiment :**

Suppose we have one risk free asset and $n = 6$ risky assets

(APPL,APA,ABBU,ALL,BA,and BK) chosen randomly from *S&P 100* and each stock has a weekly return from 1-1-2010 to 1-1-2013. Thus, each risky asset has 100 scenarios and $T = 2$ periods and we will solve the problem (3.13) according to Algorithm 3.2 with utility function $\sigma^2(y_i) - \alpha_i E[y_i]$, where σ for the first investor and σ_- for the second investor.

Now, by solving (3.13) and suppose that $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$, we will get the efficient frontier for cooperative investment in multi-period directly and in the dynamic programming case as shown in Figure 3.5.

- **Second Experiment :**

We consider a portfolio consisting of $n = 6$ risky assets, the same in the first experiment. We will solve the problem (3.13) for utility function $D^2(y_i) -$

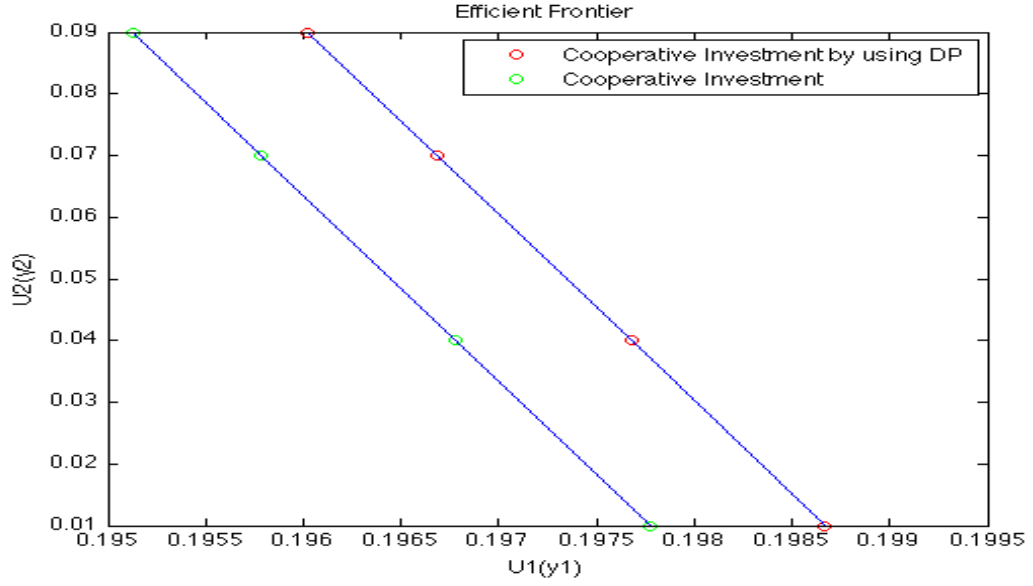


FIGURE 3.5: Comparison between efficient frontier CI and DCI (3.13) with $T = 2$ and utility function $D^2(y_i) - \alpha_i E[y_i]$ where $D = \sigma$ for the first investor and $D = \sigma_-$ for the second investor

$\alpha_i E[y_i]$ where $D = \sigma$ for the first investor and $D = \sigma_-$ for the second investor and in binary scenario tree where we have $T = 4$ periods. Note that the trading strategy $z_{i,t}$ is a function of $T - 1$ variable the arbitrary according to the optimisation problem, where $t = 2, 3, 4$. we follow the trading Algorithm 3.2 and do not need to reduce the number of scenarios since the number is not too great. With the same $r_0 = 0.000136$, $\pi_1 = 0.00016$ and $\pi_2 = 0.00019$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$. Thus, the result we get is shown in Figure 3.6.

- **Third Experiment :**

We will solve the problem (3.13) and suppose we have one risk free and 6 stocks and each observation (rate of return) for each stock has 30 scenarios. Then, we will fix the values of π_1 and π_2 which are the expected return for first and second investors, respectively. In our case we assume that risk free $r_0 = 0.00013$, $\pi_1 = 0.00016$ and $\pi_2 = 0.00019$, where $\pi_1 + \pi_2 > r_0$. Otherwise, the investor will not invest in risk assets any more if the risk free assets give them more return than risky assets and $T = 2$. By following the Algorithm 3.2 we get the Figure 3.7. Then, find trading strategy $z_{i,t,h}$ recursively until we get $z_{i,t_1} = (z_1(z_{0,t_1}), \dots, z_T(z_{0,t_1}))$ as a function of x which is the trading strategy from period $t = 0$ to $t = 1$.

- **Fourth Experiment :**

We will solve the problem (3.13) and suppose we have one risk free and 6

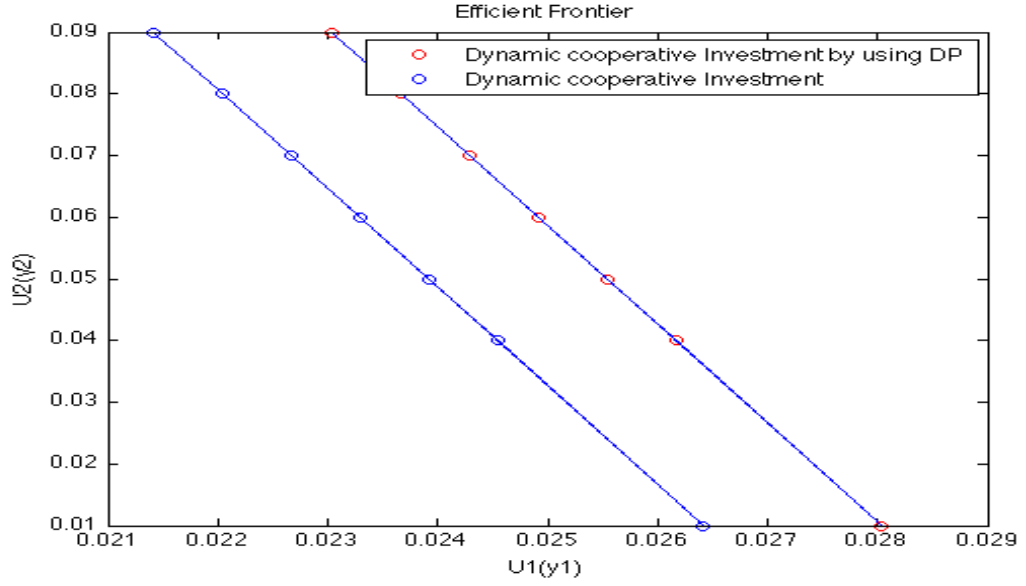


FIGURE 3.6: Comparison between efficient frontier CI and DCI (3.13) with $T = 4$ and utility function $D^2(y_i) - \alpha_i E[y_i]$ where $D = \sigma$ for the first investor and $D = \sigma_-$ for the second investor

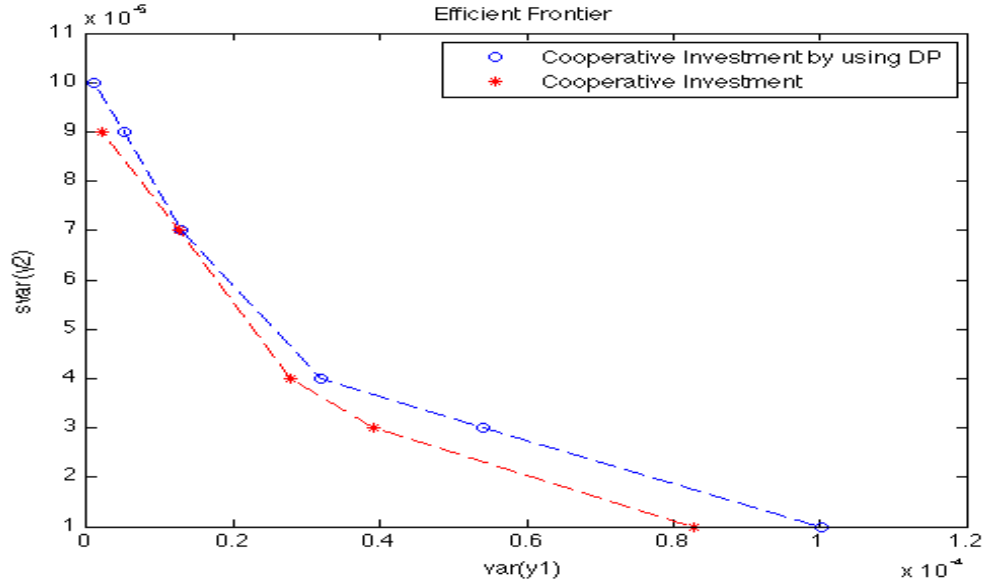


FIGURE 3.7: Comparison between efficient frontier CI and DCI (3.13) with 30 scenarios at each node and $T = 2$

stocks that arrange in a binary tree and $T = 2$ periods. The same as in the third experiment we will fix the values of π_1 and π_2 which are the expected returns for first and second investors, respectively. Note that the trading strategy $z_{i,t}$ is a function of z_0 , at t_1 variable we can select the arbitrary according to the optimisation problem. Moreover, at the end we will plug in

all the trading strategies $z_{i,t}$ to get the optimal trading strategy z_{0,t_1} between the period $t = 0$ and $t = 1$. Hence, the result is as shown in Figure 3.8.

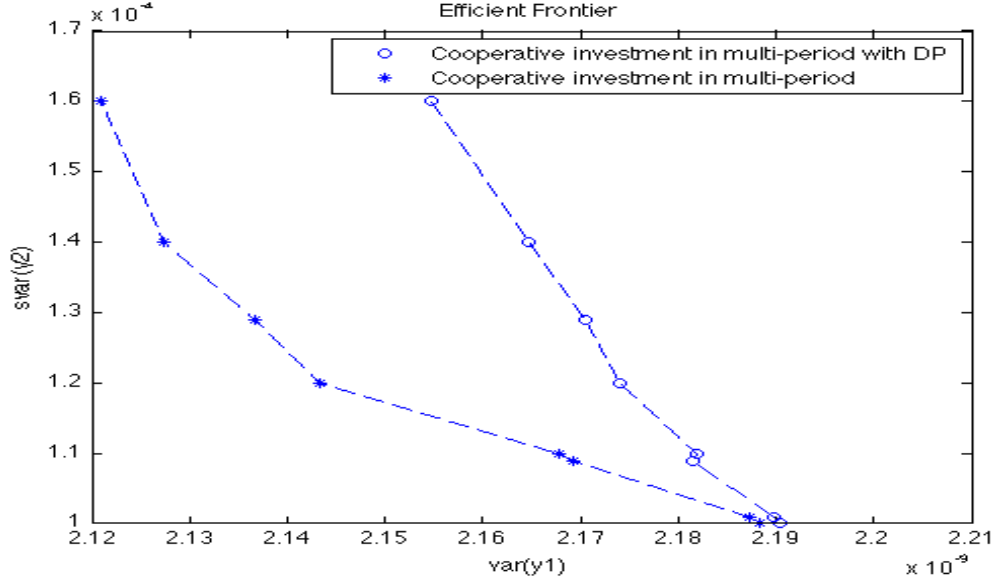


FIGURE 3.8: Comparison between efficient frontier CI and DCI (3.13) with 120 scenarios at each node and $T = 2$

- **Fifth Experiment :** We will solve the problem (3.13) and suppose we have one risk free and 6 stocks that arrange in a historical simulation and $T = 10$ periods. The same as in the third experiment, we will fix the values of π_1 and π_2 which are the expected returns for first and second investors, respectively. Note that the trading strategy $z_{i,t}$ is a function of $T - 1$ variable and we can select the arbitrary according to the optimisation problem. Moreover, at the end we will plug in all the trading strategies $z_{i,t}$ to get the optimal trading strategy z_{0,t_1} between the period $t = 0$ and $t = 1$. Hence, the result is as shown in Figure 3.9.

- **Sixth Experiment :**

We will solve the problem (3.14) and suppose we have one risk free and 6 stocks that are arranged 30 scenarios and $T = 2$ periods. The same as in the third experiment, we will fix the values of π , namely the expected returns for second investors. Moreover, π is chosen to be greater than the optimal value for the second investor when he/she solve the individual investment. By following the Algorithm 3.3 to get the optimal value for each investor and plot the efficient frontier for the optimal value for first investor C_1 which is

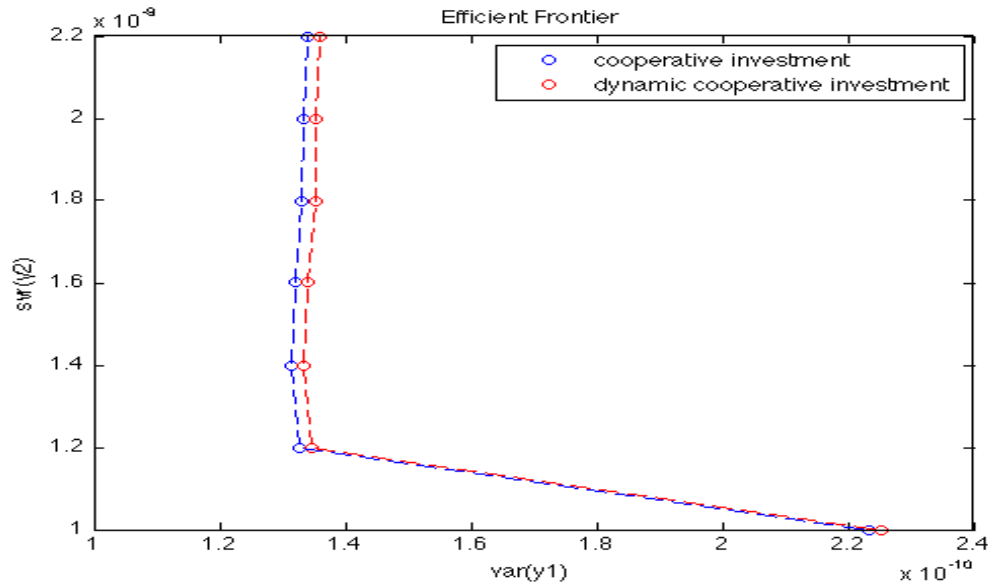


FIGURE 3.9: Comparison between efficient frontier CI and DCI (3.13) with 30 scenario at each node and $T = 10$, where the cooperative investment curve means solving cooperative investment in multi-period to find global solution, while dynamic cooperative investment curve means solve cooperative investment in in multi-period by using DP

represented on the x-axis and the optimal value for second investor C_2 which is represented on the y-axis. Hence, the result is as shown in Figure 3.10.

In addition, there are the same general notes from all experiments. We note the difference between the efficient frontiers is not very significant, which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment period.

3.3 Discussion and concluding remarks

In this chapter we formulated and solved (numerically) the cooperative investment problem (3.13) and (3.14) by following Algorithm 3.1, 3.2, and 3.3 in two ways: (i) finding an optimal pre-commitment strategy (global solution), which, however, may be unstable (time-in consistent); and (ii) finding an optimal strategy using a dynamic programming approach. This strategy is stable (time-consistent) during the investment periods. Consequently, we avoid breakdown of the contract

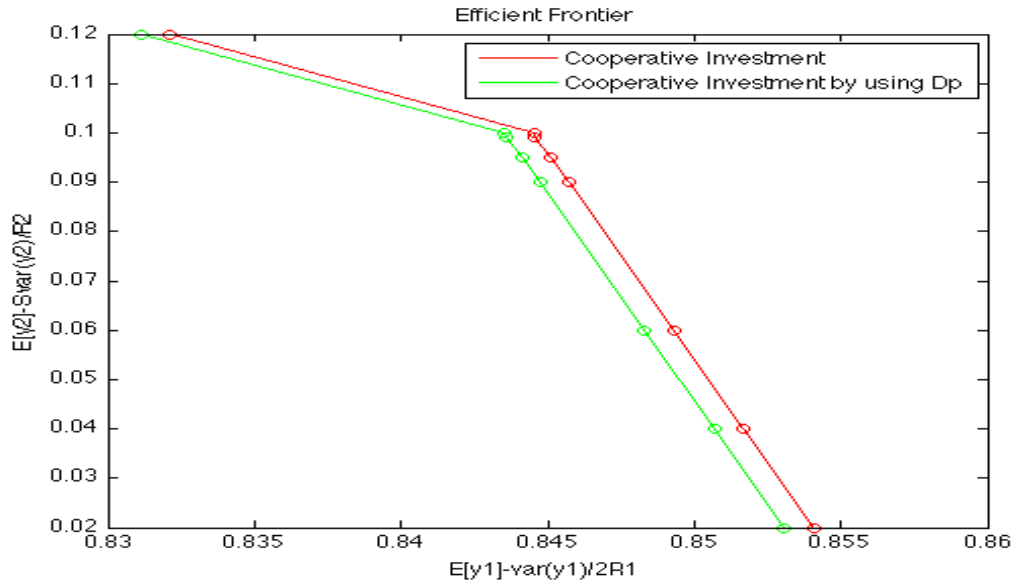


FIGURE 3.10: Comparison between efficient frontier CI and DCI (3.14) with 30 scenarios at each node and $T = 2$ with certainty equivalent

between the investors. Thus, in my contribution of this thesis we show how to solve cooperative investment in multi-period and how to get the stable trading strategies that let the investor follows until the end of investment periods.

We have observed from the Figures (3.2), (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) that the efficient frontier which comes from the dynamic programming model is relatively close to the efficient frontier coming from the global solution of cooperative investment. Note that the efficient frontier shown in Figures (3.5) and (3.6) are straight lines since we use the utility function $U(\mu, \sigma)$ which is positive homogeneous. So that in this chapter we focus on mean-variance portfolio optimisation (3.13) for risk measure $D^2(y_i)$ where $D = \sigma$ for the first investor and $D = \sigma_-$ for the second investor. Moreover, when we apply certainty equivalent to solve dynamic cooperative investment we have less computational procedure in the case of a recursive manner technique.

Chapter 4

Dynamic Cooperative Investment on GARCH Model

The majority of this Chapter has been published in Almualim [6].

In the previous chapter, all numerical examples for solving the dynamic cooperative investment (DCI) problem used a naive historical simulation method: we assumed that future rate of return of every financial instrument follows the distribution created using historical data. In this Chapter we consider a more realistic model, in which the distribution of future rates of return follow the Generalized Autoregressive Conditional Heteroskedastic (GARCH) model. The model has been introduced by Bollerslev [29], has two parameters (lags) p and q , and the usual notation is **GARCH**(\mathbf{p}, \mathbf{q}). The model has been designed in order to capture the volatility clustering effect in return. Furthermore, $GARCH(1, 1)$ is considered as one of the most common and simplest ways to produce and estimate future levels of volatilities as random process r.v. Moreover, $GARCH(1, 1)$ can model the dependence in the square returns which means the random process for variance is written in terms of square returns. In addition, in continuous-time, many economic studies have documented that financial time series tend to be highly heteroskedastic. Hence, we will develop the volatility process to have the autocorrelation function of a continuous time and the term of diffusion approximation is also existing based on Nelson [110] and Klüppelberg [89].

4.1 Problem formulation in discrete time

In this section we study how to apply the GARCH model as the economic model in order to estimate conditional variance as a random variable (r.v) and then generate future return of the stock price. Firstly, the basic idea we want to describe here is a volatility σ_t as a random variable (r.v) ‘volatility clustering of, e.g., an asset price, on day t as estimated at the end of the previous day $t - 1$ where the variance will be as a random variable rather than constant as in previous chapters. As well as studying how to do it in the most accurate and easiest way, before to begin we need to introduce same conceptions and definitions.

Definition 4.1. [35] Let $\{Y_t\}$ be a time series with $EY_t^2 < \infty$, $t = 1, 2, 3, \dots$. The mean function of Y_t is

$$\mu_Y(t) = E[Y_t]$$

The covariance function of Y_t is

$$\gamma_Y(r, s) = Cov(Y_r, Y_s) = E[(Y_r - \mu_Y(r))(Y_s - \mu_Y(s))]$$

for all integer r, s and t .

Definition 4.2. [35] $\{Y_t\}$ is weakly stationary, or just *stationary*, if

- μ_Y is independent of t ; and
- $\gamma_Y(t + h, t)$ is independent of t for each h .

Definition 4.3. [35] Let $\{Y_t\}$ be a stationary time series. The autocovariance function (ACVF) of Y_t

$$\gamma_Y(h) = Cov(Y_{t+h}, Y_t)$$

The autocorrelation function (ACF) of Y_t is

$$\rho_Y(h) = \frac{\gamma_Y(h)}{\gamma_Y(0)} = Cor(Y_{t+h}, Y_t)$$

Note that, the partial autocorrelation function (PACF) can be interpreted as a regression of the series against its past lags, which is consider only the direct correlation between Y_t and Y_{t-h} . Thus

$$PACF(k) = Cor(Y_t, Y_{t-k})$$

Definition 4.4. [35] A time series is called a white noise if it is a sequence of independent and identically distributed random variables with finite mean and variance.

In this thesis we will use the white noise model with zero mean, $WN \in N(0, \sigma^2)$.

Definition 4.5. [75] Backward shift operator is a short hand for shift backward in the time series $\beta Y_t = Y_{t-1}$, $\beta^p Y_t = Y_{t-p}$.

The time series of rates of return $\frac{p_f - p_c}{p_c}$, where p_c is the current price of the asset, and p_f is its (random) future price, is usually not stationary, and is therefore difficult to forecast. For this reason, in this section we will work with log-return $r = \log \frac{p_f}{p_c}$ instead. Let Y_t be the log-return of a financial instrument at time t . To effectively forecast Y_t , we will use the Generalized AutoRegressive Conditional Heteroscedasticity $GARCH(p, q)$ model which was described by Bollerslev [29] in 1986.

Generally, $GARCH(p, q)$ is the Generalized Autoregressive Conditional Heteroskedastic of lag p and lag q . Let the time series estimate be $Y_t = \mu_Y + \sigma_t \epsilon_t$, where $\epsilon_t \in N(0, 1)$ is a white noise, and σ_t can be found from the following formula for the variance which is a random variable and can be estimated as

$$\sigma_t^2 = \alpha + \beta_i \sum_{i=1}^p \sigma_{t-i}^2 + \gamma_j \sum_{j=1}^q a_{t-j}^2$$

where all the parameters α, β_i and γ_j are ≥ 0 for $i = 1, \dots, p$ and $j = 1, \dots, q$. Also, $\sum_{i=1}^p \beta_i + \sum_{j=1}^q \gamma_j < 1$, σ_{t-i}^2 is estimated by using EWMA, see remark 4.7 below, and $a_{t-i} = \sigma_{t-i} \epsilon_t$ is called the residual part.

Definition 4.6. [96] $GARCH(1, 1)$ is the Generalized Autoregressive Conditional Heteroskedastic of lag 1 and lag 1. Let the time series estimate be $Y_t = \mu_Y + \sigma_t \epsilon_t$, where σ_t can be found from the following formula for the variance

$$\sigma_t^2 = \alpha + \beta \sigma_{t-1}^2 + \gamma a_{t-1}^2$$

where all the parameters α, β and γ are ≥ 0 . Also, $\beta + \gamma < 1$, and $a_{t-1}^2 = (\sigma_{t-1} \epsilon_t)^2$ is called the residual part.

In our experiments, we choose $GARCH(1, 1)$ since it is the common way to estimate variance as a random process. To apply $GARCH(1, 1)$ we need to test the stationary and correlation of the observation Y_t as follows.

Firstly : Testing stationary of $\{Y_t\}$ by checking about the stationary by using Augmented Dickey Fuller test in MATLAB/ STATISTIC TOOLBOX [69].

Secondly : We can check qualitatively for correlation in the raw time series $\{Y_t\}$ in MATLAB) plot ACF (Autocorrelation Function) and PACF (Partial Autocorrelation Function) as shown in definition (4.3).

Thirdly : We need to decide whether to use ARCH or GARCH. The standard test for this is Engle's test [69]. The test always returns either 1 or 0. If the test returns 0, then you can use a simpler ARCH model. However, if the test returns 1, then the ARCH model is unacceptable, and you need to use more general GARCH model. Thus, the ARCH model is written as

$$\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2$$

and generalised this method is called the GARCH model

$$\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + \gamma a_{t-1}^2$$

Remark 4.7. In this remark we will show how to apply exponential weighting moving average (EWMA) as well as how to estimate the parameters in the GARCH formula in definition (4.6)

- **EWMA** [72] σ_{t-1}^2 can be found it by using *EWMA* (Exponential Weight Moving Average) and then use it to estimate the variance σ_t^2 by *GARCH*(1, 1). It is considered one of the alternative models in a separate class of exponential smoothing models. EWMA is introduced by λ , which is called a smooth parameter. Actually, it has some attractive properties such as it placing a great weight upon recent observations, as well as some drawbacks such as arbitrary decay factors that introduce the subject into the estimation.

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1 - \lambda)Y_{t-1}^2 \quad (4.1)$$

where $\lambda < 1$ is an input parameter. Hence today's variance is a function of yesterday's weighted variance and yesterdays' weighted, squared return: where σ_{t-1} can be calculated using the same formula (4.1): $\sigma_{t-1}^2 = \lambda\sigma_{t-2}^2 + (1 - \lambda)Y_{t-2}^2$. Substituting this into (4.1), we get

$$\sigma_t^2 = \lambda^2\sigma_{t-2}^2 + \lambda(1 - \lambda)Y_{t-2}^2 + (1 - \lambda)Y_{t-1}^2,$$

where σ_{t-2}^2 can be again estimated in the same way. After m such iterations, we get an expression of σ_t^2 as a function of σ_{t-m}^2 and Y_{t-1}, \dots, Y_{t-m} . For more explanation see David Harper [72].

- **Parmeters Estimation** : We will estimate the parameters of $GARCH(1, 1)$ by using the maximum likelihood principle, see Yang [145].

This can be done numerically by MATLAB/STATISTIC TOOLBOX [69] by using two following steps

ToEstMdl = garch(1,1);

EstMdl = estimate(ToEstMdl, {Y_t}).

Algorithm 4.1 : This algorithm explains how to apply $GARCH(1, 1)$ on data where $r_{i,1}, \dots, r_{i,T}$ are realisations of the stationary process $Y_{i,t}$ for each instrument i .

- 1- To begin we need to test the observation $r_{i,t}$ to see if it is stationary or not by using Augmented Dickey Fuller test in MATLAB/ STATISTIC TOOLBOX [69]. If it is not stationery, we conclude that this instrument is too difficult to forecast, and therefore exclude it from consideration. If it is stationary, proceed to step 2.
- 2- Test series r_{i1}, \dots, r_{iT} for correlation by plotting ACF (Autocorrelation Function) or PACF (Partial Autocorrelation Function) in MATLAB. If the correlation in r_{i1}, \dots, r_{iT} is not significant, we can check it for the series $r_{i1}^2, \dots, r_{iT}^2$ instead, see [35].
- 3- Next we decide which model to use: ARCH or GARCH by using Engle's ARCH test to detect the presence of ARCH effects. The corresponding MATLAB command is $[h, p, fstat, crit] = archtest(r_{it}, 'lags', 1)$ with more explanation in MATLAB/ STATISTIC TOOLBOX[69].
- 4- After all previous steps hold we can estimate σ_t^2 by $GARCH(1, 1)$; we use the formula

$$\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + \gamma a_{t-1}^2 \quad (4.2)$$

where σ_{t-1}^2 can be found by using EWMA from the formula $\sigma_{t-1}^2 = \lambda\sigma_{t-2}^2 + (1 - \lambda)r_{i,t-2}^2$, where σ_{t-2}^2 can be found from the same formula and so on, in a recursive way. Then, plug in the value of σ_{t-1}^2 in (4.2) to estimate σ_t^2 by $GARCH$. In addition, if the test on step 3 shows to use ARCH, we can use the same formula (4.2) with $\gamma = 0$. Thus, after we fit σ_{t-1}^2 by using EWMA, we can estimate σ_t^2 by (4.2).

- 5- Then, estimate all the parameters using likelihood method

ToEstMdl = garch(1,1);

EstMdl = estimate(ToEstMdl, r_{it}) Then, plug all of them in equation (4.2) and we get the variance σ_t^2 .

6- Estimating the $r_{i,T+t}$ for $t = 1, \dots, n$, where $r_{i,T+t} = \mu_i + \sigma_{i,T+t}\epsilon_t$, the average mean $\mu_i = \frac{1}{T} \sum_{t=1}^T r_{i,t}$ and is the white noise $\epsilon_t \in N(0, 1)$.

7- Do the same process for each instrument i .

Example 4.1. In this example, we choose the stock price BK chosen randomly from S&P 100 from 1/1/2011 to 1/1/2012. daily log return. Then, we need to check about the stationary of the observation. To solve this example:

- Firstly, we will check about the stationary time for the rate of return of stock price BK. According to Algorithm (4.1) step (1) we observed that the rate of return of stock price BK is stationary.
- Secondly, we will check about the qualitative for correlation in the raw return series by plotting the autocorrelation function (ACF) and partial-autocorrelation function(PACF), respectively. Note that the partial autocorrelation function (PACF) can be interpreted as a regression of the series against its past lags.

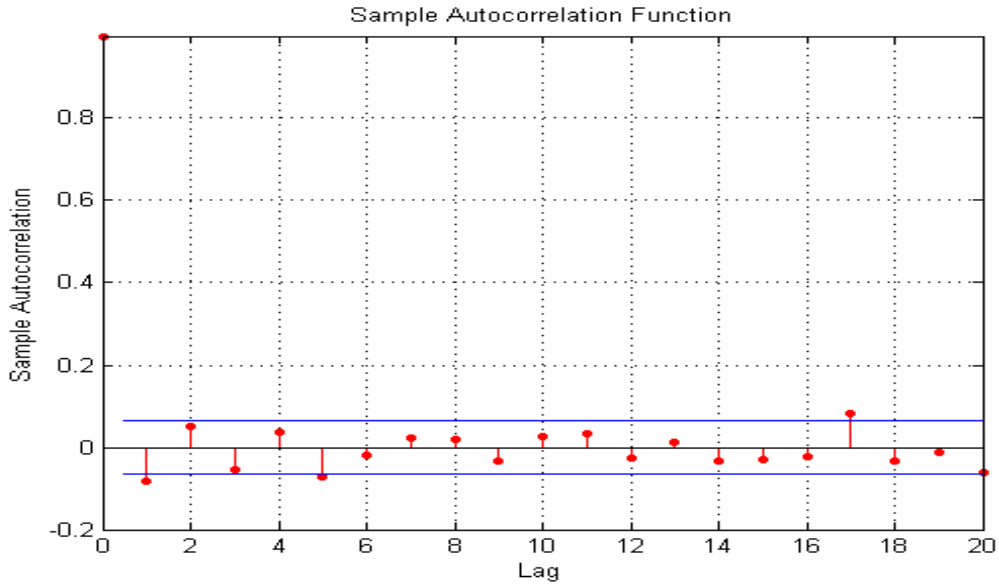


FIGURE 4.1: Sample autocorrelation function

and plot PACF (Partial Autocorrelation Function): The Autocorrelation Function (ACF) computes it and the plot of it displays the sample ACF of the daily log

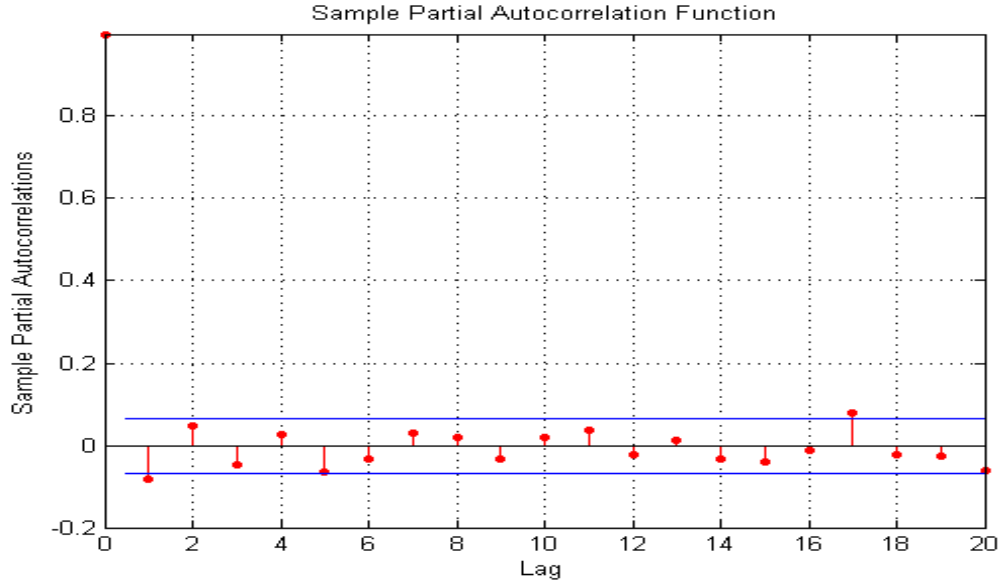


FIGURE 4.2: Sample partial autocorrelation function

returns, along with the upper and lower standard deviation confidence bounds, based on the assumption that all autocorrelations are zero beyond lag zero. Then, similarly, the plot of Partial Autocorrelation Function (PACF) displays PACF with upper and lower confidence bounds. Although the ACF of the observed returns exhibits little correlation, the ACF of the square returns may still indicate significant correlation and persistence in the second-order moments. Notice that for *ARCH* process *ARCH* effect, the process is non-correlated but, square process r^2 has significant correlation, as shown for r^2 in the Figure 4.3. The square return dies out slowly, indicating the variance process may be non-stationary. To make sure about stationary, we will use Engle *ARCH* by Engle [58]. Now we need to detect ARCH effects by using Engle's test for the presence of Arch effects by applying this test in GARCH ToolBox/MATLAB [69]. If the output $H=1$, it means that significant correlation exists. Consequently, we can complete our example to find σ_t^2 by using $GARCH(1,1)$ in order to get the volatility clustering, where the parameters are estimated by by using maximum likelihood by Yang see [145], are $\alpha = 2.5459 \times 10^{-1}$, $\beta = 0.917453$ for *GARCH* term at lag(1), $\gamma = 0.0449573$ for *ARCH* term at lag(1).

Hence, the the variance will be shown in Figure 4.4. Thus, according to Figure 4.4 homoskedasticity is satisfied which means that the computed variance of data is not constant over the time. In fact, time periods with high (low) volatility are followed by time periods with high (low) volatility.

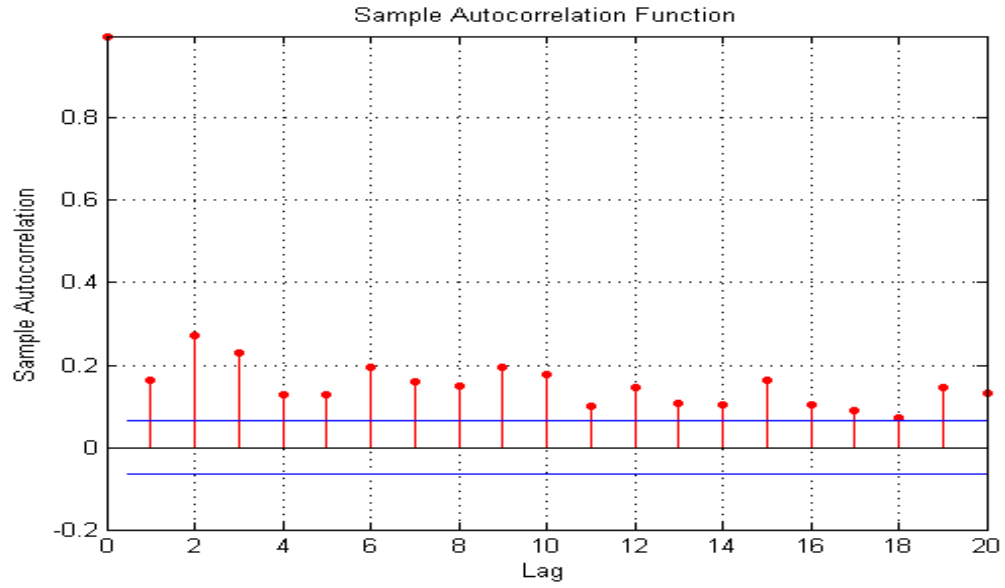


FIGURE 4.3: Sample PACF for r^2 , has significant correlation which is persistence in the second order moments.

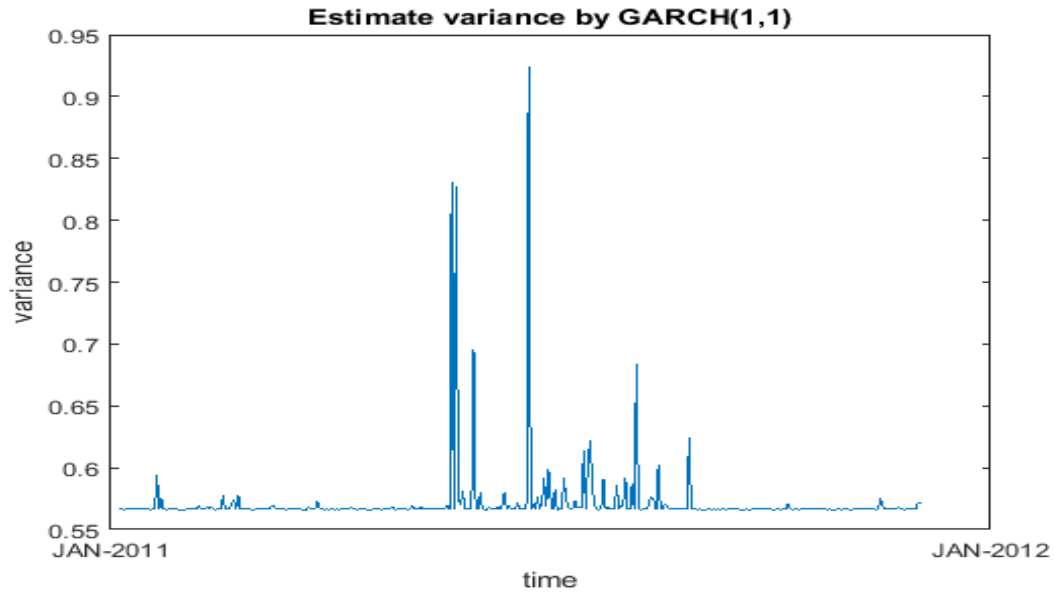


FIGURE 4.4: Estimated σ_t^2 by GARCH(1,1) model

Finally, the return for the stock price by using $GARCH(1,1)$ is shown in Figure 4.5. Now after we fit the σ_t^2 we need to plug it into r_t for each stock price. Then, we will composite the return of portfolio.

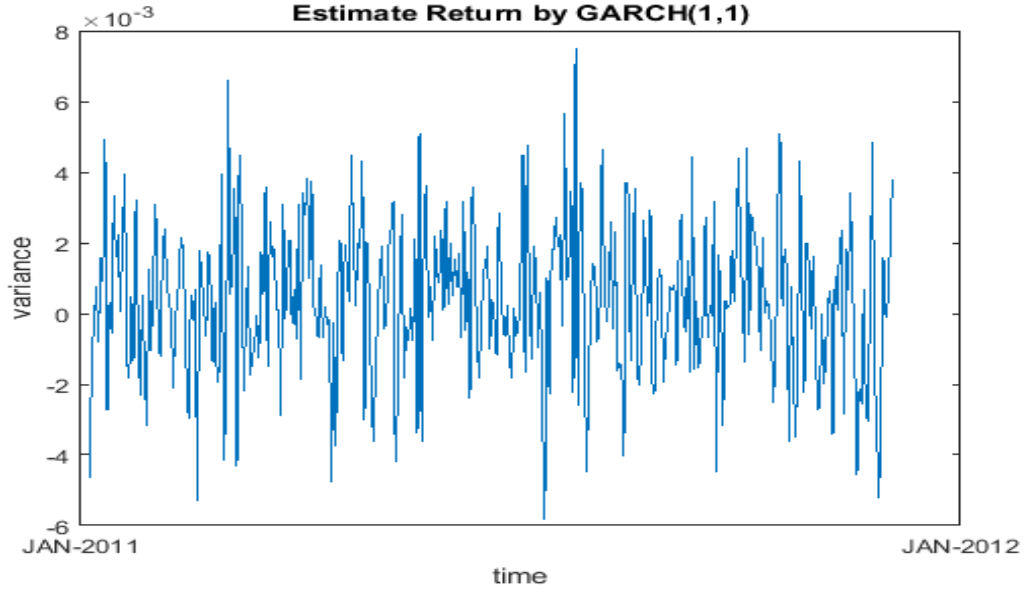


FIGURE 4.5: Estimated return r_t by GARCH(1,1) model, stock BK chosen from S&P 100 from January, 2011 to January 2012

4.1.1 Generating scenario tree

We need to generate the scenario tree after estimating the future return and conditional variance by using the $GARCH(1,1)$ model. Then, we will generate and construct a scenario tree that is a calibrated representation of the randomness in risky asset returns; see [50]. For portfolios that consist of n risky assets after we fit r_t by $GARCH$ model, we need to design a scenario tree in two ways, as follows:

- Simulation and randomised clustering approach “sequential simulation”; see Glupiner et al. [70].
 - 1) (Initialisation) create a root node, with T scenarios. Let $r_{i,t}$, $i = 1, \dots, n$, $t = 1, \dots, T$ be the rate of return of instrument i at time t which is estimated by $GARCH$ model.
 - 2) (Simulation) we use $r_{i,t}$ to generate the next-period scenario tree as follows. We assume that there are T scenarios at the first period, and then from each node T scenarios at the second period, totalling in T^2 scenarios at the second period. We denote these scenarios as (t_1, t_2) , $t_1 = 1, \dots, T$, $t_2 = 1, \dots, T$. Let $a_{(t_1, t_2), i}$ be the (predicted) rate of return of asset i under scenario (t_1, t_2) . Numbers $a_{(t_1, t_2), i}$ form a matrix A of size $T^2 \times n$. In this step, we use a historical simulation method,

and assume that

$$a_{(t_1, t_2), i} = (1 + r_{t_1, i})(1 + r_{t_2, i}) - 1, \quad \forall t_1, t_2, i.$$

Hence, we have a matrix of size $T^2 \times n$.

- **3)** (Randomised seeds) randomly choose a number of distinct scenarios b_1, \dots, b_T from matrix A around which to cluster the rest corresponding to b_j , $j = 1, \dots, T$, where the size of b_j is $1 \times n$ and n is number of risky assets.
- **4)** (clustering) group each scenario b_j , $j = 1, \dots, T$ with each scenarios from step **3** to which it is the closest. In an other words, we calculate euclidean distance between each scenario from step **3** b_j , $j = 1, \dots, T$ to the rest of scenarios in matrix A of size $T^2 \times n$. If the result is unacceptable (too far) according to Euclidean distance, return to step **3**.
- **5)** (Centroid selection) for each cluster, find the scenario which is the closest to its center ‘ random scenario chosen in step **3**’, and designate it as the centroid. In other words, we have matrix of size $T \times n$ corresponding to each scenario from step **3**.
- **6)** (Queuing) create a child scenario tree node for each cluster. This step need just calculate the average mean for each matrix $T \times n$ corresponding to each scenario from step **5** and the output, we have T scenarios in period $T = 2$.
- **7)** Now for further period, for example for period $T = 3$ at each scenario as node from step **6** we calculate μ and σ , then corresponding to each scenario create matrix with random variables by using formula $\sigma + (\mu - \sigma) * rand(T^2, n)$. At each scenario we get the matrix of size $T^2 \times n$ and then repeat the process from step **3** to **6**. Thus, output at each node from step **6** has child ‘scenarios’, which represent in a matrix of size $T \times n$. Thus, we have T scenario at first period and T scenarios at second period and T scenario at third period. The procedure of this construction to avoid growth of scenario tree, instead of getting T in first period and T^2 in second period and so on we get T^N in N period.
- **8)** Repeat process from step **7** for further periods.

- We also generate future return by using binary tree, where probability for the gain $p = \frac{e^{\frac{r_t}{n}} - d}{u - d}$ where $u = e^{\sigma\sqrt{\frac{t}{n}}}$ and in case of loss we have probability $1 - p$ and $d = e^{-\sigma\sqrt{\frac{t}{n}}}$, at time t and n risky asset. To apply this method, firstly, we estimate return vector $r_t = (r_1, \dots, r_n)$ (observation) from $GARCH(1, 1)$. Then from this vector we calculate σ as formal definition $\sigma(r_t) = \sqrt{E[(r_t - E[r_t])^2]}$ and it is variance of r_t which is constant in this case. Then use σ to construct binary tree. Hence, the root will be r_t which is estimated by $GARCH$ model and the next period we have ur_t when the return goes up and dr_t when the return goes down and the second period with four scenarios will be as follows: u^2r_t, udr_t, dur_t and d^2r_t and complete the scenario tree in the same way for more periods.

4.1.2 Dynamic cooperative investment with GARCH in discrete time

Now we apply the cooperative investment in multi-period and it will be written as an optimisation problem over the trading strategy as follows:

$$\begin{aligned}
& \min_{x, z_{i,t}} \quad \sigma^2[y_1] \\
& \text{s.t.} \\
& \sigma_-^2[y_2] \leq \beta \\
& E[y_1] \geq \pi_1 \\
& E[y_2] \geq \pi_2 \\
& y_1 + y_2 = X \\
& X \in \mathcal{F} \\
& \sum_{i=1}^n z_{i,t} = 1 \\
& -1 \leq z_{i,t} \leq 1 \text{ for } t = 1, \dots, T
\end{aligned} \tag{4.3}$$

where π_1 is the fixed return level for the first investor, and π_2 is the fixed return level for the second investor and β is the fixed σ_- level for the second investor. While, z_t is the trading strategy where for each node T scenarios and z_t of size $(1, n)$ as a function which is formulated as $z_t = \{z_{1,t}(\cdot), z_{2,t}, \dots, z_{n,t}(\cdot)\}$. Note that the number of nodes equal to the number of scenarios t , $t = 1, \dots, T$.

Alternatively, with the “certainty equivalent” approach of section 3.1.7 which are the same for cooperative investment portfolio optimisation can be written as follows:

$$\begin{aligned}
 & \max_{x, z_{i,t}} \quad \mu_1 - \frac{\sigma(y_1)^2}{2\rho_1} \\
 & s.t \\
 & \mu_2 - \frac{\sigma_-(y_2)^2}{\rho_2} \geq \pi \\
 & y_1 + y_2 = X \\
 & X \in \mathcal{F} \\
 & \sum_{i=1}^n z_{i,t} = 1 \\
 & -1 \leq z_{i,t} \leq 1 \text{ for } t = 1, ..T
 \end{aligned} \tag{4.4}$$

where π is the fixed level for utility function for the second investor, and where z_t is the trading strategy where for each node T scenarios and z_t of size $(1, n)$ as a function which is formulated as $z_t = \{z_{1,t}(\cdot), z_{2,t}, \dots, z_{n,t}(\cdot)\}$ and number of nodes equal to number of scenarios t , $t = 1, \dots, T$. Note that, problem (4.4) can be solved by using dynamic cooperative investment directly, but we choose to solve it by using certainty equivalent in a recursive manner to have less computational. In addition, the difference between the problem which written in section 3.1.7 and the problem (4.4) is just predicting the future return by historical simulation and $GARCHG(1, 1)$, respectively.

Algorithm 4.2

- Apply GARCH(1,1) model for each stock by using the process shown in the Algorithm 4.1.
- Then, solve problem (4.4) for cooperative investment directly then solve it again by using force in back technique by following the same process as in Algorithm 3.3 in Chapter 3.

Numerical experiments in discrete time

In my following experiments we show how applied Algorithm 4.2 in order to get the global solution for cooperative investment in multi-period and dynamic solution for comparative investment in multi-period and the difference between the curves are not very significant.

- We will solve the problem (4.3) and suppose we have one risk free and 6 risky assets (APPL,APA,ABBU,ALL,BA,BK) are chosen randomly from *S&P* 100 and found them by using *GARCH*(1,1) and each (daily rate of return) observation for each stock has binary scenarios, and has $T = 5$ periods. We will fix the values of π_1 and π_2 which are the expected returns for the first and second investor, respectively. The initial capital for all examples is \$100 and each investor participates with \$50. Furthermore, in our case we assume that risk-free asset $r_0 = 0.000068$, $\pi_1 = 0.0000821$ and $\pi_2 = 0.000109$, where $\pi_1 + \pi_2 > r_0$, otherwise the investor will not invest in risk assets any more if the risk free give them more return. By following the Algorithm 4.2, then solve DCI and CI as in Chapter 3. Hence, we get Figure 4.6.

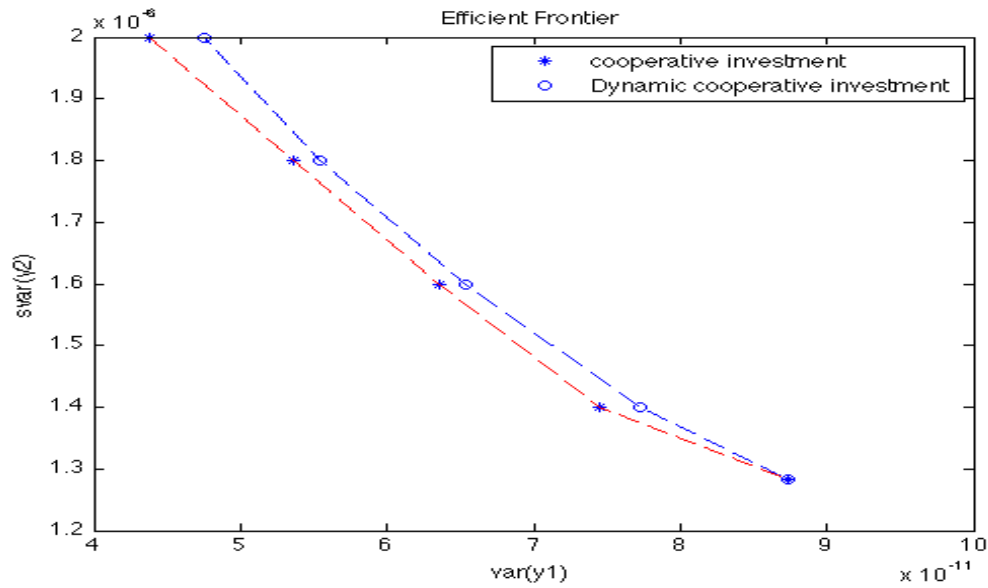


FIGURE 4.6: Efficient frontier by GARCH(1,1) with binary tree and $T = 5$ periods, where the cooperative investment curve means solving cooperative investment in multi-period to find global solution, while dynamic cooperative investment curve means solve cooperative investment in in multi-period by using DP

- In another experiment we have a portfolio consisting of one risk-free assets and 3 stocks (APA,BA,BK) are chosen randomly from *S&P* 100, and we find the return at time t by using *GARCH*(1,1). We have a daily rate of return (January.2011) - (January.2012) for each risky asset. Then, generate the scenario tree, we get the scenario tree for 3 periods which is a symmetrical tree, where at the first period each branch has $N_{t_1} = 5$ scenarios and at the T=2 period each node has $N_{t_2} = 3$ scenarios. Moreover, at the last period

$T = 3$ we have $N_{t_3} = 2$ scenarios, then we will solve problem (4.3) as a cooperative investment in multi-period directly and in dynamic programming case as shown in Chapter 3.

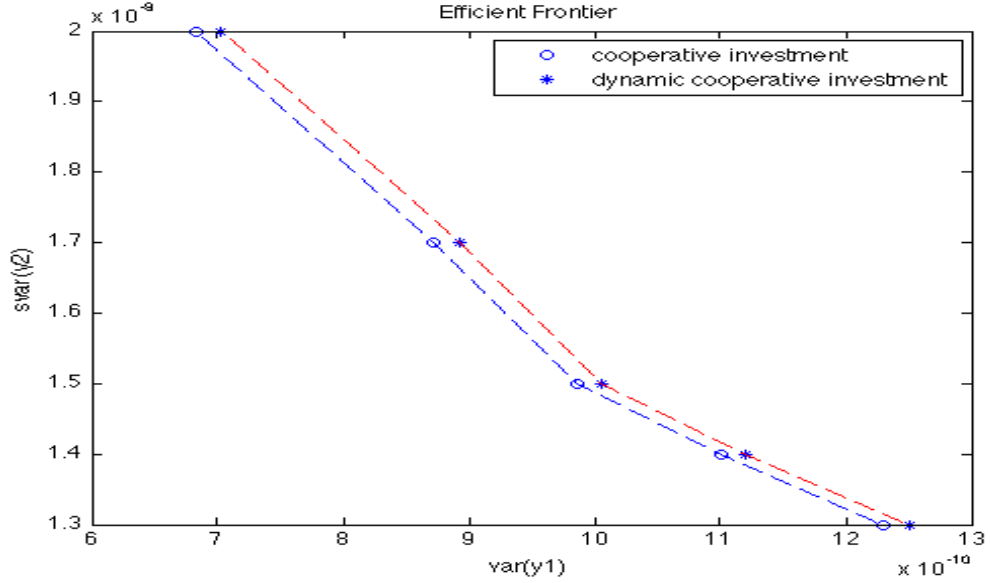


FIGURE 4.7: Efficient frontier by GARCH(1,1) and $T = 3$ periods, where the cooperative investment curve means solving cooperative investment in multi-period to find global solution, while dynamic cooperative investment curve means solve cooperative investment in in multi-period by using DP

- For the third experiments we will choose one risk-free asset and 3 risky assets (APA,BA,BK) are chosen from *S&P 100* and find the daily return (January.2011) - (January.2012) for the risky asset by using *GARCH*(1, 1). Then generate the scenario tree that has 100 scenarios at each node and $T = 30$ periods and we choose risk-free $r_0 = 0.000068$, $\pi_1 = 0.0000821$ and $\pi_2 = 0.000109$, by following Algorithm 4.2. Hence, our result is shown in Figure 4.8.
- **Secondly : Experiment in Certainty equivalent** In this experiment we will choose one risk-free asset and 3 risky assets (APA,BA,BK) are chosen from *S&P100* and find the daily return (January.2011) - (January.2012) for the risky assets by using *GARCH*(1,1). Then generate the scenario tree that has 100 scenarios at each node and $T = 30$ periods and we choose risk-free $r_0 = 0.000068$, $\pi_1 = 0.0000821$ and $\pi_2 = 0.000109$. Then, we will solve problem (4.4) by following Algorithm 4.2 and we get the Figure 4.9.

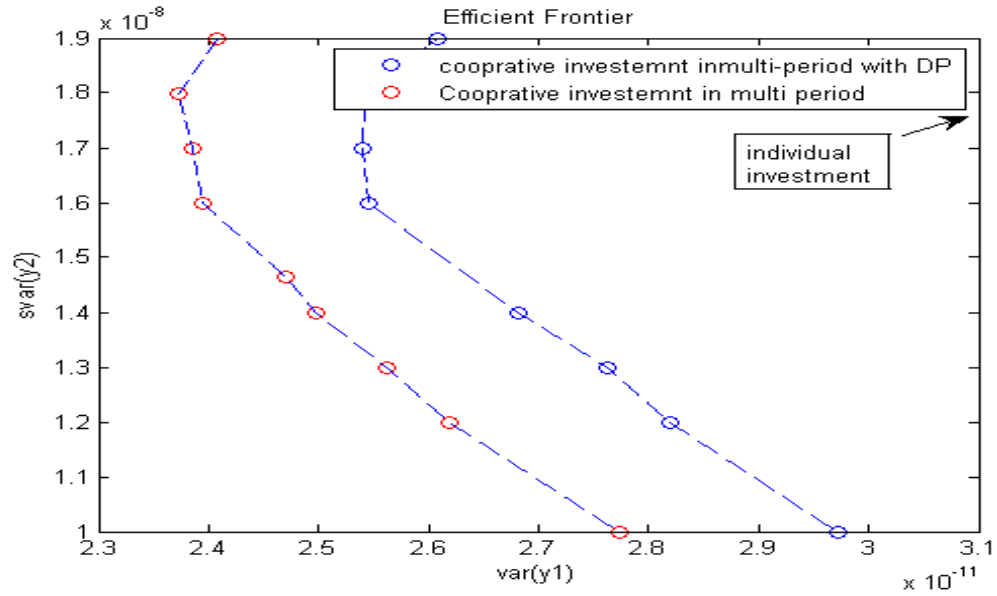


FIGURE 4.8: Efficient frontier by GARCH(1,1) with 100 scenarios at each node and $T = 30$

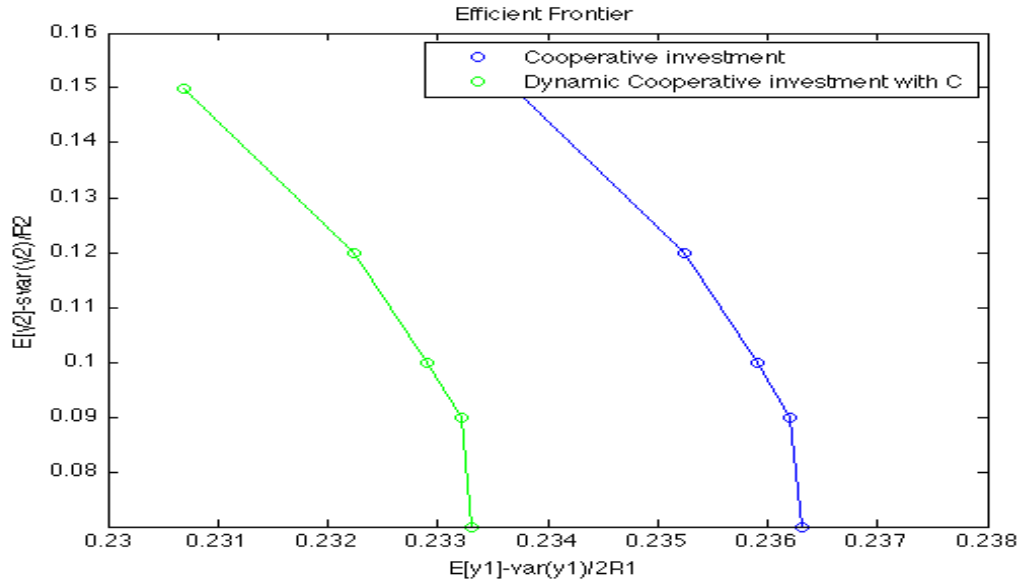


FIGURE 4.9: Efficient frontier by GARCH(1,1) with certainty equivalent by solving problem (4.4), where $R1$ in figure = $\rho_1 = 0.5$, and $R2 = \rho_2 = 0.25$ and $T = 10$

Generally, according to the experiments we get the difference between the efficient frontiers is not very significant which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment

period.

4.2 Problem formulation in continuous time

In this section we study how to apply the continuous GARCH ‘COGARCH’ model as the economic model in order to get the return of the stock price and the volatility of the return. In financial econometrics, discrete times of GARCH model are widely used to model the return of stock price at regular intervals on stocks. However, the interest in continuous time models is used to model irregularly spaced data. Consequently, the discrete time model help us to obtain a continuous time model in much the same way as Klüppelberg [89]. If we making data stationary as well as if we have a serial correlation between residual and *ARCH* effect terms, then we need to estimate σ^2 by the *GARCH* model. Hence, in continuous time it is natural to model the logarithm of the asset price itself, that is $G_t = \ln P_t$, rather than its increments as in discrete time. Then, the COGARCH(1, 1) equations are obtained by replacing the driving noise sequence $(\epsilon_n), \epsilon \in N$ by the jumps $(\Delta L_t = L_t - L_-)_{t \geq 0}$ of a *Lèvy* process, as shown in Klüppelberg [89]. Otherwise, the process has two independent Brownian motions as shown in Nelson [110].

Firstly : [7]

Notable among these attempts is the GARCH diffusion approximation of Nelson [110]. Although the GARCH process is driven by a single noise sequence, there are two independent Brownian motions $(W_t^{(1)})_t \geq 0$ and $(W_t^{(2)})_t \geq 0$. For example, the GARCH (1,1) satisfies

$$\begin{aligned} dG_t &= \sigma_t dW_t^{(1)} \\ d\sigma_t^2 &= \theta(\lambda - \sigma_t^2)dt + \rho\sigma_t^2 dW_t^{(2)} \end{aligned} \tag{4.5}$$

where $\theta \geq 0, \lambda \geq 0$ and $\rho \geq 0$ are constant. Note that the behaviour of this diffusion limit is therefore rather different from that of the GARCH process itself since the volatility process $(\sigma_t^2)_t \geq 0$ evolves independently of the process $(W_t^{(1)})_t \geq 0$ in the first equation (4.5).

Secondly : Klüppelberg [89]

In another way, to construct COGARCH(1,1) by the following the Klüppelberg [89] method who showed a continuous time analogue of the GARCH(1,1) process, denoted COGARCH(1,1). This model, based on a single background driving

jump-*Lévy* process, for more details of his method see Klüppelberg [89]. The COGARCH(1,1) model has the basic properties of discrete time GARCH process. The COGARCH(1,1) process $(G_t)_t \geq 0$ is defined in terms of its stochastic differential dG_t , such that

$$\begin{aligned} dG_t &= \sigma_t dL_t, t \geq 0 \\ d\sigma_t^2 &= \theta(\lambda - \sigma_t^2)dt + \rho\sigma_{t-}^2 d[L, L]_t, t \geq 0 \end{aligned} \quad (4.6)$$

where $\theta \geq 0, \lambda \geq 0$ and $\rho \geq 0$ are constant. Note that from proposition 3.2 in Klüppelberg [89], $(\sigma_t^2)_t \geq 0$ satisfies the stochastic differential equation (4.6) and we have

$$\sigma_t^2 = \theta\lambda + \log \rho \int_0^t \sigma_s^2 ds + (\lambda/\rho) \sum_{0 < s < t} \sigma_s^2 (\Delta L_t)^2 + \sigma_0^2, t \geq 0 \quad (4.7)$$

where Euler approximation is used for the integral $\int_0^t \sigma_s^2 ds \approx \sigma_{t-1}^2$ and $\sum_{0 < s \leq t} \sigma_s^2 (\Delta L_t)^2 \approx (G_t - G_{t-1})^2 \approx (\sigma_n \epsilon_n)^2$, where $G_t = \ln P_t$ and since ΔL_s is usually not observable, we can find that the equation (4.7) is analogue of *GARCH*(1,1) in discrete time as follows: since the discrete time *GARCH*(1,1) satisfies

$$\sigma_{n+1}^2 - \sigma_n^2 = \theta\lambda - (1 - \theta)\sigma_n^2 + \rho\sigma_n^2 \epsilon_n^2, n \in N$$

which by summation yields

$$\sigma_n^2 = \theta\lambda n - (1 - \theta) \sum_{i=0}^{n-1} \sigma_i^2 + \rho \sum_{i=0}^{n-1} \sigma_i^2 \epsilon_i^2 + \sigma_0^2$$

where the last equation is analogously to (4.7), note that Klüppelberg [89] used $(\sigma_n^2)_{n \in N}$ to denote the sequared discrete time *GARCH* volatility process, and (σ_t^2) to denote the continuous time process. Hence, we will get the value of volatility by using the equations (4.5) or (4.6) in order to use it to predict the stock returns. Now, after we fit the COGARCH(1,1) model and apply it to get the future return for each risky asset we need to observe the wealth of the portfolio for the investor in continuous time, where the portfolio consists of one risk-free asset and n risky assets. Hence,

- The price process of the risk-less assets, $S_0(t)$, is subject to the following ordinary differential equation

$$dS_0(t) = r(t)S_0 dt, t \geq 0, S_0(0) = s_0 > 0 \quad (4.8)$$

where deterministic function $r(t)$ is the interest rate, and it is easy to see the solution of this ordinary differential equation will be

$$S_0(t) = S_0 e^{(\int_0^t r(s) ds)}$$

- Besides that, the price processes of the other n risky assets, $S_1(t), S_2(t), \dots, S_n(t)$, generated setting for underlying asset price S in the black- scholes model is that S follows a Gometric Brownion motion GBM where the price process for one share of the stocks model by following stochastic differential equations (SDE) as follows:

$$dS_i(t) = S_i(t)(b_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW^j(t)), t \geq 0 \quad (4.9)$$

$$S_i(0) = s_i > 0, i = 1, 2, \dots, n$$

where $(W^1(t), \dots, W^n(t))$ is the n - dimensional Brownian motion defined on a probability space (Ω, \mathbb{F}, P) and all $b = (b_1, \dots, b_n)$, and σ_{ij} , $i, j = 1, \dots, n$ is denoted by σ , also b and σ are \mathbb{F} -predictable, see Yu [146] p.337. In addition, the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is generated by W_t . Moreover, the differential dW_t of Brownian motion W_t is called weight noise. Furthermore, we assume for constant $\delta > 0$, such that $\sigma(t)\sigma(t)' \geq \delta I_n$, where I_n is $n \times n$ identity matrix. Moreover the solution of equation (4.9) is illustrated $\forall t \in [0, T]$, where $\sigma(t) = (\sigma_{ij}(t))_{m \times n}$ and $\delta > 0$ as the black-scholes formula so that

we will rewrite the equation (4.9) as $dS_t = S_t(b dt + \sigma dW)$.

Then the solution as follows:

$$S_t = S_0 e^{((b - \frac{1}{2})t + \sigma W_t)}.$$

See Pliska [115]

Now, to define the wealth of the portfolio let $u_i(t)$, $i = 1, \dots, n$ be the amount of money which an investor invests in the risky asset at time t .

- The wealth process of the investor, $X(t)$, then satisfies the following stochastic differential equation

$$dx(t) = (r(t)x(t) + \sum_{i=1}^n (b_i(t) - r(t))u_i(t))dt + \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij}(t)u_i(t)dW^j(t) \quad (4.10)$$

$$x(0) = x_0 > 0, \quad (4.11)$$

$$dX(t) = (r(t)x(t) + \sum_{i=1}^n (b_i(t) - r(t))u_i(t))dt + \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij}(t)u_i(t)dW^j(t)$$

$$X(0) = x_0 > 0, i = 1, 2, \dots, n \quad (4.12)$$

Solve the stochastic differential equation (4.12) by numerical approximation, particularly by the Euler Maruyama method on some interval of time $[0, T]$.

Then the Euler Maruyama approximation to the true solution X is the Markov chain Y defined as follows:

- Partition the interval $[0, T]$ into N equal sub-intervals of width $\Delta t > 0$.
- $0 = t_0 < t_1 < \dots < t_N = T$ and $\Delta t = T/N$. Hence, solving SDE equation (4.12) by using Euler Maruyama methods to find the iteration formula which is set $Y_0 = x_0$.
- Recursively define Y_i for $1 \leq i \leq N$ by

$$Y_{i+1} = Y_i + a(Y_i)\Delta t_i + b(Y_i)\Delta W_i$$

applying this formula on equation (4.12) we get

$$Y_{i+1} = Y_i + (r(t_i)Y_i + \sum_{i=1}^n (b_i(t) - r(t_i))u_i(t_i))\Delta t_i + \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij}(t_i)u_i(t_i)\Delta W_i$$

where $\Delta t_i = t_i - t_{i-1}$, $\Delta W_i = z_i \sqrt{\Delta t_i}$, also $z_i \in N(0, 1)$ and it is (i.i.d).

Now we will solve the problem (4.3) by applying the continuous *GARCH*. Since problem (4.3) faced time-inconsistent, thus we need to revise the solution by using the dynamic programming model. This is referred to mean-variance portfolio

selection under pre-commitment. Hence, we will revise the solution in order to obtain the efficient frontier by using Dynamic Programming. Thus, the following algorithm explains how to solve the problem (4.3) by applying COGARCH(1,1).

Algorithm 4.3.

- Apply COGARCH(1,1) model for each stock by using Nelson method in equations (4.5). Note that $G_t = \ln P_t$, and we solve equation (4.5) as a stochastic difference equation ‘numerical simulation of SDE’. In order to find σ_t this is done in MATLAB SDE Toolbox; choose Ito SDE, and put the value of drift term and diffusion term from equation (4.5). Then we compute $\sigma_{i,j} = \sigma_t \sigma'_t$.
- Instead of constructing a scenario tree of Algorithm 3.2, we will solve the equation (4.12) by simulation SDE, ‘Euler Maruyama approximation’, which obtains the wealth $X(\omega_T)$, where T is the end of the investment period, Graphically we get many trajectories which depend on the number of partitions N and the interval $[0, T]$. This step is also done in MATLAB SDE Toolbox, similar to the previous step, by choosing the value of $[t_0, T]$, where t_0 is the start of the investment period and T is the end of investment period and N is the number of partitions in the interval $[t_0, T]$.
- Now for solving cooperative investment problem (4.3) in the case of global solutions, we solve cooperative investment in multi-periods exactly as in discrete time, since the wealth $X(\omega_T)$ solved it by (4.12) by numerical approximation for SDE over the trading strategy $u_i(t_i)$. This is done by just following Algorithm 3.2 just we need to rewrite $X(\omega_T)$ by (4.12). Thus, the result in this step is that we find $u_i(t_i)$ for partition of each trajectory and the optimal value for the first investor $\sigma(\beta)$ which is correspond to the value of β chosen to be less than optimal value for individual investment for the second investor. Repeat this process with a different value of β to get the whole efficient frontier.
- In case of solving cooperative investment problem (4.3) by dynamic programming, the only step we need is to break down each trajectory $X(\omega_T)$ and solve it the same as in Algorithm 3.2 as force in back technique note that we break down the trajectory according to $\Delta t = T/N$ as step size in order to get the best trading strategy $u_i(t_i)$ one after another until reaching the first period, and fix all the trading strategies $u_i(t_i)$ for $i = 1, \dots, T$ and

resolve again cooperative investment problem (4.3) to get the best trading strategy at first period $u_i(t_0)$ which is the same procedure as shown in Algorithm 3.2. In other words, in this Algorithm, in order to solve dynamic programming (recursive manner) we only need to partition each trajectory instead (scenario tree) into N sub-interval and hence we need to solve N sub-problems for each trajectory. Thus, in this step the result we get $u_i(t_i)$ at each partition is complete recursively until we arrive at the first period, repeating the same process for each (trajectory). Consequently, similar to Algorithm 3.2 we plug all the value of $u_i(t_i)$ and solve it over one variable $u_i(t_0)$ at the first period and get the optimal value for each investor, $\sigma(\beta)$ for the first investor and choose β the same as in the global solution, then repeat the same process for different values of β to get the whole efficient frontier in case of dynamic programming, where $\sigma(\beta)$ is represented in x-axis and β is represented in y-axis.

- Similar to the previous Algorithms (3.2) we compared between the efficient frontiers from a global solution and by using the dynamic programming.

Remark 4.8. In case if we have the closed form solution of SDE in equation (4.12), we can follow the method by Czichowsky [45] claiming ‘the justify the continuous time formulation by showing that it coincides with the continuous time limit of the discrete-time formulation’. Consequently, in discrete time, this leads to determining the optimal strategy by a backward recursion starting from the terminal date. For continuous time formulation one has to combine this recursive approach to time inconsistency with a limit argument. Moreover, the recursive optimality can be characterised by a system of partial differential equations (PDE) called Hamilton-Jacobian-Bellman equations, and one can provide verification theorems which allow to deduce that if one has a smooth solution to the PDE, it gives a solution to the optimal control problem. In our thesis we will use backward SDE approximation, see Czichowsky [45].

4.3 Numerical experiments in continuous time

In continuous time, we will solve the problem (4.3) in continuous time and suppose we have one risk free and 3 stocks

(APA,BA,BK) chosen randomly from *S&P* 100 and found them by using *COGARCH*(1,1) and $T = 10$ years. Note that we deal with a log daily of rate

of return. Then we generate the trajectory of the wealth by solving the stochastic differential equation (4.12) where $x_0 = 1$, and we will fix the values of π_1 and π_2 which are the expected returns for the first and second investors, respectively. In our case we assume that risk free $r_0 = 0.0000136$, $\pi_1 = 0.00016$ and $\pi_2 = 0.0001917$, where $\pi_1 + \pi_2 > r_0$, since the most well known examples of *Lévy* processes are Brownian motion. So in this example we model the return of stock based on the Klüppelberg formula to model the return of stocks where the portfolio consists of 3 risky assets and one risk-free asset and by following the Algorithm 4.1 these are chosen randomly from *S&P 100* (January.2011) - (January.2012), By using Yang [145] method to get the parameters of GARCH we can check the result by GARCH ToolBox/MATLAB in order to get the parameters $\theta = 2.5459 \times 10^{-1}$, $\lambda = 0.917453$, $\rho = 0.0449573$ and then the Brownian motion is shown as in Figure 4.10.

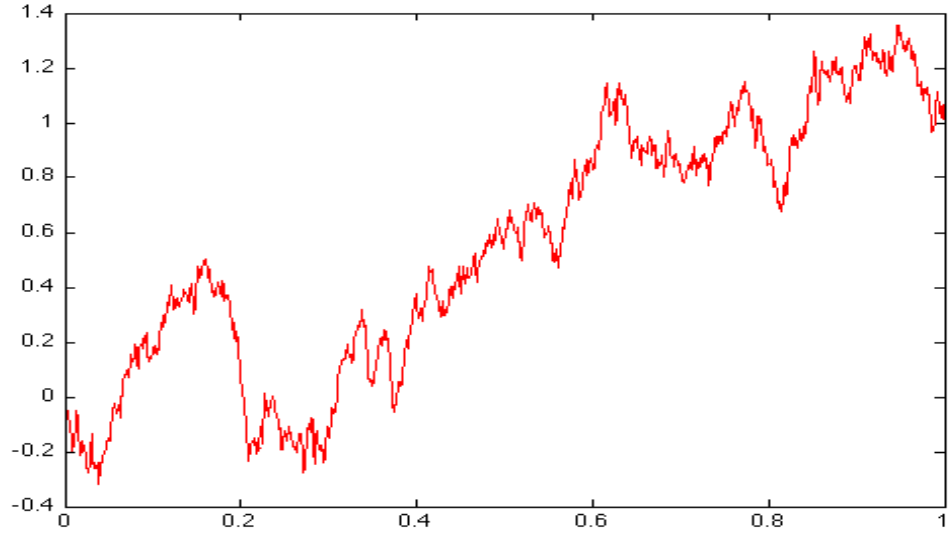


FIGURE 4.10: Brownian motion to solve COGARCH

- Now in order to get the wealth for the investor in continuous time, which can be traded continuously in the market, it can be found by solving the stochastic differential equation (4.12) by choosing the Euler Maruyama method. Hence the following example illustrates the wealth of a portfolio that consist of 3 risky assets (APA,BA,BK) and one risk-free asset. Also we chose $x_0 = 1$ and has 900 trajectories as shown in Figure 4.11. Hence, the solution of DCI and CI in multi-period of the problem (4.3) is shown in Figure 4.12. Note that, the difference between the efficient frontiers is

not very significant which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment period.

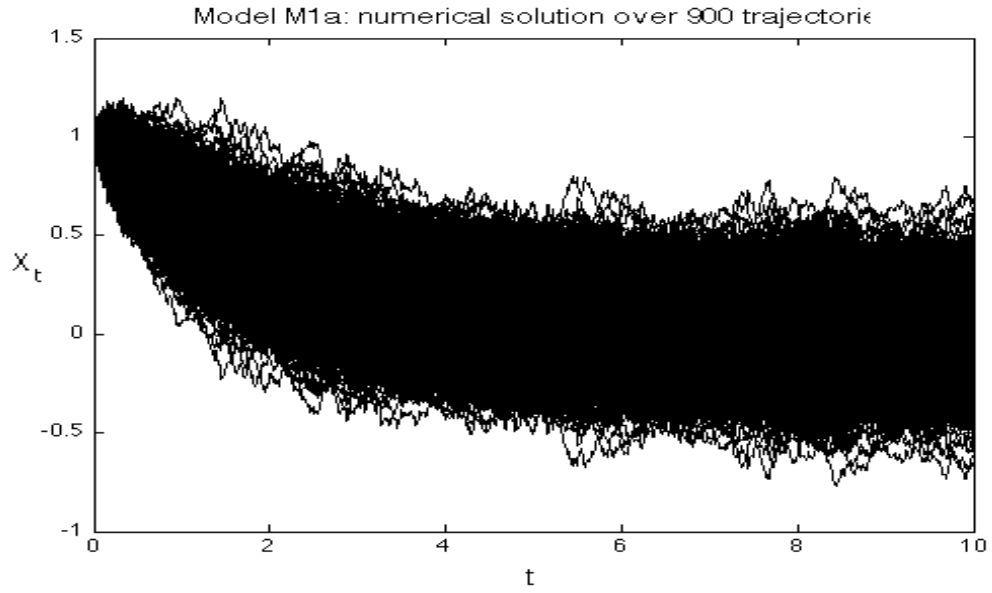


FIGURE 4.11: The wealth of 3 portfolios that consist of 3 risky assets and one risk-free asset. and chose $x_0 = 1$ and has 900 trajectory, $T = 10$

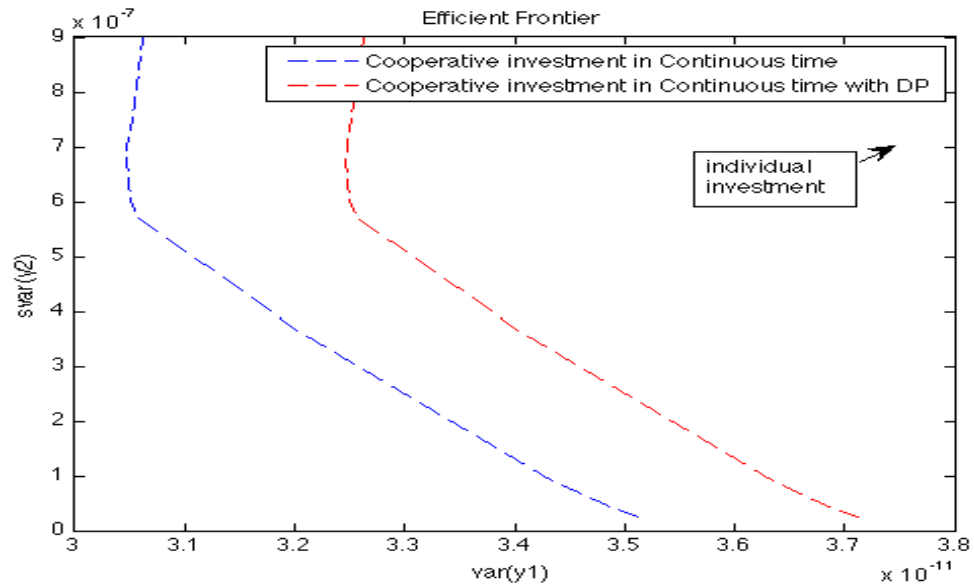


FIGURE 4.12: Efficient frontier by COGARCH(1,1), with 900 trajectory and $T = 10$

4.4 Discussion and concluding remarks

In this chapter, we applied an economic model $GARCH(1, 1)$ in order to build a more realistic model to forecast the future returns of financial instruments as a stochastic process in discrete and continuous time. Solving problem (4.3) and (4.4) where the results are shown in the Figures 4.6, 4.7, 4.8, 4.9 in discrete time and 4.12 show that the efficient frontier by using numerical approximation of problem (4.3) in the case of $COGARCH(1, 1)$ in continuous time is smother than that in discrete time. In addition, we solved the $COGARCH$ by using numerical approximation. Also, the efficient frontier which comes from force in back technique is more stable which means it is ‘time-consistent’ than efficient frontier that solves the cooperative investment in multi-period directly without recursive manner. Furthermore, we show the distance between two efficient frontiers is relatively small. Consequently the investor can follow the stable trading strategy without hesitation, which should make him/her happy.

Chapter 5

Equilibrium

In the previous chapter we have developed methods for finding optimal dynamic cooperative investment strategies, and characterising the set of all Pareto optimal allocations, which can be visualised as the efficient frontier. In this chapter we address the question of how to select a unique ‘fair’ point on the efficient frontier. We show that in many cases the efficient frontier contains a special point which is called ‘equilibrium allocation’, and suggest that namely this point should be selected. In some cases, an explicit formula for equilibrium allocation is derived, and the uniqueness and local stability for equilibrium is also shown.

Let $I = \{1, 2, \dots, m\}$ be the set of agents/investors. An *allocation* y_i , $i \in I$ is a set of r.v.s., $y_i \in \mathcal{L}^2(\Omega)$ for some probability space Ω that ensures the $\sigma(y_i)$ and $\sigma_-(y_i)$ exist and finite. An allocation y_i , $i \in I$ is called *feasible* if $X := \sum_{i \in I} y_i \in \mathcal{F}$, where \mathcal{F} is a feasible set defined in section 3.1.

A functional price P is a linear continuous functional $P : \mathcal{L}^2(\Omega) \rightarrow \mathbb{R}$, such that $P(X) > 0$ whenever $X > 0$ with probability 1. Without loss of generality, we can also assume normalisation $P(1) = 1$.

Definition 5.1. Follmer and Schied [59]

A functional price P together with a feasible allocation $(y_i)_{i \in I}$ is called Arrow-Debreu equilibrium if each y_i solves the utility maximisation problem

$$\begin{aligned} & \max U_i(y_i) \\ & s.t \\ & P(y_i) \leq 0 \end{aligned} \tag{5.1}$$

of agents i with respect to the price P . In this case, P is called the equilibrium price, and the pair $(P, (y_i)_{i \in I})$ is called Walrasian equilibrium.

Every Walrasian equilibrium is Pareto optimal according to the first Welfare theorem.

Equilibrium in Single Period

We will illustrate the general background of the equilibrium in single period. Usually, the notion of equilibrium as a ‘fair’ point from Pareto optimal set is used in the context of the risk sharing problem. In this problem, there are m agents with utility functions u_1, \dots, u_m , where $U_i(.) = E[u_i(.)]$ and initial endowments W_1, \dots, W_m , modelled as random variables. In this context, allocation $(y_i)_{i \in I}$ is called Arrow-Debreu equilibrium, for investor i and I is the set of all investors. If each y_i solves the utility maximisation problem

$$\begin{aligned} & \max U_i(y_i) \\ & s.t \\ & P(y_i) \leq P(W_i) \end{aligned} \tag{5.2}$$

In the cooperative investment problem, we suppose that no initial endowment is available, and therefore the familiar condition $P(y_i) \leq P(W_i)$ is replaced by $P(y_i) \leq 0$, hence (5.2) is reduced to (5.1). We will call the price functional P *consistent* if $P(X) = 0$ for any $X \in \mathcal{F}$. Furthermore, from definition of efficient frontier for cooperative investment, see Definition (3.4) we get the set of all Pareto optimal and from first Welfare theorem that every equilibrium allocation is Pareto optimal. Thus, we need only solve the problem (5.2) to get the equilibrium allocation and equilibrium price which are represented in a pair $(P, (y_i)_{i \in I})$, I is the set of all investor i . The pair is considered to be the fair element in the Pareto optimal set and it is also called \mathcal{F} -equilibrium for some consistent pricing functional P .

5.1 Problem formulation for Dynamic Equilibrium

The natural question now how is can we find equilibrium in the multi-period case. According to Henriksen and Spear [74] the equilibrium in multi-period must be generated recursively. Hence, an equilibrium is a sequence of allocation $(y_i)_{t \in T}$ where the time horizon is finite and a sequence of price $P_{t \in T}$ such that:

each individual solves her/ his optimisation problem subject to budget constraint

$$\begin{aligned} & \max U_i(y_{i,t}) \\ & s.t \\ & P_t(y_{i,t}) \leq 0, \end{aligned} \tag{5.3}$$

where $y_{i,t}$ is an allocation for investor i at time t , satisfying $\sum_{i \in I} y_{i,T} = X$, where $X \in \mathcal{F}$ is uncertainty outcome at the end investment period and \mathcal{F} is the feasible set.

Assuming that we have a finite number N of scenarios, any r.v. X can be written as $X = (x_1, \dots, x_N)$, where x_i is the value of X under scenario i , and the price functional P_t can be written as

$$P_t(X) = \sum_{i=1}^N P_{i,t} x_i$$

for some non-negative real number $P_{i,t}$. Normalisation $P_t(1) = 1$ implies that $\sum_{i=1}^N P_{i,t} = 1$ for every t .

In the following propositions we will derive an explicit formula for equilibrium allocation and then we will check about the feasibility of equilibrium allocation.

Proposition 5.1. *If the risk preferences of the first investor are given by*

$$U_1(y_1) = \begin{cases} -\sigma^2(y_1) & \text{if } E(y_1) \geq \pi_1 \\ -\infty & \text{if } E(y_1) \leq \pi_1 \end{cases}$$

then the explicit formula for his/her component of the equilibrium allocation at each $t \in T$ is given by

$$y_{1,t}^i = \pi_1 + \frac{(1 - P_{i,t} N_t) \pi_1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2 - 1}, \quad i = 1, \dots, N_t,$$

where N_t is the number of states of the word at time t , and $y_{1,t}^i$ is the share of investor 1 at time t under scenario i . Thus, the final equilibrium allocation for the first investor is $y_1 = (y_{1,T}^1, y_{1,T}^2, \dots, y_{1,T}^{N_T})$.

Proof. It is clear that in optimality $E(y_{1,t}) = \pi_1$, hence let $U_1(y_1) = -\sigma^2(y_1)$ if $E[y_{1,t}] \geq \pi_1$. We will solve (5.3) by using Lagrange multipliers where $L = \frac{1}{N_t} \sum_{i=1}^{N_t} (y_{1,t}^i)^2 -$

$(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i)^2 + \alpha(-\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i + \pi_1) + \lambda(\sum_{i=1}^{N_t} p_{i,t} y_{1,t}^i) = 0$. We need to find the derivative of L with respect to $y_{1,t}^i$

$$\frac{\partial L}{\partial y_{1,t}^i} = \frac{2}{N_t} y_{1,t}^i - 2\pi_1(\frac{1}{N_t}) - \frac{\alpha}{N_t} + \lambda P_{i,t} = 0 \quad (5.4)$$

now multiplying (5.4) by $\frac{N_t}{2}$ we get

$$y_{1,t}^i = \pi_1 + \frac{\alpha}{2} - \lambda P_{i,t} \frac{N_t}{2} \quad (5.5)$$

Now multiplying (5.4) by $P_{i,t}$ into sides and summing we get

$$\frac{2}{N_t} \sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i - \frac{2\pi_1}{N_t} \sum_{i=1}^{N_t} P_{i,t} - \frac{\alpha}{N_t} \sum_{i=1}^{N_t} P_{i,t} + \lambda \sum_{i=1}^{N_t} P_{i,t}^2 = 0$$

where $\sum_{i=1}^{N_t} P_{i,t} = 1$ and $\sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i = 0$. Hence, last equation reduce to

$$\lambda \sum_{i=1}^{N_t} P_{i,t}^2 = \frac{\alpha}{N_t} + \frac{2\pi_1}{N_t} \quad (5.6)$$

Now summing all the equations i in (5.4) we get

$$\frac{2}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i - 2\pi_1 - \alpha + \lambda \sum_{i=1}^{N_t} P_{i,t} = 0 \quad (5.7)$$

where $\frac{\sum_{i=1}^{N_t} y_{1,t}^i}{N_t} = \pi_1$ as well as $\sum_{i=1}^{N_t} P_{i,t} = 1$. Hence last equation implies to

$$2\pi_1 - 2\pi_1 - \alpha + \lambda(1) = 0$$

Thus,

$$\alpha = \lambda$$

now plugging $\alpha = \lambda$ in (5.6) to get the value of α

$$\alpha \sum_{i=1}^{N_t} P_{i,t}^2 - \frac{\alpha}{N_t} = \frac{2\pi_1}{N_t}$$

Thus, $\alpha(\sum_{i=1}^{N_t} P_{i,t}^2 - \frac{1}{N_t}) = \frac{2\pi_1}{N_t}$ in this case we have assumption that all $P_{i,t}$ are not the same, since if $P_{1,t} = P_{2,t} = \dots = P_{N_t,t}$ then $\sum_{i=1}^{N_t} P_{i,t}^2 = \frac{1}{N_t}$. Hence, we have the assumption that $P_{i,t}$ are not the same, so that $\alpha = \frac{2\pi_1/N_t}{(\sum_{i=1}^{N_t} P_{i,t}^2 - \frac{1}{N_t})}$. Thus,

$$\alpha = \frac{2\pi_1}{N \sum_{i=1}^{N_t} P_{i,t}^2 - 1} = \lambda$$

Now plugging the value of α and λ in (5.5) where $\alpha = \lambda$ we get

$$y_{1,t}^i = \pi_1 + \left(\frac{1}{2} - \frac{P_{i,t}N_t}{2}\right)\alpha$$

$$=$$

$$y_{i,t}^i = \pi_1 + \left(\frac{1}{2} - \frac{P_{i,t}N_t}{2}\right) \frac{2\pi_1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2 - 1}$$

Hence,

$$y_{1,t}^i = \pi_1 + \frac{(1 - P_{i,t}N_t)\pi_1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2 - 1}$$

for each $i = 1, \dots, N_t$. Then, fair allocation for first investor $y_1^* = (y_{1,t}^1, y_{1,t}^2, \dots, y_{1,t}^{N_t})$. \square

Proposition 5.2. *If the risk preferences of the second investor are given by*

$$U(y_2) = \begin{cases} -\sigma_-^2(y_2) & \text{if } E(y_2) \geq \pi_2 \\ -\infty & \text{if } E(y_2) \leq \pi_2 \end{cases}$$

then the explicit formula for his/her component of the equilibrium allocation at each $t \in T$ is given by

$$y_{2,t}^i = \pi_2 + \frac{\lambda N_t}{2}(P_{N_t,t} - P_{i,t}), \quad i = 1, \dots, m$$

$$y_{2,t}^i = \frac{-\pi_2 N_t (\frac{m}{N_t} - 1)}{(N_t - m)} - \frac{\lambda N_t P_{N_t,t}}{2(N_t - m)} + \frac{\lambda N_t (1 - (N_t - m)P_{N_t,t})}{2(N_t - m)}, \quad i = m + 1, \dots, N_t$$

where assuming the price $P_{1,t} \geq P_{2,t} \geq \dots \geq P_{N_t,t}$, N_t is the number of states of the word at time t , m is the minimal number such that $P_{m+1,t} = P_{m+2,t} = \dots = P_{N_t,t}$, and

$$\lambda = \frac{2[N_t P_{N_t,t} (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})]}{N_t [P_{N_t,t}^2 - 2P_{N_t,t}(1 - (N_t - m)P_{N_t,t}) + \sum_{i=1}^m p_{i,t}^2]}$$

$y_{2,t}^i$ is the share of investor 2 at time t under scenario i . Thus, the equilibrium allocation for the second investor is $y_2^ = (y_{2,t}^1, y_{2,t}^2, \dots, y_{2,t}^{N_t})$.*

Proof. It is clear that in optimality $E(y_{2,t}) = \pi_2$. As in the previous proposition solving the individual problem

$$\max U(y_2), \text{ s.t } P(y_2) \leq 0 \tag{5.8}$$

which can be rewritten as

$$\begin{aligned} \max U(y_2) &= \begin{cases} -\sigma_-^2(y_{2,t}) & \text{if } E(y_{2,t}) \geq \pi_2 \\ -\infty & \text{if } E(y_{2,t}) \leq \pi_2 \end{cases} \\ \text{s.t } P_t(y_{i,t}) &\leq 0 \end{aligned}$$

where the semi-variance is similar to variance; however, it only considers observations below the mean. Hence, semi variance defined as $\sigma_-^2 = \frac{1}{N_t} \sum_{i=1}^{N_t} (\max(-(y_{2,t}^i - \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i), 0))^2$ while for portfolio or asset analysis semi-variance only looks at the negative fluctuations of an asset. Thus,

$$L = \frac{1}{N_t} \sum_{i=1}^{N_t} (\max(-(y_{2,t}^i - \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i), 0))^2 + \alpha(-\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i + \pi_2) + \lambda(\sum_{i=1}^{N_t} P_{i,t} y_{2,t}^i)$$

then

$$\frac{\partial L}{\partial y_{2,t}^i} = \begin{cases} \frac{2}{N_t} y_{2,t}^i - 2\pi_2(\frac{1}{N_t}) - \frac{\alpha}{N_t} + \lambda P_{i,t} = 0 & \text{if } y_{2,t}^i \leq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i \\ 0 - \frac{\alpha}{N_t} + \lambda P_{i,t} = 0 & y_{2,t}^i > \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i \end{cases}$$

To begin the proof, let $y_{2,t}^1, \dots, y_{2,t}^{N_t}$ be the optimal solution for solving the individual problem (5.8) we assume $P_{1,t} \geq P_{2,t} \dots \geq P_{N_t,t}$. Then we have four statements we need to prove to avoid cases and get a system, where the statement are

- **1)** In optimality equalities hold $E[y_2] = \pi_2$ and $P(y_2) = 0$.
- **2)** If $P_{i,t} = P_{j,t}$ then $y_{2,t}^i = y_{2,t}^j$. If not replace $y_{2,t}^i, y_{2,t}^j$ by average $\frac{y_{2,t}^i + y_{2,t}^j}{2}$, then $E[y_2]$ would not change, $P(y_2)$ would not change, but $\sigma_-(y_2)$ would decrease (or stay the same) by proposition (3.2) in Grechuk et al. [66].
- **3)** $y_{2,t}^1 \leq y_{2,t}^2 \leq \dots \leq y_{2,t}^{N_t}$. Indeed, assume that $y_{2,t}^i > y_{2,t}^j$ for $i < j$. Then consider solution $y'_2 = (y_{2,t}^1, y_{2,t}^2, \dots, y_{2,t}^{i-1}, y_{2,t}^j, y_{2,t}^i, \dots, y_{2,t}^{j-1}, y_{2,t}^i, y_{2,t}^{j+1}, \dots, y_{2,t}^{N_t})$ with i and j interchanged. Then $\sigma_-(y'_2) = \sigma_-(y_2)$, $E[y'_2] = E[y_2] = \pi_2$, but $P(y'_2) < P(y_2) \leq 0$, which is contradictory to (1). Indeed, $P(y_2) - P(y'_2) = P_{i,t} y_{2,t}^i + P_{j,t} y_{2,t}^j - (P_{i,t} y_{2,t}^j + P_{j,t} y_{2,t}^i)$ equal to $P_{i,t}(y_{2,t}^i - y_{2,t}^j) + P_{j,t}(y_{2,t}^i - y_{2,t}^j) = (P_{i,t} - P_{j,t})(y_{2,t}^i - y_{2,t}^j) > 0$ if $P_{i,t} > P_{j,t}$ and if $P_{i,t} = P_{j,t}$ then $y_{2,t}^i = y_{2,t}^j$ by (2).
- **4)** Obviously, $y_{2,t}^{N_t} \geq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i$, then $-\frac{\alpha}{N_t} + \lambda P_{N_t,t} = 0$, thus $P_{N_t,t} = \frac{\alpha}{N_t \lambda}$ from the second terms of $\frac{\partial L}{\partial y_{2,t}^i} = 0$.

Now, let m be the minimal number such that $P_{m+1,t} = P_{m+2,t} = \dots = P_{N_t,t}$ then by (2) we have $y_{2,t}^{m+1} = \dots = y_{2,t}^{N_t}$. Then for any $i \leq m$ $P_{i,t} > P_{N_t,t} = \frac{\alpha}{N_t \lambda}$ so

$\frac{-\alpha}{N_t} + \lambda P_{i,t} \neq 0$ then $y_{2,t}^i \leq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i$ this implied the first term of $\frac{\partial L}{\partial y_{2,t}^i} = 0$ which is $\frac{2}{N_t} y_{2,t}^i - 2\pi_2(\frac{1}{N_t}) - \frac{\alpha}{N_t} + \lambda P_{i,t} = 0$.

Hence, according to four statements we have a linear system which is

- 1) $\frac{2}{N_t} y_{2,t}^i - 2\pi_2(\frac{1}{N_t}) - \frac{\alpha}{N_t} + \lambda P_{i,t} = 0$ which include m equations
- 2) $y_{2,t}^{m+1} = \dots = y_{2,t}^{N_t}$ which include $m - N_t - 1$ equations
- 3) $\frac{\alpha}{N_t \lambda} = P_{N_t,t}$ which includes one equation
- 4) $\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i = \pi_2$ which includes one equation.
- 5) $\sum_{i=1}^{N_t} P_{i,t} y_{2,t}^i = 0$ which includes one equation.

Hence, we have a system of $N_t + 2$ equation with $N_t + 2$ variables which are $y_{2,t}^1, \dots, y_{2,t}^{N_t}, \alpha, \lambda$.

Actually, this system can solve it numerically as solving a system of linear equations as in linear algebra, but we will solve this system analytically to find the explicit formula for y_2 in terms of equilibrium price P which is done as follows:

from (1) in the system we can rewrite it as follows: where the first term of $\frac{\partial L}{\partial y_{2,t}^i} = 0$ implies

$$y_{2,t}^i = \pi_2 + \frac{\alpha}{2} - \frac{\lambda N_t}{2} P_{i,t} \quad (5.9)$$

multiply (5.9) by $P_{i,t}$ and sum up to m we get

$$\sum_{i=1}^m y_{2,t}^i P_{i,t} = \pi_2 \sum_{i=1}^m P_{i,t} + \frac{\alpha}{2} \sum_{i=1}^m P_{i,t} - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2 \text{ where}$$

$$\sum_{i=1}^m y_{2,t}^i P_{i,t} = \sum_{i=1}^{N_t} y_{2,t}^i P_{i,t} - \sum_{i=m+1}^{N_t} y_{2,t}^i P_{i,t} = 0 - (N_t - m) P_{N_t,t} y_{2,t}^{N_t}, \text{ since } \sum_{i=1}^{N_t} y_{2,t}^i P_{i,t} = 0, \text{ thus } \sum_{i=1}^m y_{2,t}^i P_{i,t} = -(N_t - m) P_{N_t,t} y_{2,t}^{N_t}.$$

Also, $\sum_{i=1}^m P_{i,t} = \sum_{i=1}^{N_t} P_{i,t} - \sum_{i=m+1}^{N_t} P_{i,t}$, since $\sum_{i=1}^{N_t} P_{i,t} = 1$, thus $\sum_{i=1}^m P_{i,t} = 1 - (N_t - m) P_{N_t,t}$ so last equation is equal to

$$\sum_{i=1}^{N_t} y_{2,t}^i P_{i,t} - (N_t - m) P_{N_t,t} y_{2,t}^{N_t} = \pi_2 (1 - (N_t - m) P_{N_t,t}) + \frac{\alpha}{2} (1 - (N_t - m) P_{N_t,t}) - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2$$

$$- (N_t - m) P_{N_t,t} y_{2,t}^{N_t} = \pi_2 (1 - (N_t - m) P_{N_t,t}) + \frac{\alpha}{2} (1 - (N_t - m) P_{N_t,t}) - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2 \quad (5.10)$$

Now sum (5.9) up to m we get

$$\sum_{i=1}^m y_{2,t}^i = \pi_2 m + \frac{\alpha}{2} m - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t} \text{ which is equal to } \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i - \frac{(N_t - m)}{N_t} y_{2,t}^{N_t} = \frac{\pi_2 m}{N_t} + \frac{\alpha m}{2 N_t} - \frac{\lambda}{2} (1 - (N_t - m) P_{N_t,t})$$

which is equivalent to

$$\pi_2 - \frac{(N_t - m)}{N_t} y_{2,t}^{N_t} = \frac{\pi_2 m}{N_t} + \frac{\alpha m}{2 N_t} - \frac{\lambda}{2} (1 - (N_t - m) P_{N_t,t}) \text{ hence,}$$

$$y_{2,t}^{N_t} = \frac{-\pi_2 N_t (\frac{m}{N_t} - 1)}{(N_t - m)} - \frac{\alpha}{2(N_t - m)} + \frac{\lambda N_t (1 - (N_t - m) P_{N_t,t})}{2(N_t - m)} \quad (5.11)$$

plug the equation (5.11) in (5.10) to get the value of λ in terms of α we get

$$-(N_t - m)P_{N_t,t} \left[\frac{-\pi_2 N_t (\frac{m}{N_t} - 1)}{(N_t - m)} - \frac{\alpha}{2(N_t - m)} + \frac{\lambda N_t (1 - (N_t - m)P_{N_t,t})}{2(N_t - m)} \right] = \pi_2 (1 - (N_t - m)P_{N_t,t}) + \frac{\alpha}{2} (1 - (N_t - m)P_{N_t,t}) - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2$$

simplifying we get

$$-\pi_2 N_t (\frac{m}{N_t} - 1) P_{N_t,t} + \frac{\alpha P_{N_t,t}}{2} - \frac{\lambda N_t P_{N_t,t} (1 - (N_t - m)P_{N_t,t})}{2} = \pi_2 (1 - (N_t - m)P_{N_t,t}) + \frac{\alpha}{2} (1 - (N_t - m)P_{N_t,t}) - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2$$

more simplifying we get

$$\frac{\alpha}{2} P_{N_t,t} - \frac{\alpha}{2} (1 - (N_t - m)P_{N_t,t}) = \pi_2 N_t (\frac{m}{N_t} - 1) P_{N_t,t} + \pi_2 (1 - (N_t - m)P_{N_t,t}) - \frac{\lambda N_t}{2} \sum_{i=1}^m P_{i,t}^2 + \frac{\lambda N_t P_{N_t,t} (1 - (N_t - m)P_{N_t,t})}{2}$$

which is equal to

$$\frac{\alpha}{2} [P_{N_t,t} - (1 - (N_t - m)P_{N_t,t})] = \pi_2 [P_{N_t,t} N_t (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})] + \frac{\lambda N_t}{2} [P_{N_t,t} (1 - (N_t - m)P_{N_t,t}) - \sum_{i=1}^m P_{i,t}^2] \text{ so that we get}$$

$$\frac{\alpha}{2} = \frac{\pi_2 [P_{N_t,t} N_t (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})] + \frac{\lambda N_t}{2} [N_t P_{N_t,t} (1 - (N_t - m)P_{N_t,t}) - \sum_{i=1}^m P_{i,t}^2]}{[P_{N_t,t} - (1 - (N_t - m)P_{N_t,t})]} \quad (5.12)$$

Now, from the equation (3) in the system, we have $\frac{\alpha}{N_t \lambda} = P_{N_t,t}$ this implies

$$\alpha = N_t \lambda P_{N_t,t} \quad (5.13)$$

then plug the equation (5.13) into (5.12) to get the value of λ as follows:

$$\frac{N_t \lambda P_{N_t,t}}{2} = \frac{\pi_2 [P_{N_t,t} N_t (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})] + \frac{\lambda N_t}{2} [N_t P_{N_t,t} (1 - (N_t - m)P_{N_t,t}) - \sum_{i=1}^m P_{i,t}^2]}{[P_{N_t,t} - (1 - (N_t - m)P_{N_t,t})]}$$

multiplying two sides by $[P_{N_t,t} - (1 - (N_t - m)P_{N_t,t})]$ and simplifying we get

$$\frac{N_t \lambda}{2} [(P_{N_t,t}^2 - P_{N_t,t} (1 - (N_t - m)P_{N_t,t})) - P_{N_t,t} (1 - (N_t - m)P_{N_t,t}) + \sum_{i=1}^m P_{i,t}^2] = \pi_2 [N_t P_{N_t,t} (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})]$$

simplifying we get the value of λ

$$\lambda = \frac{2[N_t P_{N_t,t} (\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})]}{N_t [P_{N_t,t}^2 - 2P_{N_t,t} (1 - (N_t - m)P_{N_t,t}) + \sum_{i=1}^m P_{i,t}^2]} \quad (5.14)$$

since we have the value of α and λ we can get the value of $y_{2,t}^i$ from (5.9) and $y_{2,t}^{N_t}$ from (5.11) and since that $y_{2,t}^{m+1} = y_{2,t}^{m+2} = \dots = y_{2,t}^{N_t}$ from (2) in the system. Thus,

$$y_{2,t}^i = \pi_2 + \frac{\lambda N_t P_{N_t,t}}{2} - \frac{\lambda N_t}{2} P_{i,t}, \quad i = 1, \dots, m$$

and

$$y_{2,t}^i = \frac{-\pi_2 N_t (\frac{m}{N_t} - 1)}{(N_t - m)} - \frac{\lambda N_t P_{N_t,t}}{2(N_t - m)} + \frac{\lambda N_t (1 - (N_t - m)P_{N_t,t})}{2(N_t - m)}, \quad i = m + 1, \dots, N_t$$

where

$$\lambda = \frac{2[N_t P_{N_t,t}(\frac{m}{N_t} - 1) + (1 - (N_t - m)P_{N_t,t})]}{N_t[P_{N_t,t}^2 - 2P_{N_t,t}(1 - (N_t - m)P_{N_t,t}) + \sum_{i=1}^m P_{i,t}^2]}$$

Hence, $y_{2,t}^i$ is the share of investor 2 at time t under scenario i . Thus, the final equilibrium allocation for the second investor is $y_2^* = (y_{2,t}^1, y_{2,t}^2, \dots, y_{2,t}^{N_t})$. \square

Proposition 5.3. *For exponential utility function $U = E[u(X)]$, where $u(X) = 1 - \exp(-\alpha_1 X)$, is the utility of uncertain profit at end of the investment period, α_1 is the risk aversion. Then, at each $t \in T$ and let N_t is the number of states of the word. Then, there is fair allocation ‘equilibrium allocation’ for the investor in terms of equilibrium price, and the equilibrium allocation can be written in an explicit formula as follows:*

$y_{1,t}^i = \frac{1}{\alpha_1} [\sum_{i=1}^{N_t} P_{i,t} \ln(\frac{N_t}{\alpha_1} P_{i,t})] - (\frac{1}{\alpha_1} \ln(\frac{N_t}{\alpha_1} P_{i,t}))$, where $i = 1, 2, \dots, N_t$. Then $y_{1,t}^* = (y_{1,t}^1, y_{1,t}^2, \dots, y_{1,t}^{N_t})$.

Proof. To find the explicit formula for equilibrium allocation (fair allocation), we will solve

$$\max E[u(y_1)] \quad \text{s.t.} \quad P(y_1) \leq 0$$

by Lagrange multipliers as follows: then

$$\frac{\partial U}{\partial y_{1,t}^i} = \lambda P_{i,t}, \quad i = 1, \dots, N_t \quad (5.15)$$

implies

$$\frac{\alpha_1}{N_t} \exp(-\alpha_1 y_{1,t}^i) = \lambda P_{i,t}, \quad (5.16)$$

Solving (5.16) in $y_{1,t}^i$. Then, $y_{1,t}^i = -\frac{1}{\alpha_1} \ln(\frac{N_t}{\alpha_1} P_{i,t}) - \frac{1}{\alpha_1} \ln(\lambda)$ where $i = 1, \dots, N_t$, then multiplying $y_{1,t}^i$ by $P_{i,t}$, $i = 1, \dots, N_t$, taking into account that the normalisation of price function $\sum_{i=1}^{N_t} P_{i,t} = 1$ and $\sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i = 0$ it should be from the constraint and definition of price function. Then, solve these equations by taking the summation of these equations in order to get the value of λ where $\sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i = \sum_{i=1}^{N_t} -\frac{P_{i,t}}{\alpha_1} (\frac{N_t}{\alpha_1} P_{i,t}) - \frac{1}{\alpha_1} \ln(\lambda) \sum_{i=1}^{N_t} P_{i,t}$ then we will get:

$$\frac{-1}{\alpha_1} \ln(\lambda) = \sum_{i=1}^{N_t} \frac{1}{\alpha_1} [P_{i,t} \ln(\frac{N_t}{\alpha_1} P_{i,t})]$$

substituting the value of λ in $(y_{1,t}^i)$, $i = 1, \dots, N_t$

to get the value of $y_{1,t}^i$ as a function of $P_{i,t}$, $i = 1, \dots, N_t$ then

$$y_{1,t}^i = \sum_{i=1}^{N_t} \frac{1}{\alpha_1} [P_{i,t} \ln(\frac{N_t}{\alpha_1} P_{i,t})] - (\frac{1}{\alpha_1} \ln(\frac{N_t}{\alpha_1} P_{i,t})), \quad \text{where } i = 1, 2, \dots, N_t. \quad \square$$

Proposition 5.4. *Quadratic utility function of the form $u(X) = X - \alpha_1 X^2$, is the utility of uncertain profit at the end of the investment period, α_1 is the risk aversion, and let N_t be the number of scenarios at time t . Then, at each*

$t \in T$ there is fair allocation ‘equilibrium allocation’ for the investor in terms of equilibrium price, and the equilibrium allocation can be written in an explicit formula as follows:

$$y_{1,t}^i = \frac{1}{2\alpha_1} \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right], \quad i = 1, \dots, N_t$$

Proof. Similar to all previous proofs by straightforwardly using maximum utility function to solve $\max E[u(y_1)]$ s.t $P(y_1) \leq 0$, where $U = E[u(.)]$ and $U = \frac{1}{N_t} [\sum_{i=1}^{N_t} (y_{1,t}^i - \alpha_1 (y_{1,t}^i)^2)]$. Hence, by maximisation of utility function we have

$$\frac{\partial U}{\partial y_{1,t}^i} = \lambda P_{i,t}, \quad i = 1, \dots, N_t$$

implies

$$\frac{1}{N_t} [1 - 2\alpha_1 y_{1,t}^i] = \lambda P_{i,t}$$

thus,

$$y_{1,t}^i = \frac{1}{2\alpha_1} [1 - N_t \lambda P_{i,t}], \quad i = 1, \dots, N_t \quad (5.17)$$

multiplying $y_{1,t}^i$ by $P_{i,t}$ $i = 1, \dots, N_t$, taking into account that the normalisation of price function $\sum_{i=1}^{N_t} P_{i,t} = 1$ and from consistent price function we have $\sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i = 0$. Then, solve these equations by taking the summation of these equations in order to get the value of λ where,

$$\sum_{i=1}^{N_t} P_{i,t} y_{1,t}^i = \frac{1}{2\alpha_1} \sum_{i=1}^{N_t} P_{i,t} - \frac{N_t}{2\alpha_1} \lambda \sum_{i=1}^{N_t} P_{i,t}^2$$

thus,

$$\lambda = \frac{1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2}$$

plugging the value of λ into the value of $y_{1,t}^i$ from equation (5.17) we get

$$y_{1,t}^i = \frac{1}{2\alpha_1} \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right], \quad i = 1, 2, \dots, N_t.$$

Thus, $y_{1,t}^* = (y_{1,t}^1, y_{1,t}^2, \dots, y_{1,t}^{N_t})$

□

5.1.1 Certainty equivalent

In this section we will find the fair allocation point for the problem (4.4) in Chapter 4 by using certainty equivalent.

Proposition 5.5. *A fair allocation for the investor in terms of equilibrium price where the certainty equivalent for the investor is expressed as $C = U(y_1)$, where $U(y_1) = E[y_1] - \frac{\sigma^2(y_1)}{2\rho_1}$ where $E[y_1] = \frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i$ and ρ_1 is the risk aversion. Then, the explicit formula for equilibrium allocation at each $t \in T$ can be written as follows:*

$$y_{1,t}^i = \rho_1 N_t \sum_{i=1}^{N_t} P_{i,t}^2 \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right]$$

where $i = 1, \dots, N_t$. Thus, an equilibrium allocation for investor is $y_{1,t} = (y_{1,t}^1, y_{1,t}^2, \dots, y_{1,t}^{N_t})$, and N_t is the number of the states of the word.

Proof. Straightforward by using maximum of the function U which is written as

$$\max U(y_1) \text{ s.t } P(y_1) \leq 0$$

Thus, we will solve this individual problem in order to get the explicit formula for equilibrium allocation for the investor as follows: $\nabla U = \lambda P$ where $U(y_1) = E[y_1] - \frac{\sigma(y_1)^2}{2\rho_1}$ which is equal to $U(y_1) = \frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i - \frac{1}{2\rho_1} \left[\frac{1}{N_t} \sum_{i=1}^{N_t} (y_{1,t}^i)^2 - \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)^2 \right]$. Then, $\frac{\partial U}{\partial y_{1,t}^i} = \lambda P_{i,t}$ which is implied

$$\frac{1}{N_t} - \frac{1}{2\rho_1} \left[\frac{2}{N_t} y_{1,t}^i - 2 \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) \frac{1}{N_t} \right] = \lambda P_{i,t} \quad (5.18)$$

simplifying we get

$$\frac{1}{N_t} - \frac{1}{2\rho_1} \left[\frac{2}{N_t} (y_{1,t}^i - \frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i) \right] = \lambda P_{i,t}$$

multiplying the last equation by $P_{i,t}$ in two sides for each i and summing both sides, and taking into account that $\sum_{i=1}^{N_t} y_{1,t}^i P_{i,t} = 0$ is from the constraint and the definition of price function, as well as $\sum_{i=1}^{N_t} P_{i,t} = 1$ and $E[y_1] = \frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i$. Then we get

$$\frac{1}{N_t} - \frac{1}{2\rho_1} \left[\frac{2}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i P_{i,t} - \frac{2}{N_t} \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) \sum_{i=1}^{N_t} P_{i,t} \right] = \lambda \sum_{i=1}^{N_t} P_{i,t}^2$$

simplifying we get

$$\frac{1}{N_t} - \frac{1}{2\rho_1} \left[-\frac{2}{N_t} \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) \right] = \lambda \sum_{i=1}^{N_t} P_{i,t}^2$$

Thus,

$$\lambda = \frac{1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} + \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)}{N_t \rho_1 \sum_{i=1}^{N_t} P_{i,t}^2}$$

plug the value of λ into equation (5.18) we get

$$\frac{1}{N_t} - \frac{1}{2\rho_1} \left[\frac{2}{N_t} y_{1,t}^i - \frac{2 \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)}{N_t} \right] = \frac{P_{i,t}}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} + \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) P_{i,t}}{N_t \rho_1 \sum_{i=1}^{N_t} P_{i,t}^2} \quad (5.19)$$

hence,

$$\frac{y_{1,t}^i}{N_t \rho_1} = \frac{1}{N_t} + \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)}{\rho_1 N_t} - \frac{P_{i,t}}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} - \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) P_{i,t}}{N_t \rho_1 \sum_{i=1}^{N_t} P_{i,t}^2}$$

multiplying each side by $N_t \rho_1$ we get

$$y_{1,t}^i = \rho_1 + \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) - \frac{P_{i,t} \rho_1}{\sum_{i=1}^{N_t} P_{i,t}^2} - \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \quad (5.20)$$

simplifying and delete ρ_1 in the second term we get

$$y_{1,t}^i = \rho_1 + \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) - \frac{P_{i,t} \rho_1}{\sum_{i=1}^{N_t} P_{i,t}^2} - \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \quad (5.21)$$

Now, sum all last equation for each $i = 1, \dots, N_t$ and divided it by N_t we get

$$\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) = \rho_1 + \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) - \frac{\rho_1 \sum_{i=1}^{N_t} P_{i,t}}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} - \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) \sum_{i=1}^{N_t} P_{i,t}}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} \quad (5.22)$$

simplifying, where $\sum_{i=1}^{N_t} P_{i,t} = 1$ we get

$$\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) = \rho_1 + \left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right) - \frac{\rho_1}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} - \frac{\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)}{N_t \sum_{i=1}^{N_t} P_{i,t}^2} \quad (5.23)$$

solve equations (5.21) and (5.23) to delete $\left(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i \right)$ or simplifying equation (5.23) and plug it in equation (5.21), where $\frac{1}{N_t} \sum_{i=1}^{N_t} y_{1,t}^i = E[y_1]$, then we get

$$y_{1,t}^i = \rho_1 \left(N_t \sum_{i=1}^{N_t} P_{i,t}^2 - 1 \right) \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right] + \rho_1 \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right]$$

simplifying the last equation we get

$$y_{1,t}^i = \rho_1 N_t \sum_{i=1}^{N_t} P_{i,t}^2 \left[1 - \frac{P_{i,t}}{\sum_{i=1}^{N_t} P_{i,t}^2} \right]$$

Thus, the fair allocation for the investor is $y_{1,t}^* = (y_{1,t}^1, y_{1,t}^2, \dots, y_{1,t}^{N_t})$. \square

Proposition 5.6. *A fair allocation for the investor in terms of equilibrium price where the certainty equivalent for the investor is expressed as $C = U(y_2)$, where $U(y_2) = E[y_2] - \frac{\sigma^2(y_2)}{\rho_2}$ where $E[y_2] = \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i$ and ρ_2 is the risk aversion, where assuming the price $P_{1,t} \geq P_{2,t} \geq \dots \geq P_{N_t,t}$, N_t is the number of the states of the word, and m is the minimal number such that $P_{m+1,t} = P_{m+2,t} = \dots = P_{N_t,t}$. Then, the explicit formula for equilibrium allocation at each $t \in T$ can be written as follows:*

$$y_{2,t}^i = \frac{\rho_2}{2} \left\{ 1 - \frac{P_{i,t}}{P_{N_t,t}} - \frac{2[\frac{m}{2} - \frac{1}{P_{N_t,t}} + \frac{1}{2P_{N_t,t}^2} \sum_{i=1}^m P_{i,t}^2 + (N_t - m)]}{[N_t + \rho_2(m + \frac{1}{P_{N_t,t}} - (N_t - m))]} \right\}$$

this for $i = 1, \dots, m$ and

$$y_{2,t}^i = \frac{\rho_2}{2(N_t - m)} \left\{ -m + \frac{1}{P_{N_t,t}} (1 - (N_t - m)P_{N_t,t}) \right. \\ \left. + 2N_t \left(1 + \frac{m\rho_2}{N_t} \right) \frac{[\frac{m}{2} - \frac{1}{P_{N_t,t}} + \frac{1}{2P_{N_t,t}^2} \sum_{i=1}^m P_{i,t}^2 + (N_t - m)]}{[N_t + \rho_2(m + \frac{1}{P_{N_t,t}} - (N_t - m))]} \right\}$$

this for $i = m + 1, \dots, N_t$

Thus, an equilibrium allocation for investor is $y_{2,t} = (y_{2,t}^1, y_{2,t}^2, \dots, y_{2,t}^{N_t})$.

Proof. Straightforward by using maximum of the function U which is written as

$$\max U(y_2) \text{ s.t } P(y_2) \leq 0$$

Thus, we will solve this individual problem in order to get the explicit formula for equilibrium allocation for the investor as follows: $\nabla U = \lambda P$ where $U(y_2) = (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i) - \frac{\sigma^2(y_2)}{\rho_2}$ which is equal to

$U(y_2) = \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i - \frac{1}{\rho_2} [\frac{1}{N_t} \sum_{i=1}^{N_t} (\max(-(y_{2,t}^i - \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i), 0))^2]$. Then, $\frac{\partial U}{\partial y_{2,t}^i} = \lambda P_{i,t}$ which is implied

$$\frac{\partial U}{\partial y_{2,t}^i} = \begin{cases} \frac{1}{N_t} - \frac{1}{\rho_2} [\frac{2}{N_t} y_{2,t}^i - \frac{2}{N_t} (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)] - \lambda P_{i,t} = 0 & \text{if } y_{2,t}^i \leq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i \\ \frac{1}{N_t} - \lambda P_{i,t} = 0 & y_{2,t}^i > \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i \end{cases}$$

Hence, we need four statements as in proposition (5.2) to avoid cases where the first statement

(1) In optimality equation $P(y_2) = 0$ is held.

The statements (2), and (3) are the same as the statements (2), and (3) in proposition (5.2), also $E[y_2]$ is not fixed and $E[y_2] \neq \pi_2$.

4) Obviously, $y_{2,t}^{N_t} \geq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i$, then $\frac{1}{N_t} - \lambda P_{N_t,t} = 0$, thus $P_{N_t,t} = \frac{1}{N_t \lambda}$ from the second terms of $\frac{\partial L}{\partial y_{2,t}^i} = 0$. Now, let m be the minimal number such that $P_{m+1,t} = P_{m+2,t} = \dots = P_{N_t,t}$ then by (2) we have $y_{2,t}^{m+1} = \dots = y_{2,t}^{N_t}$. Then for any $i \leq m$ $P_{i,t} > P_{N_t,t} = \frac{1}{N_t \lambda}$ so $\frac{1}{N_t} - \lambda P_{i,t} \neq 0$ then $y_{2,t}^i \leq \frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i$ this implied the first term of $\frac{\partial L}{\partial y_{2,t}^i} = 0$ which is $\frac{1}{N_t} - \frac{1}{\rho_2} [\frac{2}{N_t} y_{2,t}^i - \frac{2}{N_t} (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)] - \lambda P_{i,t} = 0$. These statements imply a system of linear equations as follows:

- 1) $\frac{1}{N_t} - \frac{1}{\rho_2} [\frac{2}{N_t} y_{2,t}^i - \frac{2}{N_t} (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)] - \lambda P_{i,t} = 0$ which include m equations
- 2) $y_{2,t}^{m+1} = y_{2,t}^{m+2} = \dots = y_{2,t}^{N_t}$ which include $m - N_t - 1$ equations.
- 3) $\frac{1}{N_t} = \lambda P_{N_t,t}$ which includes one equation.
- 4) $\sum_{i=1}^{N_t} P_{i,t} y_{2,t}^i$ which includes one equation.

Thus the system include N_t+1 equation with N_t+1 variables which are $y_{2,t}^1, \dots, y_{2,t}^{N_t}, \lambda$. This system can be solved numerically as solving a system of linear equations as in linear algebra, but we will solve this system analytically to get the explicit formula for equilibrium allocation in terms of equilibrium price as follows: we have m equations from (1) in the system

$$\frac{1}{N_t} - \frac{1}{\rho_2} [\frac{2}{N_t} y_{2,t}^i - \frac{2}{N_t} (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)] - \lambda P_{i,t} = 0 \quad (5.24)$$

multiply (5.24) by $P_{i,t}$ and sum up to m we get

$$\frac{1}{N_t} \sum_{i=1}^m P_{i,t} - \frac{2}{N_t \rho_2} \sum_{i=1}^m P_{i,t} y_{2,t}^i + \frac{2}{N_t} (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i) \sum_{i=1}^m P_{i,t} - \lambda \sum_{i=1}^m P_{i,t}^2 = 0$$

where $\sum_{i=1}^m P_{i,t} = \sum_{i=1}^{N_t} P_{i,t} - \sum_{i=m+1}^{N_t} P_{i,t} = 1 - (N_t - m)P_{N_t}$ and $\sum_{i=1}^m P_{i,t} y_{2,t}^i =$

$$0 - (N_t - m)P_{N_t}y_{2,t}^{N_t} = -(N_t - m)P_{N_t}y_{2,t}^{N_t}.$$

Thus the last equation is equivalent to

$$\begin{aligned} \frac{1}{N_t}(1 - (N_t - m)P_{N_t,t}) - \frac{2}{N_t\rho_2}(-(N_t - m)P_{N_t,t}y_{2,t}^{N_t}) \\ + \frac{2}{N_t}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right)(1 - (N_t - m)P_{N_t,t}) - \lambda\sum_{i=1}^m P_{i,t}^2 = 0 \end{aligned} \quad (5.25)$$

additionally we can simplify (5.25) to write it in $y_{2,t}^{N_t}$ as follows:

$$\begin{aligned} y_{2,t}^{N_t} &= \frac{-\rho_2}{2(N_t - m)P_{N_t,t}}(1 - (N_t - m)P_{N_t,t}) \\ &\quad - \frac{\rho_2}{(N_t - m)P_{N_t,t}}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right)(1 - (N_t - m)P_{N_t,t}) \\ &\quad + \frac{N_t\rho_2\lambda}{2(N_t - m)P_{N_t,t}}\sum_{i=1}^m P_{i,t}^2 \end{aligned} \quad (5.26)$$

Now sum (5.24) up to m where $\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i = E[y_2]$, then we get

$$\frac{1}{N_t}m - \frac{2}{\rho_2 N_t}\sum_{i=1}^m y_{2,t}^i - \frac{2}{N_t}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right)m - \lambda\sum_{i=1}^m P_{i,t} = 0$$

simplifying we get

$$\frac{2}{\rho_2 N_t}\sum_{i=1}^m y_{2,t}^i = \frac{m}{N_t} - \frac{2m}{N_t}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right) - \lambda\sum_{i=1}^m P_{i,t}$$

dividing the last equation by $\frac{N_t\rho_2}{2}$ we get

$$\sum_{i=1}^m y_{2,t}^i = \frac{\rho_2 m}{2} - m\rho_2\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right) - \frac{N_t\rho_2}{2}\lambda\sum_{i=1}^m P_{i,t}$$

plug the value of $\sum_{i=1}^m y_{2,t}^i$ and divided over N_t we get

$$\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i - \frac{(N_t-m)}{N_t}y_{2,t}^{N_t} = \frac{\rho_2 m}{2N_t} - \frac{m\rho_2}{N_t}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right) - \frac{\rho_2\lambda}{2}(1 - (N_t - m)P_{N_t,t})$$

which is equal to

$$\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right) - \frac{(N_t-m)}{N_t}y_{2,t}^{N_t} = \frac{\rho_2 m}{2N_t} - \frac{m\rho_2}{N_t}\left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right) - \frac{\rho_2\lambda}{2}(1 - (N_t - m)P_{N_t,t})$$

simplifying we get

$$-\frac{(N_t-m)}{N_t}y_{2,t}^{N_t} = \frac{\rho_2 m}{2N_t} - \left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right)\left(1 + \frac{m\rho_2}{N_t}\right) - \frac{\rho_2\lambda}{2}(1 - (N_t - m)P_{N_t,t}),$$

thus

$$y_{2,t}^{N_t} = \frac{-\rho_2 m}{2(N_t - m)} + \left(\frac{1}{N_t}\sum_{i=1}^{N_t}y_{2,t}^i\right)\frac{N_t}{(N_t - m)}\left(1 + \frac{m\rho_2}{N_t}\right) + \frac{\rho_2 N_t \lambda}{2(N_t - m)}(1 - (N_t - m)P_{N_t,t}) \quad (5.27)$$

Now the equality of equations (5.26) equal to (5.27) held which we need to delete the value of $(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)$ we get

$$\begin{aligned} & \frac{-\rho_2 m}{2(N_t - m)} + (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i) \frac{N_t}{(N_t - m)} [1 + \frac{m\rho_2}{N_t}] + \frac{\rho_2 N_t \lambda}{2(N_t - m)} (1 - (N_t - m)P_{N_t,t}) \\ &= \frac{-\rho_2}{2(N_t - m)P_{N_t,t}} (1 - (N_t - m)P_{N_t,t}) - \\ & \frac{\rho_2 (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)}{(N_t - m)P_{N_t,t}} (1 - (N_t - m)P_{N_t,t}) \\ &+ \frac{N_t \rho_2 \lambda}{2(N_t - m)P_{N_t,t}} \sum_{i=1}^m P_{i,t}^2 \end{aligned}$$

simplifying last equation where $\lambda = \frac{1}{N_t P_{N_t,t}}$ and we assume $P_{N_t,t} \neq 0$ we get

$$\begin{aligned} & (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i) [\frac{N_t}{(N_t - m)} + \rho_2 (\frac{m}{(N_t - m)} + \frac{1}{(N_t - m)} P_{N_t,t} - 1)] \\ &= \rho_2 [\frac{m}{2(N_t - m)} - \frac{1}{(N_t - m)P_{N_t,t}} + \frac{1}{2(N_t - m)P_{N_t,t}^2} \sum_{i=1}^m P_{i,t}^2 + 1] \end{aligned} \quad (5.28)$$

thus, from (5.24) we have

$$y_{2,t}^i = \frac{\rho_2}{2} - (\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i) - \frac{\rho_2}{2P_{N_t,t}} P_{i,t} \quad (5.29)$$

Hence, just by plugging the expression from (5.28) multiply by $(\frac{1}{N_t} \sum_{i=1}^{N_t} y_{2,t}^i)$ in to equation (5.27) and (5.29), then the optimal sharing (equilibrium allocation in terms of equilibrium price is as follows:

$$y_{2,t}^i = \frac{\rho_2}{2} \left\{ 1 - \frac{P_{i,t}}{P_{N_t,t}} - \frac{2[\frac{m}{2} - \frac{1}{P_{N_t,t}} + \frac{1}{2P_{N_t,t}^2} \sum_{i=1}^m P_{i,t}^2 + (N_t - m)]}{[N_t + \rho_2(m + \frac{1}{P_{N_t,t}} - (N_t - m))]} \right\}$$

this for $i = 1, \dots, m$ and

$$\begin{aligned} y_{2,t}^i &= \frac{\rho_2}{2(N_t - m)} \left\{ -m + \frac{1}{P_{N_t,t}} (1 - (N_t - m)P_{N_t,t}) \right. \\ & \quad \left. + 2N_t(1 + \frac{m\rho_2}{N_t}) \frac{[\frac{m}{2} - \frac{1}{P_{N_t,t}} + \frac{1}{2P_{N_t,t}^2} \sum_{i=1}^m P_{i,t}^2 + (N_t - m)]}{[N_t + \rho_2(m + \frac{1}{P_{N_t,t}} - (N_t - m))]} \right\} \end{aligned}$$

this for $i = m + 1, \dots, N_t$.

Thus, an equilibrium allocation for the investor is $y_{2,t} = (y_{2,t}^1, y_{2,t}^2, \dots, y_{2,t}^N)$.

□

5.1.2 Finding the equilibrium price P

- Firstly, we express each equilibrium allocation as a function of equilibrium price P , we can then find P from the system of equations

$$\begin{aligned} y_1^i(P) + y_2^i(P) &= X_\beta(\omega_i), \quad i = 1, \dots, N_t \\ P(X_\beta(\omega_i)) &= 0 \\ \sigma_2^2(y_2) &= \beta \end{aligned} \tag{5.30}$$

where $X_\beta(\omega_i)$ is the uncertain outcome of the optimal portfolio at the end of the investment period.

- Secondly, The system (5.30) can be solved as a feasibility problem for the optimisation problem

$$\begin{aligned} \min_p \quad & 0 \\ \text{s.t} \quad & \\ y_1^i(P) + y_2^i(P) &= X_\beta(\omega_i), \quad i = 1, \dots, N_t \\ \sigma_2^2(y_2) &= \beta \\ P(X_\beta(\omega_i)) &= 0. \end{aligned} \tag{5.31}$$

Note that, $X_\beta(\omega_i)$ depends on β but does not depend on price P and we solve problem (5.31) over P . Also, $X_\beta(\omega_i)$, $i = 1, \dots, N_t$ is arbitrary and can be found just by solving cooperative investment problem, for example, problem (4.3) and the purpose of solving the problem (5.31) is to check the feasibility of $y_1(P)$ and $y_2(P)$. Hence we find the value of equilibrium price P and then we can calculate the value of $y_1(P)$ and $y_2(P)$ which are an equilibrium allocation ‘fair point’ for each investor. In addition, the size of P is equal to the size of uncertain outcome $X_\beta(\omega_i)$, y_1 and y_2 where the size is $1 \times N_t$, and N_t is the number of the scenario tree. Moreover, the whole

process from begin of solving cooperative investment problem until to find the ‘fair point’ equilibrium allocation will be shown in Algorithm 5.15.

Remark 5.7. Now in case of two investors, after we solve the problem (4.4) in Chapter 4 by following the Algorithm 4.2, we need to find the fair allocation and check the feasibility by solving the problem (5.31) for $m = 2$. Note that, for example the fair allocation can be written from proposition 5.5 which is $y_{1,t}^i(P)$ for the first investor and proposition 5.6 $y_{2,t}^i(P)$ for the second investor, and plug the value of $y_{1,t}^i(P)$ and $y_{2,t}^i(P)$ in (5.31) where $y_{1,t}^i + y_{2,t}^i = X_\beta(\omega_i)$ and $X_\beta(\omega_i)$ is uncertain outcome at the end of investment period at scenario i . More details to solve problem (5.31) is shown in Algorithm 5.15. Also, $X_\beta(\omega_i)$ depends on β so that we need to fix the value of $\sigma_-^2(y_2) = \beta$ in the whole curve in efficient frontier as shown in section (5.1.2).

5.2 Uniqueness and stability

Now the natural question after finding the explicit formula for equilibrium allocation is that we need to study the uniqueness and stability of equilibrium price since the equilibrium allocation is written as a function of equilibrium price. According to Welfare’s theorems, every Walrasian equilibrium allocation is a Pareto optimal. Hence, we need to prove the uniqueness of Walrasian equilibrium for cooperative investment.

Grechuk and Zabaranin [68] demonstrated the connection between Walrasian equilibrium for cooperative investment and standard Walrasian equilibrium (5.2) for the risk sharing problem. In particular, if equilibrium is unique in the risk sharing problem for *any* set of initial endowments W_i such that $\sum_{i=1}^m W_i = X$, that it is unique for the cooperative investment problem with optimal portfolio X . Motivated by this, we will study the uniqueness and stability questions in the context of problem (5.2).

We will assume the following properties for agent’s utility functions u_i , where $U_i = E[u_i(\cdot)]$

- A1) For all agents $i \in I$, u_i is continuous.
- A2) For all agents $i \in I$, u_i is increasing, i.e. $u_i(y_i) > u_i(x_i)$ whenever $P(y_i \geq x_i) = 1$ and $P(y_i > x_i) > 0$.
- A3) For all agents $i \in I$, u_i is concave.

The second Welfare theorem says that if we start from initial endowment we can redistribute the resource and set price that ensures the allocation is Walrasian equilibrium. Now, $Z(P, W)$ is the aggregate demand function, Debreu [49]. It is defined as $Z(P, W) = \sum_{i=1}^2 y_i(P) - W$, where $W \in R$ is the initial endowment, and P is the equilibrium price vector at the end of investment period, in my experiments $y_i(P)$ that comes from an explicit formula from propositions in section 5.2. Thus, $Z_i(P_{i,t}, W) = y_{1,t}^i(P) + y_{2,t}^i(P) - W_i$. Note that we start to find $y_1(P)$ and $y_2(P)$ from propositions in section 5.2 and we used them to get the equilibrium price P from solving problem (5.31), then we find $Z(P, W)$ and we use it to investigate the uniqueness of equilibrium price. In addition, in this section we assume that no initial endowment available so that we will get $y_i(P)$, $i = 1, 2$ from problem (5.3) instead of problem (5.2). Furthermore, with these general facts we will start with some basic concepts and definitions from the Welfare theorems that show the efficiency properties of equilibrium that imply the efficient condition to prove the uniqueness of equilibrium under certain conditions. Consequently, we will investigate the level of stability of equilibrium corresponding to unique equilibrium.

Proposition 5.8. *See proposition 4 in [81] and [11]*

The aggregate demand function $Z(P, W)$, $W \in R$ and price $P \in R^n$ satisfies

- i- Z is continuous on P ;*
- ii- Z is homogeneous of degree zero;*
- iii- $Z(P, W) = 0$ for all P (Walras' law);*
- iv- for some $Z > 0$, $Z_l(P, W) > -Z$ for every $l \in \mathcal{L}$ and all P ; and*
- v- if $P^n \rightarrow P$, where $P \neq 0$ and $P_l = 0$ for some l , then*
 $\max Z_l(P^n, W), \dots, Z_L(P^n, W) \rightarrow \infty$.

For more information about the details of the proof the reader may look at Levin [81], Arrow and Debreu [11].

In addition, the definition of normalisation of equilibrium price P is as follows: let $A = \Delta^N$, where $\Delta^N = \{P \in R^N : \sum_{i=1}^N P_{i,t} = 1\}$. Note that the set Δ^N is closed, bounded and convex.

Proposition 5.9. *[81] Take the aggregate excess demand function $Z(P, W)$ that fulfills all the properties of the proposition (5.8). Then there exists an equilibrium price vector P such that $Z(P, W) = 0$.*

Proof. The details of this proof can be seen in Levin [81] and Quah [117].

□

Number of equilibria [81]

Since the aggregate demand satisfies the Walras' Law, then $Z(P, W) : R^N \rightarrow R^N$ fully characterises the economy, where the economy in my case is the risk sharing.

Moreover that $\partial Z(P, W)$ be the matrix of size $(N \times N)$ which is the matrix of price effect at normalised price P_t .

Definition 5.10. [81] An equilibrium price vector $P^* = (P_1, \dots, P_N)$ is regular, if the $\partial Z(P, W)$ has full rank N . If every normalised equilibrium price vector is regular then economy is regular.

Proposition 5.11. [81] *For any regular economy 'risk sharing', the number of equilibrium is finite and odd.*

5.2.1 Uniqueness

Theorem 5.12. [81] *The excess demand function $Z(P, W)$ satisfies the weak axiom of the revealed preferences, if for any vectors P, P' we get*

$$Z(P, W) \neq Z(P', W) \text{ and } P \cdot Z(P', W) \leq 0 \text{ implies } P' \cdot Z(P, W) > 0.$$

where $P \cdot Z(P', W)$ is the inner product between vector price P and excess demand function $Z(P, W)$ at vector price P' and initial endowment W , and the size of vector price P' is the same size of P equal to $1 \times N_t$, N_t is the number of scenarios. Thus, $P \cdot Z(P', W) = \sum_{i=1}^{N_t} P_i \cdot Z_i(P', W)$.

Now we are ready to prove the uniqueness of the equilibrium.

Proposition 5.13. [81] *If Z satisfies the weak axiom, then for any rate of returns the set of equilibrium price vectors is convex. If the economy is regular, the equilibrium price is unique.*

Now in order to prove the uniqueness we need to just prove the weak axiom on aggregate demand function. According to Quah [117] the aggregate demand function satisfies the weak axioms. In particular, he studied local weak axiom and found the condition which guarantees that the economy excess demand function

obeys the local weak axioms near equilibrium price. Hence, the weak axioms is satisfied, where the weak axioms defined in Theorem (5.12). We will illustrate that in the following Remark.

The quote of the following Remark is taken from Quah [117].

Remark 5.14. Note that since the agent's individual excess demand function is derived from utility maximisation, it will satisfy strong structural properties like the weak and strong axioms. But aggregate excess demand function may not have those properties. Hence, we need to find important conditions on agent's preferences as follows:

- i) If the utility function is not strictly concave and there is no risk aversion such as $U(x) = -\sigma(x)$ if $E[X] \geq \pi$, then the sufficient condition for the local weak axioms requires $\zeta \in R^N$ to obey the differentiable weak axiom at price P which is the negative definite on the set $P^\perp = \{\zeta \in R^N : \zeta^T P = 0\}$ i.e $\zeta^T \partial_P Z(P, W) \zeta < 0$ for all $\zeta \neq 0$ in P^\perp , where $Z(P, W)$ is aggregate excess demand.
- ii) In case of the utility function $\in C^2$ and if its strictly positive, differentiable strictly concave then the utility function is called regular. Then the utility functions straightforwardly and satisfies the minimal model program (MMP). On another word, the utility function needs to verify the condition (5.32) below is called (MMP) coefficient which guarantees that the aggregate demand function obeys the weak axioms, where MMP- coefficient can be defined as

$$\psi_u(x) = -\frac{x^T \partial^2 u(x) x}{\partial u(x) x} \quad (5.32)$$

hence,

$$\psi_{\succeq}(x) = \inf_{u \in \mathcal{U}(x)} \psi_u(x) \quad (5.33)$$

where \mathcal{U} is the collection of regular and concave utility functions of $u(\cdot)$, so tht \mathcal{U} is always not empty, since u is concave, then $\psi_u(x) \geq 0$, so $\psi_{\succeq}(x) \geq 0$ for all x , see [117]. Note that there is no restriction on $\psi_{\succeq}(x)$ which guarantees the local weak axioms.

- iii) In case if the utility function is not regular ' which means the utility function is not strictly positive or not strictly concave' and defined over contingent constraint in N state of the world and u has the expected utility form $U(\cdot) = E[u(\cdot)]$. In this case, one could show that the coefficient of MMP is satisfied if the agent's

coefficient of relative risk aversion does not vary by more than 4, i.e.

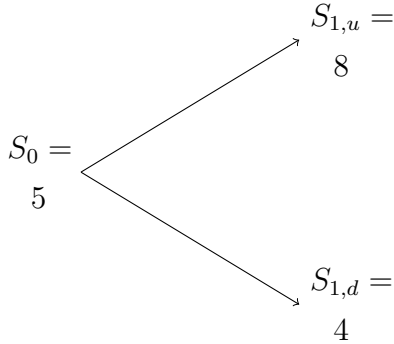
$$\max_{x>0} \left(-\frac{ru''(x)}{u'(x)} \right) - \min_{x>0} \left(-\frac{ru''(x)}{u'(x)} \right) < 4 \quad (5.34)$$

Now, we will illustrate some examples in each case in Remark (5.14).

5.2.2 Examples

Example 1:

First, we illustrate the issue using a simple example. Assume that during each period of time there exists a risk-free asset with 0 return and a single risky asset with statistically independent rate of return, with probabilities p , $1 - p$, correspondingly $0 \leq p \leq 1$, so that we invest a for risky asset with the return $\frac{8}{5}$ and $(1 - a)$ for risk-free asset with 0 return in this case the uncertainty outcome will be written as follows $X(\omega_1) = (1 - a) + \frac{8a}{5}$, thus, $X(\omega_1) = \frac{3a}{5}$. Also, in case of price going down we will invest a in risky asset with the return $\frac{4}{5}$ and $(1 - a)$ in risk-free asset with the return zero, so that we can write the uncertainty outcome $X(\omega_2) = (1 - a) + \frac{4a}{5}$. Thus, $X(\omega_2) = \frac{-a}{5}$, where $-1 \leq a \leq 1$ and a does not depend on initial endowment $W = (W_1, W_2) = 0$.



Firstly, we need to write excess demand function for the cooperative investment which in this case will be of the form: $Z_i(P, W) = y_1^i(P) + y_2^i(P) - W_i$ where $i = 1, 2$ and $W = (W_1, W_2) = (0, 0)$ which means no initial endowment exist. Suppose first investor has a variance as a risk preference and the second investor has semi-variance as a risk preference. Also, we fixed levels of return for each investor are $\pi_1 = 0.65$ and $\pi_2 = 0.26$, where the explicit formula for an individual investor uses the propositions (5.1) and (5.2) and used it in order to write the aggregate demand function for investors.

Note that, the Walras' law holds from the fact that the price function is consistent $P(X) = 0$ for all the $X \in \mathcal{F}$. Now, for the existence: suppose $\Delta^N = \{P \in R^N : \sum_{i=1}^N P_i = 1\}$ and note that the Δ^N is closed and bounded and convex, which is called dimensional unit simplex as shown from normalisation of functional price P .

Now, to check about the regularity of economy 'risk sharing' in this example we need to check about the $\partial Z(P, W)$ being 2×2 matrix of price effect at normalised price P , where

$$\partial Z(P, W) = \begin{bmatrix} \frac{\partial z_1(p, W)}{\partial P_1} & \frac{\partial z_2(p, W)}{\partial P_1} \\ \frac{\partial z_1(p, W)}{\partial P_2} & \frac{\partial z_2(p, W)}{\partial P_2} \end{bmatrix}$$

should be of rank 2 to check that we need to find

$Z_1(P, W) = y_1^1(P) + y_2^1(P) - W_1$ which is equal to

$$Z_1(P, W) = (\pi_1 + \frac{(1-2P_1)\pi_1}{2\sum_{i=1}^N P_{i,t}^2 - 1}) + \pi_2 + \frac{(P_2 - P_1)(1-2P_2)}{(P_1^2 - P_2^2 + 2P_2)} - W_1$$

and $Z_2(P, W) = y_1^2(P) + y_2^2(P) - W_2$ which is equal to

$$\begin{aligned} Z_2(P, W) &= \pi_1 + \frac{(1-2P_2)\pi_1}{2\sum_{i=1}^N P_i^2 - 1} + \pi_2 + \frac{(1-2P_2)^2}{(P_1^2 - P_2^2 + 2P_2)} - W_2 \\ \frac{\partial Z_1(p, W)}{\partial P_1} &= \frac{-2\pi_1}{2(P_1^2 + P_2^2) - 1} - \frac{4(1-2P_1)\pi_1 P_1}{(2(P_1^2 + P_2^2) - 1)^2} - \frac{(1-2P_2)}{(P_1^2 - P_2^2 + 2P_2)} - \frac{(P_2 - P_1)(1-2P_2)2P_1}{(P_1^2 - P_2^2 + 2P_2)^2} \\ \frac{\partial Z_2(p, W)}{\partial P_1} &= \frac{-4(1-2P_1)\pi_1 P_2}{(2(P_1^2 + P_2^2) - 1)^2} - \frac{2(1-2P_2)^2 P_1}{(P_1^2 - P_2^2 + 2P_2)^2} \\ \frac{\partial Z_1(p, W)}{\partial P_2} &= \frac{-4(1-2P_2)\pi_1 P_1}{(2(P_1^2 + P_2^2) - 1)^2} - \frac{1-4P_2+2P_1}{(P_1^2 - P_2^2 + 2P_2)} - \frac{(P_2 - P_1)(1-2P_2)}{(P_1^2 - P_2^2 + 2P_2)^2} \\ \frac{\partial Z_2(p, W)}{\partial P_2} &= \frac{-2\pi_1}{2(P_1^2 + P_2^2) - 1} - \frac{4(1-2P_2)\pi_1 P_2}{(2(P_1^2 + P_2^2) - 1)^2} + \frac{-4(1-2P_2)}{(P_1^2 - P_2^2 + 2P_2)} - \frac{(1-2P_2)^2(-2P_2+2)}{(P_1^2 - P_2^2 + 2P_2)^2} \end{aligned}$$

To calculate these derivatives we need the value of P_1 and P_2 so that, before checking uniqueness of equilibrium price we need to find the value of P_1 and P_2 .

Firstly, we need to solve problem (4.3) in two dimensions to find the value of X_β . From solving problem (4.3) we have $a = 0.65$ at $\beta = 0.0001$ and then plug the value of a in $X_\beta(\omega_1)$ and $X_\beta(\omega_2)$ and plug the value of $X_\beta(\omega_1)$ and $X_\beta(\omega_2)$ in the following system to check about feasibility and find P as follows:

$$\begin{aligned} &\min_p 0 \\ &s.t \\ &\pi_1 + \frac{(1-2P_1)\pi_1}{2\sum_{i=1}^2 P_i^2 - 1} + \pi_2 + \frac{(P_2 - P_1)(1-2P_2)}{(P_1^2 - P_2^2 + 2P_2)} = 1 + \frac{3a}{5} \\ &\pi_1 + \frac{(1-2P_2)\pi_1}{2\sum_{i=1}^2 P_i^2 - 1} + \pi_2 + \frac{(1-2P_2)^2}{P_1^2 - P_2^2 + 2P_2} = 1 - \frac{a}{5} \\ &(\frac{3}{5})P_1 - (\frac{1}{5})P_2 = 0 \\ &\sigma_-^2(y_2) = \beta \end{aligned} \tag{5.35}$$

Hence, solve the problem (5.35) by Lagrange multipliers, where first and second conditions comes from $y_1^i(P) + y_2^i(P) = X_\beta(\omega_i)$ for $i = 1, 2$ and third constraint from $P(X) = 0$ implies the factor of a equal to zero since $a \neq 0$. Then, let $L = 0 + \lambda_1(y_1^1(P) + y_2^1(P) - (1 + \frac{3a}{5})) + \lambda_2(y_1^2(P) + y_2^2(P) - (1 - \frac{a}{5})) + \lambda_3((\frac{3}{5})P_1 - (\frac{a}{5})P_2)$. We find $\frac{\partial L}{\partial a} = 0$ and $\frac{\partial L}{\partial \lambda_i} = 0$ where $i = 1, 2, 3$.

Then, from the condition $\frac{\partial L}{\partial a} = 0$ we get $\lambda_2 = 3 \lambda_1$ and from condition $\frac{\partial L}{\partial \lambda_3} = 0$ implies that $P_2 = 3P_1$, from normalization of vector price we have $P_1 + P_2 = 1$, hence $P = (\frac{1}{4}, \frac{3}{4})$ then plug the value of P_2 in the equations $\frac{\partial L}{\partial \lambda_1} = 0$ and $\frac{\partial L}{\partial \lambda_2} = 0$, then we solve the problems over one variable P_1 and by simplifying we get the equilibrium price vector $P = (\frac{1}{4}, \frac{3}{4})$. Hence, we plug in the values of P_1 and P_2 in $\partial Z(P, W)$ and we get $\partial Z(P, W)$ is full rank = 2 which implies that the economy is regular and there is finite number and finite odd number of the equilibrium price. Hence, there exists a local equilibrium price P^* . Then, to show the uniqueness we need to satisfy the sufficient condition in Remark (5.14) part (i) in order to check about the condition for local weak axioms. On other hand, we need to satisfy the negative definite conditions, which is the sufficient condition in order that the aggregate demand functions obey the local weak axioms, since $\partial_P Z(P, W)$ is matrix 2×2 after plugging the values of P_1 and P_2 we get $\partial_P Z(P, W) = [-0.82, +0.6002; +0.5801, -2.24]$ that is approximately $\partial_P Z(P, W) = [-0.82, +0.60; +0.60, -2.24]$ which is a symmetric matrix and has two eigenvalues, $\gamma_1 = -2.4596$ and $\gamma_2 = -0.6004$ and determinant of this matrix is positive and first component in first row is greater than zero. Thus, this matrix is negative definite and hence for any ζ we have $\zeta^T \partial_P Z(P, W) \zeta < 0$ negative definite; which satisfies the sufficient condition for local axioms which guarantees that the equilibrium price is local unique near equilibrium price P , see Remark (5.14) part (i). Hence the equilibrium allocation in our case is obtained by plugging the value of P_1 and P_2 in proposition (5.1). Then, $y_1 = (1.9506, -0.502)$,

Similarly for the second investor by plugging the value of P in proposition (5.2) in order to get $y_2 = (-0.5606, 1.372)$,

To check the value of y_1 and y_2 we will compute the expectation for each of them we get $E[y_1] = 0.6502 = \pi_1$ and $E[y_2] = 0.2637 = \pi_2$ as well as $y_1 + y_2 = (1.39, 0.87) = (1 + \frac{3a}{5}, 1 - \frac{a}{5}) = (X_\beta(\omega_1), X_\beta(\omega_2)) = X_\beta(\omega)$, at $\beta = 0.0001$.

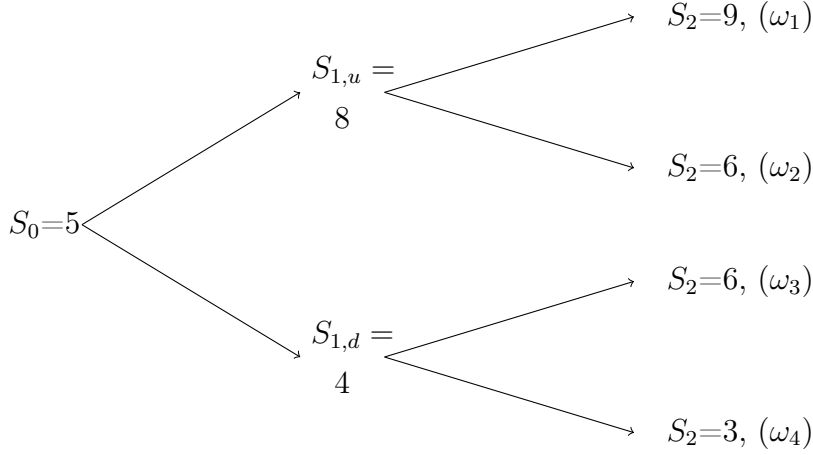
Example 2:

For the expected utility function $U(X) = E[u(X)]$, where $u(X) = 1 - \exp(-\alpha X)$ of profit X , where α is risk aversion and the utility is strictly concave and strictly positive from the Remark (5.14). The sufficient condition in order to get the

unique solution is the utility function is regular where

$u'(X) = \alpha \exp(-\alpha X)$ and $u''(X) = -\alpha^2 \exp(-\alpha X)$. Then the $u \in C^2$ and straight forward the utility satisfies the MMP as shown in Remark (5.14), since u is concave, then $\psi_u(x) \geq 0$, so $\psi_{\geq}(x) \geq 0$ for all x . Note that, there is no restriction on $\psi_{\geq}(x)$ which guarantees the local weak axioms.

Now, to solve the cooperative investment problem and find the equilibrium price and equilibrium allocation where risk aversion $\alpha_1 = 0.25$ for the first investor and $\alpha_2 = 0.5$ for the second investor for the following example.



The cooperative investment problem will be as follows:

$$\text{maximise}_{a,b,c} \quad E[u_1(y_1)] = 1/4 \sum_{k=1}^4 [1 - \exp(-\alpha_1 y_1^i)] \quad (5.36)$$

subject to

$$E[u_2(y_2)] \geq \mu$$

$$y_1^i + y_2^i = X_\mu(\omega_i)$$

where, $i = 1, 2, 3, 4$, $(X_\mu(\omega_1), X_\mu(\omega_2), X_\mu(\omega_3), X_\mu(\omega_4)) \in \mathcal{F}$ feasible set by changing of the value of $\mu \in (0, 1)$ the curve which is the efficient frontier ‘Pareto Optimal’ for the expected utility for both agents where

$$X_\mu(\omega_1) = (-a + \frac{8a}{5}) - b + (\frac{9b}{8}) ,$$

$$X_\mu(\omega_2) = (-a + \frac{8a}{5}) - b + (\frac{6b}{8}) ,$$

$$X_\mu(\omega_3) = (-a + \frac{4a}{5}) - c + (\frac{6c}{4}) ,$$

$$X_\mu(\omega_4) = (-a + \frac{4a}{5}) - c + (\frac{3c}{4}) .$$

Also, by changing the value of μ in problem (5.36), we get the whole efficient frontier curve as shown in Figure 5.1. Now, from this efficient frontier ‘Pareto

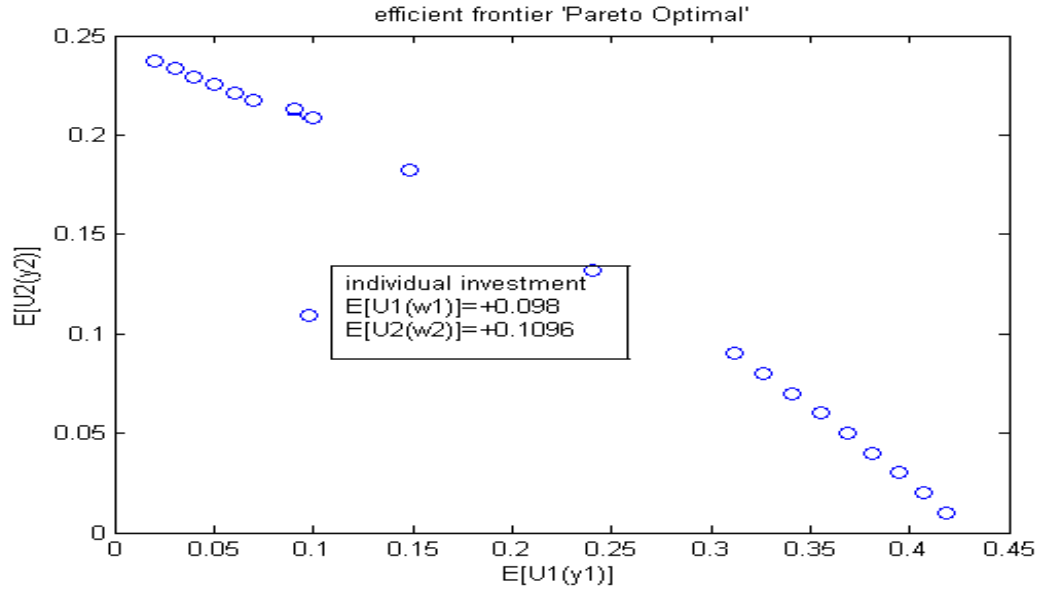


FIGURE 5.1: Solving (5.36) for cooperative investment and determine the IV point for each investor

Optimal' curve we need to find the equilibrium allocation corresponding to equilibrium price: we need to resolve the problem as follows:

First of all, we need to rewrite the allocation y_1 and y_2 as a function of equilibrium price, see proposition (5.3), where $N_t = 4$ and $P(y_1) = \sum_{i=1}^4 (P_i y_1^i)$, $P = (P_1, P_2, P_3, P_4)$, and $y_1 = (y_1^1, y_1^2, y_1^3, y_1^4)$. Then, the value of $y_1^i(P)$ in terms of equilibrium price will be

$$y_1^i = \frac{1}{\alpha_1} [\sum_{i=1}^4 P_i \ln(\frac{4}{\alpha_1} P_i)] - (\frac{1}{\alpha_1} \ln(\frac{4}{\alpha_1} P_i))$$

Similarly, the second agent(investor) has a different preference which is represented in his utility function $U_2(y_2) = 1 - \exp(-\alpha_2(y_2))$ with absolute risk aversion $\alpha_2 = 1/4$. Hence, the value of $y_2(P)$ is the same as $y_1(P)$ and replace α_2 instead of α_1 .

Now, we need to check about the feasibility of $y_1 = (y_1^1, y_1^2, y_1^3, y_1^4)$ and $y_2 = (y_2^1, y_2^2, y_2^3, y_2^4)$ by solving the feasibility problem and finding the equilibrium price $P = (P_1, P_2, P_3, P_4)$, by solving feasibility problem as follows:

$$\text{find } P \tag{5.37}$$

subject to

$$y_1^i(P) + y_2^i(P) = X_\mu(\omega_i)$$

$$\sum_{i=1}^4 (P_i X_\mu(\omega_i)) = 0$$

$$U_2(y_2) = \mu$$

where, $i = 1; 2; 3; 4$. Note that the last constraint can be rewritten as

$$P_1 X_\mu(\omega_1) + P_2 X_\mu(\omega_2) + P_3 X_\mu(\omega_3) + P_4 X_\mu(\omega_4) = 0 \quad (5.38)$$

by substituting the value of $X_\mu(\omega_i)$ in (5.38) we get

$$[(-P_1 + (\frac{8}{5}P_1) - P_2 + (\frac{8}{5}P_2) - P_3 + (\frac{4}{5}P_3) - P_4 + (\frac{4}{5}P_4)]a + [-P_1 - P_2 + (\frac{9}{8}P_1) + (\frac{6}{8}P_2)]b + [-P_3 - P_4 + (\frac{6}{4}P_3) + (\frac{3}{4}P_4)]c = 0$$

Simplifying the last equation and substituting in (5.38), and taking in account that trading strategies a, b and c do not equal zero. Then our feasibility problem reduces to

$$\min_p \quad 0 \quad (5.39)$$

subject to

$$\begin{aligned} y_1^i(P) + y_2^i(P) &= X_\mu(\omega_i) \\ \frac{3}{5}P_1 + \frac{3}{5}P_2 - \frac{1}{5}P_3 - \frac{1}{5}P_4 &= 0 \\ \frac{1}{8}P_1 - \frac{2}{8}P_2 &= 0 \\ \frac{1}{2}P_3 - \frac{1}{4}P_4 &= 0 \\ U_2(y_2) &= \mu \end{aligned}$$

where $i = 1; 2; 3; 4$. Solving problem (5.39) is just to find equilibrium price P and taking into account that there is no objective function specified, the problem is interpreted as a feasibility problem, which is the same as performing a minimisation with the objective function set to zero. Solving (5.39) by CVX in MATLAB, in this case optimal value is 0, so a feasible point is found. Then, we will get the value of equilibrium price

$$P = [0.1667, 0.0833, 0.2500, 0.500].$$

Then, we need to substitute the value of P in $y_1(P) = (y_1^1, y_1^2, y_1^3, y_1^4)$ and $y_2(P) = (y_2^1, y_2^2, y_2^3, y_2^4)$ in order to get the equilibrium allocation.

Hence, $y_1 = [1.1855, 2.5730, 0.3750, -1.0114]$ and $y_2 = [2.3709, 5.1460, 0.7499, -2.0228]$.

Then substituting the value of y_1 and y_2 in the utility function for the first agent and the second agent we get: the optimal value for the first agent (investor)= +0.1670, and the optimal value for the second agent (investor)= +0.1709 which is a fair allocation, which is strictly preferable than individual one and in this

example $\mu = +1.7$, see Figure 5.2.

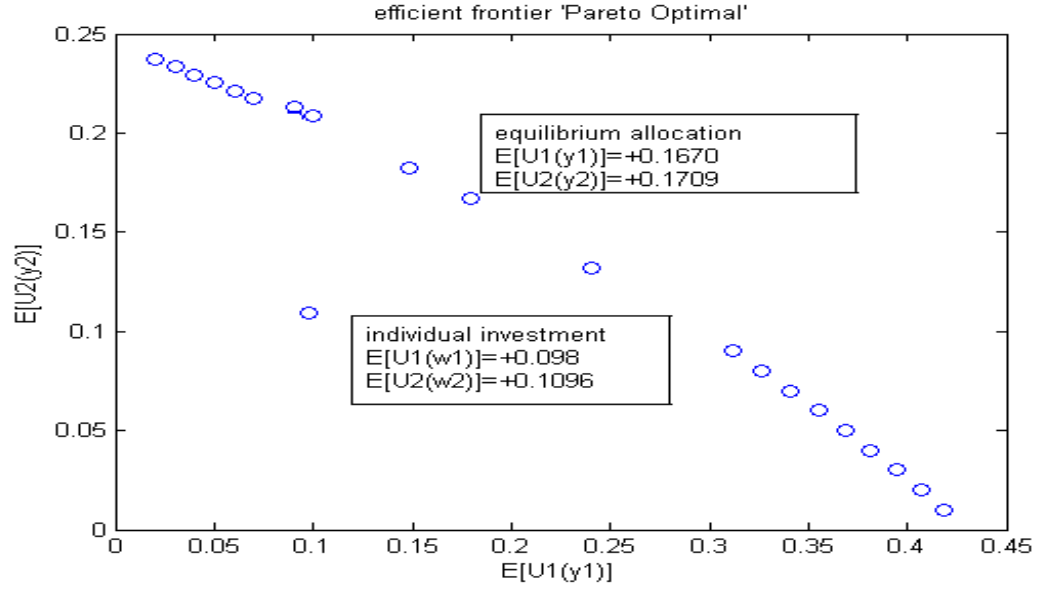


FIGURE 5.2: Solving (5.36) for DCI and determine the fair allocation for the investors in DCI and IV

Example 3: This example shows how to prove uniqueness of equilibrium in Remark (5.14) part (iii) for the Logarithm utility function $U(X) = \frac{X^{1-\rho}-1}{1-\rho}$ and $U'(X) = X^{-\rho}$ and $U''(X) = -\rho X^{-\rho-1}$. Then the sufficient condition by Remark (5.14) part (iii) and applying (5.34) we have, $\max_x(\rho) - \min_x(\rho) = 0 < 4$, where the interest rate r . Then the condition holds. Thus, the uniqueness of equilibrium is available.

5.2.3 Stability

Since the aggregate demand function satisfies the Walras law which means that there exists the equilibrium price which is the Walrasian equilibrium price such that $Z(P, w) = 0$. Then, an equilibrium price vector P can naturally be said to be locally stable if the price adjustment rule converges to P . In addition, of course $\frac{dP}{dt} = 0$, since $Z(P, w) = \frac{dP}{dt}$, for more details see Arrow et al. [12]. But the relative equilibrium price obtained by solving the feasible system (5.31) to the equilibrium price is the local stable since in multi-period we solve the equilibrium in the recursive manner.

Definition 5.15. [12] An equilibrium price P^* is said to be locally stable if there exists a neighbourhood $N(P^*)$ of P^* such that for any point P of the neighbourhood $N(P^*)$ every solution of the fixed point f converges to P^* . Consequently, at each local uniqueness equilibrium price, it is a local stable at this price.

In addition, the global and local stability was studied in the famous paper by Scarf [127] and he insure of stability of Walrasian equilibrium price by Arrow [9]. Also, Xia [143] paper about the cooperative investment game shows that the cooperative investment is not empty by using the scarf theorem, Scarf [128] where at least local stability corresponds to local unique of equilibrium.

Algorithm 5 15.

Step 1 : Solve cooperative investment problem (4.3) and find $y_1 + y_2 = X_\beta$, where X_β depends on β and $X_\beta \in \mathcal{F}$, where \mathcal{F} is feasible set. See example 2 in this chapter to be more clear.

Step 2 : Each individual solves her/his optimisation problem subject to budget constraints as in section 5.1.

$$\begin{aligned} & \max_{y_i} U_i(y_i) \\ & s.t \\ & P_t(y_i) \leq 0 \end{aligned} \tag{5.40}$$

for each time t , we find equilibrium allocation $y_i(P)$, $i = 1, 2$ in terms of equilibrium price P as an explicit formula.

Step 3 : The equilibrium allocation satisfies $y_1^i(P) + y_2^i(P) = X_\beta(\omega_i)$, where $y_1^i(P)$ and $y_2^i(P)$ as a term of equilibrium price that can be found from the propositions (5.1) and (5.2), respectively. Note that, you can change the risk preferences and then the explicit formula in terms of equilibrium price will change as in propositions in section 5.1, depending on which preference modeling that investor will choose. $X_\beta(\omega_i)$, $i = 1, \dots, N_t$ is the uncertainty outcome at the end of investment period and can be written exactly as Algorithms 3.2 or 4.3, depending on β see step 1, where for example in case of two periods $X_\beta(\omega_i) = ((W_0 - \sum_{i=1}^n x_i)r_0 + xr') - \sum_{i=1}^n z_{i,t}r_0 + z'_tr_{i,t_2}$ where W_0 is initial capital, x is trading strategy at the first period, z_t is trading strategy at the second period and r is the rate of return at the first period and r_{i,t_2} is the rate of return at scenario i at the second period.

Step 4 : Plug in the value of $X_\beta(\omega_i)$, $i = 1, \dots, N_t$ that we got from step 1 in the next step, then

Step 5 : Check about feasibility and find the equilibrium price as follows:

$$\begin{aligned}
 & \min_p \quad 0 \\
 & s.t \\
 & y_1^i(P) + y_2^i(P) = X_\beta(\omega_i), \quad i = 1, \dots, N_t \\
 & \sigma_2^2(y_2) = \beta \\
 & P(X_\beta(\omega_i)) = 0.
 \end{aligned} \tag{5.41}$$

Remark 5.16. In Algorithm (5.15), $X_\beta(\omega_i)$ depends on β but does not depend on price P and we solve problem (5.41) over P . Also, $X_\beta(\omega_i)$, $i = 1, \dots, N_t$ is arbitrary and can be found just by solving cooperative investment problem, for example, problem (4.3) and the purpose of solving the problem (5.41) is to check the feasibility of $y_1(P)$ and $y_2(P)$ and find P . Hence we find the value of equilibrium price P and then we can calculate the value of $y_1(P)$ and $y_2(P)$ which are an equilibrium allocation ‘fair point’ for each investor. In addition, the size of P is equal to the size of uncertain outcome $X_\beta(\omega_i)$, y_1 and y_2 where the size is $1 \times N_t$, and N_t is the number of the scenario tree.

5.3 Numerical experiment

In this section we complete our experiments in Chapter 4 for discrete time and continuous time cases to find the fair equilibrium allocation among all the elements from efficient frontier set.

- For the first experiments we will choose one risk-free asset and 3 risky assets chosen from *S&P 100* and find the return for the risky assets by using **GARCH(1,1)** as in Chapter 4, see numerical experiment Example 3 in discrete time. Then, generate the scenario tree for $T=30$ periods with 100 scenarios at each node where the first investor has $\sigma(y_1)$ a risk measure and the second investor has $\sigma_-(y_2)$ as a risk measure. As well we choose risk free $r_0 = 0.000068$, $\pi_1 = 0.0000821$ and $\pi_2 = 0.000109$. Hence, the fair allocation according to Algorithm (5.15) are shown in Figure 5.3. Note that, the difference between the efficient frontiers is not very significant which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in

multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment period.

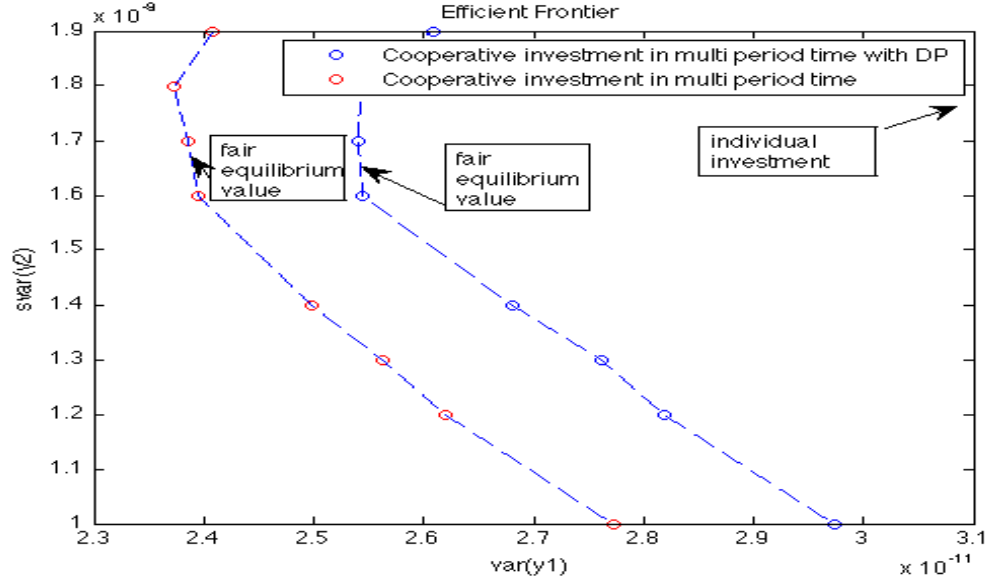


FIGURE 5.3: Finding equilibrium allocation for DCI according to Algorithm (5.15) where the equilibrium allocation for first investor from Proposition (5.1) and Proposition (5.2) for second investor and solving CI and DCI as shown in Chapter 4 for discrete time

- Solving problem (4.4) for portfolio consisting of one risk-free asset and 3 risky assets is chosen from *S&P 100* and find the return for the risky asset by using **GARCH(1,1)** as in Chapter 4, see numerical experiment in Chapter 4. Then the equilibrium allocation point will exist by following the Algorithm (5.15) and shown in the Figure 5.4. Note that, the difference between the efficient frontiers is not very significant which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment period.
- In continuous time, we will complete solving dynamic cooperative investment DCI problem (4.3) as in Chapter 4 in continuous time to find fair allocation and suppose we have one risk free and 3 stocks chosen randomly from *S&P 100* and found by using **COGARCH(1,1)**. Let $T = 10$ years where the first investor has $\sigma(y_1)$ as a risk measure and the second investor has $\sigma_-(y_2)$ as a risk measure. We will fix the value of π_1 and π_2 which are the

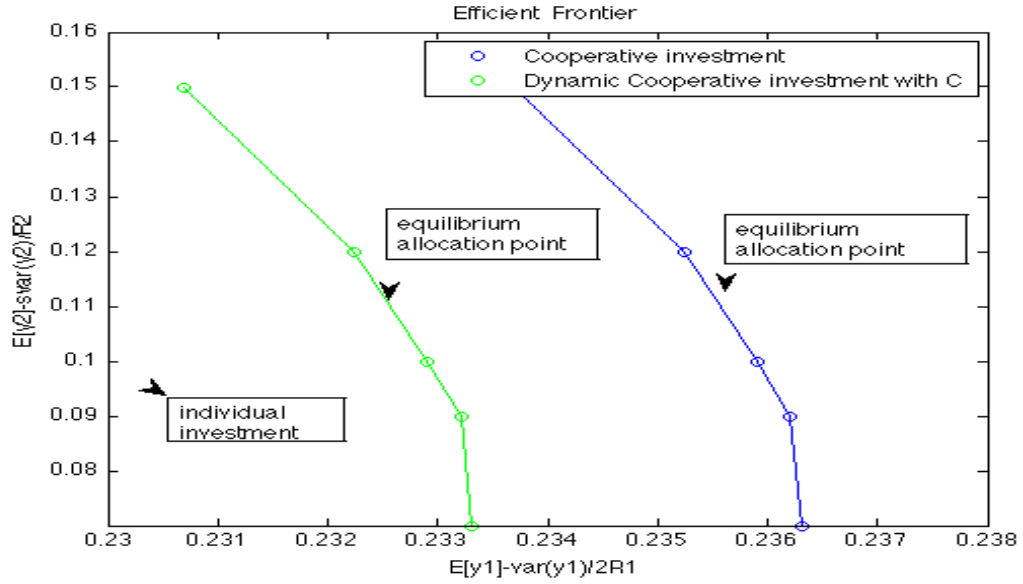


FIGURE 5.4: Finding equilibrium allocation for DCI according to Algorithm (5.15) where the equilibrium allocation for first investor from Proposition (5.5) and Proposition (5.6) for second investor and solving CI and DCI as shown in Algorithm 4.2, where R_1 in this Figure = $\rho_1 = 0.5$, and $R_2 = \rho_2 = 0.25$

expected returns for first and second investor, respectively. In our case we assume that risk free $r_0 = 0.0000136$, $\pi_1 = 0.00016$ and $\pi_2 = 0.0001917$, where $\pi_1 + \pi_2 > r_0$. Hence, the fair allocation by following Algorithm (5.15) and our result is shown in Figure 5.5. Note that, the difference between the efficient frontiers is not very significant which implies that the investor will follow the trading strategy until the end of the investment period which comes from solving cooperative investment in multi-periods with dynamic programming in order to avoid breaking down the contract between investors in the middle of the investment period.

- Certainty equivalent:** In our experiment we will choose a portfolio consisting of one-risk free asset and three risky assets chosen randomly from S&P 100. Firstly, we solve problem (4.4) to find the certainty equivalent, where each investor has \$50 a initial wealth, hence the total initial wealth they are begin with is \$100. Moreover, solve two period problem with 30 scenarios at each node and solve the problem as a global case as shown in the Algorithm 4.2. Also, we will resolve the same problem (4.4) by using force in back technique (dynamic program) as a second way, then apply the Algorithm (5.15) and comparing the result as shown in the following Table 5.1. We notice that we get the same value for the certainty equivalent or

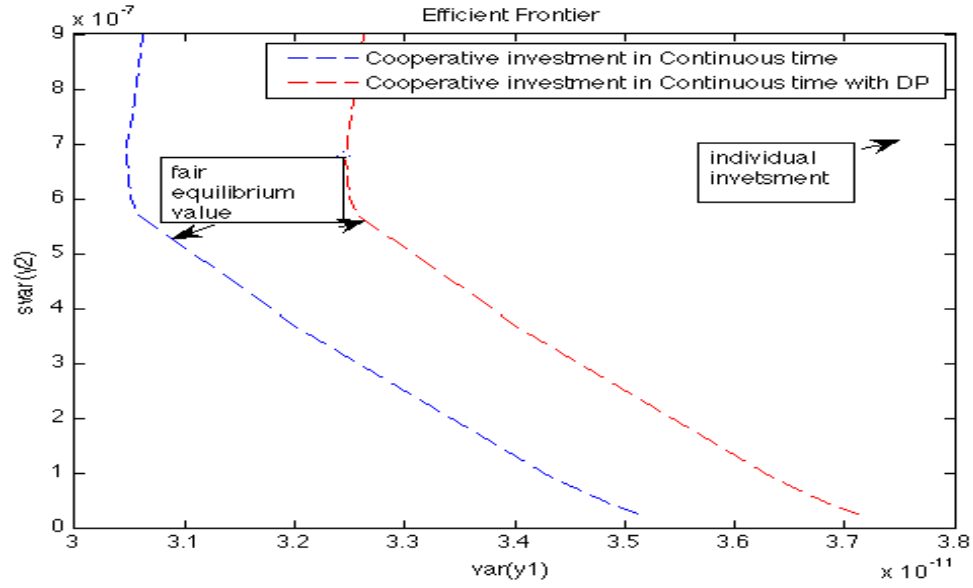


FIGURE 5.5: Finding equilibrium allocation for DCI according to Algorithm (5.15) where the equilibrium allocation for first investor from Proposition (5.1) and Proposition (5.2) for second investor and solving CI and DCI as shown in Chapter 4 for continuous time

methods	C for 1 st	C for 2 nd
global solution	+50.039	+50.18
dynamic solution	+50.007	+50.18
individual solution	+50.00035	+50.0017

TABLE 5.1: Example of certainty equivalent with $T=2$ and 30 scenarios

cash amount for the second investor since we solve the problem at fixed level for $\pi = 50.18$ in problem (4.4).

Extension period:

We solve the same problem (4.4) but in $T = 20$ periods and 50 scenarios at each node in each period and exactly the same way as in the second period we solve it in the case of a global solution as well as in the dynamic solution where we fix the $\pi = 101.2175$ as the initial wealth each investor has \$100, hence the result we get is shown in the following Table 5.2.

methods	C for 1 st	C for 2 nd
global solution	+101.06752	+101.2175
dynamic solution	+101.01311	+101.2175
individual solution	+100.0071	+100.00862

TABLE 5.2: Example of certainty equivalent with $T=20$ and 50 scenarios

Numerical solution for prospect theory:

In this experiment we will solve the problem (3.3) with ν chosen from Tversky and Kahnmanand [138] for the first investor and Prelec [116] for the second investor.

Firstly : we will solve the problem (3.3) in the two-period model and 30 scenarios at each node. In case of global solution and dynamic solution the way to process is as in the Algorithm (5.15). where the portfolio contains one risk-free asset and three risky assets chosen randomly from *S&P* 100.

Secondly :

We will solve the problem (3.3) by using force in back technique (dynamic solution), where each investor has \$50 as initial wealth and note that the reference point is the risk-free return $r_f = r_0 = 2.5$. Then our results are shown in the following Figure 5.6 and Table 5.3.

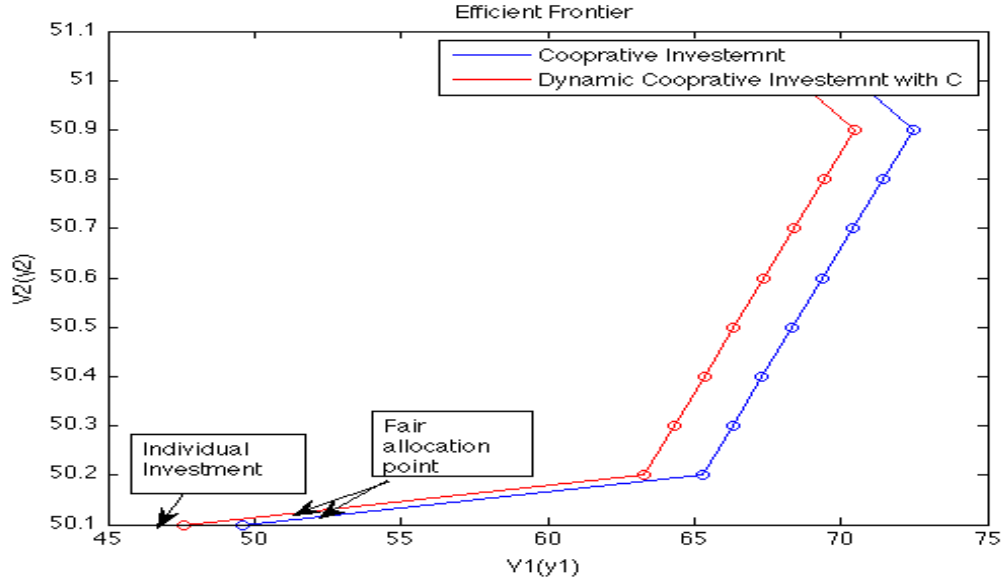


FIGURE 5.6: Finding equilibrium allocation for DCI, problem (3.3) with Certainty equivalent for prospect theory according to Algorithm (5.15)

methods	C for 1 st	C for 2 nd
global solution	+51.36407	+50.088
dynamic solution	+51.00232	+50.088
individual solution	50.224	+50.012

TABLE 5.3: Example of certainty equivalent by using prospect theory and solving problem (3.3) with $T=2$ and 30 scenarios

Extension period: In this experiment, we need to extend the problem and resolve it in $T = 20$ periods and 50 scenarios at each node, with the same

risk-free rate and the initial wealth for each investor will be \$100. Thus our result can be illustrated as in the following Table 5.4.

methods	C for 1 st	C for 2 nd
global solution	+101.4956	+100.046
dynamic solution	+101.0131	+100.046
individual solution	+100.009	+100.012

TABLE 5.4: Example of certainty equivalent by using prospect theory and solving problem (3.3) with T=20 and 50 scenarios

Note that, the experiments show that the investors having different preferences can use cooperation to achieve strictly better outcome which is presented as equilibrium allocation, compared to optimal individual investment.

5.4 Discussion and concluding remarks

In this chapter, we addressed the question of selecting a ‘fair’ point from the Pareto optimal set, and suggest that ‘fair’ allocation corresponds to an equilibrium one. In some cases, we derive an explicit formula for equilibrium allocation for each investor according to their preferences. Then, the questions of uniqueness and stability of the equilibrium price are addressed. We developed the sufficient conditions that guarantee that the local equilibrium price is unique for the cooperative investment case such as for the first investor having a standard deviation as risk preference and standard lower semi-deviation for the second investor. We also determined the unique equilibrium allocation corresponding to cooperative investment problems with risk measure preferences as well as for certainty equivalent and prospect theory formulations. In all numerical experiments, the suggested “fair” allocation is strictly better for every investor than optimal individual investment, which provides a strong motivation for cooperation.

Chapter 6

Conclusion and Future Work

The thesis studies the dynamic cooperative investment problem, in which $m = 2$ investors collect their initial capital, invest the joint capital into a trading strategy, and then divide the terminal wealth among them in an ‘optimal’ way. The case of 2 investors is the simplest non-trivial case of cooperation, which already contains many ideas which can be applied to the more general cases. It also has a practical importance, because it may be easier for an investor to find one partner whom he/she would trust, than to form a big coalition. We have demonstrated that even this simple coalition of 2 investors already can achieve strictly better results than optimal individual investment. This problem was studied before only in a few special cases, such as agents with expected utility preferences, see Parkes [113] and Xia [143], mean-deviation preferences in a one-period model, see Grechuk and Zabarankin [66], or multi-period model with drawdown constraints, see Grechuk and Zabarankin [68].

The summary of solving dynamic cooperative investment problem in this thesis consists of the following parts.

- (i) **Preference modelling.** First, understand how agents would invest individually, that is, what are their utility/risk preferences in the financial market in one-period and dynamic cases. We study these questions in Chapters 2 and 3. As a result we investigate the behaviour of agents in the context of expected utility theory, mean-deviation analysis, mean-risk analysis, and prospect theory. In the dynamic setting, we also distinguish between optimal pre-commitment strategies, which may however be time-inconsistent,

and optimal time-consistent strategies. Thus, the best treatment of this problem resolving cooperative investment by using dynamic programming.

- (ii) **Market modelling.** Second, we developed methods for forecasting the future behaviour of rates of return of financial instruments as random variables/stochastic processes. Here, we use the naive approach with historical simulation, as well as a more economically meaningful approach using simulation with the GARCH model. This model applied to numerical experiments as shown in Chapters 3 and 4.
- (iii) **Optimal trading strategies and Pareto optimal allocation.** Third, given investors' risk preferences (step (i)) and forecasts of future market behaviour (step (ii)), hence we can develop optimal trading strategies for the group of investors, and an optimal way to divide the resulting terminal wealth among them. Here, by 'optimal' we mean Pareto optimal, that is, there is no way for *all* investors to be better simultaneously. For two investors, we graphically present the corresponding 'efficient frontiers' between them. See Chapters 3 and 4.
- (iv) **Fair allocation.** Finally, we address the question of selecting a 'fair' point from the Pareto optimal set. We solve this question by using equilibrium theory. We present an explicit formula for each investor in case of standard-deviation and semi-deviation for first and second investor, respectively. The explicit formulas are presented in propositions 5.1, and 5.2 which are the most significant mathematical part in this thesis, see Chapter 5.

As a result of this thesis, the main benefit from cooperative investment is that the terminal wealth at the last stage may be divided in an *arbitrary* way, and the agent may therefore create 'shares' which are not available at the initial incomplete market. As a result, the agents can have either shares with strictly less risk compared to the optimal individual investment, or with the same risk and strictly greater expected return. This is demonstrated numerically in Chapter 2, Tables (2.3),(2.4),(2.5),(2.6),(2.7),(2.8) and (2.9) for one-period model, and in Chapters 3 and 4, Graphs (3.2),(3.5),(3.6),(3.7),(3.8),(3.9),(3.10),(4.6),(4.7),(4.8),(4.9) and (4.12) in multi-period and continuous time models.

While working in a multi-period setting in Chapter 3, we notice that optimal pre-commitment strategy is hard to compute and time-inconsistent. We have therefore solving it, complemented it by a dynamic programming technique (see Bielecki et

al. [24] and Bjork and Murgoci [25]), and, alternatively, by a new technique based on compound independent axioms of Segal [133]. In the latter method, an uncertain outcome from investment on the last period can be replaced by its certainty equivalent, and then this process is repeated until the first period is reached. The resulting strategies are worse than the pre-commitment strategy, but are dynamically stable, so that investors will not want to break down the contract between them during the process. We remark, however, that the difference between efficient frontiers resulting from optimal pre-commitment and optimal stable strategies is not very significant.

In the result in Chapter 4, we address the question of realistic market modelling, to get the realistic one we apply the $GARCH(1, 1)$ model in discrete time as well as the $COGARCH(1, 1)$ model in continuous time, see Nelson [110] and Kluppelberg [89]. Based on these models, we have designed a scenario tree as shown in Glupiner et al. [70] and binary tree. We then check that the resulting tree is arbitrage free using the method from Klassen [88].

Resulting in Chapter 5, we address the last main question in this dissertation, which is finding the fair division among the participants from the set of Pareto optimal divisions. We argue that the fair allocation should be based on the concept of equilibrium. In particular, in Chapter 5 we derive an explicit formula to find a fair equilibrium allocation according to equilibrium price for each investor. The explicit formulas are presented in propositions 5.1, and 5.2 which are the most significant mathematical part in this thesis. Using the formulas that shown in section 5.1, we were able to numerically find equilibrium prices for investors using different risk preferences. We also show that a sufficient condition for the local uniqueness of equilibrium, see Quah [117], holds in many of the cases considered in this dissertation; see Remark 5.13.

We argue that the result of the thesis may have *practical* importance. Resulting from numerical experiments Table(2.3),(2.4),(2.6),(2.7),(2.8) and (2.9) for a one-period model and Graphs (3.2),(3.5),(3.6),(3.7),(3.8),(3.9),(3.10), (4.6),(4.7),(4.8), (4.9) and (4.12) in multi-period and continuous time clearly show that it is strictly beneficial to invest cooperatively rather than individually. Most of the programmes developed in the thesis can be used by practitioners. In summary, a group of investors should (i) decide what are their corresponding utility/risk functions, and which of these should be considered in this thesis, (ii) download historical data for the financial instruments they want to invest in, and (iii) use the programmes

developed in Appendix A in the thesis to see how they should invest during each period, and, at the end, how they should distribute the final wealth among them see propositions (5.1)-(5.6). The detailed step-by-step user guide on how to do this is presented in Appendix B. In addition, all the MATLAB code used is presented in Appendix A, so that advanced users could modify it for their special needs.

In future work, we would like to investigate the effect of the model of consumption utility function as shown in Rabin [118] and Khoszegi [92] and the cooperative investment model with agents using prospect theory. In addition, we would like to use the developed methods to find a closed form solution for dynamic cooperative investment for the prospect theory case. We would also like to extend the results of this thesis to the markets with transaction cost. It would also be interesting to solve the stochastic differential equation (4.12) by using the spline interpolation method, which would lead to a more accurate solution of CI and DCI in continuous time.

Appendix A

Some codes in MATLAB for IV, CI, DCI and equilibrium

Some code solving IV CI and DCI in single period

All of the following code that solve the problem which shown in the numerical experiment in Chapter 2. We use Price matrix that download from Yahoo finance, this shows the price matrix for *S&P100* with weekly closed price of these stocks from 1/January/2011 to 1/January/2013, where the price matrix is $P_{96,100}$ matrix and Return matrix is $R_{95,100}$. Note that, between `cvx` begin and `cvx` end it is underscore symbol.

- 1) Cooperative Investment between two investors, $D1(y1)$ is standard deviation $\sigma = \sqrt{\sigma^2}$, and $D2(y2)$ is standard lower semi deviation $\sigma_- = \sqrt{\sigma_-^2}$.
function [w,val] = minvar119(P)
% P= [],pricing matrix
% variance and semi variance , cooperative investment
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1

```

for j=1:1:n
E=sum(R)./N
end
end
R=R;
E=E;
r=E
rr=sum(R);
size(R);
size(E);
c=cov(R);
v= diag(c)
vv=(v')*v
c=corrcoef(R);
s=vv*c
pi=0.6    %  $\pi > r_0$ 
n=100;
cvx-begin
variable w(n);
variable y1(n);
variable y2(n);
Erp=E*w
X=r'.*w
D1=square(y1-(sum(y1)/n)) % first investor has  $D1(y1) = \text{variance}(y1)$ 
DD1=sum(D1)/n
for i=1:1:100
D2= square-pos(sum((max(-y2(i)+((sum(y2))/100),0))))/100 % second investor has
D2(y2)= standard lower deviation.
end
% note that semi variance is square-neg but this is not working in cvx except
square-pos so that, we multiply - between -(X-EX)
Ds=max(DD1,D2)
minimize(Ds)
subject to
sum(w)==1;
w>=0;

```

```

Erp>=pi
y1+y2==X
cvx-end
y1=y1
y2=y2
W=w
end

```

```

Shortcuts  How to Add  What's New
gap := fmincon(AA)
relative gap      = 8.09e-09
actual relative gap = 7.47e-09
rel. primal infeas = 6.06e-14
rel. dual infeas   = 1.11e-09
norm(X), norm(Y), norm(Z) = 1.4e+01, 8.2e-01, 8.4e-01
norm(A), norm(B), norm(C) = 3.3e+02, 1.5e+01, 2.0e+00
Total CPU time (secs) = 3.86
CPU time per iteration = 0.12
termination code      = 0
DIMACS: 4.6e-13  0.0e+00  1.1e-09  0.0e+00  7.5e-09  8.1e-09
-----
Status: Solved
Optimal value (cvx_optval): +0.000179576

W =
0.020640972254920
0.000000092108577
0.000000418250347
0.000000147190977
0.000000150996969
0.000000506415953
0.039619420068853
0.000000137478923
0.000000291243364
0.000000226707503
0.010486041784275
0.000000143266786
0.000001011938518
0.000000442877582
0.052290726842977
0.000000142980551
0.000000103682865
0.000000136223089
0.000000111059359
0.000000160685419
0.000000104350138
0.066343135211174
0.000000112981311
0.000000209575226
0.062037157874528
0.000000130254473
0.004304565310620
0.056171474719912

```

FIGURE A.1: Result in MATLAB for CI with variance and semi variance

```

• 2) Individual investment for investor has a variance as risk preferences
function [ w,val] = minvar112( P)
% P= [],pricing matrix
% standard deviation  $\sigma=\sqrt{\sigma^2}$ ,as individual investment
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)./N;
end
end
R=R;
E=E;
r=E
rr=sum(R);
size(R);
size(E);
pi=0.03;
n=100;
cvx-begin
variable w(n);
Erp=E*w
X=r'.*w
xs=size(X)
D1=square(X-(sum(X)/n))
DD1=sum(D1)/n
minimize(DD1)
subject to
sum(w)==1;
w>=0;

```

```

Erp>=pi
cvx-end
W=w
end

```

```

Shortcuts  How to Add  What's New
18 1.000 1.000 3.7e-14 1.0e-12 1.7e-05 1.011216e-03 9.942798e-04 0:0:01 chol 1 1
19 1.000 1.000 1.7e-13 1.0e-12 1.9e-06 1.002045e-03 1.000184e-03 0:0:01 chol 1 1
20 1.000 1.000 2.4e-13 1.0e-12 8.2e-08 1.000943e-03 1.000861e-03 0:0:01 chol 1 1
21 1.000 0.999 8.3e-14 1.0e-12 1.5e-09 1.000893e-03 1.000891e-03 0:0:01
stop: max(relative gap, infeasibilities) < 1.49e-08
=====
number of iterations = 21
primal objective value = 1.00089270e-03
dual objective value = 1.00089120e-03
gap := trace(XZ) = 1.51e-09
relative gap = 1.50e-09
actual relative gap = 1.50e-09
rel. primal infeas = 8.27e-14
rel. dual infeas = 1.00e-12
norm(X), norm(y), norm(Z) = 1.0e+01, 3.5e-01, 3.7e-01
norm(A), norm(b), norm(C) = 3.2e+02, 1.1e+01, 1.1e+00
Total CPU time (secs) = 1.09
CPU time per iteration = 0.05
termination code = 0
DIMACS: 4.6e-13 0.0e+00 1.1e-12 0.0e+00 1.5e-09 1.5e-09
=====
Status: Solved
Optimal value (cvx_optval): +0.00100089

W =
0.000000016496823
0.000000002205188
0.000000006145721
0.000000003185004
0.000000003245794
0.000000006739407
0.0000000032143391
0.000000003026160
0.000000005036243
0.000000004307784
0.000000013895738
0.000000003121475
0.000000008759762
0.000000006323549
0.002560311776554
0.000000003116807
0.000000002427995
0.000000003005220

```

FIGURE A.2: Result in MATLAB for IV with variance

- **3)** Individual investment for investor has a semi variance as risk preferences function $[w, val] = \text{minsemvar1113}(P, \pi)$

```

% P=[],pricing matrix
% variance from formula, individual investment r0=1
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)./m
end
end
R=R;
E=E;
r=sum(R)
rr=sum(R);
size(R);
pi=0.020; %  $P_{i_1} > r_0$ 
cvx-begin
variable w(n);
Erp=E*w
X=r'.*w
xs=size(X)
% for i=1:1:n
% D1= square-pos(((max(-X(i)+Erp,0))))
% D1=square(X-(sum(X)/n))
D1= square-pos((-X+(sum(X)/n)))
DD1=sum(D1)/n
minimize(DD1)
subject to
sum(w)==1;

```



```

w>=0;
Erp>=pi
cvx-end
W=w
rrr=r'
EE=E'
end

```

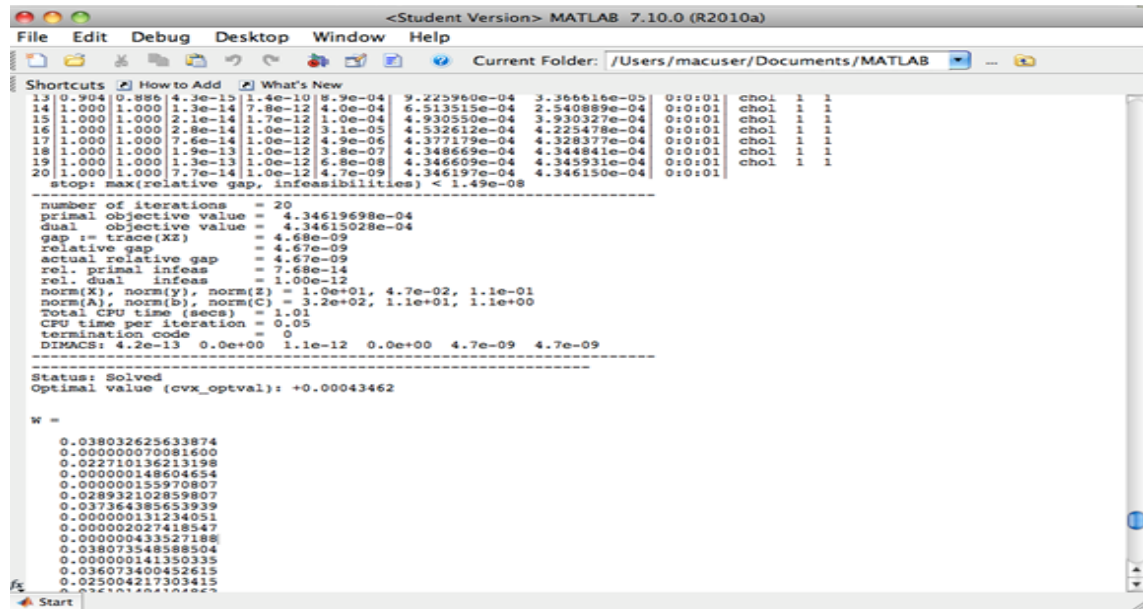


FIGURE A.3: Result in MATLAB for IV with semi variance

- 4) Cooperative investment between MAD(y1) and standard lower semi-deviation $\sigma_- = \sqrt{\sigma_-^2}$ for second investor.

```

function [ w,val] = MADmean4(P)
% P= [],pricing matrix
% variance from formula and MAD for cooperative case
%P= [],pricing matrix
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)/N
end
end
E=E;
n=100;
pi=0.05;
c=cov(R);
E=sum(R)/N
n=100;
cvx-begin
variable w(n);
variable y1(n);
variable y2(n);
Erp=E*w
X=Erp
size(w)
ab=sum(abs(y1-(sum(y1)./n)))./n % first investor D1(y1)=MAD(y1)
sab=size(ab)
MAD= sum(ab)*1/100
D1=MAD

```

File Edit Debug Desktop Window Help
 Current Folder: /Users/macuser/Documents/MATLAB
 Shortcuts How to Add What's New
 stop: max(relative gap, infeasibilities) < 1.49e-08

 number of iterations = 15
 primal objective value = 1.68866513e-10
 dual objective value = -1.94834294e-10
 gap := trace(XZ) = 5.55e-09
 relative gap = 5.55e-09
 actual relative gap = 3.64e-10
 rel. primal infeas = 1.21e-11
 rel. dual infeas = 5.29e-10
 norm(X), norm(y), norm(z) = 1.3e+01, 2.5e-02, 9.7e-01
 norm(A), norm(b), norm(C) = 2.1e+01, 2.4e+00, 2.0e+00
 Total CPU time (secs) = 0.97
 CPU time per iteration = 0.06
 termination code = 0
 DIMACS: 1.5e-11 0.0e+00 5.3e-10 0.0e+00 3.6e-10 5.5e-09

 Status: Solved
 Optimal value (cvx_optval): +1.68867e-10

 W =
 0.009784827543301
 0.009800634051441
 0.009786548802976
 0.00972266569509
 0.009791977073391
 0.009786199743192
 0.009784669160197
 0.009793105898093
 0.009787571215145
 0.009788719448487
 0.009784929625109
 0.009792387123012
 0.009785506758728
 0.009786433631891
 0.009784677357934
 0.009792611443166
 0.009797843675942
 0.009793226431911
 0.009796468178745
 0.009791322564792
 0.00979708229278
 0.00978928304612

FIGURE A.4: Result in MATLAB for CI with mean absolute deviation

```

• 5) Individual investment for MAD(X)
function [ w,val] = MADmean3(P )
% P=[],pricing matrix
% MAD formula, individual investment
% P=[],pricing matrix
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)/N
end
end
E=E;
n=100;
r=sum(R);
pi=0.03 % pi>r0
E=sum(R)/n
n=100;
cvx-begin
variable w(n);
Erp=E*w
X=r'.*w
EX=sum(X)/n
size(w)
ab=abs(EX-X)
sab=size(ab)
MAD= sum(ab)*1/n
MMAD= avg-abs-dev(X)
% note that the formula MMAD= that calculate MAD by CVX the same
exactly as we calculate it by MAD

```

```

minimize(MAD)
subject to
sum(w)==1;
w>=0;
sum(Erp)>=pi
cvx-end
W=w
end

```

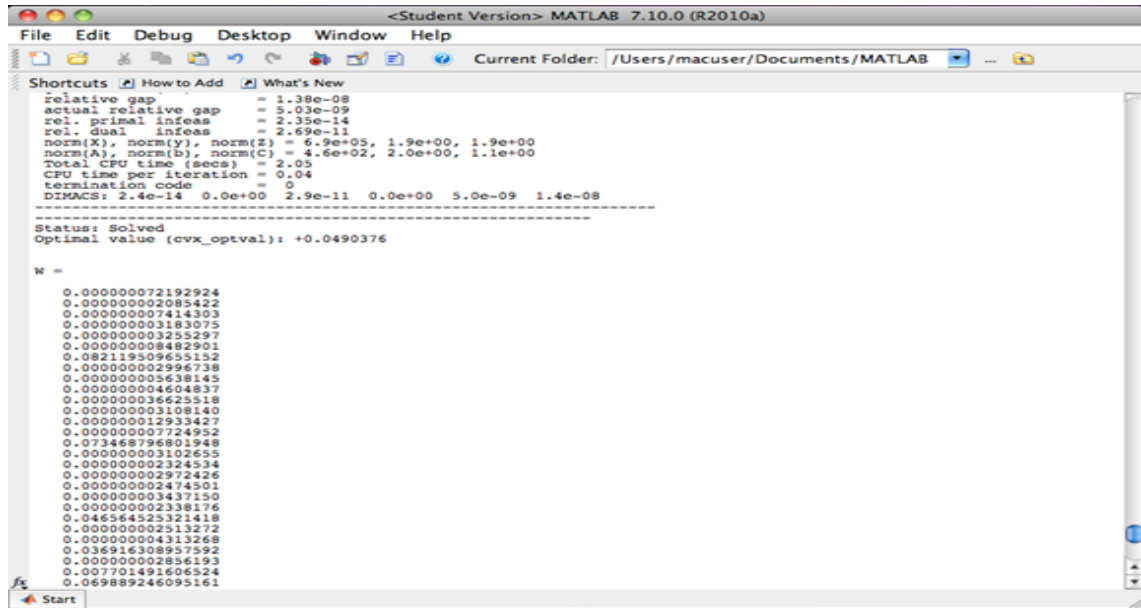


FIGURE A.5: Result in MATLAB for IV with mean absolute deviation

- **6)** Cooperative investment between $D1(y1)=E y1-\inf(y1)$, $D2(Y2)=\text{Sup}(y2)-\inf(Y2)$

```

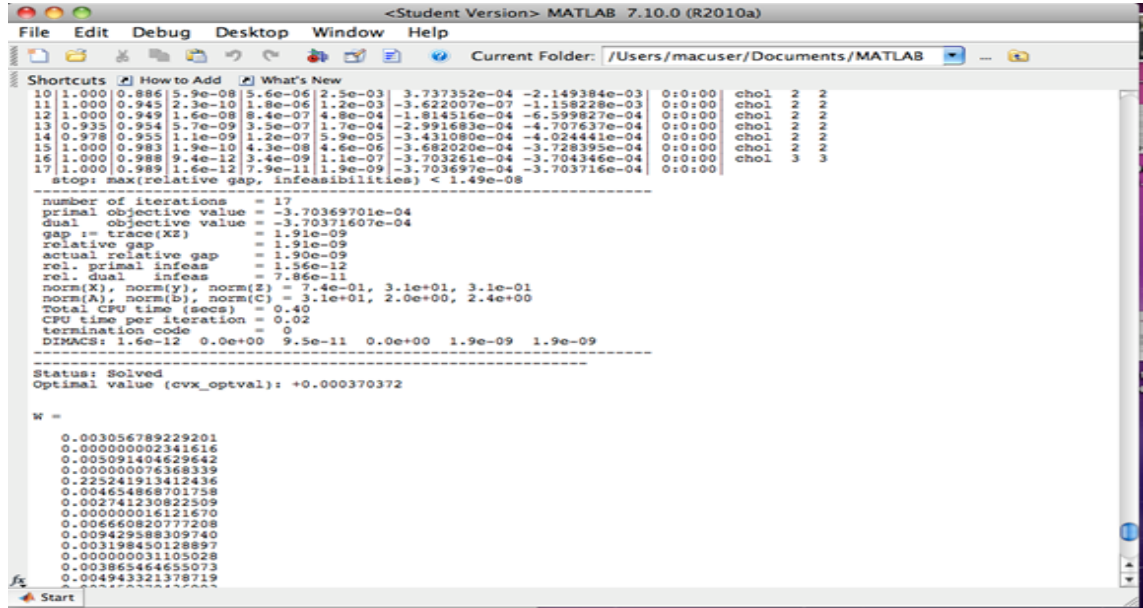
function [ w,val] = Exinfmeancoopr(P)
% P= [],pricing matrix
% variance from formula
% P= [],pricing matrix
% with D1 and D2( collation)
r0=0.01;
R=sum(P)
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)/N
end
end
E=E;
n=100;
r=sum(R)
pi=0.06 % pi>r0
r=sum(R);
cvx-begin
variable w(n);
variable y1(n);
variable y2(n);
Erp=R*w
X=r'.*w
size(w)
D1=(sum(y1)/100)-min(y1) % first investor has D1(y1)=EX-inf(X)
D2=max(y2)-min(y2) % second investor has D2(y2)= sup(X)-inf(X)

```

```

Ds=max(D1,D2) % deviation measure for coalition S, max between D1, and
D2
minimize(Ds)
subject to
sum(w)==1;
w>=0;
sum(Erp)>=pi
y1+y2==X
cvx-end
W=w
y1=y1;
y2=y2;
end

```

FIGURE A.6: Result in MATLAB for CI with $\inf(X)$ and $\sup(X)$

```

• 7) Individual investment,  $D1(X)=EX-\inf(X)$  or  $D2(X)=\sup(X)-\inf(X)$ 
function [ w,val] = Exinfmeanindvdual(P)
% P=[],pricing matrix
%P=[],pricing matrix
% with variance and semi ( collation)
%  $D1(x)=EX-\inf(x)$ ,or  $D1(X)=\sup(X)-\inf(X)$ , individual investment
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)/N
end
end
E=E;
n=100;
r=sum(R)
pi=0.02
cvx-begin
variable w(n);
Erp=E*w
X=r'.*w
size(w)
D1=(sum(X)/N)-min(X)
%  $D1=\max(X)-\min(X)$ 
minimize(D1)
subject to
sum(w)==1;
w>=0;
Erp>=pi
cvx-end

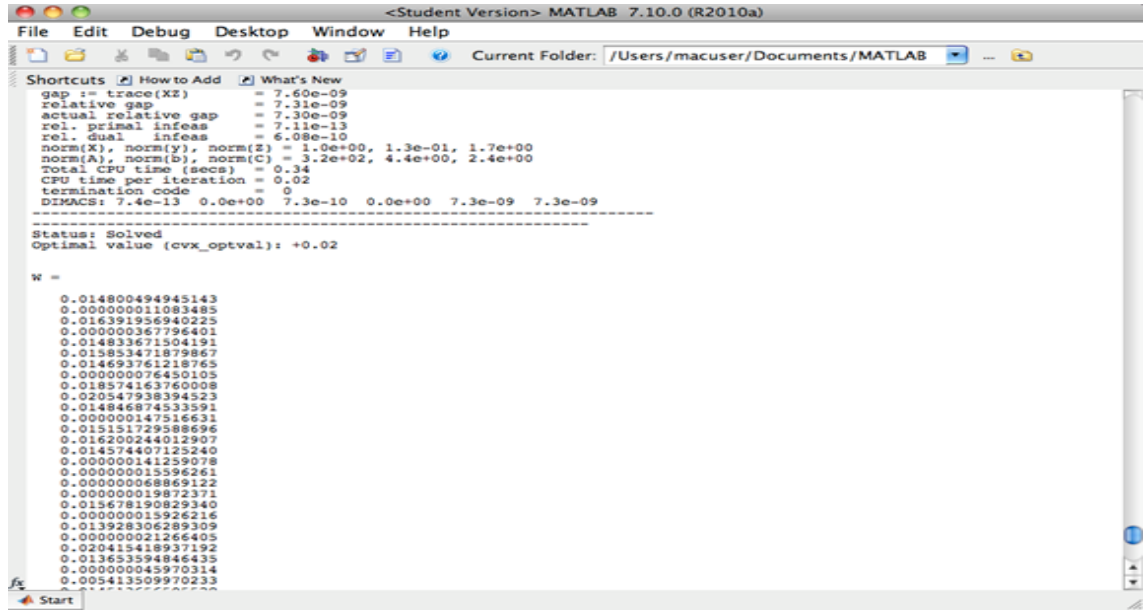
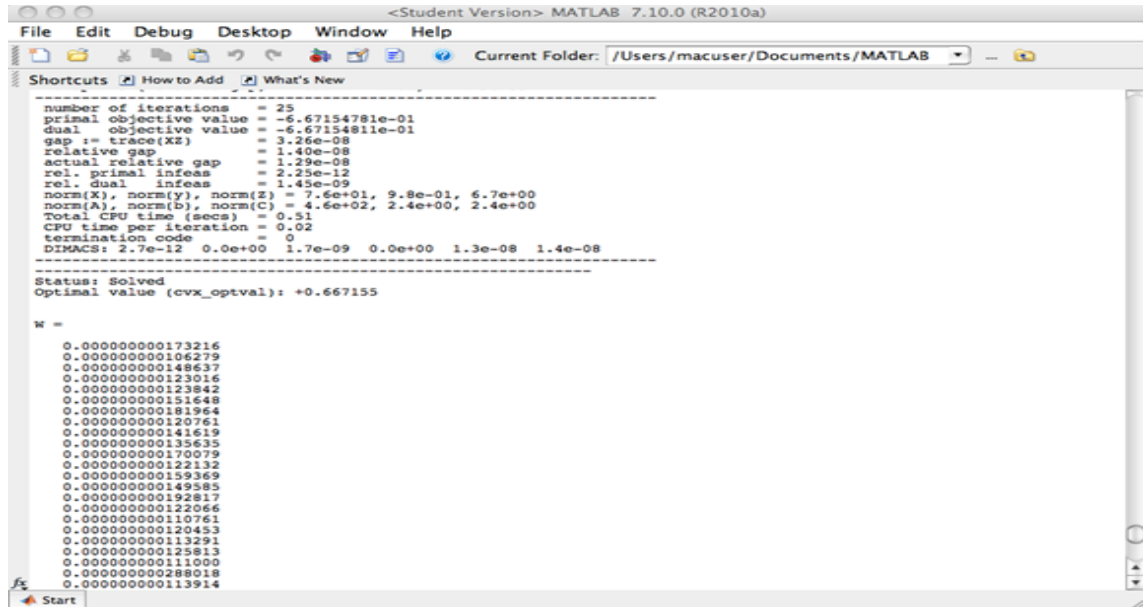
```



```

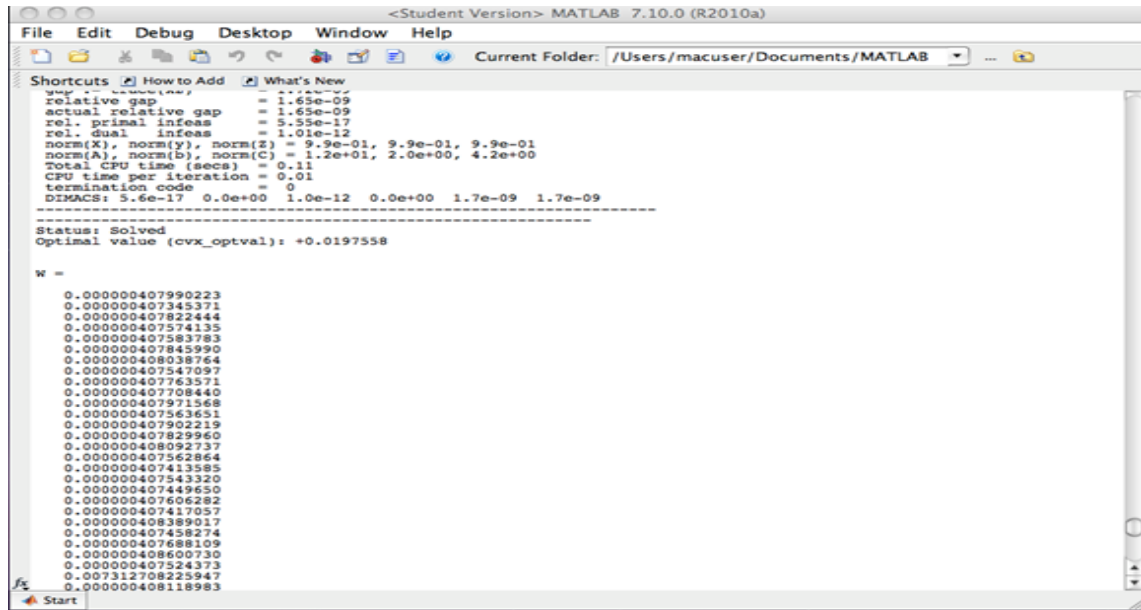
W=w
ssss=sum(w)
end

```

FIGURE A.7: Result in MATLAB for IV with $EX - \inf(X)$ FIGURE A.8: Result in MATLAB for IV with $\sup(X) - \inf(X)$

- **8)** Individual investment with risk measure is CVaR(X)
function [w,val] = MixCVARmean4(P)
% P=[],pricing matrix
% Mix CVar from formula
%P=[],pricing matrix
% mix CVar and CVar(collation)
N=3;
r0=0.01; % r0=1
[m,n]=size(P)
% For Return matrix
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
% for expected return
N=100;
for i=1:1:m-1
for j=1:1:n
E=sum(R)./N
end
end
R=R;
E=E;
r=sum(R);
pi=0.02;
n=100;
u=mean(E);
vv=var(E);
q=quantile(E,0.95);
PDF=unidpdf(E,100);
%this is probability distribution function for random variable of rate of re-
turn for portfolio and n=100,
CDF=unidcdf(E,100);
% this is Cumulative distribution function
alpha1=95;
alpha2=97;

```
cvx-begin
variable w(n);
Erp=E*w
X=r'.*w
XX=X'
size(w)
lambda1=0.25;
lambda2=0.75;
Cvar1=sum((X(1:95)))*1/n
CCvar1=(sum(X)/n)-(1/alpha1)*(Cvar1)
% second CVar
Cvar2=sum((X(1:97)))*1/n
CCvar2=(sum(X)/n)-(1/alpha2)*(Cvar2)
CCvar=(lambda1*CCvar1)+(lambda2*CCvar2)
minimize(CCvar)
subject to
sum(w)==1;
w>=0;
sum(Erp)>=pi
cvx-end
W=w
e=E'
end
```

FIGURE A.9: Result in MATLAB for IV with mix-CVAR(X)

- **9)** Cooperative case Mix-CVAR

```
function [ w,val] = MixCVARmeancooprat(P)
```

```
% P=[],pricing matrix
```

```
% Mix Cvar from formula
```

```
%P=[],pricing matrix
```

```
% CVar and Mix- Cvar ( collation)
```

```
N=3;
```

```
r0=0.01; % r0=1
```

```
[m,n]=size(P)
```

```
% For Return matrix
```

```
for i=1:1:m-1;
```

```
for j=1:1:n
```

```
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
```

```
end
```

```
end
```

```
% for expected return
```

```
N=100;
```

```
for i=1:1:m-1
```

```
for j=1:1:n
```

```
E=sum(R)./N
```

```
end
```

```
end
```

```

R=R;
E=E;
r=sum(R);
pi=0.02;
n=100;
u=mean(E);
vv=var(E);
q=quantile(E,0.95);
PDF=unidpdf(E,100);
% this is probability distribution function for random variable of rate of
return for
portfolio and n=100,
CDF=unidcdf(E,100);
% this is Cumulative distribution function
alpha1=95;
alpha2=97;
cvx-begin
variable w(n);
variable k(1);
Erp=E*w
X=r'.*w
XX=X'
y1=X-k;
y2=X-y1;
size(w)
lambda1=0.25;
lambda2=0.75;
Cvar1=sum((y1(1:95)))*1/n
CCvar1=(sum(y1)/n)-(1/alpha1)*(Cvar1)
% second CVar
Cvar2=sum((y1(1:97)))*1/n
CCvar2=(sum(y1)/n)-(1/alpha2)*(Cvar2)
CCvar=(lambda1*CCvar1)+(lambda2*CCvar2)
% second investor
alpha3=99;
Cvar3=sum((y2(1:99)))*1/n
CCvar3=(sum(y2)/n)-(1/alpha3)*(Cvar3)

```

```

% second CVar
Ds=max(CCvar,CCvar3)
minimize(Ds)
subject to
sum(w)==1;
w>=0;
sum(Erp)>=pi
y1+y2==X
cvx-end
W=w
end

```

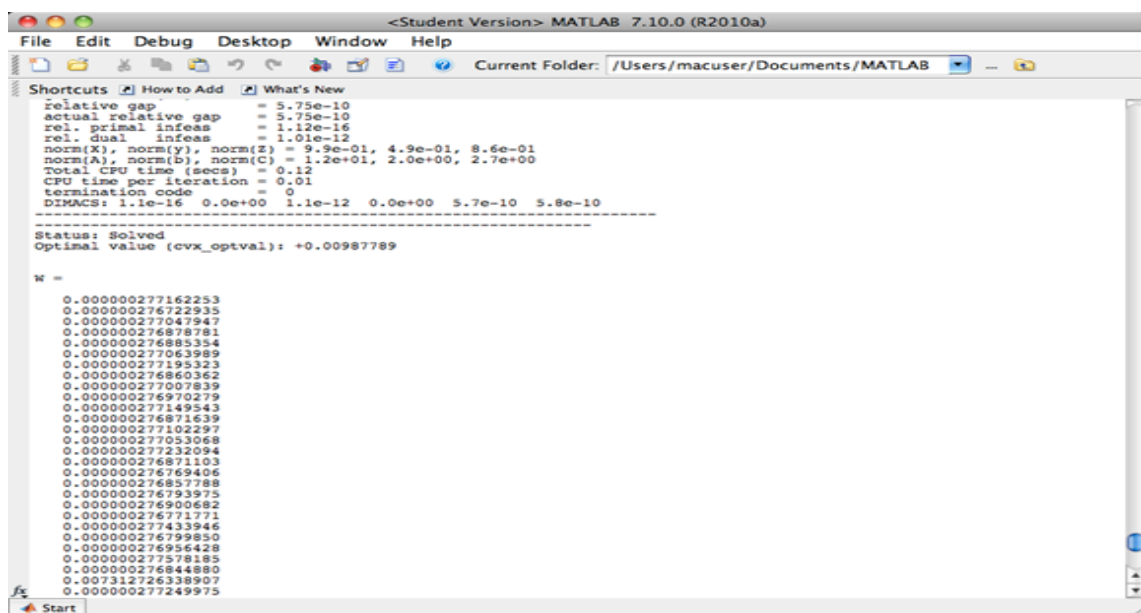


FIGURE A.10: Result in MATLAB for CI with $\text{mix-CVAR}(X)$

- **CI in multi period and DCI** The following codes are solve CI and DCI follow example in Pliska (1997) [115] as shown in the table

TABLE A.1: Observing security price

ω_k	t=0	t=1	t=2
ω_1	$S_0=5$	$S_1=8$	$S_2=9$
ω_2	$S_0=5$	$S_1=8$	$S_2=6$
ω_3	$S_0=5$	$S_1=4$	$S_2=6$
ω_4	$S_0=5$	$S_1=4$	$S_2=3$

- **10)** Cooperative investment solving problem directly
function [OPTimal] = copraminvarmeanvarsemiexample(beta,r0,pi1,pi2)
% this programme to find a fair allocation for mean-variance function ,where
Cooperative Investment between two investors, D1(y1) is standard deviation
 $\sigma = \sqrt{\sigma^2}$, and D2(y2)= is standard lower semi deviation $\sigma_- = \sqrt{\sigma_-^2}$.
N=4;
cvx-begin
variable a(1,1);
variable b(1,1);
variable c(1,1);
variable y1(N,1);
variable y2(N,1);
%i=the number of scenario of the second period , i=1,...,N, to get
%first scenario of uncertainty outcome for all
Xw11=((-1*a)+(8*a/5))-b+(b*(9/8))
Xw21=((-1*a)+(8*a/5))-b+(b*(6/8))
Xw31=((-1*a)+(4*a/5))-b+(c*(6/8))
Xw41=((-1*a)+(4*a/5))-b+(c*(3/8))
Xw=[Xw11,Xw21,Xw31,Xw41];
D1= square((y1-(sum(y1)/N)))
svary1=sum(D1)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))
D2= square-pos((-y2+(sum(y2)/N)))
EX=sum(Xw)/N
vary2=sum(D2)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))

```
D2= square((y1-(sum(y1)/N)))
D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N
Ey2=((sum(y2)/N))
minimize(vary1)
subject to
svary21<=beta
Ey1>=pi1
Ey2>=pi2
y1+y2==Xw'
a>=-1
a<=1
b>=-1
b<=1
c>=-1
c<=1
cvx-end
a=a
b=b
c=c
y1=y1
y2=y2
end
```

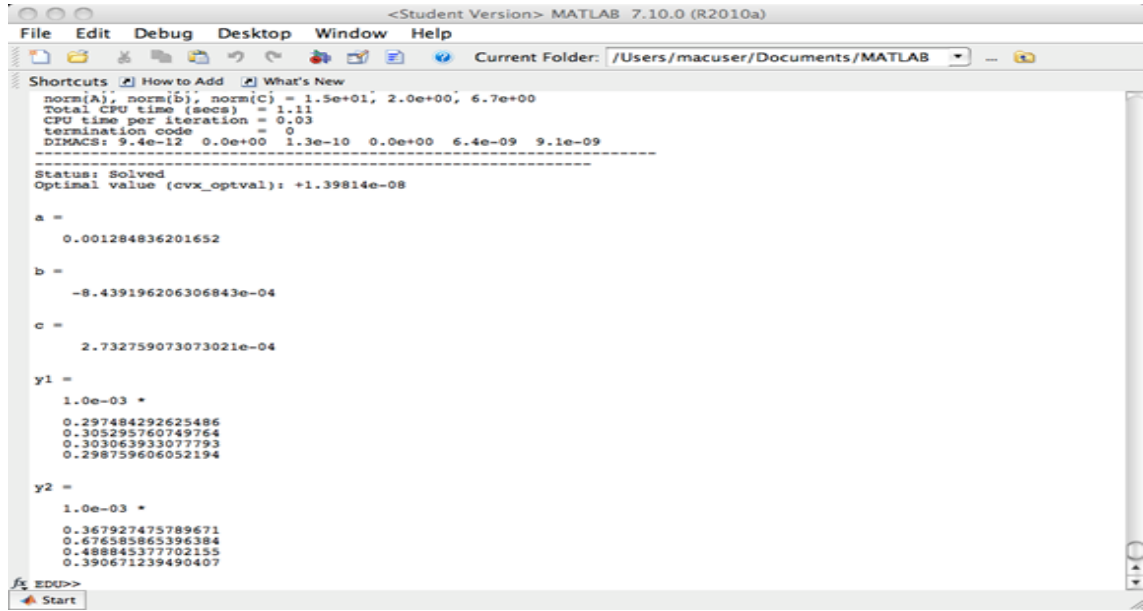



FIGURE A.11: Result in MATLAB for DCI,T=2 for global solution

- 11) Dynamic cooperative investment

The following problem used to find the optimal strategy c and b as separated code by solve cooperative investment in single period for each one. firstly by put % for variable c as well as for $Xw31$ and $Xw41$ and solve to get the optimal b . Secondly, by put

% for variable b as well as for $Xw11$ and $Xw21$. Hence find the optimal strategy c . $D1(y1)$ is standard deviation $\sigma = \sqrt{\sigma^2}$, and $D2(y2)$ is standard lower semi deviation $\sigma_- = \sqrt{\sigma_-^2}$.

function [OPTimal] = copraminvarmeanvarsemiexamply(beta,r0,pi1,pi2)

% Dynamic cooperative investment , hence we have two sub problems in

% single -period for example $pi2 = 0.3, pi1 = 0.2; N = 2$, we will use this

% program to solve dynamic programming first case we will solve over the

% variable a, b , and we will choose $Xw11$ and $Xw21$. second case to find

% trading strategy c by choose $Xw11$ and $Xw21$ and a will be arbitrary

$N=2$;

cvx-begin

variable a(1,1);

%variable b(1,1);

variable c(1,1);

variable y1(N,1);

variable y2(N,1);

%k =the arrange of scenario of the second period , $k = 1, , \dots, N$, to get

```

% the all scenarios in the second period , if we put  $k = 1$  they will get
% the first set of scenarios at  $T = 2$ , and if you put,  $k = 2$ , we get the
% second set of all scenarios at  $T = 2$ 
% first scenario for all
%  $X_{w11} = ((-1*a) + (8*a/5)) - b + (b*(9/8))$ 
%  $X_{w21} = ((-1*a) + (8*a/5)) - b + (b*(6/8))$ 
 $X_{w31} = ((-1*a) + (4*a/5)) - c + (c*(6/8))$ 
 $X_{w41} = ((-1*a) + (4*a/5)) - c + (c*(3/8))$ 
 $X_w = [X_{w31}, X_{w41}]$ ;
 $D2 = \text{square}((y1 - (\text{sum}(y1)/N)))$ 
 $\text{vary1} = \text{sum}(D2)/N$ 
 $Ey2 = ((\text{sum}(y2)/N))$ 
 $Ey1 = ((\text{sum}(y1)/N))$ 
 $D1 = \text{square-pos}((-y2 + (\text{sum}(y2)/N)))$ 
 $EX = \text{sum}(X_w)/N$ 
 $\text{vary1} = \text{sum}(D2)/N$ 
 $Ey2 = ((\text{sum}(y2)/N))$ 
 $Ey1 = ((\text{sum}(y1)/N))$ 
 $D2 = \text{square}((y1 - (\text{sum}(y1)/N)))$ 
 $D1 = \text{square-pos}((-y2 + (\text{sum}(y2)/N)))$ 
 $\text{svary21} = \text{sum}(D1)/N$ 
 $Ey2 = ((\text{sum}(y2)/N))$ 
minimize(vary1)
subject to
svary21 <= beta
 $Ey1 \geq \pi1$ 
 $Ey2 \geq \pi2$ 
 $y1 + y2 == X_w'$ 
 $a \geq -1$ 
 $a \leq 1$ 
%  $b \geq -1$ 
%  $b \leq 1$ 
 $c \geq -1$ 
 $c \leq 1$ 
cvx-end
a=a
%b=b

```

```

c=c
y1=y1
y2=y2
end

```

```

<Student Version> MATLAB 7.10.0 (R2010a)
File Edit Debug Desktop Window Help
Current Folder: /Users/macuser/Documents/MATLAB

Shortcuts How to Add What's New
stop: max(relative gap, infeasibilities) < 1.49e-08
-----
number of iterations = 27
primal objective value = -1.78523852e-07
dual objective value = -1.80533254e-07
gap := trace(XZ) = 2.53e-09
relative gap = 2.53e-09
actual relative gap = 2.01e-09
rel. primal infeas = 9.06e-14
rel. dual infeas = 2.58e-10
norm(X), norm(y), norm(Z) = 1.0e+00, 3.8e+03, 3.8e+03
norm(A), norm(b), norm(C) = 8.7e+00, 2.0e+00, 5.3e+00
Total CPU time (secs) = 0.94
CPU time per iteration = 0.03
termination code = 0
DIMACS: 9.1e-14 0.0e+00 4.5e-10 0.0e+00 2.0e-09 2.5e-09
-----
Status: Solved
Optimal value (cvx_optval): +1.80533e-07

a =
    0.577456300153528

b =
   -1.924448398415990e-07

y1 =
    1.0e-03 *
    0.300078340597588
    0.300081241585941

y2 =
    0.346173677695918
    0.346173746961746

fx EDU>>
Start

```

FIGURE A.12: Result in MATLAB for DCI,T=2 find trading strategy *b*

```

<Student Version> MATLAB 7.10.0 (R2010a)
File Edit Debug Desktop Window Help
Current Folder: /Users/macuser/Documents/MATLAB

Shortcuts How to Add What's New
stop: max(relative gap, infeasibilities) < 1.49e-08
-----
number of iterations = 27
primal objective value = -1.78629128e-07
dual objective value = -1.80498164e-07
gap := trace(XZ) = 2.36e-09
relative gap = 2.36e-09
actual relative gap = 1.87e-09
rel. primal infeas = 7.45e-14
rel. dual infeas = 2.43e-10
norm(X), norm(y), norm(Z) = 1.0e+00, 3.8e+03, 3.8e+03
norm(A), norm(b), norm(C) = 8.7e+00, 2.0e+00, 5.3e+00
Total CPU time (secs) = 0.94
CPU time per iteration = 0.03
termination code = 0
DIMACS: 7.4e-14 0.0e+00 4.3e-10 0.0e+00 1.9e-09 2.4e-09
-----
Status: Solved
Optimal value (cvx_optval): +1.80498e-07

a =
   -0.578315822565379

c =
   -2.077464978711419e-06

y1 =
    1.0e-03 *
    0.300057488652127
    0.300091679186197

y2 =
    0.115363626390673
    0.115364371249505

fx EDU>>
Start

```

FIGURE A.13: Result in MATLAB for DCI,T=2 find trading strategy *c*

- **12)** Fixed the trading strategy and solve it in recursive manner over one variable a

This programme to find optimal strategy a at first period and fix the optimal strategy b and c which are found it from previous code for DCI.

function [OPTimal] = copraminvarmeanvarsemiexamplyfix(beta,r0,b,c,pi1,pi2)

%Dynamic cooperative investment, hence we have two sub-problem in

% single – period for example 4 $pi2 = 0.3$, $pi1 = 0.2$; $N = 2$, we will use this

% program to solve dynamic programming after we fix the trading strategy

%band c and solve the problem over one variable a . $D1(y1)$ is standard deviation $\sigma = \sqrt{\sigma^2}$, and $D2(y2)$ is standard lower semi deviation $\sigma_- = \sqrt{\sigma_-^2}$.

N=2;

N=4;

cvx-begin

variable a(1,1);

variable y1(N,1);

variable y2(N,1);

% k = the arrange of scenario of the second period , $k=1,...,N$, to get

% the all scenarios in the second period , if we put $k = 1$ they will get

% the first set of scenarios at $T = 2$, and if you put, $k = 2$, we get the

%second set of all scenarios at $T = 2$

% first scneario for all

Xw11= $((-1*a)+(8*a/5))-b+(b*(9/8))$

Xw21= $((-1*a)+(8*a/5))-b+(b*(6/8))$

Xw31= $((-1*a)+(4*a/5))-b+(c*(6/8))$

Xw41= $((-1*a)+(4*a/5))-b+(c*(3/8))$

Xw=[Xw11,Xw21,Xw31,Xw41];

D2= square((y1-(sum(y1)/N)))

vary1=sum(D2)/N

Ey2=((sum(y2)/N))

Ey1=((sum(y1)/N))

D1= square-pos((-y2+(sum(y2)/N)))

EX=sum(Xw)/N

vary1=sum(D2)/N

Ey2=((sum(y2)/N))

Ey1=((sum(y1)/N))

D2= square((y1-(sum(y1)/N)))

```

D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N
Ey2=((sum(y2)/N))
minimize(vary1)
subject to
svary21<=beta
Ey1>=pi1
Ey2>=pi2
y1+y2==Xw'
a>=-1
a<=1
cvx-end
a=a
end

```

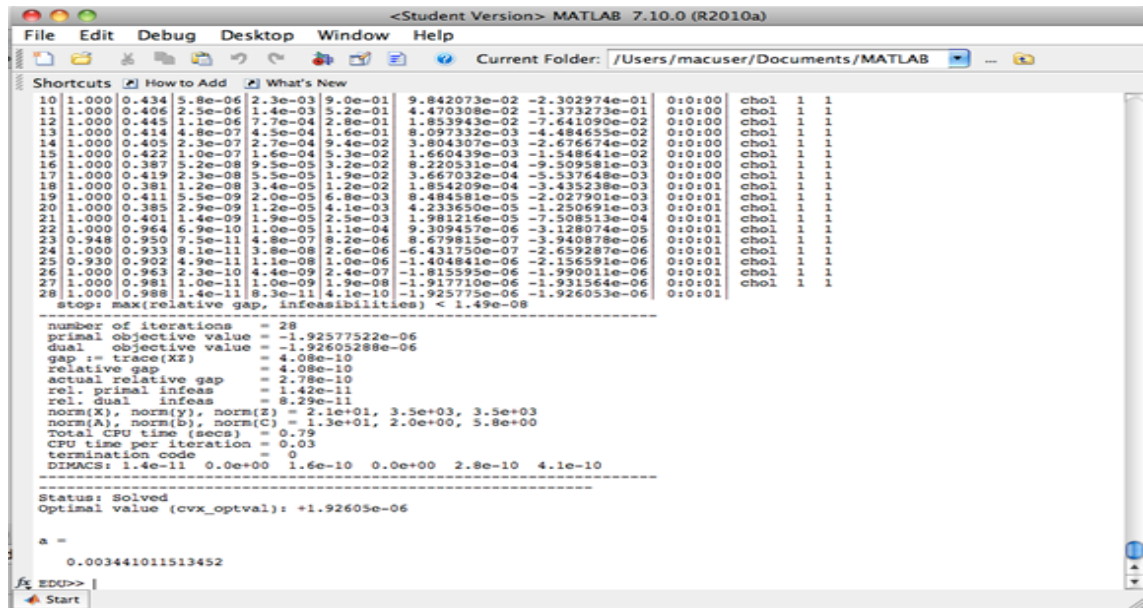


FIGURE A.14: Result in MATLAB for DCI,T=2 find trading strategy a , first period

- **13)** Equilibrium price for DCI

This code used to check feasibility problem and to find the equilibrium price for mean-variance problem.

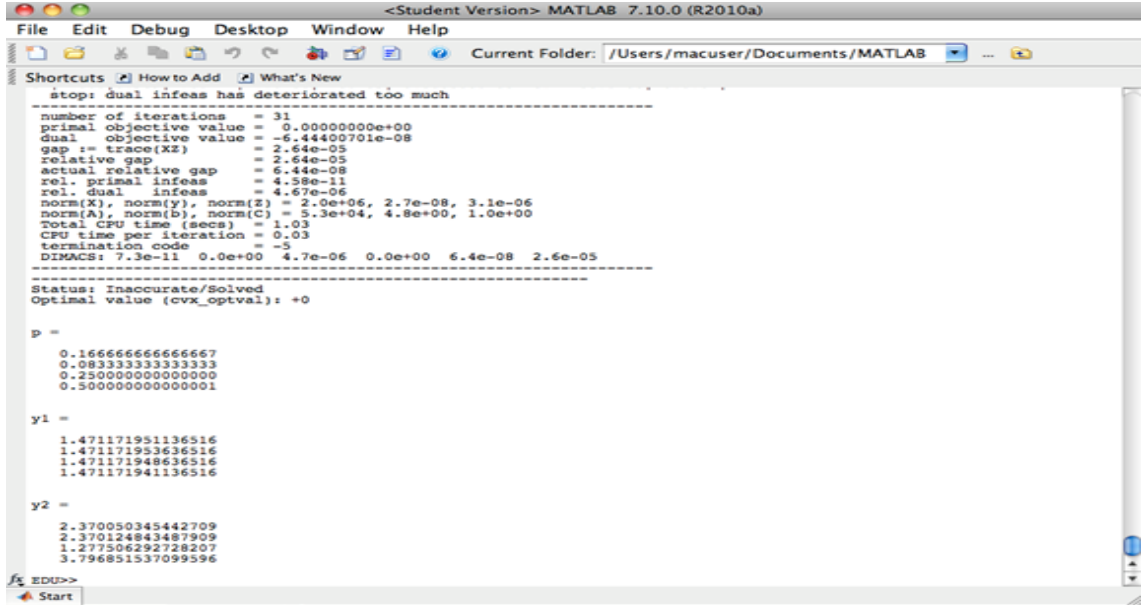
```
function [OPTimal] = equilalloprice(pi1,pi2)
%Equilibrium for feasibility
% this programe to find a fair allocation for mean variance function as
% with four random variable.D1(y1) is standard deviation  $\sigma = \sqrt{\sigma^2}$  , and
D2(y2)= is standard lower semi deviation  $\sigma_- = \sqrt{\sigma_-^2}$ .
n=1;
cvx-begin
variable p(4,n)
variable a(n)
variable b(n)
variable c(n)
% for the first agent
pq=square-pos (p)
spq=sum(pq)
ssppq= sum-square (p)
ppq=4*ssppq/pi1
y11=pi1-(pi1*p(1,1))/spq
size(y11)
y12=pi1-(pi1*p(2,1))/spq
y13=pi1-(pi1*p(3,1))/spq
y14=pi1-(pi1*p(4,1))/spq
% for the second agent
pqq=4*ssppq/pi2
y21=max (pi2-(pi2*p(1,1))/spq,0)
y22=max (pi2-(pi2*p(2,1))/spq,0)
y23=max (pi2-(pi2*p(3,1))/spq,0)
y24=max (pi2-(pi2*p(4,1))/spq,0)
% the random variable of uncertainty outcome
w1=((1-a)+8*a/5)-b+(9*b/8)
w2=((1-a)+8*a/5)-b+(6*b/8)
w3=((1-a)+4*a/5)-c+(6*c/8)
w4=((1-a)+4*a/5)-c+(3*c/8)
minimize(0)
subject to
```

```

(y11+y21)-w1==0
(y12+y22)-w2==0 % y1+y2 in F feasible set
(y13+y23)-w3==0
(y14+y24)-w4==0
p(1,1)+p(2,1)+p(3,1)+p(4,1)==1
(-p(1,1)+((8/5)*p(1,1))-p(2,1)+((8/5)*p(2,1))-p(3,1)+((4/5)*p(3,1))-p(4,1)
+((4/5)*p(4,1)))==0
(-p(1,1)-p(2,1)+((9/8)*p(1,1))+((6/8)*p(2,1)))==0
(-p(3,1)-p(4,1)+((6/4)*p(3,1))+((3/4)*p(4,1)))==0
b>-1
b<1
cvx-end

p=p
a=a
b=b
c=c
end

```

FIGURE A.15: Result in MATLAB for equilibrium allocation for σ and σ_-

Another example :

- **14)** Equilibrium for exponential utility function

This code used to check feasibility problem and to find the equilibrium price for exponential utility function.

function [Optimal] = equilallo(alpha1,alpha2) % this programme to find a fair allocation for exponential function as % utility function when $\alpha = 1/4$ % with four random variable % α is a risk aversion % $U(W) = -\exp(\alpha * W)$, the same for second investor where $\alpha_2 = 1/2$

n=1;

cvx-begin

variable p(4,n)

variable a(n)

variable b(n)

variable c(n)

%for the first agent

y11=(-2*((log(8*p(1,1))))+((log(8*p(2,1))))+((log(8*p(3,1))))
+((log(8*p(4,1)))))-(2*log(8*p(1,1)))

size(y11)

y12=(-2*((log(8*p(1,1))))+((log(8*p(2,1))))+((log(8*p(3,1))))
+((log(8*p(4,1)))))-(2*log(8*p(2,1)))

y13=(-2*((log(8*p(1,1))))+((log(8*p(2,1))))+((log(8*p(3,1))))
+((log(8*p(4,1)))))-(2*log(8*p(3,1)))

y14=(-2*((log(8*p(1,1))))+((log(8*p(2,1))))+((log(8*p(3,1))))
+((log(8*p(4,1)))))-(2*log(8*p(4,1)))

% for the second agent

y21=(-4*((log(16*p(1,1))))+((log(16*p(2,1))))+((log(16*p(3,1))))
+((log(16*p(4,1)))))-(4*log(16*p(1,1)))

y22=(-4*((log(16*p(1,1))))+((log(16*p(2,1))))+((log(16*p(3,1))))
+((log(16*p(4,1)))))-(4*log(16*p(2,1)))

y23=(-4*((log(16*p(1,1))))+((log(16*p(2,1))))+((log(16*p(3,1))))
+((log(16*p(4,1)))))-(4*log(16*p(3,1)))

y24=(-4*((log(16*p(1,1))))+((log(16*p(2,1))))+((log(16*p(3,1))))
+((log(16*p(4,1)))))-(4*log(16*p(4,1)))

% the random variable of Uncertainty outcome

w1=(-a)+8*a/5)-b+(9*b/8)

w2=(-a)+8*a/5)-b+(6*b/8)

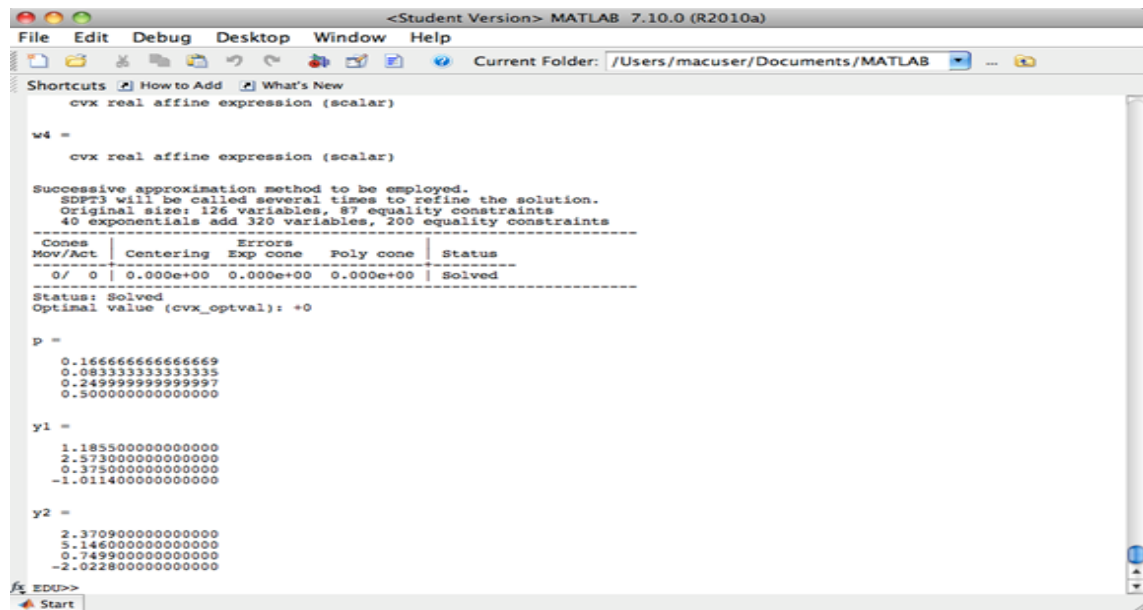
w3=(-a)+4*a/5)-c+(6*c/8)


```

w4=(((-a)+4*a/5)-c+(3*c/8)
minimize(0)
subject to
(y11+y21)-w1==0
(y12+y22)-w2==0 % y1+y2 in F fesiable set
(y13+y23)-w3==0
(y14+y24)-w4==0
(-p(1,1)+((8/5)*p(1,1))-p(2,1)+((8/5)*p(2,1))-p(3,1)+((4/5)*p(3,1))-p(4,1)
+((4/5)*p(4,1)))==0
(-p(1,1)-p(2,1)+((9/8)*p(1,1))+((6/8)*p(2,1)))==0
(-p(3,1)-p(4,1)+((6/4)*p(3,1))+((3/4)*p(4,1)))==0
p(1,1)+p(2,1)+p(3,1)+p(4,1)==1
a>=-1
a<=1
b>=-1
b<=1
c>=-1
c<=1
cvx-end

p=p
a=a
b=b
c=c
end

```

FIGURE A.16: Result in MATLAB for equilibrium allocation for $1 - \exp(-\alpha X)$

- 15) Detection of Arbitrage

```
function [OPTimal] =generalmulti2assetarbitrage6b(P)
```

```
r0=0.10
```

```
[m,n]=size(P)
```

```
for i=1:1:m-1;
```

```
for j=1:1:n
```

```
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
```

```
end
```

```
end
```

```
N=100; % for example
```

```
for i=1:1:m-1
```

```
for j=1:1:n
```

```
E=sum(R)./N
```

```
end
```

```
end
```

```
R=R;
```

```
[M,n]=size(R)
```

```
ss=1+R
```

```
cvx-begin
```

```
variable v(1,M)
```

```
maximize(0)
```

```
f=sum(v*ss)
```

```

subject to
sum(v*ss)==1
(v)>=0
cvx-end
end

```

- 16)** Solving the CI to get the Certainty equivalent that shown in Chapter 5

```

function [OPTimal] = copraminvarmeanvarsemiexemplCEGlobal5(r0,R1,R2,K)
N=4;
% where  $c_1 \approx U_1 = \mu - \frac{\sigma^2}{2R}$  and  $c_2 \approx U_2 = \mu - \frac{\sigma^2}{R}$ 
cvx-begin
variable a(1,1);
variable b(1,1);
variable c(1,1);
variable y1(4,1);
variable y2(4,1);
firstly: scenario for all at the end period
Xw11= 1+(8/5)*a+b*(8/5)
Xw21=1+(8/5)*a-b*(6/5)
Xw31=1-(6/5)*a+c*(8/5)
Xw41=1-(6/5)*a-c*(6/5)
D2=var(y1)
Xw=[Xw11,Xw21,Xw31,Xw41];
vary1=sum(D2)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))
D1= square-pos((-y2+(sum(y2)/N)))
EX=sum(Xw)/N
sdy2=sum(D1)/N
Xw=[Xw11,Xw21];
EX=sum(Xw)/N
U1=Ey1-((vary1)/2*R1)
U2=Ey2-((sdy2)/R2)
maximize(U1)
subject to
U2>=K
a>=-1
a<=1

```

```

b>=-1
b<=1
c>=-1
c<=1
y1(1,1)+y2(1,1)==Xw11
y1(2,1)+y2(2,1)==Xw21
y1(3,1)+y2(3,1)==Xw31
y1(4,1)+y2(4,1)==Xw41
cvx-end

a=a
b=b
c=c
end

```

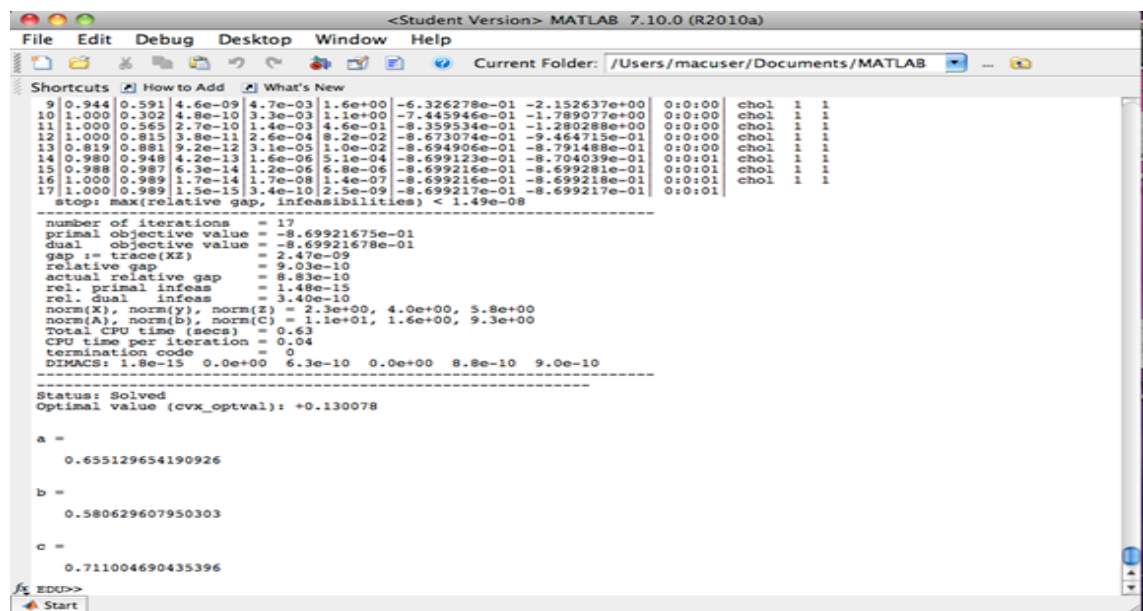


FIGURE A.17: Result in MATLAB for with certainty equivalent

In all of my following code the figures show the result for $n=6$ risky asset with $N=30$ scenario and up to $T=20$ according to my experiments in this thesis. $pi_1 = 0.003$, $pi_2 = 0.0004$, $w_0 = 100$, $r_0 = 0.0001$, where $D1(y1)$ is standard deviation $\sigma = \sqrt{\sigma^2}$, and $D2(y2)$ is standard lower semi deviation $\sigma_- = \sqrt{\sigma_-^2}$.

- Cooperative investment with real data

```
% the following codes that solve min var(y1) s.t svar(y2)
% N= number of scenario , h= number of node 1:N
% N= number of scenario , h= number of node 1:N at time T
% Note that in case of multi period we need to write the uncertainty outcome
which will written as follows
% (W0-sum(x))r0+x'*r here r is a vector m x 1 and x is trading strategies
at period t1 and w0 is the initial capital. Then, we will reinvest the whole
amount which we have in to the next period as ((w0 - sum(x))r0 + x' * r(
, 1)) - sum(zt1) * r0 + zt1' * r(m * N + 1 : 1)
% where zt1 is the trading strategy for first node for second period and so
on complete each period. in the codes below, we will reinvest the amount of
money which we have from previous period up to period T. Also, see cooprat-
meanvarco.m, coopratmeanvarcody.m, and coopratmeanvarcodyfix.m. note
that, by changing the value of beta in the code e coopratmeanvarco.m, and
coopratmeanvarcodyfix.m. we will get the whole efficient frontier.
```

- 17) Cooperative investment with real data global solution

```
function [ w,val] = coopratmeanvarco(P,beta,w0,r0, N,T,pi1,pi2)
[m,n]=size(P)
Rr=[];
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
rt1=R(1:(m-1)/2 ,:);
rt2=R(((m-1)/2)+1:m-1,:);
rt11=rt1+1
rt21=rt2+1
[k,n]=size(rt21)
for i=1:k
```

```

a(:,1)=(rt11(:,1)*rt11(i,1)')-1
A=a
end
s=sum(R);
s=sum(R);
ss=mean(R);
vv=var(R);
r=mean(ss);
v=var(vv);
[M, h]=size(R)
for i=1:1:N
rr(i,:)=R(i,:);
end
rs=size(r)
rrs=size(rr)
cvx-begin
variable x(n,1);
variable z(n,N);
variable y1(N,1);
variable y2(N,1);
for k=1:N
% xww=(((w0-sum(x))*T*r0+x'*rr(k,:)')-sum(z(:,k)))*(T-k)*r0+z(:,k)')*a((k-
1)*N+1:k*N,:)' % for many period t.
xww=(((w0-sum(x)) *(T*r0))+x'*rr(2,:))-sum(z))*(T-h)*r0
end
for k=1:1:N
sxww=sum(xww)/N
end
Xw=sxww;
EX=sum(Xw)/N
D2= square((y1-(sum(y1)/N)))
EX=sum(Xw)/N
vary1=sum(D2)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))
D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N

```

```

Ey2=((sum(y2)/N))
minimize(vary1)
subject to
svary21<=beta
Ey1>=pi1
Ey2>=pi2
y1+y2==Xw'
sum(x)==1
sum(z)==1
z>=-1
z<=1
x>=-1
x<=1
cvx-end

x=x
z1t1=z(:,1);
z2t1=z(:,2);
z3t1=z(:,3);
end

```

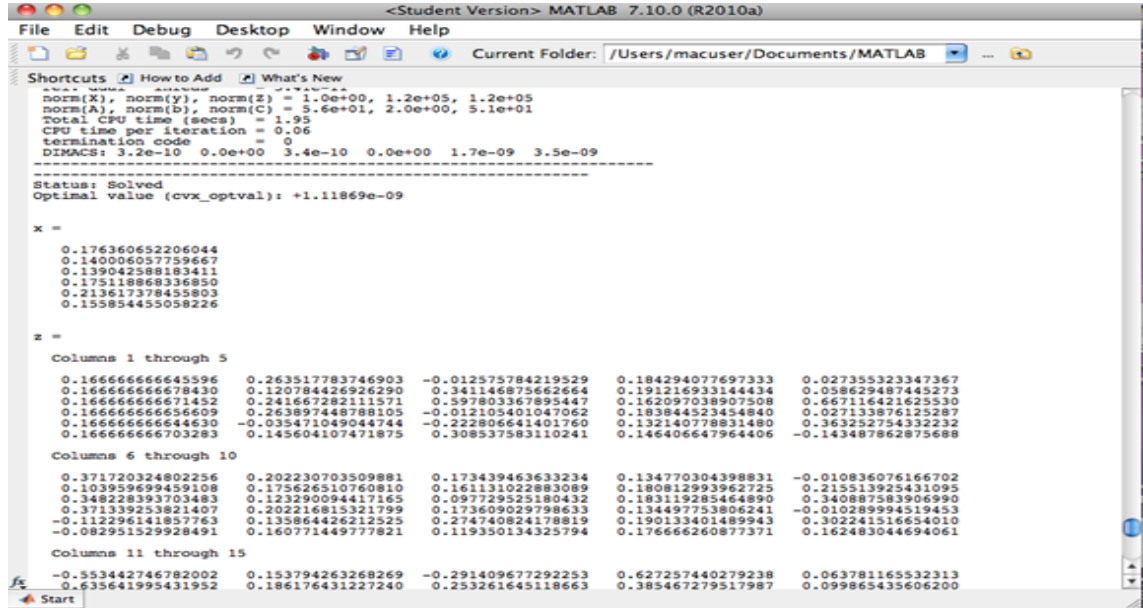


FIGURE A.18: Result in MATLAB for CI for global solution with *GARCH*, each x is the trading strategy at first period and z is a matrix of size $T \times N$ and each column of each row contains of trading strategy $z_{n,t}$ period t and node n

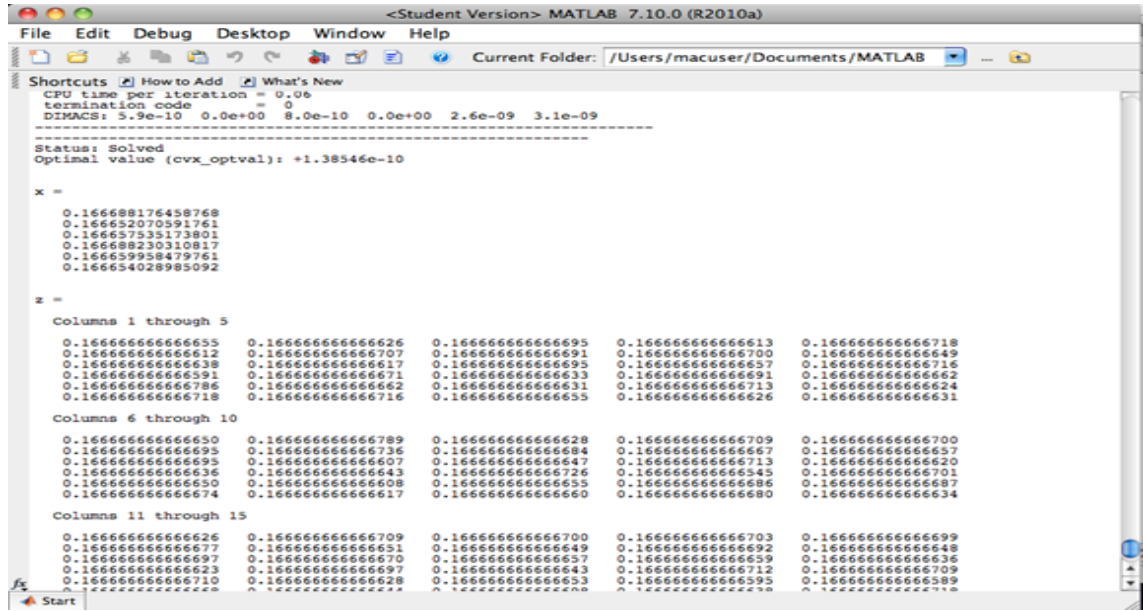


FIGURE A.19: Result in MATLAB for CI for global solution with historical data simulation, each x is the trading strategy at first period and z is a matrix of size $T \times N$ and each column of each row contains of trading strategy $z_{n,t}$ period t and node n

- **18)** Dynamic code to find each trading at each node in each period
function [w,val] = coopratmeanvarcody(P,beta,w0,r0, N,h,T,pi1,pi2)
% this is program that solve min var(y1) s.t svar(y2)
% N= number of scenario , h= number of node 1:N
% N= number of scenario , h= number of node n=1:N
s=sum(R);
% N= number of scenario , h= number of node n=1:N
% k=the arrange of scenario of the T period , k=1,...,N, to get
% the all scenarios in the T, if we put k=1 they will get
% the first set of scenarios at T, and if you put, k=2, we get the
% second set of all scenarios at T, where T begin of 2,...in my case I applied
it up to T=50
[m,n]=size(P)
Rr=[];
for i=1:m-1;
for j=1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
rt1=R(1:(m-1)/2 ,:);


```

rt2=R(((m-1)/2)+1:m-1,:);
rt11=rt1+1
rt21=rt2+1
[k, n]=size(rt21)
for i=1:k
a(:,1)=(rt11(:,1)*rt11(i,1)')-1
A=a
end
s=sum(R);
s=sum(R);
ss=mean(R);
vv=var(R);
r=mean(ss);
v=var(vv);
[M, h]=size(R)
for i=1:1:N
rr(i,:)=R(i,:);
end
rs=size(r)
rrs=size(rr)
cvx-begin
variable x(n,1);
variable z(n,1);
variable y1(N,1);
variable y2(N,1);
% Xw= (((w0-sum(x)) *(T*r0))+x'*rr(2,:))-sum(z))*(T-h)*r0
+z'*a((i*N)+1:(i+1)*N,:)' % for many period t
Xw=(((w0-sum(x)) *(T*r0))+x'*rr(2,:))-sum(z))*(T-h)*r0
D2= square((y1-(sum(y1)/N)))
vary1=sum(D2)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))
D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N
Ey2=((sum(y2)/N))
minimize(vary1)
subject to

```

```

svary21<=beta
Ey1>=pi1
Ey2>=pi2
y1+y2==Xw'
sum(x)==1
sum(z)==1
x>=-1
x<=1
z>=-1
z<=1
cvx-end
x=x
z=z;
end

```

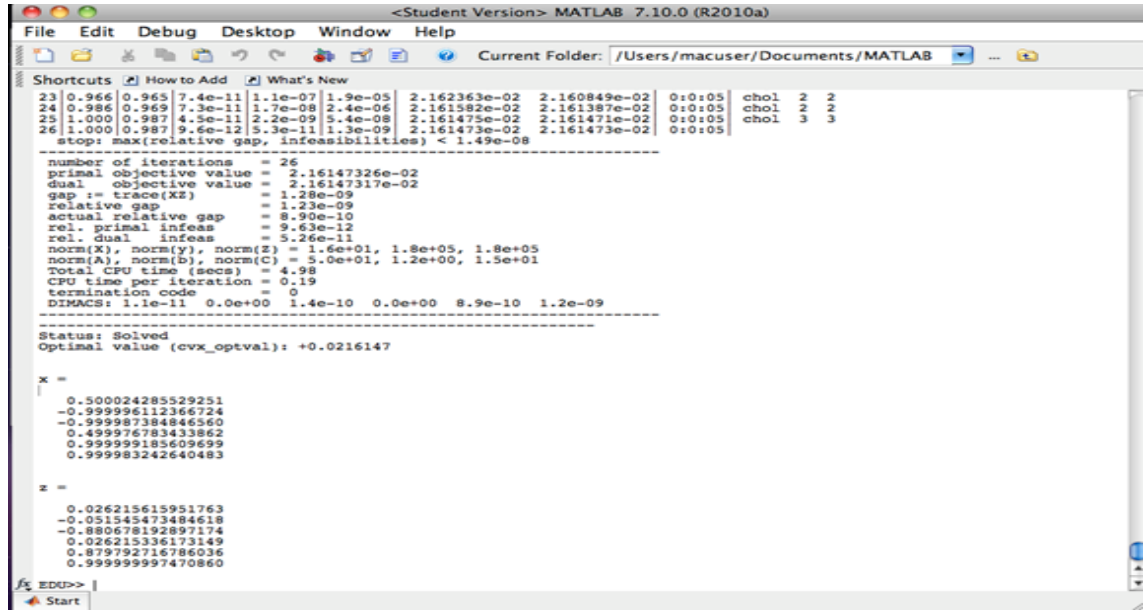


FIGURE A.20: Result in MATLAB for DCI for each node with *GARCH*, each x is the trading strategy at first period and z is a vector of size $m \times 1$ contains of trading strategy $z_{n,t}$ period t and node n

- **19)** Dynamic code at first period after fix all the trading strategy at each node in each period. then, save them in matrix z of size $T \times N$ and use it as input in the following code.

```

function [ w,val] = coopratmeanvarcofix(P,z,beta,w0,r0,N,T,pi1,pi2)
% this is program that solve min var(y1) s.t svar(y2)
% N= number of scenario , h= number of node 1:N
% N= number of scenario , h= number of node n=1:N
s=sum(R);
% N= number of scenario , h= number of node n=1:N
[m,n]=size(P)
Rr=[];
for i=1:1:m-1;
for j=1:1:n
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
end
end
rt1=R(1:(m-1)/2 ,:);
rt2=R(((m-1)/2)+1:m-1,:);
rt1l=rt1+1
rt2l=rt2+1
[k,n]=size(rt2l)
for i=1:k
a(:,1)=(rt1l(:,1)*rt1l(i,1)')-1
A=a
end
s=sum(R);
s=sum(R);
s=sum(R);
ss=mean(R);
vv=var(R);
r=mean(ss);
v=var(vv);
[M,h]=size(R)
for i=1:1:N
rr(i,:)=R(i,:);
end rs=size(r)
rrs=size(rr)

```

```

cvx-begin
variable x(n,1);
variable y1(N,1);
variable y2(N,1);
for k=1:N
xww=(((w0-sum(x)) *(T*r0))+x'*rr(2,:))-sum(z))*(T-h)*r0
% xww=(((w0-sum(x))*T*r0+x'*rr(k,:))-sum(z(:,k)))*(T-k)*r0+z(:,k))*a((k-
1)*N+1:k*N,:)' % for many period t.
end
for k=1:1:N
sxww=sum(xww)/N
end
Xw=sxww;
EX=sum(Xw)/N
D2= square((y1-(sum(y1)/N)))
EX=sum(Xw)/N
vary1=sum(D2)/N
Ey2=((sum(y2)/N))
Ey1=((sum(y1)/N))
D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N
Ey2=((sum(y2)/N))
minimize(vary1)
subject to
svary21<=beta
Ey1>=pi1
Ey2>=pi2
y1+y2==Xw'
sum(x)==1
x>=-1
x<=1
cvx-end
x=x
end

```

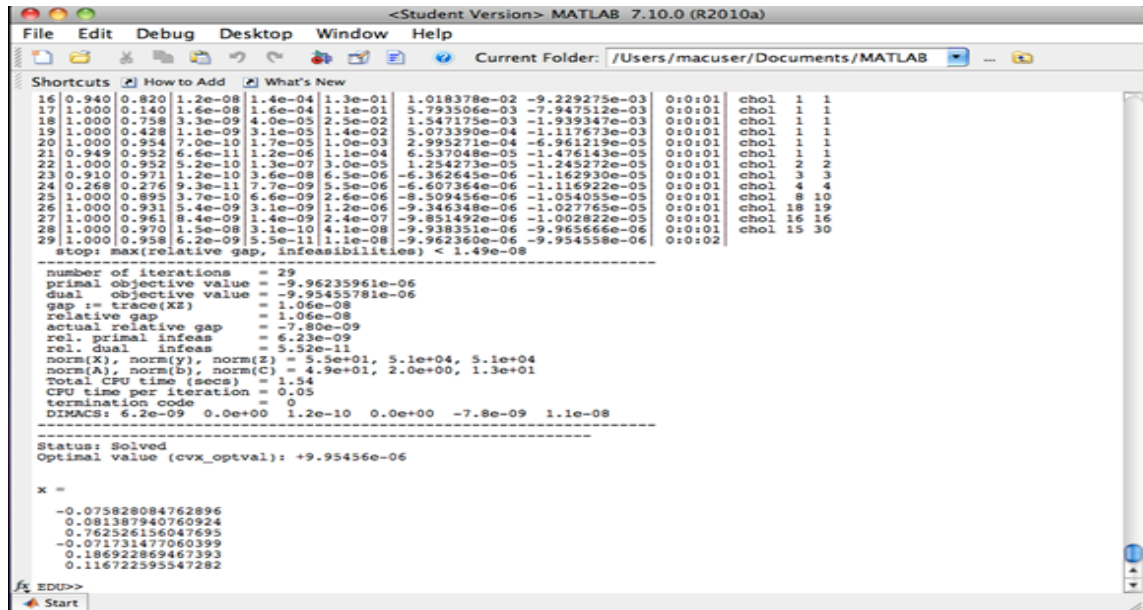


FIGURE A.21: Result in MATLAB for DCI for first period recursively with *GARCH*, each x is the trading strategy at first period

- 20) Finding equilibrium for N scenario at period T , and the fair allocation for σ and σ_- for first and second investors, respectively.

function [w,val] = equilbgeneral2(R,x,z,w0,r0, N,h,T,pi1,pi2)

% this is program that solve min var(y1) s.t svar(y2)

% N= number of scenario , h= number of node 1:N

% N= number of scenario , h= number of node n=1:N

s=sum(R);

% N= number of scenario , h= number of node n=1:N

[M,h]=size(R)

for i=1:1:N

rr(i,:)=R(i,:);

end

for i=1:1:N-1

r((i*N)+1:(i+1)*N,:)=R((i*N)+1:(i+1)*N,:);

end

rs=size(r)

rrs=size(rr)

cvx-begin

variable P(N,1);

% k=the arrange of scenario of the T period , k=1,...,N, to get

% the all scenarios in the T, if we put k=1 they will get

```

% the first set of scenarios at T, and if you put, k=2, we get the
% second set of all scenarios at T, where T begin of 2,...in my case I applied
it up to T=50
Xw = (((w0 - sum(x)) * T * r0 + x' * rr(h, :)') - sum(z))) * (T - h) * r0 +
z' * r((h - 1) * N + 1 : h * N, :)')
% Xw is a matrix consist of n rows=number of scenarios
% and one column = uncertainty outcome
% for the first agent
pq=square-pos (p)
spq=sum(pq)
ssppq= sum-square (p)
ppq=N*ssppq/pi1
for i=1:1:N
y1(i)=pi1-(pi1*p(i,1))/spq
end
ppq=N*ssppq/pi2
for i=1:1:N
y21=max (pi2-(pi2*p(i,1))/spq,0)
end
D2= square((y1-(sum(y1)/N)))
vary1=sum(D2)/N
Ey2=((sum(y2)/N))
vary1=var(y1)
Ey1=((sum(y1)/N))
D1= square-pos((-y2+(sum(y2)/N)))
svary21=sum(D1)/N
Ey2=((sum(y2)/N))
minimize(0)
subject to
(y1+y2)-Xw ==0
sum(p)==1
for i=1:1:N
((T - i) * r0) - (ones(1, N) - (T - i) * r0 * r((h - 1) * N + 1 : h * N, :)')
end
cvx-end
end

```

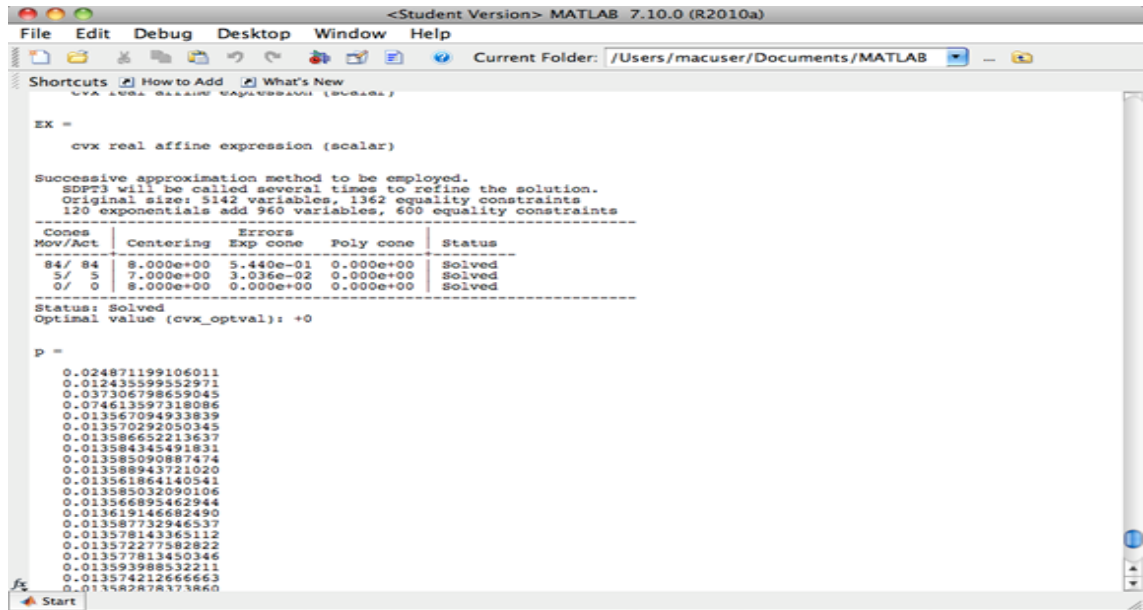


FIGURE A.22: Result in MATLAB to find fair equilibrium allocation for N nodes and T , for σ and σ_- for first and second investors, respectively.

- **21)** finding equilibrium price for real data

```
function [OPTimal] = equilbgeneral(P,N,w0,r0,T,pi1,pi2)
```

```
alpha1=pi1;
```

```
alpha2=pi2;
```

```
[m,n]=size(P)
```

```
% For Return matrix
```

```
Rr = [];
```

```
for i=1:1:m-1;
```

```
for j=1:1:n
```

```
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
```

```
end
```

```
end
```

```
% N= number of scenario , h= number of node n=1:N
```

```
s=sum(R);
```

```
[M,h]=size(R)
```

```
for i=1:1:N
```

```
rr(i,:)=R(i,:);
```

```
end
```

```
rrs=size(rr)
```

```
nm=1;
```

```
cvx-begin
```

```

variable p(N,1)
variable x(n,1);
variable z(n,N);
% for the first agent
a=alpha1*1/2
b=alpha2*1/2
for j=1:1:N
for k= 1:N
y1(j)=(-(1/a)*(sum(log((1/2*a)*p(k,1)))))-(1/a)*log((1/2*a)*p(j,1))
end
end
% for second agent
for j=1:1:N
for k= 1:N
y2(j)=(-(1/b)*(sum(log((1/2*b)*p(k,1)))))-(1/b)*(log((1/2*b)*p(j,1)))
end
end
for i=1:1:N
rr(i,:)=R(i,:);
end
for k=1:N
xww((((w0-sum(x)) *(T*r0))+x'*rr(2,:))-sum(z))*(T-h)*r0
% xww((((w0-sum(x))*T*r0+x'*rr(k,:))-sum(z(:,k)))*(T-k)*r0
+z(:,k))*a((k-1)*N+1:k*N,:))' % for many period t.
end
for k=1:1:N
sxww=sum(xww)/N
end
Xw=sxww;
EX=sum(Xw)/N
minimize(0)
subject to
(y1+y2)-sxww<=0
sum(x)==1
sum(z)==1
z>=-1
z<=1

```



```

x>=-1
x<=1
((w0-sum(x))*(T*r0))*p(N,1)==0
x'*rr(k,:)'*p(N,1)==0
sum(z))*(T-h)*r0*p(N,1)==0
sum(p)==1
p>=-1
p<=1
cvx-end
p=p
x=x
y1=y1';
y2=y2';
vy1=var(y1);
end

```

- **22)** Estimate return for risky asset by *GARCH*(1, 1)

note that, all of these steps can do it directly in command window in MATLAB

```
function [ R,val] = GARCH(P)
```

% the first few steps can do it directly in the command window in MATLAB
as

% well as can do it in Excel.

```
[m,n]=size(P)
```

% For Return matrix

```
Rr=[];
```

```
for i=1:1:m-1;
```

```
for j=1:1:n
```

```
R(i,j)=(P(i+1,j)-P(i,j))/P(i,j);
```

```
end
```

```
end
```

% check for correlation

%1- plot autocorrelation function (ACF), choose the risky asset to check
its correlation

```
autocorr(R(:,i)) % for i=1,...n
```

% plot partial-autocorrelation (PACF)

```
parcorr(R(:,1))
```

% return the same process for R^2

```

autocorr(R(:,i).^2) % for i=1,...n
autocorr(R(:,i).^2) % for i=1,...n
% detect Arch effect
[h,p,fstat,crit]archtest(R(:,i),'lags',1) % for i=1,...,n
% if I got is H=1, means that no hypothesis of effect is rejected
% now, to estimate the parameters as following or likelihood method
ToEstMdl = garch(1,1);
EstMdl = estimate(ToEstMdl,R(:,i)) % for i=1,...n
% after estimate EWMA
lamndda=94/100 % 94 %
w1=1-lambda % weight for moving average
w2=w1*(94/100) % weight for the second row
EWm1= R(1,i)*w1 % i=1, for first weighting moving average
Ewm2=R(2,i)*w2 % for i=2 for second weighting moving average and com-
plete the same
for i=3:1:n
w(i,1)=w(i-1)*w(i+1)
Ewm=w(i,1) * R(:,i)^2
end
% now to estimate conditional variance just plug in the formula of GARCH(1,1)
% fid at-1=at1
u=mean(R(:,i); %for i=1,...n
at1=R(:,i)^2 - u;
% the epsilon
et=normrand(0,1);
% now to estimate conditional variance just plug in the formula of GARCH(1,1)
v2=alph0+alpha1at1.^2 + beta * Ewm.^2
% estimate the return for each risky asset
at=sqrt(v2)*et;
rt=u+at; %for each i=1,...n , n risky asset
% find the whole return matrix for each risky asset
R = [rt(i)] % rti is the return vector for risky asset i
end

```

- **23)** Generate scenario tree by using clustering method in particularly we use euclidean distance in this code, where $N = 30$ in my experiment.

```
function [Odis] =distanceofmean2(R, N)
```

```
% this code to generate scenario tree for future return
```

% N is the number of scenario at each node in my experiment $N = 30$,
we start from return matrix for risky asset at first period $t = 0$ and e will
generate scenario tree form it

```
[m, n]=size(R)
u(i)=R(i,:) % for i=1,.....N choose N arbitrary row and calculate the eu-
clidean distance from return matrix.
for j=1:1:N
for i=1:1:m
% d(i)=abs(sum(R(i,:)-u))
d(i,j)=(R(i,:)-u(i))*(R(i,:)-u(i+1))'; D=d(i,j)'
end
end
Des=sort(D)
dd=d(i,j)';
size(dd);
size(Des)
DD=[d(i, j)];
size(DD);
A=DD;
[M, I] = min(A, [], 2);
for j=1:1:N
v(j)=A(j,I(j));
end
V=v';
Z=zeros(N,N*N);
for k=1:1:N
Z(k,I(k))=V(k);
D2=DD-Z;
size(D2)
end
for j=1:1:N
for h=1:1:N
D(h,I(h))= NaN;
A(j)=D;
[M(j), I(j)]=min(A(j), [], j);
end
end
```

```

I=[I(j)]
RR1=R(I,:);
size(RR1);
for i=1:N
ro1(i)=R(((i*N-N)+1:i*N,:));
end
for i=1:N
r(i)=mean(ro1(i))
R=[r(i)];
end
R=R;
end

```

- **24)** Finding historical return matrix

```

% P=[],pricing matrix
[m,n]=size(P)
for i=1:1:m-1;
for j=1:1:n
Rr(i,j)=P(i+1,j)-P(i,j)/P(i,j);
end
end
end

```

Appendix B

User guide for real data

- **1)** Download MATLAB to the computer and then download CVX: Matlab Software for Disciplined Convex Programming from <http://cvxr.com/cvx/download/>
- **2)** Download the price matrix $P_{i,t}$ from yahoo finance, which is the price at asset i and $i = 1, ..n$, where we have n instruments and time t weekly or daily,..etc.
- **3)** Plug the price matrix P in the following codes `coopratmeanvarco.m`, `coopratmeanvarcody.m`, `coopratmeanvarcodyfix.m`. We will get the whole efficient frontier by changing the value of β or see next step.
- **4)** Determine specific point in efficient frontier by two ways
 - 4a**– Choose specific point by yourself.
 - 4b**– Use equilibrium to determine specific point, this step by following Algorithm 5.15.

In case of equilibrium we just need to find a fair allocation y_1 and y_2 in terms of equilibrium price according to risk preferences or utility function for investor by running the code `equialloprice.m` in case of variance and semi variance. Otherwise run the code `equilallo.m` for the utility function as investor preferences. Note that, in real data run `equilbgeneral.m`. Note that, in this step you can change the risk preference and then change the explicit formula for equilibrium allocation as shown in propositions in section 5.1.

Thus, the output of the codes `equialloprice.m` or `equilallo.m` will get the fair allocation y_1 and y_2 , and the optimal value will be 0 that show the feasibility of the problem.

- **5)** Finding the trading strategy $Z_{i,t,h}$ which is the proportion of money that I will invest in risky asset i at time t node h , where $h = 1, \dots, N$.

By running the following codes `coopratmeanvarco(P,beta,w0,r0, h,T,pi1,pi2).m`, `coopratmeanvarcody(P,beta,w0,r0, N,h,T,pi1,pi2).m`, and `coopratmeanvarcodyfix((P,z,beta,w0,r0,N,T,pi1,pi2).m`. Note that, P is price matrix from step **2**, β is the determined point from step **3** which is a specific point for y_2 , w_0 is initial capital and r_0 is risk-free, and $pi1$ and $pi2$ is the fixed level of expected return for each investor, at node h , $h = 1, \dots, N$. Note that the number of scenario N is equal to the number of node h in my scenario tree for future return.

5a- Global solution, run `coopratmeanvarco.m` which is code to solve CI in multi-period directly so you will get the global solution. The output will be as follows

- i**— x is the trading strategy at first period $t = 1$
- ii**— choose number of node $h = 1, \dots, N$ and $t = 2, 3, \dots, T$. You will get the trading $z_{i,t,h}$. Then repeat the process to get the whole trading strategy $z_{i,t,h}$ at each risky asset i , time t , and node $h = 1, \dots, N$.

5b) Dynamic programming, run the two following codes:

- i-** Run the `coopratmeanvarcody.m` code to solve the DCI at each node in a recursive manner. The output is the trading strategy $z_{i,t,h}$ at each risky asset i , time t , and node $h = 1, \dots, N$. Note that, we will choose the number of node and time in order to find $z_{i,t,h}$ one after another, as in a recursive manner. Then arrange all the outputs $z_{i,t,h}$ in a matrix $Z = [z_{i,t,h}]$ in order to use it as input for the next code.

- ii-** Run `coopratmeanvarcodyfix.m`: this code solves DCI after we fix the trading strategy from $Z = [z_{i,h,t}]$ that we got from the previous code. Thus, the code will solve DCI over one variable x and, after running this code the output will be the trading strategy x at the first period.

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