DISTRIBUTED PARAMETER THEORY

IN OPTIMAL CONTROL

by

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CONTENTS

CONTENTS

CONTENTS

INDEX OF FIGURES

INDEX OF FIGURES

INDEX OF FIGURES

INDEX OF FIGURES

INDEX OF TABLES

PART I.

STATEMENT OF THE PROBLEM : GENERAL THEORY.

1. Introduction and Chemical Background.

1.1 Introduction.

Optimal control theory has been developed principally over the last twenty years, particularly in response to the requirements of space flight; but more recently it has been developed in the areas of social, economic, ecological and medical problems. Many industrial and other systems operate below their full potentialities and it is therefore desirable to create and implement control systems which enable them to achieve specified optimum goals. For example a control system may be required to ensure the highest productivity for a given expenditure of fuel and raw materials. High accuracy and high speed of operation of a system or plant are often required, beyond the capability of the human controller, and frequently self-regulating or automatic control is desired. Many systems are distributed in time and space and their dynamic behaviour may be described by partial differential equations, integral equations, differential delay equations, or functional differential equations. The development of optimal control theory for non-ordinary differential equation systems was hindered until relatively recently by the lack of a well established theory of partial differential equations and of functional differential equations {34}. Some early references in the area are the papers of Egorov {11, 12}, and the books of Butkovskiy {3} and Sage {31}. Since 1970 there has been an enormous growth of interest in optimal control theory for distributed parameter systems, notably by Lions and his coworkers {20}, and for delay equations and functional differential equations, particularly by Hale and others {4, 8, 9, 14, 34}. Useful surveys have been published by Robinson {28}, Davies {6} and Ray {26}. However, it must be said that to date relatively few applications of

1. INTRODUCTION AND CHEMICAL BACKGROUND

distributed parameter optimal control theory have been made to full-sized industrial processes, and there remains a regrettable gap between the potential user and the control theorist.

An early application of optimal control theory to a distributed parameter system in the chemical industry was made by Degtyarev and Sirazetdinov {7} , who considered the system

$$\frac{\partial \psi_1}{\partial t} + (a + bx)\frac{\partial \psi_1}{\partial x} = -(k_1 + k_2)\psi_1 \qquad (1.1)$$

$$\frac{\partial \psi_2}{\partial t} + (a + bx)\frac{\partial \psi_2}{\partial t} = k_2\psi_1 \ . \qquad (1.2)$$

The control was exercised through the reaction rates k_1 and k_2 which are temperature dependent so that the two first order p.d.e.'s are essentially uncoupled, leading to a complete resolution of the problem. However, the paper is notable since it considers in the p.d.e. connection constrained controls which operate sometimes at their boundaries and sometimes in the interiors of their ranges, and also variable end time control.

The present study considers the regulation of the outlet state of one stream of the counterflow exchanger, using the inlet state of the second stream and the flow rates as the controls. Subject to certain simplifying assumptions, the governing equations are a coupled pair of first order partial differential equations. Consideration is also given to the study of the optimal control of the general first order linear partial differential equation

$$\frac{\partial \phi}{\partial t}(x,t) + u(x,t)\frac{\partial \phi}{\partial x}(x,t) = f(u, w, \phi, x, t) \qquad (1.3)$$

which may be described as the transport or moving furnace equation, and to the restricted counterflow system in which one stream of the exchanger is assumed to be very massive.

1. INTRODUCTION AND CHEMICAL BACKGROUND

Thus the application of distributed parameter optimal control theory in this work falls under four main headings :

(i) Bang-bang optimal control of the transport equation $\frac{\partial\phi}{\partial t} + u(x,t)\frac{\partial\phi}{\partial t} = 0$

(ii) Continuous optimal control of the transport equation

$$\frac{\partial\phi}{\partial t} + u(x,t)\frac{\partial\phi}{\partial x} = k(w(x,t) - \phi(x,t))$$

(iii) Optimal control of the restricted counterflow system in which the controlling stream is so massive that it is unaffected by giving up heat or solute to the controlled stream

(iv) Optimal control of the full counterflow system, in which both streams affect each other.

Both numerical and analytic methods are applied to the various problems.

1.2 The counterflow exchanger.

Counterflow exchangers are widely used in the chemical and mechanical industries in heating, cooling and economising roles, for example as a device for cooling steam after its power cycle, for extracting the heat energy from the carbon dioxide stream which passes through the core of a gas-cooled nuclear reactor in order to generate steam, for pre-heating the intake gases in a jet engine or for pre-heating chemical reagents prior to reaction. Related cross flow systems occur biologically, for example the cooling of an elephant's body by the passage of blood through its ears, or the temperature control system (heating or cooling) of the dinosaur Stegosaurus's body by the passage of blood through vertical fins along its back {1} . The chemical exchanger , where a solute passes through a semi-permeable membrane from one solvent to another, is of almost equally wide application, and related systems also occur in biological organs such as the lung and kidney. The separating membrane may take the form of a liquid-gas interface, as in a distillation column.

A further application of the counter current principle is to the cooling of tubular reactors, in which the coolant may also be the feed material for the reaction, Figs. 1.2(a) and (b).

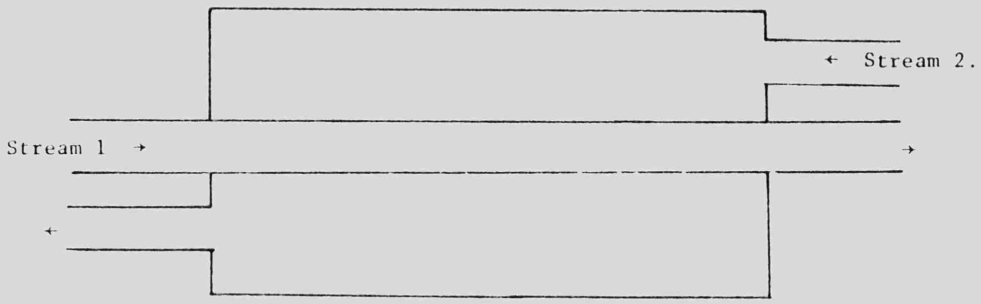

Fig. 1.1. The counterflow exchanger.

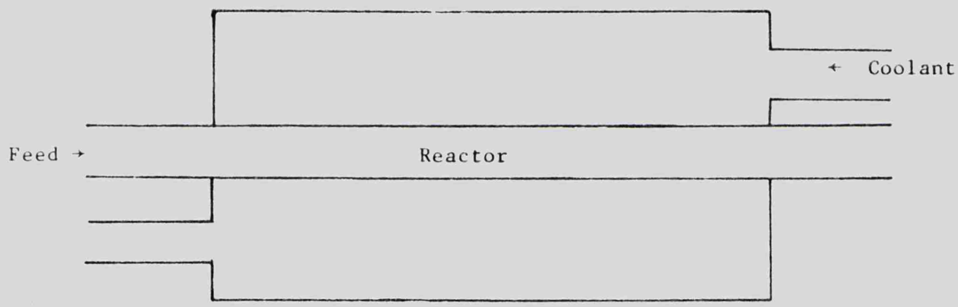

Fig. 1.2(a). The countercurrent cooled tubular reactor : independent coolant.

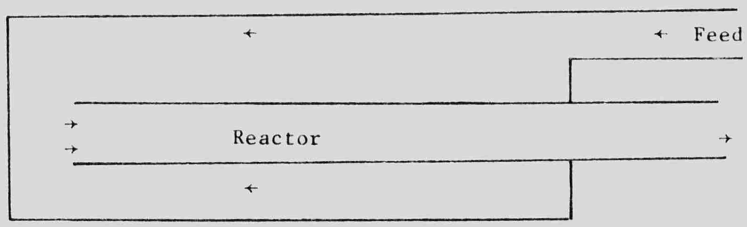

Fig. 1.2(b). The countercurrent cooled tubular reactor : feed as coolant.

1. INTRODUCTION AND CHEMICAL BACKGROUND

An early mathematical treatment of the dynamic behaviour of the counterflow exchanger is that of Jaswon and Smith {17} . However their work contains an error and is also limited to the case of a single change of a boundary input condition from one constant value to another constant value, the system being in a steady state initially.

Stafford {32} considers the optimal control of the counterflow exchanger using a reduction to lumped parameters based on finite Fourier transforms. This method suffers from the draw-back that meaningful results regarding inlet and outlet states cannot be obtained from the solutions at the end points of the spatial interval, but must be obtained from approximations taken close to the ends. However, his results agree well with results he obtains by finite differences, of which he unfortunately gives no details. Ito, Kanoh and Masubuchi {16} use a method of weighted residuals (MWR) approach based on Legendre polynomials which give good results for both open and closed loop operation of parallel flow and counterflow exchangers, using only a few terms of the expansions.

Unfortunately the counterflow system does not posses a set of eigenvalues and eigenfunctions {23} , so that the modal methods described by Brogan {2} cannot be applied to it directly.

The Taylor diffusion model introduces a simple weighted mean state which reduces the hyperbolic system to a parabolic one to which modal methods can be applied {23} . However when the counterflow system is approximated by the Taylor diffusion model superfluous boundary conditions are required in order to determine the boundary conditions of the model and it is difficult to determine the control configuration of the original distributed parameter system {16} .

The present work returns to the exact analytical approach first used by Jaswon and Smith {17} , generalising and extending it to cover the application to optimal control problems, and compares the results obtained by this method with numerical solutions obtained by using the Wendroff and Lax-Wendroff non-characteristic solutions of the hyperbolic partial differential equations. These numerical methods have been used in preference to characteristic methods to avoid the need to reconstruct the finite difference mesh in response to changes in the flow rates.

1.3 The available controls.

The counterflow exchanger has four outputs, namely the outlet temperatures or concentrations of the two streams and the residual state of the two streams at the final time, which may be subject to control ; and six inputs, the inlet temperatures or concentrations of the two streams, the flow rates, and the initial states of the two streams, all of which may be used as controls in appropriate situations.

In this work, we shall assume that the inlet state of stream 1 and the initial states of both streams are given, that the outlet state of stream 2 is immaterial (subject possibly to satisfying inequality constraints), and that the available controls are the flow rates and the inlet condition of stream 2. These will be used to influence conditions in stream 1. The possibility of feedback exists through sensing the outlet conditions of both streams, and the possibility of feed forward through sensing the inlet conditions of stream 1, Fig. 1.3(a), (b) and (c). Control mechanisms include the operation of pumps, valves, bypass circuits, and the inlet character of the controlling stream. In common with many distributed parameter systems, it will be assumed that internal monitoring of the state of the two streams inside the exchanger is not possible. The satisfaction of internal constraints on the state therefore falls within the realm of identification.

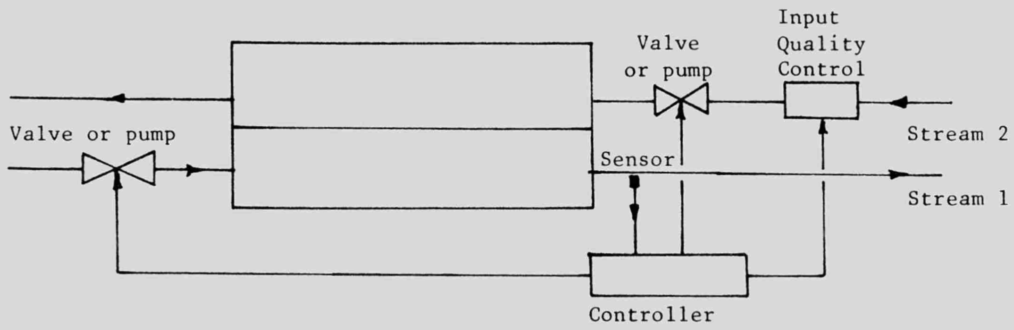

Fig. 1.3(a). Feedback control of a counterflow exchanger.

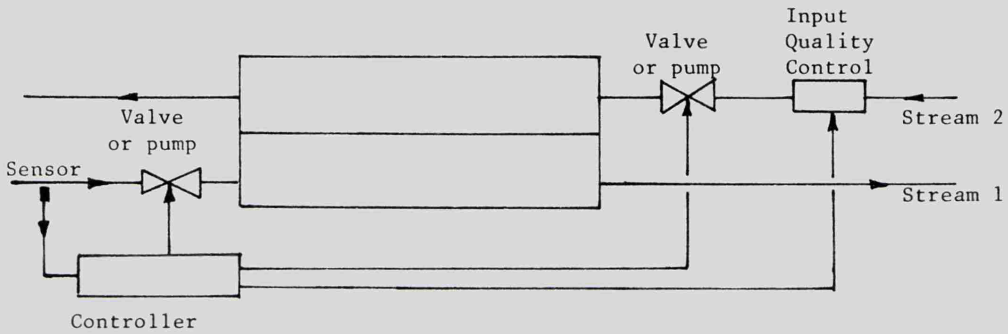

Fig. 1.3(b). Feedforward control of a counterflow exchanger.

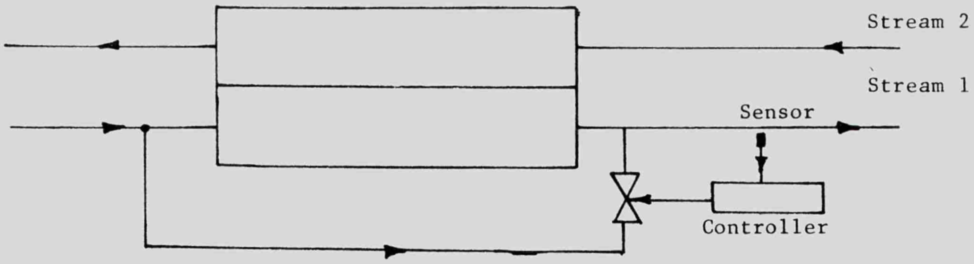

Fig. 1.3(c). Bypass control of a counterflow exchanger.

1. INTRODUCTION AND CHEMICAL BACKGROUND

1.4 Objectives.

In an exchanger, the aim is to transfer heat or a solute from one stream to another. If the objective is to maximise this transfer, we speak of maximising quantity transfer. However, if the two fluids are available at constant temperatures or solute concentrations, the flow rates being the controls, this will be achieved operating the flows as fast as possible, in order to maximise the temperature or concentration gradient across the interface. Although this will result in the largest quantity transfer, the actual change in concentration or temperature of the controlled stream will be low (the change will be "dilute") because the high flow rate will lead to low contact time in the exchanger. Hence a modification of this aim is to maximise the average quantity transfer - that is, the average temperature or concentration change in an outgoing volume of fluid is to be made as large as possible. A third aim is to achieve some desired output temperature or concentration profile for the controlled stream. This will be termed output quality regulation. In short-time or start-up problems the spatial quality distribution along the tube of either or both streams at the final time may be important; for example it may be desired to achieve a given steady state condition in minimum time. This objective is termed residual quality regulation. Further objectives are to minimise the cost of supplying and pumping fluids and the amount of heat or solute that must be supplied to one stream in order to control the other.

The above objectives will lead to the following types of term in the cost functional :

1. $$J_1 = \int_0^T F_1 \{u(t)\ \theta_1(L,t)\}\ dt$$ quantity transfer (1.4)

2. $$J_2 = \frac{\int_0^T F_2 \{u(t)\ \theta_1(L,t)\}\ dt}{\int_0^T u(t)\ dt}$$ average quantity transfer (1.5)

3. $$J_3 = \int_0^T F_3 \{\theta_1(L,t),\ \theta_1^*(t)\}\ dt$$ output quality regulation (1.6)

4. $$J_4 = \int_0^L F_4 \{\underline{\theta}(x,T),\ \underline{\tilde{\theta}}(x)\}\ dx$$ residual quality regulation (1.7)

5. $$J_5 = \int_0^T F_5 \{u(t),\ v(t),\ \theta_2(L,t)\}\ dt\ .$$ cost of supplies and pumping (1.8)

where F_1, F_2 ,...,F_5 are appropriate positive or negative definite functions. $\theta_1(x,t)$, $\theta_2(x,t)$ denote the temperatures or concentrations of the two streams, $u(t)$, $v(t)$ are flow rates, $\underline{\theta}^*$, $\underline{\tilde{\theta}}$ are desired outgoing or residual target profiles, and the domain of interest is $\mathbf{S} \equiv \{0,L\} \times \{0,T\}$. For the usual quadratic performance criterion for quality regulation we would use for example

$$F_3 = \frac{1}{2} \{\theta_1(L,t) - \theta_1^*(t)\}^2 , \qquad (1.9)$$

while for maximisation of simple quantity transfer we would use

$$F_1 = u(t)\ \theta_1(L,t). \qquad (1.10)$$

It is further possible to achieve exact control of some aspects of output whilst optimising some other cost criterion.

Other objectives such as the economic implications of design criteria, number, positions and type of sensors, and integration of the exchanger in relation to other process elements, which require an integrated approach to the plant as a whole, will not be considered here.

1.5. Constraints.

The counterflow system will usually involve constraints on both control and state variables. Constraints on the controls include the following :

(i) Flow rates Although these may become zero, they will not be allowed to become negative, and they will also be subject to certain maximum positive values. In exchangers that rely on bubbling of gas through a liquid they must be such that flooding or foaming do not occur. In general they will be assumed to take the form

$$0 \leq \underline{u}_1 \leq u(t) \leq \underline{u}_2 \quad , \quad 0 \leq \underline{v}_1 \leq v(t) \leq \underline{v}_2 \quad , \tag{1.11}$$

where u_1, u_2, v_1 and v_2 will normally be constants, although they could be time-varying.

(ii) Input temperatures or concentrations

These too can only be varied between certain available limits, giving for example an inequality constraint of the form

$$a \leq \theta_2(L,t) \leq b \tag{1.12}$$

when the input state of stream 2 is used as a control. Again a and b will normally be constants, but could be time-varying. Constraints of this type may also be of a physical nature (see constraints on the states below).

(iii) Constraints on the states

Constraints on the states will be constraints of temperature or concentration, and are frequently in the nature of a physical limitation. Temperatures in the interior of the exchanger must not become so high or so low that boiling, freezing or chemical decomposition occurs. Concentrations must not become so high as to cause saturation. Physically, a fractional concentration cannot exceed 1. There may be boundary constraints in that whilst maximising some cost criterion the outlet temperature or concentration may not be allowed to go beyond certain limits. Similarly mass output rate or total mass flow might be subject to limits.

Time is a constraint in that we may be required to achieve some desired objective in a given fixed time.

1.6 The optimal control of the counterflow exchanger.

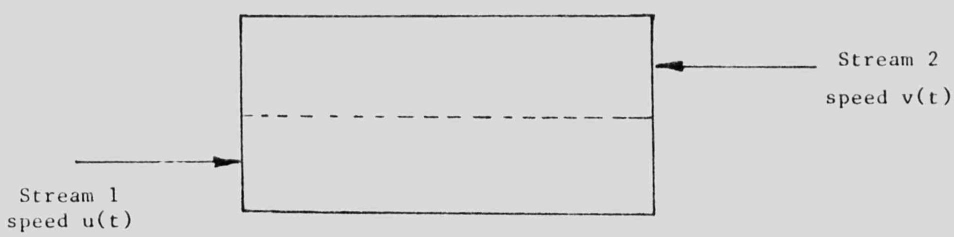

Fig. 1.4. The counterflow exchanger.

We are now in a position to set up the problem mathematically and consider the simplifying assumptions involved. As in all modelling, some simplifying assumptions must be made, and models of varying complexity are

are discussed in the literature {16-18, 22, 29, 32}. The assumptions listed below are widely used, with the exception of neglect of the absorbing capacity of the separating membrane. This assumption simplifies the detail of the mathematics without essentially altering its type {29} , and may be re-introduced if required for specific applications.

Assumptions.

(1) The fluids take up and release heat or solute without any other change in physical properties. In particular it is assumed that saturation does not occur.

(2) A linear transfer law is assumed, i.e. the diffusion rate is proportional to $h\theta_2 - \theta_1$ and is independent of the flow rate. Here h is an equilibrium constant ; for heat transfer, $h = 1$.

(3) The fluids are well mixed laterally, so that the temperature or concentration at any cross-section is the same all over that cross-section. Thus $\theta(x,t)$ describes the state of a tube completely.

(4) Motion is relatively swift, so that diffusion along the tubes can be neglected.

(5) Any absorbing capacity of the diffusing membrane for the diffusant is neglected.

(6) There is no loss of diffusant through outside walls.

(7) Any change in the flow speeds $u(t)$, $v(t)$ is transmitted instantaneously throughout the length of the exchanger. This would be approximately true, not only for incompressible liquids, but also for light gases, which may be accelerated swiftly.

1. INTRODUCTION AND CHEMICAL BACKGROUND

The Derivation of the Counterflow equations

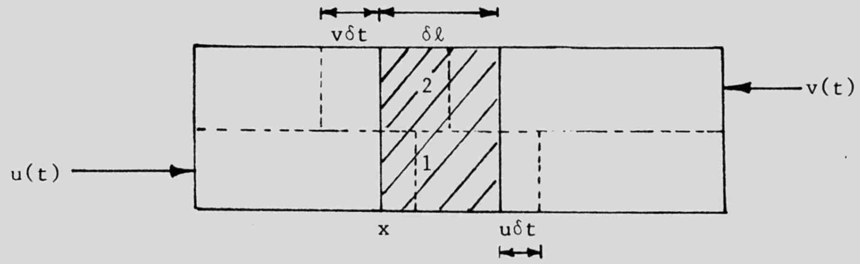

Fig. 1.5. The derivation of the counterflow equations

Referring to Fig. 1.5, let us consider elements of fluid of length $\delta\ell$ in each tube passing each other at position x and time t. Let A_1, A_2 be the cross-sectional areas of the two tubes and c_1, c_2 the volumetric capacities of the two fluids for the diffusant. After a short time δt, elements 1,2 will have moved to $x + \delta x_1$, $x + \delta x_2$ respectively where $\delta x_1 = u\delta t$, $\delta x_2 = - v\delta t$. Hence by conservation of the diffusant,

$$c_1 A_1 \delta\ell(\theta_1 (x + \delta x_1, t + \delta t) - \theta_1(x,t)) = k(h\theta_2(x,t) - \theta_1(x,t))\delta t \quad (1.13)$$

$$c_2 A_2 \delta\ell(\theta_2 (x + \delta x_2, t + \delta t) - \theta_2(x,t)) = - k (h\theta_2(x,t) - \theta_1(x,t))\delta t \quad (1.14)$$

where k is a transfer coefficient which incorporates the cross-sectional dimension of the interface bweteen the fluids. On carrying out Taylor expansions of equations (1.13), (1.14) and taking limits as δx_1, δx_2, $\delta t \to 0$, we obtain

$$C_1\left[\frac{\partial\theta_1}{\partial t} + u(t)\frac{\partial\theta_1}{\partial x}\right] = k(h\theta_2 - \theta_1) \quad (1.15)$$

$$C_2\left[\frac{\partial\theta_2}{\partial t} - v(t)\frac{\partial\theta_2}{\partial x}\right] = - k(h\theta_2 - \theta_1) \quad (1.16)$$

where C_1 , C_2 are diffusant capacities of the two fluids per unit length.

1. INTRODUCTION AND CHEMICAL BACKGROUND

Controls

Case (i) Boundary Control

The input state of stream 2, $\theta_2(L, t) = \phi(t)$, may be used to influence the state of stream 1.

Case (ii) Domain Control

The flow speeds $u(t)$, $v(t)$ may be controllable.

Case (iii)

Cases (i) and (ii) may be combined.

In each case the controls may be bounded by inequality constraints, or they may be continuous and unrestricted. The former non-coercive situation will lead to "bang-bang" controls, and the maximum principle of Egorov must be applied. In the latter coercive situation the classical calculus of variations can be used, and terms involving control will occur in the cost functional.

Boundary conditions

$\theta_1(x,0)$, $\theta_2(x,0)$, $\theta_1(0,t)$ are assumed known. $\theta_2(L,t)$ may be controllable, case (i), or known, case (ii).

The cost functional

The following general cost functional covers a variety of cases ; terms may be omitted or modified to meet individual situations.

We seek to minimise

$$J = \frac{1}{2}\int_0^T \{ \underbrace{\left[\theta_1(L,t) - \theta_1^*(t)\right]^2}_{\text{output quality regulation of stream 1.}} + a \underbrace{\left[\theta_2(L,t) - \theta_2^*(t)\right]^2}_{\text{Input quality regulation of stream 2.}}$$

$$\underbrace{+ b\left[u(t) - u^*(t)\right]^2 + c\left[v(t) - v^*(t)\right]^2}_{\text{cost of supply and pumping fluids}} \} \, dt$$

$$- d \iint_S k \left[\theta_2(x,t) - \theta_1(x,t) \right] dx\, dt + \frac{1}{2}e \int_0^L \left[(\theta_1(x,T) - \tilde{\theta}_1(x) \right]^2 dx$$

Quality transfer from stream 2 to stream 1 — Residual quality regulation of stream 1. (1.17)

a, b, c, d, e are non-negative constants. It is assumed that output quality regulation of stream 1 will always be required, hence it is included with coefficient 1. Where controls are bounded by inequality constraints, corresponding terms in the cost functional will be ommitted. Also where $\theta_2(L,t)$ is known, the term involving input quality regulation of stream 2 will be ommitted.

2. Optimal control theory.

2.1 Introduction.

Optimal control theorems for distributed parameter systems were first developed by Butkovskiy {3} and Egorov {18,12} building on the maximum principle of L.S. Pontryagin {24} . Later authors {4, 20, 25, 33} have unified the problems of control theory for different types of systems including lumped parameter, distributed parameter and differential delay systems, into a theory based on semigroup solutions to abstract evolution equations in infinite dimensional space. This approach enables the questions of controllability, observability, stability and estimation to be considered together, yielding abstract results which can be specialised for any particular problem. We give here a short description of open loop deterministic optimal control theory for a system of first order hyperbolic equations, which will be sufficient for our purposes.

2.2 A minimum principle for the open loop optimal control of a system of first order hyperbolic partial differential equations.

Let S be a closed region of the (x,t) plane, with boundary curve C. We seek to minimise the functional

$$J = \iint_S F(x,t,\, \underline{\phi},\, \underline{\phi}_x,\, \underline{u})dx\, dt + \oint_C G(x,t,\, \underline{\phi})ds \tag{2.1}$$

subject to the n state domain equations

$$\frac{\partial \underline{\phi}}{\partial t} = \underline{g}(x,t,\, \underline{\phi}\, ,\underline{\phi}_x,\, \underline{u}) \qquad \text{over } S \tag{2.2}$$

and m boundary conditions

$$\underline{M}(x,t,\, \underline{\phi},\, \underline{u}_b) = 0 \quad \text{on } C' \subseteq C\ . \tag{2.3}$$

Here $\underline{\phi}(x,t)$ is an n-dimensional state vector, $\underline{u}(x,t)$ is an r-dimensional distributed control vector belonging to some admissible set U_d and $\underline{u}_b(s)$ is a p-dimensional boundary control vector belonging to an admissible set U_b, F, G, $\underline{g}$ and $\underline{M}$ are assumed to be twice continuously differentiable with respect to their arguments. It is also assumed that for $\underline{u} \in U_d$, $\underline{u}_b \in U_b$, the boundary conditions (2.3) and the characteristic directions determined by (2.2) are such that the system is well-posed. Conditions for this require detailed consideration of the geometry of S and are discussed by Russell {30}. We introduce the n-dimensional vector of domain costate variables $\underline{\lambda}(x,t)$, the m-dimensional boundary costate vector $\underline{\mu}(s)$, and Hamiltonians H_o, H_1 defined by :

$$H_o = H_o(\underline{\phi}, \underline{\phi}_x, \underline{\lambda}, \underline{u}) = F + \underline{\lambda}^T \underline{g} \qquad (2.4)$$

$$H_1 = H_1(\underline{\phi}, \underline{u}_b, \underline{\mu}) = G + \underline{\mu}^T \underline{M} \qquad (2.5)$$

Theorem (2.1)

J is minimised when

$$\frac{\partial H_o}{\partial \underline{\phi}} - \frac{\partial}{\partial x}\left(\frac{\partial H_o}{\partial \underline{\phi}_x}\right) + \frac{\partial \underline{\lambda}}{\partial t} = 0 \qquad \text{over S,} \qquad (2.6)$$

$$\left(\frac{\partial H_1}{\partial \underline{\phi}} ds + \frac{\partial H_o}{\partial \underline{\phi}_x} dt + \underline{\lambda}\, dx\right)^T . \, \delta\underline{\phi} = 0 \qquad \text{on C,} \qquad (2.7)$$

$$\frac{\partial H_1}{\partial \underline{\mu}} = 0 \qquad \text{on C',} \qquad (2.8)$$

$$H_o = \inf_{\underline{u} \in U_d} (H_o) \qquad (2.9)$$

and $$H_1 = \inf_{\underline{u}_b \in U_b} (H_1) \quad , \qquad (2.10)$$

in the case where $\underline{u}$, $\underline{u}_b$ are bounded by inequality constraints (Theorem 2.1a).

If the controls $\underline{u}$, $\underline{u}_b$ are continuous and unrestricted, equations (2.9) and (2.10) become

$$\frac{\partial H_o}{\partial \underline{u}} = 0 \quad , \qquad (2.11)$$

$$\text{and} \quad \frac{\partial H_1}{\partial \underline{u}_b} = 0 \quad . \tag{2.12}$$

If the domain control $\underline{u}$ is a function of time only, equation (2.9) or (2.11) is replaced by

$$H_2(t) = \inf_{\underline{u}(t)\varepsilon U_d} (H_2(t)) \text{ or } \frac{\partial H_2(t)}{\partial \underline{u}(t)} = 0 \tag{2.13}$$

$$\text{where } H_2(t) = \int_{\alpha(t)}^{\beta(t)} H_o \, dx \ , \tag{2.14}$$

$x = \alpha(t)$, $x = \beta(t)$ being the boundaries of S as indicated in Fig. 2.1 (Theorem 2.1(b)).

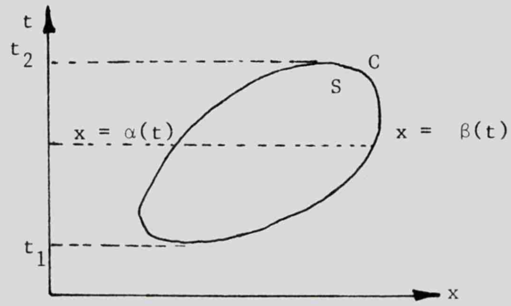

Fig. 2.1. Optimal control for the case when $\underline{u}$ is a function of t.

Proof

Consider the augmented functional

$$J^* = \iint_S \{F + \underline{\lambda}^T (\underline{g} - \frac{\partial \phi}{\partial t}) \} \, dx \, dt + \oint_C (G + \underline{\mu}^T \underline{M}) \, ds$$

$$= \iint_S (H_o - \underline{\lambda}^T \frac{\partial \phi}{\partial t}) \, dx \, dt + \oint_C H_1 \, ds \quad . \tag{2.15}$$

The first order variation in J^*, for small changes $\Delta\underline{\phi}$, $\Delta\underline{u}$, $\Delta\underline{\lambda}$ etc, is given by

$$\Delta J^* = \iint_S \{ \frac{\partial H_o^T}{\partial \underline{\phi}} \cdot \Delta\underline{\phi} + \frac{\partial H_o^T}{\partial \underline{\phi}_x} \cdot \Delta\underline{\phi}_x + \frac{\partial H_o^T}{\partial \underline{\lambda}} \cdot \Delta\underline{\lambda} + \underset{\underline{u}}{\Delta}(H_o)$$

$$- \underline{\lambda}^T . \Delta(\frac{\partial \underline{\phi}}{\partial t}) - \Delta \, \underline{\lambda}^T . \frac{\partial \underline{\phi}}{\partial t} \} \, dx \, dt$$

$$+ \oint_C \{ \frac{\partial H_1^T}{\partial \underline{\phi}} . \Delta\underline{\phi} + \underset{\underline{u}_b}{\Delta} (H_1) + \frac{\partial H_1^T}{\partial \underline{\mu}} . \Delta\underline{\mu} \} \, ds \qquad (2.16)$$

Then, after applying Green's theorem,

$$\Delta J^* = \iint_S \{ \left[\frac{\partial H_o}{\partial \underline{\phi}} - \frac{\partial}{\partial x} (\frac{\partial H_o}{\partial \underline{\phi}_x}) + \frac{\partial \underline{\lambda}}{\partial t} \right]^T . \Delta\underline{\phi} + \left[\frac{\partial H_o}{\partial \lambda} - \frac{\partial \underline{\phi}}{\partial t} \right]^T . \Delta\underline{\lambda}$$

$$+ \underset{\underline{u}}{\Delta} (H_o) \} \, dx \, dt$$

$$+ \oint_C \{ \left[\frac{\partial H_o}{\partial \underline{\phi}_x} dt + \underline{\lambda} \, dx + \frac{\partial H_1}{\partial \underline{\phi}} ds \right]^T . \Delta\underline{\phi}$$

$$+ \underset{\underline{u}_b}{\Delta} (H_1) \, ds + \frac{\partial H_1^T}{\partial \underline{\mu}} . \Delta\underline{\mu} \, ds \} \qquad (2.17)$$

Hence when $\underline{\lambda}$ and $\underline{\phi}$ satisfy (2.6), (2.7) and (2.8), the first variation of J^* vanishes except for the variation due to $\underline{u}$ and $\underline{u}_b$.

$$\therefore \; \Delta J^* = \iint_S \underset{\underline{u}}{\Delta} (H_o) \, dx \, dt + \oint_C \underset{\underline{u}_b}{\Delta} (H_1) \, ds \qquad (2.18)$$

Thus J^* is minimised when (2.9) and (2.10) or (2.11) and (2.12) (respectively (2.13)) are satisfied. Since the state, costate and boundary equations are satisfied in any solution, according to the usual classical calculus of variations theory, or the later maximum principles of Butkovskiy et al, when J^* is minimised, J is also minimised.

3. Optimisation.

3.1 Introduction

Conditions (2.6) – (2.14) are non-linear and of two point boundary value form. Hence iterative methods must be used to obtain solutions. There are three components to the problem which must be satisfied:

(i) the partial differential equations for the state and costate variables,

(ii) the state and adjoint boundary conditions,

(iii) the minimisation of the Hamiltonians.

The "gradient to the Hamiltonian" approach described by Holliday and Storey {15} satisfies (i) and (ii), using iteration to satisfy (iii). Starting from a nominal admissible control, the equations for the state and adjoint variables are solved, and the solutions used to give a correction to the control which yields an improved performance index. The method gives a steepest descent direction, which can be used directly i.e. (the steepest descent method), or incorporated into more sophisticated methods such as the conjugate gradient method or the method of Davidon-Fletcher-Powell. In each case a linear search is carried out to determine the local minimum along the search direction.

3.2 The search direction.

Equation (2.18) provides an expression for the change in J for small changes $\Delta\underline{u}$, $\Delta\underline{u}_b$ in the controls, when the state and costate equations, together with their associated boundary conditions are satisfied. (2.18) may be expressed as

$$\Delta J = \iint_S \left(\frac{\partial H_o}{\partial \underline{u}}\right)^T \cdot \Delta\underline{u}\, dx\, dt + \oint_C \left(\frac{\partial H_1}{\partial \underline{u}_b}\right)^T \cdot \Delta\underline{u}_b\, ds \quad . \tag{3.1}$$

Hence the choice

$$\Delta \underline{u} = - e \frac{\partial H_o}{\partial \underline{u}} \;, \quad (\text{respectively } -e \frac{\partial H_2(t)}{\partial \underline{u}(t)}) \;, \tag{3.2}$$

$$\Delta \underline{u}_b = - e_b \frac{\partial H_1}{\partial \underline{u}_b} \tag{3.3}$$

where e , e_b are positive step length parameters, provides a steepest descent direction subject to $\underline{u}$, $\underline{u}_b$ continuing to belong to U_d, U_b, and

$$\Delta J = - e \iint_S (\frac{\partial H_o}{\partial \underline{u}})^2 \; dx \; dt - e_b \oint_C (\frac{\partial H_1}{\partial \underline{u}_b})^2 \; ds \;. \tag{3.4}$$

In the cases of interest this must produce a decrease in J for sufficiently small changes $\underline{u}$, $\underline{u}_b$. Constraints of the types (1.11), (1.12) and may be dealt with by using

$$u_i = A_i + B_i \sin y_i \tag{3.5}$$

where u_i is any component of the control vector. Thus y_i becomes effectively a new control variable which is unrestricted, so that the optimisation problem is unconstrained. In the remainder of this chapter, the control vector $\underline{u}$ will be understood to include both domain and boundary controls, and the symbol H to include both domain and boundary Hamiltonians.

3.3 The Linear search.

The following algorithm is employed for the linear search, irrespective of the type of gradient method.

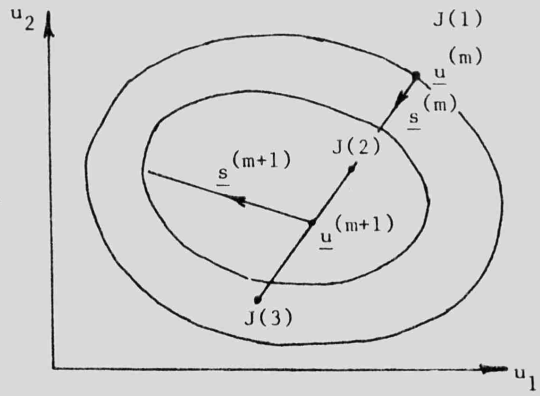

Fig. 3.1. The linear search

3. OPTIMISATION

(i) Suppose the point $\underline{u}^{(m)}$ in control space has been reached after m iterations. The state and costate equations are solved at $\underline{u}^{(m)}$ to determine the value of J(1) and the search direction $\underline{s}^{(m)}$.

(ii) Beginning with a step length parameter e, equal to e_p from the previous iteration, a step is made to the point $\underline{u}^{(m)} + e\,\underline{s}^{(m)}$, and the state equations are solved to determine J(2). If this results in a reduction of J, e is doubled (i.e. extrapolation) ; if J increases, e is halved (i.e. interpolation). †

(iii) Step (ii) is repeated until either (a) there is an increase in J following a series of decreases, or (b) vice versa. In either case suppose the final evaluation of J to be J(ℓ). The starting point $\underline{u}^{(m+1)}$ for the next linear search is given by

$$\underline{u}^{(m+1)} = \underline{u}^{(m)} + e_p\,\underline{s}^{(m)}$$

where e_p is determined by a parabolic minimum estimate based on J(1), J(ℓ-1) and J(ℓ) as follows (see Figs. 3.2(a) and 3.2(b) :

<u>Case (a) : extrapolation</u>

$$e_p = e \cdot \frac{J(\ell) + 3J(1) - 4J(\ell-1)}{4\{J(\ell) + J(1) - 2J(\ell-1)\}} \tag{3.6a}$$

<u>Case (b) : interpolation</u>

$$e_p = e \cdot \frac{J(\ell-1) + 3J(1) - 4J(\ell)}{2\{J(\ell-1) + J(1) - 2J(\ell)\}} \tag{3.6b}$$

The value of e appearing on the right hand side of equations (3.6a, b) is the most recent previous value.

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† Choices of factor other than 2 could of course be used, but this value was found to work reasonably efficiently in practice.

⊕ denotes points used in the parabolic estimate

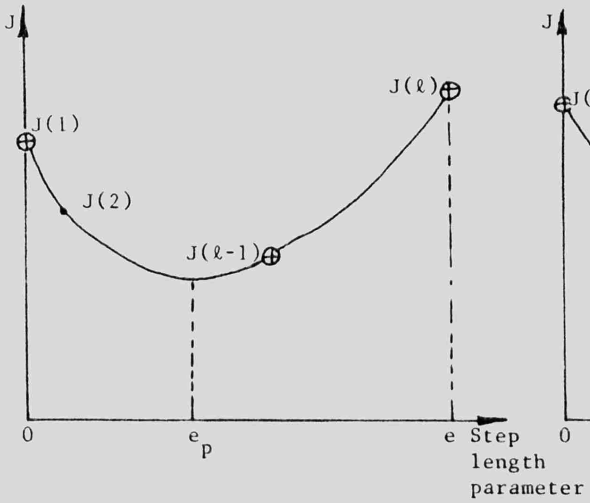

Fig. 3.2(a). The Parabolic estimate following a series of extrapolations

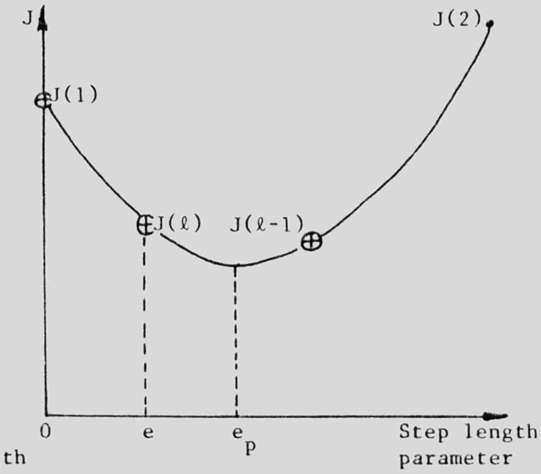

Fig. 3.2(b). The Parabolic estimate following a series of interpolations

(iv) Starting from an arbitrary initial point $\underline{u}^{(o)}$ and an initial value of e, steps (i) - (iii) are repeated until the value of J at $\underline{u}^{(m+1)}$ differs from that at $\underline{u}^{(m)}$ by less than a prescribed tolerance. We note that no tests are incorporated to ensure that J decreases at every step or that e remains positive, as it is found in practice that this type of problem rights itself best without any interference.

3.4 The steepest descent method

The search direction is simply the direction of steepest descent,

$$\underline{s} = -\underline{g} = -\frac{\partial H}{\partial \underline{u}} \tag{3.7}$$

3. OPTIMISATION

3.5 The conjugate gradient method

This method is described by Dixon {10} and was applied to optimal control problems by Lasdon et al {19} . It is a modification of the steepest descent method in which the search direction differs from the gradient direction after the first iteration in an attempt to improve the convergence rate. It depends for its success on the use of a linear search to determine a minimum along the line of search at each iteration. The search direction for the m^{th} iterate is given by

$$\underline{s}^{(m)} = - \underline{g}^{(m)} + \frac{||\underline{g}^{(m)}||^2}{||\underline{g}^{(m-1)}||^2} \underline{s}^{(m-1)} , \tag{3.8}$$

where the norm of $\underline{g}$ is defined by

$$||\underline{g}||^2 = \iint_S \underline{g}^T . \ \underline{g} \ dx \ dt \tag{3.9}$$

(with omission of integration with respect to x for those components of $\underline{g}$ which depend only on the time). Assuming that $\underline{g}$ has n components, the condition for a set of conjugate directions $\underline{t}^{(1)}, \underline{t}^{(2)}, \ldots, \underline{t}^{(n)}$ is

$$\underline{t}^{(i)T} \ G \ \underline{t}^{(j)} = 0 \ , \ i \neq j \tag{3.10}$$

where G is the matrix of second derivatives of the Hamiltonian. Referring to Fig. (3.3), the geometrical interpretation is as follows. Let $\underline{u}^{(m)}$ be a point on a surface of constant J, assumed to be approximately quadratic. Then any direction in the tangent space at $\underline{u}^{(m)}$ is conjugate to the diameter of the quadratic surface through $\underline{u}^{(m)}$. As G is the matrix of second derivatives, the step

$$d\underline{u}^{(m)} = \underline{u}^{(m+1)} - \underline{u}^{(m)} \tag{3.11}$$

produces a change in gradient

$$d\underline{g}^{(m)} = \underline{g}^{(m+1)} - \underline{g}^{(m)} \tag{3.12}$$

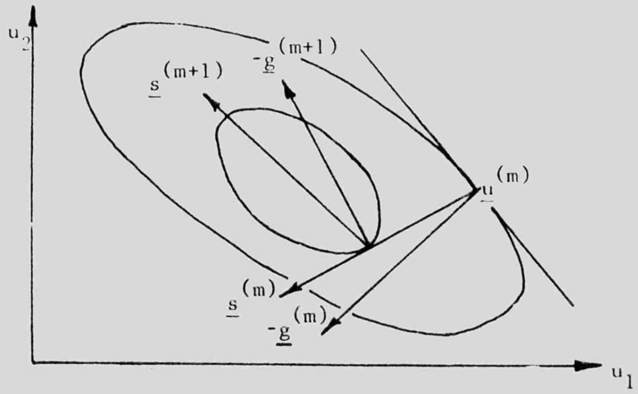

Fig. 3.3. The conjugate gradient method.

given by

$$d\underline{g}^{(m)} = G \, d\underline{u}^{(m)} . \tag{3.13}$$

If $d\underline{u}^{(m)}$ is taken in the direction $\underline{t}^{(j)}$, (3.10) becomes

$$\underline{t}^{(i)T} \cdot d\underline{g}^{(m)} = 0 \; . \tag{3.14}$$

Condition (3.14) enables a set of n conjugate directions (which are also the search directions) to be determined, starting initially from

$$\underline{t}^{(1)} = \underline{s}^{(o)} = - \underline{g}^{(o)} \tag{3.15}$$

using equation (3.8) and without determining G.

If $\underline{g}$ has n components and J is of second degree in $\underline{u}$, the conjugate gradient method converges to the minimum in at most n iterations. Where J is not quadratic, after n iterations a search is made in the direction of the gradient and the process is repeated.

3.6 The method of Davidon, Fletcher and Powell

This is one of the most powerful optimisation techniques of gradient type known, and is described by Dixon {23} . It is a further refinement of the conjugate gradient method and uses matrix iteration. The direction of search given by the Newton-Raphson technique is

$$\underline{p}^{(k)} = - G^{-1} \underline{g}^{(k)} = - H^{(k)} \underline{g}^{(k)} \tag{3.16}$$

where H is termed the inverse Hessian matrix.

Starting from the unit matrix as an initial approximation to H, so that the initial search direction is that of steepest descent, changes to H are made at each iteration using

$$H^{(k+1)} = H^{(k)} + dH^{(k)} \tag{3.17}$$

where

$$dH^{(k)} = \frac{d\underline{u}^{(k)} \, d\underline{u}^{(k)^T}}{d\underline{x}^{(k)} \, d\underline{g}^{(k)}} - \frac{H^{(k)} d\underline{g}^{(k)} \, d\underline{g}^{(k)^T} \, H^{(k)}}{d\underline{g}^{(k)^T} \, H^{(k)} \, d\underline{g}^{(k)}} \quad . \tag{3.18}$$

In the early stages of iteration, the search direction is close to steepest descent and the method is therefore robust ; whilst near the minimum the search direction approaches the Newton-Raphson direction, so that the method is efficient there.

Flow charts for the algorithms are given in Appendix 3.

4. The Numerical Solution of Hyperbolic Partial Differential Equations

4.1 Introduction.

Two widely-used formulae are described, namely those of Wendroff and Lax-Wendroff. When applied to the counterflow problem, Wendroff's formula is implicit, while that of Lax-Wendroff is explicit. The implicit formula has the advantages of guaranteed convergence and stability for all values of the flow rates but has the disadvantage that rounding errors cause a blurring where there should be a sharp cut-off in the solution along characteristic lines of discontinuity. The explicit formula is only convergent and stable for values of the flow rates ≤ 1 in normalised coordinates, but it provides a sharp cut-off along the lines of discontinuity. The use of the explicit Lax-Wendroff formula for counterflow requires the explicit use of Wendroff's formula in conjunction with it for the first cell in from each of the boundaries $x = 0$, $x = L$. Both formulae may be used explicitly for the transport equation.

4.2 The finite different formulae

Let us consider the equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad a > 0 \tag{4.1}$$

where a is a constant, subject to the initial condition $u = u_o(x)$ for $t = 0$ and $-\infty < x < \infty$, which is discussed by Mitchell {21} . Let us suppose the solution region, $t \geq 0$, $-\infty < x < \infty$, to be covered by a rectangular grid of lines parallel to the x - and t - axes and consider the portion of the grid illustrated in Fig. 4.1, in which S is the point (mh, nk) etc, h and k being the grid spacings in the x - and t - directions respectively.

4. NUMERICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

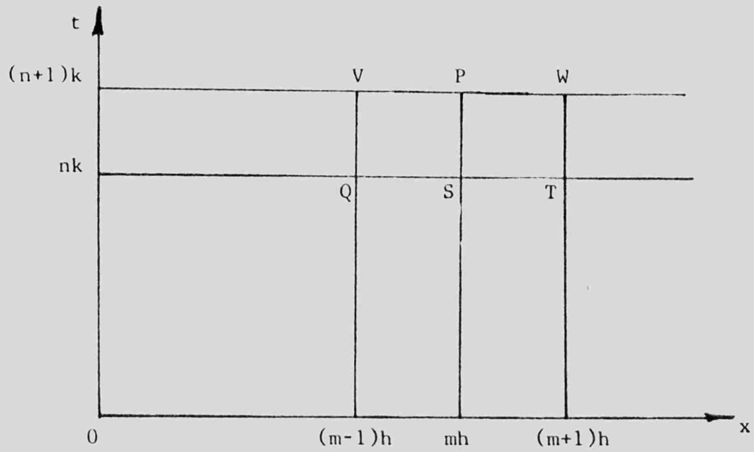

Fig. 4.1. The finite difference mesh.

Using Taylor's theorem,

$$U_p = \exp(k \frac{\partial}{\partial t}) U_s$$

$$= \exp(- ka \frac{\partial}{\partial x}) U_s \qquad (4.2)$$

But $$\frac{\partial}{\partial x} \approx \frac{1}{h} \delta_x \qquad (4.3)$$

where δ_x is the standard central difference operator, so that

$$U_p \simeq \exp(-r \delta_x) U_s$$

$$\approx (1 - r \delta_x + \tfrac{1}{2} r^2 \delta_x) U_s$$

$$\approx U_s - \tfrac{1}{2} r(U_T - U_Q) + \tfrac{1}{2} r^2 (U_T - 2U_S + U_Q)$$

$$= (1-r^2)U_S - \tfrac{1}{2} r(1-r)U_T + \tfrac{1}{2} r(1+r)U_Q \qquad (4.4)$$

where $r = \frac{ka}{h}$.

This formula, which is correct to second order, is the Lax-Wendroff formula. It operates on the inverted T-shaped molecule PQST.

Wendroff's formula may be obtained as follows. The partial differential equation (4.1) may be discretised with second order accuracy as

$$\frac{U^{n+1}_{m+\frac{1}{2}} - U^{n}_{m+\frac{1}{2}}}{k} + a\,\frac{U^{n+\frac{1}{2}}_{m+1} - U^{n+\frac{1}{2}}_{m}}{h} = 0 \tag{4.5}$$

or

$$U^{n+1}_{m+\frac{1}{2}} - U^{n}_{m+\frac{1}{2}} + r(U^{n+\frac{1}{2}}_{m+1} - U^{n+\frac{1}{2}}_{m}) = 0 \quad . \tag{4.6}$$

Now, the approximation

$$U^{n}_{m+\frac{1}{2}} \simeq \tfrac{1}{2}(U_S + U_T)$$

is correct only to first order, but the difference approximation

$$U^{n+1}_{m+\frac{1}{2}} - U^{n}_{m+\frac{1}{2}} \simeq \tfrac{1}{2}(U_P + U_W - U_S - U_T) \tag{4.7}$$

is correct to second order. Using this, with a similar result for $U^{n+\frac{1}{2}}_{m+1} - U^{n+\frac{1}{2}}_{m}$, we obtain

$$U_W = U_S + \frac{1-r}{1+r}(U_T - U_P), \tag{4.8}$$

which is Wendroff's formula. This formula operates in the rectangular molecule PWST of Fig. 4.1.

4.3 Stability analysis

There are two main sources of error in the solution of partial differential equations by finite difference methods.

(1) The exact solution to the finite difference equations may differ from the exact solution to the partial differential equation. This is the problem of convergence.

(2) Rounding errors may grow exponentially so that the actual solution to the finite difference equations differs from the exact solution. This is the problem of stability.

4. NUMERICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Let $u(x,t)$ denote the exact solution to the partial differential equation, $U(x,t)$ the exact solution to the finite difference equations, and $\tilde{U}(x,t)$ the actual solution to the finite difference equations. The finite difference equations are <u>convergent</u> if $u(x,t) - U(x,t) \rightarrow 0$ uniformly over the solution region as $h, k \rightarrow 0$ and $m, n \rightarrow \infty$ with $mh(=x)$ and $nk(=t)$ remaining fixed. They are <u>stable</u> if the error term $U_m^n - \tilde{U}_m^n$ $(=Z_m^n)$ remains bounded as n increases, for fixed k ; and they are <u>consistent</u> if

$$\frac{\text{truncation error in the finite differences formulae}}{k} \rightarrow 0 \qquad (4.9)$$

as $h, k \rightarrow 0$.

The accuracy of the finite difference schemes for linear partial differential equations is guaranteed by the stability of the scheme according to Lax's equivalence theorem {27} which states that "given a properly posed linear boundary value problem and a finite difference approximation to it which satisfies the consistency condition, stability is a necessary and sufficient condition for convergence"

The Wendroff and Lax-Wendroff finite difference schemes satisfy the consistency condition since the truncation expressions

$$\frac{1}{6}\left[h^2\frac{\partial^3 u}{\partial x^2 \partial t} + k^2\frac{\partial^3 u}{\partial t^3}\right]_m^n \quad , \quad \frac{1}{6}\left[k^2\frac{\partial^3 u}{\partial t^3} + ah^2\frac{\partial^3 u}{\partial x^3}\right]_m^n \qquad (4.10)$$

respectively tend to zero as $h, k \rightarrow 0$.

The stability of the formulae may be investigated by a number of methods, one of which is Von Neumann's method of Fourier Analysis. This method examines the propagating effect of a single row of errors along the line $t = 0$. Suppose these are represented by a finite Fourier series of the form

$$Z = \sum_j A_j e^{i\beta_j x} \qquad (4.11)$$

where the number of terms is equal to the number of mesh points on the line, and the frequencies β_j are arbitrary. It is necessary only to consider the single term $e^{i\beta x}$ where β is any real number. To investigate the error propagation as t increases, we seek a solution to the finite difference equation which reduces to $e^{i\beta x}$ when $t = 0$. Let this solution be

$$e^{\alpha t} \quad e^{i\beta x} \tag{4.12}$$

where $\alpha = \alpha(\beta)$ is in general complex. The original error component will not grow in time if $|e^{\alpha k}| \leq 1$ for all α. This is Von Neumann's criterion for stability, in which $\xi = e^{\alpha k}$ is called the amplification factor. Since U and $\tilde{U}$ both satisfy the difference equations, so also does Z.

On substituting

$$Z_m^n = e^{\alpha nk} \; e^{i\beta mh} = \xi^n \; e^{i\beta mh} \tag{4.13}$$

into Wendroff's formula, we obtain after some manipulation

$$\xi = \frac{1 - ir\tan\frac{\beta h}{2}}{1 + ir\tan\frac{\beta h}{2}} \quad , \tag{4.14}$$

so that $|\xi| = 1$. Thus Wendroff's formula is unconditionally stable.

On applying the same analysis to the Lax-Wendroff formula, we obtain

$$\xi = (1 - 2r^2 \sin^2 \frac{\beta h}{2}) - ir \sin\beta h \;, \tag{4.15}$$

so that

$$|\xi| = \{1 - 4r^2(1-r^2)\sin^4 \frac{\beta h}{2}\}^{\frac{1}{2}} \quad .$$

Hence the Lax-Wendroff method is stable if $0 \leq r \leq 1$.

An alternative condition for stability is the Courant-Friedrichs-Lewy condition which states that explicit finite difference schemes for hyperbolic equations are stable provided the region of finite difference determination lies within that of the differential equations. This may be expressed by saying that the characteristic through P must cut the line $t = nk$ at R, between Q and T, as shown in Fig. 4.2. The stability of the two schemes according to this condition agrees with that obtained above.

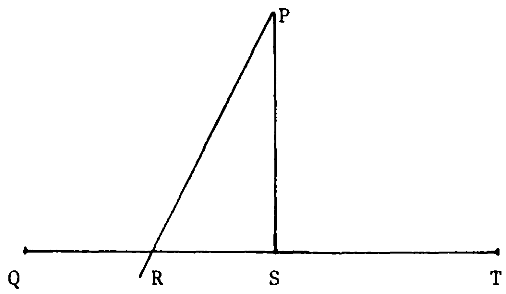

Fig. 4.2. The Courant-Friedrichs-Lewy stability condition.

4.4 Systems of hyperbolic equations.

Let us now consider the numerical solution of the hyperbolic system of equations

$$\frac{\partial \underline{u}}{\partial t} + A \frac{\partial \underline{u}}{\partial x} = 0 \tag{4.16}$$

where A is an $n \times n$ real matrix and $\underline{u}$ is an n-component column vector. If A has all real eigenvalues and n linearly independent eigenvectors, the system is hyperbolic. In the case where A is constant, Wendroff's formula is

$$\{I + \tfrac{1}{2}(I + pA)\Delta_x\} \underline{U}_m^{n+1} = \{I + \tfrac{1}{2}(I - pA)\Delta_x\} \underline{U}_m^{n} \tag{4.17}$$

and the Lax-Wendroff formula is

$$\underline{U}_m^{n+1} = \{I - \tfrac{1}{2}pA(\Delta_x + \nabla_x) + \tfrac{1}{2}p^2A^2(\Delta_x - \nabla_x)\} \underline{U}_m^{n} \tag{4.18}$$

while the formulae corresponding to (4.17) and (4.18) which maintain second order accuracy when A is a function of x and t are :

$$\{I + \tfrac{1}{2}(I + pA_{m+\frac{1}{2}}^{n+\frac{1}{2}})\Delta_x\} \underline{U}_m^{n+1} = \{I + \tfrac{1}{2}(I - pA_{m+\frac{1}{2}}^{n+\frac{1}{2}})\Delta_x\} \underline{U}_m^{n} \tag{4.19}$$

$$\underline{U}_m^{n+1} = \{ I - \tfrac{1}{2}pA_m^{n+\frac{1}{2}}(\Delta_x + \nabla_x) + \tfrac{1}{4}p^2(A_m^{n+\frac{1}{2}}\Delta_x A_m^{n+\frac{1}{2}}\nabla_x + A_m^{n+\frac{1}{2}}\nabla_x A_m^{n+\frac{1}{2}}\Delta_x\} \underline{U}_m^n \qquad (4.20)$$

In the case of counterflow, A is constant for fixed flow rates u, v, and a function of t if the flow rates are variable. However in both cases A is neither positive definite nor negative definite; in this situation, care must be taken to ensure that the boundary conditions lead to a well-posed problem. In the general case that A has k positive and n-k negative eigenvalues and a solution is required in the range $0 \leq x \leq 1$, $t \geq 0$, then k components of $\underline{u}$ must be given on the boundary x = 0 and n-k components on the boundary x = 1. All n components of $\underline{u}$ must be given on the boundary t = 0. For counterflow,

$$A = \begin{bmatrix} u & 0 \\ 0 & -v \end{bmatrix}, \qquad (4.21)$$

and so has one positive and one negative eigenvalue. θ_1 is given on the boundary x = 0, θ_2 is given on x = 1, and both θ_1 and θ_2 are given on t = 0, and thus the problem is well-posed.

As in the scalar case with constant a, the stability of the finite difference approximation at a grid point in the solution region will involve the stability of the corresponding difference equation for the values which the coefficients take at the grid points. Hence we shall now have a local stability condition, usually a limitation on the size of the ratio $\frac{k}{h}$, which will vary from point to point of the region. Thus the global stability condition is the largest mesh ratio which satisfies the stability condition at each point of the region.

If a typical Fourier term

$$\underline{U}_m^n = \underline{U}_o\, e^{i\beta x} \qquad (4.22)$$

,

where $\underline{U}_o$ is a constant vector, is substituted into the difference equations, it is found that $\underline{U}_m^{n+1}$ is of the same form as $\underline{U}_m^n$, but with $G\underline{U}_o$ replacing $\underline{U}_o$, where G is the amplification matrix. For the Lax-Wendroff method, the amplification matrix is

$$\begin{aligned} G &= I + \tfrac{1}{2}pA(e^{i\beta h} - e^{-i\beta h}) + \tfrac{1}{2}p^2A^2(e^{i\beta h} + e^{-i\beta h} - 2) \\ &= I - p^2A^2(1 - \cos\theta) - ipA\sin\theta \end{aligned} \tag{4.23}$$

where $\theta = \beta h$, while for the implicit Wendroff scheme, the amplification matrix is given by

$$\{I + ipA\tan\tfrac{1}{2}\theta\}\, G = I - ipA\tan\tfrac{1}{2}\theta \; . \tag{4.24}$$

The Von-Neumann necessary condition for stability is

$$\max_i |\mu_i| \leq 1 \quad , \quad i = 1, 2, \ldots, n \tag{4.25}$$

where the μ_i are the eigenvalues of G. This condition is satisfied for the Lax-Wendroff amplification matrix G if

$$|p\alpha_i| \leq 1 \quad , \quad i = 1, 2, \ldots, n \tag{4.26}$$

where the α_i are the eigenvalues of A. In the case of counterflow this becomes

$$pu \leq 1 \quad \text{and} \quad pv \leq 1 \quad . \tag{4.27}$$

The implicit Wendroff scheme is unconditionally stable and imposes no upper limit on p.

4.5 The extension to a non-zero right-hand side

Up to this point, the discussion of the solution of equation (4.16) has been in terms of a zero right-hand side. However the solution of the counterflow and transport equations requires a non-zero right-hand side.

4. NUMERICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

A literature search revealed no previous consideration of this problem, so that it was necessary to develop suitable formulae. Two approaches were used; one was to consider the problem with a general right-hand side in the form

$$\frac{\partial \underline{u}}{\partial t} + A\frac{\partial \underline{u}}{\partial x} = \underline{f}(x,t,\underline{u}) \ , \tag{4.28}$$

while the second was to made use of the linearity of the equations we wish to solve by considering the problem in the form

$$\frac{\partial \underline{u}}{\partial t} + A\frac{\partial \underline{u}}{\partial x} = B\underline{u} \ . \tag{4.29}$$

The first method will be referred to as the "general method" and the second as the "exponential operator method".

4.5.1. Wendroff's formula.

In the case of Wendroff's formula, the two approaches give the same result, and the derivation is as follows.

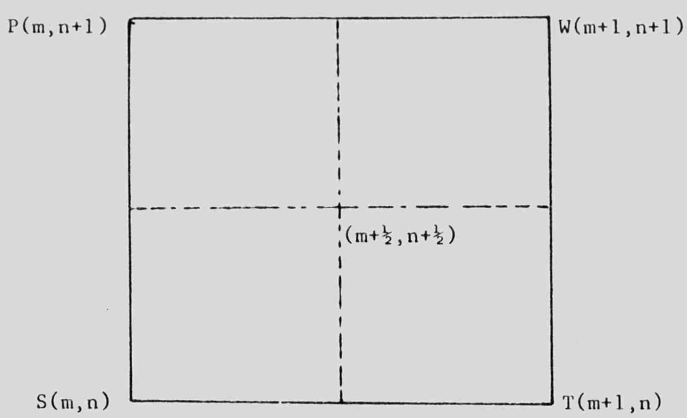

Fig. 4.3. The Wendroff finite difference molecule.

Referring to Fig 4.3, the finite difference approximation to (4.28) centred on the point $(m + \frac{1}{2},\ n + \frac{1}{2})$ is

$$\frac{\underline{U}^{n+1}_{m+\frac{1}{2}} - \underline{U}^{n}_{m+\frac{1}{2}}}{k} + A^{n+\frac{1}{2}}_{m+\frac{1}{2}} \frac{\underline{U}^{n+\frac{1}{2}}_{m+1} - \underline{U}^{n+\frac{1}{2}}_{m}}{h} = \underline{f}^{n+\frac{1}{2}}_{m+\frac{1}{2}} \ . \tag{4.30}$$

Using the replacements of $\underline{U}^{n+1}_{m+\frac{1}{2}}$ etc. described in (4.6), we have

$$\{ I + \tfrac{1}{2}(I + pA^{n+\frac{1}{2}}_{m+\frac{1}{2}})\Delta_x\} \ \underline{U}^{n+1}_{m} = \{ I + \tfrac{1}{2}(I - pA^{n+\frac{1}{2}}_{m+\frac{1}{2}})\Delta_x\} \ \underline{U}^{n}_{m} + k\underline{f}^{n+\frac{1}{2}}_{m+\frac{1}{2}}$$

$$= \{ I + \tfrac{1}{2}(I - pA^{n+\frac{1}{2}}_{m+\frac{1}{2}})\Delta_x\} \ \underline{U}^{n}_{m} + \tfrac{1}{4}k(\underline{f}_P + \underline{f}_W + \underline{f}_S + \underline{f}_T), \tag{4.31}$$

correct to second order .

4.5.2. The Lax-Wendroff formula

(i) The General Method.

With the general right-hand-side term in equation (4.28) the operator replacement for $k\frac{\partial}{\partial t}$ used in equation (4.2) is no longer posible. Instead, the following derivation leads to an equation equivalent to equation (4.20) to second order.

$$\underline{U}^{n+1}_{m} = \exp(k \tfrac{\partial}{\partial t})\underline{U}^{n}_{m}$$

$$\simeq (I + k \frac{\partial}{\partial t} + \tfrac{1}{2}k \frac{\partial^2}{\partial t^2})\underline{U}^{n}_{m}$$

$$= \left[\underline{U} + k(\underline{f} - A \frac{\partial U}{\partial x}) + \tfrac{1}{2}k^2 \frac{\partial \underline{u}}{\partial t} (\underline{f} - A \frac{\partial \underline{U}}{\partial x}) \right]^{n}_{m}$$

$$= \underline{U}^{n}_{m} + k\underline{f}^{n+\frac{1}{2}}_{m} - kA^{n+\frac{1}{2}}_{m} \frac{\partial \underline{U}^{n}_{m}}{\partial x} - \tfrac{1}{2}k^2 A^{n}_{m} \frac{\partial}{\partial x} (\underline{f} - A^{n}_{m} \frac{\partial \underline{U}^{n}_{m}}{\partial x})$$

$$= \left[\underline{U} - \tfrac{1}{2}k^2 A \frac{\partial}{\partial x} (\underline{f} - A \frac{\partial \underline{U}}{\partial x}) \right]^{n}_{m} + k\underline{f}^{n+\frac{1}{2}}_{m} - kA^{n+\frac{1}{2}}_{m} \frac{\partial \underline{U}^{n}_{m}}{\partial x}$$

$$\simeq \{I - pA^{n+\frac{1}{2}}_{m} \ \delta_x + \tfrac{1}{2}p^2 A^{n}_{m} \ \delta_x A^{n}_{m} \delta_x\} \underline{U}^{n}_{m}$$

$$+ k(f^{n+\frac{1}{2}}_{m} - \tfrac{1}{2}pA^{n}_{m} \ \delta_x f^{n}_{m}) \tag{4.32}$$

4. NUMERICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

This is the required finite difference formula.

(ii) The Exponential operator method.

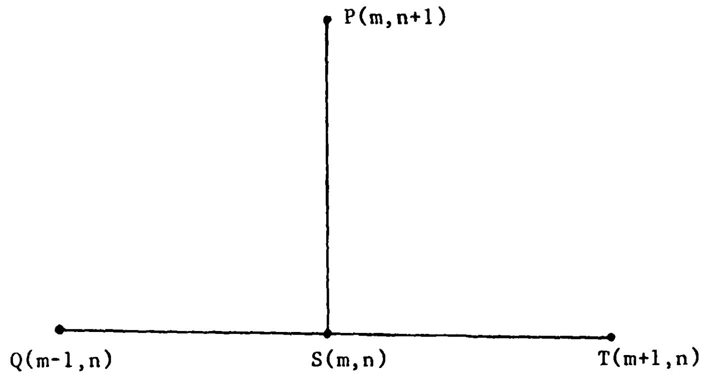

Fig. 4.4. The Lax-Wendroff finite difference molecule.

We seek to solve equation (4.29). Referring to Fig. (4.4),

$$
\begin{aligned}
\underline{U}_P &= \exp\left(k\frac{\partial}{\partial t}\right)\underline{U}_s \\
&= \exp\left(-kA\frac{\partial}{\partial x} + kB\right)\underline{U}_s \\
&\simeq \exp(-pA\,\delta_x + kB)\,\underline{U}_s \\
&\approx \left\{I - pA\,\delta_x + kB + \frac{k^2}{2}\left(B - \frac{A}{h}\,\delta_x\right)^2\right\}\underline{U}_s \\
&= \left\{I - pA\,\delta_x + kB + \frac{k^2}{2}\left(B^2 - \frac{AB+BA}{h}\,\delta_x + \frac{A^2}{h^2}\,\delta_x^2\right)\right\}\underline{U}_s \\
&= \{I + kB + \tfrac{1}{2}k^2B^2 - pA\,\delta_x + \tfrac{1}{2}p^2A^2\,\delta_x^2 - \tfrac{1}{2}pk(AB+BA)\delta_x\}\,\underline{U}_S \\
&= \underline{U}_S - \tfrac{1}{2}pA(\underline{U}_T - \underline{U}_Q) + kB(I + \tfrac{1}{2}kB)\underline{U}_S \\
&\qquad - \tfrac{1}{4}kp(AB+BA)(\underline{U}_T - \underline{U}_Q) + \tfrac{1}{2}p^2A^2(\underline{U}_T - 2\underline{U}_S + \underline{U}_Q) \\
&= (I + kB(I + \tfrac{1}{2}kB) - p^2A^2)\underline{U}_S + \tfrac{1}{2}p(A + \tfrac{1}{2}k(AB+BA) + pA^2)\underline{U}_Q
\end{aligned}
$$

$$- \tfrac{1}{2}p(A + \tfrac{1}{2}k(AB+BA) - pA^2)\underline{U}_T \ , \qquad (4.33)$$

which is the required formula.

PART II

SIMPLE MODELS

5. The transport equation.

5.1 Introduction.

This chapter is devoted to the minimisation of the following three types of cost functional :

Problem 5.4 $$J = \frac{1}{2}\int_0^T\int_0^L \phi^2(x,t)dx\,dt \tag{5.1.1}$$

Problem 5.5 $$J = \frac{1}{2}\int_0^T \{\phi(L,t) - \phi^*(t)\}^2 dt + \frac{1}{2}a\int_0^L\int_0^T \{u(x,t) - u^*(x,t)\}^2\,dx\,dt \tag{5.1.2}$$

Problem 5.6 $$J = \frac{1}{2}\int_0^T\int_0^L \{\phi^2(x,t) + a\,u^2(x,t)\}\,dx\,dt \tag{5.1.3}$$

where ϕ satisfies the transport equation

$$\frac{\partial\phi(x,t)}{\partial t} + u(x,t)\frac{\partial\phi(x,t)}{\partial x} = 0\ ,\quad 0 \leq x \leq L,\quad 0 \leq t \leq T, \tag{5.1.4}$$

together with boundary and initial conditions

$$\phi(x,0) = \phi_o(x) \tag{5.1.5}$$

$$\phi(0,t) = \phi_1(t) \tag{5.1.6}$$

satisfying the Goursat continuity condition $\phi_o(0) = \phi_1(0)$.

In Problem (5.4) the control is subject to the constraints

$$0 < u_1 \leq u\,(x,t) \leq u_2 \tag{5.1.7}$$

whilst in the other two problems $u(x,t)$ is continuous and unrestricted.

Problem (5.4) is shown to lead to an improperly posed boundary value problem. Analytic solutions are obtained to Problem (5.5) using elementary (Lagrange) methods, while Problem (5.6) is solved using a hodograph transformation leading to the Poisson-Euler-Darboux equation. Problem (5.6) is also investigated numerically in Chapter 9.

5. THE TRANSPORT EQUATION

Degtyarev and Sirazetdinov {7} have studied a similar problem arising from the optimal control of a tubular reactor, in which the coefficient u is a function of x only. However, in their case the multiplicative coefficient u is not subject to control. In the two-phase exothermic reaction which they consider the governing equations are

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial t} + (a + bx)\frac{\partial \phi_1}{\partial x} &= -(k_1 + k_2)\phi_1 \qquad (1.1) \\ \frac{\partial \phi_2}{\partial t} + (a + bx)\frac{\partial \phi_2}{\partial x} &= k_2\, \phi_1 \qquad (1.2) \end{aligned}\right\} \quad 0 \le x \le L, \quad 0 \le t \le T$$

where ϕ_1, ϕ_2 are the mass concentrations of the reacting substances and the parameters k_1 and k_2 are defined by

$$k_i = k_i^o \exp\{-E_i/R\theta(t)\} \quad , \quad i = 1, 2 \quad . \tag{5.1.8}$$

k_1^o, k_2^o, E_1, E_2 and R are constants and control is exercised through the temperature $\theta(t)$ of the reactor. θ is subject to the inequality constraints

$$0 < \theta_1 \le \theta(t) \le \theta_2 \quad . \tag{5.1.9}$$

Initial conditions at t=0 and inlet conditions at x = 0 are prescribed, and they seek to maximise the total output of ϕ_2 at x = L, namely

$$I = \int_o^T \phi_2\,(L,t)\,dt \quad . \tag{5.1.10}$$

A complete resolution of the problem is given in their paper.

The above problems belong to a more general class of furnace problems which may be stated in the following way. We seek to minimise the cost functional

5. THE TRANSPORT EQUATION

$$J = \tfrac{1}{2}\int_0^T \{\phi(L,t) - \phi^*(t)\}^2 \, dt + \tfrac{1}{2}\int_0^T\int_0^L \{a(u(x,t)-u^*(x,t))^2+bw^2(x,t)\} \, dx \, dt$$

output quality regulation cost of controls

$$- c\int_0^T\int_0^L k(w - \phi) \, dx \, dt \qquad + \tfrac{1}{2}d\int_0^L \{\phi(x,T) - \tilde{\phi}(x)\}^2 \quad dx$$

quality transfer residual regulation (5.1.11)

subject to the partial differential equation (linear furnace equation)

$$\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\frac{\partial\phi}{\partial x} = k(w(x,t) - \phi(x,t)) \qquad (5.1.12)$$

with boundary and initial conditions (5.1.5), (5.1.6), $u(x,t)$, $w(x,t)$ are controls, u corresponding to flow rate, and w to the furnace temperature.

Davies {6} has obtained results for furnace problems when the flow rate is dependent on t only, which are discussed in Section 5.2 below. The results of Davies, together with those of the present study are summarised in Table 5.1.

5. THE TRANSPORT EQUATION

Governing equation	Control	Cost functional	Results
I The problems studied by Davies			
5.1. $\phi_t+a\phi_x+k\phi=kw(x,t)$	$w(x,t)$	$\frac{1}{2}\int_0^T[\phi(L,t)-\phi^*(t)]^2dt$ $+\frac{1}{2}n\int_0^T\int_0^L w^2dxdt$ (n=constant)	Constant speed. Explicit solution available.
5.2. $\phi_t+a\phi_x+k\phi=kw(t)$	$w(t)$	$\frac{1}{2}\int_0^T\{[\phi(L,t)-\phi^*(t)]^2$ $+nw^2\}dt$ (n=constant)	Constant speed. $w(t)$ satisfies : LF:Second order linear o.d.e. SF:Second order linear delay system.
5.3. $\phi_t+u(t)\phi_x+k\phi=kw(x,t)$	$u(t)$ $w(x,t)$	$\frac{1}{2}\int_0^T\{[\phi(L,t)-\phi^*(t)]^2$ $+mu^2\}dt$ $+\frac{1}{2}n\int_0^L\int_0^T w^2dxdt$ (m,n constants)	Controlled speed $u(t)$ satisfies : LF:Second order non-linear delay differential equation SF:Second order non-linear delay differential system.
II The problems of the present study			
5.4. $\phi_t+u(x,t)\phi_x=0$	$0<u_1\leq u\leq u_2$	$\frac{1}{2}\int_0^L\int_0^T\phi^2dxdt$	Bang-bang control. Improperly posed problem.
5.5. $\phi_t+u(x,t)\phi_x=0$	$u(x,t)$	$\frac{1}{2}\int_0^T(\phi(L,t)-\phi^*)^2dt$ $+\frac{1}{2}\int_0^L\int_0^T a(u-u^*)^2dxdt$	Controlled speed $u(x,t)$ satisfies a second order non-linear partial differential equation.
5.6. $\phi_t+u(x,t)\phi_x=0$	$u(x,t)$	$\frac{1}{2}\int_0^L\int_0^T(\phi^2+au^2)dxdt$ (a=constant,>0)	"

Table (5.1) Classification of first order partial differential equations involving control.

5. THE TRANSPORT EQUATION

5.2 Furnace problems : flow rate dependent on time

Davies {6} has studied furnace problems of the general type

$$\frac{\partial\phi}{\partial t}(x,t) + u(t)\frac{\partial\phi}{\partial x}(x,t) = k(w(x,t) - \phi(x,t)) \ , \ 0 \le x \le L, \ 0 \le t \le T \ , \tag{5.2.1}$$

with boundary and initial conditions (5.1.5), (5.1.6) satisfying the Goursat continuity condition.

In these problems characteristic curves are given by

$$x - \beta(t) = \text{constant} \tag{5.2.2}$$

where

$$u(t) = \beta'(t) \ , \qquad \beta(0) = 0 \ , \tag{5.2.3}$$

and it is necessary to consider the two cases of long furnace ($L > \beta(T)$) and short furnace ($L < \beta(T)$).

We give here a brief description of the solution of problem (5.3).

This problem is to minimise

$$J = \tfrac{1}{2}\int_0^L\int_0^T \{m\,[u(t)]^2 + n\,[w(x,t)]^2\}\,dx\,dt + \tfrac{1}{2}\int_0^T [\phi(L,t) - \phi^*(t)]^2\,dt \ , \tag{5.2.4}$$

where m, n are constants, subject to the linear equation (5.2.1) together with boundary and initial conditions (5.1.5), (5.1.6). According to equations (2.4), (2.5) and (2.14), the Hamiltonians are

$$H_0 = \tfrac{1}{2}\,mu^2 + \tfrac{1}{2}\,nw^2 + \lambda\,\{k(w - \phi) - u\,\frac{\partial\phi}{\partial x}\} \ , \tag{5.2.5}$$

$$H_1 = \tfrac{1}{2}\,\{\phi(L,t) - \phi^*(t)\}^2 \qquad , \tag{5.2.6}$$

$$H_2 = \int_0^L H_0\,(\phi,\ \phi_x,\ u, w)dx \quad . \tag{5.2.7}$$

The domain co-state equation is

$$\frac{\partial\lambda}{\partial t} + u(t)\frac{\partial\lambda}{\partial x} - k\lambda = 0 \tag{5.2.8}$$

and the boundary co-state equations are

$$\phi(L,t) - \phi^*(t) - u(t) \quad \lambda(L,t) = 0 \qquad \text{on } x = L \tag{5.2.9}$$

$$\lambda(x,T) = 0 \qquad \text{on } t = T \, , \tag{5.2.10}$$

while from equation (2.13) we have the optimality condition

$$m \, L \, u(t) - \int_o^L \lambda(x,t) \, \frac{\partial \phi}{\partial x} \, (x,t) \, dx = 0 \, . \tag{5.2.11}$$

The control $w(x,t)$ must be determined from

$$nw(x,t) + k\lambda(x,t) = 0. \tag{5.2.12}$$

On introducing the characteristic function $\beta(t)$ defined by equation (5.2.3), it is found after considerable algebra that in the long furnace case illustrated in Fig. (5.2.1) $\beta(t)$ satisfies the second order non-linear differential delay equation

$$P\{\beta(t)\} \equiv Lm \, \beta''(t) + \frac{n}{4k} \, \frac{d}{dt}\{(e^{2kt} - 1) \, A_3^2(L - \beta(t))\}$$

$$- \frac{n}{k} \, \beta'(t) \, \phi_o(L - \beta(t)) \, A_3(L - \beta(t)) \quad = 0 \tag{5.2.13}$$

with $\beta(0) = 0$, $\beta'(T) = 0$, where A_3 is defined by

$$A_3(L - \beta(t)) = \frac{\phi^*(t) - e^{-kt}\phi_o(L - \beta(t))}{\sinh kt + \frac{n}{k} \, e^{kt} \, \beta'(t)} \, . \tag{5.2.14}$$

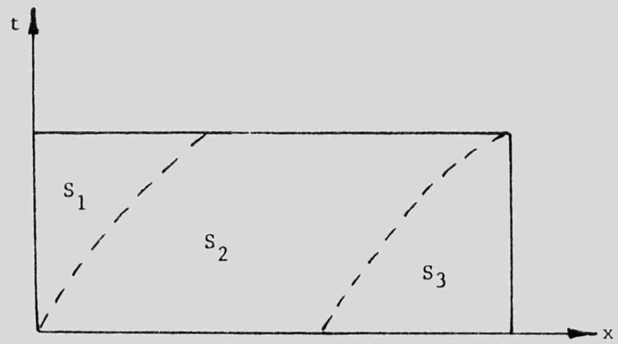

Figure 5.2.1. The solution domain for the long furnace

When $\beta(t)$ has been found from equation (5.2.13), the solution for $w(x,t)$ follows from

$$w(x,t) = \begin{cases} 0 & , \quad (x,t)\varepsilon S_1,\ S_2 \\ e^{kt} A_3\,(x-\beta(t)) & , \quad (x,t)\varepsilon\ S_3 \end{cases} \tag{5.2.15}$$

In the short furnace case, illustrated in Fig. 5.2.2, in which $0 < t_N < t_M < T$, $\beta(t)$ satisfies the following differential delay equations in the various time ranges :

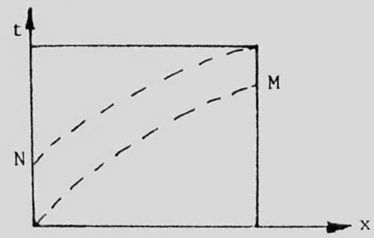

Fig. 5.2.2. The solution domain for the short furnace $(0 < t_N < t_M < T)$.

(a) $0 < t < t_N$:

$$P\{\beta(t)\} = -\tfrac{1}{2}\, n^2 e^{2kt} A_2^2(-\beta(t)) + \frac{n^2}{k} A_2(-\beta(t)) \frac{d}{dt}\{e^{kt} \phi_1(t)\} \tag{5.2.16}$$

where $P\{\beta(t)\}$ is defined in (24) and $A_2(-\beta(t))$ is defined by

$$\begin{aligned} &\tfrac{1}{2}\, e^{kt} A_2(-\beta(t)) + e^{-kt} B_2(-\beta(t)) = \phi_1(t) \\ &\{\tfrac{1}{2} + \frac{n^2}{u} \beta'(t)\}\, e^{kt} A_2(L-\beta(t)) + e^{-kt} B_2(L-\beta(t)) \equiv \phi^k(t) \end{aligned} \tag{5.2.17}$$

(b) $0 < t_N < t_M$: $P\{\beta(t)\} = 0$ (5.2.18)

(c) $t_M < t < T$:

$$m^2 L\, \beta''(t) + \frac{n^2}{4k} \frac{d}{dt}\{e^{2kt} A_2^2(L-\beta(t))\} - \frac{n^2}{k}\beta'(t)\, A_2(L-\beta(t))\, B_2'(L-\beta(t)) = 0 \tag{5.2.19}$$

where A_2 and B_2 are as defined in equation (5.2.17).

The boundary conditions for $\beta(t)$, are $\beta(0) = 0$, $\beta'(T) = 0$ with $\beta(t)$, $\beta'(t)$ continuous at t_N and t_M .

No solutions of these equations have as yet been obtained, but it is clear from the above mathematical description that the speed control will require a highly sophisticated control mechanism.

5. THE TRANSPORT EQUATION

5.3 Bang-bang control of the transport equation

This section, previously considered by the author {13}, is included to show that in distributed parameter problems non-coercive controls that affect the characteristicsof the partial differential equations lead to improperly posed problems, to non-unique controls, and to regions of the solution domain where the state variables are not defined. For purposes of illustration we consider the following first order transport equation problem :

Problem 5.4

$$\phi_t = -u\phi_x \equiv f \quad , \tag{5.3.1}$$

where $S \equiv \{0 \leq t \leq T, \quad 0 \leq x \leq 2\pi\}$, the control $u(x,t)$ satisfies $\frac{1}{2} \leq u \leq 1$, and the boundary conditions are

$$\phi = 0 \quad \text{on} \quad x = 0 \tag{5.3.2}$$

$$\phi = \sin x \text{ on } t = 0 \ . \tag{5.3.4}$$

We seek to minimise the quadratic functional

$$J = \int_0^{2\pi} \phi^2(x,T)dx = \int_0^{2\pi} G_3 \{ \phi(x,T) \} \, dx \ . \tag{5.3.5}$$

The Hamiltonian is

$$H = \lambda f = - u\phi_x\lambda \ . \tag{5.3.6}$$

J is minimised when H is minimised with respect to u, so that for

$$\phi_x\lambda > 0, \; u = u_{max} = 1 \tag{5.3.7}$$

$$\phi_x\lambda < 0, \; u = u_{min} = \tfrac{1}{2} \ , \tag{5.3.8}$$

i.e. the control is "bang-bang".

λ satisfies

5. THE TRANSPORT EQUATION

$$\frac{\partial\psi}{\partial t} = -\frac{\partial H}{\partial\phi} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\phi_x}\right) = -\lambda_x u, \tag{5.3.9}$$

that is, the same equation as for ϕ. Hence in particular characteristics for λ are the same as those for ϕ.

Boundary conditions for λ are that on $t = T$,

$$\lambda(x,T) = \frac{\partial G_3}{\partial\phi(x,T)} = 2\phi(x,T), \tag{5.3.10}$$

while on $x = 2\pi$,

$$\left[\frac{\partial H}{\partial\phi_x}\right]_{x=2\pi} = -\frac{\partial G_2}{\partial\phi(2\pi,t)} \quad , \tag{5.3.11}$$

that is, $\lambda(2\pi, t) = 0$. (5.3.12)

The characteristics of the equations are the lines

$x - ut = \text{const},$

along which ϕ and λ are constant. The slope of these lines is $\frac{1}{u}$.

The domain S is subdivided into regions of differing u.

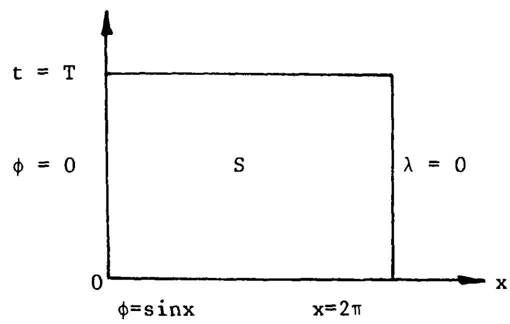

Fig. 5.3.1. Boundary conditions for example 5.4.

The following argument establishes the optimal control and shows that in some parts of the domain S at least, the control, and the state and co-state functions are non-unique.

Values of $\phi(x,t)$ are determined by transporting boundary values of ϕ forward (i.e. in the direction of increasing t) along characteristics, while values of $\lambda(x,t)$ are determined by transporting boundary values of λ back along characteristics.

Consider the neighbourhood of the point$(\frac{\pi}{2}$, 0) for a small value of the end time T. The boundary condition $\lambda(x,T) = 2\phi(x,T)$ shows that $\text{sgn}\{\lambda(x,t)\} = \text{sgn}\{\phi(x,t)\}$ and since $\phi = \sin(x - ut)$, which is positive in the neighbourhood of $t = 0$, $\lambda(x,t)$ will also be positive. Since $\frac{\partial\phi}{\partial x}(\frac{\pi}{2}, 0) = 0$, so that $\frac{\partial\phi}{\partial x}$ changes sign, we may expect a change of control along some contour emanating from the point $(\frac{\pi}{2}, 0)$ with $u = 1$ on the left where $\phi_x\lambda > 0$ and $u = \frac{1}{2}$ on the right where $\phi_x\lambda < 0$. The requirement that $\phi(x,t)$ be continuous shows this contour to be a straight line of slope $^4/_3$. Such a contour will be called a "switching curve", although it is not a curve in phase space, as in the ordinary differential equation case.

In a similar way, one would expect a switching curve emanating from $(\frac{3\pi}{2}, 0)$ where $\frac{\partial\phi}{\partial x}(x,0)$ again changes sign, while $\phi(x,0)$ (and hence $\lambda(x,t)$ for small T) is of constant sign,

These two switches require that $u(x,t)$ switches back from $\frac{1}{2}$ to 1 along some contour emanating from a point on the $t = 0$ axis lying between $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$. The point must be $(\pi, 0)$ since $\phi(x,0)$ (and hence $\lambda(x,t)$ for small T) changes sign there, while $\frac{\partial\phi}{\partial x}$ is of constant sign in the neighbourhood of this point.

We have thus established the patterm of events detailed in Tables 5.2 and 5.3 (see Fig. 5.3.2).

<u>Table 5.2 : Values of $\phi(x,T)$</u>

Line segment	Values of $\phi(x,T)$
AB	0
BC	$\sin(x-T)$
CD	$\sin(x-\frac{1}{2}T)$
DE	0
EF	$\sin(x-T)$
FG	$\sin(x-\frac{1}{2}T)$.

Table 5.3 Subdomains of S

Subdomain	$\phi(x,t)$	$\lambda(x,t)$	Control $u(x,t)$
OAB	0	0	is undetermined in ($\frac{1}{2}$, 1) so that control is non-unique.
OBCH	sin (x-t)	2 sin (x-t)	1
HCI	sin (x-t)	is undetermined but > 0.	1
ICJ	$\sin(x-\frac{1}{2}t)$	"	$\frac{1}{2}$
JCDK	$\sin(x-\frac{1}{2}t)$	$2\sin(x-\frac{1}{2}t)$	$\frac{1}{2}$
KDE	0	0	The switching curve is undetermined but the contribution to J along DE will be zero if switching occurs either along KD or along KE, so that again the control is non-unique.

Similar results hold for the remainder of S. We note that for $T \geq 2\pi$, $\phi(x,T)$ can be controlled to zero (see Fig. 5.3.3).

Conclusion.

This example illustrates that in the optimal control of distributed parameter systems, the use of bang-bang distributed controls which affect the characteristics of the partial differential equations will lead to improperly posed problems, giving non-unique controls and also regions of the solution domain in which the state and co-state functions are not uniquely determined.

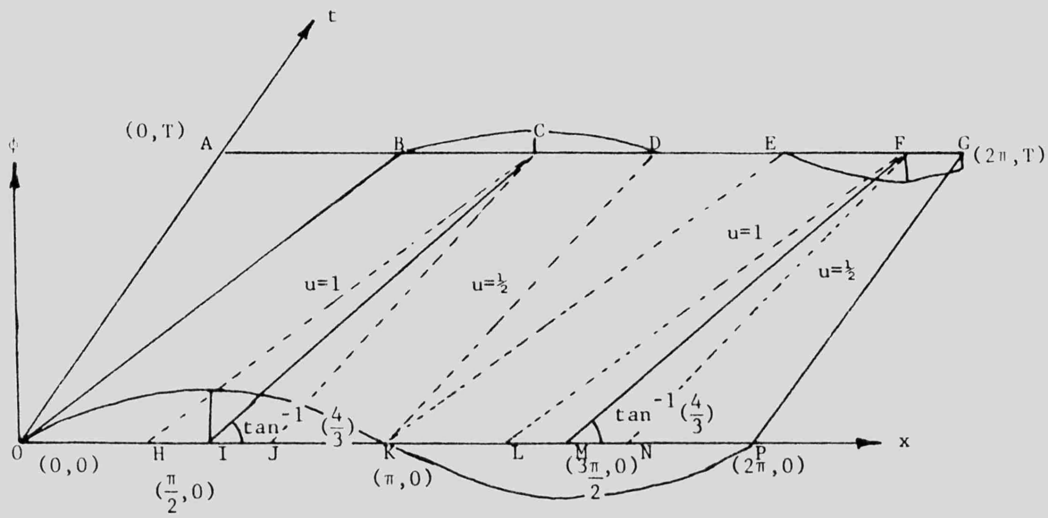

Fig. 5.3.2. The evolution of the switching curves and of the state function in example 5.4. (x, t, φ) is a three-dimensional set of coordinates.

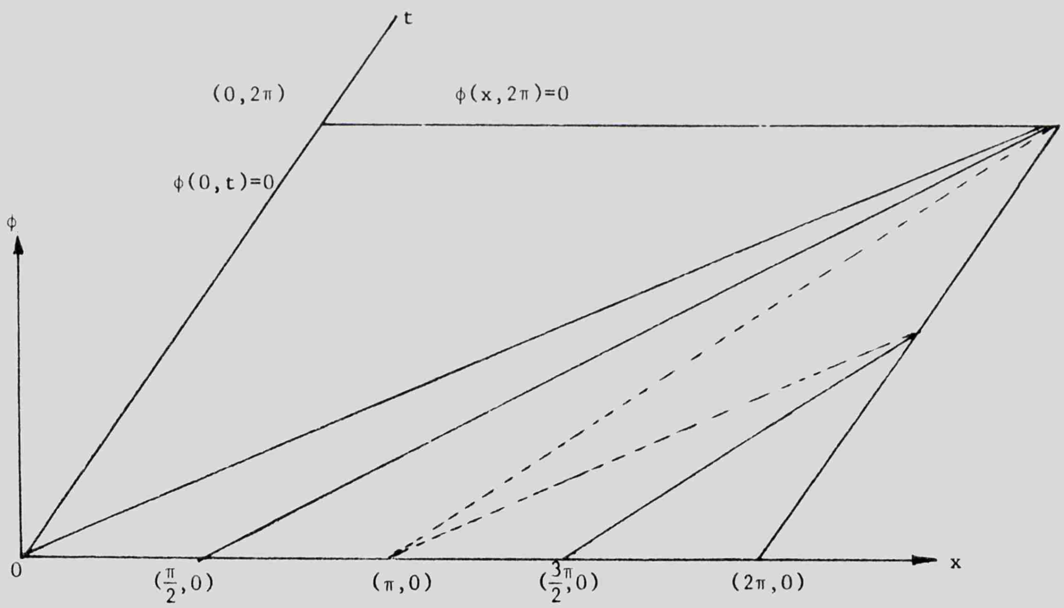

Fig. 5.3.3. The case T = 2π.

5. THE TRANSPORT EQUATION

5.4 Continuous control of the transport equation.

In this section we consider the following problem.

Problem 5.5*

We seek to minimise the cost functional

$$J = \tfrac{1}{2}\int_0^T \{\phi(L,t) - \phi^*(t)\}^2\, dt +$$

$$+ \tfrac{1}{2}\int_0^L\int_0^T \{a\,[u(x,t) - u^*(x,t)]^2 + b\,[w(x,t)]^2\}dxdt \tag{5.4.1}$$

$$+ \tfrac{1}{2}\, c\int_0^L \{\phi(x,T) - \tilde{\phi}(x)\}^2 dx$$

subject to the transport equation,

$$\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\frac{\partial\phi}{\partial x}(x,t) = k(w(x,t) - \phi(x,t)),\quad 0 \leq x \leq L,\quad 0 \leq t \leq T, \tag{5.4.2}$$

with prescribed boundary and initial conditions

$$\phi(x,0) = \phi_o(x)\,, \qquad 0 \leq x \leq L\,, \tag{5.4.3}$$

$$\phi(0,t) = \phi_1(t)\,, \qquad 0 \leq t \leq T\,, \tag{5.4.4}$$

satisfying the Goursat continuity condition $\phi_o(0) = \phi_1(0)$. a, b, c and k are non-negative constants.

The characteristics of equation (5.4.2) have slope $\frac{1}{u}$, and in the general case might enter and leave the solution domain on any of its boundaries, as indicated in Fig (5.5.1). This would lead to an improperly posed problem involving regions where ϕ could not be determined. Likewise, since the domain costate equation (5.4.8) has the same characteristics, information about λ

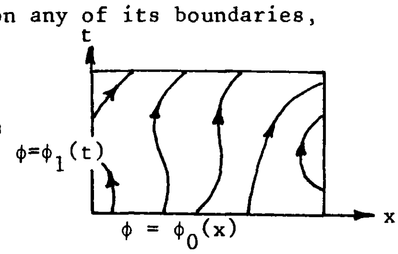

Fig. 5.4.1. The general form of characteristics of equation (5.4.2)

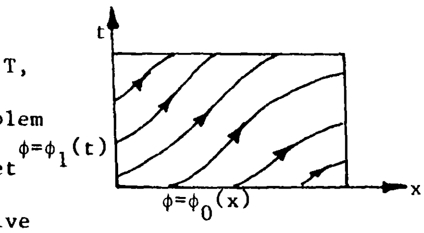

Fig.5.4.2. The assumed form of characteristics of equation (5.4.2)

being obtained from the boundaries $x = L$, $t = T$, it would also lead to an improperly posed problem for λ. Hence, in order to make progress, let us assume that the characteristics have positive slope along the edges of the solution domain as indicated in Fig. (5.4.2). This will tend to be ensured by the inclusion of the term $\frac{1}{2}a(u - u^*)$ in the cost function (5.4.1), with non-negative u^*. Furthermore the validity of the assumption can be verified, once the solutions have been obtained.

Introducing the co-state variable $\lambda(x,t)$, the Hamiltonians are

$$H_o = \lambda\{k(w - \phi) - u\phi_x\} + \tfrac{1}{2}a(u - u^*)^2 + \tfrac{1}{2}bw^2 \tag{5.4.5}$$

$$H_1 = \begin{cases} \tfrac{1}{2}(\phi - \phi^*)^2 & \text{on } x = L \\ \tfrac{1}{2}a(\phi - \hat{\phi})^2 & \text{on } t = T \ . \end{cases} \tag{5.4.6}$$

The domain costate equation is, from equation (2.6),

$$\frac{\partial\lambda}{\partial t} + \frac{\partial H}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial\phi_x}\right) = 0 \quad , \quad 0 \le x \le L, \quad 0 \le t \le T. \tag{5.4.7}$$

$$\lambda_t - k\lambda + \frac{\partial}{\partial x}(\lambda u) = 0$$

$$\lambda_t + u\,\lambda_x + \lambda u_x - k\lambda = 0 \quad . \tag{5.4.8}$$

The conditions $\frac{\partial H}{\partial u} = 0$, $\frac{\partial H}{\partial w} = 0$ yield

$$-\lambda\phi_x + a(u - u^*) = 0 \tag{5.4.9}$$

$$k\lambda + bw = 0 \ . \tag{5.4.10}$$

The boundary co-state equations are, from equation (2.7),

$$\frac{\partial H_1}{\partial\phi} + \frac{\partial H_o}{\partial\phi_x} = 0 \quad \text{on } x = L \tag{5.4.11}$$

$$\frac{\partial H_1}{\partial\phi} - \lambda = 0 \quad \text{on } t = T \tag{5.4.12}$$

which yield

$$\lambda u = c(\phi(x,T) - \tilde{\phi}(x)) \text{ on } t = T \tag{5.4.13}$$

$$\lambda = \phi(L,t) - \phi^*(t) \text{ on } x = L . \tag{5.4.14}$$

Let us consider first the solution of the domain equations. To summarise we have the following coupled set of partial differential equations

$$\phi_t + u\phi_x + k\phi - kw = 0 \tag{5.4.2}$$

$$\lambda_t + u\lambda_x + \lambda u_x - k\lambda = 0 \tag{5.4.8}$$

$$-\lambda\phi_x + a(u - u^*) = 0 \tag{5.4.9}$$

$$k\lambda + bw = 0 . \tag{5.4.10}$$

On eliminating w using (5.4.10),

$$\phi_t + u\phi_x + k\phi + \frac{k^2\lambda}{b} = 0 \tag{5.4.15}$$

$$\lambda_t + u\lambda_x + \lambda u_x - k\lambda = 0 \tag{5.4.8}$$

$$-\lambda\phi_x + a(u - u^*) = 0 . \tag{5.4.9}$$

From equations (5.4.9) and (5.4.15),

$$\phi_x u + \frac{k^2}{b}\lambda = -\phi_t - k\phi \tag{5.4.16}$$

$$au - \phi_x\lambda = au^* , \tag{5.4.17}$$

whence

$$u = \frac{\frac{a}{b}k^2u^* - \phi_x\phi_t - k\phi\phi_x}{\phi_x^2 + a\frac{k^2}{b}} \tag{5.4.18}$$

and

$$\lambda = \frac{-a(\phi_x u^* + \phi t + k\phi)}{\phi_x^2 + a\frac{k^2}{b}} . \tag{5.4.19}$$

These expressions, together with their x and t derivatives, may be substituted in (5.4.8) to yield a second order non-linear partial differential equation for ϕ. However, at this point we simplify the problem by

(i) choosing $k = 0$, so as to remove the right hand side of equation (5.4.2),

(ii) removing w from the cost function (5.4.1),

(iii) taking u^* to be constant.

(iv) removing the residual quality term from the cost functional (5.4.1).

Thus we have the following simpler problem.

Problem 5.5

We seek to minimise

$$J = \tfrac{1}{2}\int_0^T \{\phi(L,t) - \phi^*(t)\}^2 \, dt + \tfrac{1}{2}\int_0^L \int_0^T a\{u(x,t) - u^*\}^2 \, dxdt \tag{5.4.20}$$

subject to

$$\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\,\frac{\partial\phi}{\partial x}(x,t)=0, \quad 0 \le x \le L, \quad 0 \le t \le T\ , \tag{5.4.21}$$

with prescribed boundary and initial conditions (5.4.3), (5.4.4). Thus the aim is to control the output by controlling the characteristics of equation (5.4.17). Corresponding to equations (5.4.2), (5.4.8) - (5.4.10) we have the set

$$\phi_t + u\phi_x = 0 \tag{5.4.21}$$

$$\lambda_t + u\lambda_x + u_x\lambda = 0 \tag{5.4.22}$$

$$-\lambda\phi_x + a(u - u^*) = 0\ , \tag{5.4 23}$$

(■(5.4.9))

and on eliminating λ between (5.4.22) and (5.4.23) we have the pair of equations

$$\phi_t + u\phi_x = 0 \tag{5.4.21}$$

$$u_t + (3u - 2u^*)u_x = 0 \tag{5.4.24}$$

(5.4.24) integrates directly to give

$$x + (2u^* - 3u)t = \psi(u) \tag{5.4.25}$$

where ψ is an arbitrary function.

(5.4.21) has Lagrange equations

$$\frac{dt}{1} = \frac{dx}{u} = \frac{d\phi}{0} \tag{5.4.26}$$

so that one integral is

$$\phi = c_1 \quad , \tag{5.4.27}$$

and we also have the differential relationship

$$dx = u\,dt; \tag{5.4.28}$$

while from (5.4.25)

$$dx = (3u - 2u^*)dt + 3t\,du + \psi'(u)du \quad , \tag{5.4.29}$$

whence

$$\frac{dt}{du} + \frac{3t}{2(u-u^*)} = - \frac{\psi'(u)}{2(u-u^*)} \quad . \tag{5.4.30}$$

$$\therefore \quad (u-u^*)^{3/2}t = -\tfrac{1}{2}\int^u \psi'(v)\,(v - u^*)^{\frac{1}{2}}\,dv + c_2 \tag{5.4.31}$$

is a second integral. Combining these,

$$\phi = \chi\,\{(u - u^*)^{3/2}t + \tfrac{1}{2}\int^u \psi'(v)\,(v - u^*)^{\frac{1}{2}}\,dv\} \quad , \tag{5.4.32}$$

where χ is a second arbitrary function.

From (5.4. 9)

$$\lambda = \frac{a(u-u^*)}{\phi_x} \quad . \tag{5.4.33}$$

Now $\phi_x = \chi'\{(u - u^*)^{3/2}t + \tfrac{1}{2}\int^u \psi'(v)(v - u^*)^{\frac{1}{2}}\,dv\}\,.\{\tfrac{1}{2}(u - u^*)^{\frac{1}{2}}$

$$(3t + \psi'(u))\}\,u_x \tag{5.4.34}$$

while from (5.4.21),

$$u_x = \frac{1}{\psi'(u)+3t} \tag{5.4.35}$$

Hence

$$\lambda = \frac{2a(u-u^*)^{\frac{1}{2}}}{\chi'\{(u-u^*)^{3/2}t + \frac{1}{2}\int^{u} \psi'(v)(v-u^*)dv\}} \quad . \tag{5.4.36}$$

Equations (5.4.25), (5.4.32) and (5.4.36) comprise a complete solution of the system of partial differential equations (5.4.21) - (5.4.23).

5.5. The hodograph transformation and the Poisson-Euler-Darboux equations

Problem 5.6

In this section we consider the problem of minimising

$$J = \frac{1}{2}\int_o^T\int_o^L \{ [\phi(x,t)]^2 + a\,[u(x,t) - u^*]^2\}dx\,dt = \int_o^T\int_o^L F(\phi,u)\,dx\,dt \tag{5.5.1}$$

subject to $\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\,\frac{\partial\phi}{\partial x}(x,t) = 0 \quad , \quad 0 \le x \le L, \quad 0 \le t \le T$, (5.5.2)

with prescribed initial and boundary conditions

$$\phi(x,0) = \phi_o(x) \quad , \qquad 0 \le x \le L \quad , \tag{5.5.3}$$

$$\phi(0,t) = \phi_1(t) \quad , \qquad 0 \le t \le T \quad , \tag{5.5.4}$$

satisfying the Goursat continuity condition $\phi_o(0) = \phi_1(0)$. a and u^* are non-negative constants. As in Section 5.4 we shall assume that u is non-negative at the boundaries of the solution domain.

The Hamiltonian for the problem is

$$H_o = F - u\phi_x\lambda = \frac{1}{2}(\phi^2 + a(u - u^*)^2) - u\phi_x\lambda \ . \tag{5.5.5}$$

From equation (2.6) the domain co-state equation is

$$\frac{\partial\lambda}{\partial t} + \frac{\partial H_o}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial H_o}{\partial\phi_x}\right) = 0 \ , \quad 0 \le x \le L, \quad 0 \le t \le T, \tag{5.5.6}$$

5. THE TRANSPORT EQUATION

i.e. $$\lambda_t + u\lambda_x + u_x\lambda + \phi = 0 \;. \tag{5.5.7}$$

The condition $\frac{\partial H_o}{\partial u} = 0$ yields

$$-\lambda\phi_x + a(u - u^*) = 0 \qquad , \quad 0 \le x \le L, \quad 0 \le t \le T \;. \tag{5.5.8}$$

From equation (2.7) the boundary co-state equations are

$$\left[\frac{\partial H_o}{\partial \phi_x}\right]_{x=L} = 0 \tag{5.5.9}$$

and $$\left[\lambda\right]_{t=T} = 0 \tag{5.5.10}$$

which yield $$\lambda u = 0 \qquad \text{on } x=L \tag{5.5.11}$$

$$\lambda = 0 \qquad \text{on } t=T \;. \tag{5.5.12}$$

Over the solution domain S we have the coupled set of partial differential equations

$$\phi_t + u\phi_x = 0 \tag{5.5.2}$$

$$\lambda_t + u\lambda_x + u_x\lambda + \phi = 0 \tag{5.5.7}$$

$$a(u - u^*) - \phi_x\lambda = 0 \tag{5.5.8}$$

On eliminating λ from (5.5.7) and (5.5.8) we obtain the pair of equations

$$\phi_t + u\phi_x = 0 \tag{5.5.13}$$

$(\equiv(5.5.2))$

$$u_t + (3u - 2u^*)\, u_x + \phi\,\frac{\phi_x}{a} = 0 \;. \tag{5.5.14}$$

Let us now seek to find characteristic directions.

Equation (5.5.13) is already in characteristic form, and a further equation in characteristic form can be obtained by taking a linear combination of equations (5.4.13) and (5.4.14). The x and t derivatives of the equation

$$\alpha_1 \phi_t + \alpha_1(u + \frac{\alpha_2}{a} \phi) \phi_x + \alpha_2 u_t + \alpha_2(3u - 2u^*) u_x = 0 \tag{5.5.15}$$

will be in the same ratio if

$$\frac{\alpha_1}{\alpha_1 u + \frac{\alpha_2}{a} \phi} = \frac{\alpha_2}{\alpha_2(3u - 2u^*)} , \tag{5.5.16}$$

whence either $\alpha_2 = 0$, giving equation (5.4.13) again, or

$$2a \alpha_1(u - u^*) = \alpha_2 \phi. \tag{5.5.17}$$

Let us choose

$$\alpha_1 = \phi , \quad \alpha_2 = 2a(u - u^*) ; \tag{5.5.18}$$

then

$$\phi \phi_t + (3u - 2u^*)\phi \phi_x + 2a(u - u^*)(u_t + (3u - 2u^*)u_x) = 0 \tag{5.5.19}$$

$$\phi (\frac{\partial}{\partial t} + (3u - 2u^*) \frac{\partial}{\partial x})\phi + 2a(u - u^*)(\frac{\partial}{\partial t} + (3u - 2u^*) \frac{\partial}{\partial x})u = 0 \tag{5.5.20}$$

$$\phi (\frac{\partial}{\partial t} + (3u - 2u^*) \frac{\partial}{\partial x})\phi + 2a(u - u^*)(\frac{\partial}{\partial t} + (3u - 2u^*) \frac{\partial}{\partial x})(u - u^*) = 0 \tag{5.5.21}$$

(since u^* is constant)

$$\therefore \quad (\frac{\partial}{\partial t} + (3u - 2u^*) \frac{\partial}{\partial x})(\phi^2 + 2a(u - u^*)^2) = 0 . \tag{5.5.22}$$

In order to solve these equations, we use a hodograph transformation. An interpretation of (5.5.13) is that $\phi(x,t)$ is constant along the characteristic direction

$$\frac{dt}{1} = \frac{dx}{u(x,t)} ; \tag{5.5.23}$$

thus characteristics of the first kind will be solutions of the ordinary differential equation

$$\frac{dx}{dt} = u(x,t) . \tag{5.5.24}$$

In a similar manner we see from equation (5.5.22) that $\phi^2 + 2a(u - u^*)^2$ is constant along characteristics of the second kind whose ordinary differential equation is

$$\frac{dx}{dt} = 3u - 2u^* \ . \qquad (5.5.25)$$

Thus we introduce characteristic coordinates

$$r = \phi \qquad (5.5.26)$$

and $$s = \phi^2 + 2a(u - u^*)^2 \ . \qquad (5.5.27)$$

On characteristics of the first kind, $\frac{dx}{dt} = u$ and r is constant.

$$\therefore \quad dx = x_s \, ds \qquad (5.5.28)$$

$$dt = t_s \, ds \qquad (5.5.29)$$

$$\frac{dx}{dt} = u = \frac{x_s}{t_s} \qquad (5.5.30)$$

$$\therefore \quad x_s - u \, t_s = 0 \ . \qquad (5.5.31)$$

On characteristics of the second kind, $\frac{dx}{dt} = 3u - 2u^*$ and s is constant.

$$\therefore \quad dx = x_r \, dr \qquad (5.5.32)$$

$$dt = t_r \, dr \qquad (5.5.33)$$

$$\frac{dx}{dt} = 3u - 2u^* = \frac{x_r}{t_r} \qquad (5.5.34)$$

$$x_r - (3u - 2u^*) \, t_r = 0 \qquad (5.5.35)$$

where $$u = \frac{s - r^2}{2a}^{\frac{1}{2}} + u^* \ . \qquad (5.5.36)$$

We now make a further change of variable to

$$\xi = r^2 \qquad (5.5.37)$$

and $$\eta = s \ . \qquad (5.5.38)$$

Then $$\left.\begin{array}{llll} x_r = x_\xi \, 2r & , & t_r = t_\xi \, 2r & , \\ x_s = x_\eta & , & t_s = t_\eta & , \end{array}\right\} \qquad (5.5.39)$$

whence

$$x_\xi - (3u - 2u^*) \, t_\xi = 0 \qquad (5.5.40)$$

and $$x_\eta - u \, t_\eta = 0 \qquad (5.5.41)$$

where

$$u = \frac{\eta - \xi}{2a}^{\frac{1}{2}} + u^* . \tag{5.5.42}$$

On eliminating x,

$$2(u - u^*)\, t_{\xi\eta} + 3u_\eta\, t_\xi - u_\xi\, t_\eta = 0 ; \tag{5.5.43}$$

and on substituting for u and its derivatives we have

$$t_{\xi\eta} - \frac{3}{4(\xi - \eta)}\, t_\xi - \frac{1}{4(\xi - \eta)}\, t_\eta = 0 . \tag{5.5.44}$$

Eliminating t we obtain

$$x_{\xi\eta} - \frac{3u}{3u - 2u^*} \cdot \frac{x_\xi}{4(\xi - \eta)} - \frac{3u - 2u^*}{u} \cdot \frac{x_\eta}{4(\xi - \eta)} = 0. \tag{5.5.45}$$

Thus t satisfies the Poisson-Euler-Darboux equation (5.5.44) which is described in detail by Darboux {5} and is of the general form

$$E(\beta,\beta') \equiv z_{\xi\eta} - \frac{\beta' z_\xi}{\xi - \eta} + \frac{\beta z_\eta}{\xi - \eta} = 0 ; \tag{5.5.46}$$

hence t satisfies $E(-\frac{1}{4},\frac{3}{4}) = 0$.

When $u^* = 0$, the x-equation becomes

$$x_{\xi\eta} - \frac{x_\xi}{4(\xi - \eta)} - 3\,\frac{x_\eta}{4(\xi - \eta)} = 0 , \tag{5.5.47}$$

so that in this case x satisfies $E(-\frac{3}{4},\frac{1}{4}) = 0$.

When $0 < \beta < 1$, $0 < \beta' < 1$, the general solution of $E(\beta,\beta') = 0$ is

$$z(\beta,\beta') = (\eta - \xi)^{1-\beta-\beta'} \int_0^1 \Phi(\xi + (\eta - \xi)\sigma)\sigma^{-\beta} (1 - \sigma)^{-\beta'}\, d\sigma$$

$$+ \int_0^1 \Psi(\xi + (\eta - \xi)\sigma)\sigma^{\beta'-1} (1 - \sigma)^{\beta-1}\, d\sigma \tag{5.5.48}$$

where Φ and Ψ are arbitrary functions .

However, when one or other of the parameters is negative, we use instead

$$z(\beta-m,\beta'-n) = (\eta - \xi)^{m+n+1-\beta-\beta'} \frac{\partial^{m+n}}{\partial\xi^n \, \partial\eta^m} \left[\frac{z(\beta,\beta')}{(\eta - \xi)^{1-\beta-\beta'}} \right] . \tag{5.5.49}$$

Hence

$$t = (\eta - \xi)^{\frac{1}{2}} \frac{\partial}{\partial\eta} \left[\frac{z(\frac{3}{4},\frac{3}{4})}{(\eta - \xi)^{-\frac{1}{2}}} \right] \tag{5.5.50}$$

$$= (\eta - \xi)^{\frac{1}{2}} \frac{\partial}{\partial\eta} \{ I(\chi, -\tfrac{3}{4}, -\tfrac{3}{4}) + (\eta - \xi)^{\frac{1}{2}} I(\Theta, -\tfrac{1}{4}, -\tfrac{1}{4}) \} \tag{5.5.51}$$

$$= (\eta - \xi)^{\frac{1}{2}} I(\chi', \tfrac{1}{4}, -\tfrac{3}{4}) + \tfrac{1}{2} I(\chi, -\tfrac{1}{4}, -\tfrac{1}{4}) + (\eta - \xi) I(\Theta', \tfrac{3}{4}, -\tfrac{1}{4}) \tag{5.5.52}$$

where χ, Θ are arbitrary functions, and $I(\Phi, \alpha, \alpha')$ is a short-hand for the integral

$$I(\Phi, \alpha, \alpha') = \int_0^1 \Phi(\xi + (\eta - \xi)\sigma)\sigma^{\alpha} (1 - \sigma)^{\alpha'} \, d\sigma . \tag{5.5.53}$$

x may then be obtained from equation (5.5.41) as

$$x = \int^{\eta} u(\xi,\eta') \frac{\partial}{\partial\eta'} t(\xi,\eta') \, d\eta' . \tag{5.5.54}$$

No further detailed work on this general case has been carried out to date.

<u>The case $u^* = 0$.</u> Let us consider in more detail the case $u^* = 0$. Equation (5.5.45) reduces to

$$x_{\xi\eta} - \frac{x_{\xi}}{4(\xi - \eta)} - \frac{3x_{\eta}}{4(\xi - \eta)} = 0 \tag{5.5.55}$$

(≡(5.5.47))

so that x then satisfies $E(-\tfrac{3}{4},\tfrac{1}{4}) = 0$, and has solution

$$x = (\eta - \xi)^{\frac{3}{2}} \frac{\partial}{\partial\eta} \left[\frac{z(\frac{1}{4},\frac{1}{4})}{(\eta - \xi)^{\frac{1}{2}}} \right] \tag{5.5.56}$$

$$= (\eta - \xi)^{3/2} I(\Phi', \tfrac{3}{4}, -\tfrac{1}{4}) + (\eta - \xi) I(\Psi', \tfrac{1}{4}, -\tfrac{3}{4}) - \tfrac{1}{2} I(\Psi, -\tfrac{3}{4}, -\tfrac{3}{4}) \tag{5.5.57}$$

where Φ and Ψ are arbitrary functions. However, Φ, Ψ, χ and Θ must be matched by substituting the solutions (5.5.52) and (5.5.57) into equations (5.5.40) (with $u^* = 0$) and (5.5.41). This yields

5. THE TRANSPORT EQUATION

$$\Theta = 2^{\frac{1}{2}}\,\Phi, \quad \chi = 2^{\frac{1}{2}}\Psi\ , \tag{5.5.58}$$

so that x is given by equation (5.5.57) while t is given by

$$t = 2^{\frac{1}{2}}\{(\eta - \xi)^{\frac{1}{2}}\ I(\Psi'\ ,\ \tfrac{1}{4},\ -\tfrac{3}{4}) + \tfrac{1}{2}I(\Phi,\ -\tfrac{1}{4},\ -\tfrac{1}{4})$$
$$+ (\eta - \xi)I\ (\Phi',\ \tfrac{3}{4},\ \tfrac{1}{4})\} \quad . \tag{5.5.59}$$

Boundary conditions

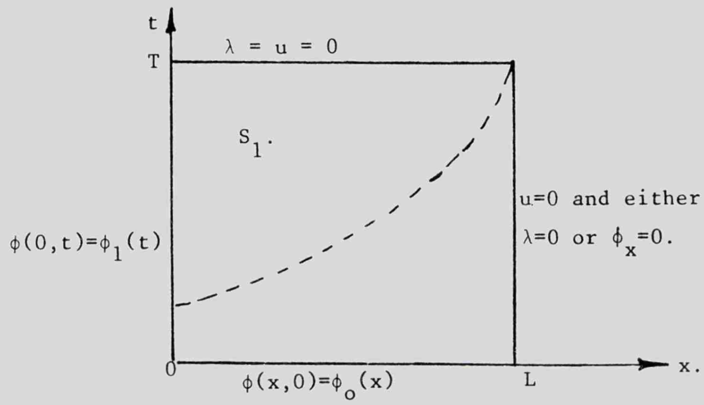

Fig. (5.5.1) Boundary conditions for the case $u^* = 0$ of problem 5.6

From equation (5.5.12) we have $\lambda = 0$ on $t = T$. Hence unless ϕ_x is infinite there, which is unlikely in a minimisation problem, from equation (5.5.8) $u = 0$ also on $t = T$. From equation (5.5.11) either $\lambda = 0$ or $u = 0$ on $x = L$. If $\lambda = 0$, then by the same argument u is likely to be zero also ; while if $u = 0$ then either $\lambda = 0$ or $\phi_x = 0$. Hence let us assume that $u = 0$ and either $\lambda = 0$ or $\phi_x = 0$, giving the set of boundary conditions illustrated in Fig. (5.5.1). Thus on $x = L$ and on $t = T$, $u = 0$, which from equation (5.4.42), implies $\xi = \eta$.

Since $u = 0$ on $x = L$, and from equation (5.5.23) $\phi(x,t)$ is constant along characteristics for which $\dfrac{dx}{dt} = u$,

$$\phi = \text{constant} = \phi_o(L) \quad \text{on} \quad x = L. \tag{5.5.60}$$

From equation (5.5.57) and (5.5.59), on $t = T$ we have

$$x = -\tfrac{1}{2}\int_o^1 \Psi(\xi)\sigma^{-\frac{3}{4}}(1-\sigma)^{-\frac{3}{4}}\,d\sigma = -\tfrac{1}{2}\Psi(\xi)\,B(\tfrac{1}{4},\tfrac{1}{4}) \tag{5.5.61}$$

$$T = 2^{-\frac{1}{2}}\int_o^1 \Phi(\xi)\sigma^{-\frac{1}{4}}(1-\sigma)^{-\frac{1}{4}}\,d\sigma = 2^{-\frac{1}{2}}\Phi(\xi)\,B(\tfrac{3}{4},\tfrac{3}{4}) \tag{5.5.62}$$

where $B(p,q)$ is the Beta function with arguments p and q.

Equation (5.5.62) shows that in a region S_1 of the solution domain bordering on $t = T$, Φ is the constant

$$\Phi = \frac{2^{\frac{1}{2}}T}{B(\frac{3}{4},\frac{3}{4})}. \tag{5.5.63}$$

Since from equations (5.5.26) and (5.5.37),

$$\xi = \{\phi(x,t)\}^2 , \tag{5.5.64}$$

from equation (5.5.61) ,

$$\Psi\{[\phi(x,T)]^2\} = \frac{-2x}{B(\frac{1}{4},\frac{1}{4})} . \tag{5.5.65}$$

On $x = L$ we have,

$$L = -\tfrac{1}{2}\int_o^1 \Psi(\xi)\sigma^{-\frac{3}{4}}(1-\sigma)^{-\frac{3}{4}}\,d\sigma = -\tfrac{1}{2}\Psi(\xi)\,B(\tfrac{1}{4},\tfrac{1}{4}) \tag{5.5.66}$$

$$t = 2^{-\frac{1}{2}}\int_o^1 \Phi(\xi)\sigma^{-\frac{1}{4}}(1-\sigma)^{-\frac{1}{4}}\,d\sigma = 2^{-\frac{1}{2}}\Phi(\xi)\,B(\tfrac{3}{4},\tfrac{3}{4}) \tag{5.5.67}$$

so that from (5.5.66)

$$\Psi\{[\phi_o(L)]^2\} = \frac{-2L}{B(\frac{1}{4},\frac{1}{4})} \tag{5.5.68}$$

while (5.5.67) yields

$$t = 2^{-\frac{1}{2}}\Phi\{[\phi_o(L)]^2\}\;B(\tfrac{3}{4},\tfrac{3}{4}) = \text{constant}, \tag{5.5.69}$$

a contradiction which indicates that the choice of $u^* = 0$ is a singular case. Fig. (5.5.2) indicates the possible behaviour of characteristics at the boundaries of the solution region. We turn to a numerical approach to this problem in Chapter 9.

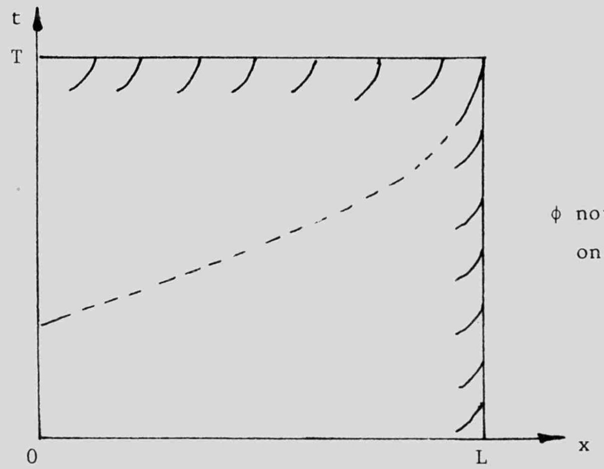

Fig. 5.5.2. Possible behaviour of characteristics of Problem 5.6 at the boundaries x = L, t = T.

6. The Optimal Control of Counterflow

6.1 General theory

In this section we consider the general problem of minimising the cost functional

$$J = \tfrac{1}{2}\int_0^T \{ [\theta_1(L,t) - \theta_1^*(t)]^2 + a[\theta_2(L,t) - \theta_2^*(t)]^2$$
$$+ b[u(t)-u^*(t)]^2 + c[v(t) - v^*(t)]^2\} \, dt$$
$$+ \tfrac{1}{2} d\int_0^L [\theta_1(x,T) - \tilde{\theta}_1(x)]^2 \, dx \qquad (6.1)$$

subject to the system equations

$$C_1\left[\frac{\partial\theta_1}{\partial t} + u(t)\frac{\partial\theta_1}{\partial x}\right] = k(h\theta_2 - \theta_1) \qquad (1.15)$$

$$C_2\left[\frac{\partial\theta_2}{\partial t} - v(t)\frac{\partial\theta_2}{\partial x}\right] = -k(h\theta_2 - \theta_1) \qquad (1.16)$$

which may alternatively be written in the form

$$\frac{\partial\theta_1}{\partial t} + u(t)\frac{\partial\theta_1}{\partial x} = k_1(h\theta_2 - \theta_1) \qquad (6.2)$$

$$\frac{\partial\theta_2}{\partial t} - v(t)\frac{\partial\theta_2}{\partial x} = -k_2(h\theta_2 - \theta_1) \qquad (6.3)$$

where $k_1 = k/C_1$, $k_2 = k/C_2$.

Let us suppose now that $u(t)$, $v(t)$, $\theta_2(L,t) \equiv \phi(t)$ are available as controls and that

$$\theta_1(x,0) = \theta_{10}(x) \qquad (6.4)$$

$$\theta_2(x,0) = \theta_{20}(x) \qquad (6.5)$$

$$\theta_1(0,t) = \theta_{11}(t) \qquad (6.6)$$

are known functions.

6. THE OPTIMAL CONTROL OF COUNTERFLOW

The optimality conditions

Introducing domain co-state variables $\lambda_1(x,t)$, $\lambda_2(x,t)$ and the boundary co-state variable $\mu(t)$, we have the following Hamiltonians :

$$H_o = \lambda_1 \{k_1(h\theta_2 - \theta_1) - u(t)\theta_{1x}\} + \lambda_2\{ -k_2(h\theta_2 - \theta_1) + v(t)\, \theta_{2x} \}$$
$$+ \frac{b}{2L}\{ u(t) - u^*(t)\}^2 + \frac{c}{2L} \{v(t) - v^*(t)\}^2 \tag{6.7}$$

$$H_1 = \begin{cases} \tfrac{1}{2} \{\theta_1(L,t) - \theta_1^*(t)\}^2 + \tfrac{1}{2} a \{\theta_2(L,t) - \theta_2^*(t)\}^2 & \\ \qquad + \mu(t) \ \{\phi(t) - \theta_2(L,t)\} & , \text{ on } x = L \\ \tfrac{1}{2} d \{\theta_1(x,T) - \tilde{\theta}_1(x)\}^2 & , \text{ on } t = T \end{cases} \tag{6.8}$$

$$H_2 = \int_o^L H_o \, dx \tag{6.9}$$

From equation (2.6), the domain co-state equations are

$$\frac{\partial \lambda_i}{\partial t} + \frac{\partial H_o}{\partial \theta_i} - \frac{\partial}{\partial x}\left(\frac{\partial H_o}{\partial \theta_{ix}}\right) = 0 \quad , \quad i = 1,\, 2 \,, \tag{6.10}$$

whence

$$\frac{\partial \lambda_1}{\partial t} + u(t) \frac{\partial \lambda_1}{\partial x} = k_1 \lambda_1 - k_2 \lambda_2 \tag{6.11}$$

$$\frac{\partial \lambda_2}{\partial t} - v(t) \frac{\partial \lambda_2}{\partial x} = - h(k_1 \lambda_1 - k_2 \lambda_2) \,. \tag{6.12}$$

From equations (2.7), (2.8) and (2.10) the boundary co-state equations are as follows :

$$\{\frac{\partial H_1}{\partial \theta_i} ds + \frac{\partial H_o}{\partial \theta_{ix}} dt + \lambda_i \, dx\} \, \delta\theta_i = 0 \ , \quad i = 1,2, \qquad \text{on } C \tag{6.13}$$

$$\frac{\partial H_1}{\partial \mu} = 0 \qquad \text{on } C' \tag{6.14}$$

$$\frac{\partial H_1}{\partial \phi} = 0 \qquad \text{on } C' \tag{6.15}$$

whence, on $x = L$ where $ds = dt$ and $dx = 0$,

$$\frac{\partial H_1}{\partial \theta_1} + \frac{\partial H_o}{\partial \theta_{1x}} = 0 \quad , \quad \therefore \ \theta_1 - \theta_1^* - u\lambda_1 = 0 \tag{6.16}$$

$$\frac{\partial H_1}{\partial \theta_2} + \frac{\partial H_o}{\partial \theta_{2x}} = 0 \quad , \quad \therefore a(\theta_2 - \theta_2^*) - \mu + v\lambda_2 = 0 \tag{6.17}$$

$$\frac{\partial H_1}{\partial \mu} = 0 \quad , \quad \therefore \ \theta_2(h,t) = \phi(t) \tag{6.18}$$

$$\frac{\partial H_1}{\partial \phi} = 0 \quad , \quad \therefore \quad \mu = 0 \ . \tag{6.19}$$

On $t = T$, where $ds = dx$ and $dt = 0$,

$$\frac{\partial H_1}{\partial \theta_1} - \lambda_1 = 0 \quad , \quad \therefore \ \lambda_1(x,T) = d(\theta_1 - \tilde{\theta}_1) \tag{6.20}$$

$$\frac{\partial H_1}{\partial \theta_2} - \lambda_2 = 0 \quad , \quad \therefore \ \lambda_2(x,T) = 0 \tag{6.21}$$

On $x = 0$, where $ds = -\,dt$, $dx = 0$, and $\delta\theta_1 = 0$,

$$\frac{\partial H_1}{\partial \theta_2} - \frac{\partial H_o}{\partial \theta_{2x}} = 0 \quad , \quad \therefore \ v(t)\ \lambda_2(0,t) = 0 \tag{6.22}$$

On $t = 0$, $\delta\theta_1 = \delta\theta_2 = 0$, so that there are no conditions on λ_1, λ_2.

In the case when the flow controls are dependent on t, from equation (2.13), we have

$$\frac{\partial H_2(t)}{\partial u(t)} = 0 \tag{6.23}$$

$$\frac{\partial H_2(t)}{\partial v(t)} = 0 \tag{6.24}$$

so that

$$b(u(t) - u^*(t)) + \int_0^L \lambda_1(x,t) \frac{\partial \theta_1}{\partial x}(x,t)\, dx = 0 \tag{6.25}$$

$$c(v(t) - v^*(t)) + \int_0^L \lambda_2(x,t) \frac{\partial \theta_2}{\partial x}(x,t)\, dx = 0 \quad . \tag{6.26}$$

Numerical procedure

Following the "gradient to the Hamiltonian" method described in Chapter 3, in the numerical work we do not use equations (6.19), (6.24) and (6.25). Instead, starting from an arbitrary choice of $\phi(t)$, $u(t)$ and $v(t)$ the state and co-state variables are solved. Let $\delta\phi$, δu , δv denote small changes in these variables. Then according to equation (3.1) the change in J is given by

$$\Delta J = \int_0^T \{ \frac{\partial H_1}{\partial \phi} \delta\phi + \frac{\partial H_2}{\partial u(t)} \delta u(t) + \frac{\partial H_2}{\partial v(t)} \delta v(t)\}\, dt \tag{6.27}$$

so that the search direction is the vector

$$\delta \underline{u} = - \begin{bmatrix} \frac{\partial H_1}{\partial \phi} \\ \frac{\partial H_2}{\partial u} \\ \frac{\partial H_2}{\partial v} \end{bmatrix} , \tag{6.28}$$

thus ensuring a decrease in the value of J.

Characteristic coordinates and scaling

From this point on, for the analytic work we shall assume that u and v are given positive constants. We shall return to the question of variable u and v in the numerical work in Chapter 9. Let us now introduce scaled variables θ_1', θ_2', λ_1', λ_2', u', v' etc which are chosen so as to simplify the equations. At the same time we will introduce characteristic coordinates ξ, η.

The equations relating the variables are

$$\theta_1' = \theta_1 \tag{6.29}$$

$$\theta_2' = h\theta_2 \tag{6.30}$$

$$\lambda_1' = k_1\lambda_1 \tag{6.31}$$

$$\lambda_2' = k_2\lambda_2 \tag{6.32}$$

$$\xi = \frac{k_2 h}{u+v}(x - ut) \tag{6.33}$$

$$\eta = \frac{k_1}{u+v}(x + vt) \tag{6.34}$$

$$\tilde{\theta}_1' = \tilde{\theta}_1 \tag{6.35}$$

$$\theta_1^{*\prime} = \theta_1^* \tag{6.36}$$

$$\theta_2^{*\prime} = h\theta_2^* \tag{6.37}$$

$$\phi' = h\phi \tag{6.38}$$

$$a' = \frac{a}{h} \tag{6.39}$$

$$u' = \frac{u}{k_1} \tag{6.40}$$

$$v' = \frac{v}{k_2} \tag{6.41}$$

$$\mu' = \mu \tag{6.42}$$

The state and co-state domain equations (6.2), (6.3), (6.11), (6.12) become

$$\frac{\partial \theta_1'}{\partial \eta} = \theta_2' - \theta_1' \tag{6.43}$$

$$\frac{\partial \theta_2'}{\partial \xi} = \theta_2' - \theta_1' \tag{6.44}$$

$$\frac{\partial \lambda_1'}{\partial \eta} = \lambda_1' - \lambda_2' \tag{6.45}$$

$$\frac{\partial \lambda_2'}{\partial \xi} \quad \lambda_1' - \lambda_2' \tag{6.46}$$

whilst the co-state boundary conditions (6.16) - (6.20) become :

on $x = L$, $$\lambda_1' = \frac{\theta_1' - \theta_1'^*}{u'} \tag{6.47}$$

$$\lambda_2' = \frac{\mu' - a'(\theta_2' - \theta_2^{*\prime})}{v'} \tag{6.48}$$

$$\theta_2' = \phi' \tag{6.49}$$

on $t = T$,

$$\lambda_1'(x,T) = d(\theta_1' - \tilde{\theta}_1') \tag{6.50}$$

$$\lambda_2'(x,T) = 0 \tag{6.51}$$

on $x = 0$, $$\lambda_2'(0,t) = 0 \ . \tag{6.52}$$

Having introduced the primes into equations (6.29) - (6.52), these will immediately be dropped in all subsequent work, on the understanding that from this point on we shall be working with scaled variables.

6.2 Some specific problems

Probelm 6.1. Restricted counterflow

In the restricted counterflow problem stream 2 is assumed to be so massive ($k_2 \to 0, c_2 \to \infty$) that it is unaffected by giving up heat or solute to stream 1. The equations for this system are (in natural variables)

$$\frac{\partial\theta_1}{\partial t} + u\,\frac{\partial\theta_1}{\partial x} = k(h\theta_2 - \theta_1) \qquad (6.2)$$

$$\frac{\partial\theta_2}{\partial t} - v\,\frac{\partial\theta_2}{\partial x} = 0\ , \qquad (6.53)$$

where $k_1 = k$ and $k_2 = 0$.

We seek to control the output of stream 1 using the input of stream 2 as the control, so that the cost functional to be minimised is

$$J = \tfrac{1}{2}\int_0^T \{\,[\theta_1(L,t) - \theta_1^*(t)]^2 + a\,[\theta_2(L,t) - \theta_2^*(t)]^2\}\ dt\ . \qquad (6.54)$$

That is, the controlled output of stream 1 and the controlling input of stream 2 are required to be as close as possible to some desired forms $\theta_1^*(t)$, $\theta_2^*(t)$. a is a parameter affecting the "cost of the control". Initial and boundary conditions for θ_1, θ_2 are given by equations (6.4) - (6.6). The solution of this problem is considered in Chapter 7.

Problem 6.2 Full counterflow

In the full counterflow problem both streams affect each other. The governing equations are equations (6.2) and (6.3) and the cost functional is given by equation (6.54). As in the problem above, boundary and initial conditions for θ_1, θ_2 are given by equations (6.4) - (6.6). Analytic methods for solving this problem form the subject of Chapter 8, and numerical methods are used in Chapter 9.

7. Restricted Counterflow

7.1 The transformation to characteristic coordinates.

In this chapter we consider the solution of Problem 6.1. Let us introduce the transformed variables

$$\theta_1' = \theta_1 \tag{6.29}$$

$$\theta_2' = h\theta_2 \tag{6.30}$$

$$\lambda_1' = k\lambda_1 \tag{7.1}$$

$$\lambda_2' = \frac{k}{h}\lambda_2 \tag{7.2}$$

$$u' = \frac{u}{k} \tag{7.3}$$

$$v' = \frac{vh}{k} \tag{7.4}$$

together with the boundary terms (6.35) - (6.39). We will also scale the distance and time variables by writing

$$x' = \frac{kh}{u + v}\, x \tag{7.5}$$

$$t' = \frac{khu}{u + v}\, t \quad , \tag{7.6}$$

so that the characteristic coordinates are

$$\xi = x' - t' \tag{7.7}$$

$$\eta = x' + st' \tag{7.8}$$

where $s = \frac{v}{u}$ is the ratio of the speeds of the two streams.
This also leads to

$$L' = \frac{kh}{u + v}\, L \quad , \quad T' = \frac{khu}{u + v}\, T \quad . \tag{7.9}$$

As before, we shall immediately drop the primes and understand that from now on we are working in scaled variables.
We shall also introduce the characteristic time

$$T_o = (1 + \frac{1}{s})\, L \quad , \tag{7.10}$$

which is the sum of the times required for a particle of fluid in each stream to travel the length of the exchanger, and a characteristic attenuation factor

$$\alpha = e^{-sT_o} \quad . \tag{7.11}$$

The domain equations corresponding to equations (6.43) - (6.46) are :

$$\frac{\partial\theta_1}{\partial\eta} = \theta_2 - \theta_1 \tag{7.12}$$

$$\frac{\partial\theta_2}{\partial\xi} = 0 \tag{7.13}$$

$$\frac{\partial\lambda_1}{\partial\eta} = \lambda_1 \tag{7.14}$$

$$\frac{\partial\lambda_2}{\partial\xi} = \lambda_1 \tag{7.15}$$

while the transformed boundary conditions corresponding to equations (6.47) - (6.52) are :

On x = L :

$$\lambda_1(L,t) = \frac{\theta_1(L,t) - \theta_1^*(t)}{u} = \psi(t) \tag{7.16}$$

$$\lambda_2(L,t) = \frac{-a(\theta_2(L,t) - \theta_2^*(t))}{v} \tag{7.17}$$

On t = T : $\lambda_1(x,T) = \lambda_2(x,T) = 0.$ (7.18)

On x = 0 :

$$\lambda_2(0,t) = 0 \tag{7.19}$$

$$\theta_1(0,t) = \theta_{11}(t) \quad . \tag{7.20}$$

On t = 0 :

$$\theta_1(x,0) = \theta_{10}(x) \tag{7.21}$$

$$\theta_2(x,0) = \theta_{20}(x) \quad . \tag{7.22}$$

7. RESTRICTED COUNTERFLOW

7.2 The solution for the control.

Referring to Fig. 7.2, the control $\phi(t)$ may be found as follows :

Assume $\phi(t)$ is known along the boundary AB.

Then from equation (7.13) θ_2 is known in $S_1 - S_4$ in terms of $\phi(t)$, and completely in S_5, S_6 from $\theta_{20}(x)$.

$\therefore$ θ_1 is known from equation (7.12) in $S_1 - S_4$ in terms of $\phi(t)$ and completely in S_5, S_6. In particular θ_1 is known along AB.

$\therefore$ Using condition (7.16), λ_1 is known in $S_2 - S_6$ in terms of $\phi(t)$.

$\lambda_1 = \lambda_2 = 0$ in S_1.

Since $\lambda_2 = 0$ along BHO, λ_2 is known in $S_2 - S_6$ in terms of $\phi(t)$, from equation (7.15) .

The condition (7.17) which holds along AB provides equations for $\phi(t)$.

Due to the different definitions of θ_2 and λ_2 in the various regions of the solution domain, there will be different matching conditions for $\phi(t)$ in the segments BF, FE, ED, and DA. In the case $T < 2T_0$, F will be internal to EA and a slightly different set of equations will result.

7. RESTRICTED COUNTERFLOW

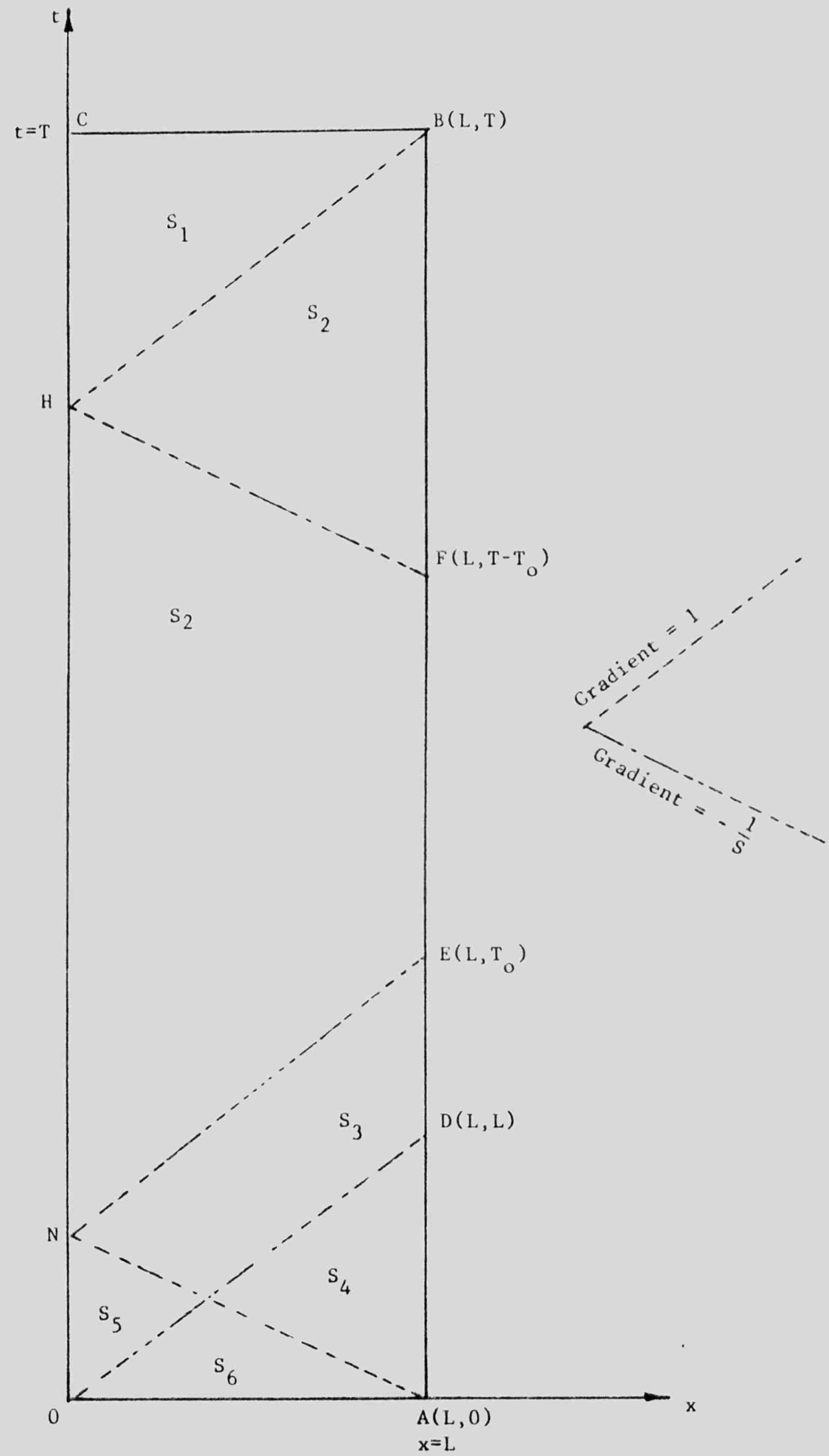

Fig. 7.1 The solution diagram in the (x,t) plane

7. RESTRICTED COUNTERFLOW

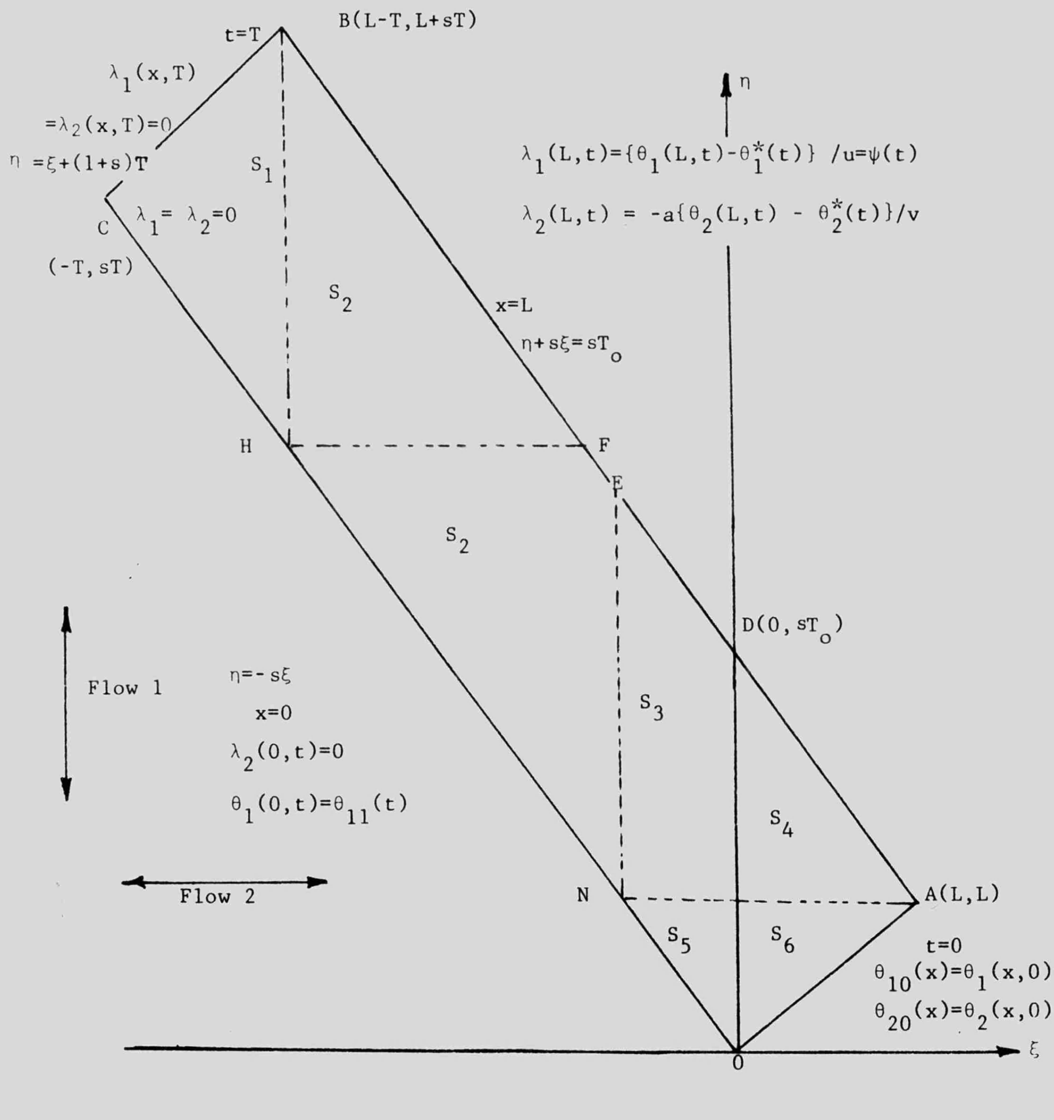

Fig. 7.2 The solution diagram in the (ξ,η) plane

7. RESTRICTED COUNTERFLOW

The solutions for θ_1 θ_2.

From equation (7.2) : $\dfrac{\partial\theta_2}{\partial\xi} = 0$,

$$\therefore \quad \theta_2 = B(\eta) = B(x + st) \tag{7.23}$$

where
$$B(u) \begin{cases} = \phi(\frac{u - L}{s}) \text{ in } S_1 - S_4 & (7.24) \\ = \theta_{20}(u) \quad \text{in } S_5, S_6 \,. & (7.25) \end{cases}$$

Hence the general solution for θ_2 is :

$$\theta_2(x,t) \begin{cases} = \phi(t + \frac{x - L}{s}) \text{ in } S_1 - S_4 & (7.26) \\ = \theta_{20}\ (x + st) \text{ in } S_5, S_6 \quad . & (7.27) \end{cases}$$

From equation (7.1), $\theta_{1\eta} + \theta_1 = \theta_2$

$$\therefore \quad \theta_1 = e^{-\eta}\{\int_0^\eta e^u\ \theta_2(\xi,u)du + A(\xi)\} \tag{7.28}$$

$$\theta_1(x,t) = e^{-(x+st)}\{\int_0^{x+st} e^u\ B(u)du + A(x-t)\} \ , \tag{7.29}$$

where A is found from boundary conditions on $x = 0$ and $t = 0$, and from making θ_1 continuous accross AN. This leads to :

$$\theta_1(x,t) = e^{-(x+st)}\int_{s(t-x)}^{x+st} e^u\ \phi(\frac{u - L}{s})\ du + \theta_{11}(t-x)e^{-x(1+s)} \text{ in } S_1 - S_2 \tag{7.30}$$

$$= e^{-(x+st)}\{\int_L^{x+st} e^u\ \phi(\frac{u - L}{s})\ du + \int_{s(t-x)}^{L} e^u\ \theta_{20}(u)\ du\} \tag{7.31}$$

$$+ \theta_{11}(t - x)e^{-x(1+s)} \quad \text{in } S_3$$

$$= e^{-(x+st)}\{\int_L^{x+st} e^u\ \phi(\frac{u - L}{s})\ du + \int_{x-t}^{L} e^u\ \theta_{20}\ du\ \} \tag{7.32}$$

$$+ \theta_{10}(x - t)\ e^{-t(1+s)} \text{ in } S_4$$

$$= e^{-(x+st)}\int_{s(t-x)}^{x+st} e^u\ \theta_{20}(u)\ du + e^{-x(1+s)}\ \theta_{11}(t-x) \quad \text{in } S_5 \tag{7.33}$$

7. RESTRICTED COUNTERFLOW

$$\theta_1(x,t) = e^{-(x+st)}\int_{x-t}^{x+st} e^u\ \theta_{20}(u)\ du + \theta_{10}(x-t)\ e^{-t(1+s)} \text{ in } S_6 \tag{7.34}$$

The solutions for λ_1, λ_2

From equation (7.3) : $\dfrac{\partial\lambda_1}{\partial\eta} = \lambda_1$

$$\lambda_1 = C'(\xi)e^{\eta} \tag{7.35}$$

$\therefore$ From equation (7.4), $\dfrac{\partial\lambda_2}{\partial\xi} = C'(\xi)e^{\eta}$ (7.36)

$$\lambda_2(\xi,\eta) = C(\xi)\ e^{\eta} + D(\eta) \tag{7.37}$$

$$\therefore \quad \lambda_1(x,t) = C'(x-t)e^{x+st} \tag{7.38}$$

$$\text{and} \quad \lambda_2(x,t) = C(x-t)e^{x+st} + D(x+st) \ . \tag{7.39}$$

In S_2 - S_6, λ_1 can be found using the boundary condition on $x = L$.

$$\lambda_1(L,t) = \psi(t) = C'(L-t)e^{L+st} \tag{7.40}$$

$$\therefore \quad C'(\xi) = \psi(L-\zeta)e^{-\{L(1+s)-s\zeta\}} = \alpha\psi(L-\zeta)e^{s\zeta} \tag{7.41}$$

where ζ is a dummy parameter.

$$\therefore \quad \lambda_1(x,t) = \alpha\psi(L-x+t)e^{(1+s)x} \quad \text{in } S_2 - S_6. \tag{7.42}$$

From equation (7.41),

$$C(\zeta) = \alpha\int_{\zeta_o}^{\zeta} \psi(L-u)e^{su}\ du \tag{7.43}$$

$$\therefore \quad \lambda_2(x,t) = \alpha e^{x+st}\int_{\zeta_o}^{x-t} \psi(L-u)e^{su}\ du + D(x+st) \ . \tag{7.44}$$

In HFAN, D can be found from the boundary condition $\lambda_2 = 0$ on $x = 0$.

$$\therefore \quad 0 = \alpha e^{st}\int_{\zeta_o}^{-t} \psi(L-u)e^{su}\ du + D(st) \tag{7.45}$$

7. RESTRICTED COUNTERFLOW

$$D(\zeta) = -\alpha e^{\zeta}\int_{\zeta_o}^{-\zeta/s} \psi(L-u)e^{su}\,du \tag{7.46}$$

$$\lambda_2(x,t) = \alpha e^{x+st}\int_{-(t+x/s)}^{x-t} \psi(L-u)e^{su}\,du \quad \text{in HFAN}\ . \tag{7.47}$$

In BHF, D can be found from equation (7.44) and from the condition $\lambda_2 = 0$ on BH, where $x - t = L - T$. Substituting,

$$0 = \alpha e^{x+st}\int_{\zeta_o}^{L-T} \psi(L-u)e^{su}\,du + D(x+st) \tag{7.48}$$

$$D(\zeta) = -\alpha e^{T}\int_{\zeta_o}^{L-T} \psi(L-u)e^{su}\,du \tag{7.49}$$

$$\therefore\ \lambda_2(x,t) = \alpha e^{x+st}\int_{L-T}^{x-t} \psi(L-u)e^{su}\,du \ \text{in BHF}\quad . \tag{7.50}$$

Finally, $\lambda_1 = \lambda_2 = 0$ in S_1. (7.51)

To summarise, the solutions for λ_1, λ_2 are as follows:

$$\lambda_1(x,t) = \alpha\psi(L-x+t)e^{(1+s)x} \quad \text{in } S_2 - S_6 \tag{7.42}$$

$$\lambda_2(x,t) = \alpha e^{x+st}\int_{-(t+x/s)}^{x-t} \psi(L-u)e^{su}\,du \ \text{in HFAN} \tag{7.47}$$

$$\lambda_2(x,t) = \alpha e^{x+st}\int_{L-T}^{x-t} \psi(L-u)e^{su}\,du \ \text{in BHF} \tag{7.50}$$

$$\lambda_1 = \lambda_2 = 0 \text{ in } S_1. \tag{7.51}$$

λ_2 is unimportant in S_6.

7. RESTRICTED COUNTERFLOW

The matching conditions for $\phi(t)$ along AB.

From equations (7.47 and (7.50) ,

$$\lambda_2(L,t) = \alpha e^{L+st}\int_{-(t+\frac{L}{s})}^{L-t} \psi(L-u)\, e^{su}\, du \qquad \text{on FA} \tag{7.52}$$

$$= \alpha e^{L+st}\int_{L-T}^{L-t} \psi(L-u)\, e^{su}\, du \qquad \text{on BF} \ . \tag{7.53}$$

Since $\psi(t) = \frac{1}{u}\{\theta_1(L,t) - \theta_1^*(t)\}$ (7.16)

$$\therefore \quad \psi(L-u) = \frac{1}{u}\{\theta_1(L,L-u) - \theta_1^*(L-u)\} \tag{7.54}$$

$$= \frac{1}{u}\left[\alpha e^{su}\int_{-su}^{sT_o-su} e^u\phi(\tfrac{u-L}{s})\, du + \theta_{11}(-u)e^{-L(1+s)} - \theta_1^*(L-u)\right] \quad \text{in } S_1\text{-}S_4 \tag{7.55}$$

$$= \frac{1}{u}\left[\alpha e^{su}\{\int_{L}^{sT_o-su} e^u\, \phi(\tfrac{u-L}{s})\, du + \int_{-su}^{L} e^u\, \theta_{20}(u)\, du\} + \theta_{11}(-u)e^{-L(1+s)} - \theta_1^*(L-u)\right] \quad \text{in } S_5 \tag{7.56}$$

$$= \frac{1}{u}\left[\alpha e^{su}\{\int_{L}^{sT_o-su} e^u\, \phi(\tfrac{u-L}{s})du + \int_{u}^{L} e^u\, \theta_{20}(u)du\} + \theta_{10}(u)e^{-(L-u)(1+s)} - \theta_1^*(L-u)\right] \quad \text{in } S_6 \ . \tag{7.57}$$

Also $\lambda_2(L,t) = -\frac{a}{v}\{\phi(t) - \theta_2^*(t)\}$ along AB. Matching these leads to the following system of integral equations for $\phi(t)$:

7. RESTRICTED COUNTERFLOW

Integral equations for $\phi(t)$

$$\int_t^T e^{-sz} \{e^{-sz} \int_{z-T_o}^{z} se^{sw} \phi(w)dw + \alpha\theta_{11}(z-L) - \theta_1^*(z)\} dz$$

$$= - \frac{a}{s} e^{-st} \{\phi(t) - \theta_2^*(t)\} , \quad T - T_o \leq t \leq T \qquad (7.58)$$

$$\int_t^{t+T_o} e^{-sz}\{e^{-sz}\int_{z-T_o}^{z} se^{sw} \phi(w)dw + \alpha\theta_{11}(z-L) - \theta_1^*(z) \}dz$$

$$= - \frac{a}{s} e^{-st} \{\phi(t) - \theta_2^*(t)\} , \quad T_o \leq t \leq T - T_o \qquad (7.59)$$

$$\int_t^{t+T_o} e^{-sz} \{ e^{-sz}\left[\int_o^{z} se^{sw} \phi(w)dw + \int_{s(z-L)}^{L} e^{v-L} \theta_{20}(v)dv\right] + \alpha\theta_{11}(z-L)$$

$$- \theta_1^*(z)\}dz = - \frac{a}{s} e^{-st}\{\phi(t) - \theta_2^*(t)\} , \quad L \leq t \leq T_o \qquad (7.60)$$

$$\int_t^{t+T_o} e^{-sz} \{ e^{-sz}\left[\int_o^{z} se^{sw} \phi(w)dw + \int_{L-z}^{L} e^{v-L} \theta_{20}(v)dv\right] + \theta_{10}(L-z) e^{-(1+s)z}$$

$$+ \theta_{10}(L-z) e^{-(1+s)z} - \theta_1^*(z) \}dz = - \frac{a}{s} e^{-st} \{\phi(t) - \theta_2^*(t)\} ,$$

$$0 \leq t \leq L \qquad (7.61)$$

Equations (7.58) - (7.61) may be rewritten

$$\int_t^T e^{-2sz} dz \int_{z-T_o}^{z} e^{sw} \phi(w)dw = - \frac{a}{s^2} e^{-st} \phi(t) + \Theta_a(t), \quad T - T_o \leq t \leq T \qquad (7.62)$$

$$\int_t^{t+T_o} e^{-2sz} dz \int_{z-T_o}^{z} e^{sw}\phi(w)dw = - \frac{a}{s^2} e^{-st} \phi(t) + \Theta_b(t), \quad T_o \leq t \leq T - T_o \qquad (7.63)$$

$$\int_t^{t+T_o} e^{-2sz}\,dz \int_o^z e^{sw}\,\phi(w)dw = -\frac{a}{s^2}\,e^{-st}\,\phi(t) + \Theta_c(t) \quad , \quad L \le t \le T_o \tag{7.64}$$

$$\int_t^{t+T_o} e^{-2sz}\,dz \int_o^z e^{sw}\,\phi(w)dw = -\frac{a}{s^2}\,e^{st}\,\phi(t) + \Theta_d(t) \quad , \quad 0 \le t \le L \tag{7.65}$$

where

$$\Theta_a(t) = -\frac{1}{s}\int_t^T (\alpha\theta_{11}(z-L) - \theta_1^*(z))\,e^{-sz}\,dz + \frac{a}{s^2}\,e^{-st}\,\theta_2^*(t) \tag{7.66}$$

$$\Theta_b(t) = -\frac{1}{s}\int_t^{t+T_o} e^{-sz}\,(\alpha\theta_{11}(z-L) - \theta_1^*(z))dz + \frac{a}{s^2}\,e^{-st}\,\theta_2^*(t) \tag{7.67}$$

$$\Theta_{cd}(t) \equiv \Theta_c(t) = -\frac{1}{s}\int_t^{t+T_o} e^{-sz}\left[e^{-sz}\int_{s(z-L)}^{L} e^{v-L}\,\theta_{20}(v)dv + \alpha\theta_{11}(z-L) - \theta_1^*(z)\right]dz$$
$$+\frac{a}{s^2}\,e^{-st}\,\theta_2^*(t) \quad , \quad L \le t \le T_o, \tag{7.68}$$

$$\Theta_{cd}(t) \equiv \Theta_d(t) = -\frac{1}{s}\int_t^{t+T_o} e^{-sz}\left[e^{-sz}\int_{L-z}^{L} e^{v-L}\,\theta_{20}(v)dv + \theta_{10}(L-z)e^{-(1+s)z}\right.$$
$$\left.-\theta_1^*(z)\right]dz + \frac{a}{s^2}\,e^{-st}\,\theta_2^*(t) \quad , \quad 0 \le t \le L, \tag{7.69}$$

7.3 The differential-delay equations

Differentiating equations (7.62) -(7.65) yields :

$$-\int_{t-T_o}^{t} e^{sw}\phi(w)dw = -\frac{a}{s^2}\,e^{st}(\phi' - s\phi) + e^{2st}\,\Theta_a'(t),\ T - T_o \le t \le T, \tag{7.70}$$

$$\alpha^2\int_{t}^{t+T_o} e^{sw}\phi(w)dw - \int_{t-T_o}^{t} e^{sw}\phi(w)dw = -\frac{a}{s^2}\, e^{st}(\phi' - s\phi) + e^{2st}\,\Theta_b'(t),$$

$$T_o \leq t \leq T - T_o, \qquad (7.71)$$

$$\alpha^2\int_{o}^{t+T_o} e^{sw}\,\phi(w)dw - \int_{o}^{t} e^{sw}\,\phi(w)dw = -\frac{a}{s^2}\, e^{st}\,(\phi' - s\phi) + e^{2st}\,\Theta_{cd}'(t),$$

$$0 \leq t \leq T_o, \qquad (7.72)$$

while differentiating again yields :

$$(1+a-\frac{a}{s^2}D^2)\phi(t) - \alpha\phi(t-T_o) = \Phi_a(t) \quad , \quad T - T_o \leq t \leq T \, , \qquad (7.73)$$

$$(1+a+\alpha^2 - \frac{a}{s^2}D^2)\,\phi(t) - \alpha\phi(t + T_o) - \alpha\phi(t - T_o) = \Phi_b(t) \quad , \quad T_o \leq t \leq T - T_o, \qquad (7.74)$$

$$(1+a-\frac{a}{s^2}D^2)\,\phi(t) - \alpha\phi(t + T_o) = \Phi_{cd}(t) \quad , \quad 0 \leq t \leq T_o, \qquad (7.75)$$

$$\text{where} \quad \Phi_i(t) = -\,e^{st}\,\frac{d}{dt}\,\{\, e^{2st}\,\Theta_i'(t)\} \qquad (7.76)$$

for i = a, b, c, d, $D = \frac{d}{dt}$, and

$$\Phi_{cd}(t) \equiv \Phi_c(t) \quad \text{for} \quad L \leq t \leq T_o \, , \qquad (7.77)$$

$$\Phi_{cd}(t) \equiv \Phi_d(t) \quad \text{for} \quad 0 \leq t \leq L \; . \qquad (7.78)$$

The solution of the differential-delay equation for $\phi(t)$

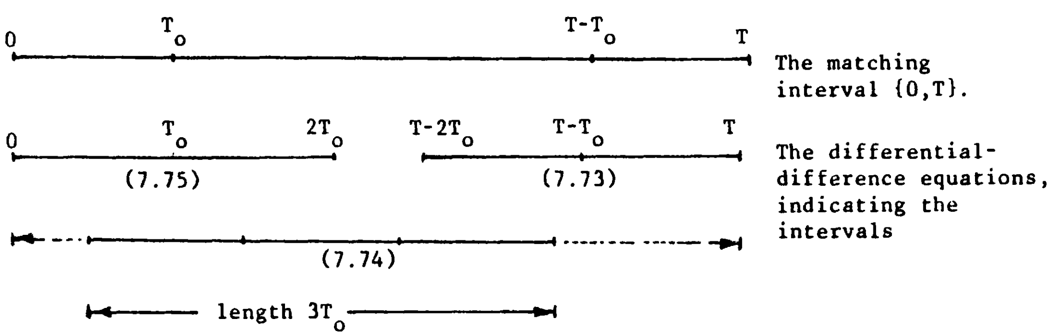

Fig. 7.3 Solution intervals for $\phi(t)$ in the general case.

7. RESTRICTED COUNTERFLOW

Fig. (7.3) illustrates how $\phi(t)$ may be determined from the differential difference equation (7.73) - (7.75) which apply to the various segments of the matching interval $\{0,T\}$.

Suppose $\phi(t)$ is known in $\{0, T_o\}$. Then from equation (7.75), given sufficient initial conditions, $\phi(t)$ is known in $\{0, 2T_o\}$, an interval of length $2T_o$. This provides sufficient initial data for equation (7.74), which can then be used to extend the solution for a further interval length T_o. Hence by repeated application of equation (7.74) $\phi(t)$ is known in the whole interval $\{0, T\}$. Equation (7.73) then provides an overlap of information of length T_o from which $\phi(t)$ in $\{0, T_o\}$, the initial assumption, can be determined. More detailed analysis of some particular cases is given below.

Case $T = 3T_o$

Equations (7.73) - (7.75) now refer to three equal intervals of length T_o. The advanced and retarded terms in equation (7.74) can be eliminated, giving a fourth-order ordinary differential equation. Writing $t + T_o$ in place of t in (7.73) gives

$$(1 + a - \frac{a}{s^2}D^2)\, \phi(t + T_o) - e^{-sT_o}\, \phi(t) = \Phi_a(t + T_o), \quad T_o \leq t \leq 2T_o. \tag{7.79}$$

Writing $t - T_o$ in place of t in (7.75) gives

$$(1 + a - \frac{a}{s^2} D^2)\, \phi(t - T_o) - e^{-sT_o}\, \phi(t) = \Phi_{cd}(t - T_o), \quad T_o \leq t \leq 2T_o . \tag{7.80}$$

On eliminating $\phi(t - T_o)$ and $\phi(t + T_o)$ between these and (7.74) there results

$$\{(1 + a - \frac{a}{s^2} D^2)(1 + a - \frac{a}{s^2} D^2 + \alpha^2) - 2\alpha^2\}\, \phi(t) = \Phi(t) \quad , \quad T_o \leq t \leq 2T_o \tag{7.81}$$

where $$\Phi(t) = \alpha\{\Phi_a(t + T_o) + \Phi_{cd}(t - T_o)\} + (1 + a - \frac{a}{s^2} D^2)\, \Phi_b(t) . \tag{7.82}$$

Equation (7.81) gives rise to four unknown constants. That no further unknown constantsarise from obtaining $\phi(t)$ in the intervals $\{0, T_o\}$, $\{2T_o, 3T_o\}$ using equations (7.73) and (7.75) is ensured by the continuity of $\phi(t)$ and $\phi'(t)$ at T_o and at $2T_o$. One of the four constants may be determined by putting $t = T$ in equation (7.58), giving $\phi(T) = \theta_2^*(T)$, but the remaining three must be obtained by substituting back into the integral equations (7.58) - (7.61).

Case $3T_o < T < 4T_o$.

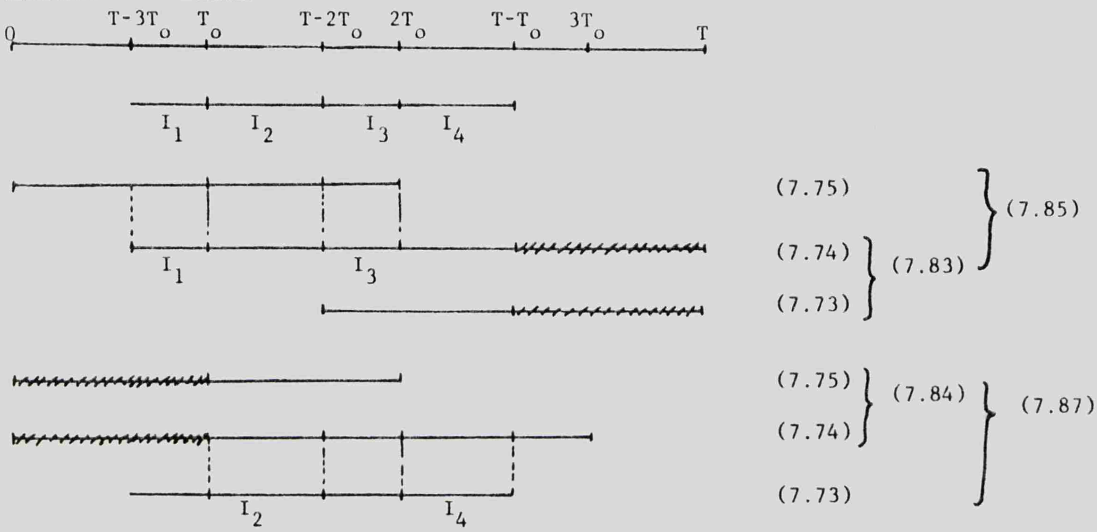

Fig. 7.4. Solution intervals for $\phi(t)$ in the case $3T_o < T < 4T_o$.

In Fig. 7.4, the intervals I_1 - I_4 are defined by

$$I_1 = \{T - 3T_o, T_o\} \qquad I_3 = \{T - 2T_o, 2T_o\}$$

$$I_2 = \{T_o, T - 2T_o\} \qquad I_4 = \{2T_o, T - T_o\} \quad .$$

The numbers in the right-hand margin refer to equations. Horizontal lines indicate intervals over which the equations operate. As all equations are differential-delay equations whose delay periods are multiples of T_o; a line of length nT_o indicates an n-term delay equation (n = 2, 3 only). Cross-hatching indicates elimination of a term to produce a new equation, which is also indicated by the equation numbers. Each new equation operates over the longer remaining interval of the two equations from which it has come. Thus, for example, equation (7.83) operates over $\{T - 3T_o, T - T_o\}$.

Eliminating $\phi(t)$ for $T - T_o \le t \le T$ between (7.73) and (7.74) yields equation (7.83), which operates over $\{T - 3T_o, T - T_o\}$. Equations (7.75) and (7.83) overlap over intervals I_1 and I_3, giving equation (7.85) for $\phi(t)$ in I_3. Eliminating $\phi(t)$ for $0 \le t \le T_o$ between (7.74) and (7.75) yields (7.84). (7.73) and (7.84) overlap over I_2 and I_4, giving equation (7.87) for $\phi(t)$ over I_2. Equation (7.85) is of fourth-order while equation (7.87) is of eigth-order, but continuity of $\phi(t)$ and its first three derivatives at $t = T - 2T_o$ ensures that no more than eight unknown constants arise. Once $\phi(t)$ is known over $I_2 \cup I_3$, which is of length T_o, $\phi(t)$ is known over the whole of $\{0, T\}$. Details of the calculation are given below.

We have

$$d\phi(t) - \alpha\phi(t - T_o) = \Phi_a(t) \quad , \quad T - T_o \le t \le T \tag{7.73}$$

$$(d + \alpha^2)\phi(t) - \alpha\phi(t + T_o) - \alpha\phi(t - T_o) = \Phi_b(t) \; , \quad T_o \le t \le T - T_o \tag{7.74}$$

$$d\phi(t) - \alpha\phi(t + T_o) = \Phi_{cd}(t) \; , \quad 0 \le t \le T_o \tag{7.75}$$

where $d = 1 + a - \frac{a}{s^2} D^2$.

Writing $t + T_o$ in place of t in (7.73) gives

$$d\phi(t + T_o) - \alpha\phi(t) = \Phi_a(t + T_o) \; , \quad T - 2T_o \le t \le T - T_o \; . \tag{7.79}$$

Eliminating $\phi(t + T_o)$ between (7.74) and (7.79) gives

$$\{d(d + \alpha^2) - \alpha^2\}\phi(t) - \alpha d\phi(t - T_o) = \alpha\Phi_a(t + T_o) + d\Phi_b(t) \; ,$$
$$T - 2T_o \le t \le T - T_o. \tag{7.83}$$

Writing $t - T_o$ in place of t in (7.75) gives

$$d\phi(t - T_o) - \alpha\psi(t) = \Phi_{cd}(t - T_o) \qquad T_o \le t \le 2T_o \; . \tag{7.80}$$

7. RESTRICTED COUNTERFLOW

Equations (7.80) and (7.83) overlap on intervals I_1 and I_3.
Eliminating $\phi(t - T_o)$ between these gives

$$\{d(d + \alpha^2) - 2\alpha^2\}\phi(t) = \alpha\{\Phi_a(t + T_o) + \Phi_{cd}(t - T_o)\} + d\Phi_b(t), \quad T - 2T_o \le t \le 2T_o$$

$$= \Phi(t), \qquad (7.85)$$

i.e. an equation for $\phi(t)$ on I_3.

Writing $t - T_o$ in place of t in (7.75) gives

$$\alpha\phi(t - T_o) - \alpha\phi(t) = \Phi_{cd}(t - T_o), \quad T_o \le t \le 2T_o. \qquad (7.80)$$

Eliminating $(t - T_o)$ between (7.74) and (7.70) gives

$$\{d(d + \alpha^2) - \alpha^2\}\phi(t) - \alpha d\phi(t + T_o) = \alpha\Phi_{cd}(t - T_o) + d\Phi_b(t), \quad T_o \le t \le 2T_o. \qquad (7.84)$$

Equations (7.73) and (7.84) overlap over I_2 and I_4. Writing $t + T_o$ in place of t in (7.73) gives

$$\{d(d + \alpha^2) - \alpha^2\}\phi(t + T_o) - \alpha d\phi(t) = \alpha\Phi_a(t + 2T_o) + d\Phi_b(t + T_o),$$

$$T - 3T_o \le t \le T - 2T_o. \qquad (7.86)$$

Eliminating $\phi(t + T_o)$ between (7.84) and (7.86) gives

$$\left[\{d(d + \alpha^2) - \alpha^2\}^2 - \alpha^2 d^2\right]\phi(t) = \alpha^2 d\Phi_a(t + T_o) + \alpha d^2\Phi_b(t+T_o)$$

$$+(d(d + \alpha^2) - \alpha^2)\{\alpha\Phi_{cd}(t - T_o) + \alpha d\Phi_b(t)\},$$

$$T_o \le t \le T-2T_o, \qquad (7.87)$$

i.e. and eighth-order equation for $\phi(t)$ on I_2.

The general case (with $T \ge 2T_o$) can be solved in a similar manner, but may lead to still higher order ordinary differential equations.

7. RESTRICTED COUNTERFLOW

7.4 Examples

In this section we consider the case of $T = T_o$ and obtain explicit solutions for two sets of initial conditions.

When $T = T_o$, there is just one integral equation for $\phi(t)$ corresponding to equations (7.62) - (7.65) :

$$\int_t^T e^{-2sz}\left(\int_o^z e^{sw}\phi(w)dw\right)dz = -\frac{a}{s^2}\phi(t)e^{-st} + \Theta_{1,2}(t),\quad 0 \le t \le T, \tag{7.4.1}$$

where
$$\Theta_1(t) = -\frac{1}{s}\int_t^T e^{-sz}\{e^{-sz}\int_{s(z-L)}^L e^{v-L}\theta_{20}(v)dv + \alpha\theta_{11}(z-L) - \theta_1^*(z)\}\,dz + \frac{a}{s^2}e^{-st}\theta_2^*(t),$$
$$L \le t \le T, \tag{7.4.2}$$

$$\Theta_2(t) = -\frac{1}{s}\int_t^T e^{-sz}\{e^{-sz}\int_{L-z}^L e^{v-L}\theta_{20}(v)dv + \theta_{10}(L-z)e^{-(1+s)z} - \theta_1^*(z)\}\,dz + \frac{a}{s^2}e^{-st}\theta_2^*(t),$$
$$0 \le t \le L. \tag{7.4.3}$$

On differentiating,

$$\int_o^t e^{sw}\phi(w)dw = \frac{a}{s^2}e^{st}(\phi' - s\phi) - \Theta_{1,2}'(t)e^{2st}, \tag{7.4.4}$$

$$(1 + a - \frac{a}{s^2}D^2)\phi(t) = -e^{-st}\frac{d}{dt}(\Theta_{1,2}'(t)e^{2st}) = \Phi_{1,2}(t). \tag{7.4.5}$$

Boundary conditions

On putting $t = T$ in equation (7.4.1), $\phi(T) = \frac{s^2}{a}e^{sT}\,\Theta_1(T)$. (7.4.6)

On putting $t = 0$ in equation (7.4.4), $\phi'(0) - s\phi(0) = \frac{s^2}{a}\Theta_2'(0)$. (7.4.7)

7. RESTRICTED COUNTERFLOW

Example 7.4.1. : Zero initial conditions

In this example we start with the zero initial and boundary conditions

$$\theta_{10}(x) = \theta_{20}(x) = \theta_{11}(t) = 0 \tag{7.4.8}$$

and seek to bring the outlet state of stream 1 as close to 1 as possible, while a term is included in the cost functional to keep the inlet state of stream 2 as near to 1 also, so that

$$\theta_1^*(t) = \theta_2^*(t) = 1 \quad . \tag{7.4.9}$$

As before, the cost functional is

$$J = \tfrac{1}{2}\int_0^T \{ \left[\theta_1(L,t) - \theta_1^*(t)\right]^2 + \tfrac{1}{2}a\left[\theta_2(L,t) - \theta_2^*(t)\right]^2\} \;dt \quad . \tag{6.54}$$

From equations (7.4.2) and (7.4.3),

$$\Theta_1(t) = \Theta_2(t) = \frac{1}{s^2}\{(1+a)e^{-st} - a\} \tag{7.4.10}$$

whence from equation (7.4.5)

$$\Phi_{12}(t) = 1 + a \tag{7.4.11}$$

so that the differential equation for $\phi(t)$ is

$$(1 + a - \frac{a}{s^2} D^2)\phi(t) = 1 + a \quad . \tag{7.4.12}$$

From equations (7.4.6) and (7.4.7), the boundary conditions are

$$\phi(T) = \frac{s^2}{a} e^{sT}\, \Theta_1(T) = 1 \quad , \tag{7.4.13}$$

$$\phi'(0) - s\phi(0) = \frac{s^2}{a}\Theta_2'(0) = - \frac{s(1+a)}{a} \quad . \tag{7.4.14}$$

The solution of equation (7.4.12) is

$$\phi(t) = A \cosh p\,t + B \sinh p\,t + 1 \tag{7.4.15}$$

where $p = s\sqrt{\frac{1+a}{a}}$. On fitting the boundary conditions ,

$$\phi(t) = \frac{s}{a} \cdot \frac{\sinh p(T - t)}{p \cosh pT + s \sinh pT} + 1 . \tag{7.4.16}$$

This is illustrated in Fig. 7.5.

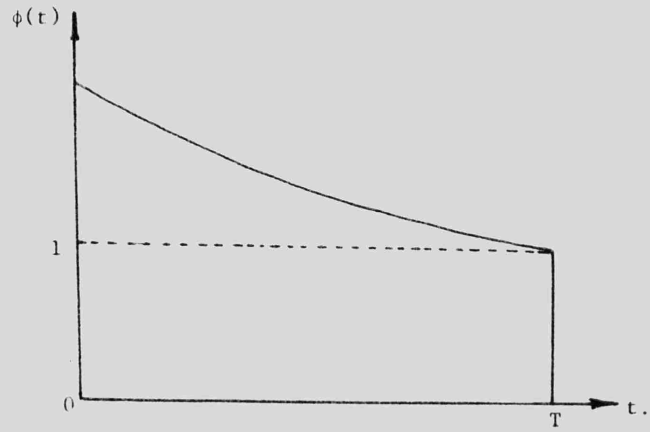

Fig. 7.5 The optimal control for example (7.4.1).

Example 7.4.2 Steady state initial conditions.

Starting from the steady state initial conditions

$$\theta_{10}(x) = e^{-(1+s)x} \tag{7.4.17}$$

$$\theta_{20}(x) = 0 \tag{7.4.18}$$

we seek to bring the outlet state of stream 1 as close to zero as possible, while the inlet state of stream 2 does not vary too much from zero, so that the target functions are

$$\theta_1^*(t) = \theta_2^*(t) = 0. \tag{7.4.19}$$

The inlet condition for stream 1 is

$$\theta_1(0,t) = \theta_{11}(t) = 1. \tag{7.4.20}$$

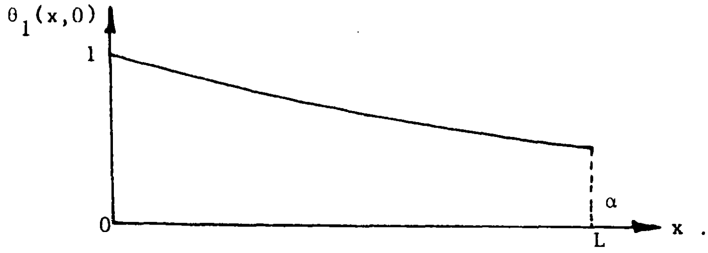

Fig. 7.6 The initial steady state profile for θ_1.

As before the cost functional is

$$J = \tfrac{1}{2}\int_0^T \{(\theta_1(L,t) - \theta_1^*(t))^2 + a\,(\theta_2(L,t) - \theta_2^*(t))^2\}dt \ . \qquad (6.54)$$

From equations (7.4.2) and (7.4.3)

$$\Theta_1(t) = \Theta_2(t) = \frac{\alpha}{s^2}\,(\alpha - e^{-st}) \ , \qquad (7.4.21)$$

so that from equation (7.4.5),

$$\Phi_{12}(t) = -\alpha \qquad (7.4.22)$$

and the differential equation for ϕ is

$$(1 + a - \frac{a}{s^2}D^2)\phi\,(t) = -\alpha \ . \qquad (7.4.23)$$

From equations (7.4.6) and (7.4.7), the boundary conditions are

$$\phi(T) = \frac{s^2}{a}e^{sT}\,\Theta_1(T) = 0 \qquad (7.4.24)$$

$$\phi'(0) - s\phi(0) = \frac{s^2}{a}\Theta_2'(0) = \alpha\frac{s}{a} \ . \qquad (7.4.25)$$

The solution for $\phi(t)$ is

$$\phi(t) = \frac{\alpha}{1+a}\left[\frac{p\cosh pt + s\sinh pt + \frac{s}{a}\sinh p(t-T)}{p\cosh pT + s\sinh pT} - 1\right] \qquad (7.4.26)$$

PART III

OPTIMAL CONTROL OF THE FULL COUNTERFLOW SYSTEM.

8. Analytic Methods.

8.1 Introduction.

In this chapter we use the method of Jaswon and Smith {17} to obtain solutions to the counterflow state and co-state equations in terms of series of Bessel functions. These expansions are fitted to boundary conditions using the method of collocation. Some exact solutions of the optimal control problem are also obtained which are of use for testing the accuracy of the numerical methods used in Chapter 9.

8.2 The General solution

From equations (6.43) - (6.46) (in scaled variables),

$$\frac{\partial\theta_1}{\partial\eta} = \theta_2 - \theta_1 \tag{6.43}$$

$$\frac{\partial\theta_2}{\partial\xi} = \theta_2 - \theta_2 \tag{6.44}$$

$$\frac{\partial\lambda_1}{\partial\eta} = \lambda_1 - \lambda_2 \tag{6.45}$$

$$\frac{\partial\lambda_2}{\partial\xi} = \lambda_1 - \lambda_2 \tag{6.46}$$

we see that

$$\frac{\partial\theta_1}{\partial\eta} = \frac{\partial\theta_2}{\partial\xi} \tag{8.1}$$

and

$$\frac{\partial\lambda_1}{\partial\eta} = \frac{\partial\lambda_2}{\partial\xi} \quad . \tag{8.2}$$

Hence we can introduce two stream functions Θ and χ such that

$$\theta_1 = \Theta_\xi \tag{8.3}$$

$$\theta_2 = \Theta_\eta \tag{8.4}$$

$$\Theta_{\xi\eta} = \Theta_\eta - \Theta_\xi \tag{8.5}$$

8. ANALYTIC METHODS

$$\lambda_1 = \chi_\xi \tag{8.6}$$

$$\lambda_2 = \chi_\eta \tag{8.7}$$

$$\chi_{\xi\eta} = \chi_\xi - \chi_\eta \tag{8.8}$$

Equations (8.5) and (8.8) may be further transformed to

$$\Phi_{\xi\eta} + \Phi = 0 \tag{8.9}$$

$$\Psi_{\xi\eta} + \Psi = 0 \tag{8.10}$$

where

$$\Phi = \Theta e^{\eta-\xi} \tag{8.11}$$

and

$$\Psi = \chi^{\xi-\eta} \quad . \tag{8.12}$$

In equation (8.9) if we seek a solution of the form

$$\Phi(\xi,\eta) = \left(\frac{\xi}{\eta}\right)^m f(\xi\,\eta) \quad , \text{ m real,} \tag{8.13}$$

we find that

$$w^2 f'' + wf' + (w - m^2)f = 0 \tag{8.14}$$

where $w = \xi\eta$.

On writing $z = 2w^{\frac{1}{2}} = 2(\xi\eta)^{\frac{1}{2}}$ and $n = 2m$, equation (8.14) becomes

$$z^2 f'' + zf' + (z^2 - n^2)f = 0 \ , \tag{8.15}$$

which is Bessel's equation of order n. Hence equation (8.13) has solutions of the type

$$\Phi(\xi,\eta) = \left(\frac{\xi}{\eta}\right)^{n/2} (C\, J_n\{2(\xi\eta)^{\frac{1}{2}}\} + DY_n\{2(\xi\eta)^{\frac{1}{2}}\}), \tag{8.16}$$

for all real values of n.

However as we shall require points where $\xi = 0$ or $\eta = 0$ to be included in the solution domain, the Bessel functions of the second kind are inadmissible since $Y_n(z)$ has a pole of order $|n|$ at the origin when $n \neq 0$, and a logarithmic singularity there when $n = 0$.

8. ANALYTIC METHODS

Let us define the function $H_n(\xi,\eta)$ by

$$H_n(\xi,\eta) = \left(\frac{\xi}{\eta}\right)^{n/2} J_n\{2(\xi\eta)^{\frac{1}{2}}\}. \tag{8.17}$$

Now

$$J_n(z) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{2}\right)^{n+2s}}{s!\ (n+s)!} , \tag{8.18}$$

so that $J_n(z)$ has a zero of order n at the origin, and

$$\lim_{\eta\to o} H_n(\xi,\eta) = \frac{\xi^n}{n!} , \qquad n > 0 , \tag{8.19}$$

$$\lim_{\xi\to o} H_n(\xi,\eta) = \frac{(-\xi)^n}{n!} , \qquad n > 0 . \tag{8.20}$$

Hence the general solution of equation (8.9) which remains finite in our solution domain is of the form

$$\Phi(\xi,\eta) = \int_{-\infty}^{\infty} C(n)\ H_n(\xi,\eta)\ dn . \tag{8.21}$$

However, satisfactory results are obtained by restricting n to integer values and fitting the boundary conditions by the method of collocation,so that we shall take $\Phi(\xi,\eta)$ to be of the form

$$\Phi(\xi,\eta) = \sum_{n=-\infty}^{\infty} C_n H_n(\xi,\eta) .$$

Similarly, the solution of equation (8.10) which we shall adopt is

$$\Psi(\xi,\eta) = \sum_{n=-\infty}^{\infty} D_n H_n(\xi,\eta) . \tag{8.23}$$

Through the recurrence relations involving derivatives of Bessel functions ,

$$\frac{\partial H_n}{\partial \xi} = H_{n-1} , \quad \frac{\partial H_n}{\partial \eta} = - H_{n+1} , \tag{8.24}$$

so that the solutions of equation (6.43) - (6.46) may be taken to be of the form

8. ANALYTIC METHODS

$$\theta_1(\xi,\eta) = e^{\xi-\eta} \sum_{-\infty}^{\infty} A_n H_n(\xi,\eta) \tag{8.25}$$

$$\theta_2(\xi,\eta) = - e^{\xi-\eta} \sum_{-\infty}^{\infty} A_{n-1} H_n(\xi,\eta) \tag{8.26}$$

$$\lambda_1(\xi,\eta) = e^{\eta-\xi} \sum_{-\infty}^{\infty} B_n H_n(\xi,\eta) \tag{8.27}$$

$$\lambda_2(\xi,\eta) = e^{\eta-\xi} \sum_{-\infty}^{\infty} B_{n-1} H_n(\xi,\eta) \tag{8.28}$$

where

$$A_n = C_n + C_{n+1} \quad , \quad B_n = - D_n + D_{n+1}. \tag{8.29}$$

To solve the general optimal control problem, these solutions must be fitted to the boundary conditions in the various regions S_1, S_2, ... of the solution domain illustrated in Fig. 7.1.

Matching conditions for the control $\phi(t) = \theta_2(L,t)$ can then be obtained along the boundary x = L, as in the case of restricted counterflow described in Chapter 7.

8.3 A specific example : Cold start-up

In this section we consider the following problem :

Problem 8.1

This problem is characterised by the choice of the final time

$$T = \frac{L}{u} \tag{8.30}$$

toegther with the initial and boundary conditions

$$\theta_1(x,0) = \theta_2(x,0) = \theta_1(0,t) = 0 \tag{8.31}$$

and the cost functional (6.54).

8. ANALYTIC METHODS

That is, it is desired to start up, in time T, a system which is initially in the zero state, so that the outlet state of stream 1 is as close as possible to θ_1^* while the inlet state of stream 2 does not differ too much from θ_2^*.

The boundary conditions for the co-state variables are given by equations (7.16) - (7.19). Referring to Fig. (8.2), † the boundary conditions on $x = 0$, $t = 0$ and $t = T$, together with the partial differential equations (6.43) - (6.46), show that

$$\lambda_1(\xi,\eta) = \lambda_2(\xi,\eta) = 0 \quad \text{in } S_1 \text{ and } S_2 \tag{8.32}$$

$$\theta_1(\xi,\eta) = \theta_2(\xi,\eta) = 0 \quad \text{in } S_2 \text{ and } S_4 \tag{8.33}$$

and that the problem may be solved by considering the region S_3 only. As there will be discontinuities of θ_2 across AK and in λ_2 across BK, the appropriate boundary conditions on AK and BK are

$$\theta_1(\xi, \frac{k_1 L}{u+v}) = 0 \qquad \text{on AK} \tag{8.34}$$

$$\lambda_2(0,\eta) = 0 \qquad \text{on BK} \quad . \tag{8.35}$$

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†Fig. (8.2) illustrates the case of $v < u$. But the same conclusions hold when $v \geq u$.

8. ANALYTIC METHODS

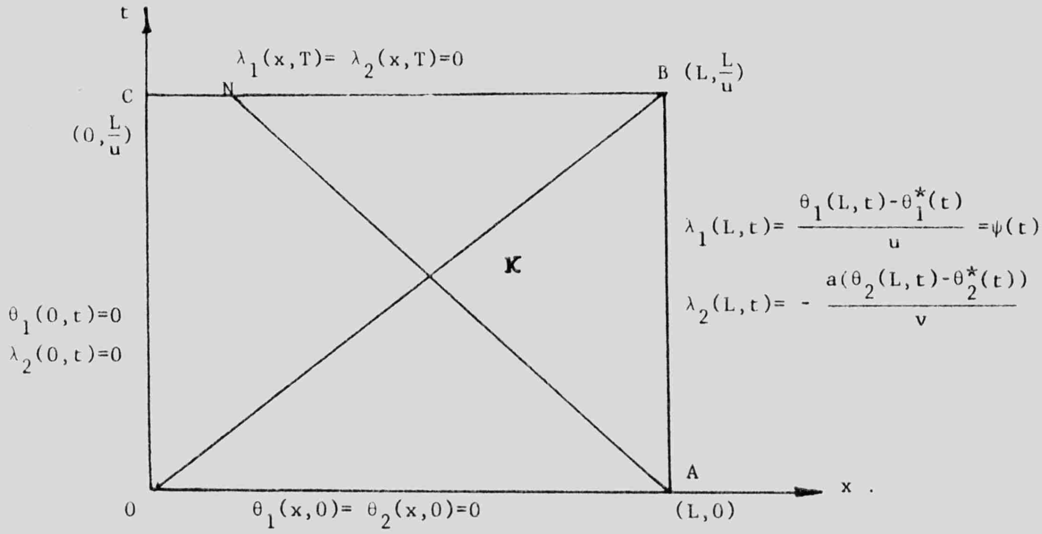


Fig. 8.1 The solution domain in the (x,t) plane for Problem 8.1

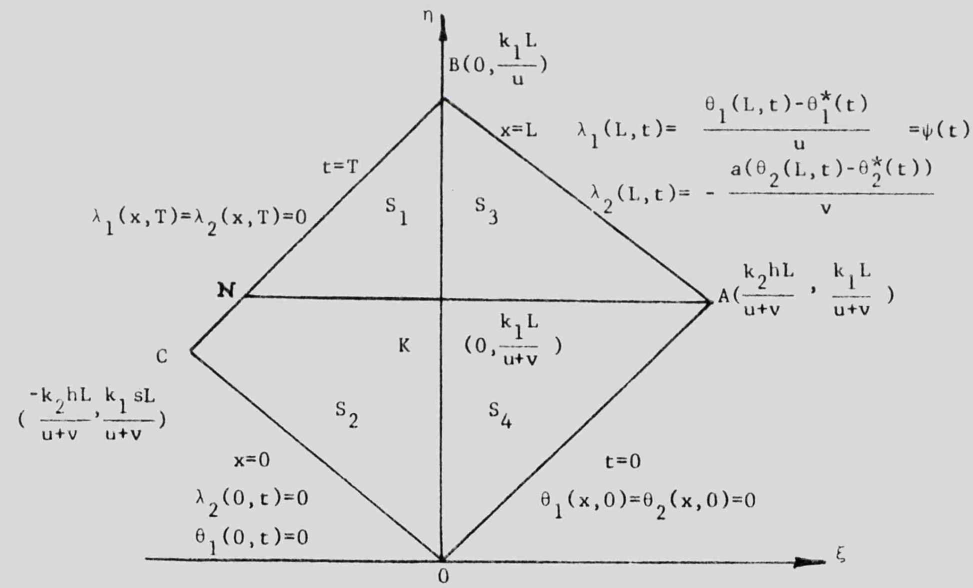


Fig. 8.2 The solution domain in the (ξ,η) plane for Problem 8.1

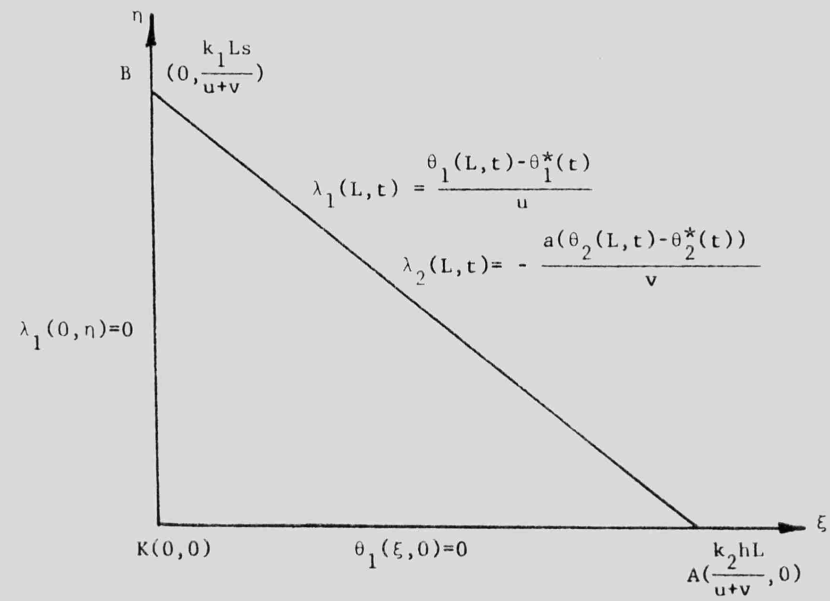


Fig. 8.3 The region S_3 in the (ξ,η) plane with the origin transferred to K

We now move the origin to the point K so that the solution region S_3 becomes as shown in Fig. 8.3 and take the solutions for $\underline{\theta}$ and $\underline{\lambda}$ to be given by (8.25) - (8.28). The boundary conditions $\theta_1(0,\xi) = 0$ on AK and $\lambda_2(\xi, 0) = 0$ on BK give, in view of the limits (8.19) and (8.20) ,

$$A_o = A_1 \quad \ldots = 0 \tag{8.36}$$

and

$$C_{-1} = C_{-2} = \ldots = 0 \tag{8.37}$$

so that

$$\theta_1(\xi,\eta) = e^{\xi-\eta} \sum_{n=1}^{\infty} A_{-n} H_{-n}(\xi,\eta) \tag{8.38}$$

$$\theta_2(\xi,\eta) = -e^{\xi-\eta} \sum_{n=1}^{\infty} A_{-n-1} H_{-n}(\xi,\eta) \tag{8.39}$$

$$\lambda_1(\xi,\eta) = e^{\eta-\xi} \sum_{n=o}^{\infty} B_n H_n(\xi,\eta) \tag{8.40}$$

$$\lambda_2(\xi,\eta) = e^{\eta-\xi} \sum_{n=o}^{\infty} B_{n-1} H_n(\xi,\eta) \quad . \tag{8.41}$$

8. ANALYTIC METHODS

Applying the boundary conditions (7.16), (7.17), which couple the expressions for $\underline{\theta}$ and $\underline{\lambda}$, enables the coefficients A_n and B_n to be determined, leading to a complete resolution of the problem.

In the numerical studies based on this method, expressions (8.38) - (8.41) are taken to m terms only and the functions are matched by the method of collocation at m equally spaced points along AB.

9. Numerical Methods

9.1 Introduction

In this chapter details are given of the numerical methods used to solve the problems discussed in the earlier chapters. A considerable number of computer programms were written in the course of working through the various methods, and a selection of these are included as Appendix 5. The counterflow programs deal with the case h = 1, heat transfer, or equilibrium constant unity, and equal capacity per unit length for the quality transferred, so that $k_1 = k_2$. As k is the symbol conventionally used for time step, the symbol c is used in this chapter for the transfer coefficient (i.e. $c = k_1 = k_2$).

9.2 The optimisation of the transport equation.

The problem of minimising the cost functional

$$J = \tfrac{1}{2}\int_0^T\int_0^L \{\phi^2(x,t) + au^2(x,t)\}\, dxdt \tag{5.1.3}$$

subject to

$$\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\,\frac{\partial\phi}{\partial x}(x,t) = 0 \tag{5.1.4}$$

over the domain $\{0 \leq x \leq 1, \quad 0 \leq t \leq 1\}$ is investigated using the steepest descent and conjugate gradient methods. The boundary and initial conditions for ϕ are

$$\phi(x,0) = \phi_0(x)\ , \tag{5.1.5}$$

$$\phi(0,t) = \phi_1(t)\ . \tag{5.1.6}$$

The partial differential equations for the state and co-state variables are solved using Wendroff's method explicitly. As stated in Chapter 4, this method is stable for all values of u(x,t).

For the state equation Wendroff's formula is used in a forward direction. Referring to Fig. (9.1) we have

9. NUMERICAL METHODS

$$\phi_W = \phi_S + \frac{1 - p\bar{u}}{1 + p\bar{u}} (\phi_T - \phi_P), \tag{9.1}$$

where $p = \frac{k}{h}$ is the ratio of step lengths in the x and t directions, and the value of $u(x,t)$ at the centre of the grid square is obtained using the average expression

$$\bar{u} = u_{m+\frac{1}{2}}^{n+\frac{1}{2}} \simeq \tfrac{1}{4}(u_S + u_T + u_P + u_W) . \tag{9.2}$$

For the co-state equation, Wendroff's formula is used in reverse :

$$\lambda_S = \lambda_W + \frac{1 - p\bar{u}}{1 + p\bar{u}} (\lambda_P - \lambda_T). \tag{9.3}$$

The solution domain is divided into (D-1) sub-intervals in the x and t directions so that $h = k = \frac{1}{D-1}$ and $p = \frac{k}{h} = 1$.

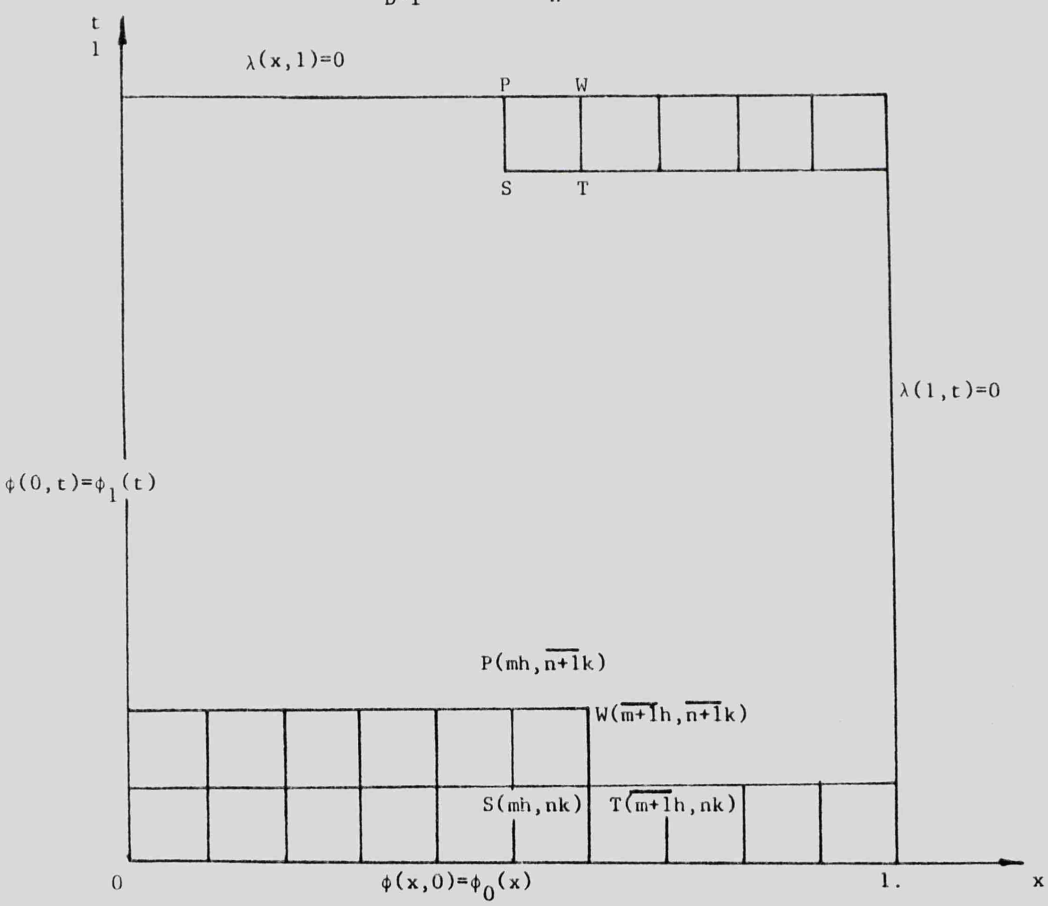


Fig. 9.1 Wendroff's explicit method for the transport equation.

9. NUMERICAL METHODS

The "gradient to the Hamiltonian" method of chapter 4 is employed. Starting from an initial value u(x,t) over the D^2 grid points, the state and co-state equations are solved to give the search direction

$$- \frac{\partial H_o}{\partial u} = \lambda \frac{\partial \phi}{\partial x} - au. \tag{9.2}$$

This is used directly in the case of steepest descent, or incorporated into formula (3.8) in the case of conjugate gradient. The minimum of J(u) is found in each linear search. Iterations are continued until J changes by less than a prescribed tolerance.

9.3 Direct solution of counterflow optimisation

The problem considered is that of minimising the cost functional (6.1) subject to equations (6.2), (6.3) with boundary and initial conditions (6.4) - (6.6). As in the previous section, the solution domain is $\{0 \leq x \leq 1, \ 0 \leq t \leq 1\}$, using (D-1) sub intervals in each direction. The problem is solved by the algorithms of steepest descent, conjugate gradient, and that of Davidon, Fletcher and Powell, using the Wendroff and Lax-Wendroff methods of solving the partial differential equations.

The flow rates u and v may be

(i) constants (with b = c = d = 0 in (6.1))

(ii) functions of t (with d = 0 in (6.1)).

In case (i) the steepest descent direction is

$$- \frac{\partial H_1}{\partial \phi} = - \mu \tag{9.5}$$

while in case (ii), the unknown flow rates u(t), v(t) at the D grid points are appended to the vector of controls, as in equation (6.28), giving the steepest descent direction

9. NUMERICAL METHODS

$$\underline{s} = - \begin{bmatrix} \dfrac{\partial H_1}{\partial \phi} \\ \dfrac{\partial H_2}{\partial u} \\ \dfrac{\partial H_2}{\partial v} \end{bmatrix} = - \begin{bmatrix} \mu(t) \\ b(u(t) - u^*(t)) + \displaystyle\int_0^1 \lambda_1 \frac{\partial \theta_1}{\partial x}\, dx \\ c(v(t) - v^*(t)) - \displaystyle\int_0^1 \lambda_2 \frac{\partial \theta_2}{\partial x}\, dx \end{bmatrix} . \tag{9.6}$$

Again, this direction is either used directly, or is available for incorporation into the more elaborate algorithms.

The Lax-Wendroff explicit method

Referring to Fig. (9.2) since θ_1 is known on $x = 0$ and on $t = 0$, the solution for θ_1 can be obtained row by row from left to right for each increment of the time using the explicit Lax-Wendroff formula for D - 2 distance steps, and the explicit Wendroff formula for the final point $(1,(n+1)k)$ on $x = 1$. At the same time, beginning with an assumed control $\theta_2(1,t) = \phi(t)$, the solution for θ_2 can be obtained row by row from right to left using the explicit Lax-Wendroff formula D - 2 steps and the explicit Wendroff formula for the final point $(0,(n+1)k)$ on $x = 0$.

Having found $\theta_1(x,t)$ and $\theta_2(x,t)$ throughout the domain, $\lambda_1(x,t)$ and $\lambda_2(x,t)$ can be found in a similar way working backwards in time using reverse versions of the formulae.

9. NUMERICAL METHODS

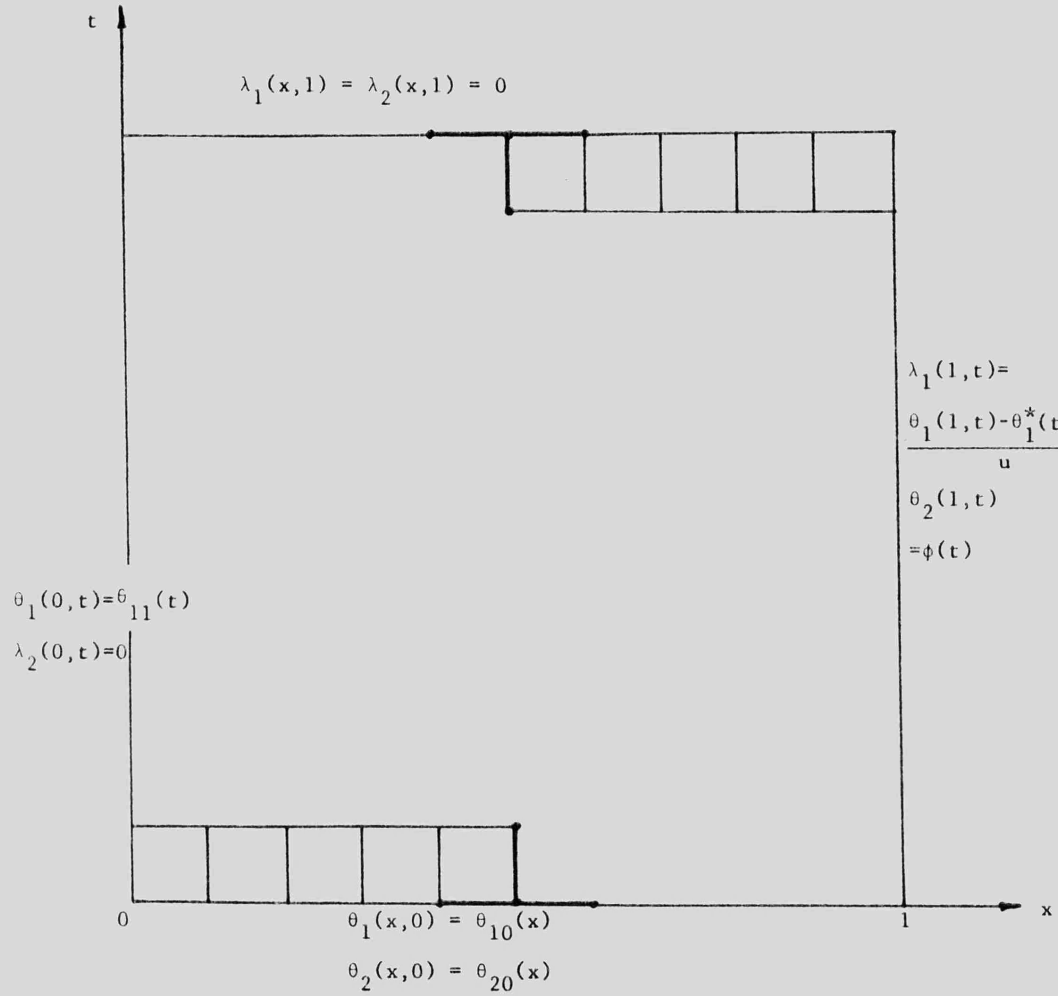


Fig. 9.2 The Lax-Wendroff explicit method for counterflow, illustrating the method of solution for θ_1 and λ_1.

9. NUMERICAL METHODS

The Wendroff implicit method.

This method has guaranteed stability for all values of the flow rates. Wendroff's formula cannot be used explicitly as in the case of the transport equation, but instead the solutions must be obtained implicitly. The vector of state variables,

$$\underline{\theta} = \begin{bmatrix} \theta_1(h,(n+1)k) \\ \theta_1(2h(n+1)k) \\ \cdot \\ \cdot \\ \cdot \\ \theta_1(Dh,(n+1)k) \\ \theta_2(0,(n+1)k) \\ \cdot \\ \cdot \\ \cdot \\ \theta_2((D-1)h,(n+1)k \end{bmatrix} , \tag{9.7}$$

is obtained in steps of increasing time by solving the system of 2D linear equations

$$W\,\underline{\theta} = \underline{B} \quad . \tag{9.8}$$

W is the matrix

$$
W = \left[\begin{array}{cccc|cccc}
x_1+1 & & & & -1 & -1 & & \\
y_1+1 & x_1+1 & & & & \ddots & \ddots & \\
& \ddots & \ddots & & & & \ddots & -1 \\
& & y_1+1 & x_1+1 & & & & -1 \\
\hline
-1 & & & & y_2+1 & x_2+1 & & \\
-1 & -1 & & & & \ddots & \ddots & \\
& \ddots & \ddots & & & & \ddots & x_2+1 \\
& & -1 & -1 & & & & y_2+1
\end{array}\right] \qquad (9.9)
$$

$$(2D - 2) \times (2D - 2)$$

in which

$$x_1 = \frac{2(1+pu)}{ck}, \quad x_2 = \frac{2(1-pv)}{ck}, \quad y_1 = \frac{2(1-pu)}{ck}, \quad y_2 = \frac{2(1+pv)}{ck} \qquad (9.10)$$

where $p = \frac{k}{h}$ is the ratio of the steps in the x and t directions. $\underline{B}$ is a vector involving known values of $\underline{\theta}$ at the n^{th} time step. The co-state variables are obtained in a similar way, working backwards in time. Further details of these equations are given in Appendix 4.

9.4 Bessel series for counterflow optimisation

This method is used to solve the problem with zero boundary and initial conditions described in Chapter 8.

Expressions (8.38) - (8.41), truncated to m terms, require to be matched to the boundary conditions (7.16), (7.17) at m collocation points along the boundary x = 1. This leads to the expressions

$$\theta_1(1,t) = e^{-(ct-Q)} \sum_{n=1}^{m} A_{-n} \left(\frac{1-ut}{vt}\right)^{-n/2} J_{-n} \{Q\sqrt{(1-ut)vt}\} \tag{9.11}$$

$$\theta_2(1,t) = -e^{-(ct-Q)} \sum_{n=o}^{m-1} A_{-n-1} \left(\frac{1-ut}{vt}\right)^{-n/2} J_{-n} \{Q\sqrt{(1-ut)vt}\} \tag{9.12}$$

$$\lambda_1(1,t) = e^{ct-Q} \sum_{n=o}^{m-1} B_n \left(\frac{1-ut}{vt}\right)^{n/2} J_n \{Q\sqrt{(1-ut)vt}\} \tag{9.13}$$

$$\lambda_2(1,t) = e^{ct-Q} \sum_{n=1}^{m} B_{n-1} \left(\frac{1-ut}{vt}\right)^{n/2} J_n \{Q\sqrt{(1-ut)vt}\} , \tag{9.14}$$

where

$$Q = \frac{c}{u+v} . \tag{9.15}$$

These are required to satisfy

$$\theta_1(L,t) - u\lambda_1(L,t) = \theta_1^*(t) \tag{9.16}$$

$$\theta_2(L,t) + \frac{v}{u}\lambda_2(1,t) = \theta_2^*(t) \tag{9.17}$$

at m equally spaced collocation points in the interval $0 \leq t \leq \frac{1}{u}$.
In the case $c = u = v = 1$, $\theta_1^*(t) = \theta_2^*(t) = 1$, the discrepancy between the left hand and right hand sides of equation (9.16), (9.17), at all points of the interval using 5 collocation points is less than .001.

Convenient exact solutions to the counterflow optimisation problem for testing against the direct numerical methods described in the previous section may be obtained by making arbitrary choices of the expansion coefficients A_{-n} and B_n.
One example of this type is programmed :

$$A_{-n} = (-1)^n \;, \qquad n = 0, 1, 2, \ldots, m-1 \;, \tag{9.18}$$

$$B_n = (-1)^{n+1}, \qquad n = 0, 1, 2, \ldots, m-1 \;. \tag{9.19}$$

9.5 Constrained optimisation of counterflow.

With reference to equation (3.5), the control $\phi(t)$ which is the input state of stream 2, is required to satisfy the inequality

$$a \leq \phi(t) \leq b \tag{9.20}$$

where a and b are constraints.

Accordingly ϕ is taken to be of the form

$$\phi(t) = \frac{a+b}{2} + \frac{b-a}{2} \sin(f(t)) \tag{9.21}$$

$$= A + B \sin(f(t)) \tag{9.22}$$

where f(t) is unconstrained.

Thus the boundary Hamiltonian is

$$H_1 = \tfrac{1}{2}(\theta_1(L,t) - \theta_1^*(t))^2 + \mu(A + B \sin(f(t)) - \theta_2(L,t)) \tag{9.23}$$

and the steepest descent direction is

$$s = -\frac{\partial H_1}{\partial f} = -\mu B \cos f(t) \;. \tag{9.24}$$

This method is programmed using the steepest descent, conjugate gradient, and Davidon - Fletcher - Powell optimisation algorithms.

PART IV

CONCLUSION.

10. Results and Conclusions.

10.1. Introduction.

This chapter contains the results and conclusions of the various numerical and analytic methods described in the preceeding chapters. The programs were run on the Burroughs B6700 computer at Leicester Polytechnic. As this is a multi-access, time-sharing machine, it is not really meaningful to quote timings ; however no program required more than 20 minutes of CPU time using up to 441 grid points, and many required less than one minute. In the tables, the numbers of iterations required for the iterative methods are given in parentheses.

10. RESULTS AND CONCLUSIONS

10.2. The transport equation.

The problems described in Section 9.2 is solved with two sets of initial and boundary conditions :

Problem 10.1. $\phi_o(x) = x, \quad \phi_1(t) = t$ (10.1)

Problem 10.2. $\phi_o(x) = x(1 - x), \quad \phi_1(t) = te^t,$ (10.2)

using Wendroff's formula explicitly to solve the partial differential equations. Values of J and the number of iterations required for different numbers of grid points and different values of the control cost parameter are given in Table 10.1. Detailed results for the control, state and costate variables over the solution domain are given with the computer program listing in the case of steepest descent in Appendix 5. Iterations are stopped when J changes by less than 10^{-6}.

10. RESULTS AND CONCLUSIONS

10.3. Coercive control and counterflow.

The problem described in Section 9.3 is solved with cold start-up initial conditions

$$\theta_1(x,0) = \theta_2(x,0) = \theta_1(0,t) = 0 \tag{10.3}$$

and with the parameter values $c = u = v = 1$. Two choices of target functions are used :

Problem 10.3. $\quad \theta_1^*(t) = \theta_2^*(t) = 1 \quad , \qquad 0 \leq t \leq 1.$ (10.4)

Problem 10.4. The coefficients A_{-n}, B_n, $n = 0, 1, \ldots, m - 1$, are given by (9.18), (9.19) so that

$$\theta_1^*(t) = \theta_1(1,t) - \lambda_1(1,t) = e^{-(t-\frac{1}{2})} \sum_{n=1}^{m} (-1)^n H_{-n}(\tfrac{1}{2}(1-t),\tfrac{1}{2}t)$$

$$-e^{(t-\frac{1}{2})} \sum_{n=o}^{m-1} (-1)^{(n+1)} H_n(\tfrac{1}{2}(1-t),\tfrac{1}{2}t),$$

$$0 \leq t \leq 1, \tag{10.5}$$

$$\theta_2^*(t) = \theta_2(1,t) + \lambda_2(1,t) = - e^{-(t-\frac{1}{2})} \sum_{n=o}^{m-1} (-1)^{n+1} H_{-n}(\tfrac{1}{2}(1-t),\tfrac{1}{2}t)$$

$$+e^{(t-\frac{1}{2})} \sum_{n=1}^{m} (-1)^n H_n(\tfrac{1}{2}(1-t),\tfrac{1}{2}t),$$

$$0 \leq t \leq 1, \tag{10.6}$$

The steepest descent, conjugate gradient and Davidon-Fletcher-Powell optimisation algorithms are used, with the Wendroff, Lax-Wendroff and analytic methods of solving the partial differential equations. Values of J and the number of iterations required in the iterative methods are given in Table 10.2. For the analytic solution to Problem 10.3,

10. RESULTS AND CONCLUSIONS

5 and 6 collocation points are used. Problem 10.4 is an exact analytic solution so that it provides a model against which the iterative methods can be compared. After some experimentation to determine an appropriate degree of precision, the programs were run with D, the number of grid points in each direction, equal to 21. The results of varying D for one of the methods are given in Table 10.3. Detailed results for the control, state and costate variables over the solution domain are given for two methods along with the computer program listings in Appendix 5.

10. RESULTS AND CONCLUSIONS

10.4. Constrained control of counterflow.

Problem 10.5. The problem of section 9.5 is solved with cold start-up initial and boundary conditions (10.3) and parameter values $c = u = v = 1$. The control $\phi(t) \equiv \theta_2(1,t)$ is subject to the constraints (9.19)

$$a \leq \phi(t), \leq b \ , \qquad 0 \leq t \leq 1 \quad , \tag{9.19}$$

and depends on the unconstrained variable $f(t)$ through equation (9.21). For the case $a = 0$, $b = 2$, the steepest descent, conjugate gradient and Davidon-Fletcher-Powell algorithms for unconstrained optimisation are programmed, together with the Wendroff and Lax-Wendroff methods of solution of the partial differential equations. The results are shown in Table 10.4 together with the numerical approximation to the exact result, since in this case the exact control can be seen to operate at its maximum value of $\phi(t) = 2$, $0 \leq t \leq 1$.

The results of varying the values of a and b for one method of solution are shown in Table 10.5.

11 grid points in each direction are used throughout.

10. RESULTS AND CONCLUSIONS

10.5. Optimal control of counterflow including variable flow rates.

Problem 10.6. The problem section 9.3(ii) is programmed using the Wendroff steepest descent and conjugate gradient methods, with various number of grid points.

The conjugate gradient method diverges, while the results for the steepest descent method are given in Table 10.6.

10. RESULTS AND CONCLUSIONS

10.6. Conclusions.

The transport equation.

As stated in Chapter 5, the constrained multiplicative control of the transport equation leads to an improperly posed problem. The analytic approach to the continuous, coercive control leads to a singular situation in the case $u^* = 0$, while the numerical approach fails to proceed towards an optimum. This may be due to the large number of points (D^2), used to describe the control. Davies {6} does not obtain solutions to the delay differential equations resulting from his analysis of the speed control u(t), and the more complicated problem of the multiplicative control which is a function of both variables remains an intractable problem.

Counterflow Systems.

In the analytic method, the excellent agreement between the results obtained using different numbers of collocation points vindicates the choice of a discrete set of solutions of the partial differential equations as a set of expansion functions.

Among the numerical methods, the Lax-Wendroff method shows moderately good agreement with the analytic results and has the advantage of sharp cut-off along characteristic discontinuities, but shows signs of oscillation. This is not surprising as in the test examples it is operating at the limit of its range of stability. The Lax-Wendroff exponential method behaves poorly and shows worse oscillation. The Wendroff method has the advantage of guaranteed stability and shows a similar degree of agreement with the analytic results. Among the optimisation methods, with the notable exception of the conjugate gradient method for variable flow rate problems, the steepest descent and conjugate gradient methods generally work satisfactorily, and the greater sophistication of the Davidon-Fletcher-Powell method does not seem to be justified. This result was also obtained by Holliday {15} .

The degree of agreement between the numerical solutions and the analytic solutions for both the test problem and for a problem with arbitrary target functions shows that the former could be applied to problems with non-zero initial conditions and longer time scales, where the methods developed in Chapter 7 indicate a greater degree of complexity for the characteristic analytic method.

As may be expected the constrained control exhibits bang-bang behaviour when the constraint limits are sufficiently wide.

The variable flow rate calculations are only partially successful. The conjugate gradient algorithm diverges while the steepest descent algorithm requires excessively long CPU times. The program failed to converge after ½ hour using a 17 × 17 grid. In the case of a 11 × 11 grid, the method is not fully satisfactory as is shown by Fig. 10.19 : the performance index would clearly be improved if $\theta_2(1,t)$ were ≥ 1 throughout the time interval and $\theta_1(1,t)$ consequently increased. This failure is probably due to the large number of variables (33) in the search procedure.

10. RESULTS AND CONCLUSIONS

10.7 Tables

Table 10.1 The transport equation.

(The number of iterations is given in parentheses.)

Problem	Optimisation algorithm	No. of grid points D	$J=\int_0^1\int_0^1\{\phi^2(x,t)+au^2(x,t)\}dxdt$ a=1.0	0.1	0.01
Problem 10.1 $\phi_o(x)=x$ $\phi_1(t)=t$	Steepest descent	11	.202453(7)	.101616(3)	.091763(3)
		17	.220850(9)	.098729(3)	.086420(3)
		21	.224140(10)	.097018(3)	.084544(2)
	Conjugate gradient	11	.575402(9)	.119319(2)	.051858(8)
		17	.581170(7)	.116920(11)	.051229(5)
		21	.582263(6)	.118079(11)	.051328(4)
Problem 10.2 $\phi_o(x)=x(1-x)$ $\phi_1(t)=te^t$	Steepest descent	11	.293165(7)	.092173(6)	.057935(6)
		17	.290577(7)	.067309(2)	.044991(1)
		21	.297165(7)	.067066(2)	.044056(1)
	Conjugate gradient	11	.437574(6)	.089785(6)	.053286(6)
		17	.446198(7)	.083835(3)	.047616(1)
		21	.447317(7)	.082354(3)	.045868(1)

10. RESULTS AND CONCLUSIONS

Table 10.2 Coercive control of counterflow.

Problem	Solution method	Algorithm	$J=\frac{1}{2}\int_0^1 \{[\theta_1(1,t)-\theta_1^*(t)]^2+a[\theta_2(1,t)-\theta_2^*(t)]^2\}dt$ a=1.0	0.1	0.01
10.3 $\theta_1^*=\theta_2^*=1$	Analytic	M=5	.322259	.245809	.100290
		M=6	.322259	.245811	.100299
	Lax-Wendroff	Steepest descent	.314452(4)	.230969(8)	.081258(8)
		Conjugate gradient	.314452(4)	.230275(7)	.080698(8)
	Lax-Wendroff Exponential	Conjugate gradient	.311884(5)	.218285(5)	.087636(5)
	Wendroff	Steepest descent	.314208(4)	.230857(7)	.078735(10)
		Conjugate gradient	.314208(4)	.229651(7)	.081658(6)
		DFP	.314208(3)	.230743(7)	.083348(13)
10.4 Analytic target data	Analytic	M=5	.416415		
	Lax-Wendroff	Steepest descent	.406207(3)		
		Conjugate gradient	.406064(6)		
	Lax-Wendroff Exponential	Conjugate gradient	.404722(4)		
	Wendroff	Steepest descent	.405483(4)		
		Conjugate gradient	.405478(4)		
		DFP	.405472(3)		

10. RESULTS AND CONCLUSIONS

Table 10.3 The Wendroff conjugate gradient method for counterflow, with various numbers of grid points.

Problem	No. of grid points in each direction D	$J=\frac{1}{2}\int_0^1\{[\theta_1(1,t)-\theta_1^*(t)]^2+a[\theta_2(1,t)-\theta_2^*(t)]^2\}dt$ a=1.0	0.1	0.01
10.3 $\theta_1^*=\theta_2^*=1$	11	.306646(4)	.212313(6)	.069245(7)
	17	.312273(4)	.225407(7)	.078000(7)
	21	.314208(4)	.229651(7)	.081658(6)
		a=1.0 Optimisation result.	Analytic result.	
10.4 Analytic target data	11	.396720(4)	.416451	
	17	.404108(4)	.416418	
	21	.405478(4)	.416415	

Table 10.4 Constrained control of counterflow by various methods.

	Solution method	Algorithm	$J=\frac{1}{2}\int_0^1\{\theta_1(1,t)-\theta_1^*(t)\}^2dt$
Problem 10.5 $0 \leq \theta_2 \leq 2$	Lax-Wendroff	Exact	.194098
		Steepest descent	.197598(6)
		Conjugate gradient	.198007(4)
	Wendroff	Exact	.193349
		Steepest descent	.193627(20)
		Conjugate gradient	.193619(13)
		DFP	.193797(8)

10. RESULTS AND CONCLUSIONS

Table 10.5 Constrained control of counterflow for various values of the constraint parameters, using the Wendroff conjugate gradient method.

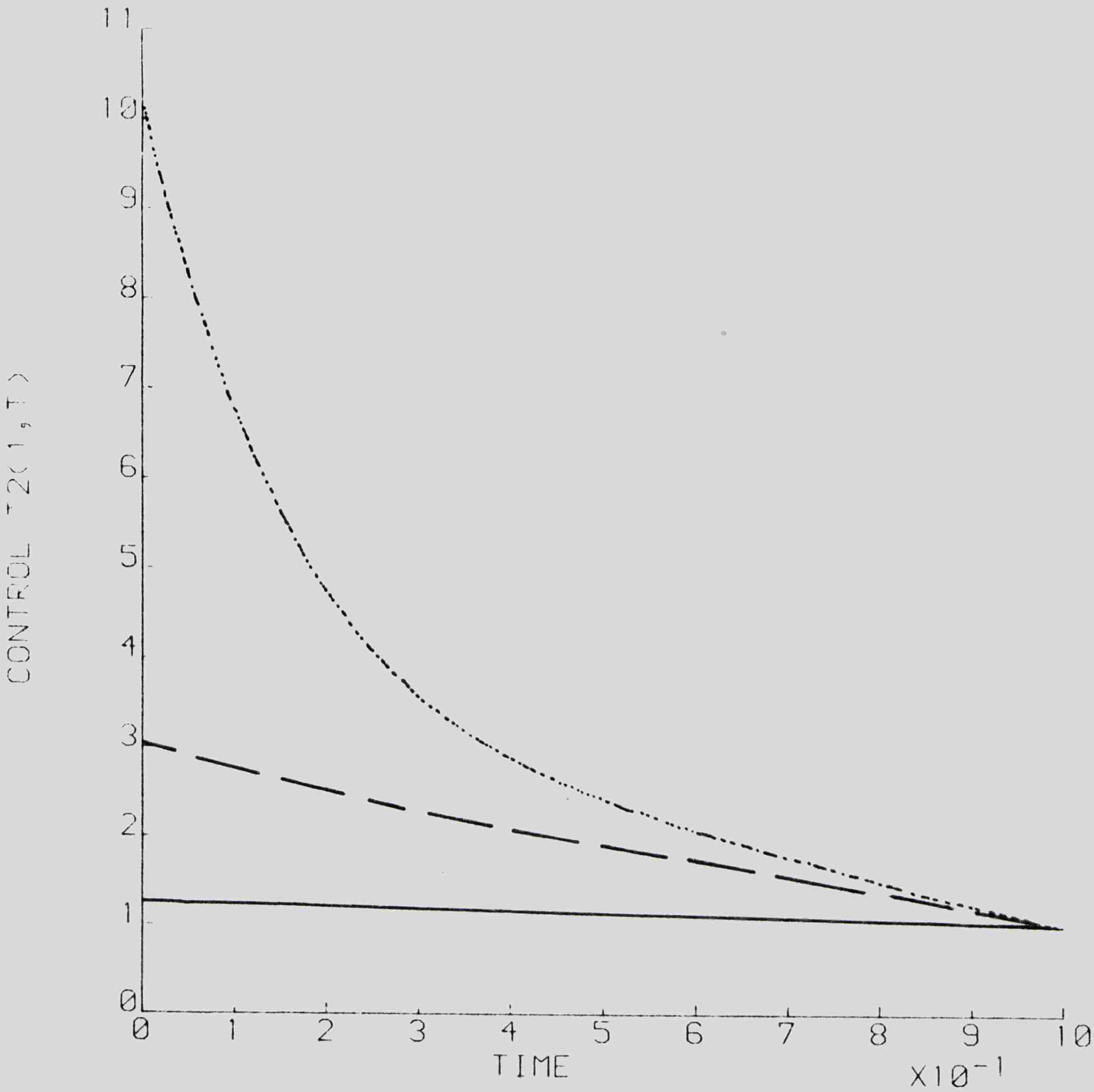
	A	B	a	b	$J=\frac{1}{2}\int_0^1\{\theta_1(L,t)-\theta_1^*(t)\}^2dt$
	1	1	0	2	.193619(13)
Problem 10.5	1	3	-2	4	.084790(19)
$a \leq \theta_2(L,t) \leq b$	1	5	-4	6	.050235(24)
	1	7	-6	8	.034693(18)

Table 10.6 Variable flow rate control of counterflow using the Wendroff steepest descent method.

Problem 10.6	No. of grid points in each direction D	$J=\frac{1}{2}\int_0^1\{[\theta_1(1,t)-\theta_1^*(t)]^2+[\theta_2(1,t)-\theta_2^*(t)]^2 + [u(t)-u^*(t)]^2+[v(t)-v^*(t)]^2\}dt$
	11	.291462(11)
	17	Failed to converge after ½ hour CPU time

10.8. Figures.

The multi-valued nature of a portion of Figure 10.15 is solely due to the cubic spline routine used to interpolate between the data points.



Key

a = 1.0

0.1

0.01

FIG 10.1 PROBLEM 10.3 USING ANALYTIC COLLOCATION

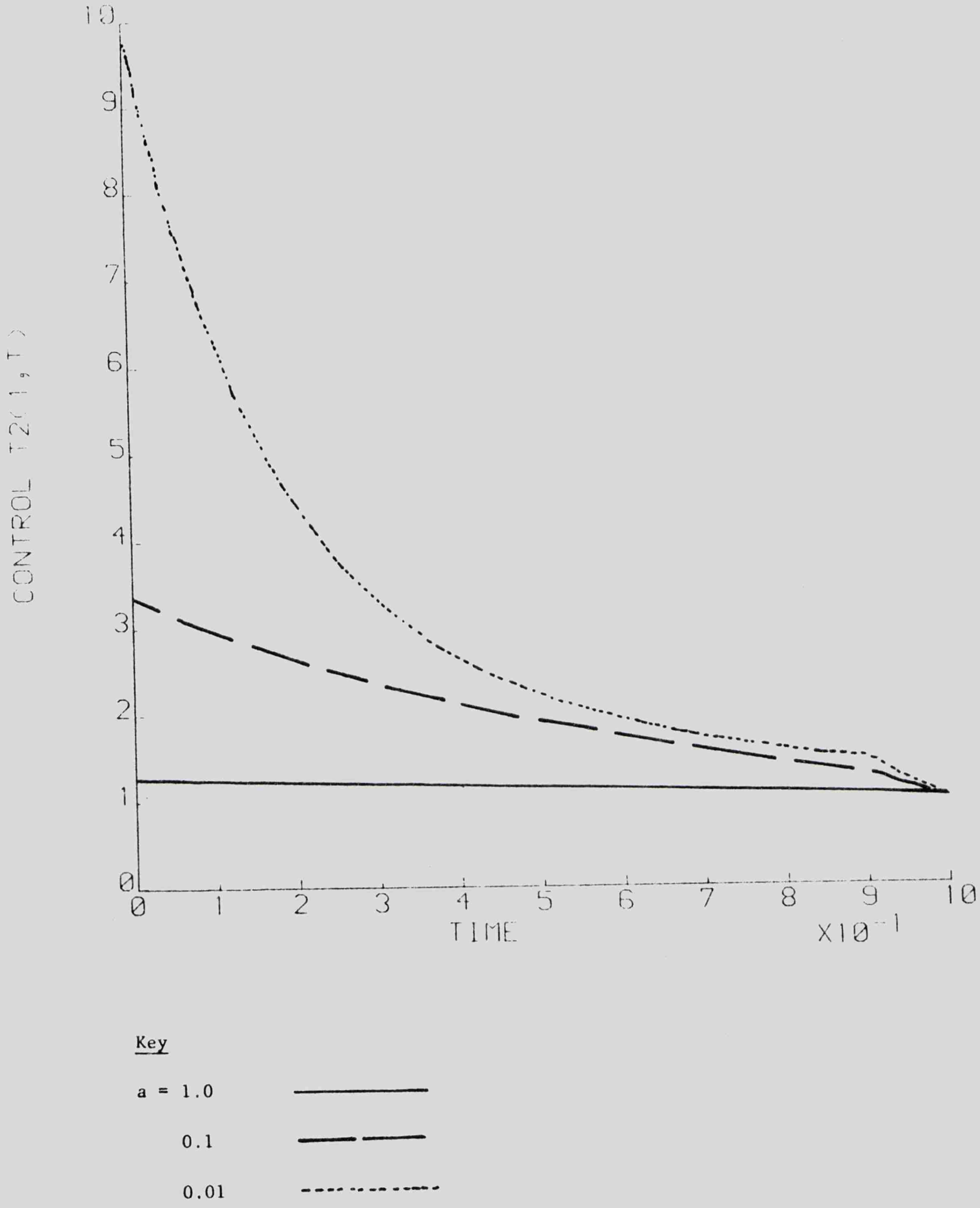


FIG 10.2 PROBLEM 10.3 LAX-WENDROFF STEEPEST DESCENT

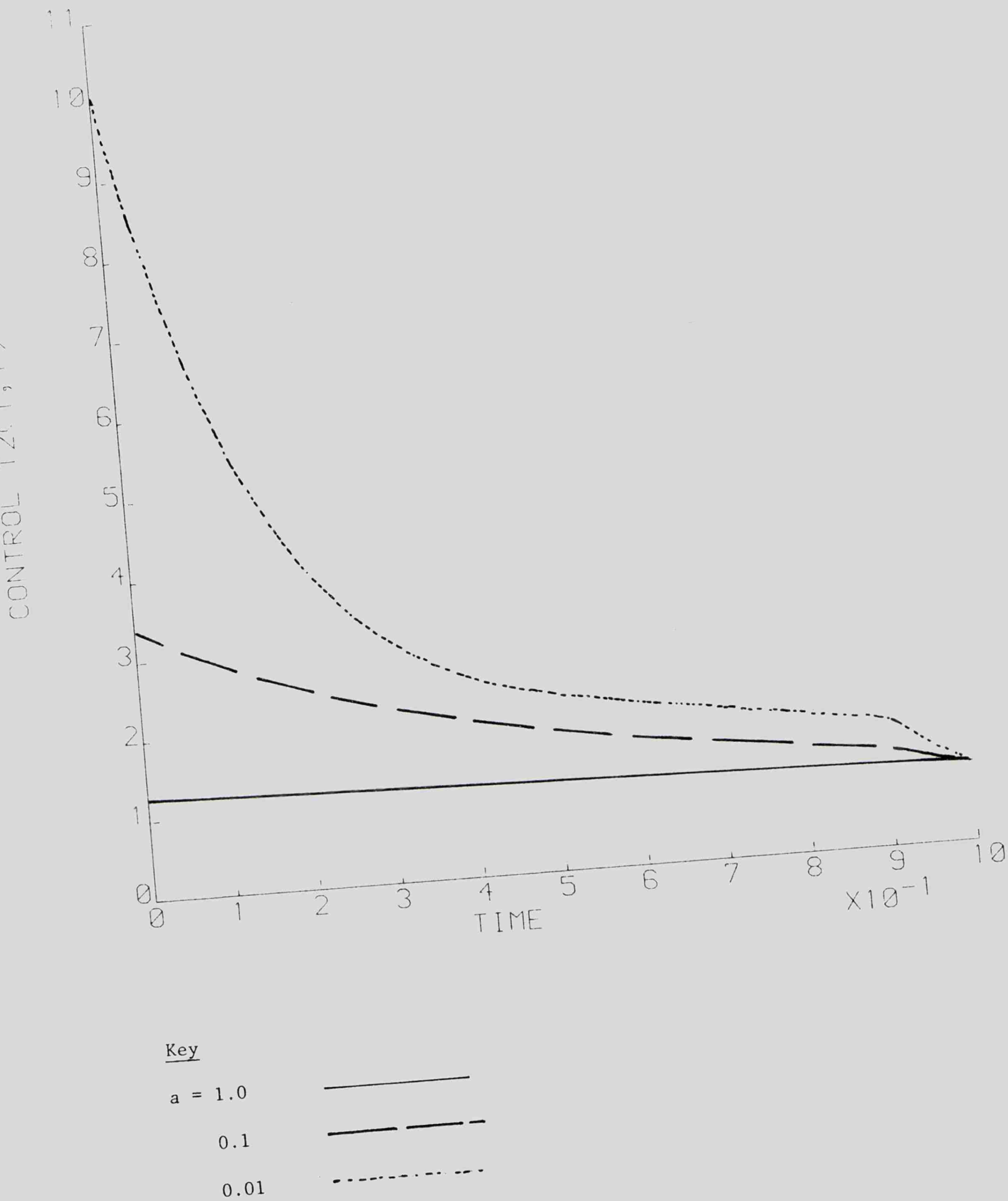
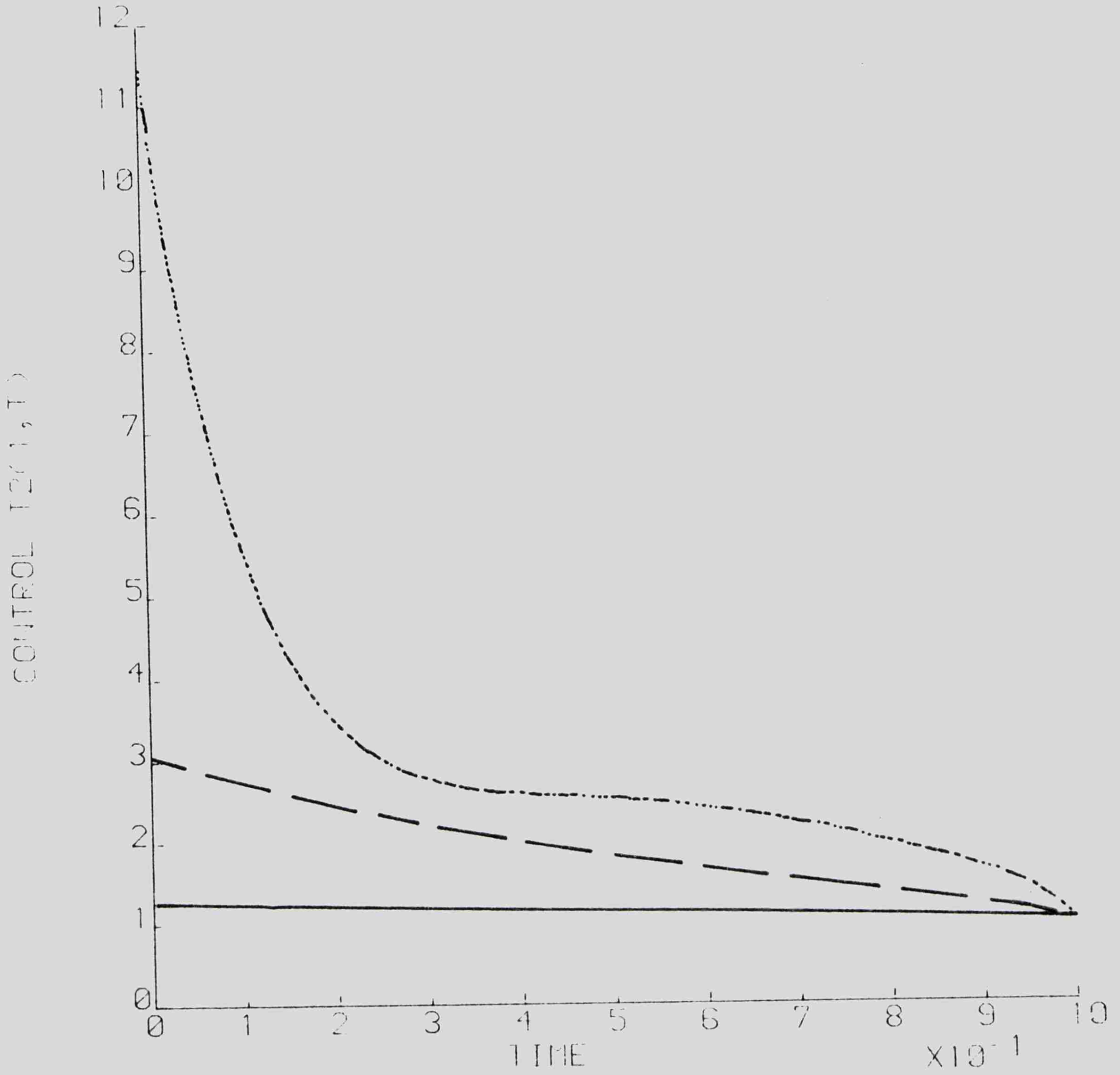


FIG 10.3 PROBLEM 10.3 LAX-WENDROFF CONJ. GRADIENT



Key

a = 1.0

0.1

0.01

FIG 10.4 PROBLEM 10.3 WENDROFF STEEPEST DESCENT

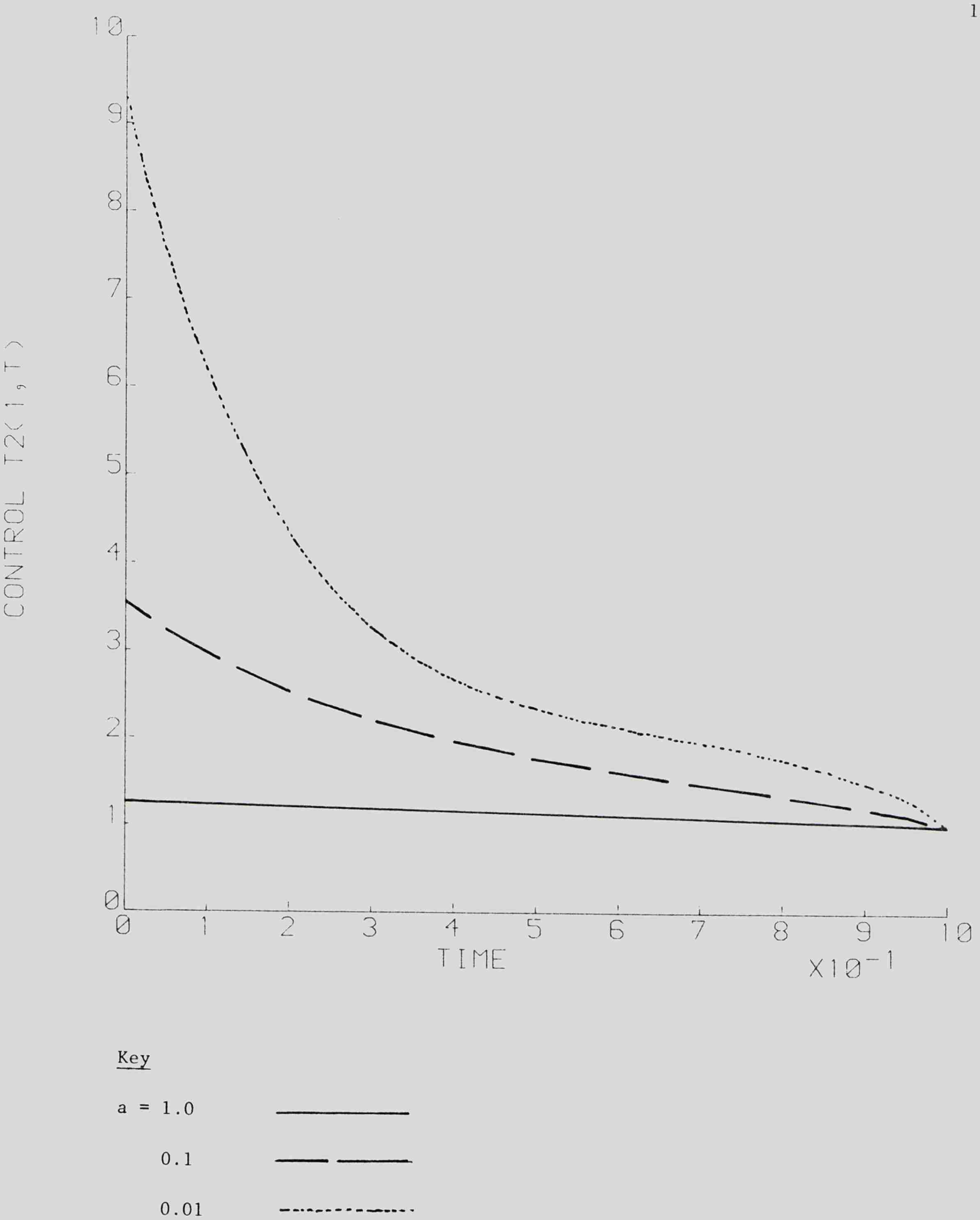
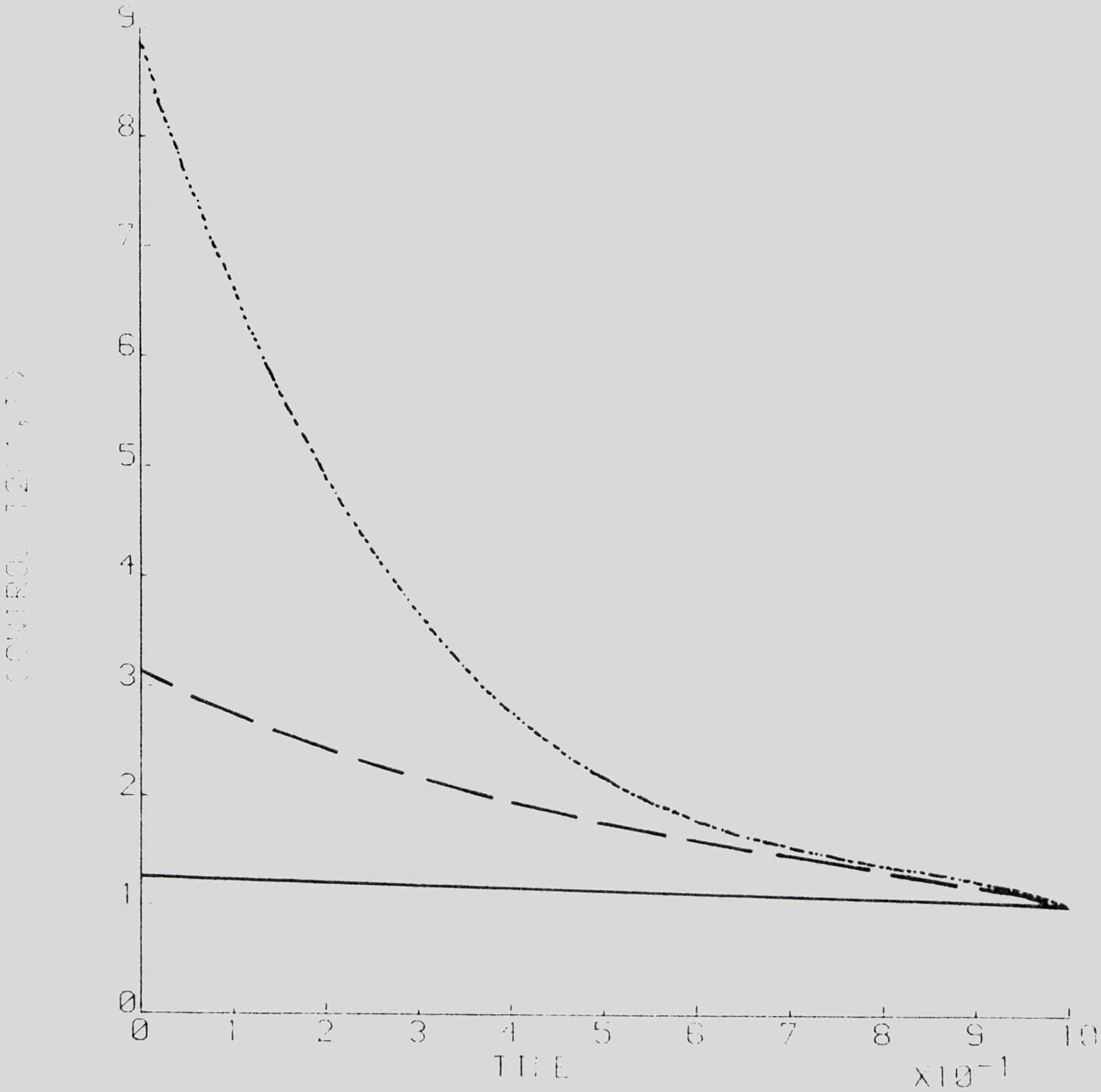


FIG 10.5 PROBLEM 10.3 WENDROFF CONJUGATE GRADIENT



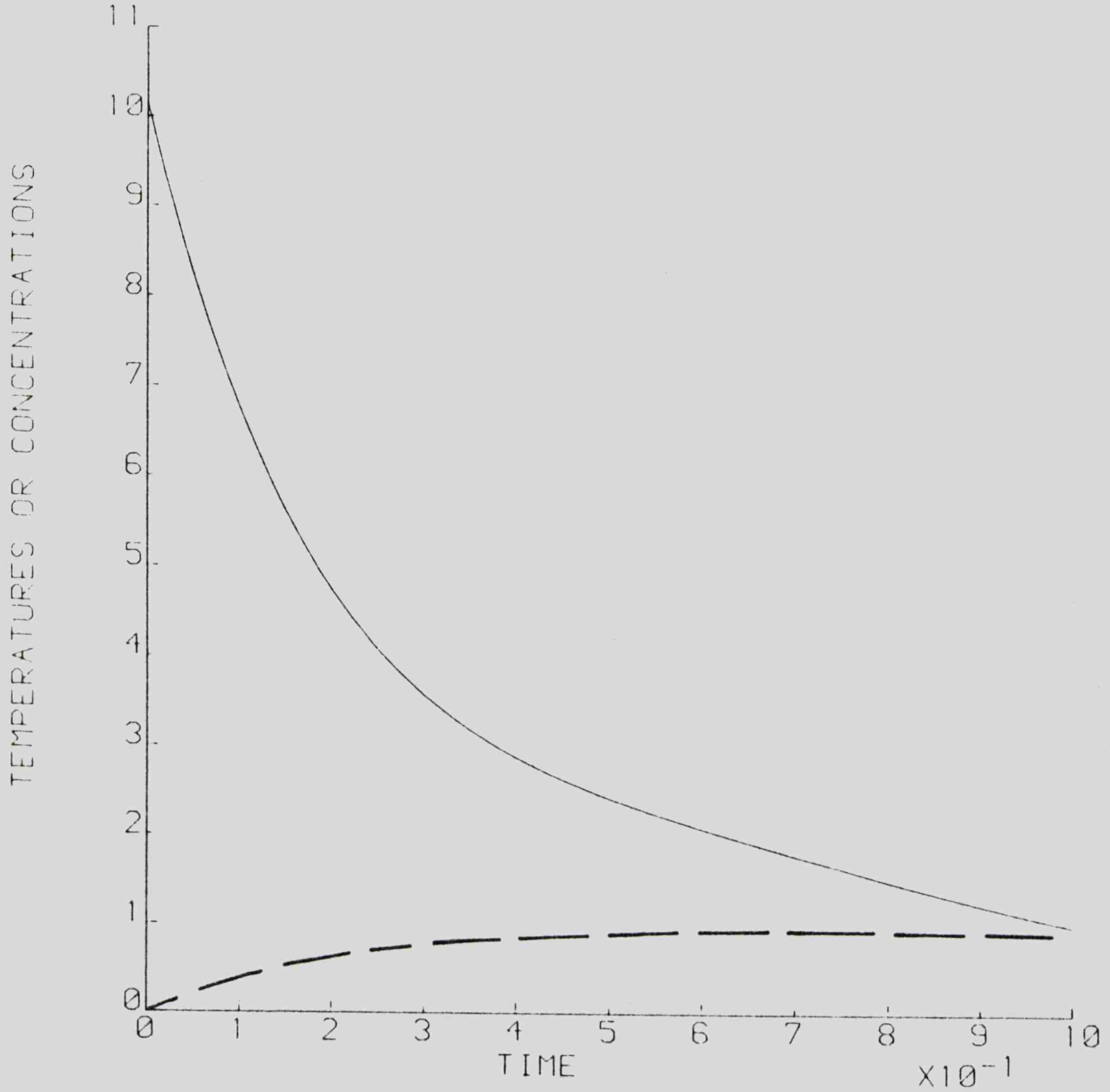
Key

a = 1.0

0.1

0.01

FIG 10.6 PROBLEM 10.3 WENDROFF D.F.P.

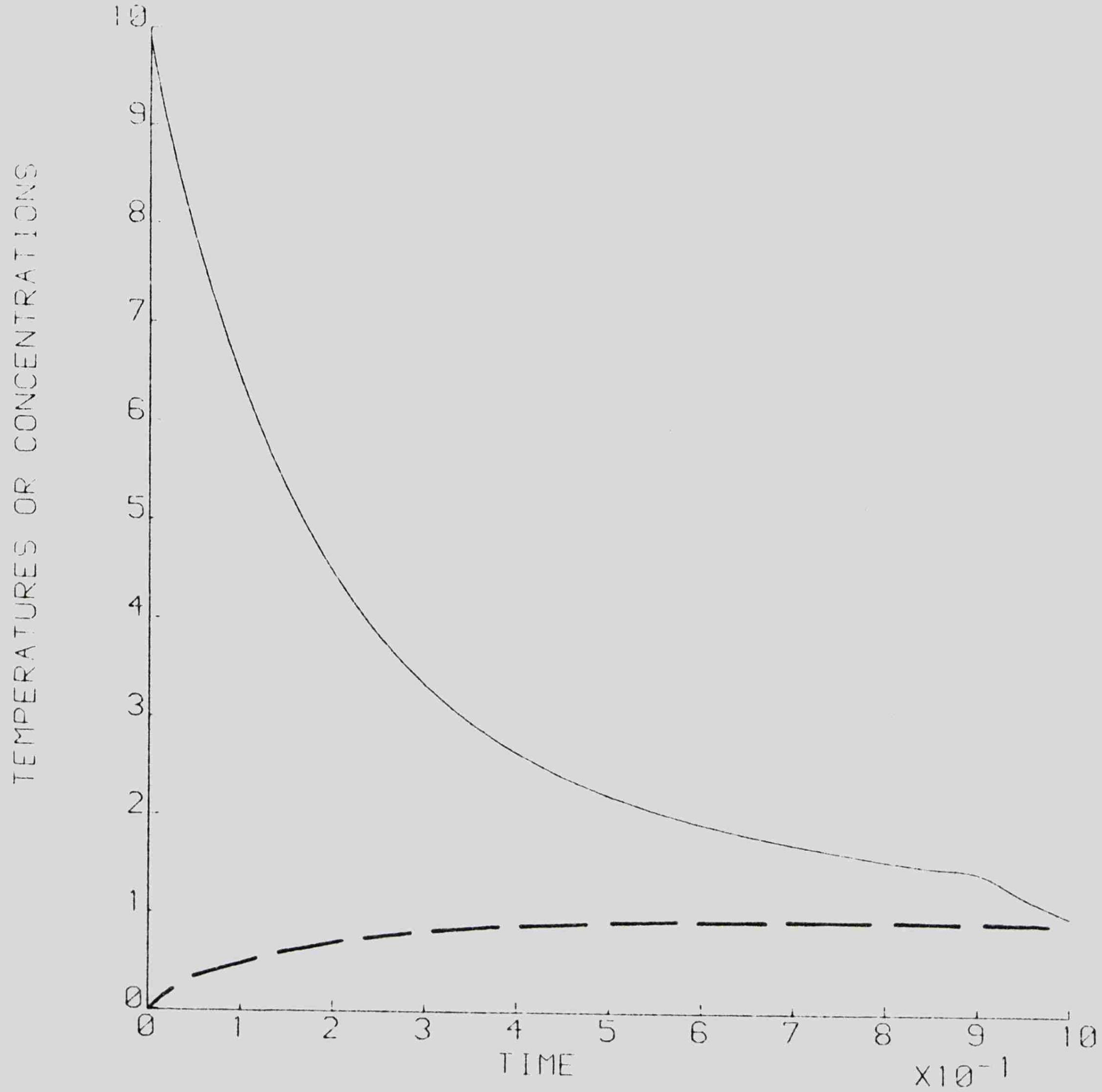


Key ($\alpha = .01$)

Control $\theta_2(1,t)$ ————

Response $\theta_1(1,t)$ — — —

FIG 10.7 PROBLEM 10.3 USING ANALYTIC COLLOCATION

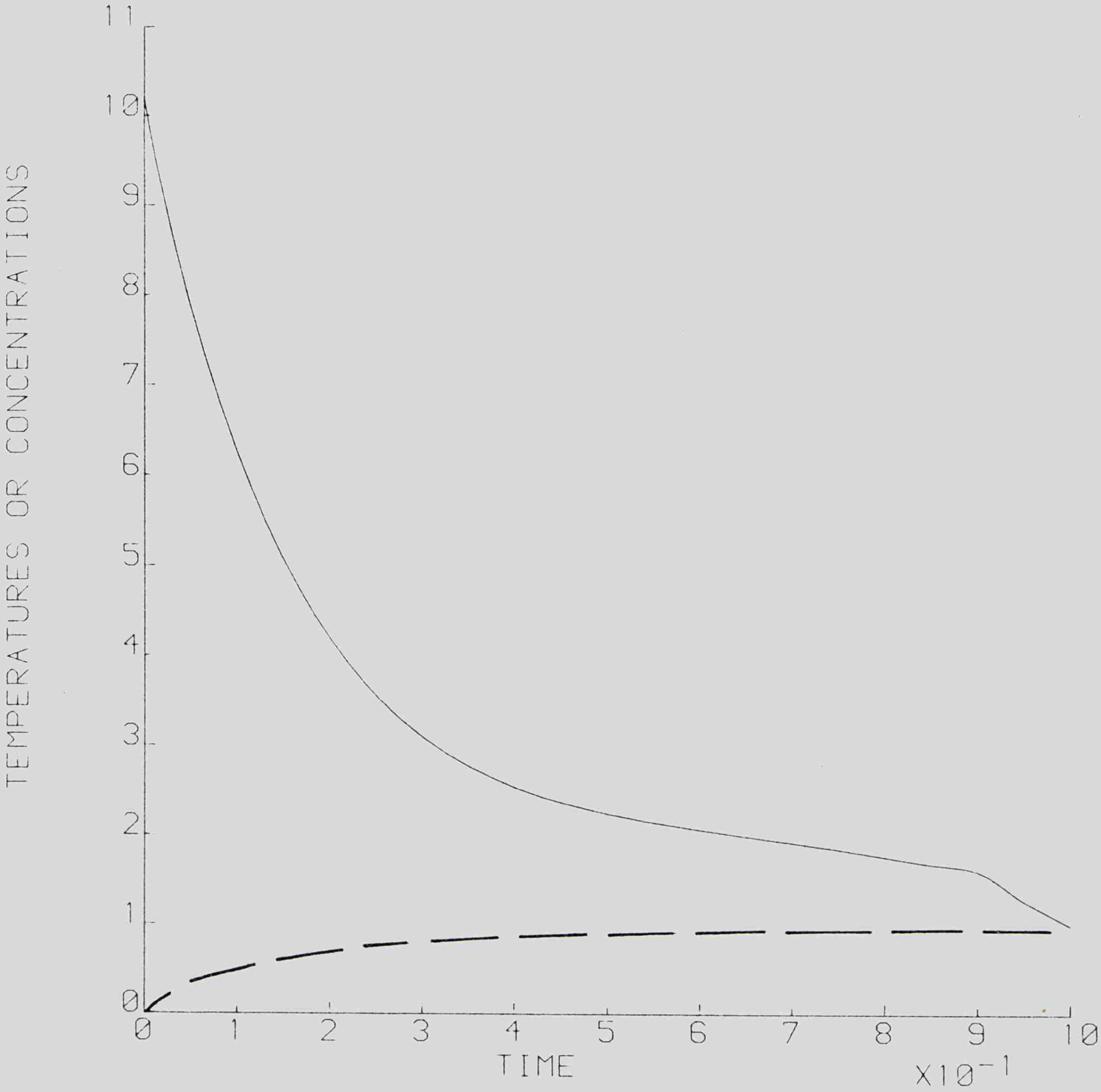


Key ($\alpha = \cdot 01$)

Control $\theta_2(1,t)$ ————————

Response $\theta_1(1,t)$ — — — —

FIG 10.8 PROBLEM 10.3 LAX-WENDROFF STEEPEST DESCENT

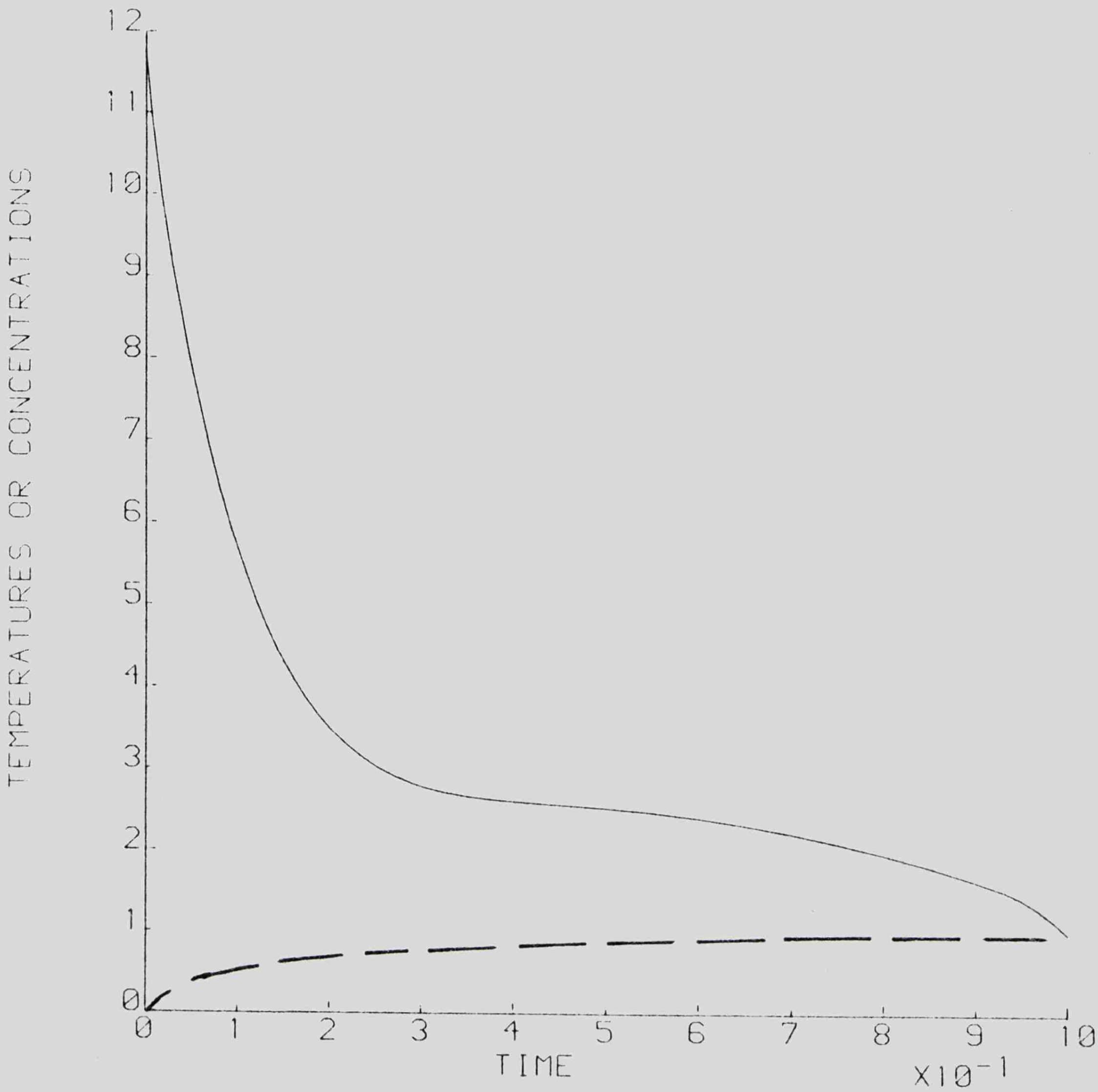


<u>Key</u> ($\alpha = \cdot 01$)

Control $\theta_2(1,t)$ ————————

Response $\theta_1(1,t)$ — — — —

FIG 10.9 PROBLEM 10.3 LAX-WENDROFF CONJ. GRADIENT

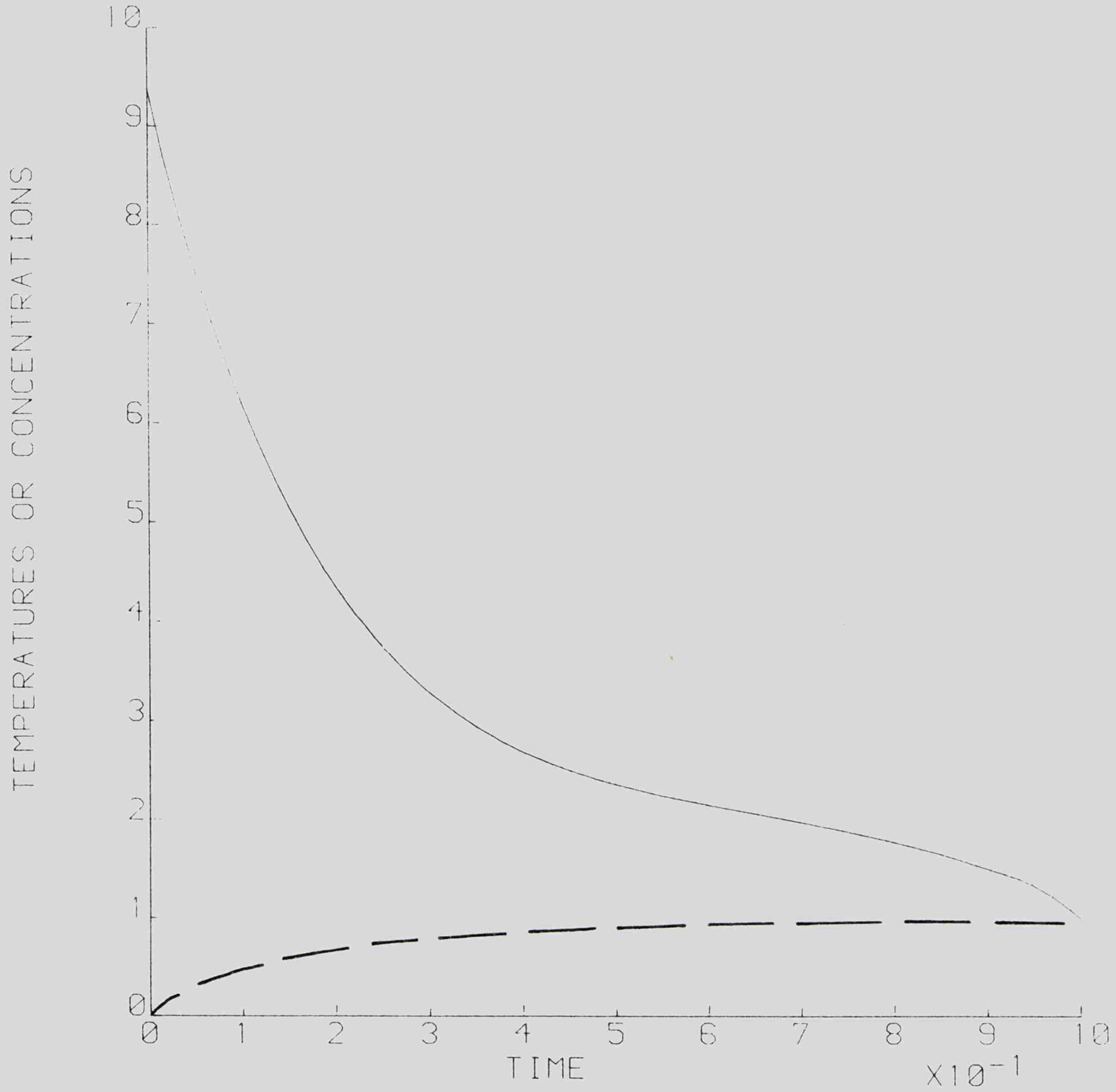


Key (a = ·01)

Control $\theta_2(1,t)$ ————————

Response $\theta_1(1,t)$ — — — —

FIG 10.10 PROBLEM 10.3 WENDROFF STEEPEST DESCENT

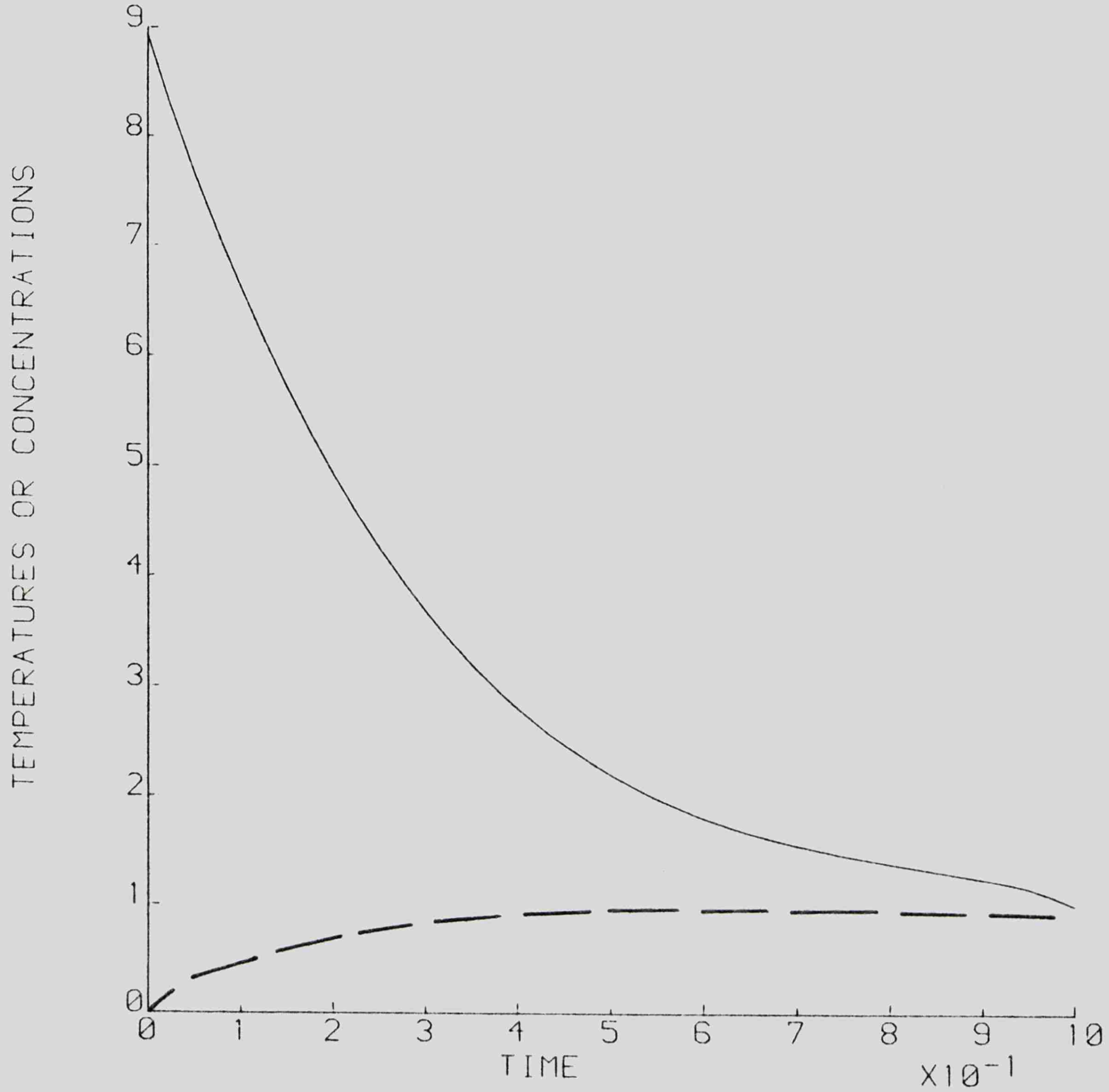


Key $(\alpha = \cdot 01)$

Control $\theta_2(1,t)$

Response $\theta_1(1,t)$

FIG 10.11 PROBLEM 10.3 WENDROFF CONJUGATE GRADIENT

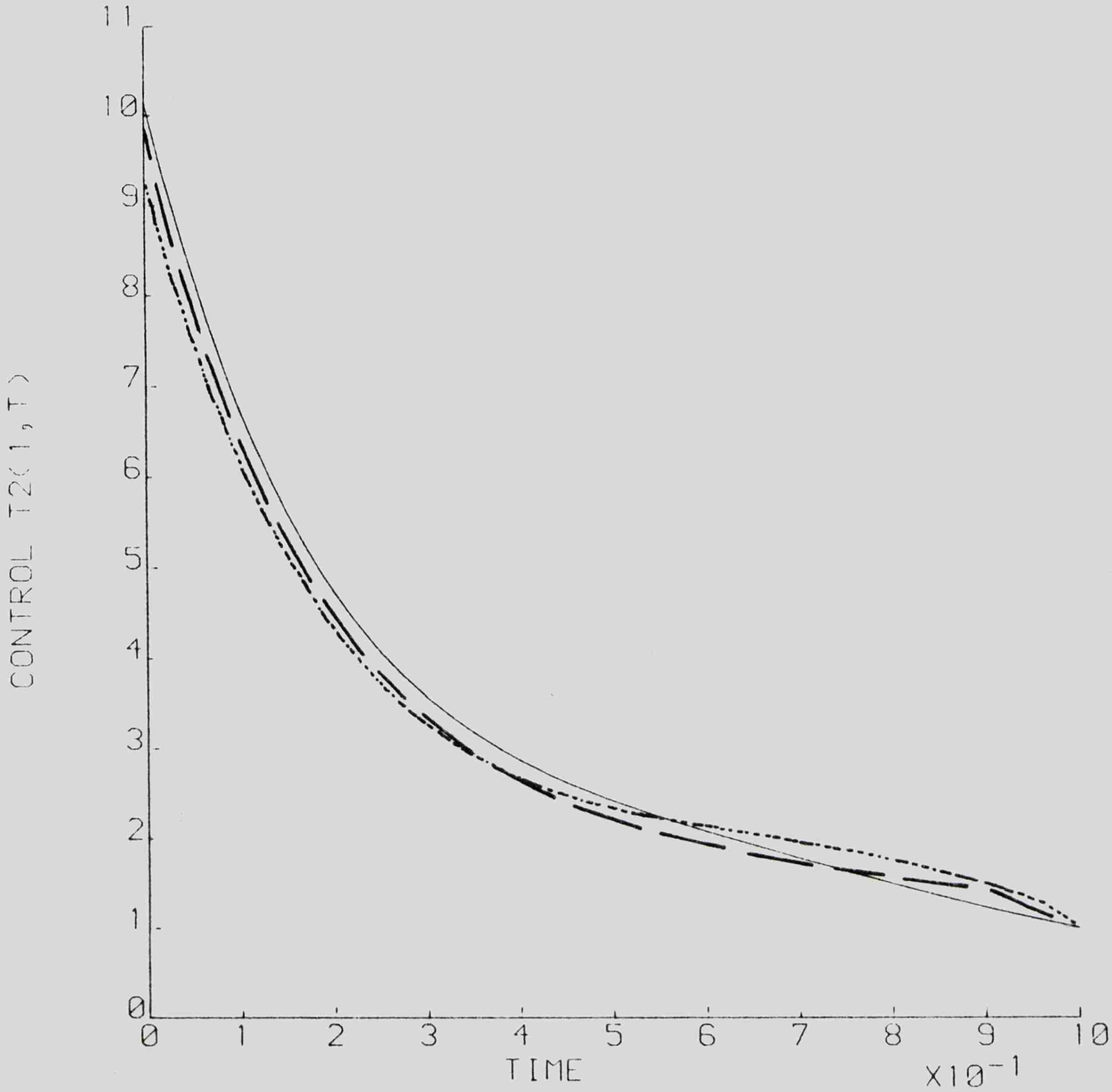


Key (a = .01)

Control $\theta_2(1,t)$ ————

Response $\theta_1(1,t)$ — —

FIG 10.12 PROBLEM 10.3 WENDROFF D.F.P.



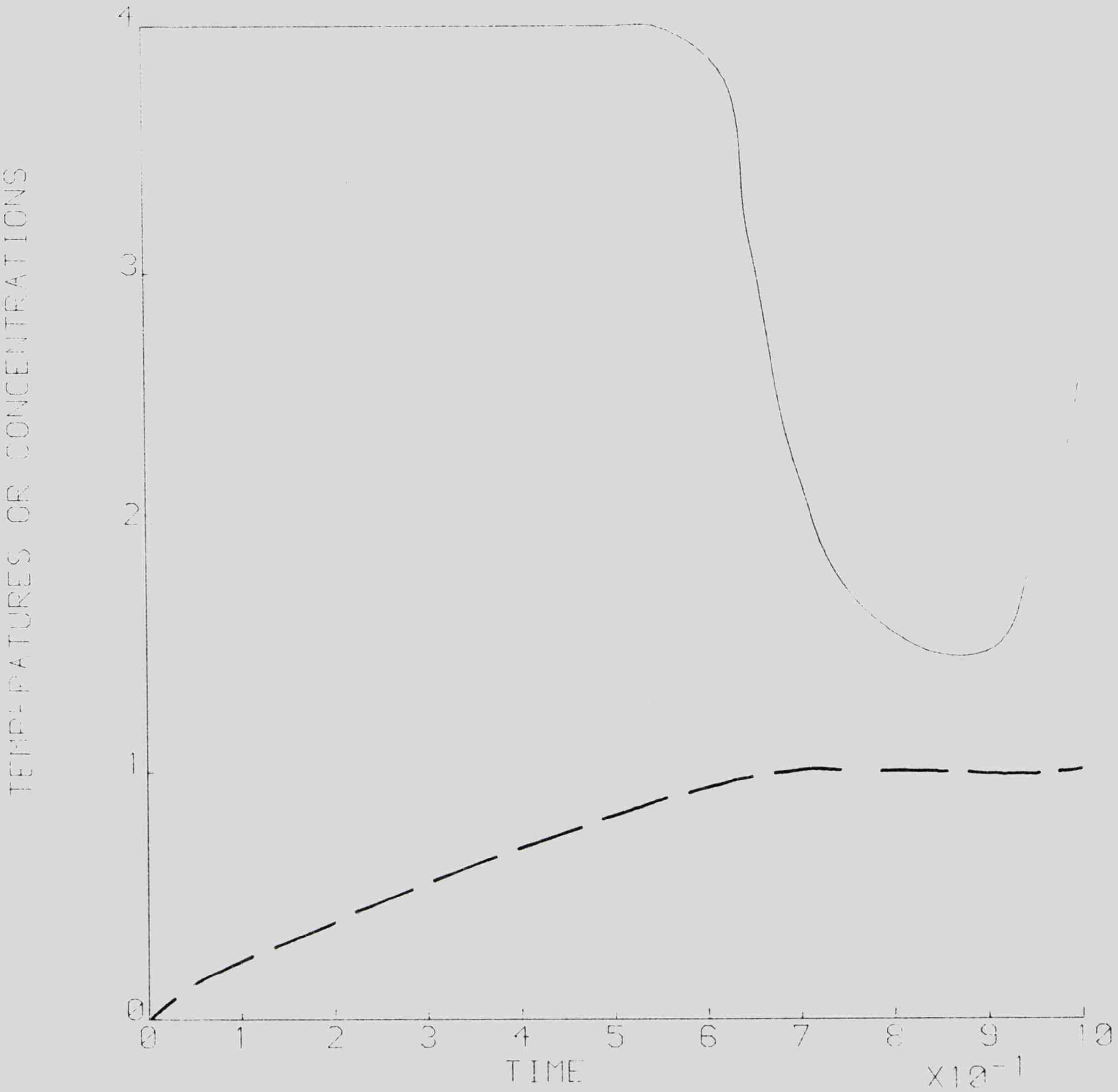
Key (a = 0.01 for each method)

Analytic collocation

Lax-Wendroff steepest descent

Wendroff conjugate gradient

FIG 10.13 PROBLEM 10.3 COMPARISON OF METHODS

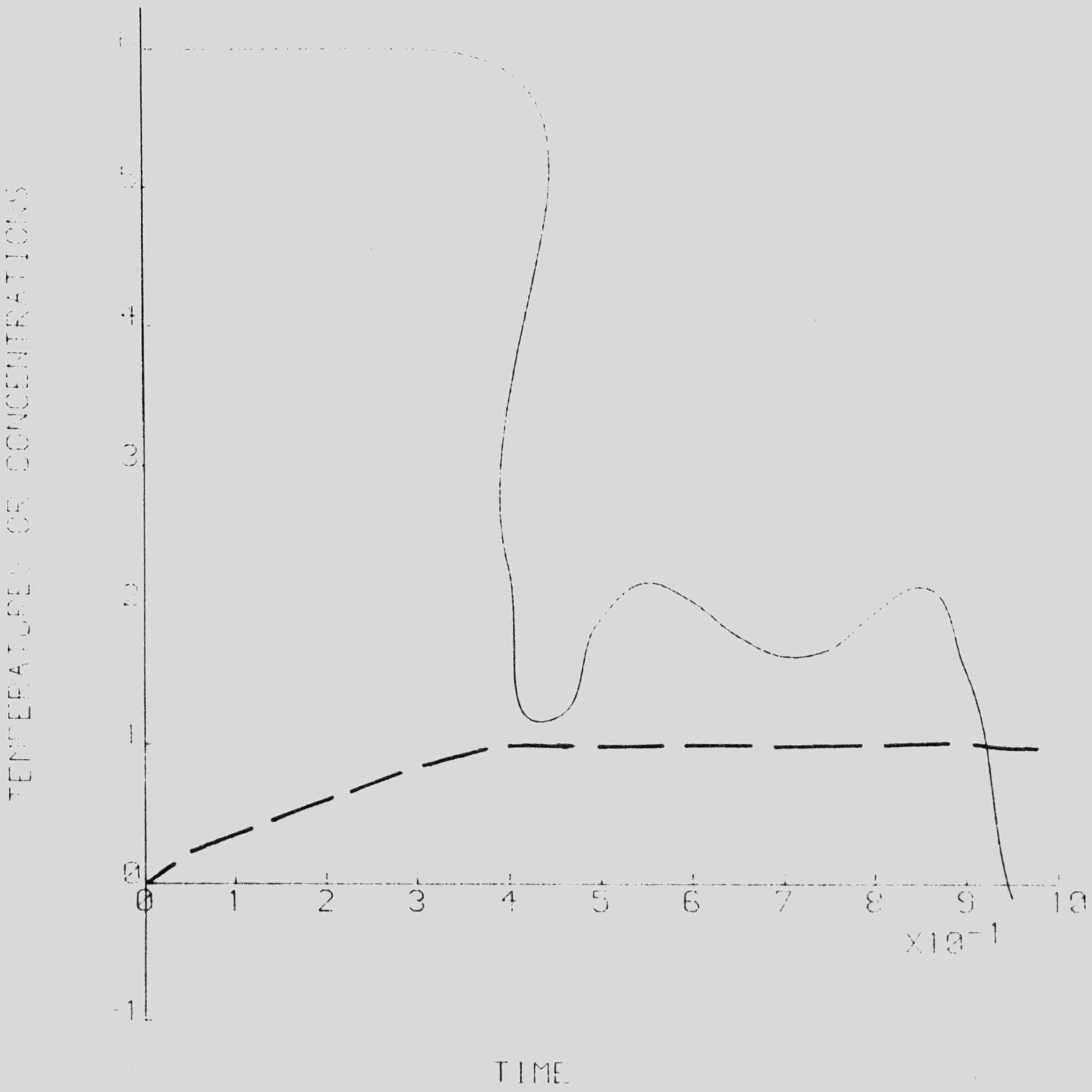


Key

Control $\theta_2(1,t)$ ————

Response $\theta_1(1,t)$ — — — —

FIG 10.14 PROBLEM 10.5 WENDROFF CONJ.GRAD. A=1,B=3

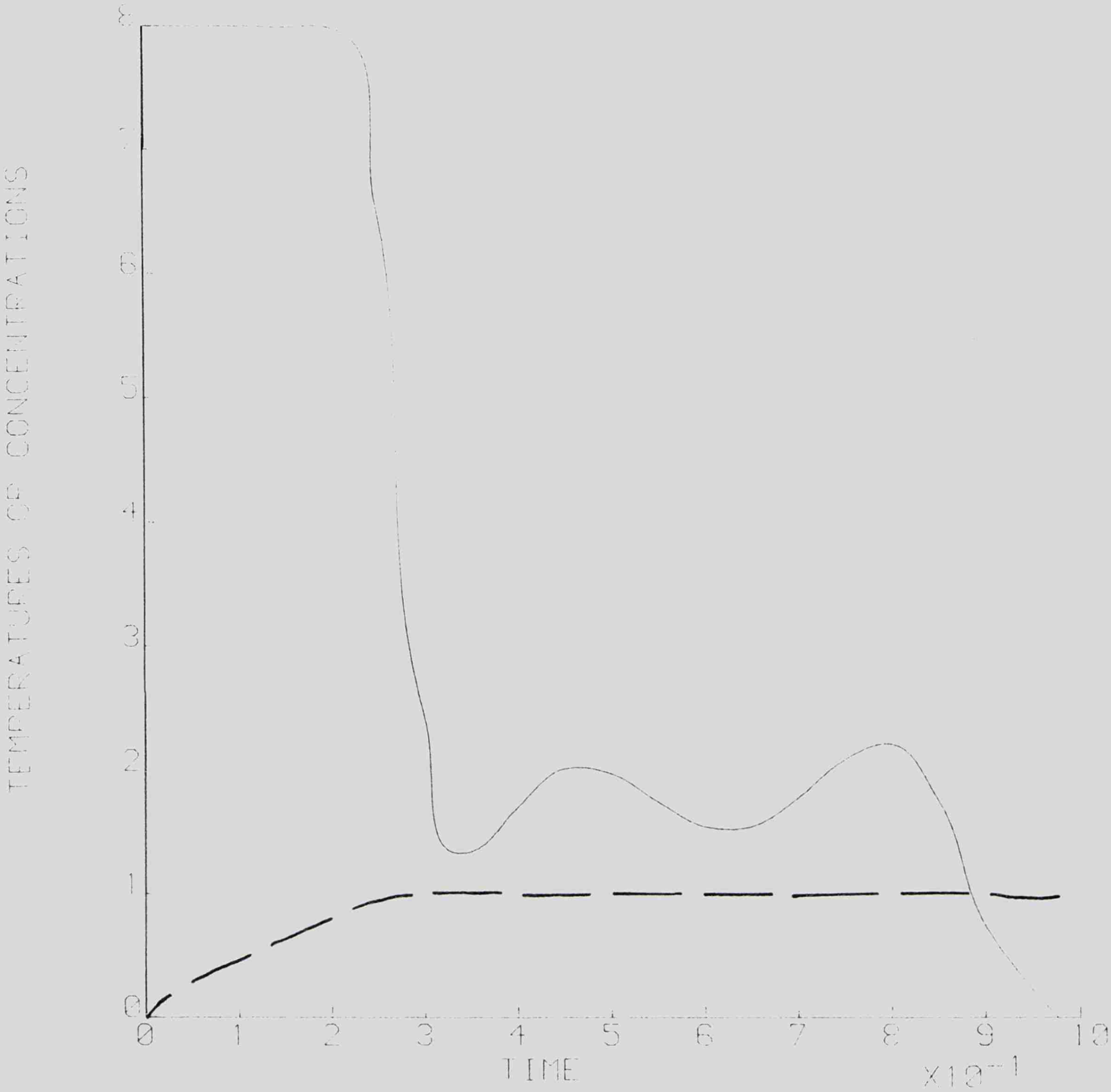


Key

Control $\theta_2(1,t)$

Response $\theta_1(1,t)$.

FIG 10.15 PROBLEM 10.5 WENDROFF CONJ.GRAD. A=1,B=5

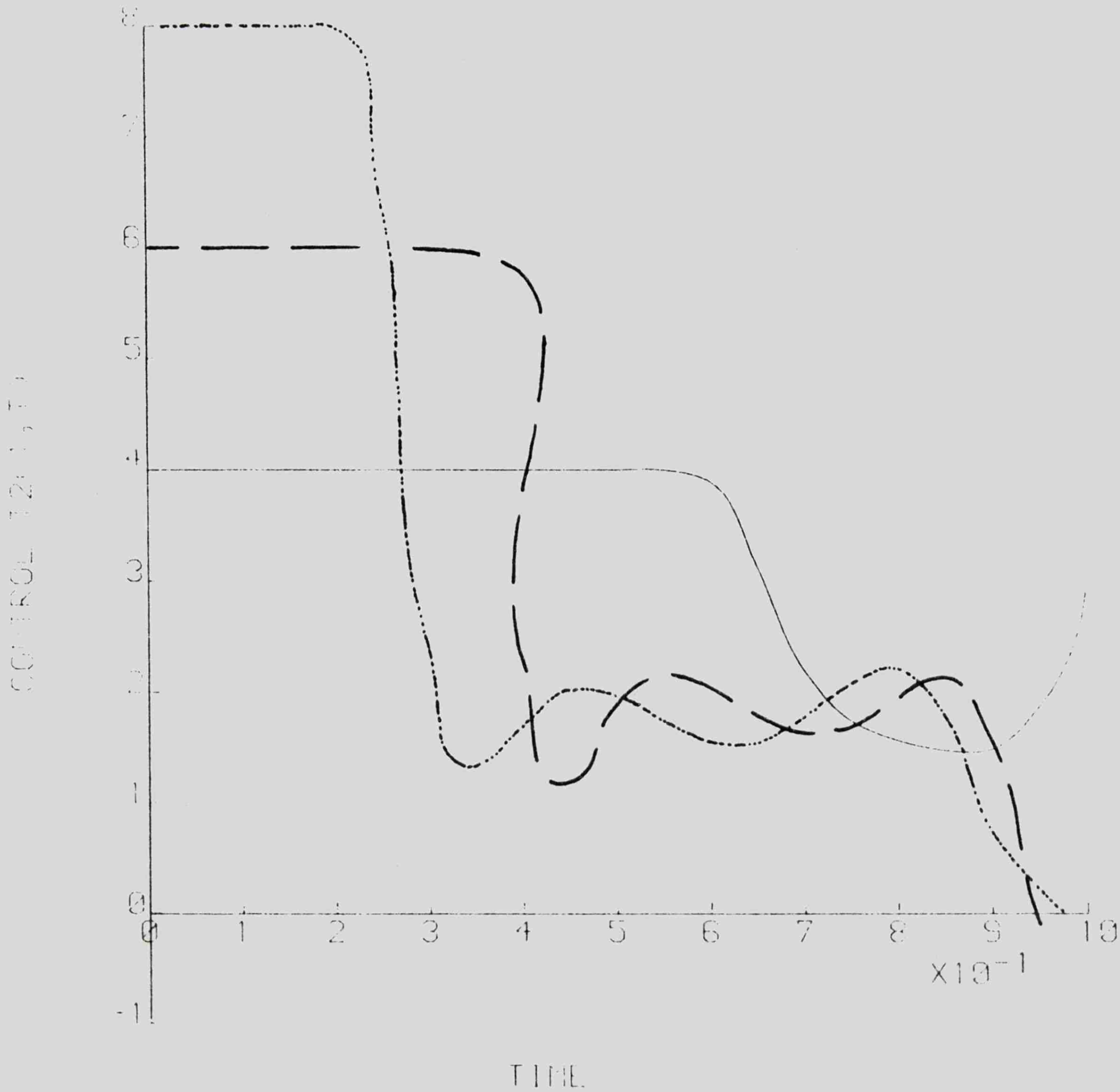


Key

Control $\theta_2(1,t)$

Response $\theta_1(1,t)$

FIG 10.16 PROBLEM 10.5 WENDROFF CONJ. GRAD. A=1,B=7



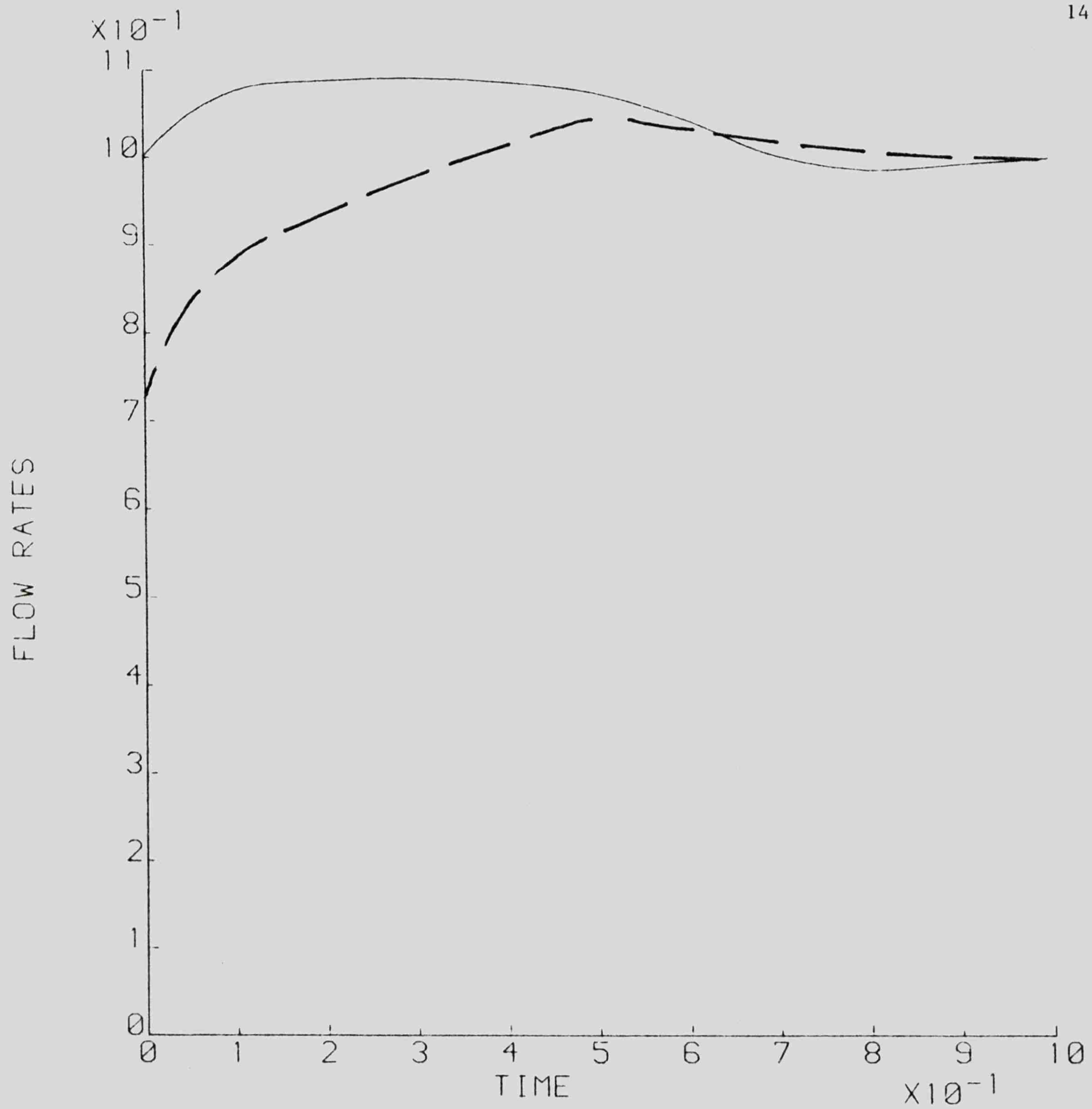
<u>Key</u> (Constrained control using Wendroff conjugate gradient)

A = 1, B = 3 ————————

A = 1, B = 5 — — — —

A = 1, B = 7 ....

FIG 10.17 PROBLEM 10.5 COMPARISON OF CONSTRAINTS

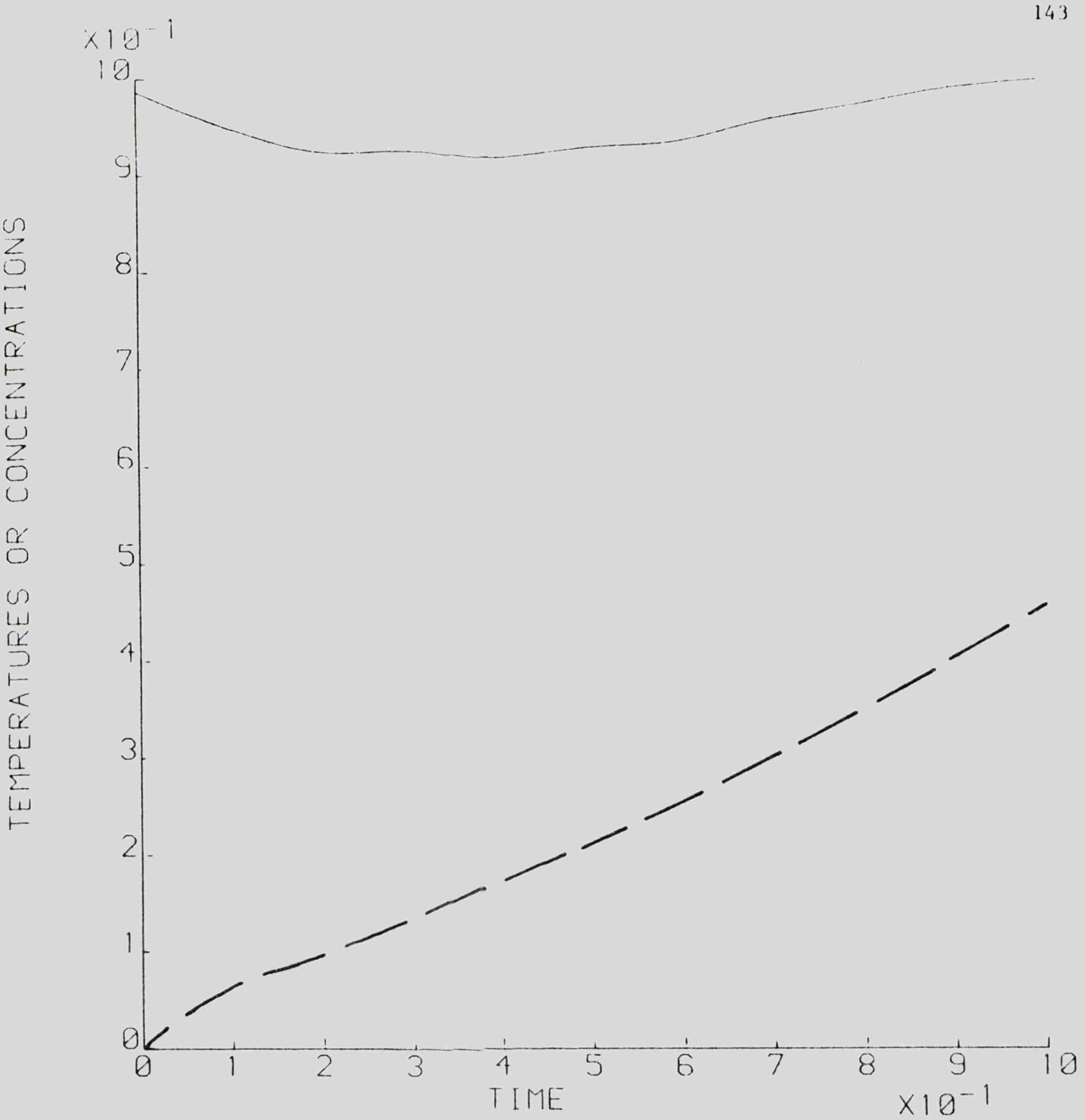


<u>Key</u> (Steepest descent method)

u(t) ————————

v(t) — — — —

FIG 10.18 PROBLEM 10.6 VARIABLE FLOW RATES



<u>Key</u> (Steepest descent method)

Control $\theta_2(1,t)$ ————————

Response $\theta_1(1,t)$ — — — —

FIG 10.19 PROBLEM 10.6 VARIABLE FLOW RATE

Appendices

Appendix 1.

Steady state solutions of the counterflow equations.

The time-independent version of equations (6.2), (6.3) (in natural coordinates) is

$$\frac{d\theta_1}{dx} = \frac{k_1}{u}(h\theta_2 - \theta_1) \tag{A1.1}$$

$$\frac{d\theta_2}{dx} = \frac{k_2}{v}(h\theta_2 - \theta_1) \tag{A1.2}$$

After transforming the coordinates according to equations (6.29) - (6.42), and dropping the primes, these equations become

$$\frac{d\theta_1}{dx} = \frac{1}{u}(\theta_2 - \theta_1) \tag{A1.3}$$

$$\frac{d\theta_2}{dx} = \frac{h}{v}(\theta_2 - \theta_1) \tag{A1.4}$$

The solutions of these equations which satisfy boundary conditions θ_1 given at $x = 0$ and θ_2 given at $x = L$ are, in the case $\beta = \frac{h}{v} - \frac{1}{u} \neq 0$,

$$\theta_1(x) = \frac{\theta_1(0)(e^{\beta x} - \frac{h}{s} e^{\beta L}) + \theta_2(L)(1 - e^{\beta x})}{1 - \frac{h}{s} e^{\beta L}} \tag{A1.5}$$

$$\theta_2(x) = \frac{\frac{h}{s}\theta_1(0)(e^{\beta x} - e^{\beta L}) + \theta_2(L)(1 - \frac{h}{s} e^{\beta x})}{1 - \frac{h}{s} e^{\beta L}} \; . \tag{A1.6}$$

In the case $\beta = 0$, they are

$$\theta_1(x) = \frac{\theta_2(L) - \theta_1(0)}{L + u} x + \theta_1(0) \tag{A1.7}$$

$$\theta_2(x) = \frac{\theta_2(L) - \theta_1(0)}{L + u} x + \frac{u\theta_2(L) + L\theta_1(0)}{L + u} \; . \tag{A1.8}$$

Appendix 2.

Steady state optimisation of the counterflow system

We seek to minimise

$$J = \tfrac{1}{2}\{(\theta_1(L) - \theta_1^*)^2 + a(\theta_2(L) - \theta_2^*)^2\} \qquad \text{(A2.1)}$$

subject to equations (A1.1) and (A1.2), and the boundary condition that $\theta_1(0)$ is given. $\phi = \theta_2(L)$ is the control.

Let us introduce co-state variables $\lambda_1(x)$, $\lambda_2(x)$, and the Hamiltonians

$$H_o = k_1 \frac{\lambda_1}{u}(h\theta_2 - \theta_1) + k_2 \frac{\lambda_2}{v}(h\theta_2 - \theta_1) \qquad \text{(A2.2)}$$

$$H_1 = \tfrac{1}{2}\{(\theta_1(L) - \theta_1^*)^2 + a(\theta_2(L) - \theta_2^*)^2\} + \mu(\phi - \theta_2(L)) \qquad \text{(A2.3)}$$

The optimality conditions are

$$\frac{d\lambda_1}{dx} = -\frac{\partial H_o}{\partial \theta_1} = \frac{k_1}{u}\lambda_1 + \frac{k_2}{v}\lambda_2 \qquad \text{(A2.4)}$$

$$\frac{d\lambda_2}{dx} = -\frac{\partial H_o}{\partial \theta_2} = -h\left(\frac{k_1\lambda_1}{u} + \frac{k_2\lambda_2}{v}\right) \qquad \text{(A2.5)}$$

$$\lambda_1(L) = \frac{\partial H_1}{\partial \theta_1} = \theta_1(L) - \theta_1^* \qquad \text{(A2.6)}$$

$$\lambda_2(L) = \frac{\partial H_1}{\partial \theta_2} = a(\theta_2(L) - \theta_2^*) - \mu \qquad \text{(A2.7)}$$

$$\lambda_2(0) = 0 \qquad \text{(A2.8)}$$

$$\frac{\partial H_1}{\partial \mu} = 0 \quad \text{so that} \quad \phi = \theta_2(L) \qquad \text{(A2.9)}$$

$$\frac{\partial H_1}{\partial \phi} = 0 \quad \text{so that} \quad \mu = 0 \qquad \text{(A2.10)}$$

APPENDIX 2

Transforming θ_1, θ_2, θ_1^*, θ_2^*, ϕ, a, u, v according to equations (6.29), (6.30) and (6.36) - (6.42), and λ_1, λ_2 according to

$$\lambda_1' = \frac{\lambda_1}{u'} \tag{A2.11}$$

$$\lambda_2' = -\frac{\lambda_2}{v'} \tag{A2.12}$$

and dropping the primes we obtain

$$\frac{d\theta_1}{dx} = \frac{1}{u}(\theta_2 - \theta_1) \tag{A1.3}$$

$$\frac{d\theta_2}{dx} = \frac{h}{v}(\theta_2 - \theta_1) \tag{A1.4}$$

$$\frac{d\lambda_1}{dx} = \frac{1}{u}(\lambda_1 - \lambda_2) \tag{A2.13}$$

$$\frac{d\lambda_2}{dx} = \frac{h}{v}(\lambda_1 - \lambda_2) \tag{A2.14}$$

with boundary conditions

$$\lambda_1(L) = \frac{\theta_1(L) - \theta_1^*}{u} = \psi \tag{A2.15}$$

$$\lambda_2(L) = \frac{\mu - a(\theta_2(L) - \theta_2^*)}{v}\ , \tag{A2.16}$$

these equations being the time-independent forms of equations (6.16) - (6.22).

From equations (A1.5) and (A1.6), in the case $\beta = \frac{h}{v} - \frac{1}{u} \neq 0$, the solution of the state equations, with the boundary conditions that $\theta_1(0)$ is given and $\theta_2(L) = \phi$, is

APPENDIX 2

$$\theta_1(x) = \frac{\theta_1(0)\ (e^{\beta x} - \frac{h}{s} e^{\beta L}) + \phi(1-e^{\beta x})}{1 - \frac{h}{s} e^{\beta L}} \tag{A2.17}$$

$$\theta_2(x) = \frac{\theta_1(0)\ \frac{h}{s}\ (e^{\beta x} - e^{\beta L}) + \phi(1 - \frac{h}{s} e^{\beta x})}{1 - \frac{h}{s} e^{\beta L}} \quad . \tag{A2.18}$$

The solution of the costate equations, with boundary conditions (A2.8) and (A2.15) is

$$\lambda_1(x) = \frac{\psi(\frac{h}{s} - e^{-\beta x})}{\frac{h}{s} - e^{-\beta L}} \tag{A2.19}$$

$$\lambda_2(x) = \frac{\psi\frac{h}{s}\ (1 - e^{-\beta x})}{\frac{h}{s} - e^{-\beta L}} \quad . \tag{A2.20}$$

On substituting these results into the boundary condition (A2.16), we obtain

$$\phi = \frac{\frac{a}{h}\theta_2^*\ (\frac{h}{s} - e^{-\beta L})^2 + \left[\theta_1(0)(1-\frac{h}{s}) + \theta_1^*(\frac{h}{s} - e^{-\beta L})\right](1-e^{-\beta L})}{(1-e^{-\beta L})^2 + \frac{a}{h}\ (\frac{h}{s} - e^{-\beta L})^2} \tag{A2.21}$$

The same result can of course be obtained by direct minimisation of expression (A2.1) for J when $\theta_1(L)$ is obtained from equation (A2.17).

In the case $\beta = 0$, we have

$$\theta_1(x) = \frac{\phi - \theta_1(0)}{L + u}\ x + \theta_1(0) \tag{A2.22}$$

$$\theta_2(x) = \frac{\phi - \theta_1(0)}{L + u}\ x + \frac{u\phi + L\theta_1(0)}{L + u} \tag{A2.23}$$

$$\lambda_2(x) = \frac{\psi}{L + u}\ (x+u) \tag{A2.24}$$

APPENDIX 2

$$\lambda_2(x) = \frac{\psi x}{L + u} \quad . \tag{A2.25}$$

On substituting into equation (A2.16) we have

$$\phi = \frac{\frac{a}{s}\,\theta_2^*(L+u)^2 + \{\theta_1^*(L+u) - \theta_1(0)u\}\,L}{L^2 + \frac{a}{s}(L+u)^2} \quad . \tag{A2.26}$$

Appendix 3.

Flow charts for optimisation algorithms.

The flow charts shown in Figures (A3.1) - (A3.4) are included as just three examples of the many flow charts used in constructing the computer programs.

List of symbols used.

Program symbol	Textual symbol	Meaning
C	$k_1=k_2$	transfer rate coefficient
EØ	-	initial step length
E	-	step length
F	ϕ	control vector
FF	ϕ	control vector at the end of each iteration
G(M)	J	cost functional
HH	H	approximate inverse Hessian matrix
L,M	-	loop counts
LF,MF	-	maximum loop sizes
T1,T2	θ_1,θ_2	state variables
TT1,TT2	θ_1^*,θ_2^*	state target functions
W1,W2	λ_1,λ_2	costate variables
Z	μ	boundary costate variable = steepest ascent direction

APPENDIX 3

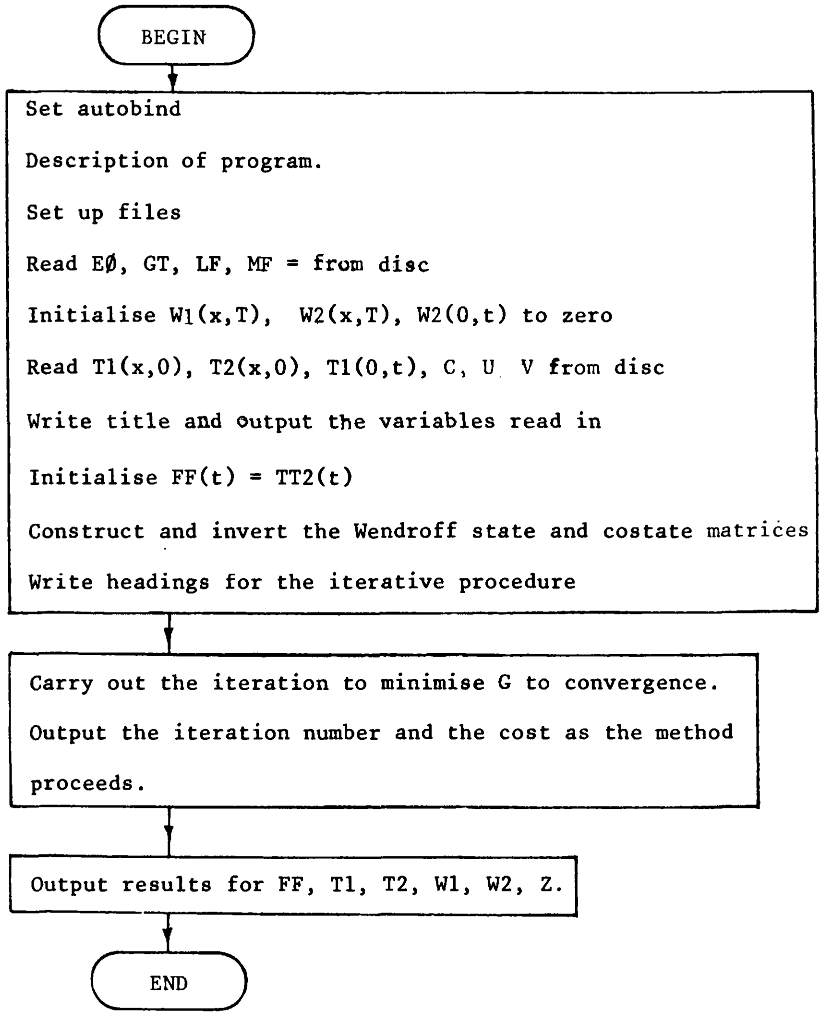


Figure A3.1 Flow chart for counterflow optimisation programs using Wendroff's method of solving the partial differential equations.

APPENDIX 3

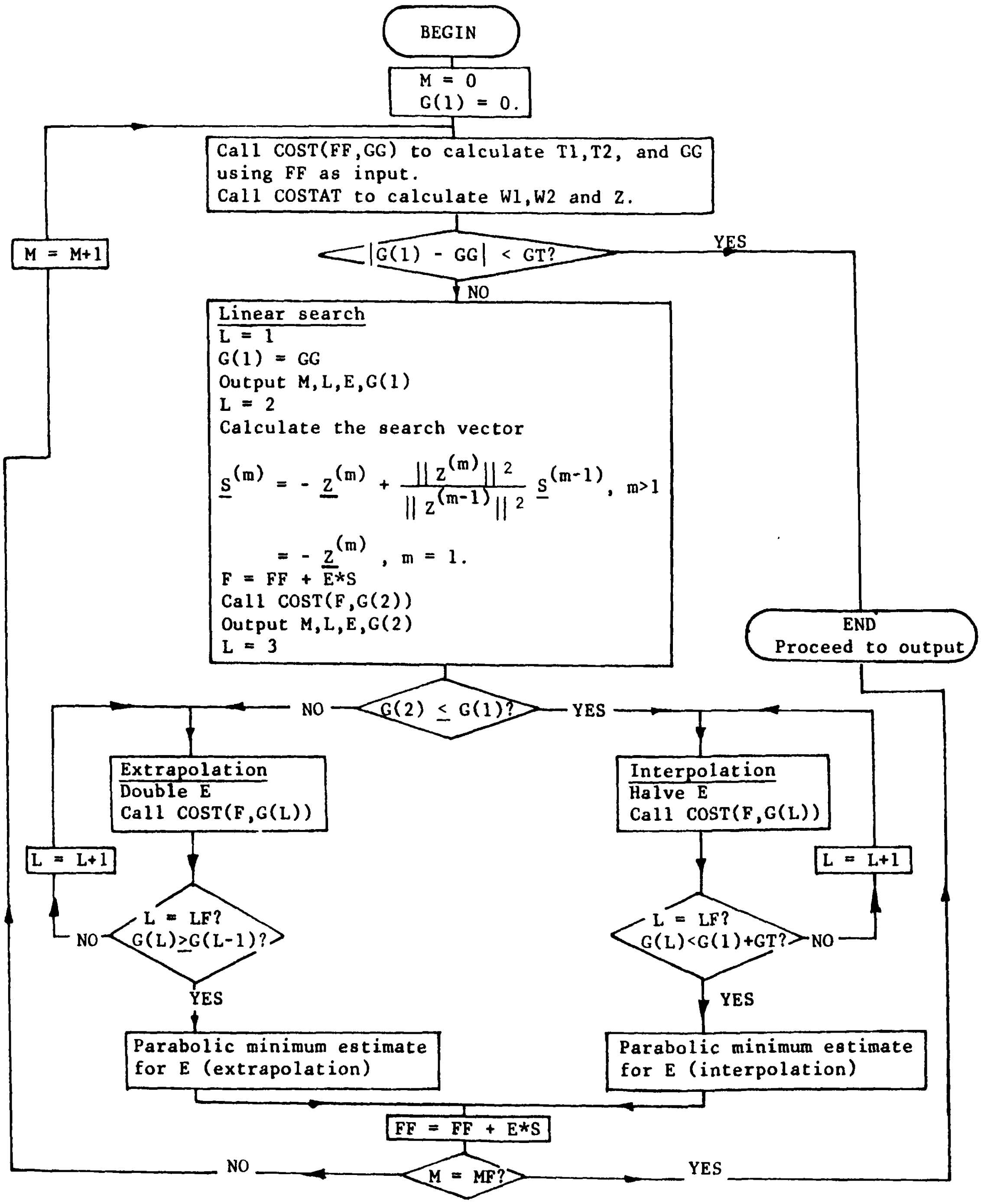


Fig. A3.2 The conjugate gradient algorithm for the optimal control of counterflow.

APPENDIX 3

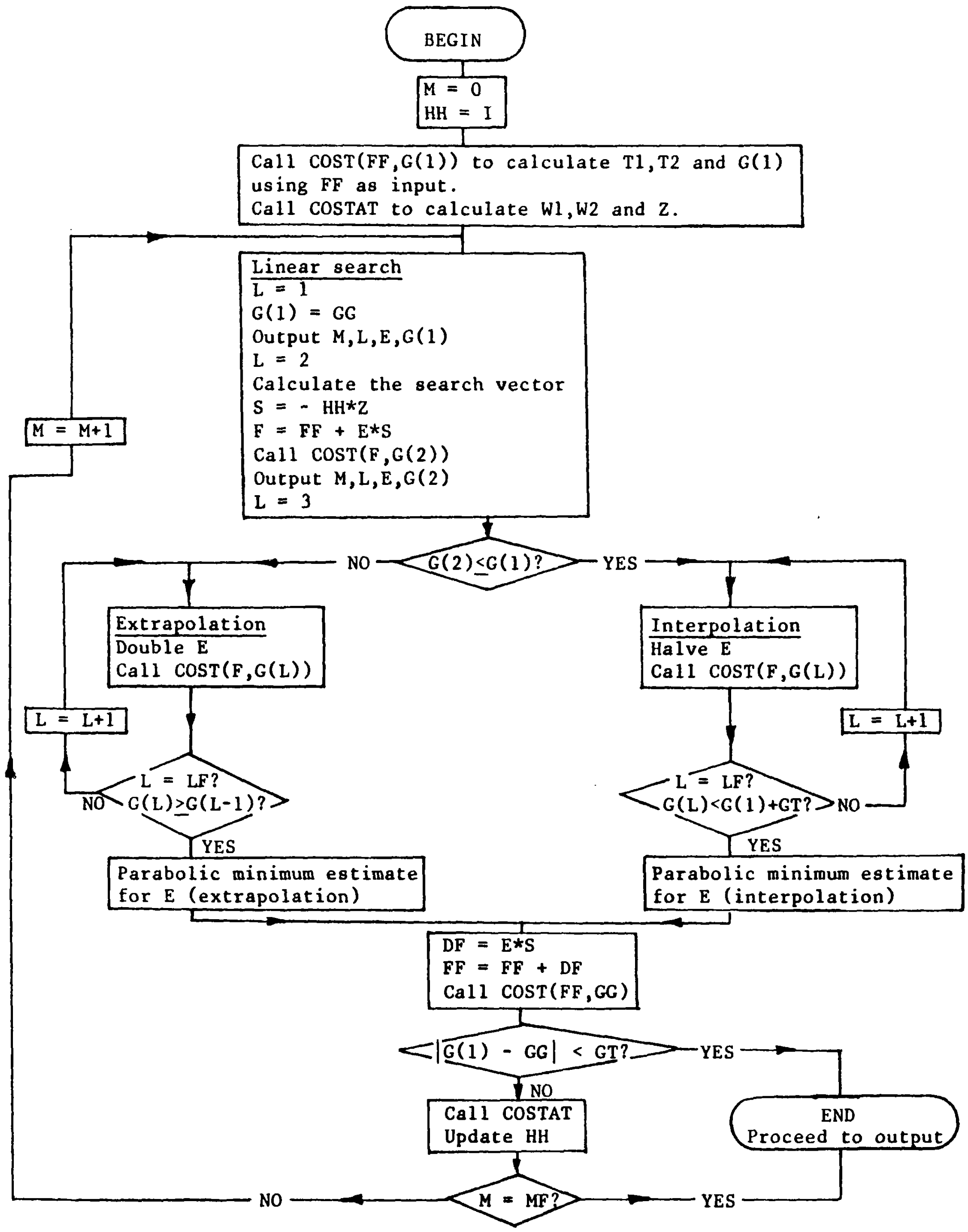


Fig. A3.3 The Davidon-Fletcher-Powell algorithm for the optimal control of counterflow.

APPENDIX 4

Equations for the implicit use of Wendroff's method.

A4.1 The implicit Wendroff method for the State equations.

$$
\left[\begin{array}{ccccc|ccccc}
x_1+1 & & & & & -1 & -1 & & & \\
y_1+1 & x_1+1 & & & & & \cdot & \cdot & & \\
 & \cdot & \cdot & & & & & \cdot & \cdot & \\
 & & \cdot & \cdot & & & & & \cdot & \cdot \\
 & & & \cdot & \cdot & & & & \cdot & -1 \\
 & & & y_1+1 & x_1+1 & & & & & -1 \\
\hline
-1 & & & & & y_2+1 & x_2+1 & & & \\
-1 & -1 & & & & & \cdot & \cdot & & \\
 & \cdot & \cdot & & & & & \cdot & \cdot & \\
 & & \cdot & \cdot & & & & & \cdot & \cdot \\
 & & & \cdot & \cdot & & & & \cdot & x_2+1 \\
 & & & -1 & -1 & & & & & y_2+1
\end{array}\right]
\left[\begin{array}{c}
\theta_1(h,(n+1)k) \\
\theta_1(2h,(n+1)k) \\
\vdots \\
\theta_1(1,(n+1)k) \\
\hline
\theta_2(0,(n+1)k) \\
\vdots \\
\theta_2((D-2)h,(n+1)k)
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
(x_1-1)\theta_1(0,nk)+(y_1-1)\theta_1(h,nk)+\theta_2(0,nk)+\theta_2(h,nk)-(y_1+1)\theta_1(0,(n+1)k) \\
(x_1-1)\theta_1(h,nk)+(y_1-1)\theta_1(2h,nk)+\theta_2(h,nk)+\theta_2(2h,nk) \\
\vdots \\
(x_1-1)\theta_1(1-h,nk)+(y_1-1)\theta_1(1,nk)+\theta_2(1-h,nk)+\theta_2(1,nk)+\theta_2(1,(n+1)k) \\
\hline
(x_2-1)\theta_2(0,nk)+(y_2-1)\theta_2(h,nk)+\theta_1(0,nk)+\theta_1(h,nk)+\theta_1(0,(n+1)k) \\
(x_2-1)\theta_2(h,nk)+(y_2-1)\theta_2(2h,nk)+\theta_1(h,nk)+\theta_1(2h,nk) \\
\vdots \\
(x_2-1)\theta_2(1-h,nk)+(y_2-1)\theta_2(1,nk)+\theta_1(1-h,nk)+\theta_1(1,nk)-(x_2+1)\theta_2(1,(n+1)k)
\end{array}\right]
$$

where $x_1 = \frac{1+pu}{1+ck}$, $x_2 = \frac{1-pv}{1+ck}$, $y_1 = \frac{1-pu}{1+ck}$, $y_2 = \frac{1+pv}{1+ck}$, $p = \frac{k}{h}$, c is the heat transfer coefficient and k is the time step.

APPENDIX 4

A4.2 The implicit Wendroff method for the co-state equations.

$$
\left[\begin{array}{cccc|cccc}
x_1+1 & y_1+1 & & & -1 & & & \\
 & \ddots & \ddots & & -1 & -1 & & \\
 & & \ddots & \ddots & & \ddots & \ddots & \\
 & & & y_1+1 & & & \ddots & \ddots \\
 & & & x_1+1 & & & -1 & -1 \\
\hline
-1 & -1 & & & y_2+1 & & & \\
 & -1 & \ddots & & x_2+1 & y_2+1 & & \\
 & & \ddots & \ddots & & \ddots & \ddots & \\
 & & -1 & -1 & & & \ddots & \ddots \\
 & & & -1 & & & x_2+1 & y_2+1
\end{array}\right]
\left[\begin{array}{c}
\lambda_1(0,(n-1)k) \\
\lambda_1(h,(n-1)k) \\
\vdots \\
\lambda_1(1-h,(n-1)k) \\
\hline
\lambda_2(h,(n-1)k) \\
\vdots \\
\lambda_2(1,(n-1)k)
\end{array}\right]
$$

$$
= \left[\begin{array}{l}
(y_1-1)\lambda_1(0,nk)+(x_1-1)\lambda_1(h,nk)+\lambda_2(0,nk)+\lambda_2(h,nk)+\lambda_2(0,(n-1)k) \\
(y_1-1)\lambda_1(h,nk)+(x_1-1)\lambda_1(2h,nk)+\lambda_2(h,nk)+\lambda_2(2h,nk) \\
\vdots \\
(y_1-1)\lambda_1(1-h,nk)+(x_1-1)\lambda_1(1,nk)+\lambda_2(1-h,nk)+\lambda_2(1,nk)-(y_1+1)\lambda_1(1,(n-1)k) \\
\hline
(y_2-1)\lambda_2(0,nk)+(x_2-1)\lambda_2(h,nk)+\lambda_1(0,nk)+\lambda_2(h,nk)-(x_2+1)\lambda_2(0,(n-1)k) \\
(y_2-1)\lambda_2(h,nk)+(x_2-1)\lambda_2(2h,nk)+\lambda_1(h,nk)+\lambda_1(2h,nk) \\
\vdots \\
(y_2-1)\lambda_2(1-h,nk)+(x_2-1)\lambda_2(1,nk)+\lambda_1(1-h,nk)+\lambda_1(1,nk)+\lambda_1(1,(n-1)k)
\end{array}\right]
$$

Appendix 5.

Computer Programs.

Two computer programs are included as typical examples of the programs written in the course of the work.

A5.1 TOSD - Transport equation optimisation using the steepest descent algorithm .

A5.2 CWCG - Counterflow optimisation using the conjugate gradient gradient algorithm and Wendroff's method of solution.

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Abstract.

Distributed Parameter Theory in Optimal Control.

by M.J. Gregson, B.A., M.Sc.

The main result of this work is the solution of open loop optimal control problems for counterflow diffusion processes, which occur very widely in chemical and mechanical engineering. In these processes two fluids pass each other moving in opposite directions separated by a membrane which is permeable to heat or a chemical solute. The membrane may also take the form of a liquid-gas interface. Subject to certain simplifying assumptions, the equations describing such processes are

$$C_1\left[\frac{\partial\theta_1}{\partial t} + u(t)\frac{\partial\theta_2}{\partial x}\right] = k(h\theta_2 - \theta_1)$$
$$C_2\left[\frac{\partial\theta_2}{\partial t} - v(t)\frac{\partial\theta_2}{\partial x}\right] = -k(h\theta_2 - \theta_1) \quad . \tag{1}$$

$\theta_1(x,t)$, $\theta_2(x,t)$ are the temperatures, or concentrations of solute, of the two fluids and $u(t)$, $v(t)$ are time dependent flow rates. k is a transfer coefficient which is assumed constant, and C_1, C_2 are thermal or solute capacities of the fluids per unit length of tube. h is an equilibrium constant; h = 1 for heat transfer. Possible controls are the inlet temperature or concentration of one stream and the flow rates, while possible objectives are the regulation of the outlet temperature or concentration of the other stream , or the maximisation of heat or solute transfer.

Subsidiary results are the optimal control of simpler but related hyperbolic systems. One of these is the restricted counterflow problem in which the controlling stream is assumed to be so massive that it is unaffected by giving up heat or solute to the controlled stream, i.e. the system is described by the equations :

$$\frac{\partial\theta_1}{\partial t} + u(t)\frac{\partial\theta_1}{\partial x} = k(h\theta_2 - \theta_1)$$
$$\frac{\partial\theta_2}{\partial t} - v(t)\frac{\partial\theta_2}{\partial x} = 0 \quad . \tag{2}$$

Another is the furnace equation

$$\frac{\partial\phi}{\partial t} + u(x,t)\frac{\partial\phi}{\partial x} = k(w(x,t) - \phi(x,t)) \tag{3}$$

ABSTRACT

in which u and w are possible controls.

Different classes of problem arise according to whether the multiplicative controls u and v are subject to rigid constraints (frequently leading to "bang-bang" controls), or whether they are constants, functions of x and t, or functions of t only.

Variational methods based on the maximum principle of A.I. Egorov are employed. Analytic solutions and numerical solutions using finite differences are obtained to the various problems. The simplifying assumptions made are probably too severe for many of the results to be directly applicable to industry. However the qualitative features of the optimal control of these processes are explained, and it is not too difficult to build more complex models.