Equivariant Hochschild Cohomology

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"Do not judge me by my successes, judge me by how many times I fell down and got back up again."

Nelson Mandela

Abstract

In this thesis our goal is to develop the equivariant version of Hochschild cohomology. In the equivariant world there is given a group G which acts on objects. First naive object which can be considered is a G-algebra, that is, an associative algebra A on which G acts via algebra automorphisms. In our work we consider two more general situations. In the first case we develop a cohomology theory for oriented algebras and in the second case we develop a cohomology theory for Green functors.

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Contents

To my family

Chapter 1

Introduction

Homological algebra is a relatively young branch of mathematics, which was first used in algebraic topology in the early 20th century. During the period 1940-1955, homological algebra had become an independent branch of algebra. In 1956, Cartan and Eilenberg published their book entitled "Homological Algebra" [3], which was truly a revolution in the subject. Cartan and Eilenberg's book yet remains a main book of reference, and the subject became standard course material at many universities. They used derived functors, defined via projective and injective resolutions of modules to define and explore the cohomology theories for groups, associative algebras and Lie algebras. Nowadays, homological algebra is a fundamental tool in many areas of mathematics.

One of the main application of homological algebra is the classical cohomology of associative algebras invented by Hochschild [10] in 1945. It is a particular case of general machinery developed by Cartan and Eilenberg. Let A be an associative k-algebra and let M be an A-A-bimodule. The low dimensional groups ($n \leq 2$) have well known interpretations of classical algebraic structures such as derivations and extensions. Moreover, Gerstenhaber [6] observed that the second Hochschild cohomology group of a finite dimensional algebra $H^2(A, A)$ has a close connection to the deformation theory of A, that is, if $H^2(A, A) = 0$ then all deformations of A are trivial. By the work of A. Connes in 80's the Hochschild cohomology plays an important role in so called noncommutative differential geometry.

Our goal is to developed the equivariant version of Hochschild cohomology. In the equivariant world there is given a group G which acts on objects. The first naive object which can be consider is a G-algebra, that is, an associative algebra A on which G acts via algebra automorphisms.

In our work we consider two more general situations. In the first case we consider an oriented algebra.

Let G be a group and $\epsilon : G \to \{\pm 1\}$ be a group homomorphism. An *oriented algebra* is an associative algebra A equipped with a G-module structure

$$(g, a) \mapsto {}^g a,$$

satisfying the conditions

$$^g(a+b) = {}^g a + {}^g b$$

$$^{gh}a = {}^g({}^h a)$$

$$^g(ab) = \begin{cases} {}^g a {}^g b & \text{if } \varepsilon(g) = +1, \\ {}^g b {}^g a & \text{if } \varepsilon(g) = -1. \end{cases}$$

$$^g(1) = 1.$$

Hence oriented algebras are more general than G-algebras as well as algebras with involutions.

In the second case we consider a Green functor.

The theory of Mackey functors was originally initiated by Green [7] in the early 1970's and later developed by numerous authors in the last four decades (J. Green [13], A. Dress [4], P. Webb [21]). The notion of Mackey functors associated to a finite group G are a standard tool for studying representations of a finite group and its subgroups.

There are at least three equivalent definitions of Mackey functors for a group G. The first definition which is due to Green [7] is based on the poset of subgroups of G. The second definition which is due to Dress [4] uses the category of G-sets. The third one is given by Thévenaz and Webb in [20]. They define Mackey functors as modules over the Mackey algebra.

Roughly speaking, a Green functor for a finite group G over the commutative ring R is a Mackey functor with a compatible ring structure. More specifically, there are two equivalent definitions of Green functors. The first definition which is due to Green [7] relies on the poset of subgroups of G. The second definition is analogous to the Dress definition of Mackey functors which is based on the category of G-sets, and is detailed in [2].

Roughly, there are analogies between Mackey functors and abelian groups as follows:

$$\mathcal{M}ack(G) \longleftrightarrow Ab,$$

$$\text{Green functors} \longleftrightarrow \text{ring with unit},$$

$$\text{modules over a Green functor } A \longleftrightarrow \text{modules over a ring } R,$$

$$\otimes,\ \mathcal{HOM} \longleftrightarrow \otimes,\ \text{Hom}.$$

In fact, if G is trivial, then Mackey functors and Green functors are nothing than ordinary abelian groups and rings.

1.1 Thesis outline

In Chapter 2, we give a brief overview of some of the fundamental terminologies of homological algebra and deformation theory. The material of this chapter is taken from [19], [15], [22], [3], [9] and [5]. We start section 1 by giving several basic definitions and basic properties of chain complexes. Then in section 2, we give the definition of homotopy of chain complexes and some of its basic properties. In section 3, we outline the construction of the derived functors Ext^n and Tor_n using projective and injective resolutions. In section 4, we provide a short overview of the homology and cohomology of groups. Then in section 5, we remind the reader of the Hochschild homology and cohomology groups of an associative algebras. In section 6, we look at spectral sequences. The last section of Chapter 2 provides an overview of the deformation theory.

Chapter 2 only provides familiar background material which will be required later. It does not contain any original work. The original work can be found in the remaining chapters of the thesis.

In Chapter 3, we introduce oriented algebras. We start Chapter 3 by stating and defining some notations for the standard chain complexes associated to groups and associative algebras. We also introduce a bicomplex which we will use through the whole Chapter 3. The main references for this part of the chapter 3 is [22]. We then define oriented algebras and give some examples. In section 2, we describe the construction relies on the possibility to mix standard chain complexes computing group and associative algebra cohomologies. In section 3, we extend the well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras to oriented algebras. In section 4, we prove some important results about such cohomologies. In the last section of Chapter 3 we extend the deformation theory of associative algebras due to Gerstenhaber [6] to oriented algebras.

In Chapter 4, we study the Hochschild cohomology of Green functors. Throughout this chapter, we restrict attention to the case when G is a cyclic group C_p of prime order p and for the general case see chapter 5. In section 1, we give the definitions of G-Mackey functors and provide some examples. Then in section 2, we provide the definitions of G-Green functors and present some examples. In section 3, we give a detailed description of C_p-Mackey functors, C_p-tensor product and C_p-$\mathcal{HOM}$. In section 4, we provide a detailed description of C_p-Green functors and modules over C_p-Green functors and we end this section by extending the definitions of Hochschild homology and Hochschild cohomology to C_p-Mackey functors. Finally, we extend the well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras to C_p-Green functors in section 5. We finish this chapter by extending the deformation theory of associative algebras due to Gerstenhaber [6] to C_p-Green functors.

The aim of Chapter 5 is to generalise the results of Chapter 4 to an arbitrary finite group G.

Chapter 2

Preliminaries

In this chapter, we provide familiar background material that is necessary for understanding what comes later. The material in this section can be found in many good books, including [19], [15], [22], [3], [9] and [5].

2.1 Chain Complexes of R-modules

The terminologies of complexes originally began in algebraic topology. Complexes have been utilised in several branches of mathematics giving us different (co)homology theories. In this section, we present some concepts concerning chain complexes. The main references for this section are [19], [15] and [22].

Definition 2.1.1. [22] *A chain complex (C,d) of R-modules is a family* $\{C_n, d_n\}_{n\in\mathbb{Z}}$ *of R-modules* C_n *and R-module maps* $d_n : C_n \longrightarrow C_{n-1}$ *such that* $d_{n-1} \circ d_n = 0$ *for all* $n \in \mathbb{Z}$. *The maps* d_n *are called the differential maps or boundary maps. The n-cycles are the elements of kernel of* $d_n : C_n \longrightarrow C_{n-1}$, *denoted by* Z_n. *The n-boundaries are the elements of image of* $d_{n+1} : C_{n+1} \longrightarrow C_n$, *denoted by* B_n. *Note that the condition* $d_{n-1} \circ d_n = 0$ *indicates that* $B_n \subset Z_n$ *for all* $n \in \mathbb{Z}$. *The* n^{th}*-homology module of C is defined by*

$$H_n(C) = Z_n/B_n.$$

Definition 2.1.2. [22] *Dually, a cochain complex (C,d) of R-modules is a family* $\{C^n, d^n\}_{n\in\mathbb{Z}}$ *of R-modules* C^n *and R-module maps* $d^n : C^n \longrightarrow C^{n+1}$ *such that* $d^{n+1} \circ d^n = 0$ *for all* $n \in \mathbb{Z}$. *The maps* d^n *are called the differential maps or* *coboundary maps. The n-cocycles are the elements of kernel of* $d^n : C^n \longrightarrow C^{n+1}$, *denoted by* Z^n. *The n-coboundaries are the elements of image of* $d^{n-1} : C^{n-1} \longrightarrow C^n$, *denoted by* B^n. *Note that the condition* $d^{n+1} \circ d^n = 0$ *means that* $B^n \subset Z^n$ *for all* $n \in \mathbb{Z}$. *The* n^{th}*-cohomology module of C is defined by*

$$H^n(C) = Z^n/B^n.$$

Definition 2.1.3. [22] *Let (C,d) and (D,d′) be two chain complexes. Then a chain map* $f : (C, d) \longrightarrow (D, d')$ *is a family of R-modules homomorphisms* $\{f_n : C_n \longrightarrow D_n\}_{n\in\mathbb{Z}}$ *such that* $f_{n-1} \circ d_n = d'_n \circ f_n$ *for all* $n \in \mathbb{Z}$. *That is, the last condition means that the following diagram commutes.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d'_{n+1}"'] & D_n \arrow[r, "d'_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Similarly, let (C,d) and (D,d′) be two cochain complexes. Then a cochain map $f : (C, d) \longrightarrow (D, d')$ *is a family of R-modules homomorphisms* $\{f^n : C^n \longrightarrow D^n\}_{n\in\mathbb{Z}}$ *such that* $f^{n+1} \circ d^n = d'^n \circ f^n$ *for all* $n \in \mathbb{Z}$. *That is, the last condition means that the following diagram commutes.*

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"] & C^n \arrow[r, "d^n"] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "d'^{n-1}"'] & D^n \arrow[r, "d'^n"'] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.1.4. [22] *A sequence*

$$\cdots \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \to \cdots$$

of R-modules and R-module maps is said to be exact at C_n *if* $Ker(f_n) = Im(f_{n+1})$. *The sequence is said to be exact if it is exact at all* C_n.

Similarly, a sequence

$$\cdots \to C^{n-1} \xrightarrow{f^{n-1}} C^n \xrightarrow{f^n} C^{n+1} \to \cdots$$

of R-modules and R-module maps is said to be exact at C^n *if* $Ker(f^n) = Im(f^{n-1})$. *The sequence is said to be exact if it is exact at all* C^n.

Definition 2.1.5. [22] *A short exact sequence is a sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where f is a monomorphism, g is an epimorphism and $Ker(g) = Im(f)$.

Definition 2.1.6. [22] *A short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called split if there exists a map $h : C \longrightarrow B$ such that $gh = id_C$.

Example 2.1.7. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-modules. The sequence is split if $B = A \oplus C$ up to isomorphism.

The following theorem is one of the fundamental results on chain complexes and the proof can be found in [22].

Theorem 2.1.8. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of chain complexes. Then there exist natural maps $\partial : H_n(C) \longrightarrow H_{n-1}(A)$, called connecting homomorphisms, such that

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.
Similarly, if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence of cochain complexes, then there are natural maps $\partial : H^n(C) \longrightarrow H^{n-1}(A)$ such that

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

Definition 2.1.9. [22] *A bicomplex (or double complex) is a family $\{C_{p,q}\}$ of modules together with a horizontal differential $d^h : C_{p,q} \longrightarrow C_{p-1,q}$ and a vertical* *differential* $d^v : C_{p,q} \longrightarrow C_{p,q-1}$ *such that* $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$ *for all* $p, q \in \mathbb{Z}$*. It is useful to draw the bicomplex as a lattice:*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C_{02} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{12} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{22} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
C_{01} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{11} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{21} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
C_{00} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{10} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{20} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots
\end{tikzcd}$$

where each column and each row is a chain complex and each square anticommutes.

Similarly, a bicomplex (or double complex) is a family $\{C^{p,q}\}$ *of modules together with a horizontal differential* $d^h : C^{p,q} \longrightarrow C^{p+1,q}$ *and a vertical differential* $d^v : C^{p,q} \longrightarrow C^{p,q+1}$ *such that* $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$ *for all* $p, q \in \mathbb{Z}$*. It is useful to draw the bicomplex as a lattice:*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C_{02} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{12} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{22} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
C_{01} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{11} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{21} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
C_{00} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{10} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{20} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots
\end{tikzcd}$$

where each column and each row is a cochain complex and each square anticommutes.

Definition 2.1.10. [22] *The total complexes Tot(C)* $= Tot^{\prod}(C)$ *and* $Tot^{\oplus}(C)$ *of a chain complex C are defined by*

$$Tot^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \qquad \text{and} \qquad Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The differential maps are given by $d = d^h + d^v$*. We note that if* C *is bounded then* $Tot^{\prod}(C) = Tot^{\oplus}(C)$*, especially if* C *is a first quadrant bicomplex. We denoted the homology modules of the bicomplex (C) by* $H_n(Tot(C))$*.*

Similarly, the total complexes Tot(C) $= Tot^{\Pi}(C)$ *and* $Tot^{\oplus}(C)$ *of a cochain complex* C *are defined by*

$$Tot^{\Pi}(C)^n = \prod_{p+q=n} C^{p,q} \qquad \text{and} \qquad Tot^{\oplus}(C)^n = \bigoplus_{p+q=n} C^{p,q}.$$

The differential maps are given by $d = d^h + d^v$. *We note that if* C *is bounded then* $Tot^{\Pi}(C) = Tot^{\oplus}(C)$, *especially if* C *is a first quadrant bicomplex. We denoted the cohomology modules of the bicomplex (C) by* $H^n(Tot(C))$.

2.2 Chain Homotopies

In this section, we state the definition of the homotopy of chain complexes and present some of its properties. The main reference for this section is [22].

Definition 2.2.1. [22] *A chain map* $f : C \longrightarrow D$ *is said to be null-homotopic if there are maps* $t_n : C_n \longrightarrow D_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[dl, "t_n"'] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[dl, "t_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d'_{n+1}"'] & D_n \arrow[r, "d'_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that

$$f = d'_D t + t d_C$$

The maps $\{t_n\}$ *are called a chain contraction of* f.

Definition 2.2.2. [22] *Two chain maps* f *and* $g : C \longrightarrow D$ *are chain homotopic, written* $f \simeq g$, *if* $f - g$ *is null homotopic, that is, there are maps* $t_n : C_n \longrightarrow D_{n+1}$ *such that*

$$f - g = d'_D t + t d_C.$$

Chain homotopy is an equivalence relation.

Definition 2.2.3. [22] *A map* $f : C \longrightarrow D$ *is a chain homotopy equivalence if there exists a map* $h : D \longrightarrow C$ *such that* $hf \simeq 1_C$, $fh \simeq 1_D$.

Lemma 2.2.4. *If* f *and* g *are chain homotopic, then they induce the same maps* $H_n(C) \longrightarrow H_n(D)$.

2.3 Derived Functors

Let R be a ring. A standard method of computing derived functors between categories of R-modules is by applying the functor to a resolution and then take the (co)-homology of the obtaining complex. In this section, we study derived functors. We define projective and injective modules, left derived functors, right derived functors, Ext and Tor functors. The following materials can be found in [22], [15], [19] and [3].

2.3.1 Projective and injective modules

In this subsection, we will state the definitions of the Projective and injective modules and present some properties.

Definition 2.3.1. [19] *An R-module P is projective if for any epimorphism $g : N \longrightarrow M$ and any map $f : P \longrightarrow M$, there exists a map $h : P \longrightarrow N$ such that $f = g \circ h$.*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\ N \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

The proof of the following result is in [19].

Proposition 2.3.2. *Let P be an R-module. The following are equivalent:*

1. *P is projective.*
2. *If*
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
is exact, then
$$0 \to Hom_R(P, A) \xrightarrow{f^*} Hom_R(P, B) \xrightarrow{g^*} Hom_R(P, C) \to 0$$
is also exact.
3. *Every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ splits.*

The proof of the following result is in [19].

Lemma 2.3.3. *A direct sum of R-modules $\bigoplus_{i\in I} P_i$ is projective if and only if each P_i is projective.*

There is a dual definition, obtained by reversing all the arrows and swapping surjective and injective.

Definition 2.3.4. [19] *An R-module I is injective if for any monomorphism $f : N \longrightarrow M$ and any map $g : N \longrightarrow I$, there exists a map $h : M \longrightarrow I$ such that $g = h \circ f$.*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

The proof of the following result is in [19].

Proposition 2.3.5. *Let I be an R-module. The following conditions are equivalent:*

1. *I is injective.*
2. *If*
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
is exact, then
$$0 \to Hom_R(C, I) \xrightarrow{g_*} Hom_R(B, I) \xrightarrow{f_*} Hom_R(A, I) \to 0$$
is also exact.
3. *Every short exact sequence $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits.*

The proof of the following result is in [19].

Lemma 2.3.6. *A direct product of R-modules $\prod_{i\in I} I_i$ is injective if and only if each I_i is injective.*

The proof of the following result is in [19].

Proposition 2.3.7. *Every R-module M can be embedded in an injective R-module.*

2.3.2 Projective and injective resolutions

In this subsection, we will define projective and injective resolutions and present some properties.

Definition 2.3.8. [19] *Let M be a R-module. A projective resolution $P = \{P_j\}$ of M is an exact sequence of R-modules*

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\theta} M \to 0.$$

such that all P_j are projective for all $j \geq 0$.

The proof of the following lemma is in [22].

Lemma 2.3.9. *Every R-module M has a projective resolution.*

The proof of the following important theorem is in [22].

Theorem 2.3.10. *Let $\boldsymbol{P}$ be a projective resolution of a module M and $\boldsymbol{Q}$ be a projective resolution of a module N. Then there is a chain map $f : \boldsymbol{P} \longrightarrow \boldsymbol{Q}$ making the completed diagram commute.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\mu"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "\theta"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"'] & Q_1 \arrow[r, "d_1"'] & Q_0 \arrow[r, "\pi"'] & N \arrow[r] & 0
\end{tikzcd}$$

Definition 2.3.11. [19] *Let M be a R-module. An injective resolution $I = \{I_j\}$ of M is an exact sequence of R-modules*

$$0 \to M \xrightarrow{\lambda} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \to \cdots .$$

such that all I_j are injective for all $j \geq 0$.

The proof of the following lemma is exactly dual to that of lemma 2.3.9.

Lemma 2.3.12. *Every R-module N has an injective resolution.*

The proof of the following important theorem is exactly dual to that of theorem 2.3.10.

Theorem 2.3.13. *Let $\boldsymbol{I}$ be an injective resolution of a module N and $\boldsymbol{H}$ be an injective resolution of a module M. Then there is a cochain map $f : \boldsymbol{H} \longrightarrow \boldsymbol{I}$ making the completed diagram commute.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\lambda"] \arrow[d, "\epsilon"] & H^0 \arrow[r, "d_0"] \arrow[d, "f_0"] & H^1 \arrow[r, "d_1"] \arrow[d, "f_1"] & H^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & \cdots \\
0 \arrow[r] & N \arrow[r, "\delta"'] & I^0 \arrow[r, "d_0"'] & I^1 \arrow[r, "d_1"'] & I^2 \arrow[r, "d_2"'] & \cdots
\end{tikzcd}$$

2.3.3 Left derived functors

In this subsection, we construct the Left derived functors and present its properties. The construction proceeds as follows.
Let A, B be commutative rings, and A-*mod*, B-*mod* denote the category of A-*modules* and B-*modules*, respectively. Let $F : A\text{-}mod \longrightarrow B\text{-}mod$ be a right exact covariant additive functor. Let M be an A-*module* and

$$\mathbf{P} \longrightarrow M \longrightarrow 0$$

be a projective resolution for M. Then, by applying F to $\mathbf{P}$ we obtain a sequence of B-*modules*

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0.$$

Definition 2.3.14. [3] *The n^{th} left derived functor of F, denoted by L_nF, is defined by*

$$L_nF(M) = H_n(F(\boldsymbol{P}))$$

for $n \geq 0$.

Note that L_nF is independent of the choice of projective resolution of M and we always obtain $L_0F(M) \cong F(M)$ since $F(P_1) \to F(P_0) \to F(M) \to 0$ is exact. Moreover, if M is projective then $L_nF(M) = 0$ for $n > 0$.
The proof of the following theorem is in [22].

Theorem 2.3.15. *The functors L_nF are additive.*

Theorem 2.3.16. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of modules. Then there are connecting morphisms

$$L_{n+1}F(C) \longrightarrow L_nF(A)$$

such that

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0.$$

is a long exact sequence.

For the proof of the above theorem, see [19].

2.3.4 Right derived functors

In this subsection, we construct the Right derived functors and present its properties.
The construction proceeds as follows.
Let A, B be commutative rings, and A-*mod*, B-*mod* denote the category of A-*modules* and B-*modules*, respectively. Let $F : A\text{-}mod \longrightarrow B\text{-}mod$ be a left exact covariant additive functor. Let M be an A-*module* and

$$0 \longrightarrow M \longrightarrow \mathbf{I}$$

be an injective resolution for M. Then, by applying F to $\mathbf{I}$ we obtain a sequence of B-*modules*

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

Definition 2.3.17. [3] *The n^{th} right derived functor of F, denoted by R^nF, is defined by*

$$R^nF(M) = H^n(F(\boldsymbol{I}))$$

for $n \geq 0$.

Note that R^nF is independent of the choice of injective resolution of M and we always obtain $R^0F(M) \cong F(M)$ since $0 \to F(M) \to F(I^0) \to F(I^1)$ is exact. Moreover, if M is injective then $R^nF(M) = 0$ for $n > 0$.
The proof of the following theorem is in [19].

Theorem 2.3.18. *The functors R^nF are additive.*

The proof of the following theorem is similar to that of theorem 2.3.16.

Theorem 2.3.19. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Then there are connecting morphisms

$$R^n F(C) \longrightarrow R^{n+1} F(A)$$

such that

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \cdots .$$

is a long exact sequence.

Remark 2.3.20. If F is a contravariant right or left exact functor, then we can construct left or right derived functors in a similar way. The only difference is that the left derived functors are computed by using an injective resolution, whilst the right derived functors are computed by using a projective resolution.

2.3.5 Ext and Tor

In this subsection, we present the most common examples of derived functors which are the functors Ext^n and Tor_n. We deal with Tor_n first.

Definition 2.3.21. [3] *Let R be a ring and let N be a left R-module. The functor $F(-) = - \otimes_R N$ is a covariant additive right exact functor. For $n \geq 0$ we define*

$$Tor_n^R(-, N) = L_n(- \otimes_R N).$$

We state a few basic properties of these functors:

1. One has $Tor_0^R(-, N) \cong (- \otimes_R N)$.
2. For any projective module M we have $Tor_n^R(M, N) = 0$ for $n \neq 0$.
3. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then there is a long exact sequence.
   $\cdots \to Tor_2(M', N) \to Tor_2(M, N) \to Tor_2(M'', N) \to$
   $\to Tor_1(M', N) \to Tor_1(M, N) \to Tor_1(M'', N) \to$
   $\to (M' \otimes_R N) \to (M \otimes_R N) \to (M'' \otimes_R N) \to 0.$

Definition 2.3.22. [3] *Let R be a ring and let M be a right R-module. The functor $F(-) = M \otimes_R -$ is a contravariant additive right exact functor. For $n \geq 0$ we define*

$$Tor_n^R(M, -) = L_n(M \otimes_R -).$$

Similarly, we state a few basic properties of these functors:

1. One has $Tor_0^R(M, -) \cong (M \otimes_R -)$.
2. For any projective module N we have $Tor_n^R(M, N) = 0$ for $n \neq 0$.
3. If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules, then there is a long exact sequence.
   $\cdots \to Tor_2(M, N') \to Tor_2(M, N) \to Tor_2(M, N'') \to$
   $\to Tor_1(M, N') \to Tor_1(M, N) \to Tor_1(M, N'') \to$
   $\to (M \otimes_R N') \to (M \otimes_R N) \to (M \otimes_R N'') \to 0.$

Next, we deal with Ext^n.

Definition 2.3.23. [3] *Let R be a ring and let M be a left R-module. The functor $F(-) = Hom_R(M, -)$ is a covariant additive left exact functor. For $n \geq 0$ we define*

$$Ext_R^n(M, -) = R^n(Hom_R(M, -)).$$

We state a few basic properties of these functors:

1. One has $Ext_R^0(M, -) \cong Hom_R(M, -)$.
2. For any injective module N we have $Ext_R^n(M, N) = 0$ for $n \neq 0$.
3. If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules, then there is a long exact sequence.
   $0 \to Hom_R(M, N') \to Hom_R(M, N) \to Hom_R(M, N'') \to$
   $\to Ext^1(M, N') \to Ext^1(M, N) \to Ext^1(M, N'') \to$
   $\to Ext^2(M, N') \to Ext^2(M, N) \to Ext^2(M, N'') \to \cdots .$

Definition 2.3.24. [3] *Let R be a ring and let N be a left R-module. The functor $F(-) = Hom_R(M, -)$ is a contravariant additive left exact functor. For $n \geq 0$ we define*

$$Ext_R^n(-, N) = R^n(Hom_R(-, N)).$$

We state a few basic properties of these functors:

1. One has $Ext_R^0(-, N) \cong Hom_R(-, N)$.
2. For any injective module M we have $Ext_R^n(M, N) = 0$ for $n \neq 0$.
3. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then there is a long exact sequence.
   $0 \to Hom_R(M'', N) \to Hom_R(M, N) \to Hom_R(M', N) \to$
   $\to Ext^1(M'', N) \to Ext^1(M, N) \to Ext^1(M', N) \to$
   $\to Ext^2(M'', N) \to Ext^2(M, N) \to Ext^2(M', N) \to \cdots$.

We record a few observations relating $\otimes$ and Hom of complexes, starting with relations between $\otimes$ and Hom on the category of *R-modules*. For *R-modules* L, M, and N, we have an adjunction

$$Hom(L \otimes M, N) \cong Hom(L, Hom(M, N)).$$

We also have a natural homomorphism

$$Hom(L, M) \otimes N \longrightarrow Hom(L, M \otimes N),$$

and this is an isomorphism if either L or N is a finitely generated projective *R-module*. Again, we have a natural map

$$Hom(L, M) \otimes Hom(L', M') \longrightarrow Hom(L \otimes L', M \otimes M'),$$

which is an isomorphism if L and L' are finitely generated and projective or if L is finitely generated and projective and $M = R$.

2.4 Group Homology and Cohomology

Homology and cohomology are concepts that are utilised in many areas of algebra and topology. Historically, the terminologies of homology and cohomology were first used in a topological sense. Algebraically, we can define the homology and cohomology via derived functors, for examples the Tor and Ext functors. In this section, we present several basic definitions and basic properties of group homology and cohomology. The following material can be found in [22] and [9].

2.4.1 Definitions via Ext and Tor groups

We start by giving the basic algebraic objects of group (co)homology which are group rings and modules over group rings.

Definition 2.4.1. [9] *Let G be a group. The group ring $\mathbb{Z}G$ is the free $\mathbb{Z}$-module with elements of G as basis and with multiplication determined by the multiplication in the group G. Thus, elements of $\mathbb{Z}G$ are formal sums*

$$\sum_{g\in G} \delta_g g,$$

where $\delta_g \in \mathbb{Z}$, and where $\delta_g = 0$ for all but finitely many $g \in G$, and the formula of multiplication of two general elements is given by

$$(\sum_{g\in G} \delta_g g)(\sum_{h\in G} \vartheta_h h) = \sum_{g,h\in G} (\delta_g \vartheta_h) gh.$$

Definition 2.4.2. [9] *Let A be $\mathbb{Z}$-module. Then, A may be regarded as a $\mathbb{Z}G$-module with trivial action, i.e. $ga = a$ for all $g \in G$ and for all $a \in A$.*

Definition 2.4.3. [22] *Let A be $\mathbb{Z}G$-module. The invariants A^G of A are the elements of the $\mathbb{Z}$-submodule,*

$$A^G = \{a \in A| \quad ga = a \quad \textit{for all} \quad g \in G, \textit{ and } \quad a \in A\}.$$

The coinvariants A_G of A are elements of the quotient $\mathbb{Z}$-module,

$$A_G = A/(ga - a \mid \quad g \in G, a \in A).$$

The proof of the following important result is in [22].

Lemma 2.4.4. *Let A be a $\mathbb{Z}G$-module. Then, there are isomorphisms*

$$A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$$

and

$$A^G \cong Hom_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Definition 2.4.5. [22] *Let G be a group and let A be $\mathbb{Z}G$-module. The n-th homology group of G with coefficients in A is the value at A of the n-th left derived* *functors:*

$$H_n(G;A) = L_n(-_G)(A);$$

by the lemma above,

$$H_n(G;A) \cong Tor_n^{\mathbb{Z}G}(\mathbb{Z}, A).$$

Similarly, The n-th cohomology group of G with coefficients in A is the value at A of the n-th right derived functors:

$$H^n(G;A) = R^n(-^G)(A);$$

by the lemma above,

$$H^n(G;A) \cong Ext^n_{\mathbb{Z}G}(\mathbb{Z}, A).$$

In particular, $H_0(G;A) = A_G$ *and* $H^0(G;A) = A^G$.

Definition 2.4.6. [22] *The augmentation ideal of* $\mathbb{Z}G$ *is the kernel* τ *of the ring homomorphism* $\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z}$ *such that*

$$\alpha(\sum_{g\in G} \delta_g g) = \sum_{g\in G} \delta_g.$$

Definition 2.4.7. *Let G be a finite group. We define the norm element N of the group ring* $\mathbb{Z}G$ *by the sum*

$$N = \sum_{g\in G} g.$$

2.4.2 Cyclic groups

In this subsection, we compute explicitly the (co)homology of cyclic groups.

2.4.2.1 Calculation

Let $G = C_m$ be the cyclic group of order m with generator ρ. The norm in $\mathbb{Z}C_m$ is the element $N = 1 + \rho + \rho^2 + \cdots + \rho^{m-1}$. So $0 = \rho^m - 1 = (\rho - 1)N$ in $\mathbb{Z}C_m$. We can form the free resolution of the trivial C_m-module $\mathbb{Z}$

$$\mathbb{Z} \xleftarrow{\theta} \mathbb{Z}C_m \xleftarrow{\rho-1} \mathbb{Z}C_m \xleftarrow{N} \mathbb{Z}C_m \xleftarrow{\rho-1} \mathbb{Z}C_m \xleftarrow{N} \cdots .$$

Indeed, since $\mathbb{Z}\cdot N = (\mathbb{Z}C_m)^{C_m}$ and $\tau = \{a \in \mathbb{Z}C_m : Na = 0\}$, there are the following exact sequences

$$0 \leftarrow \mathbb{Z}\cdot N \xleftarrow{N} \mathbb{Z}C_m \leftarrow \tau \leftarrow 0$$

and

$$0 \leftarrow \tau \xleftarrow{\rho-1} \mathbb{Z}C_m \leftarrow \mathbb{Z}\cdot N \leftarrow 0.$$

We obtain the periodic free resolution of $\mathbb{Z}$ by splicing these sequences together.

$$\begin{tikzcd}
& & & & & & 0 & & 0 \arrow[dl] & & 0 & & 0 \arrow[dl] \\
& & & & & & & \mathbb{Z}\cdot N \arrow[ul] \arrow[dl] & & & & \mathbb{Z}\cdot N \arrow[ul] \arrow[dl] & \\
0 & & \mathbb{Z} \arrow[ll] & & \mathbb{Z}C_m \arrow[ll, "\theta"'] & & \mathbb{Z}C_m \arrow[ll, "\rho-1"'] \arrow[dl, "\rho-1"] & & \mathbb{Z}C_m \arrow[ll, "N"'] \arrow[ul] & & \mathbb{Z}C_m \arrow[ll, "\rho-1"'] \arrow[dl, "\rho-1"] & & \cdots \arrow[ll, "N"'] \arrow[ul] \\
& & & & & \tau \arrow[ul] \arrow[dl] & & & & \tau \arrow[ul] \arrow[dl] & & & \\
& & & & 0 & & 0 \arrow[ul] & & 0 & & 0 \arrow[ul] & &
\end{tikzcd}$$

For any C_m-module A we then apply $-\otimes_{\mathbb{Z}C_m} A$ and $\mathrm{Hom}_{\mathbb{Z}C_m}(-, A)$ and taking homology and cohomology, we find the following result:

Theorem 2.4.8. *If A is a C_m-module for the cyclic group C_m, then*

$$H_n(C_m; A) = \begin{cases} A/(\rho-1)A & \text{if } n = 0 \\ A^{C_m}/NA & \text{if } n \text{ is odd} \\ \{a \in A : Na = 0\}/(\rho-1)A & \text{if } n \text{ is even} \end{cases}$$

$$H^n(C_m; A) = \begin{cases} A^{C_m} & \text{if } n = 0 \\ \{a \in A : Na = 0\}/(\rho-1)A & \text{if } n \text{ is odd} \\ A^{C_m}/NA & \text{if } n \text{ is even} \end{cases}$$

Example 2.4.9. *If $A = \mathbb{Z}$, then we find that*

$$H_n(C_m; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/m & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$H^n(C_m; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ \mathbb{Z}/m & \text{if } n \text{ is even} \end{cases}$$

2.4.3 The Bar Resolution

In this subsection, we will describe a particular resolution called the *bar resolution*. Consider $\mathbb{Z}$ as $\mathbb{Z}G$-module with trivial action of G. We shall now describe the following exact sequences of $\mathbb{Z}G$-modules.

$$0 \leftarrow \mathbb{Z} \xleftarrow{\theta} B_0 \xleftarrow{d_1} B_1 \xleftarrow{d_2} B_2 \leftarrow \cdots \tag{*}$$

and

$$0 \leftarrow \mathbb{Z} \xleftarrow{\theta} B_0^u \xleftarrow{d_1} B_1^u \xleftarrow{d_2} B_2^u \leftarrow \cdots \tag{**}$$

where the first exact sequence is called *normalised bar resolution* and the second exact sequence is called *unnormalised bar resolution*. Here B_0 and B_0^u are $\mathbb{Z}G$ and the map θ is given by

$$\mathbb{Z}G \longrightarrow \mathbb{Z}$$

$$g \longrightarrow 1.$$

For $n > 0$, B_n^u is the free $\mathbb{Z}G$-module generated by symbols $[g_1 \otimes \cdots \otimes g_n]$ with $g_i \in G$, while B_n is the free $\mathbb{Z}G$-module generated by symbols $[g_1|\cdots|g_n]$ with $g_i \in G - \{1\}$.

Definition 2.4.10. [22] *Define the differential $d : B_n^u \longrightarrow B_{n-1}^u$ (for $n > 0$) by*

$$\begin{aligned} d_n([g_1 \otimes \cdots \otimes g_n]) = g_1[g_2 \otimes \cdots \otimes g_n] \\ + \sum_{i=1}^{n-1} (-1)^i \, [g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n] \\ + (-1)^n [g_1 \otimes \cdots \otimes g_{n-1}]. \end{aligned}$$

Similarly, we define the differential $d : B_n \longrightarrow B_{n-1}$ (for $n > 0$) by

$$\begin{aligned} d_n([g_1|\cdots|g_n]) &= g_1[g_2|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1} (-1)^i \, [g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ &+ (-1)^n [g_1|\cdots|g_{n-1}]. \end{aligned}$$

Example 2.4.11.

$$d[g_1] = g_1[\ ] - [\ ]$$

$$d[g_1|g_2] = g_1[g_2] - [g_1 g_2] + [g_1]$$

$$d[g_1|g_2|g_3] = g_1[g_2g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2]$$

2.4.3.1 Homology

Let A be a right $\mathbb{Z}G$-module. Then, $H_n(G;A)$ is the homology of the following chain complex by applying $A \otimes_{\mathbb{Z}G} -$ to the sequence (*)

$$0 \to A \otimes_{\mathbb{Z}G} B_0 \xrightarrow{d_0} A \otimes_{\mathbb{Z}G} B_1 \xrightarrow{d_1} A \otimes_{\mathbb{Z}G} B_2 \to \cdots$$

In particular, we have that $H_1(G;\mathbb{Z})$ is the quotient of the free abelian group on the symbols $[g]$, $g \in G$, with relations that $[1] = 0$ and $[f] + [g] = [fg]$ for all $f, g \in G$. Thus, $H_1(G;\mathbb{Z}) = G/[G,G]$.

2.4.3.2 Cohomology

Let A be a left $\mathbb{Z}G$-module. Then, $H^n(G;A)$ is the cohomology of the following chain complexes by applying $Hom_{\mathbb{Z}G}(-, A)$ to the sequences (*) and (**)

$$0 \to Hom_{\mathbb{Z}G}(B_0, A) \xrightarrow{d^0} Hom_{\mathbb{Z}G}(B_1, A) \xrightarrow{d^1} Hom_{\mathbb{Z}G}(B_2, A) \to \cdots$$

$$0 \to Hom_{\mathbb{Z}G}(B_0^u, A) \xrightarrow{d^0} Hom_{\mathbb{Z}G}(B_1^u, A) \xrightarrow{d^1} Hom_{\mathbb{Z}G}(B_2^u, A) \to \cdots .$$

An *n-cochain* is a set map $f : G^n = G \times \cdots \times G \longrightarrow A$, denoted by $C^n(G;A)$, and we see that the elements of $Hom_{\mathbb{Z}G}(B_n^u, A)$ are just $C^n(G;A)$. A cochain f is *normalised* if $f(g_1, \cdots) = 0$ whenever some $g_i = 1$, where these elements are in $Hom_{\mathbb{Z}G}(B_n, A)$. The differential df is given by

$$\begin{aligned}(d^n f)(g_0, \cdots, g_n) = {} & g_0 f(g_1, \cdots, g_n) \\ & + \sum_{i=1}^{n-1} (-1)^i \, f(g_1, \cdots, g_i g_{i+1}, \cdots g_n) \\ & + (-1)^n f(g_0, \cdots, g_{n-1}).\end{aligned}$$

The n-cochains where $d^n f = 0$ are called *n-cocycles*, denoted by $Z^n(G;A) = Kerd^n$, and the n-cochains $d^n f$ are called *n-coboundaries*, denoted by $B^n(G;A) = Imd^{n-1}$. Thus, $H^n(G;A) = Z^n(G;A)/B^n(G;A)$.

Example 2.4.12. *We have that*

$$H^1(G, A) = Z^1(G, A)/B^1(G, A)$$

where

$$Z^1(G,A) = \{f : G \longrightarrow A \mid f(ab) = af(b)+f(a), \ \ for \ \ all \ \ a,b \in G\} = Der(G,A),$$

is the derivations of G in A, and

$$B^1(G,A) = \{f : G \longrightarrow A, a \in A \mid f(g) = ga - a\} = PDer(G,A),$$

is the Principal derivations of G in A.

Example 2.4.13. *We have that*

$$H^2(G,A) = Z^2(G,A)/B^2(G,A)$$

where

$$Z^2(G,A) = \{f : G \times G \longrightarrow A \mid af(a,c) - f(ab,c) + f(a,bc) - f(a,b) = 0\}$$

and

$$B^2(G,A) = \{f : G \times G \longrightarrow A \mid f(a,b) = ag(b) - g(ab) + g(a), \ \ g : G \longrightarrow A\}.$$

2.4.4 H^2 and Extensions

In this subsection, we show that H^2 classifies equivalence classes of group extensions.

Definition 2.4.14. [22] *A group extension E of G by A is a short exact sequence*

$$E : 0 \to A \to B \xrightarrow{\gamma} G \to 1$$

such that A is an abelian group.

Definition 2.4.15. [22] *An extension $E : 0 \to A \to B \xrightarrow{\gamma} G \to 1$ is called split if there is a section $\alpha : G \longrightarrow B$ such that $\gamma \circ \alpha = id_G$.*

Definition 2.4.16. [22] *Two extensions E_1 and E_2 are equivalent if there exists a group homomorphism $\varphi : B_1 \longrightarrow B_2$ such that*

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
E_1 : 0 \arrow[r] & A \arrow[r] & B_2 \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

is commutative.

Given such a section $\alpha : G \longrightarrow B$ such that $\alpha(1)$ is the identity elements of B and $\gamma\alpha(g) = g$ for all $g \in G$. Both $\alpha(gh)$ and $\alpha(g)\alpha(h)$ are elements of B mapping to $gh \in G$, thus their difference lies in A. we thus define

$$[g, h] = \alpha(g)\alpha(h)(\alpha(gh))^{-1}.$$

Definition 2.4.17. [22] *The set function* $[\ ] : G \times G \longrightarrow A$ *defined above is called the factor set depending on* B *and* α.

Definition 2.4.18. [22] *A normalised 2-cocycle is a function* $[\ ] : G \times G \longrightarrow A$ *satisfying the following conditions:*

1. $[g, 1] = [1, g] = 0$ *for all* $g \in G$.
2. $x[y, z] - [xy, z] + [x, yz] - [x, y] = 0$ *for all* $x, y, z \in G$.

The proof of the following lemma is in [22].

Lemma 2.4.19. *Let* A *be a* G*-module. A set function* $[\ ] : G \times G \longrightarrow A$ *is a factor set if and only if it is a normalised 2-cocycle, that is, an element of* $Z^2(G, A)$.

The proof of the following lemma is in [19].

Lemma 2.4.20. *Let* E_1 *be an extension of* G *by* A *with based section* $\alpha_1 : G \longrightarrow B_1$*, and let* $[\ ]$ *be the factor set depending on* α_1*. If* E_2 *is an equivalent extension, then there exists a based section* $\alpha_2 : G \longrightarrow B_2$ *of* E_2 *such that the factor set determined by* α_2 *is* $[\ ]$.

The proof of the following lemma is in [19].

Lemma 2.4.21. *Given an extension* $0 \to A \to B \xrightarrow{\gamma} G \to 0$*, two different factor sets* $[\ ]_1$ *and* $[\ ]_2$*, corresponding to choices* α_1 *and* α_2 *of based sections respectively, differ by a 2-coboundary.*

The above lemmas show that there is a well-defined map ψ from the set of the equivalence classes of extensions, denoted by $E(G, A)$ to $H^2(G; A)$.
The proof of the following lemma is in [22].

Lemma 2.4.22. *Two extensions of G by A with section maps $\alpha_i : G \longrightarrow B_i$ yield the same factor set are equivalent.*

The proof of the following lemma is in [9].

Lemma 2.4.23. *The map ψ from the set of the equivalence classes of extensions $E(G, A)$ to $H^2(G; A)$ is surjective.*

The proof of the following lemma is in [9].

Lemma 2.4.24. *The map ψ from the set of the equivalence classes of extensions $E(G, A)$ to $H^2(G; A)$ is injective.*

The main result in this subsection is the following theorem.

Theorem 2.4.25. *There is a one-to-one correspondence between the classes of extensions $E(G, A)$ and the cohomology group $H^2(G, M)$.*

Proof. We have already known that the map ψ is well-defined, and lemmas 2.4.5 and 2.4.6 say that ψ is bijective. This proves the theorem. □

2.5 Hochschild Homology and Cohomology

In 1945, Hochschild introduced the Hochschild cohomology groups of an associative algebra [10]. Hochschild cohomology of associative algebras is important in many branches of mathematics, for example ring theory, commutative algebra, representation theory, group theory, and topology. The low dimensional groups ($n \leq 2$) have well known interpretations of classical algebraic structures such as derivations and extensions. The main reference for this section is [22].
Let k be a commutative ring, A be an associative k-algebra. An A-A-*bimodule* over A is a k-module M which is both a left module and right module in such away that $(am)a' = a(ma')$ for $a, a' \in A$ and $m \in M$.

2.5.1 Hochschild homology and cohomology of associative algebras

In this subsection, we give directly the definitions of Hochschild homology and cohomology of associative algebras.

Definition 2.5.1. [22] *The Hochschild homology $H_n(A, M)$ of A with coefficients in M is the homology of the following chain complex $C_n(A, M)$:*

$$0 \leftarrow M \xleftarrow{\delta_0} M \otimes A \xleftarrow{\delta_1} M \otimes A \otimes A \xleftarrow{\delta_2} \cdots$$

where the boundary map

$$\delta_n : M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$$

is given by

$$\begin{aligned}\delta_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= (ma_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ &+ \sum_{0<i<n} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}).\end{aligned}$$

Hence, $H_n(A, M) = H_n(C_n(A, M))$ where $C_n(A, M) = M \otimes A^{\otimes n}$.

Example 2.5.2. *(The 0-Hochschild homology).*
The boundary map

$$\delta_0 : M \otimes A \longrightarrow M$$

is the k-submodule [M,A] of M generated by the elements $ma - am$, where $a \in A$ and $m \in M$. Thus, $H_0(A, M) \cong M/[M, A]$.

Definition 2.5.3. [22] *The Hochschild cohomology $H^n(A, M)$ of A with coefficients in M is the cohomology of the following cochain complex $C^n(A, M)$:*

$$0 \rightarrow M \xrightarrow{\delta^0} Hom(A, M) \xrightarrow{\delta^1} Hom(A^{\otimes 2}, M) \xrightarrow{\delta^2} \cdots$$

where the coboundary map

$$\delta^n : Hom(A^{\otimes n}, M) \longrightarrow Hom(A^{\otimes n+1}, M)$$

is given by

$$\begin{aligned}\delta^n(f)(a_1, \cdots, a_{n+1}) &= a_1 f(a_2, \cdots, a_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1}.\end{aligned}$$

Hence, $H^n(A,M) = H^n(C^n(A,M))$ *where* $C^n(A,M) = Hom(A^{\otimes n}, M)$.

Example 2.5.4. *(The 0-Hochschild cohomology).*
We have that

$$\begin{aligned} H^0(A,M) &= ker(\delta^0) \\ &= \{m \in M, \delta^0(m)(a) = am - ma = 0, \ \ \forall a \in A.\} \end{aligned}$$

In particular, $H^0(A) = Z(A)$ *the center of* A.

Example 2.5.5. *(The first Hochschild cohomology).*
We have that

$$H^1(A,M) = ker(\delta^1)/im(\delta^0)$$

where

$$\begin{aligned} ker(\delta^1) &= \{f \in Hom(A,M) \ | \ \delta^1(f)(a,b) = af(b) - f(ab) + f(a)b = 0, \ \ \forall a,b \in A\} \\ &= Der(A,M) \end{aligned}$$

are the derivations of A *in* M*, and*

$$\begin{aligned} im(\delta^0) &= \{f \in Hom(A,M) : \ \ f = \delta^0(m), \ \ m \in M\} \\ &= \{f_m \in Hom(A,M), \ \ m \in M : \ \ f(a) = am - ma\} \\ &= PDer(A,M) \end{aligned}$$

are the principal derivations of A *in* M*. Then* $H^1(A,M) \cong Der(A,M)/PDer(A,M)$.

Example 2.5.6. *(The second Hochschild cohomology).*
We have that

$$H^2(A,M) = ker(\delta^2)/im(\delta^1)$$

where

$$\begin{aligned} ker(\delta^2) &= \{f : A \otimes A \longrightarrow M \ | \ \delta^2(f) = 0\} \\ &= \{f : A \otimes A \longrightarrow M \ | \ af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0\} \end{aligned}$$

and

$$\begin{aligned} im(\delta^1) &= \{f : A \otimes A \longrightarrow M \ | \ f = \delta^1(g), \ \ g \in Hom(A,M)\} \\ &= \{f : A \otimes A \longrightarrow M \ | \ f(a,b) = ag(b) - g(ab) + g(a)b, \ \ g \in Hom(A,M)\}. \end{aligned}$$

2.5.2 H^2 and Extensions

In this subsection, we show that H^2 classifies equivalence classes of Hochschild extensions.

Definition 2.5.7. [22] *Let A be a k-algebra, M be an A-A-bimodule. A Hochschild extension E of A by M is a short exact sequence*

$$E : 0 \to M \xrightarrow{\phi} B \xrightarrow{\gamma} A \to 0$$

where γ is an epimorphism of algebras and ϕ is a monomorphism of k-modules such that

$$\phi(\gamma(b) \cdot m) = b \cdot \phi(m),$$

$$\phi(m \cdot \gamma(b)) = \phi(m) \cdot b,$$

$\forall b \in B,\ m \in M$.

Definition 2.5.8. [22] *A Hochschild extension $E : 0 \to M \xrightarrow{\phi} B \xrightarrow{\gamma} A \to 0$ is called split if there is a section $\alpha : A \longrightarrow B$ such that $\gamma \circ \alpha = id_A$.*

Definition 2.5.9. [22] *Two extension E_1 and E_2 are equivalent if there exists a morphism of algebras $\varphi : B_1 \longrightarrow B_2$ such that*

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\
E_2 : 0 \arrow[r] & M \arrow[r] & B_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Definition 2.5.10. [22] *The function $f : A \otimes A \longrightarrow M$ is called a factor set of the Hochschild extension corresponding to the splitting α.*

Definition 2.5.11. [22] *A 2-cocycle is a function $f : A \otimes A \longrightarrow M$ satisfying:*

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z = 0$$

for all $x, y, z \in A$.

We denote the set of equivalence classes of the Hochschild extensions of A by M by $E(A, M)$.

The proof of the following important result is in [22].

Theorem 2.5.12. *There is a one-to-one correspondence between the classes of Hochschild extensions E(A,M) and the Hochschild cohomology $H^2(A, M)$.*

2.6 Spectral Sequences

Jean Leray introduced spectral sequences in order to compute the (co)homology of a chain complex [12]. Spectral sequences give a fundamental computational tool in algebra, topology and homological algebra. In this section, we give a brief overview of spectral sequences. This material can be found in [22] and [19].

Definition 2.6.1. [19] *A homology spectral sequence in the category R-mod of R-modules consists of the following data:*

1. *A family $\{E^r_{pq}\}$ of R-modules for all integers p, q and $r \geq 1$.*
2. *R-maps*
$$d^r_{pq} : E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$$
that are differentials in the sense that $d^r d^r = 0$.
3. *Isomorphisms between E^{r+1}_{pq} and the homology of $E^r_{\star\star}$ at the spot E^r_{pq}:*
$$E^{r+1}_{pq} \cong Ker(d^r_{pq})/Im(d^r_{p+r,q-r+1})$$

There is a category of homology spectral sequences. A morphism $f : E \longrightarrow E'$ is a family of R-maps $f^r_{pq} : E^r_{pq} \longrightarrow E'^r_{pq}$ in R-modules with $d^r f^r = f^r d^r$, that is, f^r commutes with the differentials and each f^{r+1}_{pq} is the map induced by f^r_{pq} on homology.

Definition 2.6.2. [19] *Dually, a cohomology spectral sequence in the category R-mod of R-modules consists of the following data:*

1. *A family $\{E^{pq}_r\}$ of R-modules for all integers p, q and $r \geq 1$.*
2. *R-maps*
$$d^{pq}_r : E^{pq}_r \longrightarrow E^{p+r,q-r+1}_r$$
that are differentials in the sense that $d_r d_r = 0$.

3. *Isomorphisms between E_{r+1}^{pq} and the homology of $E_r^{\star\star}$ at the spot E_r^{pq}:*

$$E_{r+1}^{pq} \cong Ker(d_r^{pq})/Im(d_r^{p-r,q+r-1})$$

There is a category of homology spectral sequences. A morphism $f : E \longrightarrow E^{'}$ is a family of R-maps $f_r^{pq} : E_r^{pq} \longrightarrow E_r^{'pq}$ in R-modules with $d_r f_r = f_r d_r$, that is, f_r commutes with the differentials and each f_{r+1}^{pq} is the map induced by f_r^{pq} on homology.

A homology spectral sequence is said to be *bounded* if for each n there are only finitely many nonzero terms of total degree n in E_{**}^{a}. If so, then for each p and q there is an r_0 sucg that $E_{pq}^{r} = E_{pq}^{r+1}$ for all $r \geq r_0$. We write E_{pq}^{∞} for this stable value of E_{pq}^{r}.

We say that a bounded spectral sequence *converges* to H_* if we are given family of objects H_n, each having a $finite$ filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n,$$

and we are given isomorphisms $E_{pq}^{\infty} \cong F_p H_{p+q}/F_{p-1} H_{p+q}$. The traditional symbolic way of describing such a bounded convergence is like this:

$$E_{pq}^{a} \Rightarrow H_{p+q}.$$

Similarly, a cohomology spectral sequence is called *bounded* if there are only finitely many nonzero terms of total degree in E_a^{**}. In a bounded cohomology spectral sequence, we write E_{∞}^{pq} for the stable value of the terms E_r^{pq} and say the bounded spectral sequence *converges* to H^* if there is a $finite$ filtration

$$0 = F^t H^n \subseteq \cdots F^{p+1} H^n \subseteq F^p H^n \cdots \subseteq F^s H^n = H^n$$

so that $E_{\infty}^{pq} \cong F^p H^{p+q}/F^{p+1} H^{p+q}$.

The proof of the following useful result is in [22].

Lemma 2.6.3.

1. *If $E_{pq}^{1} \Rightarrow H_{p+q}$ and $E_{pq}^{1} = 0$ for $q > 0$, then $H_p = E_{p0}^{2}$ for all $p \geqslant 0$.*

2. *If additionally $E_{p0}^{2} = 0$ for $p > 0$, then $H_p = 0$ for $p > 0$.*

2.7 Deformation Theory

Algebraic deformation theory was introduced for associative algebras by Gerstenhaber in [6]. In this section, we describe the connection between deformation theory and cohomology theory. All definitions and theorems in this section can be found in [5].

Let K be a field. A one-parameter algebraic deformation of a finite dimensional K-algebra A, may be considered informally as a family of algebras A_t parameterized by K such that $A_0 \cong A$ and the multiplicative structure of A_t varies algebraically with t.

Definition 2.7.1. [5] *Let $A[[t]]$ be the $K[[t]]$-module of formal power series with coefficients in the K-module A, that is, $A[[t]] = A \otimes_K [[t]]$ as a module.*

Now we will state the formal definition of a deformation.

Definition 2.7.2. [5] *A one-parameter formal deformation of a K-algebra A is a formal power series $F = \sum_{n=0}^{\infty} f_n t^n$ with coefficients in $Hom_k(A \otimes A, A)$ and for all $a, b \in A$, $f_0(a, b) = ab$. The deformation $A[[t]]$ with the multiplication defined by F may be written as $A[[t]]_F$ or A_F. If F is finite, or at least finite for each pair $(a, b) \in A \otimes A$, the multiplication may be defined on $A[t]$ over $K[t]$.*

Definition 2.7.3. [5] *Let A be an associative K-algebra. Then the deformation A_F is called associative if*

$$F(F(a, b), c) = F(a, F(b, c)) \tag{2.1}$$

for all a, b, c in A.

If we expand both sides of the equation (2.1) and collect the coefficients of t^n we have

$$\sum_{i=0}^{n} f_i(f_{n-i}(a, b), c) = \sum_{i=0}^{n} f_i(a, f_{n-i}(b, c)) \tag{2.2}$$

Definition 2.7.4. [5] *Let f_n be the first non zero coefficient after f_0 in the expansion $F = \sum f_n t^n$. Then f_n is called the infinitesimal of F.*

Theorem 2.7.5. *If F is an associative deformation of A then the infinitesimal f_n of F is a Hochschild 2-cocycle.*

Proof. Let F be an associative deformation of A and f_n be the infinitesimal of F. We may rewrite (2.1) as

$$f_0(f_n(a,b),c) + f_n(f_0(a,b),c) = f_0(a, f_n(b,c)) + f_n(a, f_0(b,c)).$$

Since f_0 is the multiplication in A we will obtain

$$f_n(ab,c) + f_n(a,b)c = af_n(b,c) + f_n(a,bc),$$

or

$$af_n(b,c) - f_n(ab,c) + f_n(a,bc) - f_n(a,b)c = 0. \tag{2.3}$$

The left hand side of (2.3) is the Hochschild coboundary of f_n and therefore $d^2 f_n = 0$. That is, $f_n \in Kerd^2$. Thus, f_n is a Hochschild 2-cocycle. $\square$

For arbitrary m, equation (2.1) above may be written as

$$d^2 f_m(a,b,c) = \sum_{i=1}^{m-1} f_i(f_{m-i}(a,b),c) - f_i(a, f_{m-i}(b,c)). \tag{2.4}$$

The right hand side of (2.4) is the obstruction to finding f_m that extends the deformation.
The following theorem is the most important result in deformation theory and the proof can be found in [5].

Theorem 2.7.6. *The obstruction is a Hochschild* 3*-cocycle.*

The proof of the following result is in [5].

Corollary 2.7.7. *If $H^3(A,A) = 0$ then every* 2*-cocycle of A may be extended to an associative deformation of A.*

2.7.1 Equivalent and trivial deformations

Given associative deformations A_F and A_G of A, we want to know when there is an isomorphism $A_F \longrightarrow A_G$ which keeps A fixed.

Definition 2.7.8. [5] *A formal isomorphism $\Psi : A_F \longrightarrow A_G$ is a $k[[t]]$-linear map $A[[t]]_F \longrightarrow A[[t]]_G$ that may be expressed in the form*

$$\Psi(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \psi_3(a)t^3 + \cdots$$

where $\psi_0(a) = a$ for $a \in A$. Observe that it is enough to consider $a \in A$, since Ψ is defined over $K[[t]]$. We consider that each ψ_n is a k-linear map $A \longrightarrow A$. If Ψ is multiplication preserving, we say it is an algebraic isomorphism, that is,

$$G(\Psi(a), \Psi(b)) = \Psi(F(a, b))$$

for all a and b in A.

Definition 2.7.9. [5] *We say that two deformations A_F and A_G are equivalent if there exists a formal isomorphism $\Psi : A_F \longrightarrow A_G$, and we write $A_F \cong A_G$.*

Proposition 2.7.10. *Two infinitesimal deformations f_n and g_n of F and G respectively, are equivalent if they are in the same cohomlogy class. That is, they represent the same element of $H^2(A, A)$.*

The proof of the following theorem can be found in [5].

Theorem 2.7.11. *If $H^2(A, A) = 0$, then all deformations of A are isomorphic.*

Definition 2.7.12. [5] *A deformation A_F is called a trivial deformation if $A_F \cong A$. That is, $F = f_0$.*

Definition 2.7.13. [5] *An algebra A is called rigid if it has only trivial deformations.*

Corollary 2.7.14. *If $H^2(A, A) = 0$, then A is rigid.*

Chapter 3

Cohomology of Oriented Algebras

In this chapter, we develop a cohomology theory of oriented algebras. The construction is based on the possibility to mix standard chain complexes computing group and associative algebra cohomologies. We will prove several important results about such cohomologies.

We recall some notations for the standard chain complexes associated to groups and associative algebras. We also recall the bicomplex which we will use through the whole chapter.

In what follows k denotes a ground commutative ring with unit. All modules and algebras are considered over k. Furthermore, we write $\otimes$ and Hom instead of $\otimes_k$ and Hom_k. For a group G and G-module C we let $C^\bullet(G, C)$ denote the standard complex computing the group cohomology. We let

$$C^n(G, C) = \mathcal{M}aps(G^n, C)$$

and the coboundary map

$$\partial : \mathcal{M}aps(G^n, C) \longrightarrow \mathcal{M}aps(G^{n+1}, C)$$

is given by

$$\begin{aligned}(\partial\alpha)(x_1, \cdots, x_{n+1}) &= x_1\alpha(x_2, \cdots, x_{n+1}) \\ &\quad + \sum_{i=1}^{n}(-1)^i\alpha(x_1, \cdots, x_ix_{i+1}, \cdots, x_{n+1}) \\ &\quad + (-1)^{n+1}\alpha(x_1, \cdots, x_n).\end{aligned}$$

So by the definition

$$H^n(G,C) = H^n(C^\bullet(G,C)).$$

We will say that a cochain complex

$$C^\bullet = C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} \cdots$$

is a *G-complex* if each module C^n is endowed with a structure of G-module and each boundary is a G-homomorphism. If this is the case, we let $C^\bullet(G,C^\bullet)$ be the total complex of the following bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^0(G,C^2) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^1(G,C^2) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^2(G,C^2) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots \\
C^0(G,C^1) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^1(G,C^1) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^2(G,C^1) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots \\
C^0(G,C^0) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^1(G,C^0) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & C^2(G,C^0) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots
\end{tikzcd}$$

The cohomology of $C^\bullet(G,C^\bullet)$ is denoted by $H^\bullet(G,C^\bullet)$ and is called the *hypercohomology* of G with coefficients in $C^\bullet$. The spectral sequences associated to this bicomplexes have the form

$${}^1E_1^{pq} = H^q(G,C^p) \Longrightarrow H^*(G,C^\bullet)$$

and

$${}^2E_2^{pq} = H^p(G,H^q(C^\bullet)) \Longrightarrow H^*(G,C^\bullet).$$

Let A be an associative k-algebra. Recall that the Hochschild cohomology of A with coefficients in a A-bimodule M is the cohomology of the following cochain complex:

$$0 \to M \xrightarrow{\delta^0} \mathrm{Hom}(A,M) \xrightarrow{\delta^1} \mathrm{Hom}(A^{\otimes 2},M) \xrightarrow{\delta^2} \cdots$$

where the coboundary map

$$\delta^n : Hom(A^{\otimes n},M) \longrightarrow Hom(A^{\otimes n+1},M)$$

is given by

$$\begin{aligned}\delta(f)(a_1, \cdots, a_{n+1}) &= a_1 f(a_2, \cdots, a_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1}.\end{aligned}$$

Hence, $H^n(A, M) = H^n(C^n(A, M))$, where $C^n(A, M) = Hom(A^{\otimes n}, M)$.
Let us also recall that for an algebra A the category of A-bimodules is isomorphic to the category of left A^e-modules, where A^e is the *enveloping* algebra. As a module one has $A^e = A \otimes A$, while the multiplication is defined by

$$(a \otimes b)(c \otimes d) = ac \otimes db.$$

Moreover, one has an isomorphism $H^*(A, M) = Ext^*_{A^e}(A, M)$, provided A is projective as a k-module.

3.1 Oriented Algebras

In this section, we define oriented algebras and provide some examples.

Definition 3.1.1. *An orientation is a pair (G, ε), where G is a group and ε is a group homomorphism*

$$\varepsilon : G \longrightarrow \{\pm 1\}$$

If such orientation is fixed, then we called that G is an oriented group.

Example 3.1.2.

1. *Any group G can be equipped with a trivial orientation: $\varepsilon(g) = 1$ for all $g \in G$.*
2. *For more interesting examples, we could take*
   (a) *$G = \{\pm 1\}$ and $\varepsilon = id$.*
   (b) *$G = S_n$ and $\varepsilon(\sigma) = sgn(\sigma)$.*

Definition 3.1.3. *Let G be an oriented group and A be an associative algebra. An oriented action of (G, ε) on A is given by a map*

$$G \times A \to A,$$

$$(g, a) \mapsto {}^g a,$$

such that under this action A is a G-module and

$${}^g(a + b) = {}^g a + {}^g b$$

$${}^{gh} a = {}^g({}^h a)$$

$${}^g(ab) = \begin{cases} {}^g a {}^g b & \text{if } \varepsilon(g) = +1, \\ {}^g b {}^g a & \text{if } \varepsilon(g) = -1. \end{cases}$$

$${}^g(1) = 1.$$

An oriented algebra over (G, ε) is an associative algebra equipped with an oriented action of (G, ε) on A.

Example 3.1.4.

1. *Observe that if G is equipped with a trivial orientation, then G acts on A via algebra automorphisms, hence in this case an oriented algebra is nothing but a G-algebra in the classical sense.*

2. *Another interesting example is obtained when $G = \{\pm 1\}$ and $\varepsilon = id$. In this case A is nothing but an involutive algebra. Recall that an involutive algebra is an associative algebra A together with a k-linear map*

$$A \to A,$$

$$a \mapsto \overline{a},$$

such that

$$\overline{a + b} = \overline{a} + \overline{b}$$

$$\overline{ab} = \overline{b}\overline{a}$$

$$\overline{\overline{a}} = a.$$

Definition 3.1.5. *Let A and B be oriented algebras over an oriented group (G, ε). A homomorphism of G-modules $f : A \to B$ is called a homomorphism of oriented algebras provided f is a homomorphism of algebras.*

The oriented algebras and oriented algebra homomorphisms over an oriented group (G, ε) form a category denoted by (G, ε)-Alg. There is an obvious forgetful functor $U : (G, \varepsilon)$-$\mathbf{Alg} \to G$-$\mathbf{Mod}$ to the category of G-modules.

Let M be a G-module. Consider the tensor algebra

$$T^*(M) = k \oplus M \oplus M^{\otimes 2} \oplus \cdots \oplus M^{\otimes n} \oplus \cdots$$

Define an action of G on $T^*(M)$ by

$$^g(m_1 \otimes \cdots \otimes m_n) = \begin{cases} {}^gm_1 \otimes \cdots \otimes {}^gm_n, & \text{if } \varepsilon(g) = +1, \\ {}^gm_n \otimes \cdots \otimes {}^gm_1, & \text{if } \varepsilon(g) = -1. \end{cases}$$

One checks that this action on the tensor algebra defines an oriented algebra structure.

Lemma 3.1.6. *The assignment*

$$M \mapsto T^*(M)$$

*defines a functor G-**Mod** $\to (G, \epsilon)$-**Alg**, which is left adjoint to the forgetful functor $U : (G, \epsilon)$-**Alg** $\to$ **G**-**Mod**.*

Proof. Let A be an oriented algebra and let $f : M \to A$ be a G-module homomorphism. By properties of the tensor algebra the map f has a unique extension as an algebra homomorphism $T^*(M) \mapsto M$, which by abuse of the notations still is denoted by f. So

$$f(m_1 \otimes \cdots \otimes m_n) = f(m_1) \cdots f(m_n).$$

Now it is clear that the extended map is compatible on G-actions and the result follows. $\square$

3.2 Oriented Bimodules and Cohomology

Definition 3.2.1. *Let A be an oriented algebra over an oriented group (G, ε). An oriented bimodule over A is an usual bimodule X together with a G-module structure on X such that*

$$^g(ax) = \begin{cases} {}^ga{}^gx, & \textit{if } \varepsilon(g) = +1, \\ {}^gx{}^ga, & \textit{if } \varepsilon(g) = -1, \end{cases}$$

$$^g(xa) = \begin{cases} {}^gx{}^ga, & \textit{if } \varepsilon(g) = +1, \\ {}^ga{}^gx, & \textit{if } \varepsilon(g) = -1. \end{cases}$$

$$^{gh}x = {}^{g}({}^{h}x).$$

It should be clear what homomorphisms of oriented bimodules are.

Let A be an oriented algebra over an oriented group (G, ε) and let X be an oriented bimodule. For any $n \geqslant 0$ one defines an action of G on $Hom(A^{\otimes n}, X)$ by

$$({}^{g}f)(a_1, ..., a_n) = \begin{cases} {}^{g}f({}^{g^{-1}}a_1, ..., {}^{g^{-1}}a_n) & \text{if } \varepsilon(g) = +1, \\ (-1)^{\frac{(n-1)(n-2)}{2}} \, {}^{g}f({}^{g^{-1}}a_n, ..., {}^{g^{-1}}a_1) & \text{if } \varepsilon(g) = -1. \end{cases}$$

In particular, for $n = 1$ the action is independent on the parity of $\varepsilon(g)$.

Lemma 3.2.2. *With this action the Hochschild complex*

$$0 \to X \xrightarrow{\delta^0} Hom(A, X) \xrightarrow{\delta^1} Hom(A^{\otimes 2}, X) \xrightarrow{\delta^2} \cdots$$

is a G-complex.

Proof. We have to check that the equality $\delta^n({}^{g}f) = {}^{g}\delta^n(f)$, holds for all $f \in Hom(A^{\otimes n}, X)$ and $g \in G$. There are two cases to consider $\varepsilon(g) = +1$ or $\varepsilon(g) = -1$. Firstly, we deal with the first case when $\varepsilon(g) = +1$. We have

$$\begin{aligned} \delta({}^{g}f)(a_1, \cdots, a_{n+1}) &= a_1(({}^{g}f)(a_2, \cdots, a_{n+1})) \\ &+ \sum_{0<i<n+1} (-1)^i(({}^{g}f)(a_1, \cdots, a_ia_{i+1}, \cdots, a_{n+1})) \\ &+ (-1)^{n+1}(({}^{g}f)(a_1, \cdots, a_n))a_{n+1} \\ &= a_1 \, {}^{g}f({}^{g^{-1}}a_2, \cdots, {}^{g^{-1}}a_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i \, {}^{g}f({}^{g^{-1}}a_1, \cdots, {}^{g^{-1}}a_i \, {}^{g^{-1}}a_{i+1}, \cdots, {}^{g^{-1}}a_{n+1}) \\ &+ (-1)^{n+1} \, {}^{g}f({}^{g^{-1}}a_1, \cdots, {}^{g^{-1}}a_n)a_{n+1} \end{aligned}$$

We also have

$$\begin{aligned}
({}^g\delta(f))(a_1,\cdots,a_{n+1}) &= {}^g\Big(\delta(f)({}^{g^{-1}}a_1,\cdots,{}^{g^{-1}}a_{n+1})\Big)\\
&= {}^g\Big({}^{g^{-1}}a_1\, f({}^{g^{-1}}a_2,\cdots,{}^{g^{-1}}a_{n+1})\Big)\\
&+ \sum_{0<i<n+1}(-1)^i\ {}^g\Big(f({}^{g^{-1}}a_1,\cdots,{}^{g^{-1}}a_i\,{}^{g^{-1}}a_{i+1},\cdots,{}^{g^{-1}}a_{n+1})\Big)\\
&+(-1)^{n+1}\ {}^g\Big(f({}^{g^{-1}}a_1,\cdots,{}^{g^{-1}}a_n)\,{}^{g^{-1}}a_{n+1}\Big)\\
&= a_1\,{}^gf({}^{g^{-1}}a_2,\cdots,{}^{g^{-1}}a_{n+1})\\
&+ \sum_{0<i<n+1}(-1)^i\,{}^gf({}^{g^{-1}}a_1,\cdots,{}^{g^{-1}}a_i\,{}^{g^{-1}}a_{i+1},\cdots,{}^{g^{-1}}a_{n+1})\\
&+(-1)^{n+1}\,{}^gf({}^{g^{-1}}a_1,\cdots,{}^{g^{-1}}a_n)a_{n+1}.
\end{aligned}$$

Next, we deal with second case when $\varepsilon(g)=-1$. We have

$$\begin{aligned}
\delta({}^gf)(a_1,\cdots,a_{n+1}) &= a_1(({}^gf)(a_2,\cdots,a_{n+1}))\\
&+\sum_{0<i<n+1}(-1)^i({}^gf)(a_1,\cdots,a_ia_{i+1},\cdots,a_{n+1})\\
&+(-1)^{n+1}(({}^gf)(a_1,\cdots,a_n))a_{n+1}\\
&=(-1)^{\frac{(n-1)(n-2)}{2}}\ a_1\,{}^gf({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_2)\\
&+\sum_{0<i<n+1}(-1)^{\frac{(n-1)(n-2)}{2}+i}\,{}^gf({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_{i+1}\,{}^{g^{-1}}a_i,\cdots,{}^{g^{-1}}a_1)\\
&+(-1)^{\frac{(n-1)(n-2)}{2}+n+1}\,{}^gf({}^{g^{-1}}a_n,\cdots,{}^{g^{-1}}a_1)a_{n+1}
\end{aligned}$$

We also have

$$\begin{aligned}
({}^g\delta(f))(a_1,\cdots,a_{n+1}) &= (-1)^{\frac{n(n-1)}{2}}\ {}^g\Big(\delta(f)({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_1)\Big)\\
&=(-1)^{\frac{n(n-1)}{2}}\ {}^g\Big({}^{g^{-1}}a_{n+1}\,f({}^{g^{-1}}a_n,\cdots,{}^{g^{-1}}a_1)\Big)\\
&+(-1)^{\frac{n(n-1)}{2}}\sum_{0<i<n+1}(-1)^i\,{}^g\Big(f({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_{n-i+2}\,{}^{g^{-1}}a_{n-i+1},\cdots,{}^{g^{-1}}a_1)\Big)\\
&+(-1)^{\frac{n(n-1)}{2}}(-1)^{n+1}\ {}^g\Big(f({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_2)\,{}^{g^{-1}}a_1\Big)\\
&=(-1)^{\frac{n(n-1)}{2}}\ {}^gf({}^{g^{-1}}a_n,\cdots,{}^{g^{-1}}a_1)a_{n+1}\\
&+(-1)^{\frac{n(n-1)}{2}}\sum_{0<j<n+1}(-1)^{n+1-j}\,{}^gf({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_{j+1}\,{}^{g^{-1}}a_j,\cdots,{}^{g^{-1}}a_1)\\
&+(-1)^{\frac{n(n-1)}{2}+n+1}\ a_1\,{}^gf({}^{g^{-1}}a_{n+1},\cdots,{}^{g^{-1}}a_2)
\end{aligned}$$

Thus, we see that δ commutes with the group action. $\square$

Thus one can form the following bicomplex $C^*_G(A, X)$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathcal{M}aps(G^2, X) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \mathcal{M}aps(G^2, Hom(A, X)) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \mathcal{M}aps(G^2, Hom(A^{\otimes 2}, X)) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \cdots \\
\mathcal{M}aps(G, X) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \mathcal{M}aps(G, Hom(A, X)) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \mathcal{M}aps(G, Hom(A^{\otimes 2}, X)) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \cdots \\
X \arrow[u, "\partial''"] \arrow[r, "\partial'"] & Hom(A, X) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & Hom(A^{\otimes 2}, X) \arrow[u, "\partial''"] \arrow[r, "\partial'"] & \cdots
\end{tikzcd}$$

where the coboundary maps given as following:

- The coboundary of every horizontal maps $\partial^{'}$ is given by:

$$\begin{aligned}
(\partial^{'}\alpha)(g_1, \cdots, g_n, a_1, \cdots, a_{n+1}) &= a_1\alpha(g_1, \cdots, g_n, a_2, \cdots, a_{n+1}) \\
&+ \sum_{0<i<n+1} (-1)^i \alpha(g_1, \cdots, g_n, a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\
&+ (-1)^{n+1}\alpha(g_1, \cdots, g_n, a_1, \cdots, a_n)a_{n+1}.
\end{aligned}$$

- The coboundary of the first vertical maps is given by:

$$\begin{aligned}
(\partial^{''} f)(g_1, \cdots, g_{n+1}) &= g_1 f(g_2, \cdots, g_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) \\
&+ (-1)^{n+1} f(g_1, \cdots, g_n).
\end{aligned}$$

- The coboundary of the second vertical maps when $\varepsilon(g) = \pm 1$ is given by:

$$\begin{aligned}
(\partial^{''}\beta)(g_1, \cdots, g_{n+1}, a) &= {}^{g_1}\beta(g_2, \cdots, g_{n+1}, {}^{g_1^{-1}}a) \\
&+ \sum_{i=1}^{n} (-1)^i \beta(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a) \\
&+ (-1)^{n+1}\beta(g_1, \cdots, g_n, a).
\end{aligned}$$

- The coboundary of the third vertical maps when $\varepsilon(g) = +1$ is given by:

$$\begin{aligned}(\partial'' \gamma)(g_1, \cdots, g_{n+1}, a, b) &= {}^{g_1}\gamma(g_2, \cdots, g_{n+1}, {}^{g_1^{-1}}a, {}^{g_1^{-1}}b) \\ &\quad + \sum_{i=1}^{n} (-1)^i \gamma(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a, b) \\ &\quad + (-1)^{n+1} \gamma(g_1, \cdots, g_n, a, b).\end{aligned}$$

- The coboundary of the third vertical maps when $\varepsilon(g) = -1$ is given by:

$$\begin{aligned}(\partial'' \gamma)(g_1, \cdots, g_{n+1}, a, b) &= {}^{g_1}\gamma(g_2, \cdots, g_{n+1}, {}^{g_1^{-1}}b, {}^{g_1^{-1}}a) \\ &\quad + \sum_{i=1}^{n} (-1)^i \gamma(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a, b) \\ &\quad + (-1)^{n+1} \gamma(g_1, \cdots, g_n, a, b).\end{aligned}$$

The homologies of the total complex are denoted by $H^n_G(A, X)$ where $n \geqslant 0$. We will also need the following double complex $\tilde{C}^*_G(A, X)$ that is obtained by deleting the first column and reindexing.

$$\begin{tikzcd}
\vdots & \vdots & \\
\mathcal{M}aps(G^2, Hom(A, X)) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \mathcal{M}aps(G^2, Hom(A^{\otimes 2}, X)) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots \\
\mathcal{M}aps(G, Hom(A, X)) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \mathcal{M}aps(G, Hom(A^{\otimes 2}, X)) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots \\
Hom(A, X) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & Hom(A^{\otimes 2}, X) \arrow[r, "\partial'"] \arrow[u, "\partial''"] & \cdots
\end{tikzcd}$$

The homologies of the total complex of $\tilde{C}^*_G(A, X)$ are denoted by $\tilde{H}^n_G(A, X)$ where $n \geqslant 0$. These groups fit in the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0_G(A, X) \arrow[r] & H^0(G, X) \arrow[r] & \tilde{H}^0_G(A, X) \arrow[dll, rounded corners] & \\
& H^1_G(A, X) \arrow[r] & H^1(G, X) \arrow[r] & \tilde{H}^1_G(A, X) \arrow[dll, rounded corners] & \\
& H^2_G(A, X) \arrow[r] & H^2(G, X) \arrow[r] & \tilde{H}^2_G(A, X) \arrow[r] & \cdots
\end{tikzcd}$$

3.3 Classification of Singular Extension of Oriented Algebras

It is a well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras [22]. Here we obtain a similar result for oriented algebras.

Definition 3.3.1. *Let A be an oriented algebra over an oriented group (G, ε). Moreover, let X be an oriented bimodule over A. A singular extension of A by X is a k-split short exact sequence of G-modules*

$$0 \to X \xrightarrow{i} B \xrightarrow{p} A \to 0$$

where B is also an oriented algebra over an oriented group (G, ε). Furthermore, p is a homomorphism of oriented algebras and i is homomorphism of G-modules such that

$$i(x_1)i(x_2) = 0,$$

$$i(x)b = i(xp(b)),$$

$$bi(x) = i(p(b)x),$$

for all $x, x_1, x_2 \in X$, $b \in B$.

Theorem 3.3.2. *Let A be an oriented algebra over an oriented group (G, ε). Moreover, let X be an oriented bimodule over A. Then, there is a one-to-one correspondence between equivalence classes of extensions of A by X and $\tilde{H}^1_G(A, X)$.*

Before giving the proof, one can observe that $\tilde{H}^1_G(A, X) = \tilde{Z}^1_G(A, X)/\tilde{B}^1_G(A, X)$, where $\tilde{Z}^1_G(A, X)$ is the collection of pairs (α, β), where $\alpha \in \mathcal{M}aps(G, Hom(A, X))$ and $\beta \in Hom(A^{\otimes 2}, X)$ satisfying the following conditions:

$$\alpha(gh, a) = {}^g\alpha(h, {}^{g^{-1}}a) + \alpha(g, a),$$

$$a_1\alpha(g, a_2) - \alpha(g, a_1a_2) + \alpha(g, a_1)a_2 = \begin{cases} \beta(a_1, a_2) - {}^g\beta({}^{g^{-1}}a_1, {}^{g^{-1}}a_2), & \text{if } \varepsilon(g) = +1, \\ \beta(a_1, a_2) - {}^g\beta({}^{g^{-1}}a_2, {}^{g^{-1}}a_1), & \text{if } \varepsilon(g) = -1, \end{cases}$$

$$a_1\beta(a_2, a_3) - \beta(a_1a_2, a_3) + \beta(a_1, a_2a_3) - \beta(a_1, a_2)a_3 = 0.$$

Observe that the last equality simply says that β is a Hochschild 2-cocycle. Moreover, $(\alpha,\beta)\in \tilde{B}^1_G(A,X)$ if and only if there exists $\gamma\in Hom(A,X)$ such that

$$\beta(a_1,a_2)=a_1\gamma(a_2)-\gamma(a_1a_2)+\gamma(a_1)a_2$$

and

$$\alpha(g,a)={}^g\gamma({}^{g^{-1}}a)-\gamma(a).$$

Proof. Let us start with a singular extension as above. To simplify the notation we will assume that X is a submodule of B and $i(x)=x$. Choose a linear map $s:A\longrightarrow B$ such that $ps=id_A$. One defines

$$\alpha\in \mathcal{M}aps(G,Hom(A,X))$$

and

$$\beta\in Hom(A^{\otimes 2},X)$$

by

$$\alpha(g,a)=s(a)-{}^gs({}^{g^{-1}}a) \tag{3.1}$$

and

$$\beta(a_1,a_2)=s(a_1)s(a_2)-s(a_1a_2). \tag{3.2}$$

We claim that $(\alpha,\beta)\in \tilde{Z}^1_G(A,X)$. By the classical argument β is a Hochschild 2-cocycle [22]. Next, we have

$$\begin{aligned}
{}^g\alpha(h,{}^{g^{-1}}a)+\alpha(g,a) &= {}^g(s({}^{g^{-1}}a)-{}^hs({}^{h^{-1}g^{-1}}a))+s(a)-{}^gs({}^{g^{-1}}a)\\
&= {}^gs({}^{g^{-1}}a)-{}^{gh}s({}^{h^{-1}g^{-1}}a)+s(a)-{}^gs({}^{g^{-1}}a)\\
&= s(a)-{}^{gh}s({}^{h^{-1}g^{-1}}a)\\
&= \alpha(gh,a)
\end{aligned} \tag{3.3}$$

To obtain the reminder equations, we have to consider two cases. If $\varepsilon(g)=+1$ we have form (3.1)

$$\begin{aligned}
s(a_1a_2) &= {}^gs({}^{g^{-1}}a_1\,{}^{g^{-1}}a_2)+\alpha(g,a_1a_2)\\
&= {}^g(s({}^{g^{-1}}a_1)s({}^{g^{-1}}a_2)-\beta({}^{g^{-1}}a_1,{}^{g^{-1}}a_2)))+\alpha(g,a_1a_2)\\
&= {}^gs({}^{g^{-1}}a_1)\,{}^gs({}^{g^{-1}}a_2)-{}^g\beta({}^{g^{-1}}a_1,{}^{g^{-1}}a_2)+\alpha(g,a_1a_2)
\end{aligned}$$

and from (3.2) we also have

$$\begin{aligned} s(a_1a_2) &= s(a_1)s(a_2) - \beta(a_1, a_2) \\ &= ({}^g s({}^{g^{-1}}a_1) + \alpha(g, a_1))({}^g s({}^{g^{-1}}a_2) + \alpha(g, a_2)) - \beta(a_1, a_2) \\ &= {}^g s({}^{g^{-1}}a_1){}^g s({}^{g^{-1}}a_2) + a_1\alpha(g, a_2) + \alpha(g, a_1)a_2 - \beta(a_1, a_2) \end{aligned}$$

Comparing these expressions we see that

$$a_1\alpha(g, a_2) - \alpha(g, a_1a_2) + \alpha(g, a_1)a_2 = \beta(a_1, a_2) - {}^g\beta({}^{g^{-1}}a_1, {}^{g^{-1}}a_2). \tag{3.4}$$

By replacing ${}^{g^{-1}}a_1 = b_1$ and ${}^{g^{-1}}a_2 = b_2$ in (3.4) we have

$${}^g b_1\alpha(g, {}^g b_2) - \alpha(g, {}^g b_1{}^g b_2) + \alpha(g, {}^g b_1){}^g b_2 = \beta({}^g b_1, {}^g b_2) - {}^g\beta(b_1, b_2) \tag{3.5}$$

Similarly, if $\varepsilon(g) = -1$ and from (3.1) we have

$$\begin{aligned} s(a_1a_2) &= {}^g s({}^{g^{-1}}a_2{}^{g^{-1}}a_1) + \alpha(g, a_1a_2) \\ &= {}^g(s({}^{g^{-1}}a_2)s({}^{g^{-1}}a_1) - \beta({}^{g^{-1}}a_2, {}^{g^{-1}}a_1)) + \alpha(g, a_1a_2) \\ &= {}^g s({}^{g^{-1}}a_1){}^g s({}^{g^{-1}}a_2) - {}^g\beta({}^{g^{-1}}a_2, {}^{g^{-1}}a_1) + \alpha(g, a_1a_2) \end{aligned}$$

and from (3.2) we also have

$$\begin{aligned} s(a_1a_2) &= s(a_1)s(a_2) - \beta(a_1, a_2) \\ &= ({}^g s({}^{g^{-1}}a_1) + \alpha(g, a_1))({}^g s({}^{g^{-1}}a_2) + \alpha(g, a_2)) - \beta(a_1, a_2) \\ &= {}^g s({}^{g^{-1}}a_1){}^g s({}^{g^{-1}}a_2) + a_1\alpha(g, a_2) + \alpha(g, a_1)a_2 - \beta(a_1, a_2) \end{aligned}$$

Comparing these expressions we see that

$$a_1\alpha(g, a_2) - \alpha(g, a_1a_2) + \alpha(g, a_1)a_2 = \beta(a_1, a_2) - {}^g\beta({}^{g^{-1}}a_2, {}^{g^{-1}}a_1) \tag{3.6}$$

By replacing ${}^{g^{-1}}a_1 = b_2$ and ${}^{g^{-1}}a_2 = b_1$ in (3.6) we have

$${}^g b_2\alpha(g, {}^g b_1) - \alpha(g, {}^g b_2{}^g b_1) + \alpha(g, {}^g b_2){}^g b_1 = \beta({}^g b_2, {}^g b_1) - {}^g\beta(b_1, b_2) \tag{3.7}$$

Hence, we show that in fact $(\alpha, \beta) \in \tilde{Z}^1_G(A, X)$.

Conversely, starting with $(\alpha, \beta) \in \tilde{Z}^1_G(A, X)$ one can define $B = X \oplus A$ where the multiplication is given by

$$(x_1, a_1)(x_2, a_2) = (x_1a_2 + a_1x_2 + \beta(a_1, a_2), a_1a_2)$$

and

$$^{g}(x,a) = (^{g}x - \alpha(g, {^{g}a}), {^{g}a})$$

We claim that B satisfies all properties of oriented algebra and defines an extension. Firstly, we have

$$\begin{aligned} ^{g}(0,1) &= (^{g}0 - \alpha(g, {^{g}1}), {^{g}1}) \\ &= (-\alpha(g,1),1) \\ &= (0,1) \end{aligned}$$

Since α is normalised. Next, for $x_1, x_2 \in X$ and $a_1, a_2 \in A$, we have

$$\begin{aligned} ^{g}((x_1,a_1)+(x_2,a_2)) &= {^{g}(x_1+x_2, a_1+a_2)} \\ &= (^{g}x_1 + {^{g}x_2} - \alpha(g, {^{g}a_1} + {^{g}a_2}), {^{g}a_1} + {^{g}a_2}) \end{aligned}$$

and

$$\begin{aligned} ^{g}(x_1,a_1) + {^{g}(x_2,a_2)} &= (^{g}x_1 - \alpha(g, {^{g}a_1}), {^{g}a_1}) + (^{g}x_2 - \alpha(g, {^{g}a_2}), {^{g}a_2}) \\ &= (^{g}x_1 + {^{g}x_2} - \alpha(g, {^{g}a_1}) - \alpha(g, {^{g}a_2}), {^{g}a_1} + {^{g}a_2}) \end{aligned}$$

Therefore, from definition of α it follows that

$$^{g}((x_1,a_1)+(x_2,a_2)) = {^{g}(x_1,a_1)} + {^{g}(x_2,a_2)}.$$

Then, if $\varepsilon(g) = +1$ we have

$$\begin{aligned} ^{g}((x_1,a_1)(x_2,a_2)) &= {^{g}(x_1a_2 + a_1x_2 + \beta(a_1,a_2), a_1a_2)} \\ &= (^{g}x_1{^{g}a_2} + {^{g}a_1}{^{g}x_2} + {^{g}\beta(a_1,a_2)} - \alpha(g, {^{g}a_1}{^{g}a_2}), {^{g}a_1}{^{g}a_2}) \end{aligned}$$

and

$$\begin{aligned} ^{g}(x_1,a_1){^{g}(x_2,a_2)} &= (^{g}x_1 - \alpha(g, {^{g}a_1}), {^{g}a_1})(^{g}x_2 - \alpha(g, {^{g}a_2}), {^{g}a_2}) \\ &= (^{g}x_1{^{g}a_2} - \alpha(g, {^{g}a_1}){^{g}a_2} + {^{g}a_1}{^{g}x_2} \\ &\quad - {^{g}a_1}\alpha(g, {^{g}a_2}) + \beta(^{g}a_1, {^{g}a_2}), {^{g}a_1}{^{g}a_2}) \end{aligned}$$

Therefore, from (3.5) it follows that

$$^{g}((x_1,a_1)(x_2,a_2)) = {^{g}(x_1,a_1)}{^{g}(x_2,a_2)}.$$

Similarly, if $\varepsilon(g) = -1$ we have

$$\begin{aligned} {}^g((x_1, a_1)(x_2, a_2)) &= {}^g(x_1a_2 + a_1x_2 + \beta(a_1, a_2), a_1a_2) \\ &= ({}^g a_2 {}^g x_1 + {}^g x_2 {}^g a_1 + {}^g\beta(a_1, a_2) - \alpha(g, {}^g a_2 {}^g a_1), {}^g a_2 {}^g a_1) \end{aligned}$$

and

$$\begin{aligned} {}^g(x_2, a_2){}^g(x_1, a_1) &= ({}^g x_2 - \alpha(g, {}^g a_2), {}^g a_2)({}^g x_1 - \alpha(g, {}^g a_1), {}^g a_1) \\ &= ({}^g x_2 {}^g a_1 - \alpha(g, {}^g a_2){}^g a_1 + {}^g a_2 {}^g x_1 \\ &\quad - {}^g a_2 \alpha(g, {}^g a_1) + \beta({}^g a_2, {}^g a_1), {}^g a_2 {}^g a_1) \end{aligned}$$

Therefore, from (3.7) it follows that

$${}^g((x_1, a_1)(x_2, a_2)) = {}^g(x_2, a_2){}^g(x_1, a_1).$$

Finally, for $x \in X$ and $a \in A$, we have

$${}^{gh}(x, a) = ({}^{gh}x - \alpha(gh, {}^{gh}a), {}^{gh}a)$$

and

$${}^g({}^h(x, a)) = {}^g({}^h x - \alpha(h, {}^h a), {}^h a) = ({}^{gh}x - {}^g\alpha(h, {}^h a) - \alpha(g, {}^{gh}a), {}^{gh}a).$$

Comparing these expression we have that

$${}^{gh}(x, a) = {}^g({}^h(x, a)) \ \text{ if and only if } \quad \alpha(gh, {}^{gh}a) = {}^g\alpha(h, {}^h a) + \alpha(g, {}^{gh}a).$$

But this follows from (3.3) by replacing a by ${}^{gh}b$. Thus one obtains an inverse map from the cohomology to extensions. $\square$

3.4 Enveloping algebra, cohomology and *Ext*

Let A be an oriented algebra over an oriented group (G, ϵ). We let (A, G)-Bim be the category of all oriented bimodules. As we will see soon this is an abelian category with enough projective and injective objects. This follows from Lemma 3.4.1 below, which says that the category of oriented bimodules (A, G)-Bim is isomorphic to the category of left modules over an algebra $(A, G)^e$-Mod, where $(A, G)^e$ is the following associative algebra, called the *enveloping algebra* of an oriented algebra A. As a module it is $A \otimes k[G] \otimes A$, while the multiplication is given by

$$(a \otimes g \otimes b)(c \otimes h \otimes d) = \begin{cases} a\,{}^{g}c \otimes gh \otimes {}^{g}db, & \text{if } \varepsilon(g) = +1, \\ a\,{}^{g}d \otimes gh \otimes {}^{g}cb, & \text{if } \varepsilon(g) = -1. \end{cases}$$

Where $a, b, c, d \in A$ and $g, h \in G$.

Lemma 3.4.1. *The above formula defines an associative algebra structure on $(A, G)^e$. Moreover, one has an isomorphism of categories (A, G)-Bim $\sim (A, G)^e$-Mod.*

Proof. To show that $(A, G)^e$ is an associative algebra we have to check only the associativity property:

$$((a \otimes g \otimes b)(c \otimes h \otimes d))(u \otimes k \otimes v) = (a \otimes g \otimes b)((c \otimes h \otimes d)(u \otimes k \otimes v)).$$

Indeed, there are four cases to considered. First case if $\varepsilon(g) = +1$ and $\varepsilon(h) = +1$ we have

$$((a\otimes g\otimes b)(c\otimes h\otimes d))(u\otimes k\otimes v) = (a\,{}^{g}c\otimes gh\otimes {}^{g}db)(u\otimes k\otimes v) = a\,{}^{g}c\,{}^{gh}u\otimes ghk\otimes {}^{gh}v\,{}^{g}db.$$

and

$$(a\otimes g\otimes b)((c\otimes h\otimes d)(u\otimes k\otimes v)) = (a\otimes g\otimes b)(c\,{}^{h}u\otimes hk\otimes {}^{h}vd) = a\,{}^{g}c\,{}^{gh}u\otimes ghk\otimes {}^{gh}v\,{}^{g}db.$$

Second case if $\varepsilon(g) = +1$ and $\varepsilon(h) = -1$ we have

$$((a\otimes g\otimes b)(c\otimes h\otimes d))(u\otimes k\otimes v) = (a\,{}^{g}c\otimes gh\otimes {}^{g}db)(u\otimes k\otimes v) = a\,{}^{g}c\,{}^{gh}v\otimes ghk\otimes {}^{gh}u\,{}^{g}db.$$

and

$$(a\otimes g\otimes b)((c\otimes h\otimes d)(u\otimes k\otimes v)) = (a\otimes g\otimes b)(c\,{}^{h}v\otimes hk\otimes {}^{h}ud) = a\,{}^{g}c\,{}^{gh}v\otimes ghk\otimes {}^{gh}u\,{}^{g}db.$$

Third case if $\varepsilon(g) = -1$ and $\varepsilon(h) = -1$ we have

$$((a\otimes g\otimes b)(c\otimes h\otimes d))(u\otimes k\otimes v) = (a^{g}d\otimes gh\otimes {}^{g}cb)(u\otimes k\otimes v) = a\,{}^{g}d\,{}^{gh}u\otimes ghk\otimes {}^{gh}v\,{}^{g}cb.$$

and

$$\begin{aligned}(a\otimes g\otimes b)((c\otimes h\otimes d)(u\otimes k\otimes v)) &= (a\otimes g\otimes b)(c\,{}^{h}v\otimes hk\otimes\ {}^{h}ud)\\ &= a\,{}^{g}({}^{h}ud)\otimes ghk\otimes{}^{g}(c^{h}v)b\\ &= a\,{}^{g}d\,{}^{gh}u\otimes ghk\otimes\ {}^{gh}v\,{}^{g}cb.\end{aligned}$$

Last case if $\varepsilon(g)=-1$ and $\varepsilon(h)=+1$ we have

$$((a\otimes g\otimes b)(c\otimes h\otimes d))(u\otimes k\otimes v)=(a^{g}d\otimes gh\otimes{}^{g}cb)(u\otimes k\otimes v)=a\,{}^{g}d\,{}^{gh}v\otimes ghk\otimes\ {}^{gh}u\,{}^{g}cb.$$

and

$$\begin{aligned}(a\otimes g\otimes b)((c\otimes h\otimes d)(u\otimes k\otimes v)) &= (a\otimes g\otimes b)(c\,{}^{h}u\otimes hk\otimes\ {}^{h}vd)\\ &= a\,{}^{g}({}^{h}vd)\otimes ghk\otimes{}^{g}(c^{h}u)b\\ &= a\,{}^{g}d\,{}^{gh}v\otimes ghk\otimes\ {}^{gh}u\,{}^{g}cb.\end{aligned}$$

Thus, we show that $(A,G)^{e}$ is an associative algebra. To show the last assertion, it suffices to observe that if X is an oriented bimodule, then the formula

$$(a\otimes g\otimes b)x := a({}^{g}x)b$$

defines a left $(A,G)^{e}$-module structure on X and vice versa. $\square$

Let A be an oriented algebra over an oriented group (G,ϵ). Then the map $A^{e}\to(A,G)^{e}$, given by $a\otimes b\mapsto a\otimes 1\otimes b$ is an algebra homomorphism. Since

$$(a\otimes 1\otimes b)(1\otimes g\otimes 1)=a\otimes g\otimes b$$

it follows that $(A,G)^{e}$ as a left A^{e}-module is free with the basis $\{1\otimes g\otimes 1|g\in G\}$. As a consequence of this fact we will prove the following result.

Lemma 3.4.2. *Let X be a an injective oriented bimodule, then X is also injective as an usual bimodule.*

Proof. Let Y be an usual bimodule. Then X is a left $(A,G)^{e}$-module and Y is a left A^{e}-module. Hence we will have isomorphism

$$Hom_{(A,G)^{e}}((A,G)^{e}\otimes_{A^{e}}Y,X)=Hom_{A^{e}}(Y,X)$$

Since $(A,G)^e$ is free A^e-module, the left hand side is an exact functor on Y, hence the right hand side is also an exact functor on Y. Thus X is injective as a A^e-module. □

It is clear that A itself is an oriented bimodule, thus also a left $(A,G)^e$-module. Explicitly, one has

$$(a \otimes g \otimes b)c = a^g cb, \quad a,b,c \in A, g \in G.$$

This particular bimodule plays an important role, thanks to the following Theorem.

Theorem 3.4.3. *Let A be an oriented algebra over an oriented group (G,ϵ). Assume A is projective as a k-module. Then for any oriented bimodule X one has a natural isomorphism*

$$H^*_G(A,X) = Ext^*_{(A,G)^e}(A,X).$$

Proof. By the well-known axiomatic characterization of the Ext-groups [22], we need to verify the following three properties:

i) There is an isomorphism $H^0_G(A,X) = Hom_{(A,G)^e}(A,X)$.

ii) For any short exact sequence of oriented bimodules

$$0 \to X_1 \to X \to X_2 \to 0$$

there is a long cohomological sequence

$$0 \to H^0_G(A,X_1) \to H^0_G(A,X) \to H^0_G(A,X_2) \to H^1_G(A,X_1) \to \cdots$$

iii) If X is an injective oriented bimodule, then $H^n_G(A,X) = 0$, for $n > 0$.

To see i), one can observe that

$$H^0_G(A,X) = \{x \in X \mid {}^g x = x \text{ and } ax = xa \ \ a \in A, g \in G\}.$$

Next, one defines the map

$$\chi : Hom_{(A,G)^e}(A,X) \to H^0_G(A,X)$$

by $\chi(f) = f(1) = x$, where $f : A \to X$ is an oriented bimodule homomorphism. We claim that $f(1) = x \in H^0_G(A, X)$. One can observe that

$$f({}^g b) = {}^g f(b)$$

$$af(b) = f(ab)$$
$$f(b)a = f(ba).$$

$\forall b \in A$. By taking $b = 1$ and $f(1) = x$, we have

$$f({}^g 1) = {}^g f(1) \quad \Longrightarrow \quad f(1) = {}^g x \quad \Longrightarrow \quad x = {}^g x$$

and

$$\left.\begin{array}{r} af(1) = f(a) \\ f(1)a = f(a) \end{array}\right\} \Longrightarrow \quad af(1) = f(1)a \quad \Longrightarrow \quad ax = xa.$$

Hence, χ is a well-defined homomorphism.

Conversely, one defines

$$\theta : H^0_G(A, X) \to Hom_{(A,G)^e}(A, X)$$

by

$$\theta(x) : A \to X$$
$$\theta(x)(a) = ax$$

where $x \in H^0_G(A, M)$ and $a \in A$. We need to verify the following three conditions:

$$\theta(x)({}^g a) = {}^g(\theta(x)(a))$$

$$\theta(x)(ab) = a(\theta(x)(b))$$
$$\theta(x)(ab) = (\theta(x)(a))b.$$

We have

$$\theta(x)({}^g a) = {}^g(\theta(x)(a))$$
$$\Rightarrow {}^g ax = {}^g(ax)$$
$$\Rightarrow {}^g ax = {}^g a {}^g x$$
$$\Rightarrow {}^g ax = {}^g ax$$

since ${}^{g}x = x$.
Similarly,

$$\begin{aligned}\theta(x)(ab) &= a(\theta(x)(b))\\ \Rightarrow abx &= a(bx)\\ \Rightarrow abx &= abx.\end{aligned}$$

Finally,

$$\begin{aligned}\theta(x)(ab) &= (\theta(x)(a))b\\ \Rightarrow abx &= (ax)b\\ \Rightarrow a(bx) &= a(xb)\end{aligned}$$

since $bx = xb$. Hence, the map θ is the inverse of χ.

To see ii), one can observe that projectivity over k implies that one has an exacts sequence of bicomplexes

$$0 \to C^*_G(A, X_1) \to C^*_G(A, X) \to C^*_G(A, X_2) \to 0$$

and the result follows.

To see iii), one can observe that one of the spectral sequence associated to the double complex $C^*_G(A, X)$ has the form

$$E_2^{pq} = H^p(G, H^q(A, X)) \Longrightarrow H_G^{p+q}(A, X)$$

In the case, when X is an injective object in the category of oriented bimodules the group $H^q(A, X)$ vanishes provided $q > 0$, thanks to Lemma 3.4.2. Hence $H^*_G(A, X) = H^*(G, H^0(A, X))$. Next, we can assume that $X = Hom((A, G)^e, I)$, where I is an injective k-module. In this case

$$H^0(A, X) \cong Hom(k[G] \otimes A, I) \cong Hom(k[G], Hom(A, I))$$

where the first isomorphism assigns to an element $\alpha \in H^0(A, X)$ the element $\beta \in Hom(k[G] \otimes A, I)$ given by $\beta(g, a) = \alpha(1, g, a)$. Thus $H^0(A, X)$ is an coinduced G-module. Hence $H^n(G, H^0(A, X)) = 0$ if $n > 0$ and the result follows. $\square$

One easily checks that the map

$$\mu : A \otimes A \to A, \quad a \otimes b \mapsto ab,$$

is an epimorphism of $(A,G)^e$-modules, where $A \otimes A$ is considered as a left $(A,G)^e$-module by

$$(a \otimes g \otimes b)(c \otimes d) = \begin{cases} a\, {}^g c\, g^d b, & \text{if } \epsilon(g) = +1, \\ a\, {}^g d\, g^c b, & \text{if } \epsilon(g) = -1. \end{cases}$$

We let $I(A,G)$ be the kernel of μ.

Proposition 3.4.4. *Let A be an oriented algebra over an oriented group (G,ϵ). Assume A is projective as a k-module. Then for any oriented bimodule X one has a natural isomorphism*

$$\tilde{H}^*_G(A,X) = Ext^*_{(A,G)^e}(I(A,G),X).$$

Proof. As in the proof of Theorem 3.4.3 we have to check that the functors $\tilde{H}^*_G(A,-)$ satisfy three properties:

1) They coincide with $Hom_{(A,G)^e}(I(A,G),-)$.

2) They form an exact and connected sequence.

3) In positive dimensions they vanish on injective objects.

One can observe that the property 2) is obvious, the property 3) follows from the exact sequence

$$0 \to H^0_G(A,X) \to H^0(G,M) \to \tilde{H}^0_G(A,X) \to H^1_G(A,X) \to H^1(G,X) \to \cdots$$

and the fact that the groups $H^i_G(A,X)$ and $H^i(G,X)$ both vanishes for $i > 0$ and injective X, thanks to Theorem 3.4.3 and Lemma 3.4.2.

To show the property 1) one can observe that both functor in the question are left exact. Hence it suffices to consider only injective X. In that case our exact sequence has the form

$$0 \to H^0_G(A,X) \to H^0(G,X) \to \tilde{H}^0_G(A,X) \to 0.$$

On the other hand we have a short exact sequence of bimodules

$$0 \to I(A,G) \to A \otimes A \to A \to 0,$$

which gives

$$0 \to Hom_{(A,G)^e}(A, X) \to Hom_{(A,G)^e}(A \otimes A, X) \to Hom_{(A,G)^e}(I(A,G), X) \to 0.$$

The first term is isomorphic to the group $H^0_G(A, X)$ by Theorem 3.4.3, while the second term is isomorphic to $Hom_G(k, X) = H^0(G, X)$. Thus the third term is isomorphic to $\tilde{H}^0_G(A, X)$. Hence the result. □

3.5 Deformation of Oriented Algebras

The aim of this section is to extend the deformation theory of associative algebras due to Gerstenhaber [6] to oriented algebras.

Definition 3.5.1. *Let A be an oriented algebra over an oriented group (G, ϵ). A one parameter formal deformation of A is a pair (Ψ, Φ), where*

$$\Psi = \sum_{i=0}^{\infty} \psi_i t^i \quad \text{and} \quad \Phi = \sum_{i=0}^{\infty} \phi_i t^i$$

are formal power series with $\psi_n \in Hom(A \otimes A, A)$ and $\phi_n \in Maps(G, Hom(A, A))$. One requires that for all $n \geqslant 0$ the following identities hold

(i) $\psi_0(a, b) = ab$ and $\phi_0(g, a) = {}^g a$,

(ii) $\sum_{i+j=n} \psi_i(\psi_j(a, b), c) = \sum_{i+j=n} \psi_i(a, \psi_j(b, c))$,

(iii) $\phi_n(gh, a) = \sum_{i+j=n} \phi_i(g, \phi_j(h, a))$,

(iv) $\sum_{i+j=n} \phi_i(g, \psi_j(a, b)) = \begin{cases} \sum_{i+j+k=n} \psi_i(\phi_j(g, a), \phi_k(g, b)) & \text{if } \epsilon(g) = +1, \\ \sum_{i+j+k=n} \psi_i(\phi_j(g, b), \phi_k(g, a)) & \text{if } \epsilon(g) = -1. \end{cases}$

Here $g, h \in G$ and $a, b, c \in A$. The last three identities can be expressed as

$$\Psi(a, \Psi(b, c)) = \Psi(\Psi(a, b), c),$$

$$\Phi(gh, a) = \Phi(g, \Phi(h, a)),$$

$$\Phi(g, \Psi(a, b)) = \begin{cases} \Psi(\Phi(g, a), \Phi(g, b)) & \text{if } \epsilon(g) = +1, \\ \Psi(\Phi(g, b), \Phi(g, a)) & \text{if } \epsilon(g) = -1, \end{cases}$$

which shows that $A[[t]]$ becomes an oriented $k[[t]]$-algebra. If for fixed $m \geqslant 1$ there are given $\psi_n \in Hom(A \otimes A, A)$ and $\phi_n \in Maps(G, Hom(A, A))$ for $n = 0, \cdots, m$ satisfying above identities for $n = 0, \cdots, m$, then we say that there is given an m*-deformation.* For $m = 1$, we have

$$a\psi_1(b, c) + \psi_1(a, bc) = \psi_1(ab, c) + \psi_1(a, b)c,$$

$$\phi_1(gh, a) = {}^g\phi_1(h, a) + \phi_1(g, {}^h a),$$

$${}^g\psi_1(a, b) + \phi_1(g, ab) = \begin{cases} {}^g a\phi_1(g, b) + \phi_1(g, a)\, {}^g b + \psi_1({}^g a, {}^g b) & \text{if } \epsilon(g) = +1, \\ {}^g b\phi_1(g, a) + \phi_1(g, b)\, {}^g a + \psi_1({}^g b, {}^g a) & \text{if } \epsilon(g) = -1, \end{cases}$$

In this case we say there is given an *infinitesimal deformation.*

Definition 3.5.2. *Two deformations* (Ψ, Φ) *and* (Ψ', Φ') *are equivalent if there exists a formal power series* $\Omega = \sum_{n=0}^{\infty} \omega_n t^n$*, with properties*

(i) $\omega_n \in Hom(A, A)$, $n \geq 0$,

(ii) $\omega_0(a) = a$, $a \in A$,

(iii) $\sum_{i+j=n} \omega_i(\psi'_j(a, b)) = \sum_{i+j+k=n} \psi_i(\omega_j(a), \omega_k(b))$,

(iv) $\sum_{i+j=n} \omega_i(\phi'_j(g, a)) = \sum_{i+j=n} \phi_i(g, \omega_j(a))$.

Here $n \geqslant 0$, $a \in A$ and $b \in B$. The last two equations can be express also as $\Omega(\Psi'(a, b)) = \Psi(\Omega(a), \Omega(b))$ and $\Omega(\Phi'(g, a)) = \Phi(g, \Omega(a))$. In other words, Ω defines an isomorphism of oriented $k[[t]]$-algebras $(\Psi, \Phi) \to (\Psi', \Phi')$. In a same way one can define under what condition two m-deformations are equivalent.

Lemma 3.5.3. *i) Let* (Ψ, Φ) *be a one parameter formal deformation of an oriented algebra* A*. Assume* $n > 0$ *is a natural number such that* $\psi_i = \phi_i = 0$ *for* $0 < i < n$*. Then the pair* (ψ_n, ξ_n) *is a* 1*-cocycle in* $\tilde{C}^*_G(A, A)$*, where*

$$\xi_n(g, a) = \phi_n(g, {}^{g^{-1}}a)$$

In particular (ψ_1, ξ_1) *is a* 1*-cocycle in* $\tilde{C}^*_G(A, A)$*.*

ii) There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of an oriented algebra A and $\tilde{H}^1_G(A, A)$.

Proof. The part ii) easily follows from i). To prove i), we observe that these equations gives

$$\psi_n(a,b)c + \psi_n(ab,c) = a\psi_n(b,c) + \psi_n(a,bc)$$

$$\phi_n(gh,a) = {}^g\phi_n(h,a) + \phi_n(g, {}^h a).$$

$$ {}^g\psi_n(a,b) + \phi_n(g,ab) = \begin{cases} {}^g a\phi_n(g,b) + \phi_n(g,a)\,{}^g b + \psi_n({}^g a, {}^g b) & \text{if } \epsilon(g) = +1, \\ {}^g b\phi_n(g,a) + \phi_n(g,b)\,{}^g a + \psi_n({}^g b, {}^g a) & \text{if } \epsilon(g) = -1. \end{cases}$$

Next, we have

$$\begin{aligned} \xi_n(gh,a) &= \phi_n(gh, {}^{h^{-1}g^{-1}}a) \\ &= {}^g\phi_n(h, {}^{h^{-1}g^{-1}}a) + \phi_n(g, {}^{h\,h^{-1}g^{-1}}a) \\ &= {}^g\xi_n(h, {}^{g^{-1}}a) + \phi_n(g, {}^{g^{-1}}a) \\ &= {}^g\xi_n(h, {}^{g^{-1}}a) + \xi_n(g,a). \end{aligned}$$

Finally, we have

$$ {}^g\psi_n(a,b) = \begin{cases} {}^g a\xi_n(g, {}^g b) + \xi_n(g, {}^g a)\,{}^g b + \psi_n({}^g a, {}^g b) - \xi_n(g, {}^g a {}^g b) & \text{if } \epsilon(g) = +1, \\ {}^g b\xi_n(g, {}^g a) + \xi_n(g, {}^g b)\,{}^g a + \psi_n({}^g b, {}^g a) - \xi_n(g, {}^g b {}^g a) & \text{if } \epsilon(g) = -1. \end{cases}$$

Hence, the pair (ψ_n, ξ_n) is a 1-cocycle in $\tilde{C}^*_G(A, A)$.

□

Chapter 4

Hochschild Cohomology of Green Functors for Cyclic Groups of Prime Order

The aim of this chapter is to develop the Hochschild cohomology theory of G-Green functors. We start this chapter by providing definitions and examples of G-Mackey functors and G-Green functors. Throughout this chapter, R denotes a commutative ring and G denotes a finite groups.

4.1 G-Mackey Functors

There are several equivalent definitions of G-Mackey functors for a finite group G. In this section, we will state two definitions of G-Mackey functors. The first definition is due to Green [7].

Definition 4.1.1. *A G-$Mackey\ functor$ M consists of a collection of abelian groups $M(H)$ together with transfer maps $tr_K^H : M(K) \longrightarrow M(H)$ and restriction maps $res_K^H : M(H) \longrightarrow M(K)$ for all subgroups $K < H \leq G$, and conjugation maps $c_{x,H} : M(H) \longrightarrow M({}^xH)$ for $x \in G$, such that the following axioms hold:*

1. *If $T \leq K \leq H$, then $tr_K^H tr_T^K = tr_T^H$ and $res_T^K res_K^H = res_T^H$.*

2. *If $x, y \in G$ and $H \leq G$, then $c_{y,{}^xH} c_{x,H} = c_{yx,H}$.*

3. *If $x \in G$ and for all subgroups $K \leq H$, then $c_{x,H}tr_K^H = tr_{^xK}^{^xH}c_{x,K}$ and $c_{x,K}res_K^H = res_{^xK}^{^xH}c_{x,H}$. Furthermore, $c_{x,H} = Id$ if $x \in H$.*

4. *(Mackey axiom) for all subgroups $T, K \leq H$*

$$res_T^H tr_K^H = \sum_{x \in [T \backslash H/K]} tr_{T \cap {^xK}}^T c_{x, T^x \cap K} res_{T^x \cap K}^K.$$

Definition 4.1.2. *A morphism f from a Mackey functor M to a Mackey functor N consists of a collection of morphism of group homomorphisms $f_H : M(H) \longrightarrow N(H)$, for $H \leq G$, such that if $K \leq H$ and $x \in G$, the squares*

$$\begin{tikzcd} M(H) \arrow[r, "f_H"] & N(H) \\ M(K) \arrow[u, "tr_K^H"] \arrow[r, "f_K"'] & N(K) \arrow[u, "tr_K^H"'] \end{tikzcd} \quad \begin{tikzcd} M(H) \arrow[r, "f_H"] \arrow[d, "res_K^H"'] & N(H) \arrow[d, "res_K^H"] \\ M(K) \arrow[r, "f_K"'] & N(K) \end{tikzcd} \quad \begin{tikzcd} M(H) \arrow[r, "f_H"] \arrow[d, "c_{x,H}"'] & N(H) \arrow[d, "c_{x,H}"] \\ M(^xH) \arrow[r, "f_{^xH}"'] & N(^xH) \end{tikzcd}$$

are commutative.

The second definition is given by Dress [4]. Let $\widehat{G}$ be the category of finite G-sets.

Definition 4.1.3. *A G-Mackey functor M for the finite group G is a pair of functors (M_*, M^*) from $\widehat{G}$ to Ab the category of abelian groups, such that the following properties hold:*

1. *$M_*(X) = M^*(X) = M(X)$ for any G-set X.*

2. *M_* is covariant and M^* is contravariant.*

3. *If X and Y are finite G-sets, and if i_X and i_Y are the respective inclusion maps from X and Y to their disjoint union $X \sqcup Y$, then the maps $M^*(i_X) \oplus M^*(i_Y)$ and $M_*(i_X) \oplus M_*(i_Y)$ are mutual inverse R-module isomorphisms:*

$$M(X) \oplus M(Y) \xrightarrow{(M_*(i_X) \oplus M_*(i_Y))} M(X \sqcup Y) \xrightarrow{(M^*(i_X) \oplus M^*(i_Y))} M(X) \oplus M(Y)$$

4. *If*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\ S \arrow[r, "v"'] & T \end{tikzcd}$$

is a pullback diagram of finite G-sets, then we have the following commutative diagram in Ab.

$$\begin{tikzcd} M(X) \arrow[r, "M_*(f)"] & M(Y) \\ M(S) \arrow[u, "M^*(g)"] \arrow[r, "M_*(v)"'] & M(T) \arrow[u, "M^*(u)"'] \end{tikzcd}$$

For a map $f : X \longrightarrow Y$ of finite G-sets, we call $M_*(f) = f_*$ a transfer map and $M^*(f) = f^*$ a restriction map. We write $\mathcal{M}ack(G)$ for the category of G-Mackey functors.

Webb shows that these definitions are equivalent in [21].

4.1.1 Examples

In this subsection, we give two common examples of G-Mackey functors.

Example 4.1.4. *The simplest example of a G-Mackey functor is the fixed point G-Mackey functor. Let M be a group with an action of G. We will denote the fixed point G-Mackey functor of M by $\underline{M}$, and we define $\underline{M}$ by:*

$$\begin{aligned} \underline{M}(H) = M^H &= \{m \in M | h \cdot m = m \text{ for all } h \in H\} \\ &= \text{The subgroup of } H\text{-fixed points in } M. \end{aligned}$$

For all subgroups K of H, the restriction map $res_K^H : M^H \longrightarrow M^K$ is simply inclusion of fixed points, and we define the transfer map $tr_K^H : M^K \longrightarrow M^H$ by the formula:

$$tr_K^H(m) = \sum_{g \in G} g \cdot m.$$

Example 4.1.5. *The most significant example of a G-Mackey functor is the Burnside G-Mackey functor, $\underline{B}$. For all subgroups H of G, we define $\underline{B}(H)$ to be the Grothendieck group on the set of isomorphism classes of the category of finite H-sets, denoted by $\widehat{H}$, and therefore,*

$$\underline{B}(H) = \{[H];\ \ H \in \widehat{H}\},$$

such that the addition is given by disjoint union, $[U] + [V] = [U \amalg V]$. Moreover, for all subgroups K of H, the transfer map $tr_K^H : \underline{B}(K) \longrightarrow \underline{B}(H)$ is given by $tr_K^H([V]) = [H \times_K V]$ and the restriction map $res_K^H : \underline{B}(H) \longrightarrow \underline{B}(K)$ is given by

$res^H_K([U]) = [\psi_K U]$ where ψ_K is the restriction functor from $\widehat{H}$ to $\widehat{K}$. The action of G is trivial.

4.2 G-Green Functors

In this section, we provide two equivalent definitions of G-Green functors. The first is a constructive definition similar to definition 4.1.1 of a G-Mackey functor.

Definition 4.2.1. *A G-Mackey functor A is a G-Green functor if the following axioms hold:*

1. *$A(H)$ is a ring for each subgroup H of G.*
2. *If $K \leq H$ are subgroups of G, and $x \in G$, then all restriction maps $res^H_K : A(H) \longrightarrow A(K)$ and all conjugation maps $c_{x,H} : A(H) \longrightarrow A({}^xH)$ are ring homomorphisms.*
3. *A satisfies Frobenius relations: If $K \leq H$ are subgroups of G then*

$$tr^H_K(a) \cdot b = tr^H_K(a \cdot res^H_K(b))$$

$$b \cdot tr^H_K(a) = tr^H_K(res^H_K(b) \cdot a)$$

for all $a \in A(K)$ and $b \in A(H)$.

Furthermore, a G-Green functor A is commutative if every $A(H)$ is a commutative ring.

A morphism f from the Green functor A to the Green functor B is a morphism of Mackey functors such that, for any subgroup H of G, the morphism f_H is a morphism of rings.

The second is the category theoretic definition analogue of the Dress definitions of Mackey functors [2]. The two definitions are equivalent [2].

Definition 4.2.2. *Let R be a commutative ring. A G-Green functor A over R for the finite group G is a G-Mackey functor endowed for any G-sets X and Y with bilinear maps*

$$A(X) \times A(Y) \longrightarrow A(X \times Y)$$

denoted by $(a, b) \longrightarrow a \times b$, such that the following properties hold:

1. *(Bifunctoriality) If* $f : X \longrightarrow X_1$ *and* $g : Y \longrightarrow Y_1$ *are morphisms of* G*-sets, then the following diagrams*

$$\begin{tikzcd} A(X) \times A(Y) \arrow[r, "\times"] \arrow[d, "A_*(f)\times A_*(g)"'] & A(X \times Y) \arrow[d, "A_*(f\times g)"] \\ A(X_1) \times A(Y_1) \arrow[r, "\times"'] & A(X_1 \times Y_1) \end{tikzcd}$$

$$\begin{tikzcd} A(X) \times A(Y) \arrow[r, "\times"] & A(X \times Y) \\ A(X_1) \times A(Y_1) \arrow[r, "\times"'] \arrow[u, "A^*(f)\times A^*(g)"] & A(X_1 \times Y_1) \arrow[u, "A^*(f\times g)"'] \end{tikzcd}$$

are commutative.

2. *(Associativity) If* X*,* Y *and* Z *are* G*-sets, then the following diagram*

$$\begin{tikzcd} A(X) \times A(Y) \times A(Z) \arrow[r, "e_{A(X)}\times(\times)"] \arrow[d, "(\times)\times e_{A(Z)}"'] & A(X) \times A(Y \times Z) \arrow[d, "\times"] \\ A(X \times Y) \times A(Z) \arrow[r, "\times"'] & A(X \times Y \times Z) \end{tikzcd}$$

is commutative, up to identifications $(X\times Y)\times Z \simeq X\times Y\times Z \simeq X\times(Y\times Z)$

3. *(Unitarity) If* $\star$ *denotes the* G*-set with one element, then there exists an element* $\tau \in A(\star)$*, such that for any* G*-set* X *and for any* $x \in A(X)$

$$A_*(l_X)(x \times \tau) = x = A_*(k_X)(\tau \times x),$$

where l_X *is the bijective projection from* $X \times \star$ *to* X *and* k_X *is the bijective projection from* $\star \times X$ *to* X*. We will write* $\mathcal{G}reen(G)$ *for the category of* G*-Green functors.*

4.2.1 Examples

In this subsection, we give two common examples of G-Green functors.

Example 4.2.3. *A fixed point* G*-Green functor is a fixed point* G*-Mackey functor* $\underline{M}$ *if we can extend the group* M *to have a ring structure that is equipped with the action of* G*. In particular, we need that* $g \cdot (ab) = (ga) \cdot (gb)$ *and* $g1 = 1$ *for all* $g \in G$ *and* $a, b \in M$*.*

Example 4.2.4. *The Burnside G-Mackey functor inherits the structure of a G-Green functor. For all subgroups H of G, we will define $\underline{B}(H)$ to be the Grothendieck group on the set of isomorphism classes of the category of finite H-sets, where addition is given by disjoint union, $[U] + [V] = [U \amalg V]$ and multiplication is given by the Cartesian product, $[U][V] = [U \times V]$. The multiplicative unit is the isomorphism class of the single point set [H/H]. The direct product of H-sets converts $\underline{B}(H)$ into a ring, such that all restriction maps are ring homomorphisms. Transfer and restriction form a G-Green functor structure on $\underline{B}$.*

Remark 4.2.5. From now on we restrict our attention to the case when $G = C_p$ and for general case see chapter 5.

4.3 C_p-Mackey Functors

In this section, we will provide a detailed description of C_p-Mackey functors, C_p-tensor product and C_p-$\mathcal{HOM}$.

Definition 4.3.1. *A C_p-Mackey functor M consists of abelian groups $A = M(e)$ and $B = M(C_p)$, together with an action c of C_p on A by group maps, group homomorphism $tr_e^{C_p} : A \longrightarrow B$, and group homomorphism $res_e^{C_p} : B \longrightarrow A$ such that the following relations hold.*

1. $c_{g,e}^p = 1$.
2. $tr_e^{C_p} c_{g,e}(a) = tr_e^{C_p}(a)$.
3. $c_{g,e} res_e^{C_p}(b) = res_e^{C_p}(b)$.
4. $res_e^{C_p}(tr_e^{C_p}(a)) = a + c_{g,e}(a) + c_{g,e}^2(a) + \cdots + c_{g,e}^{p-1}(a)$.

We will describe a C_p-Mackey functor by the following diagram:

$$
\begin{tikzcd}
B \arrow[d, bend left, "res_e^{C_p}"] \\
A \arrow[u, bend left, "tr_e^{C_p}"] \arrow[loop below, "c"]
\end{tikzcd}
$$

4.3.1 C_p-Tensor products of C_p-Mackey Functors

In this subsection, given C_p-Mackey functors M and N, we will build a Mackey functor diagram for $M \otimes N$.

Definition 4.3.2. *Let M and N be C_p-Mackey functors describe by the following diagrams:*

$$\begin{tikzcd} M(C_p) \arrow[d, bend left, "res_e^{C_p}"] \\ M(e) \arrow[u, bend left, "tr_e^{C_p}"] \arrow[loop below, "c"] \end{tikzcd} \qquad \begin{tikzcd} N(C_p) \arrow[d, bend left, "res_e^{C_p}"] \\ N(e) \arrow[u, bend left, "tr_e^{C_p}"] \arrow[loop below, "c"] \end{tikzcd}$$

then we define $M \otimes N$ by the following digram:

$$\begin{tikzcd} M(e) \otimes N(e) \oplus M(C_p) \otimes N(C_p)/\sim \arrow[d, bend left, "res_e^{C_p}"] \\ M(e) \otimes N(e) \arrow[u, bend left, "tr_e^{C_p}"] \arrow[loop below, "c=c\otimes c"] \end{tikzcd}$$

1. *The $\sim$ is given by the following relations:*

$$a \otimes tr_e^{C_p}(y) \sim res_e^{C_p}(a) \otimes y$$

   and

$$tr_e^{C_p}(x) \otimes b \sim x \otimes res_e^{C_p}(b)$$

   for all $a \in M(C_p)$, $b \in N(C_p)$, $x \in M(e)$ and $y \in N(e)$.

2. *The action is given by $c(x \otimes y) = cx \otimes cy$. Moreover, $c^p = id$.*

3. *We denote the elements in the quotient by the classes $[a \otimes b]$ and $[x \otimes y]$, where $a \otimes b \in M(C_p) \otimes N(C_p)$ and $x \otimes y \in M(e) \otimes N(e)$.*

4. *The restriction map $res_e^{C_p}$ is a homomorphism. That is,*

$$res_e^{C_p}([a \otimes b]) = res_e^{C_p}(a) \otimes res_e^{C_p}(b).$$

5. *We define the restriction map by following formula:*

$$res_e^{C_p}([x \otimes y]) = x \otimes y + c_{g,e}(x) \otimes c_{g,e}(y) + \cdots + c_{g,e}^{p-1}(x) \otimes c_{g,e}^{p-1}(y).$$

6. *We define the transfer map by* $tr_e^{C_p}(x \otimes y) = [x \otimes y]$.

4.3.2 C_p-$\mathcal{HOM}$ (A, B)

Let $\mathcal{M}ack(G)$ be the category of C_p-Mackey functors. Recall that a morphism $f \in Hom_{\mathcal{M}ack(G)}(A, B)$ in $\mathcal{M}ack(G)$ consists of a pair (f_{C_p}, f_e) of homomorphisms of abelian group such that (f_{C_p}, f_e) commute with the transfer, restriction and conjugation maps. We can visualize f with the following commutative diagram:

$$\begin{tikzcd}
A(C_p) \arrow[rr, "f_{C_p}"] \arrow[d, bend left, "res_e^{C_p}"] & & B(C_p) \arrow[d, bend left, "res_e^{C_p}"] \\
A(e) \arrow[rr, "f_e"'] \arrow[u, bend left, "tr_e^{C_p}"] & & B(e) \arrow[u, bend left, "tr_e^{C_p}"]
\end{tikzcd}$$

Now, we are ready to define C_p-$\mathcal{HOM}(A, B)$.

Definition 4.3.3. *We define C_p-$\mathcal{HOM}(A, B)$ with the following digram:*

$$\begin{tikzcd}
Hom_{\mathcal{M}ack(G)}(A, B) \arrow[d, bend left, "R"] \\
Hom(A(e), B(e)) \arrow[u, bend left, "T"] \arrow[loop below, "c"]
\end{tikzcd}$$

such that the following holds:

1. $Hom_{\mathcal{M}ack(G)}(A, B) = \{$ (f_{C_p}, f_e) *homomorphisms of abelian group from A to B* $\}$.

2. $Hom(A(e), B(e)) = \{$ *collection of homomorphisms of abelian group from* $A(e)$ *to* $B(e)$ $\}$.

3. *We define the action of C_p on $Hom(A(e), B(e))$ by a general construction given as follows. Let V and W be G-modules. Then, the action of G on $Hom(V, W)$ is given by*

$$^{x}\varphi : V \longrightarrow W$$

$$(^{x}\varphi)(v) = x\varphi(x^{-1}v)$$

Where $\varphi \in Hom(V, W)$, $x \in G$ and $v \in V$.

4. *We define the restriction map R by $R(f_{C_p}, f_e) = f_e$.*

5. *We define the transfer map $T : Hom(A(e), B(e)) \longrightarrow Hom_{\mathcal{M}ack(G)}(A, B)$ as follows. let $\psi : A(e) \longrightarrow B(e)$ be a homomorphism of abelian groups. Then, we need to define $T(\psi)$ to be a morphism of Mackey functors from A to B. That is, from the commutative diagram:*

$$\begin{tikzcd}
A(C_p) \arrow[rrr, "\alpha"] \arrow[d, bend left, "res_A"] & & & B(C_p) \arrow[d, bend left, "res_B"] \\
A(e) \arrow[rrr, "\beta"'] \arrow[u, bend left, "tr_A"] & & & B(e) \arrow[u, bend left, "tr_B"]
\end{tikzcd}$$

we define $T(\psi)$ by

$$\alpha(a) = tr_B \psi res_A(a)$$

and

$$\beta(x) = \psi(x) + c\psi(c^{p-1}x) + \cdots + c^{p-1}\psi(cx) = \sum_{i+j=p} c^i \psi(c^j x)$$

where $a \in A(C_p)$ and $x \in A(e)$.

4.4 C_p-Green Functors

In this section, we will provide a detailed description of C_p-Green functors and modules over C_p-Green functors.

Definition 4.4.1. *A C_p-Mackey functor A is a C_p-Green functor if the following axioms hold:*

1. *$A(C_p)$ and $A(e)$ are rings.*

2. *The restriction maps $res_e^{C_p} : A(C_p) \longrightarrow A(e)$ and the conjugation maps $c : A(e) \longrightarrow A(e)$ are ring homomorphisms.*

3. *A satisfies Frobenius relations: If $e \leq C_p$ are subgroups of C_p then*

$$tr_e^{C_p}(a) \cdot b = tr_e^{C_p}(a \cdot res_e^{C_p}(b))$$

$$b \cdot tr_e^{C_p}(a) = tr_e^{C_p}(res_e^{C_p}(b) \cdot a)$$

   for all $a \in A(e)$ and $b \in A(C_p)$.

We display the C_p-Green functor with the following diagram.

$$\begin{tikzcd} A(C_p) \arrow[d, bend left=50, "res_e^{C_p}"] \\ A(e) \arrow[u, bend left=50, "tr_e^{C_p}"] \arrow[loop below, "c"] \end{tikzcd}$$

Definition 4.4.2. *A C_p-Mackey functor M is a module over the C_p-Green functor A if the following axioms hold:*

1. *The modules $M(e)$ and $M(C_p)$ have the structure of an $A(e)$-module and an $A(C_p)$-module, respectively.*

2. *If $e \leq C_p$ are subgroups of C_p, $a \in A(C_p)$, $m \in M(C_p)$, $x \in A(e)$ and $u \in M(e)$, then*

$$res_e^{C_p}(a \cdot m) = res_e^{C_p}(a) \cdot res_e^{C_p}(m)$$

   and

$$c(x \cdot u) = c(x) \cdot c(u).$$

3. *Frobenius relations: If $x \in A(e)$ and $m \in M(C_p)$, then*

$$tr_e^{C_p}(x) \cdot m = tr_e^{C_p}(x \cdot res_e^{C_p}(m))$$

   and
   if $a \in A(C_p)$ and $u \in M(e)$, then

$$a \cdot tr_e^{C_p}(u) = tr_e^{C_p}(res_e^{C_p}(a) \cdot u).$$

Remark 4.4.3. A Mackey functor has a close connection to induction theorems in representation theory, see [4] and [7].

4.4.1 Hochschild Homology of C_p-Mackey Functors

In this subsection, we will extend the definition of Hochschild homology to C_p-Mackey functors.

Definition 4.4.4. *Let A be a C_p-Green functor and M be a bimodule over the C_p-Green functor A. Then, the Hochschild homology of a C_p-Mackey functor which is again a C_p-Mackey functor is the homology of the following diagram of chain complexes:*

$$\begin{tikzcd}
M(C_p) \arrow[d, bend left, "res_e^{C_p}"] & \left\{\substack{M(e)\otimes A(e)\\ \oplus \\ M(C_p)\otimes A(C_p)}\right\}/\sim \arrow[l, "d_0'"'] \arrow[d, bend left, "res_e^{C_p}"] & \left\{\substack{M(e)\otimes A(e)\otimes A(e)\\ \oplus \\ M(C_p)\otimes A(C_p)\otimes A(C_p)}\right\}/\sim \arrow[l, "d_1'"'] \arrow[d, bend left, "res_e^{C_p}"] & \arrow[l, "d_2'"'] \\
M(e) \arrow[u, bend left, "tr_e^{C_p}"] & M(e)\otimes A(e) \arrow[l, "d_0"] \arrow[u, bend left, "tr_e^{C_p}"] & M(e)\otimes A(e)\otimes A(e) \arrow[l, "d_1"] \arrow[u, bend left, "tr_e^{C_p}"] & \arrow[l, "d_2"]
\end{tikzcd}$$

where the boundary maps are given as follows:

1. *The boundary d_n of the lower complex is given by:*

$$\begin{aligned}
d_n(m \otimes x_1 \otimes \cdots \otimes x_n) &= (m \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n) \\
&\quad + \sum_{0<i<n} (-1)^i (m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n) \\
&\quad + (-1)^n (r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}).
\end{aligned}$$

2. *The boundaries d_n' of the upper complex are given by:*

   (a) $d_0'(m \otimes a \oplus u \otimes x) = ma - am + tr_e^{C_p}(ux - xu)$.

   (b) $$\begin{aligned} d_1'(m \otimes a \otimes b \oplus u \otimes x \otimes y) &= ma \otimes b - m \otimes ab + bm \otimes a \\ &\quad + [ux \otimes y] - [u \otimes xy] + [yu \otimes x]. \end{aligned}$$

   (c) $$\begin{aligned} d_2'(m \otimes a \otimes b \otimes c \oplus u \otimes x \otimes y \otimes z) &= ma \otimes b \otimes c - m \otimes ab \otimes c \\ &\quad + m \otimes a \otimes bc - cm \otimes a \otimes b \\ &\quad + [ux \otimes y \otimes z] - [u \otimes xy \otimes z] \\ &\quad + [u \otimes x \otimes yz] - [zu \otimes x \otimes y]. \end{aligned}$$

For all $m \in M(C_p)$, a, b and $c \in A(C_p)$, $u \in M(e)$ and x, y and $z \in A(e)$. Hence,

$$H_n(A, M) = H_n(C_n(A, M)) = \begin{cases} H_{n_{C_p}}(C_{n_{C_p}}(A, M)), \\ H_{n_e}(C_{n_e}(A, M)), \end{cases}$$

where $C_n(A, M) = M \otimes A^{\otimes n}$.

4.4.2 Hochschild Cohomology of C_p-Mackey Functors

The definition of Hochschild cohomology can be extended to C_p-Mackey functors as discussed in this subsection.

Definition 4.4.5. *Let A be a C_p-Green functor and M be a bimodule over the C_p-Green functor A. Then, the Hochschild cohomology of a C_p-Mackey functor which is again a C_p-Mackey functor is the cohomology of the following diagram of cochain complexes:*

$$\begin{tikzcd}
M(C_p) \arrow[r, "b_0'"] \arrow[d, bend left, "res_e^{C_p}"] & Hom_{\mathcal{M}ack}(A, M) \arrow[r, "b_1'"] \arrow[d, bend left, "R"] & Hom_{\mathcal{M}ack}(A \otimes A, M) \arrow[r, "b_2'"] \arrow[d, bend left, "R"] & {} \\
M(e) \arrow[r, "b_0"'] \arrow[u, bend left, "tr_e^{C_p}"] & Hom(A(e), M(e)) \arrow[r, "b_1"'] \arrow[u, bend left, "T"] & Hom(A(e) \otimes A(e), M(e)) \arrow[r, "b_2"'] \arrow[u, bend left, "T"] & {}
\end{tikzcd}$$

where the coboundary maps are given as follows:

1. *The coboundary b_n of the lower complex is given by:*

$$\begin{aligned} b_n(f_e)(x_1, \cdots, x_{n+1}) &= x_1 f_e(x_2, \cdots, x_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f_e(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) \\ &+ (-1)^{n+1} f_e(x_1, \cdots, x_n) x_{n+1}. \end{aligned}$$

2. *The boundaries b_n' of the upper complex are given by:*

   (a) $\begin{cases} b_0'(m)(a) = am - ma. \\ b_0'(m)(x) = x \cdot res_e^{C_p}(m) - res_e^{C_p}(m) \cdot x. \end{cases}$

   (b) $\begin{cases} b_1'(f_{C_p})(a, b) = a f_{C_p}(b) - f_{C_p}(ab) + f_{C_p}(a) b. \\ b_1'(f_e)(x, y) = x f_e(y) - f_e(xy) + f_e(x) y. \end{cases}$

$$(c) \begin{cases} b_2'(f_{C_p})(a,b,c) = af_{C_p}(b,c) - f_{C_p}(ab,c) + f_{C_p}(a,bc) - f_{C_p}(a,b)c. \\ b_2'(f_e)(x,y,z) = xf_e(y,z) - f_e(xy,z) + f_e(x,yz) - f_e(x,y)z. \end{cases}$$

For all $m \in M(C_p)$, a, b and $c \in A(C_p)$, $u \in M(e)$ and x, y and $z \in A(e)$. Hence,

$$H^n(A,M) = H^n(C^n(A,M)) = \begin{cases} H^n_{C_p}(C^n_{C_p}(A,M)), \\ H^n_e(C^n_e(A,M)), \end{cases}$$

where $C^n(A,M) = C_p\text{-}\mathcal{HOM}(A^{\otimes n}, M)$.

4.5 Classification of Singular Extension of C_p-Green Functors

It is a well-known fact that the second Hochschild cohomology group classifies the singular extensions of associative algebras[22]. Here we obtain a similar result for C_p-Green functors.

Definition 4.5.1. *Let A be a C_p-Green functor and M be an A-bimodule. A singular extension E of A by M is an exact sequence of Mackey functors*

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0,$$

where B is a C_p-Green functor, j is a homomorphism of C_p-Green functors and i is a homomorphism of C_p-Mackey functors such that the following two sequences:

$$0 \to M(C_p) \xrightarrow{i_{C_p}} B(C_p) \xrightarrow{j_{C_p}} A(C_p) \to 0$$

and

$$0 \to M(e) \xrightarrow{i_e} B(e) \xrightarrow{j_e} A(e) \to 0$$

are singular extensions of the ring $A(C_p)$ by $M(C_p)$ and the ring $A(e)$ by $M(e)$.

Definition 4.5.2. *A singular extension $E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$ is called M-split if there is an abelian group homomorphisms:*

$$s_{C_p} = s(C_p) : A(C_p) \longrightarrow B(C_p)$$

and

$$s_e = s(e) : A(e) \longrightarrow B(e)$$

such that

1. $j_{C_p} \circ s_{C_p} = id_{A(C_p)}$ *and* $j_e \circ s_e = id_{A(e)}$.

2. s_{C_p} *and* s_e *must be compatible with transfer, restriction and conjugation maps in the following sense:*

   (a) $res_B \circ s_{C_p} = s_e \circ res_A$.

   (b) $tr_B \circ s_e = s_{C_p} \circ tr_A$.

   (c) $c_B \circ s_e = s_e \circ c_A$.

Definition 4.5.3. *Let A be a C_p-Green functor and M be an A-bimodule. A C_p-Green 2-cocycle $\mathcal{Z}^2_{C_p}(A, M)$ of A with values in M is a pair (f_{C_p}, f_e), where*

$$f_{C_p} : A(C_p) \times A(C_p) \longrightarrow M(C_p)$$

and

$$f_e : A(e) \times A(e) \longrightarrow M(e)$$

are bilinear maps, satisfying the following conditions:

1. $xf_{C_p}(y, z) + f_{C_p}(x, yz) = f_{C_p}(xy, z) + f_{C_p}(x, y)z$.

2. $af_e(b, c) + f_e(a, bc) = f_e(ab, c) + f_e(a, b)c$.

3. $c_M(f_e(a, b)) = f_e(c_A(a), c_A(b))$.

4. $res_M(f_{C_p}(x, y)) = f_e(res_A(x), res_A(y))$.

5. $f_{C_p}(tr_A(a), x) = tr_M(f_e(a, res_A(x)))$.

6. $f_{C_p}(x, tr_A(a)) = tr_M(f_e(res_A(x), a))$.

Proposition 4.5.4. *If (f_{C_p}, f_e) is C_p-Green 2-cocycle, then one can construct a C_p-Green functor $B_{f_{C_p}, f_e}$ as follows:*

1. $B_{f_{C_p}}(C_p) = M(C_p) \oplus A(C_p)$ *as an associative ring with multiplication*

$$(u, x)(v, y) = (uy + xv + f_{C_p}(x, y), xy).$$

2. $B_{f_e}(e) = M(e) \oplus A(e)$ *as an associative ring with multiplication*

$$(m, a)(n, b) = (mb + an + f_e(a, b), ab).$$

3. $c_B(m, a) = (c_M(m), c_A(a))$.

4. $tr_B(m, a) = (tr_M(m), tr_A(a))$.

5. $res_B(u, x) = (res_M(u), res_A(x))$.

We can describe the C_p*-Green functor* $B_{f_{C_p}, f_e}$ *with the following diagram.*

$$\begin{tikzcd}
B_{f_{C_p}}(C_p) \arrow[d, bend left, "res_B"] \\
B_{f_e}(e) \arrow[u, bend left, "tr_B"] \arrow[loop below, "c_B"]
\end{tikzcd}$$

Proof. We need to verify that $B_{f_{C_p}, f_e}$ satisfies all axioms of a C_p-Green functor. Observe that $B_{f_{C_p}}(C_p)$ and $B_{f_e}(e)$ are associative rings since f_{C_p} and f_e are 2-cocycles. Next, we need to check that the conjugation map c_B is a ring homomorphism. That is, for (m, a) and (n, b) be elements in $B_{f_e}(e)$ we have

$$\begin{aligned} c_B((m, a) \cdot (n, b)) &= c_B(mb + an + f_e(a, b), ab) \\ &= (c_M(m) \cdot c_A(b) + c_A(a) \cdot c_M(n) + c_M(f_e(a, b)), c_A(a) \cdot c_A(b)). \end{aligned}$$

We also have

$$\begin{aligned} c_B(m, a) \cdot c_B(n, b) &= (c_M(m), c_A(a)) \cdot (c_M(n), c_A(b)) \\ &= (c_M(m) \cdot c_A(b) + c_A(a) \cdot c_M(n) + f_e(c_A(a), c_A(b)), c_A(a) \cdot c_A(b)). \end{aligned}$$

Hence, from condition 3 in definition 4.5.3 it follows that c_B is a ring homomorphism. Similarly, we need to check that the restriction map res_B is ring homomorphism. That is, for (u, x) and (v, y) be elements in $B_{f_{C_p}}(C_p)$ we have

$$\begin{aligned} res_B((u, x) \cdot (v, y)) =& res_B(uy + xv + f_{C_p}(x, y), xy) \\ =& (res_M(u) \cdot res_A(y) + res_A(x) \cdot res_M(v) \\ &+ res_M(f_{C_p}(x, y)), res_A(x) \cdot res_A(y)). \end{aligned}$$

We also have

$$\begin{aligned} res_B(u, x) \cdot res_B(v, y) =& (res_M(u), res_A(x)) \cdot (res_M(v), res_A(y)) \\ =& (res_M(u) \cdot res_A(y) + res_A(x) \cdot res_M(v) \\ &+ f_e(res_A(x), res_A(y)), res_A(x) \cdot res_A(y)). \end{aligned}$$

Therefore, from condition 4 in definition 4.5.3 it follows that res_B is a ring homomorphism. Finally, we need to check that $B_{f_{C_p},f_e}$ satisfies the Frobenius relations. That is, for (m,a) be an element in $B_{f_e}(e)$ and (u,x) be an element in $B_{f_{C_p}}(C_p)$ we have

$$\begin{aligned} tr_B(m,a)\cdot(u,x) &= (tr_M(m), tr_A(a))\cdot(u,x) \\ &= (tr_M(m)\cdot x + tr_A(a)\cdot u + f_{C_p}(tr_A(a),x), tr_A(a)\cdot x). \end{aligned}$$

We also have

$$\begin{aligned} tr_B((m,a)\cdot res_B(u,x)) &= tr_B((m,a)\cdot(res_M(u), res_A(x))) \\ &= tr_B(m\cdot res_A(x) + a\cdot res_M(u) \\ &+ f_e(a, res_A(x)), a\cdot res_A(x)) \\ &= (tr_M(m\cdot res_A(x)) + tr_A(a\cdot res_M(u)) \\ &+tr_M(f_e(a, res_A(x))), tr_A(a\cdot res_A(x))). \end{aligned}$$

Hence, from the definition of C_p-Green functors, definition of modules over C_p-Green functors and condition 5 in definition 4.5.3 it follows that:

$$tr_B(m,a)\cdot(u,x) = tr_B((m,a)\cdot res_B(u,x)).$$

Likewise, from the definition of C_p-Green functors, definition of modules over C_p-Green functors and condition 6 in definition 4.5.3 it follows that:

$$(u,x)\cdot tr_B(m,a) = tr_B(res_B(u,x)\cdot(m,a)).$$

$\square$

Definition 4.5.5. *Let A be a C_p-Green functor and M be an A-bimodule. We define $Ext(A,M)$ to be the set of equivalence classes of M-split extensions of A by M.*

Definition 4.5.6. *Let A be a C_p-Green functor and M be an A-bimodule. We define*

$$C^1_{C_p}(A,M) = \left\{ (h_{C_p}, h_e) \;\middle|\; \begin{array}{r} \forall x\in A(C_p),\;\; res_M(h_{C_p}(x)) = h_e(res_A(x)) \\ \forall a\in A(e),\;\; c_M(h_e(a)) = h_e(c_A(a)) \\ \forall a\in A(e),\;\; tr_M(h_e(a)) = h_{C_p}(tr_A(a)) \end{array} \right\},$$

where $h_{C_p} : A(C_p) \longrightarrow M(C_p)$ and $h_e : A(e) \longrightarrow M(e)$. Moreover, there exists a map $\partial : C^1_{C_p}(A,M) \longrightarrow \mathcal{Z}^2_{C_p}(A,M)$ such that $\partial(h_{C_p}, h_e) = (\delta h_{C_p}(x,y), \delta h_e(a,b))$, *where* $\delta h_{C_p}(x,y) = xh_{C_p}(y) - h_{C_p}(xy) + h_{C_p}(x)y$ *and* $\delta h_e(a,b) = ah_e(b) - h_e(ab) + h_e(a)b$.

Definition 4.5.7. *Let* A *be* C_p*-Green functor and* M *be an* A*-bimodule. We define the second Hochschild cohomology by*

$$H^2_{C_p}(A,M) = coker\ \partial.$$

Theorem 4.5.8. *Let* A *be a* C_p*-Green functor,* M *be an* A*-bimodule and* $Ext(A,M)$ *be the set of equivalence classes of* M*-split extensions of* A *by* M*. There is a one-to-one correspondence between the elements of* $Ext(A,M)$ *and those of* $H^2_{C_p}(A,M)$*.*

Proof. To prove the theorem, we are going to follow these steps.
Step 1. Show that there is a well-defined map from $Ext(A,M)$ to $H^2_{C_p}(A,M)$.
Step 2. Show that there is a well-defined map from $H^2_{C_p}(A,M)$ to $Ext(A,M)$.
Step 3. Show that these two maps are inverse to each other.
Step 1. Consider a singular extension

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

and let

$$s_{C_p} : A(C_p) \longrightarrow B(C_p)$$

and

$$s_e : A(e) \longrightarrow B(e)$$

be an abelian group homomorphisms such that

$$j_{C_p} \circ s_{C_p} = id_{A(C_p)}$$

and

$$j_e \circ s_e = id_{A(e)}.$$

Then, for every $x,y \in A(C_p)$ and $a,b \in A(e)$, there exists a uniquely determined element $f_{C_p}(x,y) \in M(C_p)$ and $f_e(a,b) \in M(e)$ such that

$$s_{C_p}(x)s_{C_p}(y) = s_{C_p}(xy) + i_{C_p}f_{C_p}(x,y). \tag{4.1}$$

and

$$s_e(a)s_e(b) = s_e(ab) + i_e f_e(a,b). \tag{4.2}$$

For $x, y, z \in A(C_p)$,

$$\begin{aligned} s_{C_p}(x)(s_{C_p}(y)s_{C_p}(z)) &= s_{C_p}(x)(s_{C_p}(yz) + i_{C_p}f_{C_p}(y,z)) \\ &= s_{C_p}(x)s_{C_p}(yz) + s_{C_p}(x)i_{C_p}f_{C_p}(y,z) \\ &= s_{C_p}(xyz) + i_{C_p}f_{C_p}(x,yz) + s_{C_p}(x)i_{C_p}f_{C_p}(y,z). \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} (s_{C_p}(x)s_{C_p}(y))s_{C_p}(z) &= (s_{C_p}(xy) + i_{C_p}f_{C_p}(x,y))s_{C_p}(z) \\ &= s_{C_p}(xy)s_{C_p}(z) + i_{C_p}f_{C_p}(x,y)s_{C_p}(z) \\ &= s_{C_p}(xyz) + i_{C_p}f_{C_p}(xy,z) + i_{C_p}f_{C_p}(x,y)s_{C_p}(z). \end{aligned} \tag{4.4}$$

Thus, multiplication in $B(C_p)$ is associative which follows from (4.3) and (4.4) that:

$$xf_{C_p}(y,z) - f_{C_p}(xy,z) + f_{C_p}(x,yz) - f_{C_p}(x,y)z = 0$$

showing that f_{C_p} is a 2-cocycle. Similarly, for $a, b, c \in A(e)$,

$$\begin{aligned} s_e(a)(s_e(b)s_e(c)) &= s_e(a)(s_e(bc) + i_ef_e(b,c)) \\ &= s_e(a)s_e(bc) + s_e(a)i_ef_e(b,c) \\ &= s_e(abc) + i_ef_e(a,bc) + s_e(a)i_ef_e(b,c). \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} (s_e(a)s_e(b))s_e(c) &= (s_e(ab) + i_ef_e(a,b))s_e(c) \\ &= s_e(ab)s_e(c) + i_ef_e(a,b)s_e(c) \\ &= s_e(abc) + i_ef_e(ab,c) + i_ef_e(a,b)s_e(c). \end{aligned} \tag{4.6}$$

Thus, multiplication in $B(e)$ is associative which follows from (4.5) and (4.6) that:

$$af_e(b,c) - f_e(ab,c) + f_e(a,bc) - f_e(a,b)c = 0$$

showing that f_e is a 2-cocycle. Next, we know that the conjugation map $c_B : B(e) \longrightarrow B(e)$ is a ring homomorphism and by applying c_B to equation (4.2) we have

$$\begin{aligned} c_B(s_e(a)s_e(b)) &= c_B(s_e(ab) + i_ef_e(a,b)) \\ \Rightarrow c_B(s_e(a))c_B(s_e(b)) &= c_B(s_e(ab)) + c_B(i_ef_e(a,b)) \\ \Rightarrow s_e(c_A(a))s_e(c_A(b)) &= s_e(c_A(ab))) + i_e(c_M(f_e(a,b))) \end{aligned}$$

$$\Rightarrow \cancel{s_e(c_A(a)c_A(b))} + \cancel{i_e} f_e(c_A(a), c_A(b)) = \cancel{s_e(c_A(a)c_A(b))} + \cancel{i_e}(c_M(f_e(a,b)))$$
$$\Rightarrow f_e(c_A(a), c_A(b)) = c_M(f_e(a,b)).$$

Similarly, we know that the restriction map $res_B : B(C_p) \longrightarrow B(e)$ is a ring homomorphism and by applying res_B to equation (4.1) we have

$$\begin{aligned}
res_B(s_{C_p}(x)s_{C_p}(y)) &= res_B(s_{C_p}(xy) + i_{C_p}f_{C_p}(x,y)) \\
\Rightarrow res_B(s_{C_p}(x))res_B(s_{C_p}(y)) &= res_B(s_{C_p}(xy)) + res_B(i_{C_p}f_{C_p}(x,y)) \\
\Rightarrow s_e(res_A(x))s_e(res_A(y)) &= s_e(res_A(xy)) + i_e(res_M(f_{C_p}(x,y))) \\
\Rightarrow \cancel{s_e(res_A(x)res_A(y))} + \cancel{i_e}(f_e(res_A(x), res_A(y))) & \\
&= \cancel{s_e(res_A(x)res_A(y)))} + \cancel{i_e}(res_M(f_{C_p}(x,y))) \\
\Rightarrow f_e(res_A(x), res_A(y)) &= res_M(f_{C_p}(x,y)).
\end{aligned}$$

Furthermore, from (4.1) we have

$$\begin{aligned}
s_{C_p}(y)s_{C_p}(x) &= s_{C_p}(yx) + i_{C_p}f_{C_p}(y,x) \\
\Rightarrow i_{C_p}f_{C_p}(y,x) &= s_{C_p}(y)s_{C_p}(x) - s_{C_p}(yx).
\end{aligned}$$

Now, by substituting $y = tr_A(a)$ in the above equation we have

$$\begin{aligned}
i_{C_p}f_{C_p}(tr_A(a), x) &= s_{C_p}(tr_A(a))s_{C_p}(x) - s_{C_p}(tr_A(a)\cdot x) \\
\Rightarrow i_{C_p}f_{C_p}(tr_A(a), x) &= tr_B(s_e(a))s_{C_p}(x) - s_{C_p}(\overbrace{tr_A(a\cdot res_A(x))}^{\text{Frobenius relation}}) \\
\Rightarrow i_{C_p}f_{C_p}(tr_A(a), x) &= \overbrace{tr_B(s_e(a)\cdot res_B(s_{C_p}(x)))}^{\text{Frobenius relation}} - tr_B(s_e(a\cdot res_A(x))) \\
\Rightarrow i_{C_p}f_{C_p}(tr_A(a), x) &= tr_B(s_e(a)\cdot s_e(res_A(x))) - tr_B(s_e(a\cdot res_A(x))) \\
\Rightarrow i_{C_p}f_{C_p}(tr_A(a), x) &= tr_B\overbrace{(s_e(a\cdot res_A(x)) + i_e f_e(a, res_A(x)))}^{\text{from (4.2)}} - tr_B(s_e(a\cdot res_A(x))) \\
\Rightarrow i_{C_p}f_{C_p}(tr_A(a), x) &= \cancel{tr_B(s_e(a\cdot res_A(x)))} + tr_B(i_e f_e(a, res_A(x))) - \cancel{tr_B(s_e(a\cdot res_A(x)))} \\
\Rightarrow \cancel{i_{C_p}}f_{C_p}(tr_A(a), x) &= \cancel{i_{C_p}}(tr_M(f_e(a, res_A(x)))) \\
\Rightarrow f_{C_p}(tr_A(a), x) &= tr_M(f_e(a, res_A(x))).
\end{aligned}$$

Similarly, from (4.1) we have

$$\Rightarrow i_{C_p}f_{C_p}(x,y) = s_{C_p}(x)s_{C_p}(y) - s_{C_p}(xy).$$

Now, by substituting $y = tr_A(a)$ in the above equation we have

$$i_{C_p}f_{C_p}(x, tr_A(a)) = s_{C_p}(x)s_{C_p}(tr_A(a)) - s_{C_p}(x \cdot tr_A(a))$$

$$\Rightarrow i_{C_p}f_{C_p}(x, tr_A(a)) = s_{C_p}(x)tr_B(s_e(a)) - s_{C_p}\overbrace{(tr_A(res_A(x) \cdot a))}^{\text{Frobenius relation}}$$

$$\Rightarrow i_{C_p}f_{C_p}(x, tr_A(a)) = \overbrace{tr_B(res_B(s_{C_p}(x) \cdot s_e(a)))}^{\text{Frobenius relation}} - tr_B(s_e(res_A(x) \cdot a))$$

$$\Rightarrow i_{C_p}f_{C_p}(x, tr_A(a)) = tr_B(s_e(res_A(x) \cdot s_e(a))) - tr_B(s_e(res_A(x) \cdot a))$$

$$\Rightarrow i_{C_p}f_{C_p}(x, tr_A(a)) = tr_B\overbrace{(s_e(res_A(x) \cdot a)) + i_e f_e(res_A(x), a)}^{\text{from (4.2)}} - tr_B(s_e(res_A(x) \cdot a))$$

$$\Rightarrow i_{C_p}f_{C_p}(x, tr_A(a)) = \cancel{tr_B(s_e(res_A(x) \cdot a))} + tr_B(i_e f_e(res_A(x), a)) - \cancel{tr_B(s_e(res_A(x) \cdot a))}$$

$$\Rightarrow \cancel{i_{C_p}}f_{C_p}(x, tr_A(a)) = \cancel{i_{C_p}}(tr_M(f_e(res_A(x), a)))$$

$$\Rightarrow f_{C_p}(x, tr_A(a)) = tr_M(f_e(res_A(x), a)).$$

Hence, f_{C_p} and $f_e \in \mathcal{Z}^2_{C_p}(A, M)$ satisfy all conditions in definition 4.5.3. Let

$$s'_{C_p} : A(C_p) \longrightarrow B(C_p)$$

and

$$s'_e : A(e) \longrightarrow B(e)$$

be two abelian homomorphisms and let

$$g_{C_p} : A(C_p) \times A(C_p) \longrightarrow M(C_p)$$

and

$$g_e : A(e) \times A(e) \longrightarrow M(e)$$

be the 2-cocycles corresponding to choices of s'_{C_p} and s'_e. Then,

$$j_{C_p} \circ s_{C_p}(x) = x = j_{C_p} \circ s'_{C_p}(x)$$

and

$$j_e \circ s_e(a) = a = j_e \circ s'_e(a)$$

for every $x \in A(C_p)$ and $a \in A(e)$, and so there exists $C^1_{C_p}(A, M) \xrightarrow{\partial} \mathcal{Z}^2_{C_p}(A, M)$ such that

$$s'_{C_p}(x) = i_{C_p}h_{C_p}(x) + s_{C_p}(x) \tag{4.7}$$

and

$$s_e^{'}(a) = i_e h_e(a) + s_e(a), \tag{4.8}$$

where $h_{C_p} : A(C_p) \longrightarrow M(C_p)$, $h_e : A(e) \longrightarrow M(e)$, $x \in A(C_p)$ and $a \in A(e)$. Now, for $x, y \in A(C_p)$ and by substituting (4.7) in (4.1) we have

$$\begin{aligned}
i_{C_p} f_{C_p}(x,y) + s_{C_p}^{'}(xy) - i_{C_p} h_{C_p}(xy) &= (s_{C_p}^{'}(x) - i_{C_p} h_{C_p}(x))(s_{C_p}^{'}(y) - i_{C_p} h_{C_p}(y)) \\
\Rightarrow i_{C_p} f_{C_p}(x,y) + s_{C_p}^{'}(xy) - i_{C_p} h_{C_p}(xy) &= s_{C_p}^{'}(x) s_{C_p}^{'}(y) - s_{C_p}^{'}(x) i_{C_p} h_{C_p}(y) \\
&\quad - i_{C_p} h_{C_p}(x) s_{C_p}^{'}(y) + \overbrace{i_{C_p} h_{C_p}(x) i_{C_p} h_{C_p}(y)}^{=0} \\
\Rightarrow i_{C_p} f_{C_p}(x,y) + \cancel{s_{C_p}^{'}(xy)} - i_{C_p} h_{C_p}(xy) &= \cancel{s_{C_p}^{'}(xy)} + i_{C_p} g_{C_p}(x,y) \\
&\quad - s_{C_p}^{'}(x) i_{C_p} h_{C_p}(y) - i_{C_p} h_{C_p}(x) s_{C_p}^{'}(y) \\
\Rightarrow \delta h_{C_p}(x,y) = g_{C_p}(x,y) - f_{C_p}(x,y) &= x h_{C_p}(y) - h_{C_p}(xy) + h_{C_p}(x) y
\end{aligned}$$

so that f_{C_p} and g_{C_p} differ by a 2-coboundary. Likewise, for $a, b \in A(e)$ and by substituting (4.8) in (4.2) we have

$$\begin{aligned}
i_e f_e(a,b) + s_e^{'}(ab) - i_e h_e(ab) &= (s_e^{'}(a) - i_e h_e(a))(s_e^{'}(b) - i_e h_e(b)) \\
\Rightarrow i_e f_e(a,b) + s_e^{'}(ab) - i_e h_e(ab) &= s_e^{'}(a) s_e^{'}(b) - s_e^{'}(a) i_e h_e(b) \\
&\quad - i_e h_e(a) s_e^{'}(b) + \overbrace{i_e h_e(a) i_e h_e(b)}^{=0} \\
\Rightarrow i_e f_e(a,b) + \cancel{s_e^{'}(ab)} - i_e h_e(ab) &= \cancel{s_e^{'}(ab)} + i_e g_e(a,b) \\
&\quad - s_e^{'}(a) i_e h_e(b) - i_e h_e(a) s_e^{'}(b) \\
\Rightarrow \delta h_e(a,b) = g_e(a,b) - f_e(a,b) &= a h_e(b) - h_e(ab) + h_e(a) b
\end{aligned}$$

so that f_e and g_e differ by a 2-coboundary. Therefore, we show that there exists a well-defined map from $Ext(A, M)$ to $H^2_{C_p}(A, M)$.

Step 2. Let $[f_{C_p}]$ and $[f_e] \in H^2_{C_p}(A, M)$, where f_{C_p} and $f_e \in \mathcal{Z}^2_{C_p}(A, M)$. Then, we define the C_p-Green functor $B_{f_{C_p}, f_e}$ as in Proposition 4.5.4. Therefore, the extension associated to f_{C_p} and f_e is the extension

$$E_{f_{C_p}, f_e} : 0 \to M \xrightarrow{i} B_{f_{C_p}, f_e} \xrightarrow{j} A \to 0,$$

where j is a homomorphism of C_p-Green functors and i is a homomorphism of C_p-Mackey functors. Now, we need to show that $[E_{f_{C_p}}]$ and $[E_{f_e}]$ are independent of the choices of f_{C_p} and f_e. In other words, if $[f_{C_p}] = [g_{C_p}] \Leftrightarrow f_{C_p} = g_{C_p} + \delta h_{C_p}$ and $[f_e] = [g_e] \Leftrightarrow f_e = g_e + \delta h_e$.

Two extensions $E_{f_{C_p}}$ and $E_{g_{C_p}}$ are equivalent if and only if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M(C_p) \arrow[r, "i_{C_p}"] \arrow[d, equal] & B_{f_{C_p}}(C_p) \arrow[r, "j_{C_p}"] \arrow[d, "\alpha_{C_p}"] & A(C_p) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M(C_p) \arrow[r, "i'_{C_p}"'] & B_{g_{C_p}}(C_p) \arrow[r, "j'_{C_p}"'] & A(C_p) \arrow[r] & 0
\end{tikzcd}$$

with α_{C_p} a homomorphism of rings. The commutativity of this diagram implies that

$$\alpha_{C_p}(u,x) = (u + h_{C_p}(x), x)$$

for $h_{C_p} \in C^1_{C_p}(A,M)$. The fact that α_{C_p} is a ring homomorphism gives the following equation,

$$\begin{aligned} \alpha_{C_p}((u,x)(v,y)) &= \alpha_{C_p}(uy + xv + f_{C_p}(x,y), xy) \\ &= uy + xv + f_{C_p}(x,y) + h_{C_p}(x,y), xy) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \alpha_{C_p}(u,x)\alpha_{C_p}(v,y) &= (u + h_{C_p}(x), x)(v + h_{C_p}(y), y) \\ &= uy + h_{C_p}(x)y + xv + xh_{C_p}(y) + g_{C_p}(x,y), xy) \end{aligned} \tag{4.10}$$

Hence, from (4.9) and (4.10) we obtain

$$f_{C_p}(x,y) - g_{C_p}(x,y) = xh_{C_p}(y) - h_{C_p}(x,y) + h_{C_p}(x)y = \delta h_{C_p}(x,y)$$

that is, $f_{C_p} - g_{C_p}$ is a 2-coboundary. Conversely, if $f_{C_p} - g_{C_p}$ is a 2-coboundary, then $E_{f_{C_p}}$ and $E_{g_{C_p}}$ are equivalent. Likewise, two extensions E_{f_e} and E_{g_e} are equivalent if and only if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M(e) \arrow[r, "i_e"] \arrow[d, equal] & B_{f_e}(e) \arrow[r, "j_e"] \arrow[d, "\alpha_e"] & A(e) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M(e) \arrow[r, "i'_e"'] & B_{g_e}(e) \arrow[r, "j'_e"'] & A(e) \arrow[r] & 0
\end{tikzcd}$$

with α_e a homomorphism of rings. The commutativity of this diagram implies that

$$\alpha_e(m,a) = (m + h_e(a), a)$$

for $h_e \in C^1_{C_p}(A,M)$. The fact that α_e is a ring homomorphism gives the following equation,

$$\begin{aligned} \alpha_e((m,a)(n,b)) &= \alpha_e(mb + an + f_e(a,b), ab) \\ &= mb + an + f_e(a,b) + h_e(a,b), ab) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \alpha_e(m,a)\alpha_e(n,b) &= (m + h_e(a), a)(n + h_e(b), b) \\ &= mb + h_e(a)b + an + ah_e(b) + g_e(a,b), ab) \end{aligned} \tag{4.12}$$

Hence, from (4.11) and (4.12) we obtain

$$f_e(a,b) - g_e(a,b) = ah_e(b) - h_e(a,b) + h_e(a)b = \delta h_e(a,b)$$

That is, $f_e - g_e$ is a 2-coboundary. Conversely, if $f_e - g_e$ is a 2-coboundary, then E_{f_e} and E_{g_e} are equivalent. Moreover, we need to check that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[rr] & & M(C_p) \arrow[rr, "i_{C_p}"] \arrow[dl, equal] \arrow[dd, bend left, "res_M"] & & B_{f_{C_p}}(C_p) \arrow[rr, "j_{C_p}"] \arrow[dl, "\alpha_{C_p}"'] \arrow[dd, bend left, "res_B"] & & A(C_p) \arrow[r] \arrow[dl, equal] \arrow[dd, bend left, "res_A"] & 0 \\
0 \arrow[r] & M(C_p) \arrow[rr, "i'_{C_p}"] \arrow[dd, bend left, "res_M"] & & B_{g_{C_p}}(C_p) \arrow[rr, "j'_{C_p}"] \arrow[dd, bend left, "res_B"] & & A(C_p) \arrow[rr] \arrow[dd, bend left, "res_A"] & & 0 \\
0 \arrow[rr] & & M(e) \arrow[rr, "i_e"] \arrow[uu, bend left, "tr_M"] \arrow[dl, equal] & & B_{f_e}(e) \arrow[rr, "j_e"] \arrow[uu, bend left, "tr_B"] \arrow[dl, "\alpha_e"] & & A(e) \arrow[r] \arrow[uu, bend left, "tr_A"] \arrow[dl, equal] & 0 \\
0 \arrow[r] & M(e) \arrow[rr, "i'_e"'] \arrow[uu, bend left, "tr_M"] & & B_{g_e}(e) \arrow[rr, "j'_e"'] \arrow[uu, bend left, "tr_B"] & & A(e) \arrow[rr] \arrow[uu, bend left, "tr_A"] & & 0
\end{tikzcd}$$

It suffices to check that $res_B \circ \alpha_{C_p}(u,x) = \alpha_e \circ res_B(u,x)$. We have

$$res_B \circ \alpha_{C_p}(u,x) = res_B(u + h_{C_p}(x), x) = (res_M(u) + res_M(h_{C_p}(x)), res_A(x))$$

and

$$\alpha_e \circ res_B(u, x) = \alpha_e(res_M(u), res_A(x)) = (res_M(u) + h_e(res_A(x)), res_A(x)).$$

Thus, from definition 4.5.6 it follows that: $res_B \circ \alpha_{C_p}(u, x) = \alpha_e \circ res_B(u, x)$. Therefore, we show that there exists a well-defined map from $H^2_{C_p}(A, M)$ to $Ext(A, M)$.

Step 3. Let f_{C_p} and f_e be 2-cocycles. Then, we define the multiplications on $B_{f_{C_p}}(C_p)$ and $B_{f_e}(e)$ as follows:

$$(u, x)(v, y) = (uy + xv + f_{C_p}(x, y), xy)$$

and

$$(m, a)(n, b) = (mb + an + f_e(a, b), ab)$$

where $u, v \in M(C_p)$, $x, y \in A(C_p)$, $m, n \in M(e)$ and $a, b \in A(e)$. The 2-cocycle property of f_{C_p} and f_e show that the multiplications on $B_{f_{C_p}}(C_p)$ and $B_{f_e}(e)$ are associative. Thus, $B_{f_{C_p}}(C_p)$ and $B_{f_e}(e)$ are associative rings. We define the maps

$$i_{C_p} : M(C_p) \longrightarrow B_{f_{C_p}}(C_p)$$

$$i_e : M(e) \longrightarrow B_{f_e}(e)$$

$$j_{C_p} : B_{f_{C_p}}(C_p) \longrightarrow A(C_p)$$

$$j_e : B_{f_e}(e) \longrightarrow A(e)$$

as follows:

$$i_{C_p}(u) = (u, 0)$$

$$i_e(m) = (m, 0)$$

$$j_{C_p}(u, x) = x$$

$$j_e(m, a) = a$$

where i_{C_p} and i_e are homomorphisms of C_p-Mackey functors and j_{C_p} and j_e are homomorphisms of C_p-Green functors and the sequence

$$E_{f_{C_p}, f_e} : 0 \to M \xrightarrow{i} B \underset{\text{s}}{\overset{\text{j}}{\rightleftarrows}} A \to 0$$

is exact. For $x \in A(C_p)$ and $a \in A(e)$, choose $s_{C_p}(x) = (0, x)$ and $s_e(a) = (0, a)$. Then, for $x, y \in A(C_p)$,

$$\begin{aligned} s_{C_p}(x)s_{C_p}(y) = (0,x)(0,y) &= (f_{C_p}(x,y), xy) \\ &= (f_{C_p}(x,y), 0) + (0, xy) \\ &= i_{C_p}(f_{C_p}(x,y)) + s_{C_p}(xy). \end{aligned}$$

Similarly, for $a, b \in A(e)$,

$$\begin{aligned} s_e(a)s_e(b) = (0,a)(0,b) &= (f_e(a,b), ab) \\ &= (f_e(a,b), 0) + (0, ab) \\ &= i_e(f_e(a,b)) + s_e(ab). \end{aligned}$$

The choices s_{C_p} and s_e thus give the 2-cocycles f_{C_p} and f_e.

Conversely, suppose that

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

is an extension and let f_{C_p} and f_e be the 2-cocycles obtained from this extension. We must show that the extension

$$E_{f_{C_p},f_e} : 0 \to M \xrightarrow{i} B_{f_{C_p},f_e} \xrightarrow{j} A \to 0$$

associated to f_{C_p} and f_e is equivalent to the given one. Indeed, E and $E_{f_{C_p},f_e}$ are equivalent if there exists a homomorphism $\theta_{f_{C_p},f_e} : B_{f_{C_p},f_e} \longrightarrow B$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, shift right, "j"'] & A \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "i"'] & B_{f_{C_p},f_e} \arrow[r, "j"'] \arrow[u, "\theta_{f_{C_p},f_e}"'] & A \arrow[r] & 0 \end{tikzcd}$$

Now, the commutativity of this diagram implies that

$$\theta_{f_{C_p}}(u, x) = i_{C_p}(u) + s_{C_p}(x)$$

and

$$\theta_{f_e}(m, a) = i_e(m) + s_e(a)$$

where $u \in M(C_p)$, $x \in A(C_p)$, $m \in M(e)$ and $a \in A(e)$. Therefore, it remains to check that $\theta_{f_{C_p}}$ and θ_{f_e} are ring homomorphisms. Let (u, x) and (v, y) be elements in $B(C_p)$,

$$\begin{aligned}\theta_{f_{C_p}}((u,x)\cdot(v,y)) &= \theta_{f_{C_p}}(uy + xv + f_{C_p}(x,y), xy)\\ &= i_{C_p}(uy) + i_{C_p}(vx) + i_{C_p}(f_{C_p}(x,y)) + s_{C_p}(xy)\\ &= i_{C_p}(u)s_{C_p}(y) + s_{C_p}(x)i_{C_p}(v) + s_{C_p}(x)s_{C_p}(y)\end{aligned}$$

and

$$\begin{aligned}\theta_{f_{C_p}}(u,x)\cdot\theta_{f_{C_p}}(v,y)) &= (i_{C_p}(u) + s_{C_p}(x))(i_{C_p}(v) + s_{C_p}(y))\\ &= \overbrace{i_{C_p}(u)i_{C_p}(v)}^{=0} + i_{C_p}(u)s_{C_p}(y) + s_{C_p}(x)i_{C_p}(v) + s_{C_p}(x)s_{C_p}(y).\end{aligned}$$

Hence, $\theta_{f_{C_p}}$ is a ring homomorphism. Likewise, let (m, a) and (n, b) be elements in $B(e)$,

$$\begin{aligned}\theta_{f_e}((m,a)\cdot(n,b)) &= \theta_{f_e}(mb + an + f_e(a,b), ab)\\ &= i_e(mb) + i_e(an) + i_e(f_e(a,b)) + s_e(ab)\\ &= i_e(m)s_e(b) + s_e(a)i_e(n) + s_e(a)s_e(b)\end{aligned}$$

and

$$\begin{aligned}\theta_{f_e}(m,a)\cdot\theta_{f_e}(n,b)) &= (i_e(m) + s_e(a))(i_e(n) + s_e(b))\\ &= \overbrace{i_e(m)i_e(n)}^{=0} + i_e(m)s_e(b) + s_e(a)i_e(n) + s_e(a)s_e(b).\end{aligned}$$

Thus, θ_{f_e} is a ring homomorphism. This proves the theorem. □

4.6 Deformation of C_p-Green Functors

The aim of this section is to extend the deformation theory of associative algebras due to Gerstenhaber [6] to obtain a similar result for C_p-Green functors.

Definition 4.6.1. *Let A be a C_p-Green functor. A one parameter formal deformation of A is a collection $(\Psi_{C_p}, \Psi_e, \Phi_e, R, T)$, where*

$$\Psi_{C_p} = \sum_{i=0}^{\infty} \psi_{i_{C_p}} t^i$$

$$\Psi_e = \sum_{i=0}^{\infty} \psi_{i_e} t^i$$

$$\Phi_e = \sum_{i=0}^{\infty} c(a_i) t^i$$

$$R = \sum_{i=0}^{\infty} res_e^{C_p}(a'_i) t^i$$

$$T = \sum_{i=0}^{\infty} tr_e^{C_p}(a_i) t^i$$

are formal power series with $\psi_{n_{C_p}} \in Hom(A(C_p) \otimes A(C_p), A(C_p))$, $\psi_{n_e} \in Hom(A(e) \otimes A(e), A(e))$, $c : A(e) \longrightarrow A(e)$, $res_e^{C_p} : A(C_p) \longrightarrow A(e)$ and $tr_e^{C_p} : A(e) \longrightarrow A(C_p)$. One requires that for all $n \geqslant 0$ the following identities hold

(i) $\psi_{0_{C_p}}(a', b') = a'b'$ and $\psi_{0_e}(a, b) = ab$,

(ii) $\sum_{i_{C_p}+j_{C_p}=n_{C_p}} \psi_{i_{C_p}}(\psi_{j_{C_p}}(a', b'), c') = \sum_{i_{C_p}+j_{C_p}=n_{C_p}} \psi_{i_{C_p}}(a', \psi_{j_{C_p}}(b', c'))$,

(iii) $\sum_{i_e+j_e=n_e} \psi_{i_e}(\psi_{j_e}(a, b), c) = \sum_{i_e+j_e=n_e} \psi_{i_e}(a, \psi_{j_e}(b, c))$,

(iv) $c(\psi_{n_e}(a, b)) = \psi_{n_e}(c(a), c(b))$,

(v) $res_e^{C_p}(\psi_{n_{C_p}}(a', b')) = \psi_{n_e}(res_e^{C_p}(a'), res_e^{C_p}(b'))$,

(vi) $\psi_{n_{C_p}}(tr_e^{C_p}(a), b') = tr_e^{C_p}(\psi_{n_e}(a, res_e^{C_p}(b')))$,

(vii) $\psi_{n_{C_p}}(b', tr_e^{C_p}(a)) = tr_e^{C_p}(\psi_{n_e}(res_e^{C_p}(b'), a))$.

Here $a, b, c \in A(e)$ and $a', b', c' \in A(C_p)$. The last six identities can be expressed as

$$\Psi_{C_p}(a', \Psi_{C_p}(b', c')) = \Psi_{C_p}(\Psi_{C_p}(a', b'), c'),$$

$$\Psi_e(a, \Psi_e(b, c)) = \Psi_e(\Psi_e(a, b), c),$$

$$\Phi_e(\Psi_e(a, b)) = \Psi_e(\Phi_e(a), \Phi_e(b)),$$

$$R(\Psi_{C_p}(a', b')) = \Psi_e(R(a'), R(b')),$$

$$\Psi_{C_p}(T(a), b') = T(\Psi_e(a, R(b'))),$$

$$\Psi_{C_p}(b', T(a)) = T(\Psi_e(R(b'), a)),$$

which shows that $A_{C_p,e}[[t]]$ becomes a $k[[t]]$-Green functor. If for fixed $m \geqslant 1$ there are given $\psi_{n_{C_p}} \in Hom(A(C_p) \otimes A(C_p), A(C_p))$, $\psi_{n_e} \in Hom(A(e) \otimes A(e), A(e))$,

$c : A(e) \longrightarrow A(e)$, $res_e^{C_p} : A(C_p) \longrightarrow A(e)$ and $tr_e^{C_p} : A(e) \longrightarrow A(C_p)$ for $n = 0, \cdots, m$ satisfying above identities for $n = 0, \cdots, m$, then we say that there is given an *m-deformation.* For $m = 1$, one also says that there is an *infinitesimal deformation.*

Definition 4.6.2. *Two deformations $(\Psi_{C_p}, \Psi_e, \Phi_e, R, T)$ and $(\Psi'_{C_p}, \Psi'_e, \Phi'_e, R', T')$ are equivalent if there exist a formal power series $\Omega_{C_p} = \sum_{n=0}^{\infty} \omega_{n_{C_p}} t^n$ and $\Omega_e = \sum_{n=0}^{\infty} \omega_{n_e} t^n$, with properties*

(i) $\omega_{n_{C_p}} \in Hom(A(C_p), A(C_p))$ and $\omega_{n_e} \in Hom(A(e), A(e))$, $n \geq 0$,

(ii) $\omega_{0_{C_p}}(a') = a'$ and $\omega_{0_e}(a) = a$, $a' \in A(C_p)$ and $a \in A(e)$,

(iii) $\sum_{i_{C_p}+j_{C_p}=n_{C_p}} \omega_{i_{C_p}}(\psi'_{j_{C_p}}(a', b')) = \sum_{i_{C_p}+j_{C_p}+k_{C_p}=n_{C_p}} \psi_{i_{C_p}}(\omega_{j_{C_p}}(a'), \omega_{k_{C_p}}(b'))$,

(iv) $\sum_{i_e+j_e=n_e} \omega_{i_e}(\psi'_{j_e}(a, b)) = \sum_{i_e+j_e+k_e=n_e} \psi_{i_e}(\omega_{j_e}(a), \omega_{k_e}(b))$.

Here $n \geqslant 0$, $a' \in A(C_p)$, $a \in A(e)$, $b' \in B(C_p)$ and $b \in B(e)$. The last two equations can be expressed also as $\Omega_{C_p}(\Psi'_{C_p}(a', b')) = \Psi_{C_p}(\Omega_{C_p}(a'), \Omega_{C_p}(b'))$ and $\Omega_e(\Psi'_e(a, b)) = \Psi_e(\Omega_e(a), \Omega_e(b))$. In other words, Ω_{C_p} and Ω_e define an isomorphism of $k[[t]]$-Green functors $(\Psi_{C_p}, \Psi_e, \Phi_e, R, T) \to (\Psi'_{C_p}, \Psi'_e, \Phi'_e, R', T')$. In a same way one can define under what condition two m-deformations are equivalent.

Corollary 4.6.3. *i) Let $(\Psi_{C_p}, \Psi_e, \Phi_e, R, T)$ be a one parameter formal deformation of a C_p-Green functor A. Assume $n > 0$ is a natural number such that $\psi_i = 0$ for $0 < i < n$. Then the pair $(\psi_{n_{C_p}}, \psi_{n_e})$ is a 2-cocycle in $C^n(A, A)$. In particular $(\psi_{1_{C_p}}, \psi_{1_e})$ is a 2-cocycle in $C^n(A, A)$.*

ii) There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of C_p-Green functors A and $H^2_{C_p}(A, A)$.

Proof. The part ii) easily follows from i). To prove i), we observe that these equations gives

$$\psi_{n_{C_p}}(a, b)c + \psi_{n_{C_p}}(ab, c) = a\psi_{n_{C_p}}(b, c) + \psi_{n_{C_p}}(a, bc),$$

$$\psi_{n_e}(a, b)c + \psi_{n_e}(ab, c) = a\psi_{n_e}(b, c) + \psi_{n_e}(a, bc),$$

$$c(\psi_{n_e}(a, b)) = \psi_{n_e}(c(a), c(b)),$$

$$res_e^{C_p}(\psi_{n_{C_p}}(a', b')) = \psi_{n_e}(res_e^{C_p}(a'), res_e^{C_p}(b')),$$

$$\psi_{n_{C_p}}(tr_e^{C_p}(a), b') = tr_e^{C_p}(\psi_{n_e}(a, res_e^{C_p}(b'))),$$

$$\psi_{n_{C_p}}(b', tr_e^{C_p}(a)) = tr_e^{C_p}(\psi_{n_e}(res_e^{C_p}(b'), a)).$$

Hence, the pair $(\psi_{n_{C_p}}, \psi_{n_e})$ is a 2-cocycle in $C^n(A, A)$. $\square$

Chapter 5

Hochschild Cohomology of G-Green Functors

The aim of this chapter is to generalise the results of chapter 4 to an arbitrary finite group G. Throughout this chapter, R denotes a commutative ring and G denotes a finite group.

5.1 G-Tensor products of G-Mackey Functors

We will construct a Mackey functor diagram $M \otimes N$ for G-Mackey functors M and N in this section.

Definition 5.1.1. *Let M and N be G-Mackey functors in the sense of Green's definition. Then, we define $M \otimes N$ as follows. For all subgroups H of G:*

$$(M \otimes N)(H) = \bigoplus_{K \leq H} M(K) \otimes N(K) / \sim .$$

1. *The $\sim$ is given by the following relations:*

$$a \otimes tr_L^K(y) \sim res_L^K(a) \otimes y$$

$$tr_L^K(x) \otimes b \sim x \otimes res_L^K(b)$$

for $L \leq K \leq H$, $a \in M(K)$, $b \in N(K)$, $x \in M(L)$ and $y \in N(L)$.

2. *We denote the element in $(M \otimes N)(H)$ by the class $[a \otimes b]$, where $a \otimes b \in M(K) \otimes N(K)$.*

3. *The action is given by*

$$c_{x,H}([a \otimes b]) = [c_{x,H}(a) \otimes c_{x,H}(b)]$$

   where $x \in G$.

4. *We define the restriction map* $res^H_K : (M \otimes N)(H) \longrightarrow (M \otimes N)(K)$ *by* $res^H_K([a \otimes b]) = res^H_K(a) \otimes res^H_K(b)$ *for* $a \otimes b \in M(K) \otimes N(K)$ *and for all subgroups* L *and* K *in* H

$$res^H_K([m \otimes n]) = \sum_{x \in [L \backslash H/K]} x \cdot ([m \otimes n])$$

   for all $m \otimes n \in M(K) \otimes N(K)$.

5. *We define the transfer map as follows:*

$$tr^H_K([a \otimes b]) = [a \otimes b]$$

   for all $a \otimes b \in M(K) \otimes N(K)$.

5.2 G-$\mathcal{HOM}$ (A, B)

Let $\mathcal{M}ack(G)$ be the category of G-Mackey functors. Recall that a morphism $f \in Hom_{\mathcal{M}ack(G)}(A, B)$ in $\mathcal{M}ack(G)$ consists of a family of group homomorphisms $f_H : A(H) \longrightarrow B(H)$ for all subgroups H of G, such that if $K \leq H$ and $x \in G$, the squares

$$\begin{tikzcd} A(H) \arrow[r, "f_H"] & B(H) \\ A(K) \arrow[u, "tr^H_K"] \arrow[r, "f_K"'] & B(K) \arrow[u, "tr^H_K"'] \end{tikzcd} \qquad \begin{tikzcd} A(H) \arrow[r, "f_H"] \arrow[d, "res^H_K"'] & B(H) \arrow[d, "res^H_K"] \\ A(K) \arrow[r, "f_K"'] & B(K) \end{tikzcd} \qquad \begin{tikzcd} A(H) \arrow[r, "f_H"] \arrow[d, "c_{x,H}"'] & B(H) \arrow[d, "c_{x,H}"] \\ A({}^xH) \arrow[r, "f_{{}^xH}"'] & B({}^xH) \end{tikzcd}$$

are commutative.

Definition 5.2.1. *Let* A *and* B *be* G*-Mackey functors in the sense of Green's definition. Then, we define* G-$\mathcal{HOM}(A, B)$ *as follows. For all subgroups* H *of* G*:*

$$\mathcal{HOM}(A, B)(H) = Hom_{\mathcal{M}ack(H)}(Res^G_H A, Res^G_H B).$$

1. *Hence any element* f *in* $\mathcal{HOM}(A,B)(H)$ *corresponds to a collection of morphisms* $f_K : A(K) \longrightarrow B(K)$*, for* $K \leq H$*.*

2. *The action of* G *on* $\mathcal{HOM}(A,B)(H)$ *is given by*

$$({}^x f)_K(m) = x f_K(x^{-1}m)$$

   for $K \leq H$*,* $x \in G$ *and* $m \in A(K)$*.*

3. *We define the restriction map* $R^H_K : \mathcal{HOM}(A,B)(H) \longrightarrow \mathcal{HOM}(A,B)(K)$ *by* $R^H_K(f_K) = f_L$*, for* $L \leq K \leq H$*.*

4. *We define the transfer map* $T^H_K : \mathcal{HOM}(A,B)(K) \longrightarrow \mathcal{HOM}(A,B)(H)$ *to be a morphisms in* $\mathcal{HOM}(A,B)(H)$*. That is, let* $f_L : A(L) \longrightarrow B(L)$ *be a collection of morphisms in* $\mathcal{HOM}(A,B)(K)$*. Then from the commutativity of* $\mathcal{HOM}(A,B)(H)$ *we define* $T^H_K(f_L)$ *by*

$$\alpha_K(a) = tr_B f_L res_A(a)$$

   and for all subgroups L *and* K *in* H

$$\beta_K(m) = \sum_{x \in [L \backslash H / K]} x f_L(x^{-1}m)$$

   for a *and* $m \in A(K)$*.*

Remark 5.2.2. For example, we have

$$\begin{aligned} \mathcal{HOM}(A,B)(e) &= Hom(A(e), B(e)) \\ &= \{\text{group homomorphism from } A(e) \text{ to } B(e)\}. \end{aligned}$$

The proof of the following result is in [2].

Proposition 5.2.3. *Let* M*,* N *and* P *be Mackey functors for the group* G*. Then there exists an isomorphism*

$$\mathcal{HOM}(M \otimes N, P) \simeq \mathcal{HOM}(N, \mathcal{HOM}(M,P))$$

natural in M*,* N *and* P*.*

5.3 Hochschild (Co)Homology of G-Mackey Functors

The definition of Hochschild (co)homology can be extended to G-Mackey functors as discussed in this section.

Definition 5.3.1. *Let A be a G-Green functor and M be a bimodule over the G-Green functor A. Then, for every subgroup H of G, the Hochschild homology of a G-Mackey functor which is again a G-Mackey functor is the homology of the following chain complex:*

$$M(H) \xleftarrow{d_0} \bigoplus_{K \leq H} M(K) \otimes A(K)/\sim \xleftarrow{d_1} \bigoplus_{K \leq H} M(K) \otimes A(K) \otimes A(K)/\sim \xleftarrow{d_2} \cdots$$

before given the boundary map, we denote by

$$i_q : M(K) \otimes A(K)^{\otimes q} \longrightarrow M(K) \otimes A(K)^{\otimes q}/\sim$$

the canonical map where $q > 0$. The boundary map is given by

$$\begin{aligned} d_{q-1} \circ i_q(m \otimes a_1 \otimes \cdots \otimes a_q) = & \,(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_q) \\ & + \sum_{0<i<q} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q) \\ & + (-1)^q (a_q m \otimes a_1 \otimes \cdots \otimes a_{q-1}). \end{aligned}$$

Hence,

$$H_n(A, M) = H_n(C_n(A, M)),$$

where $C_n(A, M) = (M \otimes A^{\otimes n})(H)$.

Example 5.3.2.

1. $d_0 \circ i_1(m \otimes a) = ma - am.$

2. $d_1 \circ i_2(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + bm \otimes a.$

3. $d_2 \circ i_3(m \otimes a \otimes b \otimes c) = ma \otimes b \otimes c - m \otimes ab \otimes c + m \otimes a \otimes bc - cm \otimes a \otimes b.$

Definition 5.3.3. *Let A be a G-Green functor and M be a bimodule over the G-Green functor A. Then, for every subgroup H of G, the Hochschild cohomology* *of a G-Mackey functor which is again a G-Mackey functor is the cohomology of the following of cochain complex:*

$$M(H) \xrightarrow{b_0} Hom_{\mathcal{M}ack(H)}(Res^G_H A, Res^G_H M) \xrightarrow{b_1} Hom_{\mathcal{M}ack(H)}(Res^G_H A \otimes A, Res^G_H M) \xrightarrow{b_2} \cdots$$

where the coboundary map is given by

$$\begin{aligned} b_n(f_K)(a_1, \cdots, a_{n+1}) &= a_1 f_K(a_2, \cdots, a_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f_K(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} f_K(a_1, \cdots, a_n) a_{n+1}. \end{aligned}$$

Hence,

$$H^n(A, M) = H^n(C^n(A, M)),$$

where $C^n(A, M) = G\text{-}\mathcal{HOM}(A^{\otimes n}, M)(H)$ *and for* $K \leq H$.

Example 5.3.4.

1. $b_0(m)(a) = am - ma$.

2. $b_1(f_K)(a, b) = af_K(b) - f_K(ab) + f_K(a)b$.

3. $b_2(f_K)(a, b, c) = af_K(b, c) - f_K(ab, c) + f_K(a, bc) - f_K(a, b)c$.

5.4 Classification of Singular Extension of G-Green Functors

It is a well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras [22]. Here we obtain a similar result for G-Green functors.

Definition 5.4.1. *Let A be a G-Green functor and M be an A-bimodule. A singular extension E of A by M is an exact sequence of Mackey functors*

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0,$$

where B is a G-Green functor, j is a homomorphism of G-Green functors and i is a homomorphism of G-Mackey functors such that for all subgroups H of G the *following sequences:*

$$0 \to M(H) \xrightarrow{i_H} B(H) \xrightarrow{j_H} A(H) \to 0$$

are singular extensions of the ring $A(H)$ by $M(H)$.

Definition 5.4.2. *A singular extension $E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$ is called M-split if for all subgroups H of G there exist a group homomorphism:*

$$s_H = s(H) : A(H) \longrightarrow B(H)$$

such that

1. $j_H \circ s_H = id_{A(H)}$.
2. s_H *must be compatible with transfer, restriction and conjugation maps in the following sense:*
   - $res_{B_K^H} \circ s_H = s_K \circ res_{A_K^H}$,
   - $tr_{B_K^H} \circ s_K = s_H \circ tr_{A_K^H}$,
   - $c_{B_K} \circ s_K = s_K \circ c_{A_K}$,

for $K \leq H$.

Definition 5.4.3. *Let A be a G-Green functor and M be an A-bimodule. Then, for all subgroups H of G, a H-Green 2-cocycle $\mathcal{Z}_H^2(A, M)$ of A with values in M is a collection of bilinear maps*

$$f_H : A(H) \times A(H) \longrightarrow M(H)$$

such that for $K \leq H$, the following conditions hold:

- $xf_H(y, z) + f_H(x, yz) = f_H(xy, z) + f_H(x, y)z$.
- $c_{M_K}(f_K(a, b)) = f_K(c_{A_K}(a), c_{A_K}(b))$.
- $res_{M_K^H}(f_H(x, y)) = f_K(res_{A_K^H}(x), res_{A_K^H}(y))$.
- $f_H(tr_{A_K^H}(a), x) = tr_{M_K^H}(f_K(a, res_{A_K^H}(x)))$.
- $f_H(x, tr_{A_K^H}(a)) = tr_{M_K^H}(f_K(res_{A_K^H}(x), a))$.

Proposition 5.4.4. *Let f_H be a H-Green 2-cocycle, then one can construct a G-Green functor B_{f_H} as follows. For all subgroups H of G:*

- $B_{f_H}(H) = M(H) \oplus A(H)$ *as an associative ring with multiplication*

$$(u, x)(w, y) = (uy + xw + f_H(x, y), xy).$$

- $c_{B_K}(u, x) = (c_{M_K}(u), c_{A_K}(x))$.
- $tr_{B_K^H}(u, x) = (tr_{M_K^H}(u), tr_{A_K^H}(x))$.
- $res_{B_K^H}(u, x) = (res_{M_K^H}(u), res_{A_K^H}(x))$.

for $K \leq H$.

Proof. We need to verify that B_{f_H} satisfies all axioms of a G-Green functor. Observe that $B_{f_H}(H)$ is an associative ring since f_H is a 2-cocycle. Next, we need to check that the conjugation map c_{B_K} is a ring homomorphism. That is, for (m, a) and (n, b) be elements in $B_{f_K}(K)$ we have

$$\begin{aligned} c_{B_K}((m, a) \cdot (n, b)) &= c_{B_K}(mb + an + f_K(a, b), ab) \\ &= (c_{M_K}(m) \cdot c_{A_K}(b) + c_{A_K}(a) \cdot c_{M_K}(n) \\ &+ c_{M_K}(f_K(a, b)), c_{A_K}(a) \cdot c_{A_K}(b)). \end{aligned}$$

We also have

$$\begin{aligned} c_{B_K}(m, a) \cdot c_{B_K}(n, b) &= (c_{M_K}(m), c_{A_K}(a)) \cdot (c_{M_K}(n), c_{A_K}(b)) \\ &= (c_{M_K}(m) \cdot c_{A_K}(b) + c_{A_K}(a) \cdot c_{M_K}(n) \\ &+ f_K(c_{A_K}(a), c_{A_K}(b)), c_{A_K}(a) \cdot c_{A_K}(b)). \end{aligned}$$

Hence, from condition 2 in definition 5.4.3 it follows that c_{B_K} is a ring homomorphism. Similarly, we need to check that the restriction map $res_{B_K^H}$ is ring homomorphism. That is, for (u, x) and (v, y) be elements in $B_{f_H}(H)$ we have

$$\begin{aligned} res_{B_K^H}((u, x) \cdot (v, y)) =& res_{B_K^H}(uy + xv + f_H(x, y), xy) \\ =& (res_{M_K^H}(u) \cdot res_{A_K^H}(y) + res_{A_K^H}(x) \cdot res_{M_K^H}(v) \\ &+ res_{M_K^H}(f_H(x, y)), res_{A_K^H}(x) \cdot res_{A_K^H}(y)). \end{aligned}$$

We also have

$$\begin{aligned}
res_{B^H_K}(u,x)\cdot res_{B^H_K}(v,y) =&(res_{M^H_K}(u), res_{A^H_K}(x))\cdot(res_{M^H_K}(v), res_{A^H_K}(y))\\
=&(res_{M^H_K}(u)\cdot res_{A^H_K}(y) + res_{A^H_K}(x)\cdot res_{M^H_K}(v)\\
&+f_K(res_{A^H_K}(x), res_{A^H_K}(y)), res_{A^H_K}(x)\cdot res_{A^H_K}(y)).
\end{aligned}$$

Therefore, from condition 3 in definition 5.4.3 it follows that $res_{B^H_K}$ is a ring homomorphism.

Finally, we need to check that B_{f_H} satisfies the Frobenius relations. That is, for (m,a) be an element in $B_{f_K}(K)$ and (u,x) be an element in $B_{f_H}(H)$ we have

$$\begin{aligned}
tr_{B^H_K}(m,a)\cdot(u,x) &= (tr_{M^H_K}(m), tr_{A^H_K}(a))\cdot(u,x)\\
&= (tr_{M^H_K}(m)\cdot x + tr_{A^H_K}(a)\cdot u + f_H(tr_{A^H_K}(a), x), tr_{A^H_K}(a)\cdot x).
\end{aligned}$$

We also have

$$\begin{aligned}
tr_{B^H_K}((m,a)\cdot res_{B^H_K}(u,x)) &= tr_{B^H_K}((m,a)\cdot(res_{M^H_K}(u), res_{A^H_K}(x)))\\
&= tr_{B^H_K}(m\cdot res_{A^H_K}(x) + a\cdot res_{M^H_K}(u)\\
&\quad + f_K(a, res_{A^H_K}(x)), a\cdot res_{A^H_K}(x))\\
&= (tr_{M^H_K}(m\cdot res_{A^H_K}(x)) + tr_{A^H_K}(a\cdot res_{M^H_K}(u))\\
&\quad +tr_{M^H_K}(f_K(a, res_{A^H_K}(x))), tr_{A^H_K}(a\cdot res_{A^H_K}(x))).
\end{aligned}$$

Hence, from the definition of G-Green functors, definition of modules over G-Green functors and condition 4 in definition 5.4.3 it follows that:

$$tr_{B^H_K}(m,a)\cdot(u,x) = tr_{B^H_K}((m,a)\cdot res_{B^H_K}(u,x)).$$

Likewise, from the definition of G-Green functors, definition of modules over G-Green functors and condition 5 in definition 5.4.3 it follows that:

$$(u,x)\cdot tr_{B^H_K}(m,a) = tr_{B^H_K}(res_{B^H_K}(u,x)\cdot(m,a)).$$

□

Definition 5.4.5. *Let A be a G-Green functor and M be an A-bimodule. We define $Ext(A,M)$ to be the set of equivalence classes of M-split extensions of A by M.*

Definition 5.4.6. *Let A be a G-Green functor and M be an A-bimodule. For all subgroups H of G we define*

$$C^1_H(A,M) = \left\{ h_H \;\middle|\; \begin{array}{r} \forall x \in A(H),\ \ res_{M^H_K}(h_H(x)) = h_K(res_{A^H_K}(x)) \\ \forall a \in A(K),\ \ c_{M_K}(h_K(a)) = h_K(c_{A_K}(a)) \\ \forall a \in A(K),\ \ tr_{M^H_K}(h_K(a)) = h_H(tr_{A^H_K}(a)) \end{array} \right\},$$

where $h_H : A(H) \longrightarrow M(H)$ and $K \leq H$. Moreover, there exists a map

$$\partial : C^1_H(A,M) \longrightarrow \mathcal{Z}^2_H(A,M)$$

such that $\partial(h_H) = (\delta h_H(x,y))$, where

$$\delta h_H(x,y) = xh_H(y) - h_H(xy) + h_H(x)y.$$

Definition 5.4.7. *Let A be G-Green functor and M be an A-bimodule. Then, for all subgroups H of G, we define the second cohomology by*

$$H^2_H(A,M) = coker\ \partial.$$

Theorem 5.4.8. *Let A be a G-Green functor, M be an A-bimodule and $Ext(A,M)$ be the set of equivalence classes of M-split extensions A by M. There is a one-to-one correspondence between the elements of $Ext(A,M)$ and those of $H^2_H(A,M)$.*

Proof. To prove the theorem, we are going to follow these steps.
Step 1. Show that there is a well-defined map from $Ext(A,M)$ to $H^2_H(A,M)$.
Step 2. Show that there is a well-defined map from $H^2_H(A,M)$ to $Ext(A,M)$.
Step 3. Show that these two maps are inverse to each other.
Step 1. Consider a singular extension

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

and let

$$s_H : A(H) \longrightarrow B(H)$$

be an abelian group homomorphism such that

$$j_H \circ s_H = id_{A(H)}.$$

Then, for every $x, y \in A(H)$, there exists a uniquely determined element $f_H(x, y) \in M(H)$ such that

$$s_H(x)s_H(y) = s_H(xy) + i_H f_H(x, y). \tag{5.1}$$

For $x, y, z \in A(H)$,

$$\begin{aligned} s_H(x)(s_H(y)s_H(z)) &= s_H(x)(s_H(yz) + i_H f_H(y, z)) \\ &= s_H(x)s_H(yz) + s_H(x)i_H f_H(y, z) \\ &= s_H(xyz) + i_H f_H(x, yz) + s_H(x)i_H f_H(y, z). \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} (s_H(x)s_H(y))s_H(z) &= (s_H(xy) + i_H f_H(x, y))s_H(z) \\ &= s_H(xy)s_H(z) + i_H f_H(x, y)s_H(z) \\ &= s_H(xyz) + i_H f_H(xy, z) + i_H f_H(x, y)s_H(z). \end{aligned} \tag{5.3}$$

Thus, multiplication in $B(H)$ is associative which follows from (5.2) and (5.3) that:

$$x f_H(y, z) - f_H(xy, z) + f_H(x, yz) - f_H(x, y)z = 0$$

showing that f_H is a H-Green 2-cocycle. Next, we know that the conjugation map $c_{B_K} : B(K) \longrightarrow B(K)$ is a ring homomorphism and by applying c_{B_K} to equation (5.1) we have

$$\begin{aligned} c_{B_K}(s_K(a)s_K(b)) &= c_{B_K}(s_K(ab) + i_K f_K(a, b)) \\ \Rightarrow c_{B_K}(s_K(a))c_{B_K}(s_K(b)) &= c_{B_K}(s_K(ab)) + c_{B_K}(i_K f_K(a, b)) \\ \Rightarrow s_K(c_{A_K}(a))s_K(c_{A_K}(b)) &= s_K(c_{A_K}(ab))) + i_K(c_{M_K}(f_K(a, b))) \\ \Rightarrow \cancel{s_K(c_{A_K}(a)c_{A_K}(b))} + \cancel{i_K} f_K(c_{A_K}(a), c_{A_K}(b)) &= \cancel{s_K(c_{A_K}(a)c_{A_K}(b))} + \cancel{i_K}(c_{M_K}(f_K(a, b))) \\ \Rightarrow f_K(c_{A_K}(a), c_{A_K}(b)) &= c_{M_K}(f_K(a, b)). \end{aligned}$$

Similarly, we know that the restriction map $res_{B^H_K} : B(H) \longrightarrow B(K)$ is a ring homomorphism and by applying $res_{B^H_K}$ to equation (5.1) we have

$$\begin{aligned} res_{B^H_K}(s_H(x)s_H(y)) &= res_{B^H_K}(s_H(xy) + i_H f_H(x, y)) \\ \Rightarrow res_{B^H_K}(s_H(x))res_{B^H_K}(s_H(y)) &= res_{B^H_K}(s_H(xy)) + res_{B^H_K}(i_H f_H(x, y)) \\ \Rightarrow s_K(res_{A^H_K}(x))s_K(res_{A^H_K}(y)) &= s_K(res_{A^H_K}(xy)) + i_K(res_{M^H_K}(f_H(x, y))) \\ \Rightarrow \cancel{s_K(res_{A^H_K}(x)res_{A^H_K}(y))} &+ \cancel{i_K}(f_K(res_{A^H_K}(x), res_{A^H_K}(y))) \\ &= \cancel{s_K(res_{A^H_K}(x)res_{A^H_K}(y)))} + \cancel{i_K}(res_{M^H_K}(f_H(x, y))) \end{aligned}$$

$$\Rightarrow f_K(res_{A_K^H}(x), res_{A_K^H}(y)) = res_{M_K^H}(f_H(x,y)).$$

Furthermore, from (5.1) we have

$$\begin{aligned} s_H(y)s_H(x) &= s_H(yx) + i_H f_H(y,x) \\ \Rightarrow i_H f_H(y,x) &= s_H(y)s_H(x) - s_H(yx). \end{aligned}$$

Now, by substituting $y = tr_{A_K^H}(a)$ in the above equation we have

$$\begin{aligned}
i_H f_H(tr_{A_K^H}(a), x) &= s_H(tr_{A_K^H}(a))s_H(x) - s_H(tr_{A_K^H}(a)\cdot x) \\
\Rightarrow i_H f_H(tr_{A_K^H}(a), x) &= tr_{B_K^H}(s_K(a))s_H(x) - s_H(\overbrace{tr_{A_K^H}(a\cdot res_{A_K^H}(x))}^{\text{Frobenius relation}}) \\
\Rightarrow i_H f_H(tr_{A_K^H}(a), x) &= \overbrace{tr_{B_K^H}(s_K(a)\cdot res_{B_K^H}(s_H(x)))}^{\text{Frobenius relation}} - tr_{B_K^H}(s_K(a\cdot res_{A_K^H}(x))) \\
\Rightarrow i_H f_H(tr_{A_K^H}(a), x) &= tr_{B_K^H}(s_K(a)\cdot s_K(res_{A_K^H}(x))) - tr_{B_K^H}(s_K(a\cdot res_{A_K^H}(x))) \\
\Rightarrow i_H f_H(tr_{A_K^H}(a), x) &= tr_{B_K^H}\overbrace{(s_K(a\cdot res_{A_K^H}(x)) + i_K f_K(a, res_{A_K^H}(x)))}^{\text{from (5.1)}} - tr_{B_K^H}(s_K(a\cdot res_{A_K^H}(x))) \\
\Rightarrow i_H f_H(tr_{A_K^H}(a), x) &= \cancel{tr_{B_K^H}(s_K(a\cdot res_{A_K^H}(x)))} + tr_{B_K^H}(i_K f_K(a, res_{A_K^H}(x))) \\
&\quad - \cancel{tr_{B_K^H}(s_K(a\cdot res_{A_K^H}(x)))} \\
\Rightarrow \cancel{i_H} f_H(tr_{A_K^H}(a), x) &= \cancel{i_H}(tr_{M_K^H}(f_K(a, res_{A_K^H}(x)))) \\
\Rightarrow f_H(tr_{A_K^H}(a), x) &= tr_{M_K^H}(f_K(a, res_{A_K^H}(x))).
\end{aligned}$$

Similarly, from (5.1) we have

$$\Rightarrow i_H f_H(x,y) = s_H(x)s_H(y) - s_H(xy).$$

Now, by substituting $y = tr_{A_K^H}(a)$ in the above equation we have

$$\begin{aligned}
i_H f_H(x, tr_{A_K^H}(a)) &= s_H(x)s_H(tr_{A_K^H}(a)) - s_H(x\cdot tr_{A_K^H}(a)) \\
\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) &= s_H(x)tr_{B_K^H}(s_K(a)) - s_H\overbrace{(tr_{A_K^H}(res_{A_K^H}(x)\cdot a))}^{\text{Frobenius relation}} \\
\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) &= \overbrace{tr_{B_K^H}(res_{B_K^H}(s_H(x)\cdot s_K(a)))}^{\text{Frobenius relation}} - tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot a)) \\
\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) &= tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot s_K(a))) - tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot a)) \\
\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) &= tr_{B_K^H}\overbrace{(s_K(res_{A_K^H}(x)\cdot a)) + i_K f_K(res_{A_K^H}(x), a)}^{\text{from (5.1)}} - tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot a)) \\
\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) &= \cancel{tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot a))} + tr_{B_K^H}(i_K f_K(res_{A_K^H}(x), a))
\end{aligned}$$

$$
\begin{aligned}
&\quad - \cancel{tr_{B_K^H}(s_K(res_{A_K^H}(x)\cdot a))}\\
&\Rightarrow \cancel{i_H} f_H(x, tr_{A_K^H}(a)) = \cancel{i_H}(tr_{M_K^H}(f_K(res_{A_K^H}(x), a)))\\
&\quad \Rightarrow f_H(x, tr_{A_K^H}(a)) = tr_{M_K^H}(f_K(res_{A_K^H}(x), a)).
\end{aligned}
$$

Hence, $f_H \in \mathcal{Z}_H^2(A, M)$ satisfy all conditions in definition 5.4.3. Let

$$s_H^{'} : A(H) \longrightarrow B(H)$$

be an abelian homomorphism and let

$$g_H : A(H) \times A(H) \longrightarrow M(H)$$

be the 2-cocycle corresponding to choices of $s_H^{'}$. Then,

$$j_H \circ s_H(x) = x = j_H \circ s_H^{'}(x)$$

for every $x \in A(H)$, and so there exists $C_H^1(A, M) \xrightarrow{\partial} \mathcal{Z}_H^2(A, M)$ such that

$$s_H^{'}(x) = i_H h_H(x) + s_H(x) \tag{5.4}$$

where $h_H : A(H) \longrightarrow M(H)$ and $x \in A(H)$. Now, for $x, y \in A(H)$ and by substituting (5.4) in (5.1) we have

$$
\begin{aligned}
i_H f_H(x, y) + s_H^{'}(xy) - i_H h_H(xy) &= (s_H^{'}(x) - i_H h_H(x))(s_H^{'}(y) - i_H h_H(y))\\
\Rightarrow i_H f_H(x, y) + s_H^{'}(xy) - i_H h_H(xy) &= s_H^{'}(x) s_H^{'}(y) - s_H^{'}(x) i_H h_H(y)\\
&\quad - i_H h_H(x) s_H^{'}(y) + \overbrace{i_H h_H(x) i_H h_H(y)}^{=0}\\
\Rightarrow i_H f_H(x, y) + \cancel{s_H^{'}(xy)} - i_H h_H(xy) &= \cancel{s_H^{'}(xy)} + i_H g_H(x, y)\\
&\quad - s_H^{'}(x) i_H h_H(y) - i_H h_H(x) s_H^{'}(y)\\
\Rightarrow \delta h_H(x, y) = g_H(x, y) - f_H(x, y) &= x h_H(y) - h_H(xy) + h_H(x) y
\end{aligned}
$$

so that f_H and g_H differ by a 2-coboundary. Therefore, we show that there exists a well-defined map from $Ext(A, M)$ to $H_H^2(A, M)$.

Step 2. Let $[f_H] \in H_H^2(A, M)$, where $f_H \in \mathcal{Z}_H^2(A, M)$. Then, we define the H-Green functor B_{f_H} as in Proposition 5.4.4. Therefore, the extension associated to f_H is the extension

$$E_{f_H} : 0 \to M \xrightarrow{i} B_{f_H} \xrightarrow{j} A \to 0,$$

where j is a homomorphism of H-Green functors and i is a homomorphism of H-Mackey functors. Now, we need to show that $[E_{f_H}]$ is independent of the choices of f_H. In other words, if $[f_H] = [g_H] \Leftrightarrow f_H = g_H + \delta h_H$.
Two extensions E_{f_H} and E_{g_H} are equivalent if and only if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M(H) \arrow[r, "i_H"] \arrow[d, equal] & B_{f_H}(H) \arrow[r, "j_H"] \arrow[d, "\alpha_H"] & A(H) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M(H) \arrow[r, "i'_H"'] & B_{g_H}(H) \arrow[r, "j'_H"'] & A(H) \arrow[r] & 0
\end{tikzcd}$$

with α_H a homomorphism of rings. The commutativity of this diagram implies that

$$\alpha_H(u,x) = (u + h_H(x), x)$$

for $h_H \in C^1_H(A,M)$. The fact that α_H is a ring homomorphism gives the following equation,

$$\begin{aligned} \alpha_H((u,x)(v,y)) &= \alpha_H(uy + xv + f_H(x,y), xy) \\ &= uy + xv + f_H(x,y) + h_H(x,y), xy) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \alpha_H(u,x)\alpha_H(v,y) &= (u + h_H(x), x)(v + h_H(y), y) \\ &= uy + h_H(x)y + xv + xh_H(y) + g_H(x,y), xy) \end{aligned} \tag{5.6}$$

Hence, from (5.5) and (5.6) we obtain

$$f_H(x,y) - g_H(x,y) = xh_H(y) - h_H(x,y) + h_H(x)y = \delta h_H(x,y)$$

that is, $f_H - g_H$ is a 2-coboundary. Conversely, if $f_H - g_H$ is a 2-coboundary, then E_{f_H} and E_{g_H} are equivalent. Moreover, we need to check that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[rr] & & M(H) \arrow[rr, "i_H"] \arrow[dd, bend left, "res_{M^H_K}"] & & B_{f_H}(H) \arrow[rr, "j_H"] \arrow[dl, "\alpha_H"'] \arrow[dd, bend left, "res_{B^H_K}"] & & A(H) \arrow[r] \arrow[dd, bend left, "res_{A^H_K}"] & 0 \\
0 \arrow[r] & M(H) \arrow[rr, "i'_H"] \arrow[ur, Rightarrow] \arrow[dd, bend left, "tr_{M^H_K}"] & & B_{g_H}(H) \arrow[rr, "j'_H"] \arrow[dd, bend left, "res_{B^H_K}"] & & A(H) \arrow[rr] \arrow[ur, Rightarrow] \arrow[dd, bend left, "tr_{A^H_K}"] & & 0 \\
0 \arrow[rr] & & M(K) \arrow[rr, "i_K"] \arrow[uu, bend left, "res_{M^H_K}"] \arrow[dl, Rightarrow] & & B_{f_K}(K) \arrow[rr, "j_K"] \arrow[dl, "\alpha_K"'] \arrow[uu, bend left, "tr_{B^H_K}"] & & A(K) \arrow[r] \arrow[uu, bend left, "res_{A^H_K}"] \arrow[dl, Rightarrow] & 0 \\
0 \arrow[r] & M(K) \arrow[rr, "i'_K"'] \arrow[uu, bend left, "tr_{M^H_K}"] & & B_{g_K}(K) \arrow[rr, "j'_K"'] \arrow[uu, bend left, "tr_{B^H_K}"] & & A(K) \arrow[rr] \arrow[uu, bend left, "tr_{A^H_K}"] & & 0
\end{tikzcd}
$$

It suffices to check that $res_{B^H_K} \circ \alpha_H(u,x) = \alpha_K \circ res_{B^H_K}(u,x)$. We have

$$res_{B^H_K} \circ \alpha_H(u,x) = res_{B^H_K}(u + h_H(x), x) = (res_{M^H_K}(u) + res_{M^H_K}(h_H(x)), res_{A^H_K}(x))$$

and

$$\alpha_K \circ res_{B^H_K}(u,x) = \alpha_K(res_{M^H_K}(u), res_{A^H_K}(x)) = (res_{M^H_K}(u) + h_K(res_{A^H_K}(x)), res_{A^H_K}(x)).$$

Thus, from definition 5.4.6 it follows that: $res_{B^H_K} \circ \alpha_H(u,x) = \alpha_K \circ res_{B^H_K}(u,x)$. Therefore, we show that there exists a well-defined map from $H^2_H(A,M)$ to $Ext(A,M)$.

Step 3. Let f_H be 2-cocycle. Then, we define the multiplications on $B_{f_H}(H)$ as follows:

$$(u,x)(v,y) = (uy + xv + f_H(x,y), xy)$$

where $u, v \in M(H)$ and $x, y \in A(H)$. The 2-cocycle property of f_H show that the multiplication on $B_{f_H}(H)$ is associative. Thus, $B_{f_H}(H)$ is an associative ring. We define the maps

$$i_H : M(H) \longrightarrow B_{f_H}(H)$$

$$j_H : B_{f_H}(H) \longrightarrow A(H)$$

as follows:

$$i_H(u) = (u, 0)$$

$$j_H(u, x) = x$$

where i_H is homomorphisms of H-Mackey functors and j_H is homomorphisms of H-Green functors and the sequence

$$E_{f_H} : 0 \to M \xrightarrow{i} B \underset{\text{s}}{\overset{\text{j}}{\rightleftarrows}} A \to 0$$

is exact. For $x \in A(H)$, choose $s_H(x) = (0, x)$. Then, for $x, y \in A(H)$,

$$\begin{aligned} s_H(x)s_H(y) = (0, x)(0, y) &= (f_H(x, y), xy) \\ &= (f_H(x, y), 0) + (0, xy) \\ &= i_H(f_H(x, y)) + s_H(xy). \end{aligned}$$

The choice s_{C_p} thus give the 2-cocycle f_H.

Conversely, suppose that

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

is an extension and let f_H be the 2-cocycle obtained from this extension. We must show that the extension

$$E_{f_H} : 0 \to M \xrightarrow{i} B_{f_H} \xrightarrow{j} A \to 0$$

associated to f_H is equivalent to the given one. Indeed, E and E_{f_H} are equivalent if there exists a homomorphism $\theta_{f_H} : B_{f_H} \longrightarrow B$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, shift right, "j"'] & A \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "i"'] & B_{f_H} \arrow[r, "j"'] \arrow[u, "\theta_{f_H}"'] & A \arrow[r] & 0 \end{tikzcd}$$

Now, the commutativity of this diagram implies that

$$\theta_{f_H}(u, x) = i_H(u) + s_H(x)$$

where $u \in M(H)$ and $x \in A(H)$. Therefore, it remains to check that θ_{f_H} is ring homomorphism. Let (u, x) and (v, y) be elements in $B(H)$,

$$\begin{aligned}\theta_{f_H}((u,x)\cdot(v,y)) &= \theta_{f_H}(uy + xv + f_H(x,y), xy)\\ &= i_H(uy) + i_H(vx) + i_{C_p}(f_H(x,y)) + s_H(xy)\\ &= i_H(u)s_H(y) + s_H(x)i_H(v) + s_H(x)s_H(y)\end{aligned}$$

and

$$\begin{aligned}\theta_{f_H}(u,x)\cdot\theta_{f_H}(v,y)) &= (i_H(u) + s_H(x))(i_H(v) + s_H(y))\\ &= \overbrace{i_H(u)i_H(v)}^{=0} + i_H(u)s_H(y) + s_H(x)i_H(v) + s_H(x)s_H(y).\end{aligned}$$

Hence, θ_{f_H} is a ring homomorphism. This proves the theorem. □

5.5 Deformation of G-Green Functors

The aim of this section is to extend the deformation theory of associative algebras due to Gerstenhaber [6] to obtain a similar result for G-Green functors.

Definition 5.5.1. *Let A be a G-Green functors. For all subgroups H of G, a one parameter formal deformation of A is a collection $(\Psi_H, \Phi_K, R_K^H, T_K^H)$, where*

$$\Psi_H = \sum_{i=0}^{\infty} \psi_{i_H} t^i,$$

$$\Phi_K = \sum_{i=0}^{\infty} c_K(a_i) t^i,$$

$$R_K^H = \sum_{i=0}^{\infty} res_K^H(a_i') t^i,$$

$$T_K^H = \sum_{i=0}^{\infty} tr_K^H(a_i) t^i,$$

are formal power series with $\psi_{n_H} \in Hom(A(H) \otimes A(H), A(H))$, $c_K : A(K) \longrightarrow A(K)$, $res_K^H : A(H) \longrightarrow A(K)$ and $tr_K^H : A(K) \longrightarrow A(H)$, for $K \leq H$.

One requires that for all $n \geqslant 0$ the following identities hold

(i) $\psi_{0_H}(a', b') = a'b'$,

(ii) $\sum_{i_H+j_H=n_H} \psi_{i_H}(\psi_{j_H}(a',b'),c') = \sum_{i_H+j_H=n_H} \psi_{i_H}(a',\psi_{j_H}(b',c'))$,

(iii) $c_K(\psi_{n_K}(a,b)) = \psi_{n_K}(c_K(a),c_K(b))$,

(iv) $res^H_K(\psi_{n_H}(a',b')) = \psi_{n_K}(res^H_K(a'),res^H_K(b'))$,

(v) $\psi_{n_H}(tr^H_K(a),b') = tr^H_K(\psi_{n_K}(a,res^H_K(b')))$,

(vi) $\psi_{n_H}(b',tr^H_K(a)) = tr^H_K(\psi_{n_K}(res^H_K(b'),a))$,

Here $a,b \in A(K)$ and $a',b',c' \in A(H)$. The last five identities can be expressed as

$$\Psi_H(\Psi_H(a',b'),c') = \Psi_H(a',\Psi_H(b',c')),$$

$$\Phi_K(\Psi_K(a,b)) = \Psi_K(\Phi_K(a),\Phi_K(b)),$$

$$R^H_K(\Psi_H(a',b')) = \Psi_K(R^H_K(a'),R^H_K(b')),$$

$$\Psi_H(T^H_K(a),b') = T^H_K(\Psi_K(a,R^H_K(b'))),$$

$$\Psi_H(b',T^H_K(a)) = T^H_K(\Psi_K(R^H_K(b'),a)),$$

which shows that $A[[t]]$ becomes a $k[[t]]$-Green functors. If for fixed $m \geqslant 1$ there are given $\psi_{n_H} \in Hom(A(H)\otimes A(H),A(H))$, $c_K : A(K) \longrightarrow A(K)$, $res^H_K : A(H) \longrightarrow A(K)$ and $tr^H_K : A(K) \longrightarrow A(H)$ for $n = 0,\cdots,m$ satisfying above identities for $n = 0,\cdots,m$, then we say that there is given an *m-deformation*. For $m = 1$, one also says that there is an *infinitesimal deformation.*

Definition 5.5.2. *Two deformations* $(\Psi_H,\Phi_K,R^H_K,T^H_K)$ *and* $(\Psi'_H,\Phi'_K,R'^H_K,T'^H_K)$ *are equivalent if there exists a formal power series*

$$\Omega_H = \sum_{n=0}^{\infty} \omega_{n_H} t^n,$$

with properties

(i) $\omega_{n_H} \in Hom(A(H),A(H)),\ n \geq 0$,

(ii) $\omega_{0_H}(a') = a',\ a' \in A(H)$,

(iii) $\sum_{i_H+j_H=n_H} \omega_{i_H}(\psi'_{j_H}(a',b')) = \sum_{i_H+j_H+k_H=n_H} \psi_{i_H}(\omega_{j_H}(a'),\omega_{k_H}(b'))$.

Here $n \geqslant 0$, $a' \in A(H)$ and $b' \in B(H)$. The last equation can be expressed also as

$$\Omega_H(\Psi'_H(a',b')) = \Psi_H(\Omega_H(a'),\Omega_H(b')).$$

In other words, Ω_H defines an isomorphism of $k[[t]]$-Green functors

$$(\Psi_H, \Phi_K, R^H_K, T^H_K) \to ({\Psi'}_H, {\Phi'}_K, {R'}^H_K, {T'}^H_K).$$

In a same way one can define under what condition two m-deformations are equivalent.

Corollary 5.5.3. *i) Let $(\Psi_H, \Phi_K, R^H_K, T^H_K)$ be a one parameter formal deformation of a G-Green functors A. Assume $n > 0$ is a natural number such that*

$$\psi_{i_H} = 0, \ for\ 0 < i < n.$$

Then ψ_{n_H} is a 2-cocycle in $C^n(A, A)$. In particular ψ_{1_H} is a 2-cocycle in $C^n(A, A)$.

ii) There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of a G-Green functors A and $H^2_H(A, A)$.

Proof. The part ii) easily follows from i). To prove i), we observe that these equations gives

$$\psi_{n_H}(a, b)c + \psi_{n_H}(ab, c) = a\psi_{n_H}(b, c) + \psi_{n_H}(a, bc),$$

$$c_K(\psi_{n_K}(a, b)) = \psi_{n_K}(c_K(a), c_K(b)),$$

$$res^H_K(\psi_{n_H}(a', b')) = \psi_{n_K}(res^H_K(a'), res^H_K(b')),$$

$$\psi_{n_H}(tr^H_K(a), b') = tr^H_K(\psi_{n_K}(a, res^H_K(b'))),$$

$$\psi_{n_H}(b', tr^H_K(a)) = tr^H_K(\psi_{n_K}(res^H_K(b'), a)).$$

Hence, ψ_{n_H} is a 2-cocycle in $C^n(A, A)$. □

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