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Comonad cohomology of track categories

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Abstract
We define a comonad cohomology of track categories, and show that it is related via a long exact sequence to the corresponding $(\mathcal{S},\mathcal{O})$-cohomology. Under mild hypotheses, the comonad cohomology coincides, up to reindexing, with the $(\mathcal{S},\mathcal{O})$-cohomology, yielding an algebraic description of the latter. We also specialize to the case where the track category is a 2-groupoid.

Keywords Track category · Comonad cohomology · Simplicial category

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1 Introduction

One of several models for $(\infty, 1)$-categories, a central topic of study in recent years (see, e.g., [4]) is the category of *simplicial categories*; that is, (small) categories Y enriched in simplicial sets. If the object set of Y is $\mathcal{O}$, we say it is an $(\mathcal{S},\mathcal{O})$-*category*.

One may analyze a topological space (or simplicial set) X by means of its *Postnikov tower* $(P^n X)_{n=0}^{\infty}$, where the n-th *Postnikov section* $P^n X$ is an n-*type* (that is, has trivial homotopy groups in dimension greater than n). The successive sections are related through their k-*invariants*: cohomology classes in $H^{n+1}(P^{n-1}X; \pi_n X)$.

Since the Postnikov system is functorial (and preserves products), one can also define it for a simplicial category Y: $P^n Y$ is then a category enriched in n-types, and its k-invariants are expressed in terms of the $(\mathcal{S},\mathcal{O})$-cohomology of [11].

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A long-standing open problem is to find a purely "algebraic" description of Postnikov systems, both for spaces and for simplicial categories. For the Postnikov sections, there are various algebraic models of n-types—and thus of categories enriched in n-types—in the literature, using a variety of higher categorical structures. However, the problem of finding an algebraic model for the k-invariants is largely open. For this purpose, we need first an algebraic formulation of the cohomology theories used to define the k-invariants. This leads us to look for an algebraic description of the cohomology of a category enriched in a suitable algebraic model of n-types.

We here realize the first step of this program, for *track categories*—that is, categories enriched in groupoids. In the future we hope to extend this to the cohomology of n-track categories—that is, those enriched in the n-fold groupoidal models of n-types developed by the authors in [6] and [17].

In [5] the authors introduced a cohomology theory for track categories (which generalizes the Baues–Wirsching cohomology of categories—see [3]), and showed that it coincides, up to indexing with the corresponding $(\mathcal{S},\mathcal{O})$-cohomology. This was then used to describe the first k-invariant for a 2-track category.

A direct generalization of this approach is problematic, because of the difficulty of defining a full and faithful simplicial nerve of weak higher categorical structures. Instead, we use a version of André–Quillen cohomology, also known as *comonad cohomology*, since we use a comonad to produce a simplicial resolution of our track category (see [1]). We envisage a generalization to higher dimensions, using the n-fold nature of the models of n-types in [6] and [17].

Our main result (see Corollaries 5.6 and 5.8) is that under mild hypotheses on a track category X (always satisfied up to 2-equivalence), the comonad cohomology of X (Definition 5.4) coincides, up to a dimension shift, with its $(\mathcal{S},\mathcal{O})$-cohomology. This follows from Theorem 5.5, which states that any track category X has a long exact sequence relating the comonad cohomology of X, its $(\mathcal{S},\mathcal{O})$-cohomology and the $(\mathcal{S},\mathcal{O})$-cohomology of the category X_0 of objects and 1-arrows of X. When the track category X is a 2-groupoid, its $(\mathcal{S},\mathcal{O})$-cohomology coincides with the cohomology of its classifying space.

1.1 Notation and conventions

Denote by Δ the category of finite ordered sets, so for any $\mathcal{C}$, $[\Delta^{\mathrm{op}},\mathcal{C}]$ is the category of simplicial objects in $\mathcal{C}$, while $[\Delta,\mathcal{C}]$ is the category of cosimplicial objects in $\mathcal{C}$. In particular, we write $\mathcal{S}$ for the category $[\Delta^{\mathrm{op}},\mathsf{Set}]$ of simplicial sets. We write $c(A)\in[\Delta^{\mathrm{op}},\mathcal{C}]$ for the constant simplicial object on $A\in\mathcal{C}$.

For any category $\mathcal{C}$ with finite limits, we write $\mathsf{Gpd}\,\mathcal{C}$ for the category of groupoids internal to $\mathcal{C}$—that is, diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X_1\times_{X_0}X_1 \arrow[r, "m"] & X_1 \arrow[loop above, "c"] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[l, bend left, "\Delta"] \end{tikzcd}$$

satisfying the obvious identities making the composition m associative and every '1-cell' in X_1 invertible.

For a fixed set $\mathcal{O}$, we denote by $\mathsf{Cat}_{\mathcal{O}}$ the category of small categories with object set $\mathcal{O}$ (and functors which are the identity on $\mathcal{O}$). In particular, a category $\mathcal{Z}$ enriched in simplicial sets with object set $\mathcal{O}$ will be called an $(\mathcal{S},\mathcal{O})$-*category*, and the category of all such will be denoted by $(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$. Equivalently, such a category $\mathcal{Z}$ can be thought of as a simplicial object in $\mathsf{Cat}_{\mathcal{O}}$. This means $\mathcal{C}$ has a fixed object set $\mathcal{O}$ in each dimension, and all face and degeneracy functors the identity on objects.

More generally, if $(\mathcal{V}, \otimes)$ is any monoidal category, a $(\mathcal{V},\mathcal{O})$-*category* is a small category $\mathcal{C} \in \mathsf{Cat}_{\mathcal{O}}$ enriched over $\mathcal{V}$. The category of all such categories will be denoted by $(\mathcal{V},\mathcal{O})\text{-}\mathsf{Cat}$. Examples for $(\mathcal{V}, \otimes)$ include $\mathcal{S}$, Top, Gp, and Gpd, with $\otimes$ the Cartesian product. When $\mathcal{V} = \mathsf{Gpd}$, we call $\mathcal{Z}$ in $\mathsf{Track}_{\mathcal{O}} := (\mathsf{Gpd},\mathcal{O})\text{-}\mathsf{Cat}$ a *track category* with object set $\mathcal{O}$ (see Sect. 4.1 below).

Another example is pointed simplicial sets $\mathcal{V} = \mathcal{S}_*$, with $\otimes = \wedge$ (smash product). We can identify an $(\mathcal{S}_*,\mathcal{O})$-category with a simplicial pointed $\mathcal{O}$-category.

1.2 Organization

Section 2 provides some background material on the Bourne adjunction (Sect. 2.1), internal arrows (Sect. 2.2), modules (Sect. 2.3), $(\mathcal{S},\mathcal{O})$-cohomology (Sect. 2.4) and simplicial model categories (Sect. 2.5). Section 3 sets up a short exact sequence associated to certain internal groupoids (Proposition 3.5), and Sect. 4 introduces the comonad used to define our cohomology, and shows its relation to $(\mathcal{S},\mathcal{O})$-cohomology (Theorem 4.9 and Corollary 4.10). Section 5 defines the comonad cohomology of track categories, and establishes the long exact sequence relating the $(\mathcal{S},\mathcal{O})$-cohomologies of X and of X_0 and the comonad cohomology of X (Theorem 5.5). Section 6 specializes to the case of a 2-groupoid, showing that in this case $(\mathcal{S},\mathcal{O})$-cohomology coincides with that of the classifying space (Corollary 6.4). The long exact sequence of Corollary 6.5 recovers [16, Theorem 13].

2 Preliminaries

In this section we review some background material on the Bourne adjunction, the internal arrow functor, modules, $(\mathcal{S},\mathcal{O})$-cohomology, and simplicial model categories.

2.1 The Bourne adjunction

Let $\mathcal{C}$ be a category with finite limits and let $\mathsf{Spl}\,\mathcal{C}$ be the category whose objects are the split epimorphisms with a given splitting:

$$(A \underset{t}{\overset{q}{\rightleftarrows}} B) \tag{1}$$

Define $R : \mathsf{Gpd}\,\mathcal{C} \to \mathsf{Spl}\,\mathcal{C}$ by

$$RX = \ (X_1 \underset{s_0}{\overset{d_0}{\rightleftarrows}} X_0).$$

Let $H : \mathsf{Spl}\,\mathcal{C} \to \mathsf{Gpd}\,\mathcal{C}$ associate to $Y = \ (A \underset{t}{\overset{q}{\rightleftarrows}} B)$ the object

$$HY: \ A \overset{q}{\times} A \overset{q}{\times} A \xrightarrow{m} A \overset{q}{\times} A \; \begin{smallmatrix} \xrightarrow{\mathrm{pr}_0} \\ \xrightarrow{\mathrm{pr}_1} \\ \xleftarrow[\Delta]{} \end{smallmatrix} \; A$$

of $\mathsf{Gpd}\,\mathcal{C}$, where $A \overset{q}{\times} A$ is the kernel pair of q, $\Delta = (\mathrm{Id}_A, \mathrm{Id}_A)$ is the diagonal map, and

$$m = (\mathrm{pr}_0, \mathrm{pr}_2) : A \overset{q}{\times} A \overset{q}{\times} A \cong (A \overset{q}{\times} A) \times_A (A \overset{q}{\times} A) \to A \overset{q}{\times} A$$

with $\mathrm{pr}_i\, m = \mathrm{pr}_i\, \pi_i$ $(i = 0, 1)$, where π_i, pr_i are the two projections. Note that HY is an internal equivalence relation in $\mathcal{C}$ with $\Pi_0(HY) = B$. Consider the following diagram

$$\begin{tikzcd} \mathsf{Gpd}\,\mathcal{C} \arrow[r, "\mathrm{Ner}"] \arrow[d, shift left, "R"] & {[\Delta^{\mathrm{op}}, \mathcal{C}]} \arrow[d, shift right, "\mathrm{Dec}"'] \\ \mathsf{Spl}\,\mathcal{C} \arrow[r, "n"'] \arrow[u, shift left, "H"] & {\mathrm{Aug}[\Delta^{\mathrm{op}}, \mathcal{C}]} \arrow[u, shift right, "+"'] \end{tikzcd} \tag{2}$$

where $\mathrm{Aug}[\Delta^{\mathrm{op}}, \mathcal{C}]$ is the category of augmented simplicial objects in $\mathcal{C}$, $+$ is the functor that forgets the augmentation, the *décalage* functor Dec (obtained by forgetting the last face operator) is its right adjoint, and nX is the nerve of the internal equivalence relation associated to X, augmented over itself, with $\mathrm{Ner}\, H = +n$.

Diagram (2) commutes up to isomorphism—that is, there is a natural isomorphism $\alpha : \mathrm{Dec}\,\mathrm{Ner} \cong nR$. Since $+ \dashv \mathrm{Dec}$, this implies that $H \dashv R$ (see [7, Theorem 1]).

Given $X \in \mathsf{Spl}\,\mathcal{C}$ as in (1), we have $RHX = (\ A \overset{q}{\times} A \underset{\Delta}{\overset{pr_0}{\rightleftarrows}} A\ )$. The unit $\eta : X \to RHX$ of the adjunction $H \dashv R$ is given by

$$\begin{tikzcd} A \arrow[r, shift left, "q"] \arrow[d, "t_1"'] & B \arrow[l, shift left, "t"] \arrow[d, "t"] \\ A \overset{q}{\times} A \arrow[r, shift left, "pr_0"] & A \arrow[l, shift left, "\Delta"] \end{tikzcd} \tag{3}$$

where t_1 is determined by

$$\begin{tikzcd}
A \arrow[dr, "t_1"] \arrow[drr, bend left, "\mathrm{Id}"] \arrow[ddr, bend right, "tq"'] & & \\
& A \overset{q}{\times} A \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"'] & A \arrow[d, "q"] \\
& A \arrow[r, "q"'] & A
\end{tikzcd}$$

so that

$$pr_0 t_1 = tq, \quad \mathrm{pr}_1\, t_1 = \mathrm{Id}, \quad \text{and} \quad t_1 t = (t, t) = \Delta t. \tag{4}$$

This show that $\eta = (t, t_1)$ is a morphism in $\mathsf{Spl}\,\mathcal{C}$.

Finally, if μ is the counit of the adjunction $H \dashv R$, and μ' that of $+ \dashv \mathrm{Dec}$, then for any $X \in \mathsf{Gpd}\,\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ner}\,HRX \arrow[r, "\mathrm{Ner}\,\mu"] \arrow[d, equal] & \mathrm{Ner}\,X \\
+nRX \arrow[d, phantom, "\wr\|"] & \\
+\,\mathrm{Dec}\,\mathrm{Ner}\,X \arrow[uur, dashed, "\mu'_{\mathrm{Ner}\,X}"'] &
\end{tikzcd}$$

Thus $\mu = P\,\mathrm{Ner}\,\mu$ where $P \dashv \mathrm{Ner}$.

2.2 The internal arrow functor

Let $U : \mathsf{Gpd}\,\mathcal{C} \to \mathcal{C}$ be the arrow functor, so $UY = Y_1$ for $Y \in \mathsf{Gpd}\,\mathcal{C}$, and assume $\mathcal{C}$ is (co)complete with commuting finite coproducts and pullbacks. For any $X \in \mathcal{C}$ let X_s, X_t be two copies of X, with $F : X_s \coprod X_t \to X$ the fold map and $LX \in \mathsf{Gpd}\,\mathcal{C}$ the corresponding internal equivalence relation, so $(LX)_0 = X_s \coprod X_t$ and $(LX)_1 = (X_s \coprod X_t) \overset{F}{\times} (X_s \coprod X_t)$. Then $X_s \coprod X_t \underset{i_1}{\overset{F}{\rightleftarrows}} X$ is an object of $\mathsf{Spl}\,\mathcal{C}$ and

$$LX = H\big(\, X_s \textstyle\coprod X_t \underset{i_1}{\overset{F}{\rightleftarrows}} X \,\big) \tag{5}$$

(cf. Sect. 2.1), where i_1 is the coproduct structure map. Therefore,

$$\begin{aligned}
&(X_s \textstyle\coprod X_t) \overset{F}{\times} (X_s \coprod X_t) \\
&\quad = (X_s \times_X X_s) \textstyle\coprod (X_s \times_X X_t) \coprod (X_t \times_X X_s) \coprod (X_t \times_X X_t) = X_{ss} \coprod X_{st} \coprod X_{ts} \coprod X_{tt}
\end{aligned} \tag{6}$$

where $X_{ss} = X_s \times_X X_s$, $X_{st} = X_s \times_X X_t$, $X_{ts} = X_t \times_X X_s$, and $X_{tt} = X_t \times_X X_t$. Under the identification (6) the face and degeneracy maps of LX are as follows:

s_0 includes $X_s \coprod X_t$ into $X_{ss} \coprod X_{tt}$; d_0 sends X_{ss} and X_{st} to X_s, and X_{ts} and X_{tt} to X_t; and d_1 sends X_{ss} and X_{ts} to X_s, and X_{st} and X_{tt} to X_t.

To see that L is left adjoint to U, given $f : X \to Y_1 = UY$, its adjoint $\tilde{f} : LX \to Y$ is given by $\tilde{f}_1 : X_{ss} \coprod X_{st} \coprod X_{ts} \coprod X_{tt} \to Y_1$ (determined by $s'_0 d'_0 f : X_{ss} \to Y_1$, $f : X_{st} \to Y_1$, $f \circ \tau : X_{ts} \to Y_1$, and $s_0 d_0 f : X_{tt} \to Y_1$), and $\tilde{f}_0 : X_s \coprod X_t \to Y_0$ (determined by $d'_0 f : X_s \to Y_0$ and $d'_1 f : X_t \to Y_0$). Here $\tau : X_{ts} \to X_{st}$ is the switch map, with $\tau \circ \tau = \mathrm{Id}$.

Conversely, given $g : LX \to Y$ with $g_1 : X_{ss} \coprod X_{st} \coprod X_{ts} \coprod X_{tt} \to Y_1$ and $g_0 : X_s \coprod X_t \to Y_0$, its adjoint $\hat{g} : X \to Y_1$, has $\hat{g}_0$ determined by $d_0 f : X_s \to Y_0$ and $d_1 f : X_t \to Y_0$, while $\hat{g}_1$ is determined by $s_0 d_0 f : X_{ss} \to Y_1$ and $s_0 d_1 f : X_{tt} \to Y_1$ (where $f : X_{st} \to Y_1$ is the composite $X_{st} \xrightarrow{i} X_{ss} \coprod X_{st} \coprod X_{ts} \coprod X_{tt} \xrightarrow{g_1} Y_1$).

2.3 Modules

Recall that an abelian group object in a category $\mathcal{D}$ with finite products is an object G equipped with a unit map $\sigma : * \to G$ (where $*$ is the terminal object), and *inverse map* $i : G \to G$, and a *multiplication map* $\mu : G \times G \to G$ which is associative, commutative and unital. We require further that

$$\mu \circ (\mathrm{Id}, i) \circ \Delta \;=\; \sigma \circ c_*, \tag{7}$$

where $\Delta : G \to G \times G$ is the diagonal, and c_* the map to $*$.

Definition 2.1 Given an object X_0 in a category $\mathcal{C}$, we denote by $(\mathsf{Gpd}\,\mathcal{C}, X_0)$ the subcategory of $\mathsf{Gpd}\,\mathcal{C}$ consisting of those Y with $Y_0 = X_0$ (and groupoid maps which are the identity on X_0). For $X \in (\mathsf{Gpd}\,\mathcal{C}, X_0)$, an X-*module* is an abelian group object M in the slice category $(\mathsf{Gpd}\,\mathcal{C}, X_0)/X$. Since the terminal object of $\mathcal{D} = (\mathsf{Gpd}\,\mathcal{C}, X_0)/X$ is $\mathrm{Id}_X : X \to X$, and the product of $\rho : M \to X$ with itself in $\mathcal{D}$ is $\rho p_1 = \rho p_2 : M \overset{\rho}{\times} M \to X$, a unit map for $\rho : M \to X$ is given by a section $\sigma : X \to M$ (with $\rho\sigma = \mathrm{Id}$), and the multiplication and inverse have the forms

$$\begin{tikzcd} M \overset{\rho}{\times} M \arrow[rr, "\mu"] \arrow[dr] & & M \arrow[dl, "\rho"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "\rho"'] & & M \arrow[dl, "\rho"] \\ & X, & \end{tikzcd}$$

respectively. Note that (7) applied to $\rho : M \to X$ implies that

$$\mu(\mathrm{Id}, i) \circ \Delta_M \;=\; \sigma\rho, \tag{8}$$

for diagonal $\Delta_M : M \to M \overset{\rho}{\times} M$ and $\sigma : X \to M$ the zero map of $\rho : M \to X$.

Remark 2.2 Suppose that $X = HY$, for $H : \mathsf{Spl}\,\mathcal{C} \to \mathsf{Gpd}\,\mathcal{C}$ as in Sect. 2.1 and

$$Y = \ (X_0 \underset{t}{\overset{q}{\rightleftarrows}} \pi_0) \ \in \mathsf{Spl}\,\mathcal{C}. \tag{9}$$

Thus $X_1 = X_0 \overset{q}{\times} X_0$, and an X-module $M \to X$ is given by

$$\begin{tikzcd}
M_1 \arrow[r, "\rho_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & X_0 \overset{q}{\times} X_0 \arrow[d, shift right, "\mathrm{pr}_0"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\
X_0 \arrow[r, "\mathrm{Id}"] & X_0
\end{tikzcd}$$

with $\rho_1 = (d_0, d_1)$. Note that the fiber $M(a, b)$ of ρ_1 over each $(a, b) \in X_0 \overset{q}{\times} X_0$ is an abelian group, with zero $\sigma_1(a, b)$, and the zero map $\phi : X \to M$ is given by

$$\begin{tikzcd}
X_0 \overset{q}{\times} X_0 \arrow[r, "\phi_1"] \arrow[d, shift right, "pr_0"'] \arrow[d, shift left, "pr_1"] & M_1 \arrow[r, "\rho_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & X_0 \times X_0 \arrow[d, shift right, "pr_0"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\
X_0 \arrow[u, bend right=60, "\Delta"'] \arrow[r, equal] & X_0 \arrow[u, bend right=60, "s_0"'] \arrow[r, equal] & X_0 \arrow[u, bend right=60, "\Delta"']
\end{tikzcd}$$

with $\rho_1\phi_1 = \mathrm{Id}$. Thus for Y as in (9), the adjoint $\hat{\phi} \in \mathrm{Hom}_{\mathsf{Spl}\,(\mathcal{C})/RHY}\,(Y, R(M))$ is the composite

$$\begin{tikzcd}
X_0 \arrow[r, "t_1"] \arrow[d, shift right, "q"'] & X_0 \times X_0 \arrow[r, "\phi_1"] \arrow[d, shift right, "pr_0"'] & M_1 \arrow[d, shift right, "d_0"'] \\
\pi_0 \arrow[r, "t"'] \arrow[u, shift right, "t"'] & X_0 \arrow[r, equal] \arrow[u, shift right, "\Delta"'] & X_0 \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

where $\eta = (t_1, t)$ is the unit of the adjunction $H \dashv R$, as in (3).

Definition 2.3 The idempotent map in $\mathsf{Spl}\,\mathcal{C}$

$$\begin{tikzcd}
X_0 \arrow[r, "tq"] \arrow[d, shift right, "q"'] & X_0 \arrow[d, shift right, "q"'] \\
\pi_0 \arrow[r, "\mathrm{Id}"'] \arrow[u, shift right, "t"'] & \pi_0 \arrow[u, shift right, "t"']
\end{tikzcd}$$

induces an idempotent $e = H(tq) : X = H(Y) \to X$ in $\mathsf{Gpd}\,\mathcal{C}$ (for Y as in (9)).

Note that $e_0 = tq$ and $e_1 = (e_0, e_0)$. We therefore obtain an idempotent operation $\underline{e}$ on $\mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, M)$, taking $f : X \to M$ to $fe : X \to M$. We write $fe = \underline{e}(f)$. This sends $(a, b) \in X_0 \times_q X_0$ to $f(tqa, tqb)$. Let e^*M be the pullback

$$\begin{tikzcd} e^*M \arrow[r, "r"] \arrow[d, "\rho'"'] & M \arrow[d, "\rho"] \\ X \arrow[r, "e"'] & X \end{tikzcd} \tag{10}$$

in $\mathsf{Gpd}\,\mathcal{C}$. If we denote the fiber of ρ' at $(a, b) \in X_1$ by $(e^*M)_1(a, b)$, we have $(e^*M)_1(a, b) = (a, b)\times_{(ta,tb)}M_1(ta, tb)$, which is isomorphic under $r_1(a, b)$ to $M_1(ta, tb)$. The unit map σ' of e^*M is given

$$\begin{tikzcd} X \arrow[dr, "\sigma'"] \arrow[drr, bend left, "\sigma e"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\ & e^*M \arrow[r, "r"] \arrow[d, "\rho'"'] & M \arrow[d, "\rho"] \\ & X \arrow[r, "e"'] & X \end{tikzcd} \tag{11}$$

where $\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "\mathrm{Id}"'] & & M \arrow[dl, "\rho"] \\ & X & \end{tikzcd}$ is the unit of $\rho : M \to X$, so in particular

$$\sigma_1 \Delta_{X_0} = s_0 : X_0 \to M_1. \tag{12}$$

Thus for each $(a, b) \in X_1$ we have $\sigma'(a, b) = ((a, b), (\sigma e)(a, b)) = ((a, b), \sigma(tqa, tqb))$.

The multiplication $(e^*M)_1\times_{X_1}(e^*M)_1 \xrightarrow{\mu'_1} (e^*M)_1$ on $((a, b), m), ((a, b), m')$ by

$$\mu'_1(((a, b), m), ((a, b), m')) = ((a, b), \mu_1(m, m')), \tag{13}$$

so identifying $(e^*M)_1\times_{X_1}(e^*M)_1$ with $X_1\times_{X_1}(M_1\times_{X_1}M_1)$, we have $\mu'_1 = (\mathrm{Id}, \mu_1)$. Finally, the zero map of e^*M is given on $((a, b), m) \in (e^*M)_1$ by

$$\sigma'_1\rho'_1((a, b), m) = \sigma'_1(a, b) = \{(a, b), \sigma(tqa, tqb)\} = \{(a, b), \sigma\rho_1(m)\}$$

since $\rho_1(m) = (tqa, tqb)$. Thus $O_{(e^*M)_1} = \sigma'_1\rho'_1 = (\mathrm{Id}, \sigma\rho) = (\mathrm{Id}, O_{M_1})$.

2.4 $(\mathcal{S},\mathcal{O})$-Categories

In [9, §1], Dwyer and Kan define a simplicial model category structure on $(\mathcal{S},\mathcal{O})$-Cat, also valid for $(\mathcal{S}_*,\mathcal{O})$-Cat (see Sect. 1.1 and [15, Prop. 1.1.8]), in which a map $f : \mathcal{X} \to \mathcal{Y}$ is a fibration (respectively, a weak equivalence) if for each $a, b \in \mathcal{O}$, the induced map $f_{(a,b)} : \mathcal{X}(a, b) \to \mathcal{Y}(a, b)$ is such.

The cofibrations in $(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ or $(\mathcal{S}_*,\mathcal{O})\text{-}\mathsf{Cat}$ are not easy to describe. However, for any $\mathcal{K} \in \mathsf{Cat}_{\mathcal{O}}$, the constant simplicial category $c(\mathcal{K}) \in [\Delta^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}] \cong (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ has a cofibrant replacement defined as follows:

Recall that a category $Y \in \mathsf{Cat}_{\mathcal{O}}$ is *free* if there exists a set S of non-identity maps in Y (called generators) such that every non-identity map in Y can uniquely be written as a finite composite of maps in S. There is a forgetful functor $U : \mathsf{Cat}_{\mathcal{O}} \to \mathsf{Graph}_{\mathcal{O}}$ to the category of directed graphs, with left adjoint the free category functor $F : \mathsf{Graph}_{\mathcal{O}} \to \mathsf{Cat}_{\mathcal{O}}$ (see [13] and compare [9, §2.1]). Both U and F are the identity on objects.

Similarly, an $(\mathcal{S},\mathcal{O})$-category $X \in [\Delta^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}] = (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ is *free* if for each $k \in \Delta$, $X_k \in \mathsf{Cat}_{\mathcal{O}}$ is free, and the degeneracy maps in X send generators to generators. Every free $(\mathcal{S},\mathcal{O})$-category is cofibrant (cf. [9, § 2.4]). Moreover, for any $\mathcal{K} \in \mathsf{Cat}_{\mathcal{O}}$, a canonical cofibrant replacement $\mathcal{F}_\bullet\mathcal{K}$ for $c(\mathcal{K})$ in $(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat} = [\Delta^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}]$ (Sect. 1.1) is obtained by iterating the comonad $FU : \mathsf{Cat}_{\mathcal{O}} \to \mathsf{Cat}_{\mathcal{O}}$ (so $\mathcal{F}_n\mathcal{K} := (FU)^{n+1}\mathcal{K}$). The augmentation $\mathcal{F}_\bullet\mathcal{K} \to \mathcal{K}$ induces a weak equivalence $\mathcal{F}_\bullet\mathcal{K} \simeq c(\mathcal{K})$ in $[\Delta^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}] \cong (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$. If $\mathcal{K}$ is pointed, $\mathcal{F}_\bullet\mathcal{K}$ is a $(\mathcal{S}_*,\mathcal{O})$-category.

More generally, if X is any $(\mathcal{S},\mathcal{O})$-category, thought of as a simplicial object in $\mathsf{Cat}_{\mathcal{O}}$, its *standard Dwyer–Kan resolution* is the cofibrant replacement given by the diagonal $\operatorname{Diag}\overline{\mathcal{F}}_\bullet X$ of the bisimplicial object $\overline{\mathcal{F}}_\bullet X \in [\Delta^{2\,\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}]$ obtained by iterating FU in each simplicial dimension.

Definition 2.4 The *fundamental track category* of an $(\mathcal{S},\mathcal{O})$-category Z is obtained by applying the fundamental groupoid functor $\hat{\pi}_1 : \mathcal{S} \to \mathsf{Gpd}$ to each mapping space $\mathcal{Z}(a,b)$ (see [12, §I.8]. When Z is fibrant, $\Lambda := \hat{\pi}_1 Z$ has a particularly simple description: for each $a, b \in \mathcal{O}$, the set of objects of $\Lambda(a,b)$ is $Z(a,b)_0$, and for $x, x' \in Z(a,b)_0$, the morphism set $(\Lambda(a.b))(x,x')$ is $\{\tau \in Z(a,b)_1 : d_0\tau = x, d_1\tau = x'\}/{\sim}$, where $\sim$ is determined by the 2-simplices of X. Since $\hat{\pi}_1$ commutes with cartesian products for Kan complexes, it extends to $(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ (after fibrant replacement).

A *module* over a track category $\Lambda \in (\mathsf{Gpd},\mathcal{O})\text{-}\mathsf{Cat}$ is an abelian group object M in $(\mathsf{Gpd},\mathcal{O})\text{-}\mathsf{Cat}/\Lambda$ (see Sect. 2.3). For example, given a (fibrant) $(\mathcal{S},\mathcal{O})$-category Z, for each $n \geq 2$ we obtain a $\hat{\pi}_1 Z$-module by applying $\pi_n(-)$ to each mapping space of Z.

For each track category $\Lambda \in (\mathsf{Gpd},\mathcal{O})\text{-}\mathsf{Cat}$, Λ-module M and $n \geq 1$, we have a twisted Eilenberg–Mac Lane $(\mathcal{S},\mathcal{O})$-category $E = E^{\Lambda}(M,n)$ over Λ, with $\pi_n E \cong M$ and $\pi_i E \cong 0$ for $2 \leq i \neq n$ (see [10, §1] and [11, §1.3(iv)]).

Given $\Lambda \in (\mathsf{Gpd},\mathcal{O})\text{-}\mathsf{Cat}$, a Λ-module M, and an object $Z \in (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ equipped with a *twisting map* $p : \hat{\pi}_1 Z \to \Lambda$, the n-th $(\mathcal{S},\mathcal{O})$-*cohomology group* of Z *with coefficients in* M is

$$\begin{aligned} H^n_{\mathrm{SO}}(Z/\Lambda; M) &:= [Z, E^{\Lambda}(M,n)]_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/B\Lambda} \\ &= \pi_0 \operatorname{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/B\Lambda}(Z, E^{\Lambda}(M,n)), \end{aligned}$$

where $\operatorname{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/A}(Z,Y)$ is the sub-simplicial set of $\operatorname{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}}(Z,Y)$ consisting of maps over a fixed base A (cf. [11, §2]) Typically, $\Lambda = \hat{\pi}_1 Z$, with p a weak equivalence; if in addition $Z \simeq B\Lambda$, we denote $H^n_{\mathrm{SO}}(Z/\Lambda; M)$ simply by $H^n_{\mathrm{SO}}(\Lambda; M)$.

2.5 Simplicial model categories

Recall that a *simplicial* model category $\mathcal{M}$ is a model category equipped with functors $X \mapsto X \otimes K$ and $X \mapsto X^K$, natural in $K \in \mathcal{S}$, satisfying appropriate axioms (cf. [14, Definition 9.1.6]).

For example, $\mathcal{S}$ itself is a simplicial model category, with $X \otimes K := X \times K$ and $X^K := \mathrm{map}(K, X)$, where $\mathrm{map}(K, X) \in \mathcal{S}$ has $\mathrm{map}(K, X)_n := \mathrm{Hom}_{\mathcal{S}}(K \times \Delta[n], X)$. Similarly, $(\mathcal{S},\mathcal{O})$-$\mathsf{Cat}$ is also a simplicial model category (see [9, Proposition 7.2]).

Definition 2.5 Let $\mathcal{M}$ be a simplicial model category. The *realization* $|X|$ of $X \in [\Delta^{\mathrm{op}}, \mathcal{M}]$ is defined to be the coequalizer of the maps

$$\coprod_{(\sigma:[n]\to[k])\in\Delta} X_k \otimes \Delta[n] \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{n\geq 0} X_n \otimes \Delta[n],$$

where on the summand indexed by $\sigma : [n] \to [k]$, ϕ is the composite of $\sigma^* \otimes 1_{\Delta[k]} : X_k \otimes \Delta[n] \to X_n \otimes \Delta[n]$ with the inclusion into the coproduct, and ψ is the composite of $1_{X_k} \otimes \sigma_* : X_k \otimes \Delta[n] \to X_k \otimes \Delta[k]$ with the same inclusion (see [12, §VII.3]).

Similarly, if $X \in [\Delta, \mathcal{M}]$ is a cosimplicial object in $\mathcal{M}$, its *total object* $\mathrm{Tot}\, X$ is the equalizer of

$$\prod_{[n]\in Ob\,\Delta} (X^n)^{\Delta[n]} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma:[n]\to[k])\in\Delta} (X^k)^{\Delta[n]}$$

with ϕ and ψ defined dually (cf. [14, Def. 18.6.3]).

The following is a straightforward generalization of [8, XII 4.3]:

Lemma 2.6 *If* $\mathcal{M} = [\Delta^{\mathrm{op}}, \mathcal{D}]$ *is a simplicial model category and* $X \in [\Delta^{\mathrm{op}}, \mathcal{M}] \cong [\Delta^{2^{\mathrm{op}}}, \mathcal{D}]$*, then* $\mathrm{Diag}\, X \cong |X|$ *and* $\mathrm{map}_{\mathcal{M}}(\mathrm{Diag}\, X, K) \cong \mathrm{Tot}\,\mathrm{map}_{\mathcal{M}}(X, K)$.

3 Short exact sequences

We now associate to any internal groupoid of the form $X = HY$ (cf. Sect. 2.1) and X-module M a certain short exact sequence of abelian groups (see Proposition 3.5). When $\mathcal{C} = \mathsf{Cat}_{\mathcal{O}}$, this can be rewritten in a more convenient form (see Proposition 5.3); however, in this section we present it in a more general context, which may be useful in future work. A similar short exact sequence appears in [19, Theorem 3.5] for $\mathcal{C}$ an algebraic category (with a different description of the third term). When $\mathcal{C} = \mathsf{Gp}$, it reduces to [16, Lemma 6], though the method of proof there is different.

Definition 3.1 Let $X = H\,(X_0 \overset{q}{\underset{t}{\rightleftarrows}} \pi_0)$, with $\mathrm{d}X_0$ the discrete internal groupoid on X_0. We define $j\colon\ \mathrm{d}X_0 \to X$ to be the map

$$\begin{tikzcd}
X_0 \arrow[r, "\Delta_{X_0}"] \arrow[d, shift right=3, "\mathrm{Id}"'] \arrow[d, shift right=1, "\mathrm{Id}"] & X_0 \times_q X_0 \arrow[d, shift right=3, "pr_0"'] \arrow[d, shift right=1, "\mathrm{pr}_1"] \\
X_0 \arrow[r, "\mathrm{Id}_{X_0}"'] \arrow[u, bend right, "\mathrm{Id}"'] & X_0 \arrow[u, bend right, "\Delta_{X_0}"']
\end{tikzcd}$$

in $(\mathsf{Gpd}\,\mathcal{C},\, X_0)$. Consider the pullback

$$\begin{tikzcd}
j^*M \arrow[r, "k"] \arrow[d, "\lambda"'] & M \arrow[d, "\rho"] \\
\mathrm{d}X_0 \arrow[r, "j"'] & X
\end{tikzcd} \tag{14}$$

in $\mathsf{Gpd}\,\mathcal{C}/X$, where $d_0 = d_1 = \lambda_1 : (j^*M)_1 \to X_0$, since $\mathrm{d}X_0$ is discrete. Because (14) induces

$$\begin{tikzcd}
(j^*M)_1 \arrow[r, "k_1"] \arrow[d, "\lambda_1"'] & M_1 \arrow[d, "\rho_1"] \\
X_0 \arrow[r, "j_1=\Delta_{X_0}"'] & X_0 \times_q X_0 = X_1
\end{tikzcd} \tag{15}$$

(a pullback in $\mathcal{C}$), we shall denote $(j^*M)_1$ by $j_1^*M_1$.

We shall use the following abbreviations for the relevant Hom groups:

$$\begin{cases}
\mathrm{Hom}(\pi_0,\ t^*j_1^*M_1) := & \mathrm{Hom}_{\mathcal{C}/\pi_0}\,((\pi_0 \xrightarrow{\mathrm{Id}} \pi_0),\ (t^*j_1^*M_1 \xrightarrow{r_1} \pi_0)),\\
\mathrm{Hom}(X_0,\ j_1^*M_1) := & \mathrm{Hom}_{\mathcal{C}/X_0}\,((X_0 \xrightarrow{tq} X_0)),\ (j_1^*M_1 \xrightarrow{\lambda_1} X_0)),\\
\mathrm{Hom}(X_1, e^*M_1) := & \mathrm{Hom}_{\mathcal{C}/X_1}\,((X_1 \xrightarrow{\mathrm{Id}} X_1)),\ (((e^*M)_1 \xrightarrow{\rho_1} X_1))),\\
\mathrm{Hom}(X, e^*M) := & \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}\,((X \xrightarrow{\mathrm{Id}} X),\ (e^*M \xrightarrow{\rho'} X)).
\end{cases} \tag{16}$$

Definition 3.2 In the situation described in Sect. 3.1, given a map

$$\begin{tikzcd}
X_0 \arrow[rr, "f"] \arrow[dr, "e_0=tq"'] & & j_1^*M_1 \arrow[dl, "\lambda_1"] \\
& X_0 &
\end{tikzcd}$$

in $\mathcal{C}/X_0$, we have $\rho_1 k_1 f = \Delta_{X_0}\lambda_1 f = \Delta_{X_0}e_0 = e_1\Delta_{X_0}$, since the square

$$\begin{tikzcd}
X_0 \arrow[r, "e_0"] \arrow[d, "\Delta_{X_0}"'] & X_0 \arrow[d, "\Delta_{X_0}"] \\
X_1 \arrow[r, "e_1"'] & X_1
\end{tikzcd} \tag{17}$$

commutes. Now let $v : X_0 \to (e^*M)_1$ be given by

$$\begin{tikzcd}
X_0 \arrow[dr, "v"] \arrow[drr, bend left, "k_1 f"] \arrow[ddr, bend right, "\Delta_{X_0} e_0 = e_1 \Delta_{X_0}"'] & & \\
& (e^*M)_1 \arrow[r, "r"] \arrow[d, "\rho'_1"'] & M_1 \arrow[d, "\rho_1"] \\
& X_1 \arrow[r, "e_1"'] & X_1
\end{tikzcd} \tag{18}$$

where $\rho_1 k_1 f = \Delta_{X_0} \lambda_1 f = \Delta_{X_0} e_0 = e_1 \Delta_{X_0} = e_1 e_1 \Delta_{X_0}$ by (14) and (17).

Since $e_0 = tq$, the following diagram commutes:

$$\begin{tikzcd}
X_0 \arrow[r, "q"] \arrow[dd, "v"'] & \pi_0 \arrow[d, "t"] \\
& X_0 \arrow[d, "\Delta_{X_0}"] \\
(e^*M)_1 \arrow[r, "\rho'_1"'] & X_1
\end{tikzcd} \tag{19}$$

so v induces $(v, v) : X_1 = X_0 \times_q X_0 \to (e^*M)_1 \times_{X_1} (e^*M)_1$. In the notation of (16), we may therefore define $\vartheta : \mathrm{Hom}(X_0, \ j_1^* M_1) \to \mathrm{Hom}(X_1, e^* M_1)$ by letting $\vartheta_1(f) : X_1 \to (e^*M)_1$ be the composite

$$X_1 \xrightarrow{(v,v)} (e^*M)_1 \times_{X_1} (e^*M)_1 \xrightarrow{(\mathrm{Id}, i_1)} (e^*M)_1 \times_{X_1} (e^*M)_1 \xrightarrow{\mu_1} (e^*M)_1 \tag{20}$$

where $i : e^*M \to e^*M$ is the inverse map for the abelian group structure on e^*M.

Lemma 3.3 *The map $\vartheta = (e_0, \vartheta_1)$ lands in $\mathrm{Hom}(X, e^*M)$, for $e_0 = tq$ and ϑ_1 as in (20).*

Proof By (8) we have $\sigma'_1 \rho'_1 = \mu'_1 (\mathrm{Id}, i'_1) \Delta_{(e^*M)_1}$, so

$$\begin{tikzcd}
X_0 \arrow[r, "v"] \arrow[dr, "e_1 \Delta_{X_0}"'] & (e^*M)_1 \arrow[r, "\Delta_{(e^*M)_1}"] \arrow[d, "\rho'_1"] & (e^*M)_1 \times_{X_1} (e^*M)_1 \arrow[r, "(\mathrm{Id}, i_1)"] & (e^*M)_1 \times_{X_1} (e^*M)_1 \arrow[r, "\mu_1"] & (e^*M)_1 \\
& X_1 \arrow[urrr, "\sigma'_1"'] & & &
\end{tikzcd} \tag{21}$$

commutes, as does

$$\begin{tikzcd} X_0 \arrow[r, "\Delta_{X_0}"] \arrow[d, "v"'] & X_1 \arrow[d, "(v,v)"] \\ (e^*M)_1 \arrow[r, "\Delta_{(e^*M)_1}"'] & (e^*M)_1 \times_{X_1} (e^*M)_1 \end{tikzcd} \tag{22}$$

so by (21), (22), (12), and the definition of $\vartheta_1(f)$ we see that

$$\vartheta_1(f)\Delta_{X_0} = \mu_1(\mathrm{Id}, i_1)(v, v)\Delta_{X_0} = \sigma_1' e_1 \Delta_{X_0} = \sigma_1' \Delta_{X_0} e_0 = s_0' e_0 \tag{23}$$

By (13), for each $(a, b) \in X_0 \times_q X_0$ (with $tqa = tqb$) we have

$$\vartheta_1(f)(a, b) = ((tqa, tqa), \mu_1(k_1 fa, i_1 k_1 fb)), \tag{24}$$

so $d_0' \vartheta_1(f)(a, b) = tqa = e_0 \,\mathrm{pr}_0(a, b)$ and $d_1' \vartheta_1(f)(a, b) = tqb = e_0 \,\mathrm{pr}_1(a, b)$. Thus

$$d_i \vartheta_1(f) = e_0 \,\mathrm{pr}_i \tag{25}$$

for $i = 0, 1$. Thus (23) and (25) show that

$$\begin{tikzcd} X_1 \arrow[r, "\vartheta_1(f)"] \arrow[d, shift right, "\mathrm{pr}_0"'] \arrow[d, shift left, "\mathrm{pr}_1"] & (e^*M)_1 \arrow[d, shift right, "d_0'"'] \arrow[d, shift left, "d_1'"] \\ X_0 \arrow[r, "e_0"'] \arrow[u, bend right, "\Delta_{X_0}"'] & X_0 \arrow[u, bend right, "s_0'"'] \end{tikzcd} \tag{26}$$

commutes. Now, given $(a, b, c) \in X_0 \times_q X_0 \times_q X_0$ (with $tqa = tqb = tqc$), we have

$$c'(\vartheta_1(f)(a, b) \times \vartheta_1(f)(b, c)) = ((tqa, tqa), \mu_1(k_1 fa, i_1 k_1 fb) \circ \mu_1(k_1 fb, i_1 k_1 fc))$$

while $\vartheta_1(f)c''(a, b, c) = \vartheta_1(f)(a, c) = ((tqa, tqa), \mu_1(k_1 fa, i_1 k_1 fc))$, where we denoted by $c'' : X_1 \times_{X_0} X_1 \to X_1$ and $c' : (e^*M)_1 \times_{X_0} (e^*M)_1$ the compositions.

Since μ_1 is a map of groupoids, by the interchange rule we see that

$$\mu_1(k_1 fa, i_1 k_1 fb) \circ \mu_1(k_1 fb, i_1 k_1 fc) = \mu_1(k_1 fa, i_1 k_1 fc)$$

where $\circ$ the groupoid composition.

Finally, $\rho_1' \circ \vartheta_1(f) = e_1$, so $(e_0, \vartheta_1(f))$ is indeed a morphism in $\mathsf{Gpd}\,\mathcal{C}/X$. $\square$

Definition 3.4 The pullback

$$\begin{tikzcd} t^* j^* M \arrow[r, "l"] \arrow[d, "r"'] & j^* M \arrow[d, "\lambda"] \\ d\pi_0 \arrow[r, "\mathrm{d}t"'] & \mathrm{d}X_0 \end{tikzcd} \tag{27}$$

in $\mathsf{Gpd}\,\mathcal{C}/d\pi_0$ gives rise to a pullback

$$\begin{tikzcd} t^* j_1^* M_1 \arrow[r, "l_1"] \arrow[d, "r_1"'] & j_1^* M_1 \arrow[d, "\lambda_1"] \\ \pi_0 \arrow[r, "t"'] & X_0 \end{tikzcd} \tag{28}$$

in $\mathcal{C}$, so we may define

$$\xi : \mathrm{Hom}(\pi_0,\ t^* j_1^* M_1) \to \mathrm{Hom}(X_0,\ j_1^* M_1) \tag{29}$$

by sending $\begin{tikzcd} \pi_0 \arrow[rr, "f"] \arrow[dr, "\mathrm{Id}"'] & & t^* j_1^* M_1 \arrow[dl, "r_1"] \\ & \pi_0 & \end{tikzcd}$, to the map $\xi(f)$ given by

$$\begin{tikzcd} X_0 \arrow[rr, "l_1 f q"] \arrow[dr, "tq"'] & & j_1^* M_1 \arrow[dl, "\lambda_1"] \\ & X_0 & \end{tikzcd} \tag{30}$$

Proposition 3.5 *Given $X = H(Y) \in \mathsf{Gpd}\,\mathcal{C}$ as in* (9) *and $M \in [(\mathsf{Gpd}\,\mathcal{C}, X_0)/X]_{\mathrm{ab}}$, there is a short exact sequence of abelian groups*

$$0 \ \to \ \mathrm{Hom}(\pi_0,\ t^* j_1^* M_1) \ \xrightarrow{\xi} \ \mathrm{Hom}(X_0, j^* M_1) \ \xrightarrow{\vartheta} \ \mathrm{Hom}(X, e^* M) \ \to \ 0,$$

in the notation of (16)*, where ξ is as in* (29) *and ϑ is as in Lemma 3.3.*

Proof We first show that $\mathrm{Im}\,\xi \subseteq \ker\vartheta$: Given $f' : \pi_0 \to t^* j_1^* M_1$ in $\mathrm{Hom}(\pi_0,\ t^* j_1^* M_1)$, the map $\xi(f') \in \mathrm{Hom}(X_0,\ j^* M_1)$ is given by the composite $X_0 \xrightarrow{q} \pi_0 \xrightarrow{f'} t^* j_1^* M_1 \xrightarrow{l_1} j_1^* M_1$

By (24) and (30), for each $(a, b) \in X_1$ we have

$$\vartheta_1(\xi(f'))(a, b) = \{(tqa, tqb), \mu_1(k_1 l_1 f' q(a), i_1 k_1 l_1 f' q(b))\}.$$

Since $q(a) = q(b)$, we have $\vartheta_1(\xi(f'))(a, b) = ((tqa, tqb), 0)$, which is the zero map of $\mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, e^* M)$. This shows that $\mathrm{Im}\ \xi \subseteq \ker\vartheta$.

Given $g' : X_0 \to j_1^* M_1$ (as in (30)) in $\ker \vartheta_1$, for all $(a, b) \in X_1$ we have

$$\vartheta(g')(a, b) = \{(tqa, tqa), \mu_1(k_1 g'(a), i_1 k_1 g'(b)\} = \{(tqa, tqb), 0\}.$$

Thus $k_1 g' pr_0(a, b) = k_1 g' a = k_1 g' b = k_1 g' pr_1(a, b)$, so that

$$k_1 g' pr_0 = k_1 g' pr_1. \tag{31}$$

Since $X_1 \underset{pr_1}{\overset{pr_0}{\rightrightarrows}} X_0 \xrightarrow{q} \pi_0$ is a coequalizer, it follows from (31) that there is a map $f : \pi_0 \to M_1$ with $fq = k_1 g'$, and thus $f = fqt = k_1 g' t$, so $\rho_1 f = \rho_1 k_1 g' t = \Delta_{X_0} \lambda_1 g' t = \Delta_{X_0} tqt = \Delta_{X_0} t$. Hence there is $\overline{f} : \pi_0 \to j_1^* M_1$ defined by $f \top t : \pi_0 \to M_1 \times X_0$ into the pullback (15). Since $\lambda_1 \overline{f} = t$, there is also a map $f' : \pi_0 \to t^* j_1^* M_1$ defined by $\overline{f} \top \mathrm{Id} : \pi_0 \to j_1^* M_1 \times \pi_0$ into the pullback (28).

By (31) and the above we have $k_1 g' = fq = k_1 \overline{f} q$. Since k_1 is monic, this implies that $g' = \overline{f} q$, and since $\overline{f} = l_1 f'$, also $g' = l_1 f' q = \vartheta_1(f')$. This shows that $\ker \vartheta \subseteq \operatorname{Im} \xi$. In conclusion, $\ker \vartheta = \operatorname{Im} \xi$.

To show that ξ is monic, assume given $f, g \in \operatorname{Hom}(\pi_0, t^* j_1^* M_1)$ with $\xi f = \xi g$. Then $l_1 f q = l_1 g q$, which implies that

$$l_1 f = l_1 g, \tag{32}$$

since q is epic. Also, $\rho_1 k_1 l_1 f = \Delta_{X_0} t s_1 f$ and $\rho_1 k_1 l_1 g = \Delta_{X_0} t s_1 g$, so by (32) we have $\Delta_{X_0} t s_1 f = \Delta_{X_0} t s_1 g$ and therefore $s_1 f = q pr_0 \Delta_{X_0} t s_1 f = q pr_0 \Delta_{X_0} t s_1 g = s_1 g$. By the definition of $t^* j_1^* M_1$ as the pullback (28), together with (32) this implies that $f = g$. Thus ξ is a monomorphism.

To show that ϑ is onto, assume given $\phi \in \operatorname{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, e^* M)$ (so that $\rho' \circ \phi = e$). The adjoint of ϕ in $\operatorname{Hom}_{\mathsf{Spl}\,\mathcal{C}/Y}(Y, R(e^* M))$, for Y as in (9), is given by a commuting triangle in $\mathsf{Spl}\,\mathcal{C}$:

$$\begin{tikzcd}
X_0 \arrow[rr, "g"] \arrow[d, shift right, "q"'] \arrow[drr, "{(t,tq)}"] & & (e^*M)_1 \arrow[d, shift right, "d_0"'] \arrow[ddl, "\rho'_1"] \\
\pi_0 \arrow[rr, "t"'] \arrow[u, shift right, "t"'] \arrow[ddr, "t"'] & & X_0 \arrow[u, shift right, "s_0"'] \arrow[ddl, "\mathrm{Id}"] \\
& X_1 \arrow[d, shift right, "pr_0"'] & \\
& X_0 \arrow[u, shift right, "\Delta_{X_0}"'] &
\end{tikzcd} \tag{33}$$

The adjoint ϕ of (g, t) is given by postcomposing with the counit of $H \dashv R$, so ϕ is the horizontal composite in:

$$
\begin{tikzcd}
X_0 \times_q X_0 \arrow[r, "{(g,g)}"] \arrow[d, shift right, "pr_0"'] \arrow[d, shift left, "pr_1"] \arrow[ddr, "g"] & (e^*M)_1 \times_{d'_0} (e^*M)_1 \arrow[r, "\mu_1"] \arrow[d, shift right, "pr_0"'] \arrow[d, shift left, "pr_1"] & (e^*M)_1 \arrow[d, shift right, "d'_0"'] \arrow[d, shift left, "d'_1"] \arrow[ddl, "\rho'_1"] \\
X_0 \arrow[r, "g"'] \arrow[u, bend left=60, "\Delta_{X_0}"] \arrow[ddr, "e_0"'] & (e^*M)_1 \arrow[r, "d'_1"'] \arrow[u, bend right=60, "\Delta_{(e^*M)_1}"'] & X_0 \arrow[u, bend right=60, "s'_0"'] \arrow[ddl, "\mathrm{Id}"] \\
& X_0 \times_q X_0 \arrow[d, shift right, "pr_0"'] \arrow[d, shift left, "pr_1"] & \\
& X_0 \arrow[u, bend right=60, "\Delta_{X_0}"'] &
\end{tikzcd}
\tag{34}
$$

By Sect. 2.1, the counit μ is $\mu_1\{((a, b), m), ((a, b), m')\} = ((a, b), m \circ m')$. Since $t_1 tq = \Delta tq$ by (4),) by (33) we have $\rho'_1 g = t_1 tq = \Delta tq = \Delta e_0$.

We have a map $g : X_0 \to (e^*M)_1$ into the pullback square of (18) given by $\sigma_1 j_1 \top t_1 tq : X_0 \to M_1 \times X_1$. This in turn defines $g' : X_0 \to j^*(e^*M)_1$, given by another pullback:

$$
\begin{tikzcd}
X_0 \arrow[dr, "g'"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "tq"'] & & \\
& j^*(e^*M)_1 \arrow[r, "k_1"] \arrow[d] & (e^*M)_1 \arrow[d, "\rho'_1"] \\
& X_0 \arrow[r, "\Delta_{X_0}"'] & X_1
\end{tikzcd}
$$

We define $g'_{M_1} : X_0 \to j^*M_1$ into the pullback (15) by $\sigma_1 j_1 \top tq : X_0 \to M_1 \times X_0$, since $\Delta_{X_0} tq = \rho_1 \sigma_1 j_1$. By the definitions of g, g'_{M_1}, and ϑ, for each $(a, b) \in X_1$ we have $\vartheta(g'_{M_1})(a, b) = \{(tqa, tqb), \mu_1(g'_{M_1}(a), i_1 k_1 g'_{M_1}(b))\}$, while by (34) we have:

$$
\begin{aligned}
\phi(a, b) &= \{(tqa, tqb), \eta_1((tqa, tqb), k_1 g'_{M_1}(a), (tqa, tqb), k'_1 g'_{M_1}(b)\} \\
&= ((tqa, tqb), g'_{M_1}(a) \circ i_1 k_1 g'_{M_1}(b)),
\end{aligned}
$$

with $g'_{M_1}(a) \circ i_1 k_1 g'_{M_1}(b) \in M_1(tqa, tqb)$. By the Eckmann–Hilton argument, the abelian group structure on $M_1(tqa, tqb)$ is the same as the groupoid structure, so $g'_{M_1}(a) \circ i_1 k_1 g'_{M_1}(b) = \mu_1(k_1 g'_{M_1}(a), i_1 k_1 g'_{M_1}(b))$. We conclude that $\vartheta(g')(a, b) = \phi(a, b)$ for each $(a, b) \in X_1$, so $\vartheta(g') = \phi$, as required. $\square$

We now rewrite the map ϑ of Proposition 3.5 in a different form (see Lemma 3.7). This will be used in Proposition 5.3 in the case $\mathcal{C} = \mathsf{Cat}_{\mathcal{O}}$:

Definition 3.6 Given $X = H(Y) \in \mathsf{Gpd}\,\mathcal{C}$ as in (9), we define a map

$$\vartheta' : \mathrm{Hom}_{\mathcal{C}/X_0}(X_0,\ j_1^* M_1) \to \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, M) \tag{35}$$

as follows: the pullback square (15) implies that a map $X_0 \to j_1^* M_1$ is given by $f : X_0 \to M_1$ with $\rho_1 f = \Delta_{X_0}$. We then define $\bar{\vartheta}(f) : X_1 \to M_1$ by $\bar{\vartheta}(f)(a, b) = \sigma(a, b) \circ f(b) - f(a) \circ \sigma(a, b)$, where $\circ$ is the groupoid composition in M and the subtraction is that of the abelian group object M. This gives rise to a map

$$\begin{tikzcd}
X_1 \arrow[r, "\bar{\vartheta}(f)"] \arrow[d, shift right, "pr_0"'] & M_1 \arrow[d, shift right, "d_0"'] \\
X_0 \arrow[r, "\mathrm{Id}"'] \arrow[u, shift right, "\Delta_{X_0}"'] & X_0 \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

in $\mathsf{Spl}\,\mathcal{C}$, where for each $(a, b) \in X_1$, with $d_0 \bar{\vartheta}(f)(a, b) = (\mathrm{Id} \circ p_0)(a, b) = a$ we have

$$\bar{\vartheta}(f)\Delta_{X_0}(a) = \bar{\vartheta}(f)(a, a) = \sigma(a, a) f(a) - f(a)\sigma(a, a) = O_M = \sigma(a, a) = s_0(a).$$

Here $\sigma \Delta_{X_0} = s_0 : X_0 \to M_1$ because $(\sigma, \mathrm{Id}) : X \to M$ is a map of groupoids. Thus we have a map in $\mathrm{Hom}_{\mathsf{Spl}\,\mathcal{C}/RHY}(Y,\ R(M))$ given by the composite

$$\begin{tikzcd}
X_0 \arrow[r, "t_1"] \arrow[d, shift right, "q"'] & X_1 \arrow[r, "\bar{\vartheta}(f)"] \arrow[d, shift right, "pr_0"'] & M_1 \arrow[d, shift right, "d_0"'] \\
\pi_0 \arrow[r, "t"'] \arrow[u, shift right, "t"'] & X_0 \arrow[r, "\mathrm{Id}"'] \arrow[u, shift right, "\Delta_{X_0}"'] & X_0 \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

By Sect. 2.1 this corresponds to the map $\vartheta'(f)$ in $\mathrm{Hom}_{\mathsf{Gpd}\mathcal{C}/X}(X, M)$ with $\vartheta'(f)_0 =$ Id. and $\vartheta'(f)_1 = \bar{\vartheta}(f)$. We define

$$\phi : \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}((X \xrightarrow{\mathrm{Id}} X), (M \xrightarrow{\rho} X)) \to \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}((X \xrightarrow{e} X), (e^* M \xrightarrow{\rho'} X)) \tag{36}$$

as follows. By Sect. 2.1, an element of $\mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}((X \xrightarrow{\mathrm{Id}} X), (M \xrightarrow{\rho} X))$ is determined by the map

$$\begin{tikzcd}
X_1 \arrow[r, "g"] \arrow[d, shift right, "pr_0"'] & M_1 \arrow[d, shift right, "d_0"'] \\
X_0 \arrow[r, "\mathrm{Id}"'] \arrow[u, shift right, "\Delta_{X_0}"'] & X_0 \arrow[u, shift right, "s_0"']
\end{tikzcd} \tag{37}$$

in $\mathsf{Spl}\,\mathcal{C}$. We associate to this another map in $\mathsf{Spl}\,\mathcal{C}$:

$$\begin{tikzcd} X_1 \arrow[r, "\widetilde{\phi}(g)"] \arrow[d, shift right, "pr_0"'] & e^*M_1 \arrow[d, shift right, "d'_0"'] \\ X_0 \arrow[r, "e_0"'] \arrow[u, shift right, "\Delta_{X_0}"'] & X_0 \arrow[u, shift right, "s'_0"'] \end{tikzcd} \tag{38}$$

where $\widetilde{\phi}(g)$ into the pullback square of (18) is given by $\psi_g \top e_1 : X_1 \to M_1 \times X_1$, with $\psi_g(a,b) = \sigma(tqa, a)g(a,b)\sigma(b, tqb)$. Note that, since $\rho_1(m) = (\partial_0 m, \partial_1 m)$:

$$\rho_1\psi_g(a,b) = \rho_1\sigma_1(tqa,a)\rho_1 g(a,b)\rho_1\sigma(b,tqb) = (tqa,a)(a,b)(b,tqb) = e_1(a,b).$$

We may check the commutativity of (38), using (11). The map $(e_0, \widetilde{\phi}(g))$ in (38) then defines an element $\phi(g)$ in $\mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}((X \xrightarrow{e} X), (e^*M \xrightarrow{\rho'} X))$.

Lemma 3.7 *For ϑ' as in (35) and ϕ as in (36), the diagram*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}/X_0}((X \xrightarrow{\mathrm{Id}} X), (j^*M \xrightarrow{\lambda} X_0)) \arrow[r, "\vartheta'"] \arrow[d, "(tq)^*"] & \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, M) \arrow[d, "\phi"] \\ \mathrm{Hom}_{\mathcal{C}/X_0}((X \xrightarrow{tq} X), (j^*M \xrightarrow{\lambda} X_0)) \arrow[r, "\vartheta"] & \mathrm{Hom}_{\mathsf{Gpd}\,\mathcal{C}/X}(X, e^*M) \end{tikzcd} \tag{39}$$

commutes, with vertical isomorphisms, where $(tq)^*(\mathrm{Id}, f) = (tq, \hat{f})$ *for* $(\mathrm{Id}, f) \in \mathrm{Hom}_{\mathcal{C}/X_0}(X_0, j_1^* M_1)$ *and* $\hat{f}(a) = \sigma(tqa, a) f(a)\sigma(a, tqa)$.

Proof For each $(a, b) \in X_1$ we have $\theta'(\mathrm{Id}, f)(a,b) = \sigma(a,b)f(b) - f(a)\sigma(a,b)$, so

$$\begin{aligned} \phi\vartheta'(\mathrm{Id}, f)(a,b) &= \sigma(tqa,a)\{\sigma(a,b)f(b) - f(a)\sigma(a,b)\}\sigma(tqb,b) \\ &= \sigma(tqa,a)\sigma(a,b)f(b)\sigma(b,tqb) - \sigma(tqa,a)f(a)\sigma(a,b)\sigma(b,tqb) \\ &= \hat{f}(b) - \hat{f}(a) = \vartheta(\hat{f})(a,b) = \vartheta(tq)^*(\mathrm{Id}, f)(a,b) \end{aligned}$$

Thus (39) commutes. The map $(tq)^*$ is an isomorphism, with inverse sending $(tq, g) : X_0 \to e^* j_1^* M_1$ to $(\mathrm{Id}, \widetilde{g}) : X_0 \to j_1^* M_1$, where $\widetilde{g}(a) = \sigma(a, tqa)g(a)\sigma(tqa, a)$. The map ϕ is an isomorphism by construction. ◻

4 Comonad resolutions for $(\mathcal{S},\mathcal{O})$-cohomology

We now define the comonad on track categories which is used to construct functorial cofibrant replacements, yielding a formula for computing the $(\mathcal{S},\mathcal{O})$-cohomology of a track category. This will provide crucial ingredients (Theorem 4.9 and Corollary 4.10) for our main result, Theorem 5.5.

4.1 Track categories

Track categories, the objects of $\mathsf{Track}_{\mathcal{O}} = \mathsf{Gpd}(\mathsf{Cat}_{\mathcal{O}})$ of Sect. 1.1, are strict 2-categories X with object set $\mathcal{O}$ (and functors which are identity on objects) such that for each $a, b \in \mathcal{O}$ the category $X(a, b)$ is a groupoid. The double nerve functor provides an embedding $N_{(2)} : \mathsf{Track}_{\mathcal{O}} \hookrightarrow [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$

The lower right corner of $N_{(2)}X$ (omitting degeneracies), appears as follows, with the vertical direction groupoidal, and the horizontal categorical:

$$\begin{tikzcd}[column sep=large]
\cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[d, shift right=4] \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] \arrow[d, leftarrow, shift left=4] & X_{11} \times_{X_{10}} X_{11} \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[d, shift right=4] \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] \arrow[d, leftarrow, shift left=4] & \mathcal{O} \arrow[d, shift right=4] \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] \arrow[d, leftarrow, shift left=4] \\
X_{11} \times_{\mathcal{O}} X_{11} \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] & X_{11} \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] & \mathcal{O} \arrow[d, shift right=2] \arrow[d] \arrow[d, leftarrow, shift left=2] \\
X_{10} \times_{\mathcal{O}} X_{10} \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, leftarrow, shift right=4] & X_{10} \arrow[r, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & \mathcal{O}
\end{tikzcd}$$

There is a functor $\Pi_0 : \mathsf{Track}_{\mathcal{O}} \to \mathsf{Cat}_{\mathcal{O}}$ given by dividing out by the 2-cells: that is,$(\Pi_0 X)_0 = \mathcal{O}$ and $(\Pi_0 X)_1 = qX_1$, where X_1 is the groupoid of 1- and 2-cells in X and $q : \mathsf{Gpd} \to \mathsf{Set}$ is the connected component functor.

Definition 4.1 We say $X \in \mathsf{Track}_{\mathcal{O}}$ is *homotopically discrete* if, for each $a, b \in \mathcal{O}$, the groupoid $X(a, b)$ is an equivalence relation, that is, a groupoid with no non-trivial loops.

Remark 4.2 By taking nerves in the groupoid direction we define

$$I : \mathsf{Track}_{\mathcal{O}} = \mathsf{Gpd}(\mathsf{Cat}_{\mathcal{O}}) \to [\Delta^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{O}}] = (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat} \tag{40}$$

If $F : X \to Y$ in $\mathsf{Track}_{\mathcal{O}}$ is a 2-equivalence (so for each $a, b \in \mathcal{O}$, $F(a, b)$ is an equivalence of groupoids), IF is a Dwyer–Kan equivalence of the corresponding $(\mathcal{S},\mathcal{O})$-categories. In particular, if $X \in \mathsf{Track}_{\mathcal{O}}$ is homotopically discrete, and $d\,\Pi_0 X$ is a track category with only identity 2-cells), this holds for the obvious $F : X \to \mathrm{d}\Pi_0 X$.

4.2 The comonad $\mathcal{K}$

Taking $\mathcal{C} = \mathsf{Cat}_{\mathcal{O}}$ in Sect. 2.2 yields a pair of adjoint functors

$$L : \mathsf{Cat}_{\mathcal{O}} \leftrightarrows \mathsf{Gpd}(\mathsf{Cat}_{\mathcal{O}}) = \mathsf{Track}_{\mathcal{O}} : U.$$

Let $\mathsf{Graph}_{\mathcal{O}}$ be the category of reflexive graphs with object set $\mathcal{O}$ and morphisms which are identity on objects (where a reflexive graph is a diagram $X_1 \underset{s_0}{\overset{d_0}{\underset{d_1}{\rightrightarrows\atop\leftarrow}}} X_0$

with $d_0 s_0 = d_1 s_0 = \mathrm{Id}$). There are adjoint functors $F : \mathsf{Graph}_{\mathcal{O}} \rightleftarrows \mathsf{Cat}_{\mathcal{O}} : V$, where V is the forgetful functor and F is the free category functor. By composition, we obtain a pair of adjoint functors

$$LF : \mathsf{Graph}_{\mathcal{O}} \rightleftarrows \mathsf{Track}_{\mathcal{O}} : VU, \tag{41}$$

and therefore a comonad $(\mathcal{K}, \varepsilon, \delta)$, $\mathcal{K} = LFVU : \mathsf{Track}_{\mathcal{O}} \to \mathsf{Track}_{\mathcal{O}}$, where ε is the counit of the adjunction (41), $\delta = LF(\eta)VU$, and η the unit of the adjunction (41). For each $X \in \mathsf{Track}_{\mathcal{O}}$ we obtain a simplicial object $\mathcal{K}_\bullet X \in [\Delta^{\mathrm{op}}, \mathsf{Track}_{\mathcal{O}}]$ with $\mathcal{K}_n X = \mathcal{K}^{n+1} X$ and face and degeneracy maps given by

$$\partial_i = \mathcal{K}^i \varepsilon \mathcal{K}^{n-i} : \mathcal{K}^{n+1} X \to \mathcal{K}^n X \quad \text{and} \quad \sigma_i = \mathcal{K}^i \delta \mathcal{K}^{n-i} : \mathcal{K}^{n+1} X \to \mathcal{K}^{n+2} X.$$

The simplicial object $\mathcal{K}_\bullet X$ is augmented over X via $\varepsilon : \mathcal{K}_\bullet X \to X$, and $\mathcal{K}_\bullet X$ is a simplicial resolution of X (see [20]).

Remark 4.3 The augmented simplicial object $VU(\mathcal{K}_\bullet X) \xrightarrow{VU\varepsilon} VUX$ is aspherical (see for instance [20, Proposition 8.6.10]).

Remark 4.4 Given $X \in \mathsf{Cat}_{\mathcal{O}}$ by (5) we see that LX is a homotopically discrete track category (Definition 4.1), with $\Pi_0 LX = X$. There are two canonical splittings $\Pi_0 LX = X \to (LX_0) = X_s \coprod X_t$, given by the inclusion in the s or in the t copy of X. Since $\mathcal{K}Y = L(FVUY)$ for each $Y \in \mathsf{Track}_{\mathcal{O}}$, the same holds for $\mathcal{K}Y$.

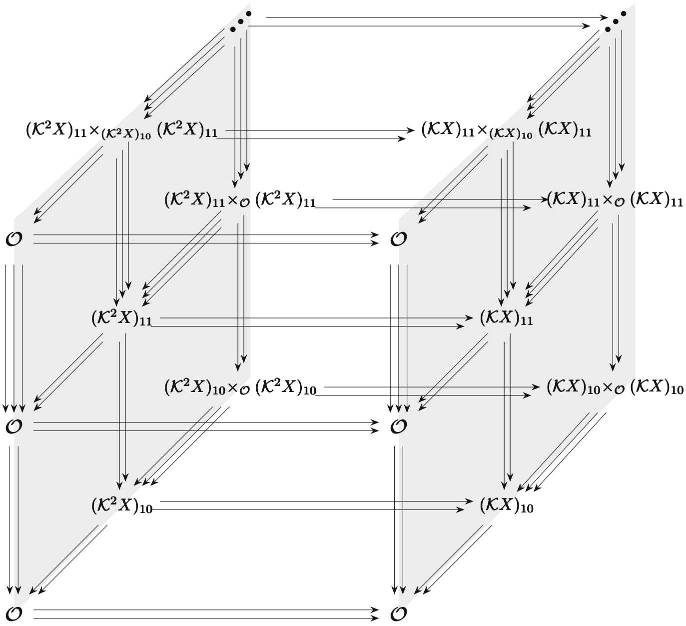
Furthermore, since $FVUY$ is a free category, so is $\Pi_0 \mathcal{K}Y = FVUY$. Since F preserves coproducts (being a left adjoint), $(\mathcal{K}Y)_0 = FVUY \coprod FVUY = F(VUY \coprod VUY)$ and, using (6), $(\mathcal{K}Y)_1 = F(VUY \coprod VUY \coprod VUY \coprod VUY)$. Thus both $(\mathcal{K}Y)_0$ and $(\mathcal{K}Y)_1$ are free categories. Similarly, $(\mathcal{K}Y)_r$ is a free category for each $r > 1$.

4.3 The comonad resolution

Let $\overline{I} : [\Delta^{\mathrm{op}}, \mathsf{Track}_{\mathcal{O}}] \to [\Delta^{\mathrm{op}}, (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}]$ be the functor obtained by applying the internal nerve functor I of (40) levelwise in each simplicial dimension—so for $X \in \mathsf{Track}_{\mathcal{O}}$, $\overline{I}\mathcal{K}_\bullet X \in [\Delta^{\mathrm{op}}, (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}]$. Similarly, $\overline{N} : (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat} \to [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ is obtained by applying the nerve functor in the category direction; applying this levelwise to $\overline{I}\mathcal{K}_\bullet X$ yields

$$W = \overline{N}\,\overline{I}\mathcal{K}_\bullet X \in [\Delta^{3^{\mathrm{op}}}, \mathsf{Set}]. \tag{42}$$

Below is a picture of the corner of W, in which the horizontal simplicial direction is given by the comonad resolution, the vertical is given by the nerve of the groupoid in each track category, and the diagonal is given by the nerve of the category in each track category.



(43)

Note that the augmentation $\epsilon : \mathcal{K}_\bullet X \to X$ induces a map in $[\Delta^{3^{\mathrm{op}}}, \mathsf{Set}]$:

$$W \to \overline{N}\,\overline{I}cX. \tag{44}$$

Now let $Z = W^{(2)}$ be W thought of as a simplicial object in $[\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ along the direction appearing diagonal in the picture, that is

$$Z \in [\Delta^{\mathrm{op}}, [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]], \tag{45}$$

with Z_0 the constant bisimplicial set at $\mathcal{O}$, Z_1 given by

$$\begin{array}{ccccc}
 & & \vdots & & \\
(\mathcal{K}^3X)_{11} \times_{(\mathcal{K}^3X)_{10}} (\mathcal{K}^3X)_{11} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & (\mathcal{K}^2X)_{11} \times_{(\mathcal{K}^2X)_{10}} (\mathcal{K}^2X)_{11} & \rightrightarrows & (\mathcal{K}X)_{11} \times_{(\mathcal{K}X)_{10}} (\mathcal{K}X)_{11} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots (\mathcal{K}^3X)_{11} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & (\mathcal{K}^2X)_{11} & \rightrightarrows & (\mathcal{K}X)_{11} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
(\mathcal{K}^3X)_{10} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & (\mathcal{K}^2X)_{10} & \rightrightarrows & (\mathcal{K}X)_{10}
\end{array}$$

with

$$Z_k \cong Z_1 \times_{Z_0} \overset{k}{\cdots} \times_{Z_0} Z_1, \tag{46}$$

for each $k \geq 2$.

By applying the diagonal functor $\mathrm{Diag} : [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$ dimensionwise to Z (viewed as in (45)), we obtain $\overline{\mathrm{Diag}}Z \in [\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$.

To show that $\overline{\mathrm{Diag}}Z$ is an $(\mathcal{S},\mathcal{O})$-category, we must show that it behaves like the nerve of a category object in simplicial sets in the outward simplicial direction: this means that $(\overline{\mathrm{Diag}}Z)_0$ is the "simplicial set of objects"—and indeed it is the constant simplicial set at $\mathcal{O}$. Similarly, $(\overline{\mathrm{Diag}}Z)_1$ is the "simplicial set of arrows".

Since Diag preserves limits, by (46) we have

$$\begin{aligned}(\overline{\mathrm{Diag}}Z)_k &\cong \mathrm{Diag}\, Z_k \cong \mathrm{Diag}\, Z_1 \times_{\mathrm{Diag} Z_0} \overset{k}{\cdots} \times_{\mathrm{Diag} Z_0} \mathrm{Diag}\, Z_1 \\ &= (\overline{\mathrm{Diag}}Z)_1 \times_{(\overline{\mathrm{Diag}}Z)_0} \overset{k}{\cdots} \times_{(\overline{\mathrm{Diag}}Z)_0} (\overline{\mathrm{Diag}}Z)_1.\end{aligned}$$

for each $k \geq 2$. Thus $\overline{\mathrm{Diag}}Z$ is 2-coskeletal, with unique fill-ins for inner 2-horns (the composite) so it is indeed in $(\mathcal{S},\mathcal{O})$-$\mathsf{Cat}$, and the map (44) induces a map $\alpha : \overline{\mathrm{Diag}}Z \to IX$ for I as in (40).

Lemma 4.5 *For Z and* $\mathrm{Diag}\, Z$ *as above, $\alpha : \overline{\mathrm{Diag}}Z \to IX$ is a Dwyer–Kan equivalence in $(\mathcal{S},\mathcal{O})$-$\mathsf{Cat}$.*

Proof We need to show that, for each $a, b \in \mathcal{O}$

$$(\overline{\mathrm{Diag}}Z)(a,b) \to X(a,b) \tag{47}$$

is a weak homotopy equivalence. Note that $(\overline{\mathrm{Diag}}Z)(a,b)$ is the diagonal of

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \cdots\ (\mathcal{K}^3X)_{11}(a,b) & \substack{\to\\\to\\\to} & (\mathcal{K}^2X)_{11}(a,b) & \rightrightarrows & (\mathcal{K}X)_{11}(a,b) \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \cdots\ (\mathcal{K}^3X)_{10}(a,b) & \substack{\to\\\to\\\to} & (\mathcal{K}^2X)_{10}(a,b) & \rightrightarrows & (\mathcal{K}X)_{10}(a,b), \end{array} \tag{48}$$

and we have a map from (48) to the horizontally constant bisimplicial set

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \cdots\ X_{11}(a,b) & \substack{\to\\\to\\\to} & X_{11}(a,b) & \rightrightarrows & X_{11}(a,b) \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \cdots\ X_{10}(a,b) & \substack{\to\\\to\\\to} & X_{10}(a,b) & \rightrightarrows & X_{10}(a,b) \end{array} \tag{49}$$

inducing the map (47) on diagonals.

We shall show that this map of bisimplicial sets from (48) to (49) is a weak equivalence of simplicial sets in each vertical dimension. The corresponding map of diagonals (47) is then a weak equivalence by [12, Prop.1.7].

Consider first vertical dimension 1. We must show that the map of simplicial sets

$$(\mathcal{K}_\bullet X)_{11}(a,b) \to cX_{11}(a,b) \tag{50}$$

is a weak equivalence, where $cX_{11}(a,b)$ denotes the constant simplicial set at $X_{11}(a,b)$. Note that $W_{11}=(VUW)_1$ for each $W\in\mathsf{Track}_{\mathcal{O}}$, where U is the internal arrow functor and V is the underlying graph functor (with $(VY)_1=Y_1$ for each $Y\in\mathsf{Cat}_{\mathcal{O}}$), so $W_{11}(a,b)=(VUW)_1(a,b)$ and thus $(\mathcal{K}_\bullet X)_{11}(a,b)=(VU\mathcal{K}_\bullet X)_1(a,b)$ for each $a,b\in\mathcal{O}$.

By Remark 4.3, the simplicial object $VU\mathcal{K}_\bullet X$ is aspherical, so $(VU\mathcal{K}_\bullet X)_1(a,b)$ is, too, and is thus weakly equivalent to $c(VUX)_1(a,b)=cX_{11}(a,b)$. Thus (50) is a weak equivalence.

In vertical dimension 0, from the vertical simplicial structure of (48) we see

$$(\mathcal{K}_\bullet X)_{10}(a,b)\to cX_{10}(a,b) \tag{51}$$

is a retract of (50), so it is also a weak equivalence.

In vertical dimension 2, we must show that

$$(\mathcal{K}_\bullet X)_{11}(a,b)\times_{(\mathcal{K}_\bullet X)_{10}(a,b)}(\mathcal{K}_\bullet X)_{11}(a,b)\to cX_{11}(a,b)\times_{cX_{10}(a,b)}cX_{11}(a,b) \tag{52}$$

is a weak equivalence. This is the induced map of pullbacks of the diagram

$$\begin{tikzcd}
(\mathcal{K}_\bullet X)_{11}(a,b) \arrow[r, "\partial_0^\bullet"] \arrow[d] & (\mathcal{K}_\bullet X)_{10}(a,b) \arrow[d] & (\mathcal{K}_\bullet X)_{11}(a,b) \arrow[l, "\partial_1^\bullet"'] \arrow[d] \\
cX_{11}(a,b) \arrow[r, "\partial_0"] & cX_{10}(a,b) & cX_{11}(a,b) \arrow[l, "\partial_1"']
\end{tikzcd} \tag{53}$$

in $\mathcal{S}$. By the above discussion, the vertical maps in (53) are weak equivalences.

By definition of $\mathcal{K}$ there is a pullback

$$\begin{tikzcd}
(\mathcal{K}_\bullet X)_{11}(a,b) \arrow[r, "\partial_1^\bullet"] \arrow[d, "\partial_0^\bullet"'] & (\mathcal{K}_\bullet X)_{10}(a,b) \arrow[d, "\nabla"] \\
(\mathcal{K}_\bullet X)_{10}(a,b) \arrow[r, "\nabla"'] & \Pi_0(\mathcal{K}_\bullet X)_{10}(a,b)
\end{tikzcd} \tag{54}$$

in $\mathcal{S}$, where

$$\nabla:(\mathcal{K}_\bullet X)_{10}(a,b)=\Pi_0(\mathcal{K}_\bullet X)_{10}(a,b)\coprod\Pi_0(\mathcal{K}_\bullet X)_{10}(a,b)\to\Pi_0(\mathcal{K}_\bullet X)_{10}(a,b).$$

is the fold map. To see that ∇ is a fibration, let $Y_\bullet=\Pi_0(\mathcal{K}_\bullet X)_{10}(a,b)$. For any commuting diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "\alpha"] \arrow[d, hook, "j"'] & Y_\bullet\coprod Y_\bullet \arrow[d, "\nabla"] \\
\Delta[n] \arrow[r, "\beta"'] & Y_\bullet
\end{tikzcd}$$

in $\mathcal{S}$, α factors through $i_t : Y_\bullet \hookrightarrow Y_\bullet \coprod Y_\bullet$ $(t = 1, 2)$, since $\Lambda^k[n]$ is connected, so

$$\begin{tikzcd} \Lambda^k[n] \arrow[rr, "\alpha"] \arrow[dr, "\alpha'"'] & & Y_\bullet \coprod Y_\bullet \\ & Y_\bullet \arrow[ur, "i_t"'] & \end{tikzcd}$$

commutes, and thus $\nabla i_t \beta = \beta$ and $i_t \beta j = i_t \nabla \alpha = i_t \nabla i_t \alpha' = i_t \alpha' = \alpha$.

The maps $\partial_0^\bullet$ and $\partial_1^\bullet$ are fibrations, since they are pullbacks of such by (54). The bottom horizontal maps in (53) are fibrations since their target is discrete. We conclude that the induced map of pullbacks (52) is a weak equivalence.

Vertical dimension $i > 2$ is completely analogous. □

Lemma 4.6 *For any $X \in$ Track$_\mathcal{O}$, $\overline{\text{Diag}} N \overline{I} \mathcal{K}_\bullet X$ is a free $(\mathcal{S},\mathcal{O})$-category (see Sect. 2.4).*

Proof By Remark 4.4, for each track category $\mathcal{K}_r X$ the nerve in the groupoid direction is a free category in each simplicial degree. Thus for $Z := \overline{N}\, \overline{I} \mathcal{K}_\bullet X$, $\overline{\text{Diag}} Z$ is also a free category in each simplicial degree. By Sect. 4.2, the degeneracies $\sigma_i : \mathcal{K}^{n+1} X \to \mathcal{K}^{n+2} X$ are given by $\sigma_i = \mathcal{K}^i LF(\eta) VU \mathcal{K}^{n-i}$, where $\eta : \text{Id} \to VULF$ is the unit of the adjunction. Therefore, σ_i sends generators to generators and so the same holds for the degeneracies of $\overline{\text{Diag}} Z$, so it is a free $(\mathcal{S},\mathcal{O})$-category. □

Corollary 4.7 *For any $X \in$ Track$_\mathcal{O}$, $\overline{\text{Diag}} N \overline{I} \mathcal{K}_\bullet X$ is a cofibrant replacement of IX in $(\mathcal{S},\mathcal{O})$-Cat.*

Proof By Lemma 4.5, the map $\overline{\text{Diag}} Z \to IX$ is a weak equivalence. □

We now show how to use the comonad resolution of a track category to compute its $(\mathcal{S},\mathcal{O})$-cohomology:

Proposition 4.8 *For $X \in$ Track$_\mathcal{O}$, let $Z = \overline{N}\, \overline{I} \mathcal{K}_\bullet X$ and let M be a Dwyer–Kan module over X. Then*

$$H^{n-i}_{\text{SO}}(IX; M) \;=\; \pi_i \operatorname{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathit{Cat}/IX}(\overline{\text{Diag}} Z, \mathcal{K}_X(M, n)). \tag{55}$$

Proof Let $\phi : \text{Diag}\, \overline{\mathcal{F}}_\bullet X \to IX$ be the Dwyer–Kan standard free resolution of Sect. 2.4. Then, by definition,

$$H^{n-i}_{\text{SO}}(IX; M) \;=\; \pi_i \operatorname{map}_{(\mathcal{S},\mathcal{O})\text{-Cat}/IX}(\text{Diag}\, \overline{\mathcal{F}}_\bullet X,\ \mathcal{K}_X(M, n)), \tag{56}$$

where $\mathcal{K}_X(M, n)$ is the twisted Eilenberg–Mac Lane $(\mathcal{S},\mathcal{O})$-category of Sect. 2.4.

By Corollary 4.7, $\alpha : \overline{\text{Diag}} Z \to IX$ is a cofibrant replacement for IX, so given a commuting diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & \text{Diag}\, \overline{\mathcal{F}}_\bullet X \arrow[d, "\phi"] \\ \overline{\text{Diag}} Z \arrow[r, "\alpha"'] & IX, \end{tikzcd}$$

there is a lift $\psi : \overline{\mathrm{Diag}} Z \to \mathrm{Diag}\, \overline{\mathcal{F}}_\bullet X$ with $\phi\psi = \alpha$ and ψ a weak equivalence. Hence

$$\pi_i \,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/IX}(\overline{\mathrm{Diag}} Z, \mathcal{K}_X(M,n)) \cong \pi_i \,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/IX}(\mathrm{Diag}\, \overline{\mathcal{F}}_\bullet X, \mathcal{K}_X(M,n))$$

Thus (55) follows from (56). ◻

Theorem 4.9 *Let $X \in \mathit{Track}_{\mathcal{O}}$, and M a module over X; then for each $s \geq 0$, $H^s_{\mathrm{SO}}(IX; M) = \pi^s C^\bullet$ for the cosimplicial abelian group*

$$C^\bullet \;:=\; \pi_1 \,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathit{Cat}/X}(\overline{I}\mathcal{K}_\bullet X, M). \tag{57}$$

Proof Since $\overline{\mathrm{Diag}} Z = \mathrm{Diag}\, \overline{I}\mathcal{K}_\bullet X$, by Proposition 4.8 and Lemma 2.6 we have

$$\begin{aligned} H^{n-i}_{\mathrm{SO}}(IX; M) &= \pi_i \,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/IX}(\overline{\mathrm{Diag}} Z, \mathcal{K}_X(M,n) \\ &= \pi_i \,\mathrm{Tot}\; \mathrm{map}_{[\Delta^{2^{\mathrm{op}}}, \mathsf{Cat}_{\mathcal{O}}]/IX}(\overline{I}\mathcal{K}_\bullet X, \mathcal{K}_X(M,n)). \end{aligned} \tag{58}$$

the homotopy spectral sequence of the cosimplicial space

$$W^\bullet = \mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}}(\overline{I}\mathcal{K}_\bullet X, \mathcal{K}_X(M,n))$$

(see [8, X6]) has $E_2^{s,t} = \pi^s \pi_t W^\bullet \Rightarrow \pi_{t-s} \,\mathrm{Tot}\; W^\bullet$ with

$$E_1^{s,t} \;=\; \pi_t \,\mathrm{map}(I\mathcal{K}^s X, \mathcal{K}_X(M,n)) = H^{n-t}_{\mathrm{SO}}(I\mathcal{K}^s X; M).$$

Here we used the fact that $I\mathcal{K}^s X$ is a cofibrant $(\mathcal{S},\mathcal{O})$-category, since it is free in each dimension and the degeneracy maps take generators to generators.

By Remark 4.4, $I\mathcal{K}^s X$ is homotopically discrete, so $I\mathcal{K}^s X \to I\,\mathrm{d}\Pi_0\mathcal{K}^s X$ is a weak equivalence. Hence $I\mathcal{K}^s X$ is a cofibrant replacement of $I\,\mathrm{d}\Pi_0\mathcal{K}^s X$, and so

$$H^{n-t}_{\mathrm{SO}}(I\mathcal{K}^s X; M) = H^{n-t}_{\mathrm{SO}}(I\,\mathrm{d}\Pi_0\mathcal{K}^s X; M). \tag{59}$$

Recall from [2, Theorem 3.10] that $H^s_{\mathrm{SO}}(\mathrm{d}\mathcal{C}; M) = H^{s+1}_{\mathrm{BW}}(\mathcal{C}, M)$ for any category $\mathcal{C}$ and $s > 0$, where $H^\bullet_{\mathrm{BW}}$ is the Baues–Wirsching cohomology of [3]. Hence if $\mathcal{C}$ is free, $H^s_{\mathrm{SO}}(\mathrm{d}\mathcal{C}; M) = 0$ for each $s > 0$, by [3, Theorem 6.3].

If $n \neq t$, it follows from by (59) that $E_1^{s,t} = H^{n-t}_{\mathrm{SO}}(I\mathcal{K}^s X; M) = 0$, since $\Pi_0 I\mathcal{K}^s X$ is a free category, so the spectral sequence collapses at the E_1-term, and

$$H^s_{\mathrm{SO}}(IX; M) \;=\; \pi_{n-s} \,\mathrm{Tot}\; W^\bullet \;=\; \pi^s \pi_n \,\mathrm{map}(\overline{I}\mathcal{K}_\bullet X, \mathcal{K}_X(M,n)). \tag{60}$$

Since $\pi_n \,\mathrm{map}(\overline{I}\mathcal{K}_\bullet X, \mathcal{K}_X(M,n)) = H^0_{(\mathcal{S},\mathcal{O})}(I\,\mathrm{d}\Pi_0\mathcal{K}_\bullet X, M)$, this is independent of n. We deduce from (60) that $H^s_{\mathrm{SO}}(IX; M) = \pi^s C^\bullet$ for $C^\bullet$ as in (57). ◻

Corollary 4.10 *For any $X \in \mathsf{Track}_{\mathcal{O}}$, X-module M, and $s \geq 0$ we have*

$$H^s_{\mathrm{SO}}(I\,\mathrm{d}X_0;\, j^*M) \;\cong\; \pi^s\widehat{C}^\bullet, \tag{61}$$

where $\widehat{C}^\bullet$ is the cosimplicial abelian group $\pi_1\,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/X}(\overline{I}(\mathcal{K}_\bullet X)_0,\, j^*M)$.

Proof Let $Z = N\overline{I}\,\mathrm{d}(\mathcal{K}_\bullet X)_0$. Then $\overline{\mathrm{Diag}}Z \to IX_0$ is a cofibrant replacement, by Corollary 4.7. Therefore, by Lemma 2.6:

$$\begin{aligned} H^{n-i}_{\mathrm{SO}}((I\,\mathrm{d}X_0;\, j^*M) &= \pi_i\,\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}/I\,\mathrm{d}X_0}(\overline{\mathrm{Diag}}Z, \mathcal{K}_{X_0}(j^*M, n)) \\ &= \pi_i\,\mathrm{Tot}\,\mathrm{map}_{[\Delta^{2\mathrm{op}},\mathsf{Cat}_{\mathcal{O}}]}(\overline{I}d(\mathcal{K}_\bullet X)_0,\, j^*M). \end{aligned}$$

The homotopy spectral sequence for $W^\bullet = \mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}}(\overline{I}\mathrm{d}(\mathcal{K}_\bullet X)_0, \mathcal{K}_{X_0}(j^*M, n))$ again collapses at the E_1-term, since $(\mathcal{K}^s X)_0$ is a free category, yielding (61). □

5 $(\mathcal{S},\mathcal{O})$-cohomology of track categories and comonad cohomology

We now use the comonad of Sect. 4.2 to define the comonad cohomology of a track category, rewrite the short exact sequence of Proposition 3.5 for $\mathcal{C} = \mathsf{Cat}_{\mathcal{O}}$, in terms of mapping spaces, and use it to prove our main result, Theorem 5.5.

5.1 Mapping spaces

Given maps $f : A \to B$ and $g : M \to B$ in a simplicial model category $\mathcal{C}$, we let $\mathrm{map}_{\mathcal{C}/B}(A, M)$ be the homotopy pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}/B}(A, M) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(A, M) \arrow[d, "g_*"] \\
\{f\} \arrow[r] & \mathrm{map}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

If A is cofibrant and g is a fibration, so is g_*. Moreover, if $h : B \to D$ is a trivial fibration, so is $h_* : \mathrm{map}(A, B) \to \mathrm{map}(A, D)$.

Thus we obtain a map $\bar{h}_* : \mathrm{map}_{\mathcal{C}/B}(A, M) \to \mathrm{map}_{\mathcal{C}/D}(A, M)$ defined by

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}/B}(A, M) \arrow[rr] \arrow[d] \arrow[dr, "\bar{h}_*"'] & & \mathrm{map}_{\mathcal{C}}(A, M) \arrow[d] \arrow[dr, "\mathrm{Id}"] & \\
\{f\} \arrow[rr] \arrow[dr] & \mathrm{map}_{\mathcal{C}/D}(A, M) \arrow[rr] \arrow[d] & \mathrm{map}_{\mathcal{C}}(A, B) \arrow[dr, "h_*"'] & \mathrm{map}_{\mathcal{C}}(A, M) \arrow[d] \\
& \{hf\} \arrow[rr] & & \mathrm{map}_{\mathcal{C}}(A, D)
\end{tikzcd}$$

Since h_* is a weak equivalence, so is $\bar{h}_*$.

When $\mathcal{C} = \mathsf{Track}_{\mathcal{O}}$, we will use this construction several times for $X = HY$ as in (9) and e^*M, j^*M, and t^*j^*M be as in (10), (14), and (27), respectively:

(i) The diagram

$$\begin{tikzcd}
\{\mathrm{Id}_X\} \arrow[r] \arrow[d] & \mathrm{map}(X, X) \arrow[d, "q_*"] & \mathrm{map}(X, M) \arrow[l, "\rho_*"'] \arrow[d, "\mathrm{Id}"] \\
\{q\} \arrow[r] & \mathrm{map}(X, d\pi_0) & \mathrm{map}(X, M) \arrow[l, "q_*\rho_*"]
\end{tikzcd}$$

induces a map $\bar{q}_* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}\pi_0}(X, M)$, where M is a track category over $\mathrm{d}\pi_0$ via the map $q : X \to \mathrm{d}\pi_0$, which is a weak equivalence (since X is homotopically discrete) and a fibration (since $\mathrm{d}\pi_0$ is discrete). Hence $\bar{q}_*$ is a weak equivalence.

(ii) The diagram

$$\begin{tikzcd}
\{\Delta_{X_0}\} \arrow[r] \arrow[d] & \mathrm{map}(\mathrm{d}X_0, X) \arrow[d, "t^*"] & \mathrm{map}(\mathrm{d}X_0, j^*M) \arrow[l] \arrow[d, "t^*"] \\
\{\Delta_{X_0}t\} \arrow[r] & \mathrm{map}(\mathrm{d}\pi_0, X) & \mathrm{map}(\mathrm{d}\pi_0, t^*j^*M) \arrow[l]
\end{tikzcd}$$

induces $\bar{t}^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, j^*M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}\pi_0}(\mathrm{d}\pi_0, t^*j^*M)$.
Since $q^*t^* = e^*$, we have a diagram

$$\begin{tikzcd}
\{\Delta_{X_0}t\} \arrow[r] \arrow[d] & \mathrm{map}(d\pi_0, X) \arrow[d, "q^*"] & \mathrm{map}(d\pi_0, t^*j^*M) \arrow[l, "(\rho k e)_*"] \arrow[d, "q^*"] \\
\{\Delta_{X_0}t\} \arrow[r] & \mathrm{map}(\mathrm{d}X_0, X) & \mathrm{map}(\mathrm{d}\pi_0, e^*j^*M) \arrow[l]
\end{tikzcd}$$

inducing $\bar{q}^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}\pi_0}(\mathrm{d}\pi_0, t^*j^*M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}\pi_0}(\mathrm{d}\pi_0, e^*j^*M)$.

(iii) The diagram

$$\begin{tikzcd}
\{\mathrm{Id}_X\} \arrow[r] \arrow[d] & \mathrm{map}(X, X) \arrow[d, "j^*"] & \mathrm{map}(X, M) \arrow[l, "\rho_*"'] \arrow[d, "j^*"] \\
\{j\} \arrow[r] & \mathrm{map}(\mathrm{d}X_0, X) & \mathrm{map}(\mathrm{d}X_0, j^*M) \arrow[l, "\lambda_*"']
\end{tikzcd}$$

induces $\bar{j}^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, j^*M)$.

(iv) The diagram

$$\begin{tikzcd}
\{q\} \arrow[r] \arrow[d] & \mathrm{map}(X, d\pi_0) \arrow[d, "t^*"] & \mathrm{map}(X, M) \arrow[l, "q_*\rho_*"'] \arrow[d, "t^*j^*"] \\
\{qjt\} = \{\mathrm{Id}_{d\pi_0}\} \arrow[r] & \mathrm{map}(d\pi_0, d\pi_0) & \mathrm{map}(\mathrm{d}\pi_0, t^*j^*M) \arrow[l, "q_*\rho_*"]
\end{tikzcd}$$

induces $\bar{t}^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(X, M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(d\pi_0, t^* j^* M)$. Since t^* is a trivial fibration, $\bar{t}^*$ is a weak equivalence.

(v) The diagram

$$\begin{tikzcd}
\{\mathrm{Id}_X\} \arrow[r] \arrow[d] & \mathrm{map}(X, X) \arrow[d, "e^*"] & \mathrm{map}(X, M) \arrow[l, "\rho_*"'] \arrow[d, "e^*"] \\
\{e\} \arrow[r] & \mathrm{map}(X, X) & \mathrm{map}(X, e^*M) \arrow[l, "\rho'_*"]
\end{tikzcd}$$

induces $\bar{e}^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, e^*M)$. Moreover, $e : X \to X$ is a weak equivalence (since X is homotopically discrete), so $\bar{e}^*$ is, too.

(vi) Using the fact that $q_* e^* = q_*$ (since $qe = qtq = q$), we have

$$\begin{tikzcd}
\{e\} \arrow[r] \arrow[d] & \mathrm{map}(X, X) \arrow[d, "q_*"] & \mathrm{map}(X, e^*M) \arrow[l, "\rho'_*"'] \arrow[d, "r_*"] \\
\{q\} \arrow[r] & \mathrm{map}(X, \mathrm{d}\pi_0) & \mathrm{map}(X, M) \arrow[l]
\end{tikzcd}$$

which induces $\bar{q}'_* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, e^*M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}\pi_0}(X, M)$, and

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \arrow[rr, "e^*"] \arrow[dr, "\bar{q}_*"'] & & \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, e^*M) \arrow[dl, "\bar{q'}_*"] \\
& \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(X, M) &
\end{tikzcd} \tag{62}$$

commutes, since $\bar{q'}_* e^* = q_*$.

(vii) Finally, the diagram

$$\begin{tikzcd}
\{e\} \arrow[r] \arrow[d] & \mathrm{map}(X, X) \arrow[d, "j^*"] & \mathrm{map}(X, e^*M) \arrow[l, "\rho'_*"'] \arrow[d, "j^*"] \\
\{jtq\} \arrow[r] & \mathrm{map}(\mathrm{d}X_0, X) & \mathrm{map}(\mathrm{d}X_0, e^* j^* M) \arrow[l, "\rho^*"']
\end{tikzcd}$$

induces $(\bar{j'})^* : \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, e^*M) \to \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, e^* j^* M)$.

Lemma 5.1 *For any map $f : A \to B$ in $\mathsf{Cat}_{\mathcal{O}}$, with A free, there is an isomorphism*

$$\pi_1\, \mathrm{map}_{Track_{\mathcal{O}}/B}(A, M) = H^0_{\mathrm{SO}}(A; M) \cong \mathrm{Hom}_{Cat_{\mathcal{O}}/B}(A, M_1).$$

Proof Since A is free, $c(A) \in s\mathcal{O}\text{-}\mathsf{Cat} = (\mathcal{S}, \mathcal{O})\text{-}\mathsf{Cat}$ is cofibrant, and is its own fundamental track category. A module M, as an abelian group object in $\mathsf{Track}_{\mathcal{O}}/B$, is an Eilenberg–Mac Lane object $E^B(M_1, 1)$, which explains the first equality. By [18, II, §2], the n-simplices of the cosimplicial abelian group $\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/B}(A, M)$ are maps of categories over B of the form $\sigma : A \otimes \Delta[n] \to M_n$ where $A \otimes \Delta[n]$

is the coproduct in $\mathsf{Cat}_{\mathcal{O}}$ of one copy of A for each n-simplex of $\Delta[n]$. Since $M_0 = B$, $\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/B}(A, M)_0$ is the singleton $\{f\}$. The non-degenerate part of $\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/B}(A, M)_1$ is $\mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}}(A, M_1)$, so the 1-cycles are $\mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/B}(A, M_1)$. Since M, and thus $\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/B}(A, M)$, is a 1-Postnikov section, the 1-cycles are equal to $\pi_1\, \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/B}(A, M)$. ◻

Lemma 5.2 *For $X \in \mathit{Track}_{\mathcal{O}}$ of the form HY and M an X-module, the diagram*

$$\begin{tikzcd}
\pi_1\, \mathrm{map}_{\mathit{Track}_{\mathcal{O}}/X}(X, M) \arrow[r, "j^*"] \arrow[d, "\wr\|"'] & \mathrm{Hom}_{\mathit{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{\mathrm{Id}} X_0),\ (j_1^* M_1 \to X_0)) \arrow[d, "\wr\|"'] \\
\mathrm{Hom}_{\mathit{Cat}_{\mathcal{O}}/\pi_0}(\pi_0,\ t^* j_1^* M_1) \arrow[r, "q^*"'] & \mathrm{Hom}_{\mathit{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{e_0} X_0),\ (e^* j_1^* M_1 \to X_0))
\end{tikzcd} \tag{63}$$

commutes, and there is an isomorphism

$$\begin{aligned}
\omega : \mathrm{Hom}_{\mathit{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{e_0} X_0),\ (e^* j_1^* M_1 \xrightarrow{\lambda_1} X_0)) \\
\to\ \mathrm{Hom}_{\mathit{Cat}_{\mathcal{O}}/X_0}\,(X_0 \xrightarrow{e_0} X_0),\ j_1^* M_1 \xrightarrow{\lambda_1} X_0))
\end{aligned}$$

such that $\omega q^* = \xi$, for ξ as in (29).

Proof We have a commuting diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \arrow[r, "\bar{j}^*"] \arrow[d, "\bar{e}^*"] \arrow[dd, bend right, "\bar{q}^*"'] & \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, j^* M) \arrow[d, "\tilde{e}^*"] \\
\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, e^* M) \arrow[r, "(\bar{j}')^*"] \arrow[d, "\bar{q}'_*"] & \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, e^* j^* M) \\
\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(X, M) \arrow[d, "\bar{t}^*"'] & \\
\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(d\pi_0, t^* j^* M) \arrow[uur, "q^*"'] &
\end{tikzcd} \tag{64}$$

in which the vertical maps are weak equivalences by Sect. 5.1. Applying π_1 yields

$$\begin{tikzcd}
\pi_1\, \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X, M) \arrow[r, "j^*"] \arrow[d, "\wr\|"'] & \pi_1\, \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0, j^* M) \arrow[d, "\wr\|"', "\chi"] \\
\pi_1\, \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(d\pi_0, t^* j^* M) \arrow[r, "q_*"'] & \pi_1\, \mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}\,((\mathrm{d}X_0 \xrightarrow{\mathrm{d}e_0} \mathrm{d}X_0),\ e^* j^* M)
\end{tikzcd} \tag{65}$$

with vertical isomorphisms. By Lemma 5.1,

$$\begin{aligned}\pi_1\,\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0,\,j^*M) &\cong \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((\mathrm{d}X_0 \xrightarrow{\mathrm{Id}} \mathrm{d}X_0),\ (j_1^*M_1 \xrightarrow{\lambda_1} \mathrm{d}X_0))\\ \pi_1\,\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/d\pi_0}(d\pi_0,\,t^*j^*M) &\cong \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/\pi_0}(\pi_0,\,t^*j_1^*M_1),\\ \pi_1\,\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(\mathrm{d}X_0,\,e^*j^*M) &\cong \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}(X_0,\,q^*t^*j_1^*M_1).\end{aligned}$$

Hence from (65) we obtain

$$\begin{tikzcd} \pi_1\,\mathrm{map}_{\mathsf{Track}_{\mathcal{O}}/X}(X,M) \arrow[r, "j^*"] \arrow[d, "\wr\|"'] & \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{\mathrm{Id}} X_0),\ (j_1^*M_1 \to X_0)) \arrow[d, "\chi"'] \\ \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/\pi_0}(\pi_0,\,t^*j_1^*M_1) \arrow[r, "q^*"'] & \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{\mathrm{Id}} X_0),\ (e^*j_1^*M_1 \to X_0)) \end{tikzcd}$$

The right vertical map χ in (65) sends $\bar{f}: X_0 \to j_1^*M_1$ to $\bar{f}e: X_0 \to e^*j_1^*M_1$, where $\bar{f}$ is given by $\mathrm{Id}\top f$ into the pullback (15) defining j^*M_1. We now rewrite

$$\begin{aligned}\chi:\ &\mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{\mathrm{Id}} X_0),\,(j_1^*M_1 \to X_0))\\ &\to \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{e_0} X_0),\,(e^*j^*M \xrightarrow{\lambda_1} X_0))\end{aligned}$$

in a different form, in order to show that it is an isomorphism. Note that the map of groupoids $\rho = (\rho_1, \mathrm{Id}): M \to X$ satisfies $\rho_1(m) = (\partial_0(m), \partial_1(m))$ for all $m \in M_1$, where $M = \ M_1 \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} X_0$. Thus the map $\bar{f} = \mathrm{Id}\top f$ into (15) has $\rho_1 f = j = \Delta_{X_0}$ so $\rho f(a) = (a,a) = (\partial_0 f(a), \partial_1 f(a))$, and thus f takes X_0 to $\coprod_{a\in X_0} M(a,a)$. Similarly, a map $\bar{g}: X_0 \to e^*j^*M_1$ into the pullback defining $e^*j^*M_1$ is given by $g\top e_0: X_0 \to M_1 \times X_0$ with $\Delta_{X_0}e_0e_0 = \Delta_{X_0}e_0 = \rho_1 g$, so $g: X_0 \to \coprod_{a\in X_0} M(tqa,tqa)$. For each $a \in X_0$ there is an isomorphism $\omega: M(a,a) \to M(tqa,tqa)$ taking $m \in M(a,a)$ to $\sigma(tqa,a)\circ m\circ\sigma(a,tqa)$, whose inverse $\omega^{-1}: M(tqa,tqa) \to M(a,a)$ takes m' to $\sigma(a,tqa)\circ m'\circ\sigma(tqa,a)$.

Given $f: X_0 \to \coprod_a M(a,a)$, there is a commuting diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "e_0"'] & \coprod_a M(a,a) \arrow[d, "\omega"] \\ X_0 \arrow[r, "f"'] & \coprod_a M(tqa,tqa) \end{tikzcd}$$

so that $\chi(\bar{f}) = (e_0, fe_0) = (e_0, \omega f)$. Thus χ is an isomorphism with inverse given by $\chi^{-1}(g) = (\mathrm{Id}, \omega^{-1}g)$.

Finally, the isomorphism

$$\omega : \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((\mathrm{Id}_{X_0}),\ (q^*t^*j_1^*M_1 \xrightarrow{\lambda_1''} X_0)) \to \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((tq),\ (j_1^*M_1 \xrightarrow{\lambda_1} X_0))$$

sends h to $\omega(h) = l_1 v_1 h$, where the maps l_1 and v_1 are given by

$$\begin{tikzcd} q^*t^*j_1^*M_1 \arrow[r, "v_1"] \arrow[d, "\lambda_1''"] & t^*j_1^*M_1 \arrow[r, "l_1"] \arrow[d, "\lambda_1'"] & j_1^*M_1 \arrow[r, "k_1"] \arrow[d, "\lambda_1"] & M_1 \arrow[d, "\rho_1"] \\ X_0 \arrow[r, "q"'] & \pi_0 \arrow[r, "t"'] & X_0 \arrow[r, "\Delta_{X_0}"'] & X_1 \end{tikzcd} \tag{66}$$

Define f' in $\mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}((X_0 \xrightarrow{tq} X_0),\ (j^*M \xrightarrow{\lambda_1} X_0))$ by $tq \top f$ into the pullback (15). Thus $\rho_1 f = \Delta_{X_0} tq$, which also implies $\rho_1 f = \Delta_{X_0}(tq)(tq)$. It follows that there is a map f'' making the following diagram commute

$$\begin{tikzcd} X_0 \arrow[dr, "f''"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "tq"'] & & \\ & q^*t^*j^*M_1 \arrow[r, "k_1 l_1 v_1"] \arrow[d, "\lambda_1''"'] & M_1 \arrow[d, "\rho_1"] \\ & X_0 \arrow[r, "\Delta_{X_0} tq"'] & X_1 \end{tikzcd}$$

We claim that $f' = l_1 v_1 f'' = \omega(f'')$. In fact, $k_1 f' = f = k_1 l_1 v_1 f''$, while by (66) we have $\lambda_1 f' = tq = tq\,tq = tq\lambda_1'' f'' = t\lambda_1' v_1 f'' = \lambda_1 l_1 v_1 f''$. Together these imply that $f' = l_1 v_1 f''$, as claimed. Thus ω is surjective.

Assume given $h, g \in \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}(X_0,\ q^*t^*j_1^*M_1)$ with $\omega(h) = \omega(g)$, (i.e., $l_1 v_1 h = l_1 v_1 g$). Then $k_1 l_1 v_1 h = k_1 l_1 v_1 g$ and $\lambda_1'' h = tq = \lambda_1'' g$, so $h = g$. This shows that ω is injective, and thus an isomorphism.

To see that $\omega q^* = \xi$, Let $h \in \mathrm{Hom}_{\mathsf{Cat}_{\mathcal{O}}/X_0}(\pi_0, t^*j_1^*M_1)$ be given by $f \top \mathrm{Id} : \pi_0 \to M_1 \times \pi_0$ in the pullback (27). Then $k_1 l_1 h = f$ and $\lambda_1' h = \mathrm{Id}$, and so $\rho_1 f q = \rho_1 k_1 l_1 h q = \Delta_{X_0} t \lambda_1' h q = \Delta_{X_0} tq = \Delta_{X_0}(tq)(tq)$. Hence there is a map h' making

$$\begin{tikzcd} X_0 \arrow[dr, "h'"] \arrow[drr, bend left, "fq"] \arrow[ddr, bend right, "tq"'] & & \\ & q^*t^*j^*M_1 \arrow[r, "k_1 l_1 v_1"] \arrow[d, "\lambda_1''"'] & M_1 \arrow[d, "\rho_1"] \\ & X_0 \arrow[r, "\Delta_{X_0} tq"'] & X_1 \end{tikzcd}$$

commute, and $h' = q^*h$, so that $\omega(q^*h) = \omega(h') = l_1 v_1 h'$. Moreover, $k_1 l_1 v_1 h' = fq = k_1 l_1 hq$ and

$$\Delta_{X_0} t \lambda_1' v_1 h' = \Delta_{X_0} \lambda_1 l_1 v_1 h' = \rho_1 k_1 l_1 v_1 h' = \rho_1 fq = \Delta_{X_0} tq\, tq = \Delta_{X_0} tq$$

so $\lambda_1' v_1 h' = q \Delta_{X_0} t \lambda_1' v_1 h' = q \Delta_{X_0} tq = q = \lambda_1' hq$, since $q \Delta_{X_0} t = \mathrm{Id}$ and $\lambda_1' h = \mathrm{Id}$. This implies that $v_1 h' = hq$. We deduce that $\omega q^*(h) = l_1 hq = \xi(h)$, so $\omega q^* = \xi$. $\square$

Proposition 5.3 *For any $X = HY$ as in (9) and X-module M, there is a short exact sequence*

$$\begin{aligned} 0 \to \pi_1 \operatorname{map}_{\mathit{Track}_{\mathcal{O}}/X}(X, M) &\xrightarrow{j^*} \pi_1 \operatorname{map}_{\mathit{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(dX_0, j^*M) \\ &\xrightarrow{\vartheta'} \operatorname{Hom}_{\mathit{Track}_{\mathcal{O}}/X}(X, M) \to 0 \end{aligned} \tag{67}$$

Proof This follows by taking $\mathcal{C} = \mathsf{Cat}_{\mathcal{O}}$ in Proposition 3.5, together with (39) and the top right corner of (63) identified with $\pi_1 \operatorname{map}_{\mathsf{Track}_{\mathcal{O}}/\,\mathrm{d}X_0}(dX_0, j^*M)$. $\square$

Let $X \in \mathsf{Track}_{\mathcal{O}}$ and $M \in [(\mathsf{Track}_{\mathcal{O}}, X_0)/X]_{\mathrm{ab}}$ and

$$\bar{I}\mathcal{K}_\bullet X \in [\Delta^{\mathrm{op}}, (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}\,/IX]$$

be as in Sect. 4.3. Note that the augmentation $\varepsilon : \mathcal{K}_\bullet X \to X$ can be thought of as a map to the constant simplicial object, so a compatible sequence of maps $\varepsilon_n : \mathcal{K}_n X \to X$ in $\mathsf{Track}_{\mathcal{O}}$ allowing us to pull back M to $\varepsilon_n^* M$.

Definition 5.4 For each $X \in \mathsf{Track}_{\mathcal{O}}$ and $M \in [(\mathsf{Track}_{\mathcal{O}}, X_0)/X]_{\mathrm{ab}}$, the comonad cohomology of X with coefficients in M is defined by

$$H^s_{\mathrm{C}}(X, M) = \pi^s \operatorname{Hom}_{\mathsf{Track}_{\mathcal{O}}/X}(\mathcal{K}_\bullet X, M).$$

Theorem 5.5 *Assume given $X \in \mathsf{Track}_{\mathcal{O}}$ and $M \in [(\mathsf{Track}_{\mathcal{O}}, X_0)/X]_{\mathrm{ab}}$. Then there is a long exact sequence of abelian groups*

$$\begin{aligned} \cdots \to H^n_{\mathrm{SO}}(IX; M) \to H^n_{\mathrm{SO}}(I\,\mathrm{d}X_0; j^*M) \to H^n_{\mathrm{C}}(X, M) \to \\ \to H^{n+1}_{\mathrm{SO}}(IX; M) \to \cdots \end{aligned}$$

Proof By Remark 4.4. $\mathcal{K}_n X$ is homotopically discrete for each n, and we can choose a splitting $\Pi_0 \mathcal{K}_n X \xrightarrow{t_n} \mathcal{K}_n X$ to $\mathcal{K}_n X \xrightarrow{q_n} \Pi_0 \mathcal{K}_n X$, because $\mathcal{K}_n X = LA$ with $A = FUVX_{n-1}$ and $q_n : A_s \coprod A_t \to A$ (see Sect. 2.2). By Proposition 5.3 there is a short exact sequence of cosimplicial abelian groups

$$\begin{aligned} 0 \to \pi_1 \operatorname{map}_{\mathsf{Track}_{\mathcal{O}}/X}(I\mathcal{K}_\bullet X, M) &\xrightarrow{j^*} \pi_1 \operatorname{map}_{\mathsf{Track}_{\mathcal{O}}/dX_0}(I\mathrm{d}(\mathcal{K}_\bullet X)_0, j^*M) \to \\ &\to \operatorname{Hom}_{\mathsf{Track}_{\mathcal{O}}/X}(\mathcal{K}_\bullet X, M) \to 0 \end{aligned}$$

where we use the augmentation $\varepsilon_\bullet : \mathcal{K}_\bullet X \to X$ to pull back M to $\varepsilon_\bullet M$.

We therefore obtain a corresponding long exact sequence

$$\to \pi^s\pi_1 \,\mathrm{map}_{\mathsf{Track}_\mathcal{O}/X}(\mathcal{K}_\bullet X, M) \xrightarrow{j_*} \pi^s\pi_1 \,\mathrm{map}_{\mathsf{Track}_\mathcal{O}/dX_0}(\mathrm{d}(\mathcal{K}_\bullet X)_0, j^*M) \to \\ \to \pi^s \,\mathrm{Hom}_{\mathsf{Track}_\mathcal{O}/X}(\mathcal{K}_\bullet X, M) \to \cdots .$$

By definition, $\pi^s \,\mathrm{Hom}_{\mathsf{Track}_\mathcal{O}/X}(\mathcal{K}_\bullet X, M) = H^s_C(X, M)$, with

$$\pi^s\pi_1 \,\mathrm{map}_{\mathsf{Track}_\mathcal{O}/X}(\bar{I}\mathcal{K}_\bullet X, M) = H^s_{\mathrm{SO}}(IX; M)$$

and $\pi^s\pi_1 \,\mathrm{map}_{\mathsf{Track}_\mathcal{O}/\,\mathrm{d}X_0}(\mathrm{d}(\mathcal{K}_\bullet X)_0, j^*M) = H^s_{\mathrm{SO}}(I\mathrm{d}X_0; j^*M)$ by Theorem 4.9 and Corollary 4.10. □

Corollary 5.6 *For $X \in \mathsf{Track}_\mathcal{O}$ with X_0 a free category and $M \in [(\mathsf{Track}_\mathcal{O}, X_0)/X]_{\mathrm{ab}}$*

$$H^{n+1}_{\mathrm{SO}}(IX; M) \cong H^n_{\mathrm{C}}(X, M) \tag{68}$$

for each $n \geq 1$.

Proof Recall from [2, Theorem 3.10] that $H^n_{\mathrm{SO}}(I\mathrm{d}X_0; j^*M) \cong H^{n+1}_{\mathrm{BW}}(IX_0, j^*M)$, and, since X_0 is free, $H^{n+1}_{\mathrm{BW}}(IX_0, j^*M) = 0$ by [3, Theorem 6.3]. Thus the long exact sequence of Theorem 5.5 yields (68) for each $n \geq 1$. □

Lemma 5.7 *There is a functor $S : \mathsf{Track}_\mathcal{O} \to \mathsf{Track}_\mathcal{O}$ such that $(s_X)_0$ is a free category, for each $X \in \mathsf{Track}_\mathcal{O}$, with a natural 2-equivalence $s_X : S(X) \to X$*

Proof Given $X \in \mathsf{Gpd}\,\mathcal{C}$ and a map $f_0 : Y_0 \to X_0$ in $\mathcal{C}$, consider the pullback

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, "f_1"'] & Y_0 \times Y_0 \arrow[d, "f_0\times f_0"] \\ X_1 \arrow[r, "(\partial_0,\partial_1)"'] & X_0 \times X_0 \end{tikzcd} \tag{69}$$

in $\mathcal{C}$. Then there is $X(f_0) \in \mathsf{Gpd}\,\mathcal{C}$ with $(X(f_0))_0 = Y_0$ and $X(f_0))_1 = Y_1$, such that $(f_0, f_1) : X(f_0) \to X$ is an internal functor.

Now let $\varepsilon_{X_0} : FVX_0 \to X_0$ be the counit of the adjunction $F \dashv V$ of Sect. 4.2, and let $SX := X(\varepsilon_{X_0})$, where $(\varepsilon_{X_0}) \in \mathsf{Track}_\mathcal{O}$ is the construction (69). Then $(SX)_0 = FVX_0$ is a free category, and there is a map $s_X : SX \to X$ in $\mathsf{Track}_\mathcal{O}$. We wish to show that it is a 2-equivalence.

Since s_X is the identity on objects, to it suffices to show that for each $a, b \in \mathcal{O}$, the map $s_X(a, b) : S(X)(a, b) \to X(a, b)$ is an equivalence of categories. The pullback

$$\begin{tikzcd} (S(X))_1 \arrow[r] \arrow[d, "s_{X1}"'] & FVX_0 \times FVX_0 \arrow[d, "\varepsilon_{X_0}\times\varepsilon_{X_0}"] \\ X_1 \arrow[r] & X_0 \times X_0 \end{tikzcd} \tag{70}$$

in Cat induces a pullback of sets

$$\begin{tikzcd}
\{S(X)(a,b)\}_1 \arrow[r] \arrow[d, "\{s_X(a,b)\}_1"'] & \{FVX_0(a,b) \times FVX_0(a,b)\}_1 \arrow[d, "\varepsilon_{X_0} \times \varepsilon_{X_0}"] \\
\{X(a,b)\}_1 \arrow[r] & \{X_0(a,b) \times X_0(a,b)\}_1
\end{tikzcd}$$

Thus for each $(c,d) \in \{FVX_0(a,b) \times FVX_0(a,b)\}_1$, the map $\{s_X(a,b)\}(c,d)$ is a bijection. Thus $s_X(a,b)$ is fully faithful. Since $(FVX_0)_1 \to X_{01}$ is surjective, as is $(FVX_0)(a,b) \to X_0(a,b)$, $s_X(a,b)$ is surjective on objects, so it is an equivalence of categories. $\square$

We finally use our previous results to conclude that the $(\mathcal{S},\mathcal{O})$-cohomology of a track category can always be calculated from a comonad cohomology.

Corollary 5.8 *For $X \in \mathsf{Track}_{\mathcal{O}}$, and M an X-module, $H^{n+1}_{\mathrm{SO}}(IX; M) \cong H^n_C(S(X), M)$ for each $n > 1$, where $S(X)$ is as in Lemma 5.7.*

Proof By Lemma 5.7 the map $s_X : S(X) \to X$ is a 2-equivalence in $\mathsf{Track}_{\mathcal{O}}$, since $(s_X)_0 = \varepsilon_{X_0}$ is bijective on objects. Hence Is_X is a Dwyer–Kan equivalence in $(\mathcal{S},\mathcal{O})$-$\mathsf{Cat}$, so $H^n_{\mathrm{SO}}(IX; M) \cong H^n_{\mathrm{SO}}(IS(X); M)$. By Lemma 5.7, SX satisfies the hypotheses of Corollary 5.6, so also $H^{n+1}_{\mathrm{SO}}(IS(X); M) \cong H^n_{\mathrm{C}}(S(X), M)$. $\square$

6 The groupoidal case

A 2-groupoid is a special case of a track category in which all cells are (strictly) invertible. The category of such is denoted by $2\text{-}\mathsf{Gpd}_{\mathcal{O}} = \mathsf{Gpd}(\mathsf{Gpd}_{\mathcal{O}})$, with $i : 2\text{-}\mathsf{Gpd}_{\mathcal{O}} \hookrightarrow \mathsf{Track}_{\mathcal{O}}$ the full and faithful inclusion.

For $X \in 2\text{-}\mathsf{Gpd}_{\mathcal{O}}$ and $W = \overline{N}\,\overline{I}\mathcal{K}_\bullet X \in [\Delta^{3^{\mathrm{op}}}, \mathsf{Set}]$ as in Sect. 4.3, let $S = W^{(1)} \in [\Delta^{\mathrm{op}}, [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]]$ be W thought of as a simplicial object along the horizontal direction. Thus for each $i \geq 0$, S_i is the bisimplicial set

$$\begin{array}{ccccc}
\cdots & \Rrightarrow & (\mathcal{K}^{i+1}X)_{11} \times_{(\mathcal{K}^{i+1}X)_{10}} (\mathcal{K}^{i+1}X)_{11} & \rightrightarrows & \mathcal{O} \\
\downdownarrows\!\downarrow & & \downdownarrows\!\downarrow & & \downdownarrows\!\downarrow \\
(\mathcal{K}^{i+1}X)_{11} \times_{\mathcal{O}} (\mathcal{K}^{i+1}X)_{11} & \Rrightarrow & (\mathcal{K}^{i+1}X)_{11} & \rightrightarrows & \mathcal{O} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
(\mathcal{K}^{i+1}X)_{10} \times_{\mathcal{O}} (\mathcal{K}^{i+1}X)_{10} & \Rrightarrow & (\mathcal{K}^{i+1}X)_{10} & \rightrightarrows & \mathcal{O}
\end{array}$$

Applying $\mathrm{Diag} : [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}] = \mathcal{S}$ dimensionwise to $W^{(1)}$ we obtain $\overline{\mathrm{Diag}}S \in [\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$, with

$$\mathrm{Diag}^{(3)} W = \mathrm{Diag}\,\overline{\mathrm{Diag}}S = \mathrm{Diag}\,\overline{\mathrm{Diag}}Z \tag{71}$$

for $Z := W^{(2)} \in [\Delta^{\mathrm{op}}, [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]]$ as in (45).

Definition 6.1 The *classifying space* of $X \in 2\text{-}\mathsf{Gpd}_{\mathcal{O}}$ is $BX = \mathrm{Diag}\, N_{(2)} X$, where $N_{(2)} : 2\text{-}\mathsf{Gpd}_{\mathcal{O}} \to [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ is the double nerve functor.

Remark 6.2 By Lemma 4.5, $\overline{\mathrm{Diag}} Z \to IX$ is a Dwyer–Kan equivalence, where we think of $\overline{\mathrm{Diag}} Z$ as an $(\mathcal{S}, \mathcal{O})$-category by the discussion preceding the Lemma 4.5. Conversely, we may also think of IX as a bisimplicial set (implicitly, by applying the nerve functor in the category direction), with $(\overline{\mathrm{Diag}} Z)_0 = (IX)_0 = c(\mathcal{O})$ (cf. Sect. 1.1). Moreover, $(\overline{\mathrm{Diag}} Z)_j \to (IX)_j$ is a weak homotopy equivalence for all $j \geq 0$. Hence $\overline{\mathrm{Diag}} Z \to IX$ induces a weak homotopy equivalence of diagonals $\mathrm{Diag}\, \overline{\mathrm{Diag}} Z \simeq \mathrm{Diag}\, IX$. Since $\mathrm{Diag}\, IX = \mathrm{Diag}\, N_{(2)} X = BX$, by (71) we have

$$BX \simeq \mathrm{Diag}^{(3)} W. \tag{72}$$

The cohomology groups of $X \in 2\text{-}\mathsf{Gpd}_{\mathcal{O}}$ with coefficients in an X-module M are defined to be $H^{n-t}(BX, M) = \pi_t\, \mathrm{map}_{[\Delta^{\mathrm{op}}, \mathsf{Set}]}(BX, \mathcal{K}(M, n))$. By (72) this can be written as

$$\pi_t\, \mathrm{map}_{[\Delta^{\mathrm{op}}, \mathsf{Set}]}(\mathrm{Diag}^{(3)} W, \mathcal{K}(M, n)).$$

Lemma 6.3 *Given $\mathcal{C} \in \mathsf{Cat}_{\mathcal{O}}$, with $N\mathcal{C} \in \mathcal{S}$ viewed as a discrete $(\mathcal{S}, \mathcal{O})$-category, and M be a $\mathcal{C}$-module, we have $H^n(B\mathcal{C}, M) \cong H^n_{\mathrm{SO}}(N\mathcal{C}; M)$ for each $n \geq 0$.*

Proof Since $N\mathcal{C}$ is a discrete $(\mathcal{S}, \mathcal{O})$-category, we see that $\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}}(N\mathcal{C}, \mathcal{K}(M, n))$, is $\mathrm{map}_{\mathcal{S}}(N\mathcal{C}, \mathcal{K}(M, n))$, so $H^n_{\mathrm{SO}}(\mathcal{C}; M) = \pi_0\, \mathrm{map}_{\mathcal{S}}(N\mathcal{C}, \mathcal{K}(M, n)) = H^n(B\mathcal{C}, M)$. □

Proposition 6.4 *For $X \in 2\text{-}\mathsf{Gpd}_{\mathcal{O}}$ and M an X-module, $H^s(BX, M) = H^s_{\mathrm{SO}}(X; M)$ for any $s \geq 0$*

Proof By (71) and Lemma 2.6

$$\begin{aligned} \mathrm{map}_{\mathcal{S}}(\mathrm{Diag}^{(3)} W, \mathcal{K}(M, n)) &= \mathrm{map}_{\mathcal{S}}(\mathrm{Diag}\, \overline{\mathrm{Diag}} S, \mathcal{K}(M, n)) \\ &\cong \mathrm{Tot}\; \mathrm{map}_{\mathcal{S}}(\overline{\mathrm{Diag}} S, \mathcal{K}(M, n)). \end{aligned}$$

Therefore, the homotopy spectral sequence for $W^\bullet = \mathrm{map}_{[\Delta^{\mathrm{op}}, \mathsf{Set}]}(\overline{\mathrm{Diag}} S, \mathcal{K}(M, n))$, with $E_2^{s,t} = \pi^s \pi_t W^\bullet \Rightarrow \pi_{t-s}\, \mathrm{Tot}\, W^\bullet$, has $E_1^{s,t} = \pi_t\, \mathrm{map}((\overline{\mathrm{Diag}} S)_s, \mathcal{K}(M, n))$. But $(\overline{\mathrm{Diag}} S)_s = \mathrm{Diag}\, I\mathcal{K}^{s+1} X \simeq I\Pi_0 \mathcal{K}^{s+1} X$, since $\mathcal{K}^{s+1} X$ is homotopically discrete, so

$$E_1^{s,t} = H^{n-t}(BI\Pi_0 \mathcal{K}^{s+1} X, M) = H^{n-t}_{\mathrm{SO}}(I\Pi_0 \mathcal{K}^{s+1} X; M),$$

by Lemma 6.3. Since $\mathcal{K}^{s+1} X$ is free, by [2, Theorem 3.10] we have

$$E_1^{s,t} = H^{n-t}_{\mathrm{SO}}(I\Pi_0 \mathcal{K}^{s+1} X; M) = H^{n-t+1}_{\mathrm{BW}}(I\Pi_0 \mathcal{K}^{s+1} X, M) = 0$$

for $n \neq t$. Thus the spectral sequence collapses at the E_1-term and

$$H^s(BX, M) = \pi_{n-s}\,\mathrm{Tot}\,W^\bullet = \pi^s\pi_n W^\bullet = \pi^s\pi_n\,\mathrm{map}_{\mathcal{S}}(\overline{\mathrm{Diag}}S, \mathcal{K}(M,n)). \quad (73)$$

Since $(\overline{\mathrm{Diag}}S)_s \to I\Pi_0\mathcal{K}^{s+1}X$ is a weak equivalence for all $s \geq 0$,

$$\pi_n\,\mathrm{map}_{[\Delta^{\mathrm{op}},\mathsf{Set}]}(\overline{\mathrm{Diag}}S, \mathcal{K}(M,n)) = H^0(B\,\overline{\mathrm{Diag}}S, M) \cong H^0(B\,I\,\Pi_0\mathcal{K}_\bullet X, M)$$

Thus $H^s(BX, M) = \pi^s H^0(B\,I\,\Pi_0\mathcal{K}_\bullet X, M)$ by (73). On the other hand, in the proof of Theorem 4.9 we showed that $H^s_{\mathrm{SO}}(X; M) = \pi^s H^0_{\mathrm{SO}}(I\,\Pi_0\mathcal{K}_\bullet X; M)$, while by Lemma 6.3 we have $H^0(B\,\Pi_0\mathcal{K}_\bullet X, M) = H^0_{\mathrm{SO}}(I\,\Pi_0\mathcal{K}_\bullet X; M)$. It follows that $H^s(BX, M) = H^s_{\mathrm{SO}}(X; M)$. □

From Theorem 5.5 and Proposition 6.4 we deduce:

Corollary 6.5 *Any $X \in 2\text{-}\mathsf{Gpd}_{\mathcal{O}}$ and X-module M have a long exact sequence*

$$\to H^n(BX, M) \to H^n(BX_0, j^*M) \to H^n_C(X, M) \to H^{n+1}(BX, M) \to \cdots.$$

Remark 6.6 A 2-groupoid with a single object is an internal groupoid in the category of groups, equivalent to a crossed module. It can be shown that in this case the long exact sequence of Corollary 6.5 recovers [16, Theorem 13].

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