THE EFFECTS OF HEAT TRANSFER ON OCEAN/ATMOSPHERE GENERAL CIRCULATION MODELS

by

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A thesis submitted to the
UNIVERSITY OF LEICESTER
for the degree of
DOCTOR OF PHILOSOPHY
in the Faculty of Science
1979

ACKNOWLEDGMENTS

The author wishes to thank Professor T. V. Davies for the opportunity to work in the Department of Mathematics, University of Leicester and for his unfailing help and encouragement throughout the whole of this work.

The receipt of a maintenance grant from the Natural Environmental Research Council is gratefully acknowledged.

Thanks must also go to Dr. J. C. Moore, Mr. R. J. Mobbs and Miss A. Bagnall.

ABSTRACT

We investigate in three problems some effects of heat transfer in linked ocean/atmosphere models. In all the problems the term involving vertical thermal conduction is retained in the heat transfer equation and both molecular and eddy values for the conductivity are considered.

In Part 1 we look at a two layer model, ignoring all macroscopic motion; the governing equation for both layers is therefore the heat transfer equation. With suitable boundary conditions the 'phase lag' between a heat source in the upper layer and the temperature at the inteface of the layers (the sea surface) is studied.

In Part 2 we consider a one layer model. A perturbation model due to Blinova is extended to include the heat transfer equation. One boundary condition introduces a time dependent heat source at the bottom of the layer, simulating a heating at the sea surface. The stream function is obtained at the bottom of the layer.

Finally, in Part 3, the stability of a two layer liquid model is examined. Macroscopic motion in the lower layer is ignored. The perturbation equations for the two layers are solved and homogeneous boundary equations yield an equation of consistency for the system which leads to criteria for stability. These criteria are found using difference methods and, following Meksyn we produce first order correction terms to Eady's well known stability results. Using Meksyn's methods once more, the model is extended to include a variable coriolis parameter and a stability equation is found.

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CHAPTER 0.1

GENERAL INTRODUCTION

This thesis presents three mathematical models dealing with the responses of the ocean/atmosphere system to differential heating. It is useful therefore to describe first the main heating features of the two media.

THE OCEAN AND ATMOSPHERE

Over half the solar radiation reaching the surface of the Earth is absorbed by the oceans (1). The oceans are comparatively opaque to radiation of all wave lengths and all but one percent of the incoming radiation is absorbed in the first 100m of water (1). Since heated water is less dense than colder water the resulting situation is stable and very little convection occurs. The deep waters of the oceans experience little variation in temperature and these slow changes are due to deep water currents (2). The surface waters which respond to the radiation are called the thermocline. The thermocline can be thought of as a thermal reservoir, capable of storing large amounts of heat and in general the sea surface is warmer than the atmosphere above (1). Also, since the oceans have a large specific heat we can see that the oceans are sluggish in their temperature responses and must act so as to moderate climatic change.

In contrast to the oceans, the absorption spectrum of the atmosphere is complex since the atmosphere is composed of many gases and particles. The most important features of the spectrum are the absorption of short wave radiation directly from the sun by ozone (3) and the absorption of long wave radiation emitted from the Earth by water vapour and to a lesser extent by carbon dioxide, dust and clouds (3,4). The troposphere is well mixed by eddies or turbulence (5); this process is important for an understanding of the temperature profile in the lower atmosphere. In addition the specific heat of the atmosphere is lower than that of the oceans and so the atmosphere responds more quickly to climatic change.

We will now introduce some particular observations concerning the oceans and atmosphere which we have tried to simulate into our models.

TIME LAGS BETWEEN TEMPERATURE RESPONSES IN THE OCEAN AND ATMOSPHERE

We have already mentioned how the oceans are much more sluggish than the atmosphere to temperature change. This phenomenon is seen in the diurnal and seasonal temperatures. The diurnal maximum temperature of the oceans occurs three hours (one eighth the period of one day) behind the maximum temperature of the atmosphere (6,2). The seasonal maximum temperature of the ocean is approximately three months (one quarter the period of one year) behind the maximum temperature of the atmosphere (6,2); in the northern hemisphere these temperatures occur in mid-September and mid-June respectively.

SEA SURFACE TEMPERATURE ANOMALIES

Anomalies of sea surface temperatures over large areas, significantly warmer or cooler than average values lasting for many years in the Pacific Ocean, have been investigated by Namias (7). The pattern for the decade 1948-1957 was characterised in the winter by anomalously warm water in the North Central Pacific and cold water off the west coast of the United States. During the spring of 1957 the pattern began to change and by the following winter the warm and cold waters had become interchanged. The effect on the atmosphere was to force a change in the position of the jet stream. The warm winters in the east and cold winters in the west of the United States reversed to cold winters in the east and warm winters in the west. By 1972 the pattern had reversed again and the present climate of 1979 in the United States is that of the 1950's.

Similar anomalies have been observed in the Atlantic (1) with warm or cold anomalous waters off Newfoundland. These pools of warmer or colder water effect the position of the Gulf Stream and the climate in North Europe.

AIMS OF THE PRESENT WORK

In our models we have incorporated some of the properties of the oceans and atmosphere discussed above in order to reproduce the phenomena observed.

In all three models we retain the thermal conductivity term in the heat transfer equation because thermal conductivity is one of the principal agents of heat transfer. Another important agent of heat transfer in the troposphere and thermocline is turbulence which mixes these layers and this process is modelled here by increasing the value of the molecular thermal conductivity by an appropriate amount to what is then called the 'eddy' conductivity (5). In the three models studied here, consideration is given to both values for the thermal conductivity and we assume that the 'eddy' values are 10^5 times as large as the molecular values.

We have seen that the thermocline is a 'thermal reservoir' and as the atmosphere absorbs long wave radiation from the Earth the surface waters of the oceans can in one sense be thought of as a heat source. In a similar way the presence of water vapour in the atmosphere can be modelled as a heat source. The changes of the amount of radiation due to variations of the Earth's orbit and changes in the amount of water vapour can both be modelled as a time dependent heat source in the atmosphere.

In Part 1 we neglect all macroscopic motion and assume that conduction alone is the agent for heat transfer. We introduce a heat source in the atmosphere simulating the presence of water vapour in a two layer model and study the 'phase lag' between the heat source and the temperature at the interface of the two layers (the sea surface).

In Part 2 we study the response of the atmosphere to the 'thermal reservoir' created in the thermocline. We represent this situation as a one layer model simulating the atmosphere along with a heat source at the base of the layer.

In Part 3 our goal was more ambitious. From the work of Namias it could be inferred that two stable regimes can exist in the Pacific corresponding to the two patterns of the decades 1948-1957 and 1958-1971. A shallow water non viscous theory model of Davies (8) had been able to reproduce two distinct regions of stability. His model consisted of two layers of liquid in a circular cylinder rotating at a steady angular velocity, W. The densities and angular velocilities relative to the axes, (r, θ, z) of the upper and lower layers are ρ', ρ, Ω' and Ω respectively. A small disturbance was introduced to the flow pattern and the homogeneous boundary conditions of zero normal velocity at the boundaries of the cylinder lead to an equation of consistency. From here, criteria for stability were found and are presented in Figs. 1-3. The three diagrams are for the different regions $0<\mu^2\nu^2<1$, $1<\mu^2\nu^2<4$ and $\mu^2\nu^2>4$ where the non dimensional parameter $\mu\nu$ is defined as

$$\mu\nu = \frac{4W^2}{k^2 g[(H_0-h_0)h_0]^{1/2}}$$

where k is the wave number and H_0 and h_0 are the undisturbed heights of the upper free surface and the interface of the two layers on the axis r = 0.

We aim to improve the model of Davies in the following ways: the two layer model is taken on a rotating sphere, the heat transfer equation is introduced and heat and temperature continuity are included in the boundary conditions. The homogeneous boundary conditions again lead to an equation of consistency from which criteria for stability are found.

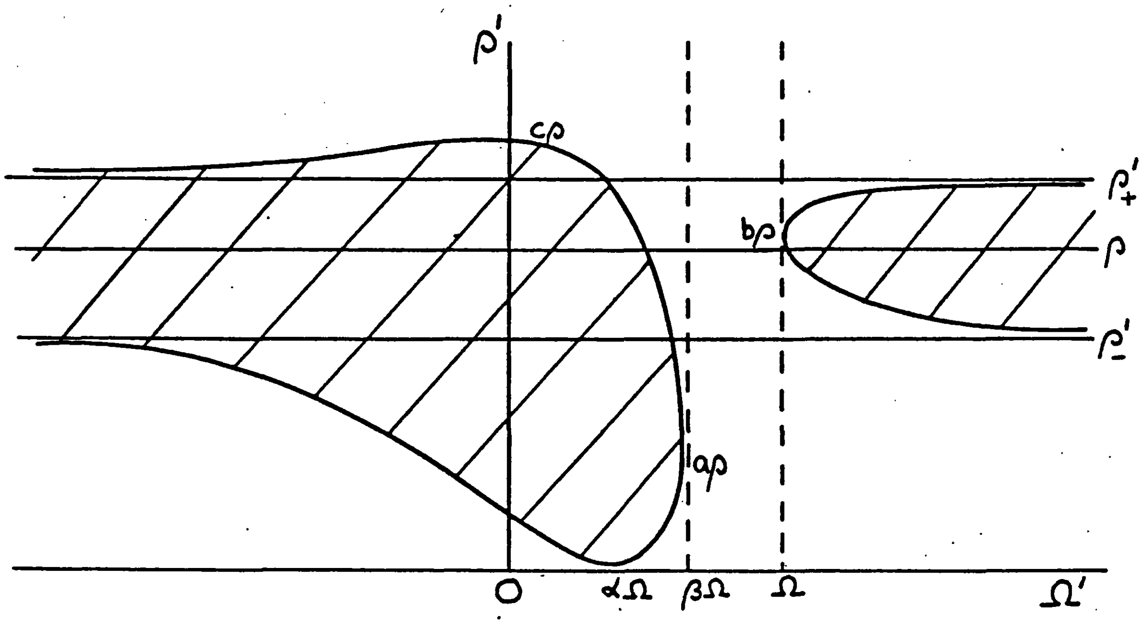

Figure 1: Regions of stability for $0 < \mu^2\nu^2 < 1$

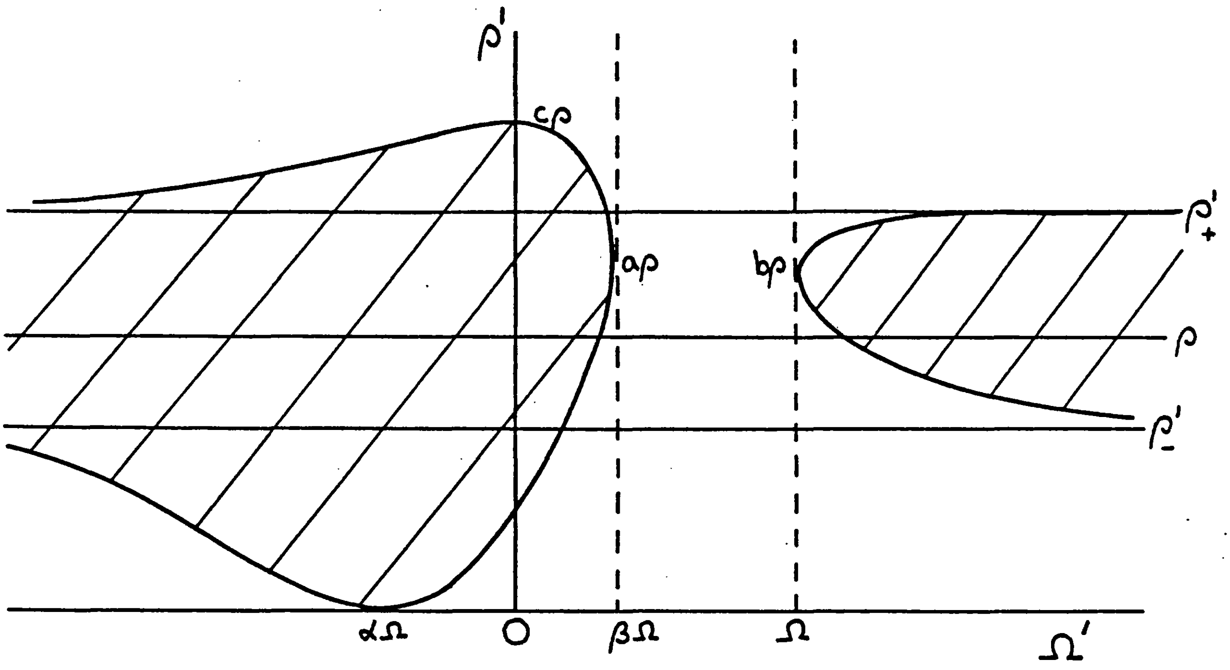

Figure 2: Regions of stability for $1 < \mu^2\nu^2 < 4$

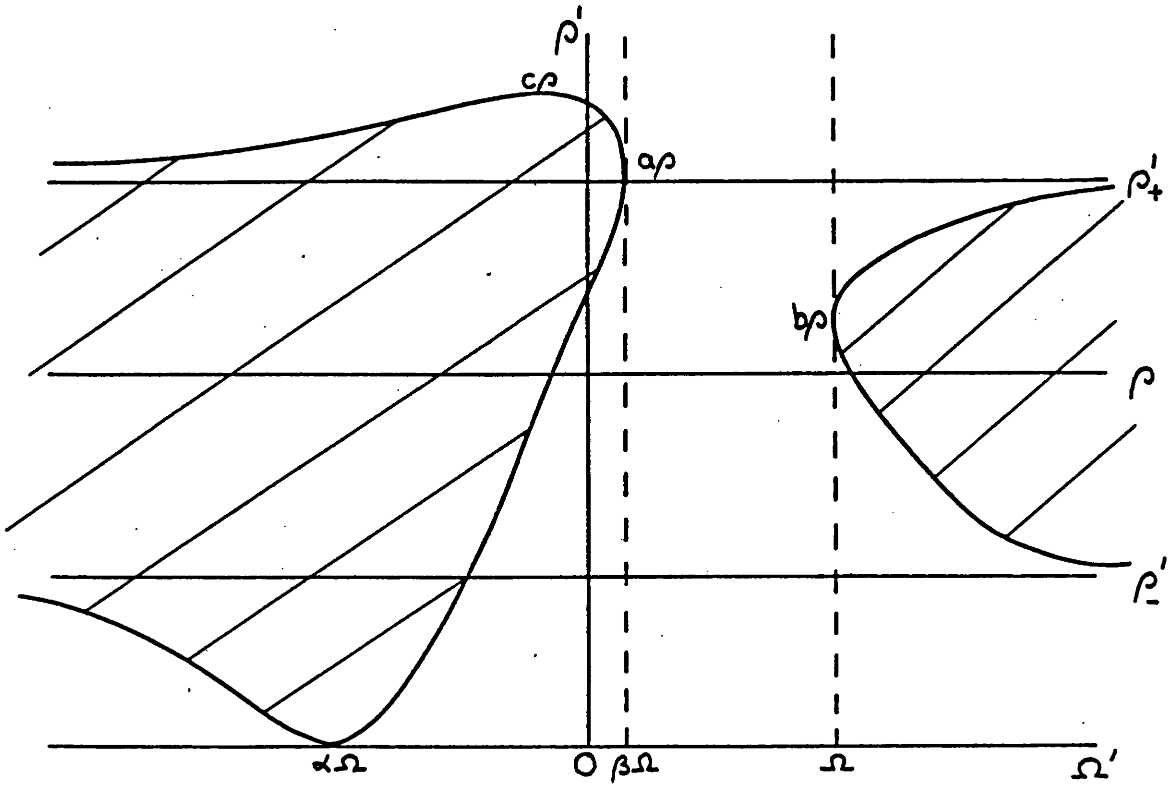

Figure 3: Regions of stability for $\mu^2\nu^2 > 4$

In all the figures the shaded areas are the stability regions and a, b, c, α, β, ρ'_+ and ρ'_- are defined as

$$a = \nu^2(1+\mu^2)\ , \quad b = 1+\mu^2\ , \quad c = (1+\nu^2)(1+\mu^2)$$

$$\alpha = (1-\mu^2\nu^2)/(1+\nu^2)\ , \quad \beta = 1/(1+\nu^2+\nu^2\mu^2)\ ,$$

$$\rho'_+ = \rho\left[(1+\delta^2)^{1/2}+\delta\right]\ , \quad \rho'_- = \rho\left[(1+\delta^2)^{1/2}-\delta\right]\ , \quad \delta = \nu\mu/(1+\nu^2)^{1/2}$$

PART 1

THE PRODUCTION OF HEATING LAGS DUE TO THE OCEAN IN A LINKED OCEAN/ATMOSPHERE MODEL

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CHAPTER 1.1

INTRODUCTION

In Part I we will study a two layer model in which the lower layer is assumed to simulate the ocean and the upper layer a fluid atmosphere. The macroscopic motion of both layers is ignored and heat flow is assumed to take place entirely by the process of thermal conduction. We investigate the phase lage between a heat source in the upper layer and the temperature at the boundary of the two layers i.e. the sea surface temperature.

The heat transfer equations for the two layers produce second order ordinary differential equations for the temperature distributions. Solutions are found subject to four boundary conditions at the boundaries of the layers. The phase lag for the sea surface temperature is then found for heat sources of different periods in the upper layer. The variation of the phase lag due to different values for thermal conductivity is also investigated

CHAPTER 1.2

FORMULATION OF THE MODEL

We choose spherical polar co-ordinates (θ, λ, r) where θ and λ are the angles of colatitude and longitude respectively and r is the distance from the centre of the Earth. The radius of the Earth is taken to be a. Since this model is static it is the heat transfer equation which governs both the upper and lower layers. This is discussed in detail in the following sections.

THE HEAT TRANSFER EQUATION FOR THE UPPER LAYER

We shall take the general form of the heat transfer equation (9) for the upper layer to be

$$\frac{dQ}{dt} = \frac{1}{\rho_1} \operatorname{div} (k_1 \operatorname{grad} T) + q^* \tag{1.1}$$

where Q is the heat content per unit mass, ρ_1 the density, T the temperature and k_1 the thermal conductivity of the upper layer. The quantity q^* is any external heating source and, for example, could arise from the absorption of solar or terrestrial radiation by water vapour. As solar radiation varies diurnally, seasonally and over longer periods due to variations in the Earth's orbit, q^* can be represented by a variable periodic function. The operator $\frac{d}{dt}$ represents differentiation following the motion but in this static model will reduce to $\frac{\partial}{\partial t}$, differentiation with respect to time.

In general for a fluid(10)we have

$$\delta Q = c_v \delta T - \frac{p}{\rho_1^2} \delta \rho_1 \tag{1.2}$$

where p is the pressure and c_v the specific heat at constant volume for the upper layer. We can replace (1.1) by

$$c_v \frac{\partial T}{\partial t} - \frac{p}{\rho_1^2} \frac{\partial \rho_1}{\partial t} = \frac{k_1}{\rho_1} \nabla^2 T + q^* \tag{1.3}$$

assuming that k_1 is constant. In a class of heat transfer problems, for example the Benard convection problem (9) , it has been found that the contribution of the $\frac{\partial \rho}{\partial t}$ term is very much smaller than the $\frac{\partial T}{\partial t}$ term and the conductivity term. We shall assume here also that the $\frac{\partial \rho_1}{\partial t}$ term is negligable. The heat transfer equation can therefore be approximated to

$$c_v \frac{\partial T}{\partial t} = \frac{k_1}{\rho_1} \nabla^2 T + q^* \tag{1.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \operatorname{cosec}^2\theta \frac{\partial^2}{\partial \lambda^2}\right].$$

A general heating function can always be expressed, using Fourier theory (11), in the terms of spherical harmonics. Thus we will take q^* and T to consist of linear combination of terms,

$$\left.\begin{aligned} q^*(\theta,\lambda,r,t) &= \sum_m \sum_n q^*_{mn}(\theta,\lambda,r,t) \;, \\ T(\theta,\lambda,r,t) &= \sum_m \sum_n T_{mn}(\theta,\lambda,r,t) \;; \end{aligned}\right\} \tag{1.5}$$

the functions q^*_{mn} and T_{mn} will be of the form

$$\left.\begin{aligned} q^*_{mn}(\theta,\lambda,r,t) &= Re\left\{Q_1(r)\, P_n^m(\cos\theta)\cos m\lambda\, e^{iqt}\right\}, \\ T_{mn}(\theta,\lambda,r,t) &= Re\left\{T_1(r)\, P_n^m(\cos\theta)\cos m\lambda\, e^{iqt}\right\}, \end{aligned}\right\} \tag{1.6}$$

where m and n are positive integers or zero. Note that $Q_1(r)$ and $T_1(r)$ will be dependent on m and n. We substitute (1.5) and (1.6) into (1.4) to obtain a second order differential equation for $T_1(r)$, namely

$$\frac{iq\rho_1 c_v}{k_1} T_1(r) = \frac{d^2T_1(r)}{dr^2} + \frac{2}{r}\frac{dT_1(r)}{dr} - \frac{n(n+1)}{r^2} T_1(r) + \frac{\rho_1}{k_1} Q_1(r). \tag{1.7}$$

THE HEAT TRANSFER EQUATION FOR THE LOWER LAYER

The heat transfer equation for the lower layer in the absence of a heating function is

$$\frac{dQ^*}{dt} = \frac{1}{\rho_2} \operatorname{div}(k_2 \operatorname{grad} T^*)$$

where Q^* is the heat per unit mass, ρ_2 is the density, T^* the temperature and k_2 the thermal conductivity of the lower layer. For a liquid (10)

$$\delta Q^* = c_2 \, \delta T^*$$

where c_2 is the specific heat of the ocean. Assuming k_2 is constant the heat transfer equation reduces to

$$c_2 \frac{\partial T^*}{\partial t} = \frac{k_2}{\rho_2} \nabla^2 T^*. \tag{1.8}$$

The ocean occupies the region $a - h_2 < r < a$ where the ocean depth $h_2 << a$. If we define a new variable z as $z = r - a, \ 0 < z < h_2$, we can approximate ∇^2 to

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{a^2}\left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \operatorname{cosec}^2\theta \frac{\partial^2}{\partial \lambda^2} \right].$$

For consistency between the two layers, we will take T* to consist of a linear combination of terms,

$$T^*(\theta, \lambda, z, t) = \sum_m \sum_n T^*_{mn}(\theta, \lambda, z, t) \tag{1.9}$$

where T^*_{mn} will take the form

$$T^*_{mn}(\theta, \lambda, z, t) = Re\left\{ T_2(z) P_n^m(\cos\theta) \cos m\lambda \, e^{iqt} \right\}. \tag{1.10}$$

The function $T_2(z)$ is therefore dependent on m and n. We substitute (1.10) into (1.8) to obtain a differential equation for $T_2(z)$,

$$\frac{d^2 T_2(z)}{dz^2} - \frac{n(n+1)}{a^2} T_2(z) = \frac{iq \rho_2 c_2}{k_2} T_2(z). \tag{1.11}$$

THE BOUNDARY CONDITIONS

Equations (1.7) and (1.11) are second order differential equations for $T_1(r)$ and $T_2(z)$. We will now chose four boundary conditions as follows:

(i) The Sommerfield radiation condition (12), which in this model will imply that for large r, heat is radiating away from the Earth, since there is a finite heat source in the atmosphere. (1.12)

(ii) Continututy of temperature at the ocean surface,

$$T_1(r)\Big|_{r=a} = T_2(z)\Big|_{z=0} \quad , \tag{1.13}$$

(iii) Continuity of heat transfer at the ocean surface,

$$k_1 \frac{dT_1(r)}{dr}\Big|_{r=a} = k_2 \frac{dT_2(z)}{dz}\Big|_{z=0} \quad , \tag{1.14}$$

(iv) No heat transfer at the bottom of the ocean,

$$\frac{dT_2(z)}{dz}\Big|_{z=0} = 0 \quad . \tag{1.15}$$

We now have a complete system of equations where T_1 and T_2 must satisfy (1.7) and (1.11) respectively subject to (1.12) - (1.15).

VALUES FOR CONSTANTS IN THIS MODEL

We present average values for the constants (13, 14, 15) appearing in this work. We will use C.G.S. units throughout.

$a = 6 \times 10^8$ cms

LOWER LAYER	UPPER LAYER
$h_2 = 5 \times 10^5$ cms	
$c_2 = 4.22$ J/gm °c	$c_v = 0.7$ J/gm °c
$\rho_2 = 0.1$ gm/cms³	$\rho_1 = 1.28 \times 10^{-3}$ gm/cms³

(1.16)

We will use two values for both thermal conductivities (5, 13). The smaller values are the molecular conductivities and the larger values are the 'eddy' conductivities. The eddy values for the conductivities may be more realistic ones in the simulation of the atmosphere and ocean.

J/cms °c s	MOLECULAR	EDDY
k_1	2.4×10^{-4}	24.1
k_2	5.61×10^{-3}	5.61×10^{2}

(1.17)

CHAPTER 1.3

A SIMPLIFIED PROBLEM, NEGLECTING INERTIA

Before proceeding to solve (1.7) and (1.11) subject to (1.12) to (1.15), we shall first consider the simplified problem obtained by neglecting the time depedent term on the left hand side of (1.17) (this term being referred to as the inertial term). This approximation is justified if q^* varies sufficiently slowly with time. The two equations to be solved for $T_1(r)$ and $T_2(z)$ are:

$$\left.\begin{aligned} &\frac{d^2T_1}{dr^2} + \frac{2}{r}\frac{dT_1}{dr} - \frac{n(n+1)}{r^2}T_1 = -\frac{\rho_1(r)}{k_1}Q_1(r), \quad a \leq r < \infty, \\ &\frac{d^2T_2}{dz^2} - \left[\frac{n(n+1)}{a^2} + \frac{iqc_2\rho_2}{k_2}\right]T_2 = 0 \quad, 0 \leq z \leq h_2 . \end{aligned}\right\} \tag{1.18}$$

SOLUTIONS FOR $T_1(r)$ AND $T_2(z)$

We can find the general solution for $T_1(r)$ by using the method of variation of parameters (16). The solution of

$$\frac{d^2T_1}{dr^2} + \frac{2}{r}\frac{dT_1}{dr} - \frac{n(n+1)}{r^2}T_1 = 0$$

is

$$T_1 = Ar^n + Br^{-n-1}$$

where A and B are arbitary constants. We will look for a particular solution for $T_1(r)$ in the form

$$T_1 = \alpha(r)r^n + \beta(r)r^{-n-1} \tag{1.19}$$

where $\alpha(r)$ and $\beta(r)$ are at our disposal. We take

$$r^n\frac{d\alpha(r)}{dr} + r^{-n-1}\frac{d\beta(r)}{dr} = 0 \tag{1.20}$$

so that (1.16) reduces to

$$n r^{n-1} \frac{d\alpha(r)}{dr} - (n+1) r^{-n-1} \frac{d\beta(r)}{dr} = \frac{-r\rho_1}{k_1} Q_1 \ . \tag{1.21}$$

Thus (1.18) and (1.19) define the functions $\alpha(r)$ and $\beta(r)$ and the general solution for T_1 takes the form

$$T_1(r) = \frac{-r^n}{(2n+1)k_1} \int_\infty^r \frac{\rho_1(\eta) Q_1(\eta)}{\eta^{n-1}} \, d\eta + \frac{r^{-n-1}}{(2n+1)k_1} \int_\infty^r \rho_1(\eta) Q_1(\eta) \, \eta^{n+2} \, d\eta$$
$$+ A r^n + B r^{-n-1} \ . \tag{1.22}$$

The function $T_1(r)$ must satisfy the Sommerfeld boundary condition (1.12), which we will take in the form

$$\lim_{r\to\infty} r^2 \frac{dT_1(r)}{dr} \longrightarrow \text{constant.} \tag{1.23}$$

Thus, $T_1(r)$ defined in (1.22) will satisfy (1.23) if $A = 0$ and $Q_1(r)$ is chosen appropriately. For convenience we will take $B = B_1 a^{n+1}$ so that $T_1(r)$ reduces to

$$T_1(r) = \frac{-r^n}{(2n+1)k_1} \int_\infty^r \rho_1(\eta) Q_1(\eta) \, \eta^{-n+1} \, d\eta + \frac{r^{-n-1}}{(2n+1)k_1} \int_\infty^r \rho_1(\eta) Q_1(\eta) \, \eta^{n+2} \, d\eta$$
$$+ B_1 \left(\frac{a}{r}\right)^{n+1} . \tag{1.24}$$

The solution for $T_2(z)$ satisfying (1.16) is

$$T_2(z) = \lambda_1 \exp\left\{ (\alpha+i\beta) z \right\} + \lambda_2 \exp\left\{ -(\alpha+i\beta) z \right\}$$

where λ_1 and λ_2 are arbitary constants and

$$(\alpha+i\beta)^2 = \frac{n(n+1)}{a^2} + \frac{iq\,\rho_2 c_2}{k_2} \ . \tag{1.25}$$

However, for our puposes, we shall find it more convenient tc write the solution of T_2 which satisfies the boundary condition (1.15) in the form

$$T_2(z) = \frac{B_2 \cosh\{(\alpha+i\beta)(z+h_2)\}}{\cosh\{(\alpha+i\beta)h_2\}}$$

(1.26)

where $B_2 = T_2(0)$, is an arbitary constant.

The conditions (1.13) and (1.14) yield the following two relations between B_1 and B_2:

$$B_2 = B_1 - \frac{a^n}{(2n+1)k_1}\int_\infty^a \rho_1 Q_1 \eta^{-n+1} d\eta + \frac{a^{-n-1}}{(2n+1)k_1}\int_\infty^a \rho_1 Q_1 \eta^{n+2} d\eta ,$$

(1.27)

$$B_2 k_2 (\alpha+i\beta)\tanh[(\alpha+i\beta)h_2] = -\frac{(n+1)}{a}k_1 B_1 - \frac{n a^{n-1}}{(2n+1)}\int_\infty^a \rho_1 Q_1 \eta^{-n+1} d\eta - \frac{(n+1)a^{-n-2}}{(2n+1)}\int_\infty^a \rho_1 Q_1 \eta^{n+2} d\eta .$$

(1.28)

We have assumed the integrals in (1.27) and (1.28) exist and are convergent.

We find that

$$B_1 = \frac{a^n}{(2n+1)k_1\chi}\left\{-\frac{n k_1}{a} + (\alpha+i\beta)k_2\tanh[(\alpha+i\beta)h_2]\right\}\int_\infty^a \rho_1 Q_1 \eta^{-n+1} d\eta - \frac{a^{-n-1}}{(2n+1)k_1}\int_\infty^a \eta^{n+2}\rho_1 Q_1 d\eta$$

(1.29)

and

$$B_2 = -\frac{a^{n-1}}{\chi}\int_\infty^a \rho_1(\eta) Q_1(\eta)\eta^{-n+1} d\eta$$

(1.30)

where we define χ as

$$\chi = \frac{(n+1)}{a}k_1 + k_2(\alpha+i\beta)\tanh[(\alpha+i\beta)h_2] .$$

(1.31)

THE SEA SURFACE TEMPERATURE

Of particular interest here is the sea surface temperature, $T^*(\theta,\lambda,0,t)$. The integral defined in (1.28) is real and χ is a complex quantity. It is therefore χ which produces the difference in phase between the heat source, and the sea surface temperature; if we write χ in the form

$$\chi = A_0 e^{i\theta}$$

then we have from (1.9) and (1.10)

$$T^*(\theta,\lambda,0,t) = \sum_m \sum_n \mathrm{Re}\left\{ -\frac{a^{n-1}}{A_0} e^{i(qt-\theta_0)} P_n^m(\cos\theta)\cos m\lambda \int_\infty^a \rho_1 Q_1 \eta^{-n+1} d\eta \right\}. \tag{1.32}$$

It is clear from (1.30) that there is a phase lag between $T^*(\theta,\lambda,0,t)$ and $q^*(\theta,\lambda,r,t)$ of magnitude θ_0/q. In addition, (1.29) and (1.23) reveal that χ is dependent on n but independent of m so that the longitudinal heating structure plays no part in the phase lag. Also χ is independent of $Q_1(r)$, the variation of the heat source in the vertical direction.

To evaluate θ_0 we rewrite χ in the form A+iB. It can be seen that χ is a complicated complex function and to simplify (1.29) we wish to approximate $(\alpha+i\beta)$ and $\tanh[(\alpha+i\beta)h_2]$. It is therefore necessary to look at orders of magnitude of the various terms. Using the quantities in (1.16) and (1.17) we can see that from (1.23)

$$(\alpha+i\beta)^2 h_2^2 \sim \begin{cases} 10^{-6} + iq\times 10^{13} & \text{(molecular)} \\ 10^{-6} + iq\times 10^{8} & \text{(eddy)} \end{cases}$$

where $q = \frac{2\pi}{P}$, P being the heating period in seconds. It will be more convenient to rewrite q in the form

$$q \sim 10^{6-L} \tag{1.33}$$

so that for L = 1 the heating period is one year, for L = 2 the heating period is ten years and so on. Thus

$$(\alpha+i\beta)^2 h_2^2 = \begin{cases} 10^{-6} + i\,10^{19-L} & \text{(molecular)} \\ 10^{-6} + i\,10^{14-L} & \text{(eddy)} \end{cases} \tag{1.34}$$

We shall now consider the cases of L being small and large separately.

1) <u>The case $L<6$ (molecular) and $L<1$ (eddy)</u>

In this region we can approximate (1.25) to

$$(\alpha+i\beta)^2 = \frac{iq\rho_2 c_2}{k_2}$$

so that

$$\alpha+i\beta = (1+i)\left[q\rho_2 c_2/k_2\right]^{1/2}.$$

We can approximate χ from (1.29) to

$$\chi \sim (\alpha+i\beta)k_2 \tanh\left[(\alpha+i\beta)h_2\right]$$

and since $(\alpha+i\beta)h_2$ is large it follows that

$$\chi \sim (\alpha+i\beta)k_2 = e^{i\pi/4}\left[q\rho_2 c_2/k_2\right]^{1/2}. \tag{1.35}$$

We have then $\theta_0 = \pi/4$, in other words the time lag between the heating function and the sea surface temperature is an eighth of the period of heating. It is clear that θ_0 is independent of L and n and is, therefore, the same for all heating functions with periods in this region.

2) <u>The case $L>6$ (molecular) and $L>1$ (eddy)</u>

For these larger values of L we can approximate (1.31) to

$$\chi = \frac{(n+1)}{a}k_1 + k_2 h_2 (\alpha+i\beta)^2$$

$$= \frac{(n+1)}{a}\left[k_1 + \frac{n}{a}k_2 h_2\right] + iqc_2\rho_2 h_2 . \tag{1.36}$$

It follows that θ_0 is defined as

$$\tan\theta_0 = \frac{q\rho_2 c_2 h_2 a}{(n+1)[k_1 + nk_2h_2/a]} \quad . \tag{1.37}$$

When we substitute values for the constants in (1.37) from (1.16) and (1.17) we find

$$\tan\theta_0 \sim \begin{cases} \dfrac{4\times 10^{12-L}}{(n+1)} \quad , & \text{(molecular)} \\[2ex] \dfrac{4\times 10^{7-L}}{(n+1)} \quad . & \text{(eddy)} \end{cases}$$

Thus for the ranges $6 < L \leq 12$ (molecular) and $1 < L < 7$ (eddy) we have $\theta_0 \sim \pi/2$ so that the phase lag is one quarter of the heating period. For larger values of n, θ_0 decreases from $\pi/2$ to zero. These results are represented in graph form in Fig. 1.

CHAPTER 1.4

THE GENERAL INERTIA MODEL

We will now return to the original model formulated in Chapter 1.2 which includes the time dependent term on the left hand side of (1.7). The two equations to be solved for $T_1(r)$ and $T_2(z)$ are therefore (1.7) and (1.11), namely

$$\left.\begin{aligned} r^2\frac{d^2T_1}{dr^2} + 2r\frac{dT_1}{dr} - \left[n(n+1) + iqAr^2\right]T_1 &= -A_1Q_1(r)r^2, \\ \frac{d^2T_2}{dz^2} - \left[\frac{n(n+1)}{a^2} + \frac{iq\rho_2 c_2}{k_2}\right]T_2 &= 0 , \end{aligned}\right\} \tag{1.38}$$

where $A = \frac{\rho_1 c_v}{k_1}$ and $A_1 = \frac{\rho_1}{k_1}$.

SOLUTIONS FOR $T_1(r)$ AND $T_2(z)$

We will assume that the density of the upper layer, ρ_1 is constant to simplify the solution of $T_1(r)$. We will again use the method of variation of parmeters(16)to find the general solution of $T_1(r)$. The solutions of

$$r^2\frac{d^2T_1}{dr^2} + 2r\frac{dT_1}{dr} - \left[n(n+1) + iqAr^2\right]T_1 = 0$$

are spherical Bessel functions of the third kind(17), thus $T_1(r)$ takes the form

$$T_1(r) = D_1 h_n^{(1)}(sr) + D_2 h_n^{(2)}(sr)$$

where

$$s = e^{i\pi/4}(qA)^{1/2} \tag{1.39}$$

and D_1 and D_2 are arbitary constants. The functions $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ are defined in terms of Hankel functions (Bessel functions of the third kind) and Bessel functions as follows:

$$h_n^{(1)}(z) = (\pi/2z)^{1/2} H_{n+1/2}^{(1)}(z) = (\pi/2z)^{1/2}\left[J_{n+1/2}(z) + iY_{n+1/2}(z)\right],$$

$$h_n^{(2)}(z) = (\pi/2z)^{1/2} H_{n+1/2}^{(2)}(z) = (\pi/2z)^{1/2}\left[J_{n+1/2}(z) - iY_{n+1/2}(z)\right].$$

Their series expansion takes the form

$$h_n^{(1)}(z) = i^{-n-1} z^{-1} e^{iz} \sum_0^n (n+\tfrac{1}{2}, k)(-2iz)^{-k},$$

$$h_n^{(2)}(z) = i^{n+1} z^{-1} e^{-iz} \sum_0^n (n+\tfrac{1}{2}, k)(2iz)^{-k}.$$

To find a particular solution, we assume $T_1(r)$ is of the form

$$T_1 = C_1(sr)\, h_n^{(1)}(sr) + C_2(sr)\, h_n^{(2)}(sr). \tag{1.40}$$

We choose C_1 and C_2 so that

$$\frac{dC_1}{dr} h_n^{(1)}(sr) + \frac{dC_2}{dr} h_n^{(2)}(sr) = 0. \tag{1.41}$$

Taking the form of T_1 as in (1.40) and using the relation (1.41) we substitute T_1 back into (1.38) to obtain a second equation between C_1 and C_2, namely

$$\frac{dC_1}{dr}\frac{dh_n^{(1)}(sr)}{dr} + \frac{dC_2}{dr}\frac{dh_n^{(2)}(sr)}{dr} = -A_1 Q_1(r). \tag{1.42}$$

From (1.41) and (1.42), using the Wronskian (17),

$$W\left\{ h_n^{(1)}(z), h_n^{(2)}(z) \right\} = -\frac{2i}{z^2},$$

we find that

$$\left.\begin{aligned} \frac{dC_1}{dr} &= -\frac{A_1}{2i} s r^2 Q_1(r)\, h_n^{(2)}(sr), \\ \frac{dC_2}{dr} &= \frac{A_1}{2i} s r^2 Q_1(r)\, h_n^{(1)}(sr). \end{aligned}\right\} \tag{1.43}$$

The complete solution for $T_1(r)$ is therefore:

$$\begin{aligned} T_1(r) = {} & \frac{sA_1}{2i}\int_a^r \xi^2 Q_1(\xi)\left[h_n^{(1)}(s\xi)\, h_n^{(2)}(sr) - h_n^{(1)}(sr)\, h_n^{(2)}(s\xi)\right] d\xi \\ & + D_1 h_n^{(1)}(sr) + D_2 h_n^{(2)}(sr). \end{aligned} \tag{1.44}$$

For $T_1(r)$ to satisfy (1.12), the Sommerfeld condition that for large r heat is radiating away from Earth, we look at the asymptotic expansions of $h_n^{(1)}(sr)$ and $h_n^{(2)}(sr)$ (17), namely

$$h_n^{(1)}(sr) \sim 2\pi/sr \exp\{i(sr - \tfrac{1}{2}n\pi - \pi/4)\},$$

$$h_n^{(2)}(sr) \sim 2\pi/sr \exp\{-i(sr - \tfrac{1}{2}n\pi - \pi/4)\};$$

from the definition of T_1 in (1.6) we have for large r terms of the form $\exp\{i(sr+qt)\}$ and $\exp\{i(-sr+qt)\}$. Therefore to satisfy the Sommerfeld condition we need the coefficients of the term $\exp\{i(sr+qt)\}$, i.e. $h_n^{(1)}(sr)$, to be zero as $r \to \infty$. Therefore we take

$$D_1 = \frac{sA_1}{2i}\int_a^{\infty} \xi^2 Q_1(\xi)\, h_n^{(2)}(s\xi)\, d\xi. \tag{1.45}$$

The solution for T_2 will be the same as in the non-inertia model, namely

$$T_2(z) = B_2 \frac{\cosh\{(\alpha+i\beta)(z+h_2)\}}{\cosh\{(\alpha+i\beta)h_2\}} \tag{1.46}$$

where as before

$$(\alpha+i\beta)^2 = \frac{n(n+1)}{a^2} + \frac{iq\rho_2 c_2}{k_2}.$$

The two remaining boundary conditions, (1.13) and (1.14) yield two equations between D_2 and B_2,

$$B_2 = D_1 h_n^{(1)}(sa) + D_2 h_n^{(2)}(sa) \tag{1.47}$$

and

$$\frac{k_2}{k_1}(\alpha+i\beta)\tanh[(\alpha+i\beta)h_2]\,B_2 = D_1 H_n^{(1)}(sa) + D_2 H_n^{(2)}(sa) \tag{1.48}$$

where

$$H_n^{(i)}(sa) = \left.\frac{dh_n^{(i)}(sr)}{dr}\right|_{r=a}, \quad i=1,2. \tag{1.49}$$

Since $T_2(0) = B_2$, it is the constant B_2 which is of most interest here, thus we eliminate D_2 between (1.47) and (1.48) and using (1.45) we find

$$B_2 = \frac{A_1 \int_a^\infty \xi^2 Q_1(\xi)\, h_n^{(2)}(s\xi)\, d\xi}{a^2\left[H_n^{(2)}(sa) - h_n^{(2)}(sa)\, \frac{k_2}{k_1}(\alpha+i\beta)\tanh[(\alpha+i\beta)h_2]\right]} . \tag{1.50}$$

THE SEA SURFACE TEMPERATURE

As in Chapter 1.3 we will now investigate the structure of the sea surface temperature, $T^*(\theta,\lambda,0,t)$. We will write the complex quantity B_2 in the form

$$B_2 = A_0 e^{i\theta_0}. \tag{1.51}$$

Using (1.9) and (1.10), the sea surface temperature distribution takes the form,

$$T^*(\theta,\lambda,0,t) = \sum_m \sum_n \mathrm{Re}\left\{ A_0 e^{i(qt-\theta_0)} P_n^m(\cos\theta) \cos m\lambda \right\} .$$

There is again a phase lag between $T^*(\theta,\lambda,0,t)$ and $q^*(\theta,\lambda,r,t)$, the heat source of magnitude θ_0/q .

Unlike the simple case in Chapter 1.3, the integral in (1.50)is now a complex quantity. It is therefore necessary to define $Q_1(r)$ more precisely. The heating function is dependent upon the water vapour distribution in the atmosphere. The greatest amounts of water vapour are found well below ten kilometres and fall off rapidly with height. Therefore $Q_1(r)$ should be a decreasing function with increasing height and a reasonable choice for $Q_1(r)$ is

$$Q_1(r) = Q_0 e^{-pr}. \tag{1.52}$$

We will calculate a value for p by assuming that the heating function decreases to one tenth of its surface value at a height of ten kilometres.

From (1.52) we find that

$$p \sim 2.3 \times 10^{-6}\ \mathrm{cms^{-1}}. \tag{1.53}$$

We will take the following form for the spherical Bessel function $h_n^{(2)}(sr)$:

$$h_n^{(2)}(sr) = \frac{i^{n+1}}{sr} e^{isr} \sum_{k=0}^{n} \frac{\Gamma(1+n+k)}{k!\,\Gamma(1+n-k)} (-2isr)^{-k} \tag{1.54}$$

so that

$$B_2 = \frac{A_1 Q_0 i^{n+1} \sum_{k=0}^{n} \frac{\Gamma(1+n+k)}{k!\,\Gamma(1+n-k)} \int_a^{\infty} \zeta e^{(is-p)\zeta} (-2is\zeta)^{-k} d\zeta}{a^2 s\left[H_n^{(2)}(sa) - h_n^{(2)}(sa) \frac{k_2}{k_1} (\alpha+i\beta) \tanh[(\alpha+i\beta)h_2]\right]} \tag{1.55}$$

B_2 is dependent on the parameter n and so we must choose particular values of n to find the heating lag. We have taken the values n = 0, 1, 2, 3, for relatively simple functions, $h_n^{(2)}$. The even values of n imply a heating source symmetric about the equator.

The integrals encountered for the above values of n are of the form

$$K_n = \int_a^{\infty} e^{(is-p)\zeta} \zeta^{1-n} d\zeta \ , \quad n = 0, 1, 2, 3 \ , \tag{1.56}$$

where

$$s = (1+i)(q\rho_1 c_v / 2k_1)^{1/2}.$$

When n = 0,1,the integral K_n can easily be evaluated but for n = 2, 3, K_n takes the form of an Exponential Integral (18) ,

$$K_n = E_{n-2}(a(p-is)) \ , \quad n = 2, 3 \tag{1.57}$$

where in general

$$E_n(z) = \int_1^{\infty} t^{-n} e^{-zt} dt \ , \quad n = 0, 1, 2 \cdots\cdots . \tag{1.58}$$

Substituting average values for the constants from (1.16), (1.17) and

(1.53) we find

$$a(p-is) \sim \begin{cases} 10 + 10^6(1-i)q^{1/2} & \text{(molecular)} \\ 10 + 10^4(1-i)q^{1/2} & \text{(eddy)} \end{cases} \tag{1.59}$$

Therefore for all values of q, $a(p-is)$ will be large and we can use the asymptotic expansion for $E_{n-2}(a(p-is))$, namely

$$E_n(z) \sim \frac{e^{-z}}{z}\left\{1 - \frac{n}{z} + \frac{n(n+1)}{z^2} + \cdots\cdots\right\} \tag{1.60}$$

and only the first term in the series is used. Thus the formulae for B_2 take the following form in the four cases n = o, 1, 2, and 3:

<u>n=0</u> : $h_0^{(2)}(z) = \frac{i}{z}e^{iz}$, $(\alpha+i\beta)^2 = iq\rho_2c_2/k_2$,

$$B_2 = \frac{-A_1Q_0ie^{-pa}\left[a/(is-p) - 1/(is-p)^2\right]}{s^2\left[1+sa+aW\right]}, \tag{1.61}$$

<u>n=1</u> : $h_1^{(2)}(z) = \left[-\frac{1}{z} - \frac{i}{z^2}\right]e^{iz}$, $(\alpha+i\beta)^2 = 2/a^2 + iq\rho_2c_2/k_2$,

$$B_2 = \frac{A_1Q_0e^{-pa}\left[a/(is-p) + 1/s(is-p) - 1/(is-p)^2\right]}{a^2s^3\left[-\frac{is^2}{a} + \frac{2s}{a^2} + \frac{2i}{a^3}\left(\frac{s}{a}+\frac{i}{a^2}\right)W\right]}, \tag{1.62}$$

<u>n=2</u> : $h_2^{(2)}(z) = \left[-\frac{i}{z} + \frac{3}{z^2} + \frac{3i}{z^3}\right]e^{iz}$, $(\alpha+i\beta)^2 = 6/a^2 + iq\rho_2c_2/k_2$,

$$B_2 = \frac{-A_1Q_0ie^{-pa}\left[-a + 1/(is-p) - 3i/s + 3/s^2a\right]}{(is-p)\left[i - \frac{3}{sa} - \frac{9i}{s^2a^2} + sa + 3i + \left[ia - 3/s + \frac{3i}{s^2a}\right]W\right]}, \tag{1.63}$$

<u>n=3</u> : $h_3^{(2)}(z) = \left[\frac{1}{z} + \frac{10i}{z^2} - \frac{90}{z^3} - \frac{630i}{z^4}\right]$, $(\alpha+i\beta)^2 = 12/a^2 + iq\rho_2c_2/k_2$,

$$B_2 = \frac{A_1Q_0e^{-pa}\left[1/s(is-p)^2 - 10i/s^2(is-p) - a/s(is-p) + 90/s^3a(is-p) - 630i/as^4\right]}{a^2\left[1/a - \frac{11}{sa^2} - \frac{110i}{s^2a^3} + \frac{900}{s^3a^4} + \frac{2520i}{s^4a^5} + \left[\frac{-1}{sa} - \frac{10i}{s^2a^2} + \frac{90}{s^3a^3} + \frac{630i}{s^4a^4}\right]W\right]}, \tag{1.64}$$

where

$$W = (\alpha+i\beta)\tanh\left[(\alpha+i\beta)h_2\right].$$

As in Chapter 1.3 we will introduce a variable L such that

$$q \sim 10^{6-L}$$

which implies that L = 1 corresponds to a heating period of one year, L = 2 corresponds to a heating period of ten years and so on. We approximate $\tanh[(\alpha+i\beta)h_2]$ to 1 for $L<6$ (molecular) and $L<1$ (eddy) and for larger values of L we take $\tanh[(\alpha+i\beta)h_2]$ to be $(\alpha+i\beta)h_2$. The formulae are manipulated into the form

$$B_2 = A_0 e^{-i\theta_0}$$

where θ_0 is a function of L. Values of θ_0 were computed for different values of L and the results are presented in graph form in Figs. 2 - 5. The turning points of θ_0 were checked with the analytical formulae and gave good agreement.

CHAPTER 1.5

DISCUSSION

It will be noticed that the graphs of the results, Figs. 1-5, have time scales up to L = 20, i.e. periods of 10^{19} years. The age of the Earth is approximately 5×10^{9} years and it is not supposed that the results for $L > 8$ have any physical meaning but were included for completeness only.

It is of interest to find that the result of replacing eddy conduc--tivities with molecular values in all the graphs is to displace the curve in the direction of increased L while preserving the shape completely. The formula (1.37) for θ_0 for the non-inertial model is of the form

$$\tan\theta_0 = \frac{\lambda_1 q}{k_1 + \lambda_2 k_2} = \frac{\lambda_1 10^{6-L}}{k_1 + \lambda_2 k_2}$$

where λ_1 and λ_2 are constants. By changing k_1 and k_2 from eddy to molecular values we are therefore multiplying q by 10^5 or replacing L by (L+5). The quantities k_1 and k_2 appear in the expressions of $\tan\theta_0$ for the inertia model in a similar way and so the same argument applies. Thus if we are interested in larger values for thermal conductivity it is only necessary to displace the curves still further. We will now discuss the graphs in detail and it will be sufficient to discuss only the molecular curves.

A comparison of the curve for the non-inertial model and the four curves of the inertial model reveal several similarities. For small L ($1 \leq L \leq 5$) θ_0 is approximately $\pi/4$ and for the region $6 \leq L \leq 11$, θ_0 is approximately $\pi/2$. It is for large L ($L > 11$) that we see marked differences in the graphs, where the limits of θ_0 vary from 0 to 2π.

We can discuss the physical significance of the above results as follows. For eddy values for the conductivities, in the region $0 < L < 1$ θ_0 is approximately $\pi/4$ and so for diurnal heating, the maximum temperatures for the sea surface should be an eighth of the heating period, i.e. three hours behind that of the atmosphere. This corresponds well with observed temperature distributions. For $1 < L < 7$ the period lag is a quarter of the heating period. Thus for seasonal changes the maximum temperature for the sea surface should be three months behind maximum values for the atmosphere which again corresponds well to observed values as discussed in the General Introduction. For ice ages the period is about 20,000 years corresponding to L = 4. The period lag should again be a quarter of the heating period which would be 5,000 years.

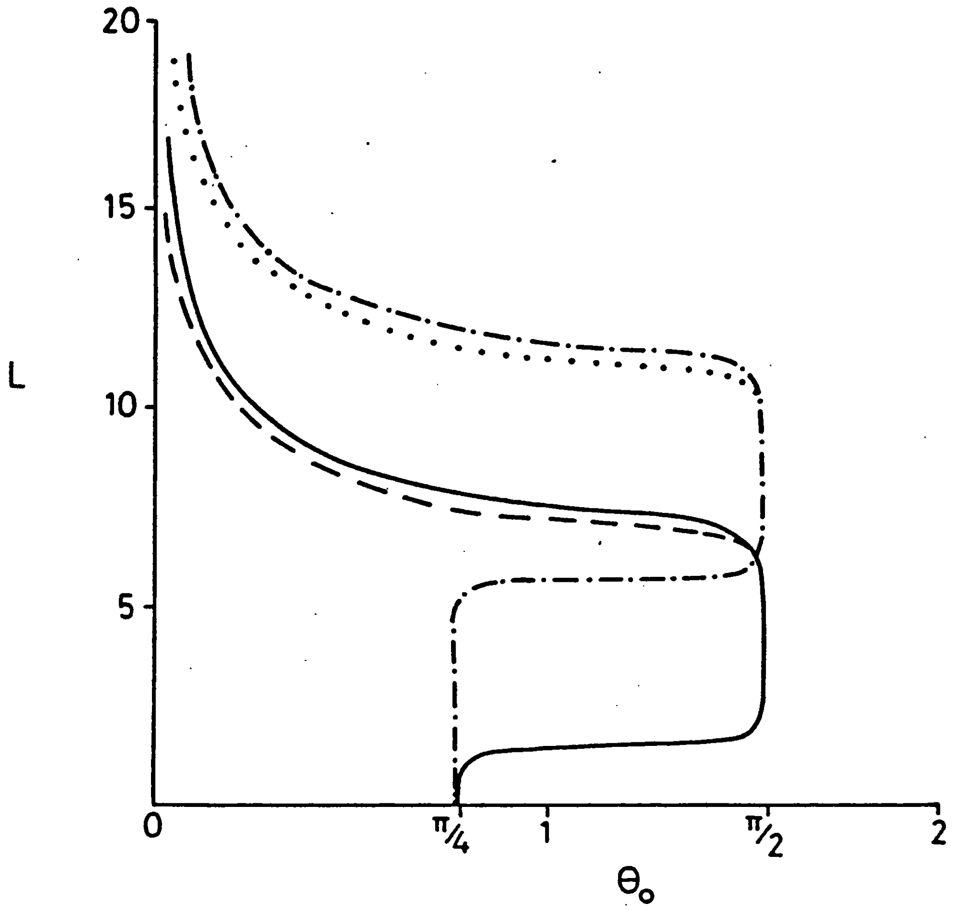
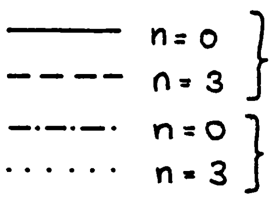


Figure 1: Variation of θ_o with L for the non-inertial model



eddy thermal conductivity

molecular thermal conductivity

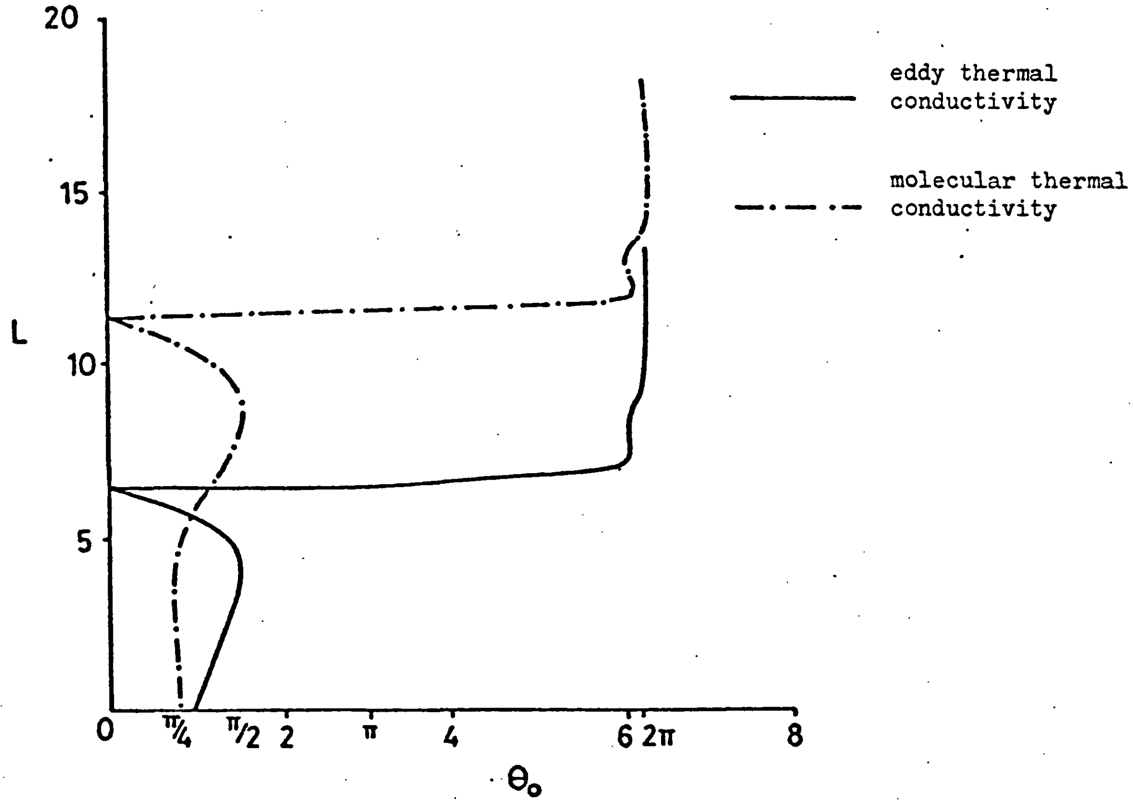


Figure 2: Variation of θ_0 with L for n = 0 in the inertial model

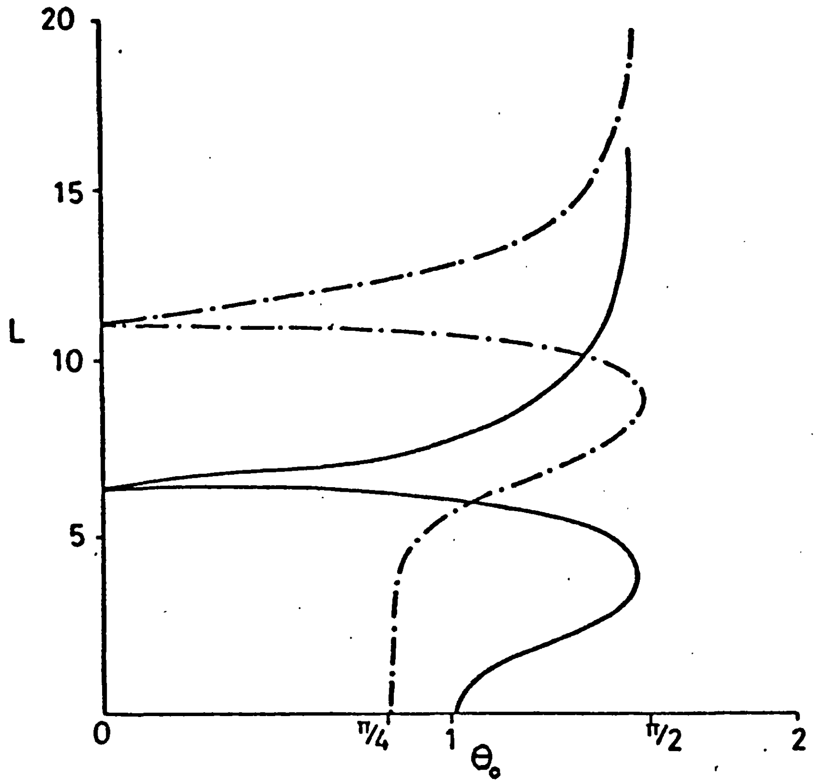


Figure 3: Variation of θ_0 with L for n = 1 in the inertial model

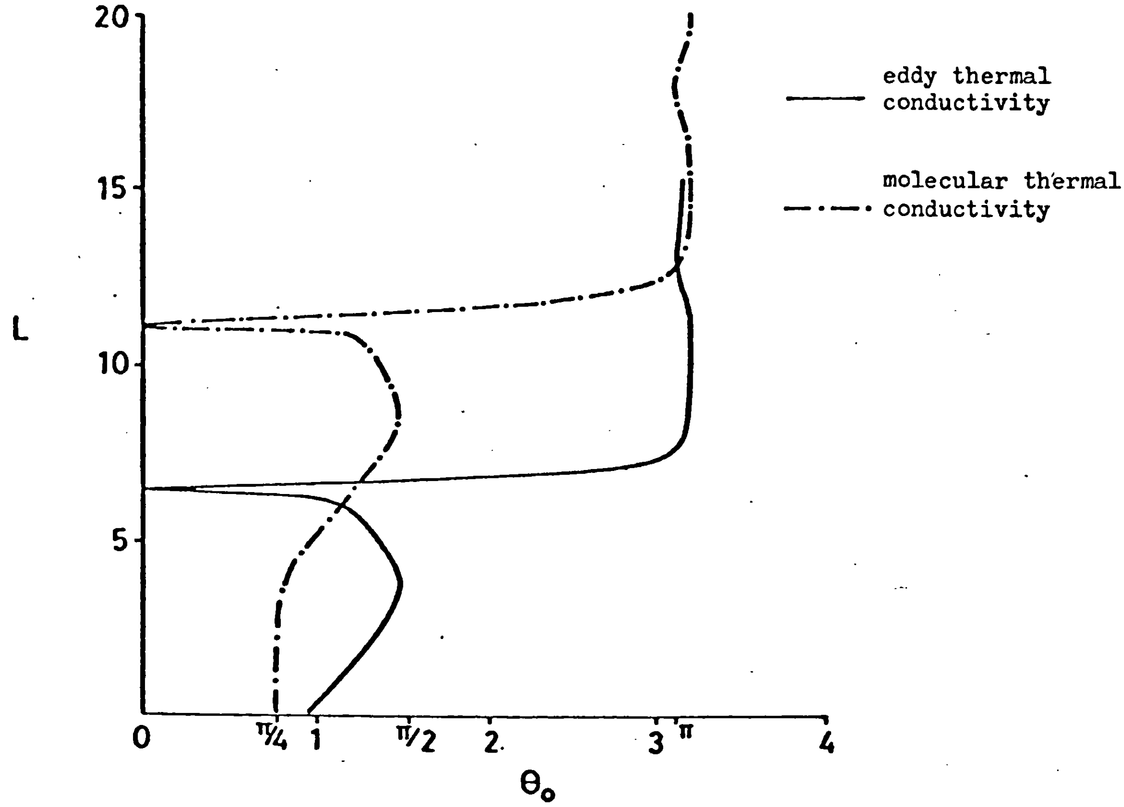


Figure 4: Variation of θ_0 with L for n = 2 in the inertial model

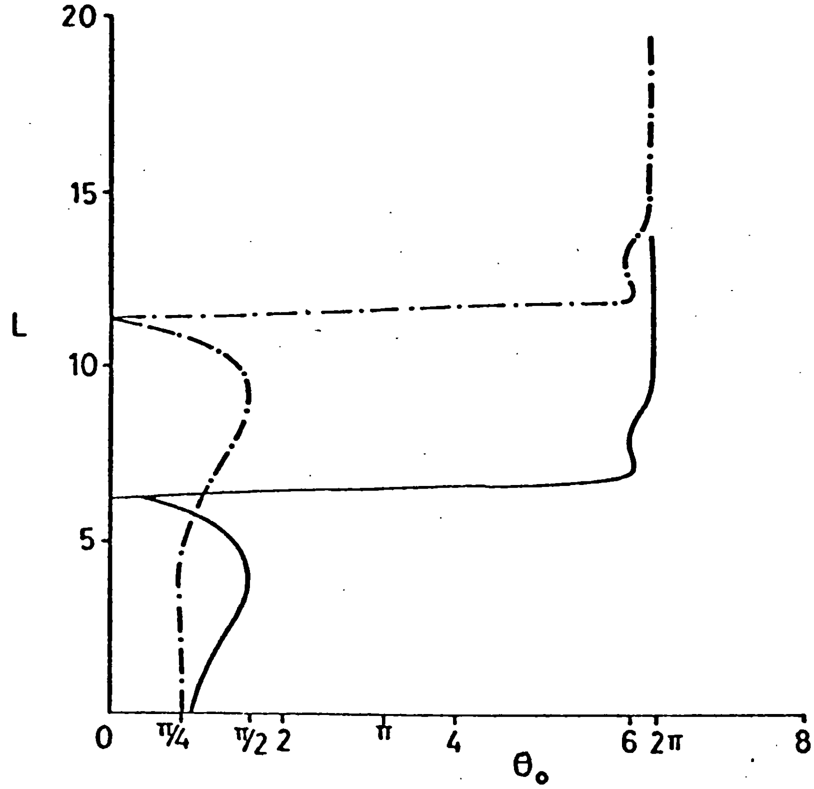


Figure 5: Variation of θ_0 with L for n = 3 in the inertial model

PART 2

THE RESPONSE OF AN ATMOSPHERE MODEL TO A TIME DEPENDENT HEAT SOURCE AT THE SEA SURFACE

PART 2

THE RESPONSE OF AN ATMOSPHERE MODEL TO A TIME DEPENDENT HEAT SOURCE AT THE SEA SURFACE

CHAPTER 2.1

INTRODUCTION

In Part 1 we looked at a static two layer model which represented the ocean and atmosphere. We introduced a heat source into the upper layer and investigated the phase lag between the sea surface temperature and the heat source. In this second model we allow motion in the upper layer, we omit the lower layer but retain a time dependent heating function at the lower boundary of the layer, which is thought of as a heat source at the sea surface.

In general our model extends a perturbation scheme due to Blinova (19) (which has no differential heating) by including the heat transfer equation in which the vertical thermal conductivity term is retained. This produces a fourth order partial differential equation for the stream function. Four boundary conditions are introduced: the vertical velocity vanishes at both boundar ies of the layer, there is no heat transfer at the top of the layer and a time dependent heat source exists at the bottom of the layer. Fourier transforms with respect to time are taken and an approximate solution for the resulting fourth order ordinary differential equation is obtained following methods of Heisenberg, Lin and others (20). Finally, the inverse Fourier transforms are found for the stream function at the lower boundary of the layer representing the sea surface.

CHAPTER 2.2

FORMULATION OF THE MODEL

We take a spherical coordinate system, (θ,λ,r) fixed on a rotating Earth with a being the radius of the Earth and t as the time dependent variable. This is a one layer model and we will first introduce the six governing equations, namely the three equations of motion, the equation of continuity, the gas equation and the heat transfer equation.

THE GOVERNING EQUATIONS

The velocity vector for the layer will be $\underline{V} = (u,v,w)$ where u, v, and w are the components along the θ, λ and r directions. We take the horizontal equations of motion for the atmosphere in the form

$$\left.\begin{aligned} \frac{du}{dt} - v(\zeta + 2\Omega\cos\theta) &= \frac{-1}{r\rho}\frac{\partial p}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) , \\ \frac{dv}{dt} + u(\zeta + 2\Omega\cos\theta) &= \frac{-1}{r\rho\sin\theta}\frac{\partial p}{\partial\lambda} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right), \end{aligned}\right\} \qquad (2.1)$$

where p is the pressure and ρ the density of the layer, Ω is the angular velocity of the Earth and $\frac{d}{dt}$ is differentiation following the motion, namely $\frac{\partial}{\partial t} + \frac{u}{r}\frac{\partial}{\partial\theta} + \frac{v}{r\sin\theta}\frac{\partial}{\partial\lambda} + \frac{w\partial}{\partial r}$. The vorticity component in the vertical direction, ζ is defined by

$$\zeta = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v\sin\theta) - \frac{\partial u}{\partial\lambda}\right]. \qquad (2.2)$$

In (2.1) we have ignored the horizontal components of the centrifugal and gravitational forces since these are small compared with the pressure terms. We have also neglected frictional terms. Since the basic mass of the atmosphere is contained in a layer whose thickness is insignificant compared with the radius of the Earth (a) we can replace r by a in the coefficients of our equations and the derivative $\frac{\partial}{\partial r}$ by $\frac{\partial}{\partial z}$, where z = r-a. The horizontal equations of motion (2.1) are replaced by

$$\frac{du}{dt} - v(\zeta + 2\Omega\cos\theta) = \frac{-1}{a\rho}\frac{\partial p}{\partial\theta} - \frac{1}{a}\frac{\partial}{\partial\theta}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) \qquad (2.3)$$

and

$$\frac{dv}{dt} + u(\zeta + 2\Omega\cos\theta) = -\frac{1}{a\sin\theta\rho}\frac{\partial p}{\partial\lambda} - \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) \tag{2.4}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a}\frac{\partial}{\partial\theta} + \frac{v}{a\sin\theta}\frac{\partial}{\partial\lambda} + w\frac{\partial}{\partial z} \tag{2.5}$$

and the vorticity, ζ will also be approximated to

$$\zeta = \frac{1}{a\sin\theta}\left[\frac{\partial}{\partial\theta}(v\sin\theta) - \frac{\partial u}{\partial\lambda}\right]. \tag{2.6}$$

The vertical equation of motion can be taken as the hydrostatic equation, namely

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{2.7}$$

Similar arguments can be used to approximate the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\underline{V}) = 0$$

to the form

$$\frac{1}{a\sin\theta}\left[\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial\lambda}\right] + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho w) = 0 \tag{2.8}$$

where it is also assumed that variations of density, ρ with time are small compared with the density itself. In addition, we have Clapeyron's law

$$p = R\rho T \tag{2.9}$$

where T is the temperature of the atmosphere.

Finally we introduce the heat transfer equation (9) which is not present in Blinova's theory. We take the equation in the form

$$\frac{dQ}{dt} = \frac{1}{\rho}\operatorname{div}(k_1 \operatorname{grad} T) \tag{2.10}$$

where Q is the heat content per unit mass and k_1 is the thermal conductivity of the layer. For a fluid (10)

$$\delta Q = c_p\,\delta T - \frac{1}{\rho}\delta p$$

where c_p is the specific heat at constant pressure of the layer. We assume that it is the vertical variation, $k_1\frac{\partial^2 T}{\partial z^2}$ which is the dominant part of the conduction term in the heat transfer equation, assuming that k_1, is constant. We can therefore approximate (2.10) to

$$c_p \frac{dT}{dt} - \frac{dp}{dt} = \frac{k_1}{\rho} \frac{\partial^2 T}{\partial z^2} \tag{2.11}$$

where $\frac{d}{dt}$ is defined in (2.5).

THE PERTURBATION EQUATIONS

The governing equations for the model are therefore (2.3), (2.4), (2.7), (2.8), (2.9) and (2.11). We will now introduce a perturbation to the model in three stages. At the first stage we shall take the standard valuesof pressure, density and temperature for a static atmosphere, i.e. $\bar{p}(z)$, $\bar{\rho}(z)$ and $\bar{T}(z)$. It follows from (2.7) and (2.9) that

$$0 = -\frac{1}{\bar{\rho}} \frac{d\bar{p}}{dz} - g \,, \tag{2.12}$$

$$\bar{p} = R\bar{\rho}\bar{T} \,. \tag{2.13}$$

It is easy to show

$$\frac{1}{\bar{p}} \frac{d\bar{p}}{dz} = \frac{-g}{R\bar{T}}$$

and therefore we have

$$\bar{p}(z) = \bar{p}(0) \exp\left[-g/R \int_0^z \frac{dz}{\bar{T}} \right] \,. \tag{2.14}$$

At the second stage we introduce a steady zonal current, from west to east, which has no longitudinal variations. We take the dependent variables in the form

$$\left.\begin{aligned} p &= \bar{p}(z) + p_0(z,\theta) \\ \rho &= \bar{\rho}(z) + \rho_0(z,\theta) \\ T &= \bar{T}(z) + T_0(z,\theta) \\ u &= 0 \\ v &= v_0(z,\theta) \\ w &= 0 \end{aligned}\right\} \tag{2.15}$$

The variables with the suffix o are of the same order of magnitude and are small compared with $\bar{p}(z)$, $\bar{\rho}(z)$ and $\bar{T}(z)$. We will now proceed to transform the expression $\frac{1}{\rho}\frac{\partial p}{\partial \theta}$;

$$\frac{1}{\rho}\frac{\partial p}{\partial \theta} = \frac{RT}{p}\frac{\partial p}{\partial \theta} = RT\frac{\partial}{\partial \theta}\left[\log(\bar{p}(z) + p_o(z,\theta))\right] = RT\frac{\partial}{\partial \theta}\left[\log \bar{p} + \log(1 + p_o/\bar{p})\right].$$

Since $\bar{p}$ is independent of θ, by expanding $\log(1 + p_o/\bar{p})$ as a power series and retaining only the first term, we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial \theta} = RT\frac{\partial}{\partial \theta}\left(\frac{p_o}{\bar{p}}\right). \tag{2.16}$$

To the same order we may write

$$\frac{1}{\rho}\frac{\partial p}{\partial \lambda} = \frac{1}{\rho}\frac{\partial}{\partial \lambda}(\bar{p}(z) + p_o(z,\theta)) = RT\frac{\partial}{\partial \lambda}\left(\frac{p_o}{\bar{p}}\right) = 0 \tag{2.17}$$

and in a similar manner,

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{gT}{\bar{T}} + RT\frac{\partial}{\partial z}\left(\frac{p_o}{\bar{p}}\right). \tag{2.18}$$

We can now write the equations of motion, (2.3) and (2.7) in the form

$$-V_o(\zeta + 2\Omega\cos\theta) = -\frac{R\bar{T}}{a}\frac{\partial}{\partial \theta}\left(\frac{p_o}{\bar{p}}\right) - \frac{1}{a}\frac{\partial}{\partial \theta}\left(\frac{V_o^2}{2}\right), \tag{2.19}$$

$$0 = -g\frac{T_o}{\bar{T}} + R\bar{T}\frac{\partial}{\partial z}\left(\frac{p_o}{\bar{p}}\right), \tag{2.20}$$

where, in the small terms $\frac{\partial}{\partial \theta}(p_o/\bar{p})$ and $\frac{\partial}{\partial z}(p_o/\bar{p})$, T has been approximated to $\bar{T}(z)$. We will simplify (2.19) to the geostrophic approximation (21)

$$2\Omega\cos\theta\, V_o = \frac{R\bar{T}}{a}\frac{\partial}{\partial \theta}\left(\frac{p_o}{\bar{p}}\right). \tag{2.21}$$

We will follow Blinova and postulate that the zonal flow $V_o(z,\theta)$ is of the form

$$V_o(z,\theta) = a\,\alpha(z)\sin\theta \tag{2.22}$$

where $\alpha(z)$ is an increasing function of z which produces angular velocities which increase with increasing elevation. Thus from (2.21) we have

$$\frac{\partial}{\partial\theta}\left(\frac{p_0}{\bar{p}}\right) = \frac{2\Omega a^2\alpha(z)}{R\bar{T}(z)} \sin\theta\cos\theta$$

so that

$$p_0/\bar{p} = \frac{\Omega a^2\alpha(z)}{R\bar{T}(z)} \sin^2\theta + f(z)$$

where $f(z)$ is a general function of z. We can therefore write

$$p_0(z,\theta) = \Omega a^2\alpha(z)\bar{\rho}(z)\sin^2\theta + p_0(z,0) \ . \tag{2.23}$$

It should be noted that the variation of p_0 at the poles, $p_0(z,0)$ is assumed to be a known quantity. When $\alpha(z)>0$, the pressure $p_0(z,\theta)$ increases from the poles to the equator. Using (2.19) and (2.22) we can show that

$$T_0(z,\theta) = \bar{T}^2\frac{\Omega a^2}{g}\frac{d}{dz}\left[\frac{\alpha(z)}{\bar{T}(z)}\right]\sin^2\theta + \frac{R\bar{T}^2}{g}\frac{d}{dz}\left[\frac{p_0(z,0)}{\bar{p}(z)}\right] \ . \tag{2.24}$$

At the third and final stage we introduce a perturbation which can be written as

$$\left.\begin{aligned}
p &= \bar{p}(z) + p_0(z,\theta) + p'(z,\theta,\lambda,t)\\
\rho &= \bar{\rho}(z) + \rho_0(z,\theta) + \rho'(z,\theta,\lambda,t)\\
T &= \bar{T}(z) + T_0(z,\theta) + T'(z,\theta,\lambda,t)\\
u &= u'(z,\theta,\lambda,t)\\
v &= V_0(z,\theta) + v'(z,\theta,\lambda,t)\\
w &= w'(z,\theta,\lambda,t)
\end{aligned}\right\} \tag{2.25}$$

It is assumed that the first column of variables is dominant and the second and third columns are of decreasing importance in numerical magnitude. The two equations of motion (2.1) and (2.2) are

$$\frac{du}{dt} - v(2\Omega\cos\theta + \zeta) = -\frac{RT}{a}\frac{\partial}{\partial\theta}(\log p) - \frac{1}{a}\frac{\partial}{\partial\theta}(\tfrac{1}{2}q^2), \tag{2.26}$$

$$\frac{dv}{dt} + u(2\Omega\cos\theta + \zeta) = -\frac{RT}{a}\frac{\partial}{\partial\lambda}(\log p) - \frac{1}{a}\frac{\partial}{\partial\lambda}(\tfrac{1}{2}q^2), \tag{2.27}$$

where $q^2 = \underline{V}\cdot\underline{V}$. From (2.26) and (2.27) we deduce that

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\theta}(v\sin\theta) - \frac{\partial u}{\partial\lambda}\right] + u\sin\theta\frac{\partial}{\partial\theta}(2\Omega\cos\theta + \zeta)$$
$$+ v\frac{\partial}{\partial\lambda}(2\Omega\cos\theta + \zeta) + (2\Omega\cos\theta + \zeta)\left[\frac{\partial}{\partial\theta}(u\sin\theta + \frac{\partial v}{\partial\lambda}\right]$$
$$= -\frac{R}{a}\left\{\frac{\partial T}{\partial\theta}\frac{\partial}{\partial\lambda}(\log p) - \frac{\partial T}{\partial\lambda}\frac{\partial}{\partial\theta}(\log p)\right\}.$$

Using the definition of ζ in (2.3) and (2.8) we have

$$\frac{\partial\zeta}{\partial t} + \frac{u}{a}\frac{\partial}{\partial\theta}(2\Omega\cos\theta + \zeta) + \frac{v}{a\sin\theta}\frac{\partial\zeta}{\partial\lambda} - (2\Omega\cos\theta + \zeta)\frac{1}{\rho}\frac{\partial}{\partial z}(\rho w)$$
$$= \frac{-R}{a^2\sin\theta}\left[\frac{\partial T}{\partial\theta}\frac{\partial}{\partial\lambda}(\log p) - \frac{\partial T}{\partial\lambda}\frac{\partial}{\partial\theta}(\log p)\right]. \tag{2.28}$$

The terms in this equation are of similar orders of magnitude and so it is relatively simple to form the linearised version of (2.28). We ignore ζ compared with $2\Omega\cos\theta$ (22) in the term $(2\Omega\cos\theta + \zeta)$ and the linear form of (2.28) is therefore

$$\frac{\partial\zeta'}{\partial t} - \frac{2\Omega\sin\theta\, u'}{a} + \frac{v_0}{a\sin\theta}\frac{\partial\zeta'}{\partial\lambda} - 2\Omega\cos\theta\frac{1}{\bar\rho}\frac{\partial}{\partial z}(\bar\rho w')$$
$$= \frac{-R}{a^2\sin\theta}\left[\frac{\partial T_0}{\partial\theta}\frac{\partial}{\partial\lambda}(\log p) - \frac{\partial T'}{\partial\lambda}\frac{\partial}{\partial\theta}(\log p)\right] \tag{2.29}$$

where $\zeta' = \frac{1}{a\sin\theta}\left[\frac{\partial}{\partial\theta}(v'\sin\theta) - \frac{\partial u'}{\partial\lambda}\right]$. It can easily be shown that, as in (2.16)

$$\frac{\partial}{\partial\lambda}\log p = \frac{\partial}{\partial\lambda}\left(\frac{p'}{\bar p}\right). \tag{2.30}$$

Using (2.30), (2.23) and (2.24) the right hand side of (2.29) can be simplified and we obtain

$$\frac{\partial \zeta'}{\partial t} - \frac{2\Omega}{a}\sin\theta\, u' + \alpha(z)\frac{\partial \zeta'}{\partial \lambda} - \frac{2\Omega\cos\theta}{\bar{\rho}}\frac{\partial}{\partial z}(\bar{\rho}w')$$

$$= -2\Omega\cos\theta\left[\frac{\bar{T}}{g\bar{\rho}}\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)\frac{\partial p'}{\partial \lambda} - \frac{\alpha}{\bar{T}}\frac{\partial T'}{\partial \lambda}\right]. \tag{2.31}$$

Using the power series method described preceeding (2.16) we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = R[\bar{T}+T_0+T'][-\frac{g\bar{\rho}}{\bar{p}} + \frac{\partial}{\partial z}\left(\frac{p_0}{\bar{p}}\right) + \frac{\partial}{\partial z}\left(\frac{p'}{\bar{p}}\right)]$$

and retaining the third order terms, from the hydrostatic equation we obtain

$$\frac{\partial}{\partial z}\left(\frac{p'}{\bar{p}}\right) = \frac{g}{R\bar{T}^2}T'. \tag{2.32}$$

We will now linearise the heat transfer equation (2.11), and it is easily shown that from the third order terms we find

$$\frac{dT}{dt} = \left[\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial \lambda}\right]T' + w'\frac{d\bar{T}}{dz} + \frac{u'}{a}\frac{\partial T_0}{\partial \theta}$$

and using (2.24), the expression for $\frac{dT}{dt}$ becomes

$$\frac{dT}{dt} = \left[\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial \lambda}\right]T' + w'\frac{d\bar{T}}{dz} - \frac{\bar{T}^2}{g\bar{\rho}}\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)\frac{\partial p'}{\partial \lambda}. \tag{2.33}$$

We also have

$$\frac{dP}{dt} = \frac{\partial p'}{\partial t} - w'\frac{d\bar{p}}{dz} \tag{2.34}$$

and thus the heat transfer equation takes the form

$$\bar{\rho}c_p\left\{\left[\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial \lambda}\right]T' + w'\frac{d\bar{T}}{dz} - \frac{\bar{T}^2}{g\bar{\rho}}\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)\frac{\partial p'}{\partial \lambda}\right\} - \frac{\partial p'}{\partial t} + w'\frac{d\bar{p}}{dz} = k_1\frac{d^2T'}{dz^2}. \tag{2.35}$$

We assume that (2.27) can be replaced by the geostrophic equation and using (2.30) reduces to

$$2\Omega u'\sin\theta\cos\theta = -\frac{R\bar{T}}{a\bar{p}}\frac{\partial p'}{\partial \lambda}. \tag{2.36}$$

The dominant terms in the equation of continuity, (2.8) are the two horizontal terms and so we can make the approximation

$$\frac{\partial}{\partial\theta}(u'\sin\theta) + \frac{\partial v'}{\partial\lambda} = 0 . \tag{2.37}$$

We can therefore introduce a stream function, ψ such that

$$u'\sin\theta = -\frac{1}{a}\frac{\partial\psi}{\partial\lambda} , \tag{2.38}$$

$$v' = \frac{1}{a}\frac{\partial\psi}{\partial\theta} . \tag{2.39}$$

With $\zeta' = 1/(a\sin\theta)\left[\frac{\partial}{\partial\theta}(v'\sin\theta) - \frac{\partial u'}{\partial\lambda}\right]$, we find

$$\zeta' = \frac{1}{a^2}\left[\frac{\partial^2\psi}{\partial\theta^2} + \cot\theta\frac{\partial\psi}{\partial\theta} + \operatorname{cosec}^2\theta\frac{\partial^2\psi}{\partial\lambda^2}\right] = \frac{1}{a^2}\nabla^2\psi . \tag{2.40}$$

If we combine (2.36) and (2.38) we deduce that

$$\cos\theta\frac{\partial\psi}{\partial\lambda} = \frac{1}{2\Omega\bar{\rho}}\frac{\partial p'}{\partial\lambda}$$

and there is no loss of generality in writing

$$\psi\cos\theta = \frac{1}{2\Omega\bar{\rho}} p' . \tag{2.41}$$

We substitute for u', ζ' and p' in (2.31) to obtain

$$\frac{1}{a^2}\left[\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial\lambda}\right]\nabla^2\psi + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} + \frac{4\Omega^2\cos^2\theta\,\bar{T}}{g}\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)\frac{\partial\psi}{\partial\lambda} - 2\Omega\cos\theta\frac{\alpha}{\bar{T}}\frac{\partial T'}{\partial\lambda} = \frac{2\Omega\cos\theta}{\bar{\rho}}\frac{\partial}{\partial z}(w\bar{\rho}) . \tag{2.42}$$

We will look at the ratio of the coefficients of $\frac{\partial\psi}{\partial\lambda}$ in the second and third terms on the left hand side of (2.42). The ratio is $(2\Omega\cos^2\theta\,\bar{T}a^2)\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)/g$ which can be approximated to $\Omega a^2\alpha/gh$ for $\theta = 60^{\circ}$. Taking average values, a = 6 x 10 m, h = 16 x 10^3m (height of troposphere), $a\alpha = 10\text{ms}^{-1}$ and g = 9.8ms^{-2} (14, 15) we find $\Omega a^2\alpha/gh \sim 1/40$ and therefore we can ignore the third term on the left hand side of (2.42), and the simplified form is

$$\frac{1}{a^2}\left[\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial\lambda}\right]\nabla^2\psi + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} - 2\Omega\cos\theta\frac{\alpha}{\bar{T}}\frac{\partial T'}{\partial\lambda} = \frac{2\Omega\cos\theta}{\bar{\rho}}\frac{\partial}{\partial z}(w\bar{\rho}) . \tag{2.43}$$

Substituting for $T' = \frac{2\Omega}{g}\cos\theta\,\bar{T}^2\frac{\partial}{\partial z}(\psi/\bar{T})$ into (2.43) and (2.37) we have

$$\left.\begin{aligned}&\frac{1}{a^2}\left(\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial\lambda}\right)\nabla^2\psi+\frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}-\frac{l^2\alpha\bar{T}}{g}\frac{\partial^2}{\partial\lambda\partial z}\left(\frac{\psi}{\bar{T}}\right)=\frac{l}{\bar{\rho}}\frac{\partial}{\partial z}(w'\bar{\rho}),\\&\frac{k_1}{c_p}\frac{\partial^2}{\partial z^2}\left(\bar{T}^2\frac{\partial}{\partial z}\left(\frac{\psi}{\bar{T}}\right)\right)-\bar{\rho}\bar{T}^2\left[\left(\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial z}\left(\frac{\psi}{\bar{T}}\right)-\frac{d}{dz}\left(\frac{\alpha}{\bar{T}}\right)\frac{\partial\psi}{\partial\lambda}\right]-\frac{g}{c_p}\bar{\rho}\frac{\partial\psi}{\partial t}=\frac{Ag}{l}\bar{\rho}w';\end{aligned}\right\} \tag{2.44}$$

where $l = 2\Omega\cos\theta$ and $A = \frac{d\bar{T}}{dz}+g/c_p$ (23), is associated with stability of the layer. Here l and A are both assumed to be constants. Therefore (2.44) yields two equations between ψ and w' for a prescribed $\bar{T}(z)$.

Blinova has only the first equation of (2.44) since the second expression is from the heat transfer equation. Blinova proceeds by integrating (2.32) from z = 0 to z = ∞ to obtain an average for ψ (z, θ, λ, t).

By including the heat transfer equation we have a closed system of equations and we proceed by eliminating w' between the two equations of (2.44). We shall take ψ and w' as a sum of linear terms

$$\left.\begin{aligned}\psi(z,\theta,\lambda,t)&=\sum_m\sum_n\psi_{mn}(z,\theta,\lambda,t),\\ w'(z,\theta,\lambda,t)&=\sum_m\sum_n w'_{mn}(z,\theta,\lambda,t),\end{aligned}\right\} \tag{2.45}$$

where ψ_{mn} and w'_{mn} are spherical harmonics of the form,

$$\begin{aligned}\psi_{mn}(z,\theta,\lambda,t)&=\mathrm{Re}\left\{\phi(z,t)P_n^m(\cos\theta)e^{im\lambda}\right\},\\ w'_{mn}(z,\theta,\lambda,t)&=\mathrm{Re}\left\{W(z,t)P_n^m(\cos\theta)e^{im\lambda}\right\},\end{aligned} \tag{2.46}$$

in which m and n are zero or positive integers. Note that $\phi(z, t)$ and $W(z, t)$ depend on m and n. We now take $\bar{T}(z)$ to be a constant, T_1. Thus T' will take the form

$$T'(z,\theta,\lambda,t)=\sum_m\sum_n\mathrm{Re}\left\{\frac{lT_1}{g}\frac{\partial\phi}{\partial z}P_n^m(\cos\theta)e^{im\lambda}\right\}. \tag{2.47}$$

Substituting (2.45) into (2.44) we obtain

$$\left.\begin{aligned}&-\left(\frac{\partial}{\partial t}+im\alpha\right)n(n+1)\phi+2\Omega mi\phi-\frac{l^2a^2}{g}im\alpha\frac{\partial\phi}{\partial z}=\frac{la^2}{\bar{\rho}}\frac{\partial}{\partial z}(\bar{\rho}W),\\&\frac{k_1}{c_p}\frac{\partial^3\phi}{\partial z^3}-\bar{\rho}\left[\left(mi\alpha+\frac{\partial}{\partial t}\right)\frac{\partial\phi}{\partial z}+\left(\frac{A}{T_1}\frac{\partial}{\partial t}-mi\alpha'\right)\phi\right]=\frac{A\bar{\rho}g}{lT_1}W.\end{aligned}\right\} \tag{2.48}$$

We can proceed to eliminate W in order to obtain a fourth order differential equation for $\phi(z, t)$,

$$\frac{k_1}{c_p\bar{\rho}}\frac{\partial^4\phi}{\partial z^4} - \left(im\alpha + \frac{\partial}{\partial t}\right)\frac{\partial^2\phi}{\partial z^2} + \left[B\frac{\partial}{\partial t} + \frac{\bar{\rho}'}{\bar{\rho}}\left(im\alpha + \frac{\partial}{\partial t}\right) + im\alpha B\right]\frac{\partial\phi}{\partial z}$$

$$+\left[\frac{n(n+1)Bg}{l^2a^2}\left(im\alpha + \frac{\partial}{\partial t}\right) - \frac{B\Omega img}{l^2a^2} + \frac{\bar{\rho}'}{\bar{\rho}}\left(im\alpha' - B\frac{\partial}{\partial t}\right) + im\alpha''\right]\phi = 0 \tag{2.49}$$

where $B = {}^{A}/_{T_1}$. From (2.13) and (2.14), we have since $\overline{T} = T_1$

$$\bar{\rho} = \rho_1 e^{-\beta z} \tag{2.50}$$

where $\beta = g/RT_1$.

THE BOUNDARY CONDITIONS

We have a fourth order partial differential equation for ϕ so it is necessary to have four boundary conditions to close the system. We will take the following equations:

(1) $W = 0$ at $z = 0$,

(2) $W = 0$ at $z = h$,

(3) It is assumed that there is a time dependent heat source $\nu_0(\theta, \lambda, t)$ at the lower surface of the layer so that

$$\frac{\partial T'}{\partial z} = \nu_0(\theta, \lambda, t) \text{ at } z = 0,$$

(4) It is assumed that there is zero heat transfer at the top of the atmosphere, i.e. $\frac{\partial T'}{\partial z} = 0$ at $z = h$.

The first two boundary conditions, that vertical velocity is zero at the top and bottom of the atmosphere were also used by Blinova. We will take $\nu_0(\theta, \lambda, t)$ as a linear combination of terms

$$\nu_0(\theta, \lambda, t) = \sum_m \sum_n \nu_0^{mn}(\theta, \lambda, t)$$

where ν_0^{mn} will take the form of spherical harmonics, namely

$$\nu_0^{mn}(\theta, \lambda, t) = Re\left\{\frac{g}{lT_1}\nu(t)\, P_n^m(\cos\theta)\, e^{imt}\right\}.$$

Note that m and n are positive integers and that $\nu(t)$ will depend on m and n. By using (2.48) and (2.46) the boundary conditions can be expressed as conditions on ϕ, namely,

$$\left.\begin{aligned}
&(1)\quad \frac{k_1}{c_p\rho}\frac{\partial^3\phi}{\partial z^3} - \left[im\alpha + \frac{\partial}{\partial t}\right]\frac{\partial\phi}{\partial z} + \left[im\alpha' - B\frac{\partial}{\partial t}\right]\phi = 0 \quad \text{at } z=0,\\
&(2)\quad \frac{k_1}{c_p\rho}\frac{\partial^3\phi}{\partial z^3} - \left[im\alpha + \frac{\partial}{\partial t}\right]\frac{\partial\phi}{\partial z} + \left[im\alpha' - B\frac{\partial}{\partial t}\right]\phi = 0 \quad \text{at } z=h,\\
&(3)\quad \frac{\partial^2\phi}{\partial z^2} = \vartheta(t) \quad \text{at } z=0,\\
&(4)\quad \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{at } z=h.
\end{aligned}\right\} \tag{2.52}$$

FOURIER TRANSFORMS WITH RESPECT TO TIME

To remove the time dependence from the problem we will now take fourier transforms with respect to time of (2.49) and the boundary conditions (2.52). We take

$$\left.\begin{aligned}
\bar\phi(z,s) &= \int_{-\infty}^{\infty} \phi(z,t)\, e^{ist}\, dt,\\
\bar\vartheta(s) &= \int_{-\infty}^{\infty} \vartheta(t)\, e^{ist}\, dt
\end{aligned}\right\} \tag{2.53}$$

and remembering that the transform of $\frac{\partial\phi}{\partial t}$ is $-is\bar\phi$ the system of equations reduce to

$$\frac{\varepsilon}{\rho}\frac{d^4\bar\phi}{dz^4} - i(\alpha-\mu)\frac{d^2\bar\phi}{dz^2} + i(\alpha-\mu)(B+\beta)\frac{d\bar\phi}{dz}$$
$$+ i\left[\gamma^2(\alpha-\mu) - \frac{B\Omega g}{l^2a^2} - \beta(\alpha'+B\mu) + \alpha''\right]\bar\phi = 0 \tag{2.54}$$

subject to the boundary conditions

$$\left.\begin{aligned}
&\frac{\varepsilon}{\rho}\frac{d^3\bar\phi}{dz^3} - i\left[(\alpha-\mu)\frac{d\bar\phi}{dz} + (\alpha'+B\mu)\bar\phi\right] = 0 \quad \text{at } z=0,h,\\
&\frac{d^2\bar\phi}{dz^2} = \bar\vartheta(\mu) \quad \text{at } z=0,\\
&\frac{d^2\bar\phi}{dz^2} = 0 \quad \text{at } z=h,
\end{aligned}\right\} \tag{2.55}$$

where

$$\varepsilon = \frac{k_1}{c_p m}, \quad \gamma^2 = \frac{n(n+1)Bg}{l^2a^2} \quad \text{and} \quad \mu = \frac{s}{m}. \tag{2.56}$$

At this stage we simplify the model slightly. We ignore the terms $\left[-\frac{B\Omega q}{L^2a^2} - \beta(\alpha' + B\mu)\right]\bar{\phi}$. This is equivalent to ignoring variations of the Coriolis parameter in the northward direction when using rectangular coordinates and is equivalent to the neglect of β in the model on Part 3 (see Chapter 1, (3.35)). In Part 3 in the final chapter we look at the case for $\beta \neq 0$. In many models this approximation is acceptable and we therefore feel justified in neglecting these terms. We now compare the $\frac{d\bar{\phi}}{dz}$ term with a typical $\bar{\phi}$ term, $\gamma^2(\alpha-\mu)\bar{\phi}$. With an average value of γh taken to be 2.4, the ratio of the two terms, Bh/γ^2h^2 is approximately $1/80$. We are therefore justified in ignoring the $\frac{d\bar{\phi}}{dz}$ term. For the equations to be consistent the terms $-iB\mu\bar{\phi}$ must be omitted from the first two boundary conditions in (2.55). Thus the system of equations reduces to

$$\frac{\varepsilon}{\bar{\rho}}\frac{d^4\bar{\phi}}{dz^4} - i(\alpha(z)-\mu)\left[\frac{d^2\bar{\phi}}{dz^2} - \gamma^2\bar{\phi} - \frac{\alpha''(z)}{(\alpha(z)-\mu)}\bar{\phi}\right] = 0 \qquad (2.57)$$

subject to the boundary conditions

$$\left.\begin{aligned} &\frac{\varepsilon}{\bar{\rho}}\frac{d^3\bar{\phi}}{dz^3} - i\left[(\alpha(z)-\mu)\frac{d\bar{\phi}}{dz} - \alpha'(z)\bar{\phi}\right] = 0 \quad \text{at } z=0,h\,, \\ &\frac{d^2\bar{\phi}}{dz^2} = \nu(\mu) \quad \text{at } z=0 \\ &\frac{d^2\bar{\phi}}{dz^2} = 0 \qquad \text{at } z=h \end{aligned}\right\} \qquad (2.58)$$

VALUES FOR CONSTANTS IN THIS MODEL

It is clear from (2.22) that $a\alpha(z)$ is a steady zonal current and hence we will assume that $\alpha(z)$ is a linear function of z ,

$$\alpha(z) = U_0 + \left(\frac{z}{h} - \tfrac{1}{2}\right)U_1 \qquad (2.59)$$

We shall take the wind speed at the bottom of the layer to be 600 cms^{-1} (24) and when we take h to be the height of the tropopause (h = 16 x 10^5cms), a fair estimate of the speed of the wind at the top of the layer is

$1400 \mathrm{cms}^{-1}$ (24).

We will also take the average values (13)

$$\left.\begin{array}{ll} c_p = 1\ \mathrm{J\,gm^{-1}\,{}^{o}C^{-1}} , & \\ k_1 = 2{\cdot}41 \times 10^{-4}\ \mathrm{J\,cm^{-1}\,s^{-1}\,{}^{o}C^{-1}} & \text{(molecular).} \end{array}\right\} \qquad (2.60)$$

As in Part 1 the eddy conductivity would be approximately 10^5 times as large as the molecular value.

CHAPTER 2.3

APPROXIMATE SOLUTION OF THE STREAM FUNCTION

The quantity ε , defined in (2.56) as $k_1/c_p m$ is small and of order k_1. This fact will be exploited to find a solution for $\bar{\phi}(z)$.

FOUR SOLUTIONS FOR THE STREAM FUNCTIONS, $\bar{\phi}$

The stream function, $\bar{\phi}$ must satisfy (2.57), namely

$$\frac{\varepsilon}{\bar{\rho}}\frac{d^4\bar{\phi}}{dz^4} - i\,(\alpha(z)-\mu)\left[\frac{d^2\bar{\phi}}{dz^2} - \gamma^2\bar{\phi}\right] = 0\,.$$

There will be four independent solutions, $\bar{\phi}_s(z,\mu)$, s = 1-4 so that

$$\bar{\phi}(z,\mu) = A_1(\mu)\bar{\phi}_1(z,\mu) + A_2(\mu)\bar{\phi}_2(z,\mu) + A_3(\mu)\bar{\phi}_3(z,\mu) + A_4(\mu)\bar{\phi}_4(z,\mu) \tag{2.61}$$

where the arbitrary functions $A_s(\mu)$ will be determined from the boundary conditions, (2.58). We will follow Heisenberg, Lin and others (20) in the determination of the four solutions for $\bar{\phi}(z,\mu)$.

For two solutions, we assume $\bar{\phi}$ can be written as a power series in ε ,

$$\bar{\phi} = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \cdots\cdots$$

Substituting for $\bar{\phi}$ into (2.57) we have the following expressions for q_0 and q_0 ;

$$q_0'' - \gamma^2 q_0 = 0 \quad ,$$

$$q_1'' - \gamma^2 q_1 = \frac{q_0\bar{\rho}}{i\,(\alpha-\mu)} \quad .$$

To a first order approximation we may take

$$\left.\begin{aligned} \bar{\phi}_1(z,\mu) &= e^{\gamma z} \quad , \\ \bar{\phi}_2(z,\mu) &= e^{-\gamma z} \quad . \end{aligned}\right\} \tag{2.62}$$

For the remaining two functions, we look for solutions in the form

$$\bar{\phi} = \exp\left\{\varepsilon^{-1/2}Q(z,\mu)\right\}\left[f_0 + \varepsilon^{1/2}f_1 + \cdots\right]\,. \tag{2.63}$$

We substitute (2.63) into (2.57) and by comparing the coefficients of ε^{-1} and $\varepsilon^{-\frac{1}{2}}$ obtain equations for $Q(z,\mu)$ and $f_0(z,\mu)$ which result in the solutions,

$$\frac{d^2Q}{dz^2} = [i(\alpha(z)-\mu)\bar{\rho}]^{1/2} , \tag{2.64}$$

$$f_0(z,\mu) = [(\alpha(z)-\mu)\bar{\rho}]^{-5/4} . \tag{2.65}$$

We can now take, to first order approximation,

$$\left.\begin{aligned} \bar{\phi}_3 &= [(\alpha(z)-\mu)\bar{\rho}]^{-5/4} \exp\{-\varepsilon^{-1/2} Q(z,\mu)\} \\ \bar{\phi}_4 &= [(\alpha(z)-\mu)\bar{\rho}]^{-5/4} \exp\{\varepsilon^{-1/2} Q(z,\mu)\} \end{aligned}\right\} \tag{2.66}$$

where

$$Q(z,\mu) = \int_{z_c}^{z} [i(\alpha(z)-\mu)\bar{\rho}(z)]^{1/2} dz$$

and z_c is the value of z for which $\alpha(z) = \mu$; z_c is therefore a branch point of $\bar{\phi}$ and the correct value must be taken when $0 < z_c < h$. We follow Lin (20) in the discussion of this "Crossing substitution" problem and take a path from 0 to h in the complex z plane such that the real part of Q increases monotonically. In this case the path should be below the critical point z_c for nearly real values of μ. Thus, for real μ we have

$$\left.\begin{aligned} \alpha-\mu &= |\alpha-\mu| , \quad \arg Q = \pi/4 \quad \text{for} \quad z > z_c , \\ \alpha-\mu &= |\alpha-\mu| e^{-i\pi}, \quad \arg Q = -5\pi/4 \quad \text{for} \quad z < z_c . \end{aligned}\right\} \tag{2.67}$$

Therefore, $\bar{\phi}_3$ decreases exponentially and $\bar{\phi}_4$ increases exponentially as z increases along the real axis.

DETERMINATION OF THE ARBITRARY CONSTANTS

We now substitute for $\bar{\phi}$ from (2.61) into the boundary conditions in (2.58) using (2.62) and (2.66) to obtain four equations for A_1, A_2, A_3 and A_4. Writing $M \equiv (\alpha-\mu)\frac{d}{dz} - \alpha'$ and using the notation $\bar{\phi}_{n1} = \bar{\phi}_n(0)$, $\bar{\phi}_{n2} = \bar{\phi}_n(h)$, $\rho_0 = \bar{\rho}(0)$ and $\rho_h = \bar{\rho}(h)$ the equations in (2.58) reduce to

$$\begin{pmatrix} \frac{\epsilon}{\rho_0}\bar{\phi}_{11}''' - iM\bar{\phi}_{11} & \frac{\epsilon}{\rho_0}\bar{\phi}_{21}''' - iM\bar{\phi}_{21} & \frac{\epsilon}{\rho_0}\bar{\phi}_{31}''' - iM\bar{\phi}_{31} & \frac{\epsilon}{\rho_0}\bar{\phi}_{41}''' - iM\bar{\phi}_{41} \\ \frac{\epsilon}{\rho_h}\bar{\phi}_{12}''' - iM\bar{\phi}_{12} & \frac{\epsilon}{\rho_h}\bar{\phi}_{22}''' - iM\bar{\phi}_{22} & \frac{\epsilon}{\rho_h}\bar{\phi}_{32}''' - iM\bar{\phi}_{32} & \frac{\epsilon}{\rho_h}\bar{\phi}_{42}''' - iM\bar{\phi}_{42} \\ \bar{\phi}_{11}'' & \bar{\phi}_{21}'' & \bar{\phi}_{31}'' & \bar{\phi}_{41}'' \\ \bar{\phi}_{12}'' & \bar{\phi}_{22}'' & \bar{\phi}_{32}'' & \bar{\phi}_{42}'' \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \bar{\nu} \\ 0 \end{pmatrix} .$$

(2.68)

For a non zero contribution of $\bar{\phi}_3$ and $\bar{\phi}_4$ to the first two boundary conditions it is necessary to calculate $\left[\frac{\epsilon}{\beta}\frac{d^3}{dz^3} - iM\right]\bar{\phi}_s$, $s = 3,4$, to second order. We take

$$\phi = \exp\left[\epsilon^{-1/2}Q\right]\left[f_0 + \epsilon^{1/2}f_1\right]$$

and we can show that, to second order approximation

$$\left[\frac{\epsilon}{\beta}\frac{d^3}{dz^3} - iM\right]\phi = \left[i(\alpha-\mu)\left[\frac{2f_0'}{f_0} + \frac{3}{2}\frac{\beta'}{\beta}\right] + \frac{5i\alpha'}{2}\right] f_0 \exp\left[\epsilon^{-1/2}Q\right]. \tag{2.69}$$

Using the crossing substitutions in (2.67) we have,

$$\frac{\bar{\phi}_{32}}{\bar{\phi}_{31}} = \frac{\exp\left[-\epsilon^{-1/2}|Q(h,\mu)|e^{-i\pi/4}\right] f_0(h,\mu)}{\exp\left[-\epsilon^{-1/2}|Q(0,\mu)|e^{-5i\pi/4}\right] f_0(0,\mu)}$$

and as $\mathrm{Re}\{e^{i\pi/4}\} > 0$ and $\mathrm{Re}\{e^{5i\pi/4}\} < 0$,

$$\frac{\bar{\phi}_{32}}{\bar{\phi}_{31}} \sim e^{-a\epsilon^{-1/2}} \frac{f_0(h,\mu)}{f_0(0,\mu)} \quad , \ a > 0 .$$

Clearly, we can take $\bar{\phi}_{32} = 0$. Since to a first order approximation, $\bar{\phi}_{32}'' = \epsilon^{-1}Q'^2(h,\mu)\,\bar{\phi}_{32}$ the same argument as above allows us to assume $\bar{\phi}_{32}'' = 0$. The contribution to the two boundary conditions at $z = h$ for $\bar{\phi}_3$ can therefore be approximated to zero. Using exactly the same arguments we can show that $\bar{\phi}_{41} = \bar{\phi}_{41}'' = 0$ and so the contribution to the two boundary conditions at $z = 0$ for $\bar{\phi}_4$ can be approximated to zero. We can now write (2.68) as

$$\begin{pmatrix} -iM\bar{\phi}_{11} & -iM\bar{\phi}_{21} & iN_1\bar{\phi}_{31} & 0 \\ -iM\bar{\phi}_{12} & -iM\bar{\phi}_{22} & 0 & iN_2\bar{\phi}_{42} \\ \bar{\phi}_{11}'' & \bar{\phi}_{21}'' & \bar{\phi}_{31}'' & 0 \\ \bar{\phi}_{12}'' & \bar{\phi}_{22}'' & 0 & \bar{\phi}_{42}'' \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \bar{\nu} \\ 0 \end{pmatrix} \tag{2.70}$$

in which all the terms are to first order approximation apart from $iN_1\bar{\phi}_{31}$ and $iN_2\bar{\phi}_{42}$ which are to second order approximation in ϵ and where $N(z, \mu)$ is defined as

$$N(z,\mu) = (\alpha-\mu)\left[\frac{2f_0'}{f_0} + \frac{3}{2}\frac{\beta'}{\beta}\right] + \frac{5\alpha'}{2} \tag{2.71}$$

and $N_1 = N(0,\mu)$ and $N_2 = N(h,\mu)$. If we write

$$\left.\begin{aligned} B_3 &= A_3 \exp\{-\varepsilon^{-1/2} Q(0,\mu)\}, \\ B_4 &= A_4 \exp\{\cdot\varepsilon^{-1/2} Q(h,\mu)\}, \end{aligned}\right\} \tag{2.72}$$

and substitute into (2.70) the values for $\bar{\phi}_{ij}$ (i = 1, -4, j = 1,2) we have

$$\begin{pmatrix} -\gamma(\alpha_0-\mu)+\alpha_0' & \gamma(\alpha_0-\mu)+\alpha_0' & N_1 f_0(0,\mu) & 0 \\ (-\gamma(\alpha_h-\mu)+\alpha_h')e^{\gamma h} & (\gamma(\alpha_h-\mu)+\alpha_h')e^{-\gamma h} & 0 & N_2 f_0(h,\mu) \\ \gamma^2 & \gamma^2 & i\varepsilon^{-1}(\alpha_0-\mu)\rho_0 f_0(0,\mu) & 0 \\ \gamma^2 e^{\gamma h} & \gamma^2 e^{-\gamma h} & 0 & i\varepsilon^{-1}(\alpha_h-\mu)\rho_h f_0(h,\mu) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \bar{\nu} \\ 0 \end{pmatrix} \tag{2.73}$$

where $\alpha_0 = \alpha(0)$ and $\alpha_h = \alpha(h)$.

We can generalise the coefficients of A_1, A_2, B_3 and B_4 and write (2.73) in the form

$$\left.\begin{aligned} \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 B_3 &= 0 \\ \beta_1 A_1 + \beta_2 A_2 + \beta_3 B_4 &= 0 \\ \gamma_1 A_1 + \gamma_2 A_2 + \varepsilon^{-1}\gamma_3 B_3 &= \bar{\nu} \\ \delta_1 A_1 + \delta_2 A_2 + \varepsilon^{-1}\delta_3 B_4 &= 0 \end{aligned}\right\} \tag{2.74}$$

where α_s, β_s, γ_s and δ_s (s = 1,2,3) are independent of ε. The solutions for A_1, A_2, B_3 and B_4 are therefore

$$A_1 \times |A| = \begin{vmatrix} 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \beta_2 & 0 & \beta_3 \\ \bar{\nu} & \gamma_2 & \varepsilon^{-1}\gamma_3 & 0 \\ 0 & \delta_2 & 0 & \varepsilon^{-1}\delta_3 \end{vmatrix}, \qquad A_2 \times |A| = \begin{vmatrix} \alpha_1 & 0 & \alpha_3 & 0 \\ \beta_1 & 0 & 0 & \beta_3 \\ \gamma_1 & \bar{\nu} & \varepsilon^{-1}\gamma_3 & 0 \\ \delta_1 & 0 & 0 & \varepsilon^{-1}\delta_3 \end{vmatrix},$$

$$B_3 \times |A| = \begin{vmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & \beta_3 \\ \gamma_1 & \gamma_2 & \bar{\nu} & 0 \\ \delta_1 & \delta_2 & 0 & \epsilon^{-1}\delta_3 \end{vmatrix}, \qquad B_4 \times |A| = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ \gamma_1 & \gamma_2 & \epsilon^{-1}\gamma_3 & \bar{\nu} \\ \delta_1 & \delta_2 & 0 & 0 \end{vmatrix},$$

where $|A|$ is defined as

$$|A| = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ \beta_1 & \beta_2 & 0 & \beta_3 \\ \gamma_1 & \gamma_2 & \epsilon^{-1}\gamma_3 & 0 \\ \delta_1 & \delta_2 & 0 & \epsilon^{-1}\delta_3 \end{vmatrix} .$$

There will be singularities in A_1, A_2, B_3 and B_4 at the points where $|A| = 0$. We expand $|A|$ by the last two columns to obtain

$$|A| = \delta_3\gamma_3\epsilon^{-2} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} + \delta_3\alpha_3\epsilon^{-1} \begin{vmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}$$

$$+ \beta_3\gamma_3\epsilon^{-1} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \delta_1 & \delta_2 \end{vmatrix} + \alpha_3\beta_3 \begin{vmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{vmatrix} .$$

To a first approximation therefore the singularities will occur when

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \tag{2.75}$$

When we substitute for α_1, α_2, β_1 and β_2 from (2.73) we find that the above equation (2.75) results in

$$(U_0 - \mu)^2 = \frac{U_1^2}{\gamma^2 h^2}\left[\frac{\gamma^2 h^2}{4} + 1 - \gamma h \coth \gamma h \right] . \tag{2.76}$$

The equation (2.75), is exactly the equation of consistency found by Eady and is discussed in detail in Chapter 3.3. This equation yields two values for μ ,

$$\left.\begin{aligned} \mu_1 &= U_0 + \frac{U_1}{\gamma h}\eta_1 \\ \mu_2 &= U_0 - \frac{U_1}{\gamma h}\eta_1 \end{aligned}\right\} \tag{2.77}$$

where η_1 is defined as

$$\eta_1 = \left[\frac{\gamma^2h^2}{4} + 1 - \gamma h \coth \gamma h\right]^{1/2} \tag{2.78}$$

It is found that η_1 is purely real or imaginary as follows:

$$\left.\begin{aligned} \eta_1 &= i\eta \quad , \quad 0 < \gamma h < 2{\cdot}4 \, , \\ \eta_1 &= \eta \quad , \quad 2{\cdot}4 < \gamma h \quad , \end{aligned}\right\} \tag{2.79}$$

where

$$\eta = \left[\left|\frac{\gamma^2h^2}{4} + 1 - \gamma h \coth \gamma h\right|\right]^{1/2}. \tag{2.80}$$

To approximate the values of A_1, A_2, B_3 and B_4 we first note that these constants can be expressed as power series in ε. From the third equation of (2.74) we can see that the first non zero term of the expansion of B_3 is of the order ε , thus

$$B_3 = \varepsilon b_3 + \varepsilon^2 b_3' + \cdots\cdots \tag{2.81}$$

The fourth equation of (2.74) implies that the first non zero term for B_4 will be at least of order ε and so we can take

$$\left.\begin{aligned} A_1 &= a_1 + \varepsilon a_1' + \varepsilon^2 a_1^2 + \cdots\cdots , \\ A_2 &= a_2 + \varepsilon a_2' + \varepsilon^2 a_2^2 + \cdots\cdots , \\ B_4 &= \qquad \varepsilon b_4' + \varepsilon^2 b_4^2 + \cdots\cdots . \end{aligned}\right\} \tag{2.82}$$

The first two equations in (2.74) lead to two equations for a_1 and a_2 , namely

$$a_1\alpha_1 + a_2\alpha_2 = 0 \, ,$$

$$a_1\beta_1 + a_2\beta_2 = 0 \, ,$$

resulting in the equation of consistency, (2.75). Since (2.75) only holds for two particular values of μ , we will therefore take $a_1 = a_2 = 0$. From the last equation in (2.74) we find $b_4' = 0$ and solving for a_1' ,

a'_2 , b^2_4 and b_3 leads to

$$\left.\begin{aligned} a'_1 &= \frac{-\bar{V}\alpha_3\beta_2}{\gamma_3(\alpha_1\beta_2-\alpha_2\beta_1)} \\ a'_2 &= \frac{\bar{V}\alpha_3\beta_1}{\gamma_3(\alpha_1\beta_2-\alpha_2\beta_1)} \\ b_3 &= \bar{V}/\gamma_3 \\ b^2_4 &= \frac{\bar{V}[\beta_2\delta_1-\beta_1\delta_2]}{\gamma_3\delta_3(\alpha_1\beta_2-\alpha_2\beta_1)} \end{aligned}\right\} \tag{2.83}$$

Note that three coefficients do have singularities when $(\alpha_1\beta_2-\alpha_2\beta_1) = 0$, as was found in (2.75).

When we substitute for α_s, β_s, γ_s and δ_s (s = 1,2,3) from (2.73) we find that to a first order approximation $\bar{\phi}(z,\mu)$ takes the form

$$\begin{aligned} \bar{\phi}(z,\mu) = \frac{\varepsilon\bar{V}(\mu)}{\rho_1}\Bigg\{ & \frac{i\beta h^2(D_1-\gamma\mu)}{(\mu-\mu_1)(\mu-\mu_2)}e^{\gamma z} - \frac{i\beta h^2 e^{2\gamma h}(D_2+\gamma\mu)}{(\mu-\mu_1)(\mu-\mu_2)}e^{-\gamma h} \\ & \frac{-i(\alpha_0-\mu)^{1/4}}{[e^{-\beta z}(\alpha-\mu)]^{5/4}}\exp\left\{-\varepsilon^{-1/2}\int_0^z (i\bar{\rho}(\alpha-\mu))^{1/2}dz\right\} \\ & \frac{-2\varepsilon\gamma^3h^2(\alpha_h-\mu)^{5/4}[\rho_h(\alpha_0-\mu)]^{1/4}}{[e^{-\beta z}(\alpha-\mu)]^{5/4}(\mu-\mu_1)(\mu-\mu_2)}\exp\left\{\varepsilon^{-1/2}\int_h^z (i\bar{\rho}(\alpha-\mu))^{1/2}dz\right\}\Bigg\} \end{aligned} \tag{2.84}$$

where

$$\left.\begin{aligned} &\alpha(z) = U_0 + (z/h-1/2)U_1\ ,\quad \alpha_0 = \alpha(0)\ ,\quad \alpha_h = \alpha(h), \\ &D_1 = U_1/h + \gamma(U_0+U_1/2)\ ,\quad D_2 = U_1/h - \gamma(U_0+U_1/2). \end{aligned}\right\} \tag{2.85}$$

INVERSE FOURIER TRANSFORMS

In order to return to $\phi(z, t)$ we use the inverse transform

$$\phi(z,t) = \frac{m}{2\pi}\int_{-\infty}^{\infty}\bar{\phi}(z,\mu)e^{-i\mu t}d\mu, \tag{2.86}$$

remembering that $s = \mu m$. In general $\bar{\phi}(z,\mu)$ of (2.84) is a complicated function but a fairly simple result can be found for $\bar{\phi}(0,\mu)$. Since $\bar{\phi}_4(0,\mu)$ is small we can approximate $\bar{\phi}(0,\mu)$ to

$$\bar{\phi}(0,\mu) = A_1\bar{\phi}_1(0,\mu) + A_2\bar{\phi}_2(0,\mu) + A_3\bar{\phi}_3(0,\mu);$$

from (2.84) we find that $\bar{\phi}(0,\mu)$ takes the form

$$\bar{\phi}(0,\mu) = \frac{i\varepsilon\bar{V}(\mu)}{\rho_1}\left\{\frac{\beta h\left[U_1(1-e^{2\gamma h})+\gamma h(1+e^{2\gamma h})(U_0+U_1/2-\mu)\right]}{(\mu-\mu_1)(\mu-\mu_2)} - \frac{1}{(\alpha_0-\mu)}\right\} \tag{2.87}$$

To illustrate the theory as simply as possible we now choose $\bar{V}$ (μ) as follows:

$$\bar{V}(\mu) = \frac{D_0}{\mu^2+D^2} \tag{2.88}$$

The inverse transformations are dealt with in detail in Appendix A. We find that from (A.3) the inverse transform of $\bar{V}(\mu)$ is

$$V(t) = \frac{D_0 m}{2D}\, e^{-mD|t|} \tag{2.89}$$

Therefore V (t) has a maximum at t = 0 of $D_0 m/2D$ and as $|t|\to\infty$, V (t)$\to$ 0. The constants D_0 and D are at our disposal. Also from Appendix A, (A.13), (A.14) and (A.23), we have the inverse transform of $\bar{\phi}(0,\mu)$ as follows:

$$\underset{0<\gamma h<2\cdot 4}{\phi(0,t)} = -\frac{\varepsilon m D_0}{2\rho_1}\left\{\frac{(\lambda_1-(-1)^n iD\lambda_2)\,e^{-mD|t|}}{iD((-1)^n iD+\mu_1)((-1)^n iD+\mu_2)} - \frac{e^{-mD|t|}}{iD(\alpha_0+(-1)^n iD)}\right.$$
$$\left. -\frac{e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)} + \frac{2(\lambda_1+\mu_n\lambda_2)\,e^{-im\mu_n t}}{(-1)^n(D^2+\mu_n^2)(\mu_2-\mu_1)}\right\}, \tag{2.90}$$

$$\underset{2\cdot 4<\gamma h}{\phi(0,t)} = -\frac{\varepsilon m D_0}{2\rho_1}\left\{\frac{(\lambda_1-(-1)iD\lambda_2)e^{-mD|t|}}{iD((-1)^n iD+\mu_1)((-1)^n iD+\mu_2)} - \frac{e^{-mD|t|}}{iD(\alpha_0+(-1)^n iD)} - \frac{e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)}\right.$$
$$\left. +\frac{(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\mu_1-\mu_2)} + \frac{(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(D^2+\mu_2^2)(\mu_2-\mu_1)}\right\}, \tag{2.91}$$

with

$n=2$ for $t>0$ and $n=1$ for $t<0$

and where λ_1 and λ_2 are defined as

$$\left.\begin{aligned} \lambda_1 &= \beta h\left[(1-e^{2\gamma h})U_1 + \gamma h(1+e^{2\gamma h})(U_0+U_1/2)\right], \\ \lambda_2 &= \beta\gamma h^2(1+e^{2\gamma h}). \end{aligned}\right\} \tag{2.92}$$

CHAPTER 2.4

DISCUSSION

The stream function $\psi(\sigma,\theta,\lambda,t)$ defined in (2.45) and (2.46) contains terms of the form $\phi(\sigma,t)P_n^m(\cos\theta)e^{im\lambda}$ and thus, from the definition of $\phi(\sigma,t)$ in (2.90) and (2.91), ψ is a combination of waves of varying amplitudes and speeds. In the complete range for γh there is a progressive wave term,

$$\frac{\varepsilon m D_0}{2\beta_1(D^2+\alpha_0)^2}e^{-im(\alpha_0 t-\lambda)}.$$

The speed of the wave is $\alpha_0 = U_0 - U_1/2$; thus the wave travels with the thermal wind speed at z = 0. There is also a damped wave term for all values of γh, namely

$$A\,e^{-mD|t|+im\lambda}.$$

In the range $0 < \gamma h < 2.4$ there are also damped wave terms of the form

$$A_1 \exp\left\{-\left(\frac{U_1\eta}{\gamma h}+iU_0\right)mt+im\lambda\right\},\ t>0,$$

$$A_2 \exp\left\{\left(-\frac{U_1\eta}{\gamma h}+iU_0\right)m|t|+im\lambda\right\},\ t<0.$$

However in the range $\gamma h > 2.4$ there are two extra progressive wave terms,

$$A_3 \exp\left\{-i\left(U_0+\frac{U_1\eta}{\gamma h}\right)mt+im\lambda\right\},\ \forall t,$$

$$A_4 \exp\left\{-i\left(U_0-\frac{U_1\eta}{\gamma h}\right)mt+im\lambda\right\},\ \forall t.$$

The amplitudes A_3 and A_4 are defined in (2.91) and their speeds are $U_0+U_1\eta/\gamma h$ and $U_0-U_1\eta/\gamma h$ respectivly.

The amplitudes of all the waves contain a factor (εD_0). The amplitudes are directly proportional to the value of the thermal conductivity. We discussed in the General Introduction how we use a larger value than the molecular conductivity to model the turbulence in the atmosphere. When we increase k_1 therefore, we directly increase the amplitudes of the progressive waves by the same amount. Similarly the amplitudes are proportional to D_0 and are therefore dependent on the size of the heating function $\mathcal{V}(t)$ at the surface of the ocean. The speeds of the waves are proportional to the thermal wind speeds.

APPENDIX A

INVERSE FOURIER TRANSFORMS FOR $\bar{V}(\mu)$ AND $\bar{\phi}(0,\mu)$

We use contour integral methods (25) to find the inverse transforms of $\bar{V}(\mu)$ and $\bar{\phi}(0,\mu)$ defined in (2.88) and (2.87) respectively.

THE INVERSE TRANSFORM OF $\bar{V}(\mu)$

We have chosen $\bar{V}(\mu)$ in (2.88) to take the form

$$\bar{V}(\mu) = \frac{D_0}{\mu^2+D^2} \tag{A.1}$$

where D_0 and D are constants. The inverse transform $V(t)$ is therefore

$$\begin{aligned} V(t) &= \frac{m}{2\pi}\int_{-\infty}^{\infty} \bar{V}(\mu)\, e^{-im\mu t}\, d\mu \\ &= \frac{D_0 m}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-im\mu t}}{\mu^2+D^2}\, d\mu. \end{aligned} \tag{A.2}$$

We take a closed curve Γ in the complex plane of μ, consisting of the real axis and a semicircle above or below the real axis. For $\mu = \mu_0 + i\mu_1$, we have

$$V(t) = \frac{D_0 m}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\mu^2+D^2} \exp\left[m\mu_1 t - im\mu_0 t\right] dt$$

and therefore for zero contribution on the semicircle we choose Γ to be above the real axis for negative t and below the real axis for positive t. For $t > 0$ the only singularity inside Γ is a simple pole at $\mu = -iD$ and

$$V(t) = \frac{D_0 m e^{-mDt}}{2D}, \quad t > 0 \quad ;$$

for $t < 0$, there is a simple pole at $\mu = iD$ within Γ so that

$$V(t) = \frac{D_0 m e^{mDt}}{2D}, \quad t > 0 \quad .$$

We therefore find that

$$V(t) = \frac{D_0 m}{2D} e^{-mD|t|}, \quad \forall t. \tag{A.3}$$

THE INVERSE TRANSFORM OF $\bar{\phi}(0,\mu)$

From (2.87) it can be seen that $\bar{\phi}(0,\mu)$ is of the form

$$\bar{\phi}(0,\mu)=\frac{i\varepsilon\bar{V}(\mu)}{\rho_1}\left[\beta h\frac{\left[U_1(1-e^{2\gamma h})+\gamma h(U_0+U_1/2-\mu)(1+e^{2\gamma h})\right]}{(\mu-\mu_1)(\mu-\mu_2)}-\frac{1}{(\alpha_0-\mu)}\right] \tag{A.4}$$

where μ_1 and μ_2 are defined as

$$\left.\begin{aligned}&\mu_1=U_0+\frac{iU_1\eta}{\gamma h}, && \mu_2=U_0-\frac{iU_1\eta}{\gamma h}, && 0<\gamma h<2.4,\\ &\mu_1=U_0+\frac{U_1\eta}{\gamma h}, && \mu_2=U_0-\frac{U_1\eta}{\gamma h}, && 2.4<\gamma h,\end{aligned}\right\} \tag{A.5}$$

and $\bar{V}(\mu)$ is defined in (A.1). The inverse transform of $\bar{\phi}(0,\mu)$ is

$$\phi(0,t)=\frac{m}{2\pi}\int_{-\infty}^{\infty}\bar{\phi}(0,\mu)e^{-im\mu t}\,d\mu\,;$$

by generalising (A.4) and substituting for $\bar{V}(\mu)$ from (A.1) we can see that the integrals to be evaluated are of the type,

$$I=\int_{-\infty}^{\infty}\frac{(\lambda_1+\mu\lambda_2)e^{-im\mu t}}{(\mu^2+D^2)(\mu-\mu_1)(\mu-\mu_2)(\alpha_0-\mu)}\,d\mu=\int_{-\infty}^{\infty}J(\mu)\,d\mu\,. \tag{A.6}$$

We will evaluate I using contour integration methods and take a closed contour Γ along the real axis and a semicircle above the real axis for $t<0$ and below the real axis for $t>0$. For the range $0<\gamma h<2.4$ there is one singularity ($\mu=\alpha_0$) on the real axis, two above ($\mu=iD, \mu=\mu_1$) and two below ($\mu=-iD, \mu=\mu_2$) the real axis. For the range $2.4<\gamma h$, there are three singularities ($\mu=\alpha_0$, $\mu=\mu_1$, $\mu=\mu_2$) on the real axis, one above ($\mu=iD$) and one below ($\mu=-iD$) the real axis. The contour Γ must have indentations around the singularities on the real axis. As in Lin (20) for the singularities μ_1 and μ_2 we must go below the real axis (the crossing substitution) and for convenience we will take all indentations on the real axis below the axis. We will evaluate $\phi(0,t)$ in the two regions, $0<\gamma h<2.4$ and $2.4<\gamma h$ separately.

(1) $\phi(0,t)$ IN THE RANGE $0<\gamma h<2.4$

In this region we have only one singularity along the real axis.

Thus we have for $t>0$

$$\frac{\pi(\lambda_1-iD\lambda_2)e^{-mDt}}{D(\alpha_0+iD)(-iD-\mu_1)(-iD-\mu_2)} + \frac{2\pi i(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(\mu_2^2+D^2)(\alpha_0-\mu_2)(\mu_2-\mu_1)} = K_1 + \left\{\left[\int_{-R}^{\alpha_0-\varepsilon} + \int_{\alpha_0+\varepsilon}^{R}\right] J(\mu)\,d\mu\right\} \tag{A.7}$$

and for $t<0$

$$\frac{\pi(\lambda_1+iD\lambda_2)e^{-mD|t|}}{D(\alpha_0-iD)(iD-\mu_1)(iD-\mu_2)} + \frac{2\pi i(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(\mu_1^2+D^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)} = K_1 + \left\{\left[\int_{-R}^{\alpha_0-\varepsilon} + \int_{\alpha_0+\varepsilon}^{R}\right] J(\mu)\,d\mu\right\} \tag{A.8}$$

where K_1 is the contribution to the integral around the singularity at $\mu=\alpha_0$. We take the indentation around $\mu=\alpha_0$ to be a semicircle below the real axis of radius ε ; writing $\mu = \alpha_0+\varepsilon e^{i\theta}$ K_1 takes the form

$$K_1 = \int_{\pi}^{2\pi} \frac{(\lambda_1+(\alpha_0+\varepsilon e^{i\theta})\lambda_2)\exp[-im(\alpha_0+\varepsilon e^{i\theta}]\,d\theta}{-\varepsilon e^{i\theta}(D^2+(\alpha_0+\varepsilon e^{i\theta})^2)(\alpha_0+\varepsilon e^{i\theta}-\mu_1)(\alpha_0+\varepsilon e^{i\theta}-\mu_2)} . \tag{A.9}$$

In the limit as $\varepsilon\to 0$ we find

$$K_1 = \frac{-\pi i(\alpha_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\alpha_0-\mu_2)} . \tag{A.10}$$

In the limit as $R\to\infty$ and $\varepsilon\to 0$ for $t>0$ from (A.7) we have

$$I = i\pi\left\{\frac{(\lambda_1-iD\lambda_2)e^{-mDt}}{iD(\alpha_0+iD)(-iD-\mu_1)(-iD-\mu_2)} + \frac{2(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(\mu_2^2+D^2)(\alpha_0-\mu_2)(\mu_2-\mu_1)} + \frac{(\lambda_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)}\right\} \tag{A.11}$$

and for $t<0$, from (A.8) we have

$$I = i\pi\left\{\frac{(\lambda_1+iD\lambda_2)e^{-mD|t|}}{iD(\alpha_0-iD)(iD-\mu_1)(iD-\mu_2)} + \frac{2(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(\mu_1^2+D^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)} + \frac{(\lambda_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)}\right\} \tag{A.12}$$

Using the formulation of $\bar{\phi}(0,\mu)$ in(A.4) and (A.11) and (A.12) we obtain the following expressions for $\phi(0,t)$:

$$\underset{t>0}{\phi(0,t)} = -\frac{\varepsilon m D_0}{2\rho_1}\left\{\frac{(\lambda_1-iD\lambda_2)e^{-mDt}}{iD(-iD-\mu_1)(-iD-\mu_2)} - \frac{e^{-mDt}}{iD(\alpha_0+iD)} - \frac{e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)} + \frac{2(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(D^2+\mu_2^2)(\mu_2-\mu_1)}\right\} \tag{A.13}$$

$$\underset{t<0}{\phi(0,t)} = -\frac{\varepsilon m D_0}{2\rho_1}\left\{\frac{(\lambda_1+iD\lambda_2)e^{-mD|t|}}{iD(iD-\mu_1)(iD-\mu_2)} - \frac{e^{-mD|t|}}{iD(\alpha_0-iD)} - \frac{e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)} + \frac{2(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\mu_1-\mu_2)}\right\} \tag{A.14}$$

where λ_1 and λ_2 are defined as

$$\left.\begin{aligned} \lambda_1 &= \beta h\left[(1-e^{2\gamma h})U_1 + (1+e^{2\gamma h})(U_0+U_1/2)\gamma h\right] \\ \lambda_2 &= -\beta h^2\gamma(1+e^{2\gamma h}) \end{aligned}\right\} \tag{A.15}$$

(2) $\emptyset(0,t)$ IN THE RANGE $2.4<\gamma h$

In this region we have three singularities on the real axis. As before we find that for $t>0$,

$$\frac{\pi(\lambda_1-iD\lambda_2)e^{-mDt}}{D(\alpha_0+iD)(iD-\mu_1)(-iD-\mu_2)} = K_1+K_2+K_3+\left\{\left[\int_{-R}^{\mu_2-\varepsilon}+\int_{\mu_2+\varepsilon}^{\alpha_0-\eta}+\int_{\alpha_0+\eta}^{\mu_1-\delta}+\int_{\mu_1+\delta}^{R}\right]J(\mu)\,d\mu\right\} \tag{A.16}$$

and for $t<0$

$$\frac{\pi(\lambda_1+iD\lambda_2)e^{-mD|t|}}{D(\alpha_0-iD)(iD-\mu_1)(iD-\mu_2)} = K_1+K_2+K_3+\left\{\left[\int_{-R}^{\mu_2-\varepsilon}+\int_{\mu_2+\varepsilon}^{\alpha_0-\eta}+\int_{\alpha_0+\eta}^{\mu_1-\delta}+\int_{\mu_1+\delta}^{R}\right]J(\mu)\,d\mu\right\} \tag{A.17}$$

where K_1, K_2, and K_3 are the contributions to the integral at the singu--larities $\mu=\mu_2$, $\mu=\alpha_0$ and $\mu=\mu_1$ respectively. Following the same method as in the previous section we find, when $\varepsilon\to 0$, $\eta\to 0$ and $\delta\to 0$,

$$K_1 = \frac{-i\pi(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(\alpha_0-\mu_2)(D^2+\mu_2^2)(\mu_2-\mu_1)}, \tag{A.18}$$

$$K_2 = \frac{-i\pi(\lambda_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\alpha_0-\mu_2)}, \tag{A.19}$$

$$K_3 = \frac{-i\pi(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)}. \tag{A.20}$$

In the limit as $R\to\infty$, $\varepsilon\to 0$, $\eta\to 0$ and $\delta\to 0$ we have from (A.16) for $t>0$

$$I = \pi i\left\{\frac{(\lambda_1-iD\lambda_2)e^{-mDt}}{iD(\alpha_0+iD)(-iD-\mu_1)(-iD-\mu_2)} + \frac{(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(D^2+\mu_2^2)(\alpha_0-\mu_2)(\mu_2-\mu_1)} + \frac{(\lambda_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\alpha_0-\mu_2)} + \frac{(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)}\right\} \tag{A.21}$$

and from (A.17) we have for $t<0$

$$I = \pi i\left\{\frac{(\lambda_1+iD\lambda_2)e^{-mD|t|}}{iD(\alpha_0-iD)(iD-\mu_1)(iD-\mu_2)} + \frac{(\lambda_1+\mu_2\lambda_2)e^{-im\mu_2 t}}{(D^2+\mu_2^2)(\alpha_0-\mu_2)(\mu_2-\mu_1)} + \frac{(\lambda_1+\alpha_0\lambda_2)e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)(\alpha_0-\mu_1)(\alpha_0-\mu_2)} + \frac{(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\alpha_0-\mu_1)(\mu_1-\mu_2)}\right\}. \tag{A.22}$$

From the definition of $\bar{\phi}(0,\mu)$ in (A.4) and using (A.21) and (A.22) we find the following expression for $\phi(0,t)$:

$$\underset{\forall t}{\phi(0,t)} = -\frac{\varepsilon m D_0}{2\rho_1}\left\{ \frac{(\lambda_1-(-1)^n iD\lambda_2)e^{-mD|t|}}{iD((-1)^n iD-\mu_1)((-1)^n iD-\mu_2)} - \frac{e^{-mD|t|}}{iD(\alpha_0+(-1)^n iD)} + \frac{(\lambda_1+\mu_1\lambda_2)e^{-im\mu_1 t}}{(D^2+\mu_1^2)(\mu_1-\mu_2)} + \frac{(\lambda_1+\mu_2)e^{-im\mu_2 t}}{(D^2+\mu_2^2)(\mu_2-\mu_1)} - \frac{e^{-im\alpha_0 t}}{(D^2+\alpha_0^2)} \right\} \tag{A.23}$$

where

$$n=2 \text{ for } t>0 \text{ and } n=1 \text{ for } t<0$$

and λ_1 and λ_2 are defined in (A.15).

PART 3

THE INFLUENCE OF TEMPERATURE FIELDS UPON STABILITY IN A LINKED OCEAN/ ATMOSPHERE MODEL

PART 3

THE INFLUENCE OF TEMPERATURE FIELDS UPON STABILITY IN A LINKED OCEAN/ ATMOSPHERE MODEL

CHAPTER 3.1

INTRODUCTION

In Part 3 we consider a two layer liquid model; the lower layer is assumed to simulate a static ocean and the upper layer a dynamic atmosphere. In both Parts 1 and 2 an external heating function was introduced; however, in Part 3 the emphasis is different. Here, we investigate the stability of the model, retaining thermal conductivity but omitting molecular viscosity.

We choose the standard system of rectangular coordinates, widely used in meterological wave problems. The upper layer equations of motion and heat transfer are linearised upon a basic West-East thermal wind. This thermal wind is produced by a basic temperature field which varies linearly in the northward direction and in the vertical direction. It is shown that the resulting perturbation equations lead to a fourth order ordinary differential equation for the perturbation pressure. The lower layer is assumed to be in a stationary state but capable of thermal conduction; thus the heat transfer equation results in a second order ordinary differential equation for the perturbation temperature. There are six boundary conditions; two of them arise from the vanishing of the vertical velocity at the boundaries of the upper layer and four are heating conditions at the boundaries of both layers. The solution for the perturbation temperature in the lower layer can be determined so that the problem is reduced to the solution of a fourth order differential equation for the perturbation pressure in the upper layer subject to four boundary conditions.

It is shown that in the absence of the lower layer and the neglect of thermal conductivity of the upper layer the model reduces to the well known Eady problem. Criteria for the onset of instability are found following Eady and by using difference methods.

For the two layer model we first use difference methods to find approximate analytical and numerical formulae for the onset of instability. A second approach to the two layer model, following Meksyn (26) also results in analytical criteria for instability, producing first order correction terms to the Eady results. Finally, using the methods of Meksyn again the model is extended to include a variable coriolis parameter and an approximate stability equation is found.

It is clear that a gaseous model for the upper layer would be more realistic but for simplicity we have chosen a liquid model. In Appendix B we aim to show that the main characteristics of a liquid model will carry over to a gaseous model.

CHAPTER 3.2

FORMULATION OF THE TWO LAYER LIQUID MODEL

We will use a rectangular coordinate system (x, y, z) with x increasing eastwards, y northwards and z vertically upward. This is mathematically permissible when the length of the wave is small compared with the circumference of the zonal circle along which the wave moves. However, in practice it has been found to be far more flexible than the mathematics would suggest and has been widely used by Rossby (27), Eady (28) and Charney (24). The governing equations for the two layers are as follows.

THE GOVERNING EQUATIONS FOR THE LOWER LAYER

The lower layer is in a static state so that the only equation governing the layer is the heat transfer equation (9), which with no external heating function takes the form

$$\frac{dQ^*}{dt} = \frac{1}{\rho_2}\operatorname{div}(k_2 \operatorname{grad} T^*)$$

where Q^* is the heat content per unit mass, ρ_2 is the constant density, T^* the temperature and k_2 the thermal conductivity of the lower layer. For a liquid (10) we have

$$\delta Q^* = c_2 \delta T^*$$

where c_2 is the specific heat of the layer. We assume that k_2 is constant and that it is the vertical component, $k_2\frac{\partial^2 T^*}{\partial z^2}$ which is the important part of the conduction term in the heat transfer equation. Accordingly the heat transfer equation may be approximated to

$$\frac{\partial T^*}{\partial t} = \frac{k_2}{\rho_2 c_2}\frac{\partial^2 T^*}{\partial z^2} \quad . \tag{3.1}$$

THE GOVERNING EQUATIONS FOR THE UPPER LAYER

We take the horizontal equations of motion in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - lv = -\frac{1}{\rho_1}\frac{\partial p}{\partial x}, \tag{3.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + lu = -\frac{1}{\rho_1}\frac{\partial p}{\partial y}, \tag{3.3}$$

where u and v are the velocity components in the x and y directions, p is the pressure, ρ_1 is a mean density of the upper layer, and where l is defined by

$$l = 2\Omega\sin\theta \tag{3.4}$$

with Ω as the angular velocity of the Earth and θ as the geographical latitude. In (3.2) and (3.3) we have ignored viscous terms which would increase the order of our final differential equation for the perturbation pressure from four to ten, making the solution far more difficult. Viscosity can cause instability (29) and it is not clear without a detailed study how it would effect our conclusions if these terms were retained (40,41). The vertical equation of motion is approximated to the hydrostatic equation,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{3.5}$$

For a liquid the equation of continuity is approximated to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.6}$$

where w is the velocity component in the z direction.

The heat transfer equation (9), with no external heat sources can be written as

$$\frac{dQ}{dt} = \frac{1}{\rho}\operatorname{div}(k_1 \operatorname{grad} T)$$

where Q is the heat content per unit mass, T the temperature and k_1 the thermal conductivity of the upper layer. As for the lower layer (10) we have

$$\delta Q = c_1\delta T$$

where c_1 is the specific heat of the upper layer. Again we assume the dominant variations of temperature are in the vertical direction and assuming that k_1 is constant we approximate the heat transfer equation to

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k_1}{c_1\rho_1}\frac{\partial^2 T}{\partial z^2} . \tag{3.7}$$

The Boussinesq approximation (9) has been used in (3.2), (3.3) and (3.7) to replace ρ by a mean density ρ_1 and in (3.6) to ignore variations of ρ. We will now form the linearised perturbation equations for the two layers.

THE PERTURBATION EQUATION FOR THE LOWER LAYER

We assume that T* comprises of a basic steady temperature distribution, $\bar{T}_2 + (A_2z-B_2y)$ and a small departure from the basic field, $T_2'(x, y, z, t)$, so that

$$T^* = \bar{T}_2 + (A_2 z - B_2 y) + T_2'(x,y,z,t) . \tag{3.8}$$

In the above expression

$$\bar{T}_2 >> (A_2 z - B_2 y) >> T_2'(x,y,z,t) ;$$

$\bar{T}_2$ is the dominant term with (A_2z-B_2y) and T_2' being of decreasing importance in numerical magnitude. The orders of magnitude will be referred to as O(1), O(2) and O(3) for $\bar{T}_2$, (A_2z-B_2y) and T_2' respectively. The linearised heat transfer equation for T_2' reduces to

$$\frac{\partial T_2'}{\partial t} = \frac{k_2}{c_2\rho_2}\frac{\partial^2 T_2'}{\partial z^2} \tag{3.9}$$

We will look for solutions of T_2' in the form

$$T_2'(x,y,z,t) = \phi(z)\, e^{ik(x-ct)} . \tag{3.10}$$

It will be noticed that T_2' is not dependent on y so that there will be no variation in the northward direction. This is a simplifying device which has been used by both Eady and Charney (28,24). Substituting for T_2' in (3.9) results in the second order differential equation for $\phi(z)$,

$$\frac{k_2}{c_2\rho_2}\frac{d^2\phi}{dz^2} + ikc\,\phi = 0 . \tag{3.11}$$

THE PERTURBATION EQUATIONS FOR THE UPPER LAYER

As in the above section we will assume that there is a basic steady state and a small departure from that state. We will assume that the complete temperature distribution T takes the form

$$T = \underset{O(1)}{\bar{T}} + \underset{O(2)}{(Az - By)} + \underset{O(3)}{T'(x,y,z,t)} \tag{3.12}$$

where the constants A and B will be related to A_2 and B_2 as defined in (3.8) due to continuity of temperature and heat transfer at the interface of the two layers. The complete density distribution will be defined as

$$\rho = \underset{O(1)}{\rho_1} - \underset{O(2)}{\alpha(Az - By)} - \underset{O(3)}{\alpha T'(x,y,z,t)} \tag{3.13}$$

where ρ_1 is the constant density as introduced in (3.2) and (3.3) and α is an experimentally determined constant, being the coefficient of cubical expansion (13). Similarly, the pressure function will be taken in the form

$$p = \underset{O(1)}{\bar{p}(z)} + \underset{O(2)}{p_0(y,z)} + \underset{O(3)}{p'(x,y,z,t)} \tag{3.14}$$

where p(z) is the hydrostatic pressure given by $p(z) = -g\rho_1 z$ + constant. The velocity field will be as follows

$$u = \qquad U(z) + u(x,y,z,t) , \tag{3.15}$$

$$v = \qquad v(x,y,z,t) , \tag{3.16}$$

$$w = \qquad w(x,y,z,t) . \tag{3.17}$$

$$\qquad O(1) \qquad O(2) \qquad O(3)$$

The O(2) term, U(z) is referred to as the thermal wind as is due to the temperature field (Az-By) and is determined as follows. When O(2) terms are retained in (3.2) and (3.3) we obtain

$$\left.\begin{aligned} lU(z) &= -\frac{1}{\rho_1}\frac{\partial p_0}{\partial y} , \\ \frac{\partial p_0}{\partial z} &= g\alpha(Az - By). \end{aligned}\right\} \tag{3.18}$$

We eliminate p_0 between the two equations to produce

$$l\frac{dU}{dz} = -\frac{1}{\rho_1}\frac{\partial}{\partial y}\left[g\alpha(Az-By)\right]$$

and hence

$$U(z) = \frac{g\alpha B z}{\rho_1 l} + \text{constant}. \tag{3.19}$$

We now consider the O(3) terms. Using the O(2) results, (3.2) and (3.3) may be rewritten in the linearised form

$$\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} + w\frac{dU(z)}{dz} - lv = -\frac{1}{\rho_1}\frac{\partial p'}{\partial x}, \tag{3.20}$$

$$\frac{\partial v}{\partial t} + U(z)\frac{\partial v}{\partial x} + lu = -\frac{1}{\rho_1}\frac{\partial p'}{\partial y}. \tag{3.21}$$

In addition (3.5) becomes

$$g\alpha T' = \frac{\partial p'}{\partial z} \tag{3.22}$$

and for the equation of continuity we may write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3.23}$$

In (3.7) we replace the term $w\frac{\partial T}{\partial z}$ by wA and $v\frac{\partial T}{\partial y}$ by $-vB$ using the approximation $T \sim \bar{T}(z) + Az - By$ and hence

$$\frac{\partial T'}{\partial t} + U(z)\frac{\partial T'}{\partial x} - vB + wA = \frac{k_1}{c_1\rho_1}\frac{\partial^2 T'}{\partial z^2}. \tag{3.24}$$

Thus (3.20)-(3.24) are the basic equations for the O(3) terms.

Now proceeding to eliminate p' between (3.20) and (3.21) we obtain

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + l\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + v\frac{\partial l}{\partial y} - U'(z)\frac{\partial w}{\partial y} = 0$$

and by using (3.23) the equation can be rewritten as

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \beta v = l\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}U'(z) \tag{3.25}$$

where $\beta = \frac{\partial l}{\partial y}$. This is the vorticity equation and the terms $l\frac{\partial w}{\partial z}$ and $U'(z)\frac{\partial w}{\partial y}$ on the right hand side are 'baroclinic' additions to the equation. We now introduce the geostrophic approximation (21) to (3.20) and (3.21), namely,

$$\left.\begin{aligned} u &= -\frac{1}{l\rho_1}\frac{\partial p'}{\partial y}, \\ v &= \frac{1}{l\rho_1}\frac{\partial p'}{\partial x}; \end{aligned}\right\} \tag{3.26}$$

we follow Charney (24) in the use of this approximation, thus

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{1}{l\rho_1}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) . \tag{3.27}$$

Using (3.26) and (3.27) in (3.25) results in the expression

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + \beta\frac{\partial p'}{\partial x} = \rho_1 l^2 \frac{\partial w}{\partial x} + \rho_1 l U(z)\frac{\partial w}{\partial y} . \tag{3.28}$$

We now have in (3.28) a relation between p' and w. A second expression between p' and w can be found from (3.24), using (3.22), (3.26) and (3.27), of the form

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right] \frac{\partial p'}{\partial z} - \frac{g\alpha B}{l\rho_1}\frac{\partial p'}{\partial x} + g\alpha A w = \frac{k_1}{c_1\rho_1}\frac{\partial^3 p'}{\partial z^3} . \tag{3.29}$$

We look for solutions of p' (x, y, z, t) and w(x, y, z, t) in the same form as T_2' (x, y, z, t) as defined in (3.10) for consistency between the two layers; thus we take

$$\left.\begin{aligned} p'(x,y,z,t) &= P(z)\, e^{ik(x-ct)}, \\ w(x,y,z,t) &= W(z)\, e^{ik(x-ct)}, \end{aligned}\right\} \tag{3.30}$$

where $k = 2\pi/L$ with L being the wave length in the x direction. We substitute for P' and w into (3.28) and (3.29) to obtain

$$ik\left[-k^2(U(z)-c) + \beta\right] P = \rho_1 l^2 \frac{dW}{dz} , \tag{3.31}$$

$$ik\left[(U(z)-c)\frac{dP}{dz} - \frac{g\alpha B}{l\rho_0}P\right] - \frac{k_1}{c_1\rho_1}\frac{d^3P}{dz^3} = -g\alpha A W . \tag{3.32}$$

We can now eliminate W between (3.31) and (3.32) to form a fourth order differential equation for P, namely

$$\frac{k_1}{c_1\rho_1}\frac{d^4P}{dz^4} - ik(U(z)-c)\left\{\frac{d^2P}{dz^2} - \frac{g\alpha A k^2}{\rho_1 l^2}P\right\} - \frac{ik\beta g\alpha A}{\rho_1 l^2}P = 0 . \tag{3.33}$$

For the earlier part of the work we take $\beta = 0$, thus ignoring variations in the coriolis parameter but in Chapter 3.6 we consider the case when $\beta \neq 0$. At this stage, however (3.33) reduces to

$$\frac{k_1}{c_1\rho_1}\frac{d^4P}{dz^4} - ik(U(z)-c)\left[\frac{d^2P}{dz^2} - \frac{g\alpha A k^2}{\rho_1 l^2}P\right] = 0 . \tag{3.34}$$

THE BOUNDARY CONDITIONS

Thus (3.11) and (3.34) are the two basic equations we wish to solve. To complete the system of equations we need to introduce six boundary conditions and we choose the following:

(1) The vertical velocity is zero at the atmosphere/ocean interface,

$$W = 0 \quad \text{at } z = 0. \tag{3.35}$$

(2) The vertical velocity is zero at the top of the atmosphere,

$$W = 0 \quad \text{at } z = h. \tag{3.36}$$

(3) Continuity of temperature at the atmosphere/ocean interface,

$$T = T^* \qquad \text{at } z = 0. \tag{3.37}$$

(4) Continuity of heat transfer at the atmosphere/ocean interface,

$$k_1 \frac{\partial T}{\partial z} = k_2 \frac{\partial T^*}{\partial z} \quad \text{at } z = 0. \tag{3.38}$$

(5) It is assumed that there is no heat transfer at the bottom of the ocean, $z = -h_2$,

$$\frac{\partial T^*}{\partial z} = 0 \quad \text{at } z = -h_2 . \tag{3.39}$$

(6) It is assumed that there is no heat lost at the top of the atmosphere,

$$\frac{\partial T}{\partial z} = 0 \quad \text{at } z = h. \tag{3.40}$$

Using (3.32) the first two boundary conditions become

$$(1) \quad \frac{k_1}{c_1 \rho_1} \frac{d^3 P}{dz^3} - ik\left[(U(z) - c) \frac{dP}{dz} - U'(z) P \right] = 0 \quad \text{at } z = 0 , \tag{3.41}$$

$$(2) \quad \frac{k_1}{c_1 \rho_1} \frac{d^3 P}{dz^3} - ik\left[(U(z) - c) \frac{dP}{dz} - U'(z) P \right] = 0 \quad \text{at } z = h . \tag{3.42}$$

Using (3.22), boundary conditions (3.37) and (3.38) and (3.40) become

(3) $$\frac{1}{g\alpha}\frac{dP}{dz} = \phi \quad \text{at } z=0 \quad , \tag{3.43}$$

(4) $$\frac{k_1}{g\alpha}\frac{d^2P}{dz^2} = k_2\frac{d\phi}{dz} \quad \text{at } z=0 \, , \tag{3.44}$$

(6) $$\frac{d^2P}{dz^2} = 0 \quad \text{at } z=h \, . \tag{3.45}$$

THE SOLUTION FOR THE LOWER LAYER

We can now solve (3.11) and using the boundary condition (3.39) we obtain the solution

$$\phi(z) = D\cosh(\varsigma(z+h_2)) \tag{3.46}$$

where D is an arbitary constant and

$$\varsigma^2 = -kc\rho_2 c_2/k_2 \, . \tag{3.47}$$

By substituting (3.46) into the boundary conditions (3.43) and (3.44) we obtain

$$\left.\begin{aligned} \frac{1}{g\alpha}\frac{dP}{dz}\Big|_{z=0} &= D\cosh(\varsigma h_2) \, , \\ \frac{k_1}{g\alpha}\frac{d^2P}{dz^2}\Big|_{z=0} &= k_2\varsigma D\sinh(\varsigma h_2) . \end{aligned}\right\} \tag{3.48}$$

Thus D can be eliminated between the two equations in (3.48) to produce one boundary equation for P, namely

$$\frac{d^2P}{dz^2} - \frac{k_2\varsigma}{k_1}\tanh(\varsigma h_2)\frac{dP}{dz} = 0 \quad \text{at } z=0 . \tag{3.49}$$

We now have a fourth order differential equation,(3.34) and four boundary conditions, (3.41), (3.42), (3.45) and (3.49) for P. We will define

$$\varepsilon = k_1/ck \quad , \tag{3.50}$$

$$\gamma^2 = g\alpha A k^2/\rho_1 l^2 \quad , \tag{3.51}$$

$$H = k_2\varsigma\tanh(\varsigma h_2)/k_2 \, , \tag{3.52}$$

and thus the system of equations to be solved is

$$\frac{\varepsilon}{\rho_1}\frac{d^4P}{dz^4} - i\,(U(z)-c)\left[\frac{d^2P}{dz^2} - \gamma^2 P\right] = 0 \tag{3.53}$$

subject to the boundary conditions

$$\left.\begin{aligned} &\frac{\varepsilon}{\rho_1}\frac{d^3P}{dz^3} - i\left[(U(z)-c)\frac{dP}{dz} - U'(z)P\right] = 0 \quad \text{at } z=0,h \\ &\frac{d^2P}{dz^2} - H\frac{dP}{dz} = 0 \qquad \text{at } z=0 \\ &\frac{d^2P}{dz^2} = 0 \qquad \text{at } z=h \end{aligned}\right\} \tag{3.54}$$

VALUES FOR CONSTANTS IN THE MODEL

In the following work we take the height of the upper layer to be the height of the troposphere. We have taken the following average values for the constants in the model (13, 15):

UPPER LAYER	LOWER LAYER
$h = 16\times10^5$ cm	$h_2 = 5\times10^5$ cm
$\rho_1 = 1{\cdot}28\times10^{-3}$ gcm^{-3}	$\rho_2 = 0{\cdot}1$ gcm^{-3}
$c_1 = 1$ J gm^{-1} °c^{-1}	$c_2 = 4{\cdot}22$ J gm^{-1} °c^{-1}
$k_1 = 2{\cdot}41\times10^{-4}$ Jcm^{-1}s^{-1}°c^{-1}	$k_2 = 5{\cdot}61\times10^{-3}$ J cm^{-1}s^{-1}°c^{-1}

(3.55)

We have used here the molecular values of thermal conductivity. For eddy conductivities the values of k_1 and k_2 are approximately 10^5 times as large (5). It is clear that ε , being the same order of magnitude as k_1 is a small quantity and thus will be exploited in the methods used to solve (3.53). We have found that $U(z)$ is a linear function of z. Using average values, $U(0) = 600\text{cms}^{-1}$, $U(h) = 1400\text{cms}^{-1}$ (24) we can formulate $U(z)$ in the form

$$U(z) = U_0 + (z/h - \tfrac{1}{2}U_1) \tag{3.56}$$

where

$$\left.\begin{aligned} U_0 &= 1000 \text{ cms}^{-1} \\ U_1 &= 800 \text{ cms}^{-1} \end{aligned}\right\} \qquad (3.57)$$

From (3.52) and (3.47) we find that H takes the form

$$H = \frac{k_2}{k_1}(1-i)\left[kc\rho_2 c_2/2k_2\right]^{1/2} \tanh\left[(1-i)h_2\left[kc\rho_2 c_2/2k_2\right]^{1/2}\right].$$

Using (3.50) we can rewrite H as

$$H = (1-i)\varepsilon^{-1/2}c^{1/2}\left[\rho_2 c_2 k_2/2k_1 c_1\right]^{1/2} \tanh\left[(1-i)h_2\left[kc\rho_2 c_2/2k_2\right]^{1/2}\right]. \qquad (3.58)$$

For large wavelengths k, we can assume $\left(h_2\left[kc\rho_2 c_2/2k_2\right]^{1/2}\right)$ is large and the asymptotic expansion for tanh can be used. Since k_1 and k_2 are of similar orders of magnitude, $\left[\rho_2 c_2 k_2/2k_1 c_1\right]^{1/2}$ will not be a large quantity. We may theefore take H as

$$H = \varepsilon^{-1/2}c^{1/2}(1-i)\lambda \qquad (3.59)$$

where

$$\lambda = \left[\rho_2 c_2 k_2/2k_1 c_1\right]^{1/2}. \qquad (3.60)$$

CHAPTER 3.3

THE EADY MODEL

Before solving the two layer problem defined by (3.53) subject to (3.54) we will look at a simplified situation, studied by Eady (28).

FORMULATION OF THE MODEL

If we consider only the upper layer of the model the governing equations will be (3.2), (3.3), (3.5), (3.6) and (3.7). Further if we ignore thermal conductivity, the heat transfer equation (3.7) will reduce to

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = 0. \qquad (3.61)$$

Following the same procedure as in Chapter 3.2 we find that the pressure function P(z) satisfies the second order ordinary differential equation

$$\frac{d^2P}{dz^2} - \gamma^2 P = 0 \qquad (3.62)$$

where γ^2 is defined in (3,51). This is the model investigated by Eady. The two boundary conditions chosen by Eady were

$w = 0$ at $z = 0$ and $z = h$.

The simplified form of (3.33) for this problem is

$$-g\alpha A W = ik\left[(U(z)-c)\frac{dP}{dz} - U'(z)P\right];$$

thus the boundary conditions become

$$(U(z)-c)\frac{dP}{dz} - U'(z)P = 0 \quad \text{at } z=0,h.$$

The Eady problem thus reduces to the solution of

$$\left.\begin{array}{l} \dfrac{d^2P}{dz^2} - \gamma^2 P = 0 \\ \text{subject to} \\ (U(z)-c)\dfrac{dP}{dz} - U'(z)P = 0 \quad \text{at } z=0,h. \end{array}\right\} \qquad (3.63)$$

THE SOLUTION FOLLOWING EADY

The solution of the differential equation in (3.63) is

$$P(z) = Ce^{\gamma z} + De^{-\gamma z} \qquad (3.64)$$

where C and D are arbitary constants. If we substitute (3.64) into the boundary conditions we obtain two equations for C and D, namely

$$\left.\begin{aligned} &C\left[\gamma(U_0-U_1/2-c)-U_1/h\right] + D\left[-\gamma(U_0-U_1/2-c)-U_1/h\right]=0,\\ &Ce^{\gamma h}\left[\gamma(U_0+U_1/2-c)-U_1/h\right] + De^{-\gamma h}\left[-\gamma(U_0+U_1/2-c)-U_1/h\right]=0,\end{aligned}\right\} \tag{3.65}$$

where

$$U = U_0 + U_1(z/h - 1/2).$$

The two equations for C and D in (3.65) leads to an equation of consistency, namely

$$e^{-\gamma h}\left[-\gamma(U_0+U_1/2-c)-U_1/h\right]\left[\gamma(U_0-U_1/2-c)-U_1/h\right] = e^{\gamma h}\left[\gamma(U_0+U_1/2-c)-U_1/h\right]\left[-\gamma(U_0-U_1/2-c)-U_1/h\right]$$

After simplification this can be written as

$$(U_0-c)^2 = \frac{U_1^2}{\gamma^2h^2}\left\{\frac{\gamma^2h^2}{4}+1-\gamma h\coth\gamma h\right\}. \tag{3.66}$$

Before discussing (3.66) we will look at another method which yields the same equation of consistency.

THE SOLUTION BY DIFFERENCE METHODS

We will show that difference methods yield the consistency equation, (3.66) without solving the differential equation defined in (3.63). Using central differencing formulae (30) for the boundary conditions at z = 0 and z = h respectively, we have the following (n + 1) equations for P_i, i = 0, 1, ...n,

$$\left.\begin{aligned} &(U_0-c+U_1(1/n-1/2))P_0-(U_0-c-U_1/2)P_1=0,\\ &P_i-(2+\gamma^2h^2/n^2)P_{i+1}+P_{i+2}=0, \qquad 0\le i\le n-1,\\ &(-U_0+c-U_1/2)P_{n-1}+(U_0-c+U_1(1/2-1/n))P_n=0.\end{aligned}\right\} \tag{3.67}$$

The (n + 1) equations thus result in the equation of consistency which can be written in the form

$$\begin{vmatrix} U_0-c+U_1(\frac{1}{n}-\frac{1}{2}) & -U_0+c+U_1/2 & 0 & 0 & \cdots & & \\ 1 & -(2+\gamma^2h^2/n^2) & 1 & 0 & \cdots & & \\ 0 & 1 & -(2+\gamma^2h^2/n^2) & 1 & \cdots & & \\ & & \ddots & \ddots & \ddots & & \\ \cdots & & 0 & 1 & -(2+\gamma^2h^2/n^2) & \cdots & 1 \\ \cdots & & 0 & 0 & -U_0+c-U_1/2 & & U_0-c+U_1(\frac{1}{2}-\frac{1}{n}) \end{vmatrix} = 0. \tag{3.68}$$

The smallest value n can take is 2, and we will show later that in the limit as $n \to \infty$ we obtain the exact consistency equation, (3.66).

For n = 2, (3.68) reduces to

$$-\frac{\gamma^2 h^2}{4}(U_0-c)^2 = 0$$

so that the solution for c is

$$c = U_0 .$$

For n = 3 the equation of consistency reduces to

$$(U_0-c)^2 = \frac{U_1^2}{36}\frac{(-2+\gamma^2h^2/9)}{(2+\gamma^2h^2/9)}$$

For $0 < \gamma h < 4.28$ c will be a complex quantity and for $4.28 < \gamma h$, c is purely real. With n = 4 we find that

$$(U_0-c)^2 = \frac{U_1^2}{16}\frac{(1+\gamma^2h^2/16)^2-3}{(2+\gamma^2h^2/16)^2-1}$$

so that for $0 < \gamma h < 3.42$ c is complex and $3.42 < \gamma h$ c is real. Finally for n = 5,

$$(U_0-c)^2 = \frac{U_1^2}{100}\frac{(9\lambda^3-12\lambda^2-26\lambda+8)}{\lambda(\lambda^2-2)}$$

where $\lambda = 2 + \gamma^2h^2/25$ and so for $0 < \gamma h < 2.5$ c is complex and for $2.5 < \gamma h$ c is real. When we discuss (3.66) we will find that the values for c at n = 5 are very accurate approximations to the exact results.

We will now look at the general equation of consistency (3.68). Using the notation that Δ_{n+1} is the whole determinant and Δ_s is the determinant of the first s rows and columns in the lower right hand corner, the following equations can be deduced:

$$\left.\begin{aligned}
\Delta_{n+1} &= (U_0-c+U_1(\tfrac{1}{n}-\tfrac{1}{2}))\Delta_n + (U_0-c-U_1/2)\Delta_{n-1} = 0,\\
\Delta_n &= -(2+\gamma^2h^2/n^2)\Delta_{n-1} - \Delta_{n-2},\\
&\;\;\vdots\\
\Delta_{s+1} &= -(2+\gamma^2h^2/n^2)\Delta_s - \Delta_{s-1},\\
&\;\;\vdots\\
\Delta_3 &= -(2+\gamma^2h^2/n^2)\Delta_2 - \Delta_1,\\
\Delta_2 &= -(2+\gamma^2h^2/n^2)\Delta_1 + (U_0-c+U_1/2),\\
\Delta_1 &= U_0-c+U_1(\tfrac{1}{2}-\tfrac{1}{n}).
\end{aligned}\right\} \qquad (3.69)$$

A solution for Δ_s can be found in the form

$$\Delta_s = A\cosh s\theta + B\sinh s\theta. \tag{3.70}$$

Using the result that

$$\Delta_{s+1} + \Delta_{s-1} = 2\cosh\theta\,\Delta_s ,$$

we can define θ from (3.69) as

$$\theta = \cosh^{-1}\left[-1-\gamma^2h^2/2n^2\right]. \tag{3.71}$$

The equations for Δ_1 and Δ_2 will define A and B. From the equation for Δ_2 we have

$$A\cosh 2\theta + B\sinh 2\theta = 2\cosh\theta\left[A\cosh\theta + B\sinh\theta\right] + U_0 - c + U_1/2.$$

and hence

$$A = -U_0 + c - U_1/2. \tag{3.72}$$

The equation for Δ_1 leads to an equation between A and B,

$$A\cosh\theta + B\sinh\theta = U_0 - c + U_1\left(\tfrac{1}{2}-\tfrac{1}{n}\right)$$

and by substituting for A and cosh θ from (3.71) and (3.72) we obtain

$$B\sinh\theta = -\frac{\gamma^2h^2}{2n^2}\left(U_0 - c + U_1/2\right) - U_1/n \tag{3.73}$$

where from (3.71)

$$\sinh\theta = -\frac{\gamma h}{n}\left[1+\frac{\gamma^2h^2}{4n^2}\right]^{1/2}.$$

The equation of consistency, $\Delta_{n+1} = 0$ can be written as

$$\left(U_0 - c + U_1\left(\tfrac{1}{n}-\tfrac{1}{2}\right)\right)\left[A\cosh n\theta + B\sinh n\theta\right]$$

$$+\left(U_0 - c - U_1/2\right)\left[A\cosh(n-1)\theta + B\sinh(n-1)\theta\right] = 0 . \tag{3.74}$$

When we substitute for A, B and θ from (3.71), (3.72) and (3.73), (3.74) becomes an equation for c.

When we substitute n = 2, 3, 4 an d 5 (3.74) will reduce to the formulae found earlier in this chapter. We now wish to find the values of c when $n \rightarrow \infty$.

From (3.74) we have

$$A\left\{(U_0-c-U_1/2)\left[(1+\cosh\theta)\cosh n\theta-\sinh\theta\sinh n\theta\right]+U_1/n\cosh n\theta\right\}$$

$$+B\left\{(U_0-c-U_1/2)\left[(1+\cosh\theta)\sinh n\theta-\sinh\theta\cosh n\theta\right]+U_1/n\sinh n\theta\right\}=0$$

(3.75)

It is necessary to find thelimits of $\cosh(n\theta)$ and $\sinh(n\theta)$ as $n\to\infty$.

From (3.71) we obtain

$$e^{\theta}+e^{-\theta}=2(1+\gamma^2h^2/2n^2)$$

and we multiply the equation by e^{θ} to produce a quadratic equation for e^{θ} ; taking the positive root we find

$$e^{\theta}=1+\frac{\gamma^2h^2}{2n^2}+\frac{\gamma h}{n}\left[1+\frac{\gamma^2h^2}{4n^2}\right]^{1/2}.$$

When n is sufficiently large we can now write

$$e^{\theta}=1+\frac{\gamma h}{n}+O\left(\frac{1}{n^2}\right),$$

$$e^{-\theta}=1-\frac{\gamma h}{n}+O\left(\frac{1}{n^2}\right).$$

With the result that

$$\lim_{n\to\infty}(1+x/n)^n=e^x$$

we have

$$\lim_{n\to\infty}e^{n\theta}=\lim_{n\to\infty}\left[1+\frac{\gamma h}{n}+O\left(\frac{1}{n^2}\right)\right]^n=e^{\gamma h}$$

and hence

$$\left.\begin{aligned}\lim_{n\to\infty}\cosh n\theta&=\cosh\gamma h,\\ \lim_{n\to\infty}\sinh n\theta&=\sinh\gamma h.\end{aligned}\right\}\qquad(3.76)$$

Substituting into (3.75) the values of A and B from (3.72) and (3.73) and using the limits (3.76), retaining terms of order $\frac{1}{n}$ yields

$$(-U_0+c-U_1/2)\left[(U_0-c-U_1/2)\gamma h\tanh\gamma h+U_1\right]+U_1(U_0-c-U_1/2)+\frac{U_1}{\gamma h}\tanh\gamma h=0$$

which simplifies to

$$(U_0 - c)^2 = \frac{U_1^2}{\gamma^2 h^2}\left(1 + \frac{\gamma^2 h^2}{4} - \gamma h \coth \gamma h\right). \tag{3.77}$$

This is the same equation found by solving the differential equation.

INVESTIGATION OF THE EQUATION OF CONSISTENCY

The equation of consistency, (3.66) found by both methods can be written as

$$(U_0 - c)^2 = \frac{U_1^2}{\gamma^2 h^2}\left[\frac{\gamma h}{2} - \coth\frac{\gamma h}{2}\right]\left[\frac{\gamma h}{2} - \tanh\frac{\gamma h}{2}\right] \tag{3.78}$$

where U_0 is the mean velocity of the zonal current and $U_1 = g\alpha B h/\rho_1 l$. The right hand side of (3.78) has one zero at approximately $\gamma h = 2.4$. When $0 < \gamma h < 2.4$ the right hand side of (3.78) is negative and when $2.4 < \gamma h$ the right hand side is positive. Thus if we write

$$d(\gamma h) = \left[\,\left|1 + \gamma^2 h^2/4 - \gamma h \coth \gamma h\right|\,\right]^{1/2} \tag{3.79}$$

we have the following values for c

$$c = U_0 \pm \frac{iU_1}{\gamma h} d(\gamma h), \quad 0 < \gamma h < 2.4, \tag{3.80}$$

$$c = U_0 \pm \frac{U_1}{\gamma h} d(\gamma h), \quad 2.4 < \gamma h. \tag{3.81}$$

To interpret (3.80) and (3.81) we return to the formulation of the perturbation pressure, $P'(x, z, t)$ which in (3.30) we defined as

$$P'(x, z, t) = P(z)\, e^{ik(x - ct)}.$$

When c has a positive imaginary term, the resulting wave term grows exponentially and is therefore unstable. For the region $2.4 < \gamma h$, c is purely real and so only progressive waves are formed. For the range $0 < \gamma h < 2.4$ however c is complex. There are two values for c; when $c = U_0 - i\frac{U_1}{\gamma h} d(\gamma h)$ the wave is damped and tends to a zero amplitude with increasing time but when $c = U_0 + i\frac{U_1}{\gamma h} d(\gamma h)$ an unstable wave exists.

The unstable wave will double its amplitude in a time t_1 given by

$$\frac{kU_1}{\gamma h} d(\gamma h) t_1 = 0\cdot 693 . \tag{3.82}$$

The function $d(\gamma h)$ is zero at $\gamma h = 0$ and $\gamma h = 2.4$ and attains a maximum at $\gamma h = 1.61$ of 0.3. The maximum growth rate is associated with a wavelength k_0, where

$$\frac{k_0 h}{l} \left[g\alpha A/\rho_1 \right]^{1/2} = 1\cdot 61 \tag{3.83}$$

since the definition of γ from (3.51) is

$$\gamma = \frac{k}{l} \left[g\alpha A/\rho_1 \right]^{1/2} .$$

CHAPTER 3.4

AN APPROXIMATE SOLUTION OF THE STABILITY PROBLEM USING DIFFERENCE METHODS

From Chapter 3.3 the method of differencing has been successful in obtaining a consistency equation and so the same method will be used here on the two layer model formulated in Chapter 3.2. The system of equations to be solved is

$$\frac{\epsilon}{\rho_1}\frac{d^4P}{dz^4} - i(U(z)-c)\left[\frac{d^2P}{dz^2} - \gamma^2 P\right] = 0, \quad 0 \leq z \leq h, \tag{3.84}$$

subject to

$$\left.\begin{aligned}
&\frac{\epsilon}{\rho_1}\frac{d^3P}{dz^3} - i\left[(U(z)-c)\frac{dP}{dz} - U'(z)P\right] = 0 \quad \text{at } z=0,h, \\
&\frac{d^2P}{dz^2} - H\frac{dP}{dz} = 0 \quad \text{at } z=0, \\
&\frac{d^2P}{dz^2} = 0 \quad \text{at } z=h.
\end{aligned}\right\} \tag{3.85}$$

As before we take U(z) in the form

$$U(z) = U_0 + U_1(z/h - 1/2). \tag{3.86}$$

THE DIFFERENCE METHOD

We divide the space z = 0 to z = h into n intervals and using the central differencing (30) formulae for P_i^{IV} and for P_i'',

$$\left.\begin{aligned}
P_i^{IV} &= [P_{i+2} - 4P_{i+1} + 6P_i - 4P_{i-1} + P_{i-2}]/(h/n)^4, \\
P_i'' &= [P_{i+1} - 2P_i + P_{i-1}]/(h/n)^2,
\end{aligned}\right\} \tag{3.87}$$

we transform (3.84) into (n-3) equations for P_j, $2 \leq j \leq n-1$,

$$\frac{\epsilon}{\rho_0}\left[P_{j+2} - 4P_{j+1} + 6P_j - 4P_{j-1} + P_{j-2}\right] - \frac{ih^2}{n^2}\left[U_0 - c + U_1\left[j/n - \tfrac{1}{2}\right]\right]\left[P_{j+1} - \left(2 + \frac{\gamma^2 h^2}{n^2}\right)P_j + P_{j-1}\right] = 0, \quad 2 \leq j \leq n-1. \tag{3.88}$$

We use the following forward difference formulae (30)

$$\left.\begin{aligned} P_i''' &= [P_{i+3} - 3P_{i+2} + 3P_{i+1} - P_i]/(\tfrac{h}{n})^3, \\ P_i'' &= [P_{i+2} - 2P_{i+1} + P_i]/(\tfrac{h}{n})^2, \\ P_i' &= [P_{i+1} - P_i]/(\tfrac{h}{n}), \end{aligned}\right\} \tag{3.89}$$

to transform the two boundary conditions z = 0 to

$$P_2 - 2P_1 + P_0 - \frac{Hh}{n}[P_1 - P_0] = 0, \tag{3.90}$$

$$\frac{\epsilon}{\rho_0}[P_3 - 3P_2 + 3P_1 - P_0] - \frac{ih^2}{n^2}(U_0 - \frac{U_1}{2} - c)[P_1 - P_0] + \frac{ih^3}{n^3}\frac{U_1}{h}P_0 = 0. \tag{3.91}$$

Similarly using the backward difference formulae(30),

$$\left.\begin{aligned} P_i''' &= [P_i - 3P_{i-1} + 3P_{i-2} - P_{i-3}]/(\tfrac{h}{n})^3, \\ P_i'' &= [P_i - 2P_{i-1} + P_{i-2}]/(\tfrac{h}{n})^2, \\ P_i' &= [P_i - P_{i-1}]/(\tfrac{h}{n}), \end{aligned}\right\} \tag{3.92}$$

the boundary conditions at z = h can be written in the form

$$\frac{\epsilon}{\rho_0}[P_n - 3P_{n-1} + 3P_{n-2} - P_{n-3}] - \frac{ih^2}{n^2}(U_0 + \frac{U_1}{2} - c)[P_n - P_{n-1}] + \frac{ih^3}{n^3}\frac{U_1}{h}P_n = 0, \tag{3.93}$$

$$P_n - 2P_{n-1} + P_{n-2} = 0. \tag{3.94}$$

We now have (n + 1) equations in (n + 1) unknowns, namely P_0, P_1, ...P_n. The equations can be written in a matrix form,

$$A_n \underset{\sim}{P} = \underset{\sim}{0}$$

where A_n is the (n + 1) x (n + 1) matrix formed by the coefficinets of in the equations (3.88), (3.90), (3.91), (3.93) and (3.94). The system is homogeneous, producing an eigen value type problem. For the equations to be consistent,

$$|A_n| = 0. \tag{3.95}$$

Thus there are only a finite set of values for c for which the equation

(3.95) holds. Using the notation

$$\left.\begin{aligned} U_j &= (u_0 + (j/n - \tfrac{1}{2})u_1 - c) \\ \delta &= 2 + \frac{\gamma^2 h^2}{n^2} \\ m &= h^2/n^2 \\ E &= \epsilon/\rho_1 \end{aligned}\right\} \qquad (3.96)$$

we can write (3.95) as

$$\begin{vmatrix} 1+\frac{Hh}{n} & -2-\frac{Hh}{n} & 1 & 0 & 0 & \cdots & & & & \\ -E+im(u_0-\frac{u_1}{2}+\frac{u_1}{n}-c) & 3E-im(u_0-\frac{u_1}{2}-c) & -3E & E & 0 & \cdots & & & & \\ E & -4E-imU_2 & 6E-im\delta U_2 & -4E-imU_2 & E & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & E & -4E-imU_j & 6E-im\delta U_j & -4E-imU_j & E & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & E & -4E-imU_{n-2} & 6E-im\delta U_{n-2} & -4E-imU_{n-2} & E \\ & & & & \cdots & 0 & -E & 3E & -3E+im(u_0+\frac{u_1}{2}-c) & E-im(u_0+\frac{u_1}{2}-\frac{u_1}{n}-c) \\ & & & & \cdots & 0 & 0 & 1 & -2 & 1 \end{vmatrix} = 0. \qquad (3.97)$$

As in Chapter 3.3 we can now take small values of n in (3.97) to determine approximate values for c. The smallest value of n in this case is 4, producing a (5 x 5) determinant. The cases n = 4 and n = 5 are dealt with in Appendix C. We will look at a general case immediately and look for solutions for c for general n. We will be able to check these formulae for c with the particular roots found in Appendix C.

TWO PROPERTIES OF THE DETERMINANT $|A_n|$

Equation (3.97) gives the general form of $|A_n|$, an (n + 1) x (n + 1) determinant. The determinant should produce a term of the form $\alpha\varepsilon^{n-1}$. However it is easy to show that $\alpha \equiv 0$. This term is found by retaining the first and last rows and only terms containing ε in the remaining rows. This determinant, A_n' is of the form

$$A_n' = \left(\frac{\varepsilon}{\rho_1}\right)^{n-1} \begin{vmatrix} 1+\frac{hH}{n} & -2-\frac{hH}{n} & 1 & 0 & 0 & \cdots & & & \\ -1 & 3 & -3 & 1 & 0 & \cdots & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ \cdots & & & & 0 & -1 & 3 & -3 & 1 \\ \cdots & & & & 0 & 0 & 1 & -2 & 1 \end{vmatrix} . \tag{3.98}$$

It can easily be seen that row 2 + (row 3 + ... + row n-1) = row n and thus $A_n' = 0$.

The second result may be obtained in the following way. We consider now the term independent of ε , ignoring the ε dependence in H. This term, denoted as A_n'' takes the form

$$A_n'' = U_2 U_3 \cdots U_{n-2} \begin{vmatrix} 1+\frac{hH}{n} & -2-\frac{Hh}{n} & 1 & 0 & \cdots & & & \\ U_0-c+U_1\left(\frac{1}{n}-\frac{1}{2}\right) & -U_0+c+\frac{U_1}{2} & 0 & 0 & \cdots & & & \\ 0 & -1 & \delta & -1 & 0 & \cdots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ \cdots & & & 0 & -1 & \delta & -1 & 0 \\ \cdots & & & & 0 & 0 & U_0-c+\frac{U_1}{2} & -U_0+c+U_1\left(\frac{1}{2}-\frac{1}{n}\right) \\ \cdots & & & & 0 & 1 & -2 & 1 \end{vmatrix} . \tag{3.99}$$

The determinant in the above expression, (3.99) is a quadratic in $(U_0 - c)$ and it can be shown that the coefficient of $(U_0 - c)^2$ is independent of H. We substitute $U_1 = 0$ into the determinant and find the coefficient of $(U_0 - c)^2$, denoted by A_n''', takes the form

$$A_n''' = \begin{vmatrix} 1+\frac{hH}{n} & -2-\frac{hH}{n} & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & \delta & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & \ddots & \ddots & \ddots & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & \delta & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -2 & 1 \end{vmatrix} . \tag{3.100}$$

By adding columns 1 and 2 it can easily be shown that A_n''' is independent of H.

TWO SERIES EXPANSIONS FOR c

When $|A_n|$ is expanded, it can be written as a series in powers of $\varepsilon^{1/2}$ remembering that $H = (1-i)\lambda\varepsilon^{-1/2}c^{1/2}$. We can write (3.97) in the form

$$\left(\frac{ih^2}{n^2}\right)^{n-1}\left[Q_1^{n-1}(c) + \varepsilon^{-1/2}c^{1/2}Q_2^{n-2}(c)\right] + \left(\frac{ih^2}{n^2}\right)^{n-2}\frac{\varepsilon}{\rho_1}\left[Q_3^{n-2}(c) + \varepsilon^{-1/2}c^{1/2}Q_4^{n-2}(c)\right] + \cdots$$
$$\cdots\cdots + \frac{ih^2}{n^2}\left(\frac{\varepsilon}{\rho_1}\right)^{n-2}\left[Q_{2n-3}^{1}(c) + \varepsilon^{-1/2}c^{1/2}Q_{2n-2}^{1}(c)\right] = 0 \tag{3.101}$$

where $Q_i^r(c)$ is a polynomial of order r in c. Note that the two properties shown above have been used in writing (3.101). The first two terms in the expansion (3.101) can be written as

$$Q_1^{n-1}(c) + \varepsilon^{-1/2}c^{1/2}Q_2^{n-2}(c) = U_2U_3\cdots U_{n-2}\Big[b_1(U_0-c)^2 + b_2(U_0-c)U_1 + b_3U_1^2$$
$$+\varepsilon^{1/2}c^{1/2}(1-i)\frac{\lambda h}{n}\left(d_1(U_0-c)U_1 + d_2U_1^2\right)\Big] .$$

When we substitute for these terms in (3.101) the equation in c becomes

$$\left(\frac{ih^2}{n^2}\right)^{n-1} U_2U_3\cdots U_{n-2}\left[b_1(U_0-c)^2+b_2(U_0-c)U_1+b_3U_1^2+\varepsilon^{-1/2}c^{1/2}(1-i)\frac{\lambda h}{n}\left(d_1(U_0-c)U_1+d_2U_1^2\right)\right]$$

$$+\left(\frac{ih^2}{n^2}\right)^{n-2}\frac{\varepsilon}{\rho_1}\left[Q_3^{n-2}(c)+\varepsilon^{-1/2}c^{1/2}Q_4^{n-2}(c)\right]+\cdots\cdots+\frac{ih^2}{n^2}\left(\frac{\varepsilon}{\rho_1}\right)^{n-2}\left[Q'_{2n-3}(c)+\varepsilon^{-1/2}c^{1/2}Q'_{2n-2}(c)\right]=0 \quad . \tag{3.102}$$

We will now consider the roots of c. The solutions fall into two classes. In the first place there are solutions for c in the form

$$c = c_0 + \varepsilon^{1/2}c_1 + \varepsilon c_2 + \cdots\cdots \tag{3.103}$$

There are (n-1) values for c_0 found by equating the term $\varepsilon^{-1/2}$ to zero in (3.102),

$$\left.\begin{array}{l}
c_0^1 = 0 \quad , \\
c_0^2 = U_2 = U_0 - (\frac{2}{n}-\frac{1}{2})U_1 \; , \\
\vdots \qquad \vdots \qquad \vdots \\
c_0^i = U_i = U_0 - (\frac{i}{n}-\frac{1}{2})U_1 \; , \\
\vdots \qquad \vdots \qquad \vdots \\
c_0^{n-2} = U_{n-2} = U_0 - (\frac{n-2}{n}-\frac{1}{2})U_1 \; , \\
c_0^{n-1} = U_0 + \frac{d_2}{d_1}U_1 \; .
\end{array}\right\} \tag{3.104}$$

In the second place there are large roots for c found by equating $Q_1^{n-1}(c)$ and $\varepsilon^{-1/2}c^{1/2}Q_2^{n-2}(c)$ in (3.101). The series for c takes the form

$$c = \frac{a_0}{\varepsilon} + \frac{a_1}{\varepsilon^{1/2}} + a_2 + \cdots\cdots \tag{3.105}$$

Thus a_0 is found by equating the term ε^{-n} to zero in (3.102) and we obtain

$$a_0 = \left[\frac{(1-i)\lambda h d_1 U_1}{n b_1}\right]^2 . \tag{3.106}$$

The solution $a_0 = 0$ does not produce a new root for c as it can easily be shown by equating the term $\varepsilon^{-n+1/2}$ in (3.102) corresponding to $a_0 = 0$, that the next term in the expansion of c, namely a_1 is also zero. This value of c now reduces to the root corresponding to $c = c_0 + \varepsilon^{1/2}c_1 + \cdots$

We have found n values for c for which $|A_n| = 0$. Using a simple example in the form of (3.102) we aim to show that these are all roots. We choose an equation of the form

$$(c-\alpha)^3 - \varepsilon^{-1/2} c^{1/2} (c-\beta)(c-\gamma) = 0. \tag{3.107}$$

From the expansion (3.103), $c = c_0 + \varepsilon^{1/2} c_1 + \ldots\ldots$, we have three values for c_0 which are β, γ and 0 and from the expansion (3.105), $c = \frac{a_0}{\varepsilon} + \frac{a_1}{\varepsilon^{1/2}} + \cdots\cdot$ we have one new root with $a_0 = 1$. If we square (3.107) we produce a sixth order equation for c with roots at approximately 0, $1/\varepsilon$, β (twice) and γ (twice). We can see now that we have indeed found all the roots of (3.107). By the same argument an equation of the form

$$(c-d)^r - \varepsilon^{-1/2} c^{1/2} (c-f_1)(c-f_2)\cdots\cdot(c-f_{r-1}) = 0$$

has (r + 1) roots.

GENERAL FORMULAE FOR THE ROOTS OF c

We will now proceed to find the general formulae for the n roots of c.

(1) The (n-2) Values for c_0

To find expressions for c_0 we need to look at $|A_n|$ with $\varepsilon = 0$, ignoring the dependence of H on ε. This term, denoted by Δ_n will take the form

$$\Delta_n = U_2 U_3 \cdots\cdot U_{n-2} \left[\frac{ih^2}{n^2}\right]^{n-1} \begin{vmatrix} 1+\frac{hH}{n} & -2-\frac{hH}{n} & 1 & 0 & \cdots & & & & \\ a & b & 0 & 0 & \cdots & & & & \\ 0 & -1 & \delta & -1 & 0 & \cdots & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ \cdots & & & & 0 & -1 & \delta & -1 & 0 \\ \cdots & & & & & 0 & 0 & \alpha & \beta \\ \cdots & & & & & 0 & 1 & -2 & 1 \end{vmatrix} \tag{3.108}$$

where

$$\left.\begin{aligned} a &= \frac{U_1}{n} - b \,, \\ b &= -(U_0 - c) + U_1/2 \,, \\ \alpha &= (U_0 - c) + U_1/2 \,, \\ \beta &= -\alpha + U_1/n \,. \end{aligned}\right\} \tag{3.109}$$

It is the H dependent terms of Δ_n, denoted by Δ_n', which when equated to zero produce the (n-1) roots, c_0^1, $c_0^2 \ldots\ldots c_0^{n-1}$.

$$\Delta_n' = U_2 U_3 \cdots U_{n-2} \left(\frac{ih^2}{n^2}\right)^{n-1} \varepsilon^{-1/2} c^{1/2} \frac{\Delta h}{n}(1-i) \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & & & & \cdots \\ a & b & 0 & 0 & \cdots & & & & \cdots \\ 0 & -1 & \delta & -1 & 0 & \cdots & & & \cdots \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ \cdots & & & & 0 & -1 & \delta & -1 & 0 \\ \cdots & & & & & 0 & 0 & \alpha & \beta \\ \cdots & & & & & 0 & 1 & -2 & 1 \end{vmatrix} = 0.$$

The first (n-2) roots are found by equating the factors outside the determinant in (3.109) to zero as shown in (3.104). The root c_0^{n-1} is found by equating the determinant in (3.109), referred to as D_n' to zero. Expanding by the first two rows D_n' reduces to

$$D_n' = (a+b) \begin{vmatrix} \delta & -1 & 0 & \cdots & & & & \cdots \\ -1 & \delta & -1 & 0 & \cdots & & & \cdots \\ & \ddots & \ddots & \ddots & \ddots & & & \\ \cdots & & & 0 & -1 & \delta & -1 & 0 \\ \cdots & & & & 0 & 0 & \alpha & \beta \\ \cdots & & & & 0 & 1 & -2 & 1 \end{vmatrix} = \frac{U_1}{n} |D_{n-2}| \tag{3.110}$$

where $|D_{n-2}|$ is of order (n-1) x (n-1).

We can evaluate $|D_{n-2}|$ by using recurrence relationships. Using the notation that $|D_s|$ is the (s + 1) x (s + 1) determinant in the bottom right hand corner of $|D_{n-2}|$, we can form the following recurrence relationships:

$$\left.\begin{array}{l} |D_{n-2}| = \delta|D_{n-3}| - |D_{n-4}|, \\ \quad\vdots \\ |D_{s+1}| = \delta|D_s| - |D_{s-1}|, \\ \quad\vdots \\ |D_2| = \delta|D_1| - \beta, \\ |D_1| = \alpha + 2\beta . \end{array}\right\} \tag{3.111}$$

We will assume that

$$|D_s| = A\cosh s\theta + B\sinh s\theta \ , \quad 1 \leq s \leq n-2 . \tag{3.112}$$

Since

$$|D_{s+1}| + |D_{s-1}| = A\left[\cosh(s+1)\theta + \cosh(s-1)\theta\right] + B\left[\sinh(s+1)\theta + \sinh(s-1)\theta\right]$$

$$= 2\cosh\theta\,|D_s|$$

we can take $\delta = 2\cosh\theta$ so that

$$\cosh\theta = 1 + \gamma^2h^2/2n^2 . \tag{3.113}$$

Using the equation for $|D_2|$ in (3.111) we find

$$A = \beta \tag{3.114}$$

and from the equation for $|D_1|$ in (3.111) and (3.114) we deduce

$$B = (\alpha + 2\beta - \beta\cosh\theta)/\sinh\theta . \tag{3.115}$$

Using (3.113), it can be shown that

$$\sinh\theta = \frac{\gamma h}{n}\left[1 + \frac{\gamma^2h^2}{4n^2}\right]^{1/2} . \tag{3.116}$$

We finally have for $|D_{n-2}|$,

$$|D_{n-2}| = \beta\cosh(n-2)\theta + (\alpha + 2\beta - \beta\cosh\theta)\frac{\sinh(n-2)\theta}{\sinh\theta} . \tag{3.117}$$

Substituting for α and β in (3.109) produces

$$|D_{n-2}| = \left[-U_0+c+U_1\left(\tfrac{1}{n}-\tfrac{1}{2}\right)\right]\cosh(n-2)\theta + \left[\frac{U_1}{n} - \frac{\gamma^2h^2}{2n^2}\left(-U_0+c+U_1\left(\tfrac{1}{n}-\tfrac{1}{2}\right)\right)\right]\frac{\sinh(n-2)\theta}{\sinh\theta} \tag{3.118}$$

Since c_0^{n-1} is the value of c for which $|D_{n-2}| = 0$ we have

$$c_0^{n-1} = U_0 + \frac{U_1\left[\left(\frac{1}{2}-\frac{1}{n}\right)\cosh(n-2)\theta - \left(\frac{1}{n}+\frac{\gamma^2h^2}{2n^2}\left(\frac{1}{2}-\frac{1}{n}\right)\right)\sinh(n-2)\theta/\sinh\theta\right]}{\cosh(n-2)\theta - \frac{\gamma^2h^2}{2n^2}\sinh(n-2)\theta/\sinh\theta}. \tag{3.119}$$

(2) <u>The (n-1) Values for c_1</u>

To find the values for c_1 we equate the term in ε^0 to zero in (3.102). It is easily shown that the value of c_1 corresponding to c_0^1, c_0^2, ... c_0^{n-3} and c_0^{n-2} is zero and so it only remains to find the value of c_1, referred to as c_1^{n-1} corresponding to c_0^{n-1}. From (3.102) we can see that

$$c_1^{n-1} = \frac{\left[b_1(U_0-c_0^{n-1})^2 + b_2(U_0-c_0^{n-1})U_1 + b_3U_1^2\right]}{(c_0^{n-1})^{1/2}\, d_1 U_1 \lambda h(1-i)/n}. \tag{3.120}$$

Since, from (3.102) $|D_{n-2}|$ corresponds to the term $(d_1(U_0-c) + d_2U_1)$ we can easily find d_1 from (3.118) as it is the coefficient of $(-c)$ and thus

$$d_1 = \frac{1}{n}\left[-\cosh(n-2)\theta + (\gamma^2h^2/2n^2)\sinh(n-2)\theta/\sinh\theta\right]. \tag{3.121}$$

To find an expression for the numerator in (3.120) we first take the H independent terms of the determinant in (3.108), namely

$$|d_n| = \begin{vmatrix} 1 & -2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & b & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & \delta & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & \delta & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \alpha & \beta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -2 & 1 \end{vmatrix}. \tag{3.122}$$

When we expand by the first two rows we find

$$|d_n| = (b+2a)|d_{n-2}| - a|d_{n-3}|$$

but for $r \leq n-2$, $|d_r| \equiv |D_r|$ and therefore

$$|d_n| = (b+2a)|D_{n-2}| - a|D_{n-3}| \tag{3.123}$$

For the value of the numerator of c_1^{n-1} in (3.120) we substitute $c = c_0^{n-1}$ into $|d_n|$. Since $|D_{n-2}|$ is zero when $c = c_0^{n-1}$ we only need to evaluate $-a\,|D_{n-3}|$ at $c = c_0^{n-1}$. Thus, the formula for c_1^{n-1} is

$$c_1^{n-1} = \frac{U_1\left(\frac{d_2}{d_1}+\frac{1}{2}-\frac{1}{n}\right)\left[\left(\frac{d_2}{d_1}+\frac{1}{n}-\frac{1}{2}\right)\cosh(n-3)\theta+\left(\frac{1}{n}-\frac{\gamma^2h^2}{2n^2}\left(\frac{d_2}{d_1}+\frac{1}{n}-\frac{1}{2}\right)\right)\sinh(n-3)\theta/\sinh\theta\right]}{d_1(1-i)\lambda h\left(U_0+\frac{d_2}{d_1}U_1\right)^{1/2}/n} \tag{3.124}$$

where d_2/d_1 is the coefficient of U_1 in (3.119).

(3) <u>The Value of a_0</u>

From (3.106) we find that a_0 is defined by

$$a_0 = \left[(1-i)\lambda h d_1 U_1/b_1 n\right]^2.$$

We already have an expression for d_1 in (3.121) so it only remains to find b_1. Since $|d_n|$ is the term $[b_1(U_0-c)^2 + b_2(U_0-c)U_1 + b_3 U_1^2]$ in (3.102) we can find b_1 by substituting $U_1 = 0$ in $|d_n|$. Using (3.123) and (3.118) we find that

$$b_1 = \cosh(n-3)\theta - \cosh(n-2)\theta - \frac{\gamma^2h^2}{2n^2\sinh\theta}\left[\sinh(n-3)\theta - \sinh(n-2)\theta\right] \tag{3.125}$$

and therefore

$$a_0 = \left[\frac{(1-i)\lambda h U_1\left(-\cosh(n-2)\theta + (\sigma^2h^2/2n^2)\sinh(n-2)\theta/\sinh\theta\right)}{n\left[\cosh(n-3)\theta - \cosh(n-2)\theta - \frac{\sigma^2h^2}{2n^2\sinh\theta}\left(\sinh(n-3)\theta - \sinh(n-2)\theta\right)\right]}\right]^2 \tag{3.126}$$

(4) <u>The Value of a_1</u>

The next term in the expansion for c in (3.105) is found by equating the term of $\varepsilon^{-n+\frac{1}{2}}$ to zero. It is easily shown that a_1 is zero.

We have now found the first two terms of the expansions of the n roots of c. These can be summarised as follows

$$\left.\begin{aligned}
c^1 &= && && O(\varepsilon)\,, \\
c^2 &= U_0 - \left(\tfrac{2}{n}-\tfrac{1}{2}\right)U_1 && && + O(\varepsilon)\,, \\
&\ \ \vdots && && \ \ \vdots \\
c^i &= U_0 - \left(\tfrac{i}{n}-\tfrac{1}{2}\right)U_1 && && + O(\varepsilon)\,, \\
&\ \ \vdots && && \ \ \vdots \\
c^{n-2} &= U_0 - \left(\tfrac{n-2}{n}-\tfrac{1}{2}\right)U_1 && && + O(\varepsilon)\,, \\
c^{n-1} &= c_0^{n-1} && + \varepsilon^{1/2} c_1^{n-1} && + O(\varepsilon)\,, \\
c^n &= a_0 \varepsilon^{-1} && && + O(\varepsilon^0)\,,
\end{aligned}\right\} \qquad (3.127)$$

where c_0^{n-1}, c_1^{n-1} and a_0 are defined in (3.119), (3.124) and (3.126) respectively. The formulae of (3.127) are checked by substituting n = 4 and n = 5 and comparing the results with Appendix C.

LIMITING VALUES OF THE ROOTS FOR c

Following the method introduced in Chapter 3.3 the next step would be to look at the coefficients in (3.127) in the limit as $n \to \infty$. The n-3 roots of c, c_0^2, ... c_0^{n-2} will tend to the continuous function,

$$c = U(z)$$

which is the root produced by ignoring the fourth order differential term in the equation for P,

$$\frac{\varepsilon}{\rho_1}\frac{d^4P}{dz^4} - i\,(U(z)-c)\left[\frac{d^2P}{dz^2} - \gamma^2 P\right] = 0\,.$$

Unfortunately the limiting process for c^{n-1} and c^n produced difficulties, and although a limit for c_0^{n-1} could be found, namely

$$\lim_{n\to\infty} c_0^{n-1} = U_0 + \frac{U_1}{\gamma h}\left[\frac{\gamma h}{2} - \tanh\gamma h\right],$$

c_1^{n-1} and a_0 did not have finite limits. We found

$$\lim_{n\to\infty} c_1^{n-1} = \infty\,, \qquad \lim_{n\to\infty} a_0 = 0\,.$$

The fact that $a_0 \to 0$ may imply that this root does not exist for the exact equations but it is hard to explain the limit of c_1^{n-1}. However for finite n we can use the formulae in (3.127) for approximate solutions of c.

RESULTS AND DISCUSSION

We have seen that the method of differencing produces n roots for c. To investigate stability of the system we return to the formulation of P' and w in (3.30). Both functions contain the factor

$e^{-ikct} = e^{-ikc_R t} e^{kc_I t}$ where $c = c_R + ic_I$. Thus when $c_I > 0$, an unstable wave exists. Looking at the roots in (3.127) the roots c^2 , ... c^{n-2} are to the order $\varepsilon^{1/2}$ purely real and therefore do not produce instabilities. From (3.126), a_0 is a negative imaginary term which would produce a damped wave; the only root to produce an unstable wave is therefore $c^{n-1} = c_0^{n-1} + \varepsilon^{1/2} c_1^{n-1} + \cdots$.

From (3.119) we see that c_0^{n-1} is real and from (3.124) c_1^{n-1} is complex. Writing c^{n-1} as,

$$c^{n-1} = c_R^{n-1} + i\, c_I^{n-1}$$

it can easily be seen that c_R^{n-1} is approximately c_0^{n-1}.

For n = 4 and n = 5, the analytical solutions for c^{n-1} are relatively simple and following Eady we investigate the value of c_I^{n-1} as a function of γh. Graphs of the two functions c_I^3 and c_I^4 are shown in Fig. 1 and 2 using constants as in (3.58) with wave lengths of order 1000kms - an average wave length for such disturbances. These values are for molecular thermal conductivities. For eddy values ε is 10^5 times larger, thus c_R^{n-1} will be unchanged and c_I^{n-1} which is of order $\varepsilon^{1/2}$ will increase by a factor of $10^{5/2}$. Unlike the Eady model instability occurs for all γh with a maximum value at $\gamma h = 3$ for n = 4 and $\gamma h = 2.5$ for n = 5. It must be remembered that these unstable waves grow slowly as they are produced by second order terms. The waves travel at a speed, c_R^{n-1} which is of the same order as the thermal wind.

For larger values of n the computer is needed to calculate c^{n-1} and we now have a choice of method. The formulae from (3.119) and (3.124) can be employed immediately or one could return to the initial formulation of the consistency equation (3.97) and use computer methods to reduce the determinant and find the roots. Both methods were used and the results from the two methods were the same within our needs of accuracy. The details of the determinant method are given in Appendix D. Values for c^{n-1} are presented in the Tables 1 and 2 for n = 5-11 and for various values of γh and wavelength $L = 2\pi/k$. The main points to notice from these tables are:

(1) For all n, the speed of the wave, c_R^{n-1} increases with γh and is independent of L and is of the order of U_0, the speed of the thermal wind.

(2) For all n, there is a maximum of c_I^{n-1} for a small value of γh, (n = 5, maximum at $\gamma h = 5$, for n = 10 maximum at $\gamma h = 1$) and the value of γh for the maximum decreases with increasing n.

(3) As n increases, more maxima and minima occur for c_I^{n-1} and some negative values do occur.

(4) As L increases, c_I^{n-1} increases.

Thus to sum up, we have found unstable waves for all values of γh, unlike the Eady model. However the growth rates are very much slower as the imaginary part of c is found in second order terms. We have not found the Eady unstabilities in the region $0 < \gamma h < 2.4$ appearing as first order terms in the series expansion of c. This is perhaps to be expected as the equation of consistency for this model, (3.97) cannot be reduced to the Eady equation of consistency, (3.68) merely by substituting $\epsilon = 0$. In the next chapter we follow methods of Meksyn and find that the Eady results there are first order terms in a series expansion of c.

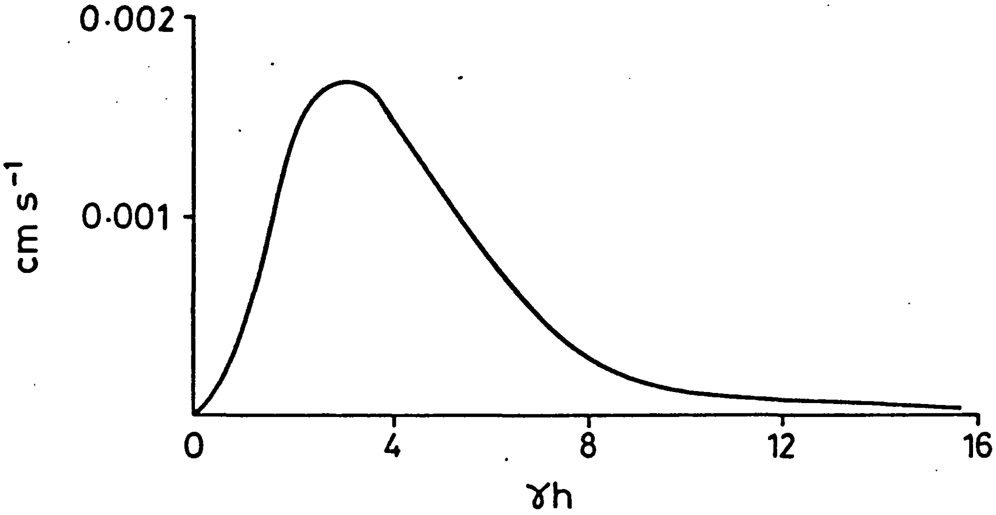


Figure 1: Variation of c_I^3 with γh (L = 1000kms)

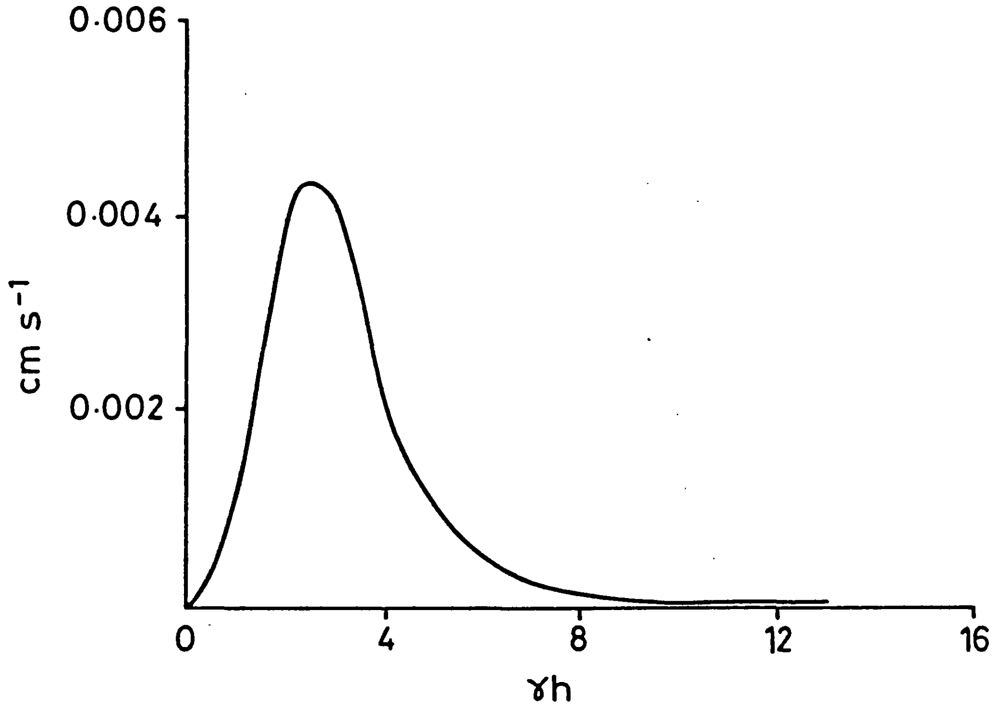


Figure 2: Variation of c_I^4 with γh (L = 1000kms)

γh	n = 5		n = 6		n = 7		n = 8		n = 9		n = 10	
	c_R^4	c_I^4	c_R^5	c_I^5	c_R^6	c_I^6	c_R^7	c_I^7	c_R^8	c_I^8	c_R^9	c_I^9
0.0	800.0	0.0	760.0	0.0	733.3	0.0	714.3	0.0	700.0	0.0	688.9	0.0
1.0	811.8	4.0^{-4}	789.0	1.1^{-3}	778.9	2.0^{-3}	774.2	2.8^{-4}	772.2	3.6^{-4}	777.8	5.5^{-5}
2.0	840.0	9.7^{-4}	850.5	1.9^{-3}	866.9	1.2^{-3}	882.4	3.0^{-4}	895.7	3.4^{-4}	906.9	3.7^{-4}
2.5	856.2	1.1^{-3}	881.4	1.7^{-3}	907.4	2.0^{-4}	929.1	2.1^{-4}	946.6	2.2^{-4}	960.8	2.3^{-4}
3.0	872.0	1.1^{-3}	909.1	1.3^{-3}	941.8	1.4^{-4}	967.4	1.4^{-4}	987.5	1.3^{-4}	1003.5	1.3^{-4}
5.0	922.0	6.8^{-4}	984.0	4.1^{-4}	1027.9	2.7^{-5}	1059.3	3.7^{-5}	1082.8	1.2^{-5}	1100.9	1.1^{-5}
6.0	938.5	4.8^{-4}	1005.6	2.1^{-4}	1051.3	1.2^{-5}	1083.7	9.1^{-6}	1107.7	1.2^{-5}	1126.2	2.0^{-6}
10.0	972.4	1.3^{-4}	1046.9	3.5^{-5}	1095.9	3.0^{-6}	1130.3	3.5^{-6}	1155.7	4.3^{-6}	1175.2	5.2^{-6}
50.0	998.7	1.9^{-5}	1078.4	2.9^{-5}	1131.4	4.1^{-6}	1169.3	5.5^{-6}	1197.5	7.1^{-6}	1219.4	8.9^{-6}

Table 1. Variation of c^{n-1} cm/s ($c^{n-1} = c_R^{n-1} + ic_I^{n-1}$) with γh with L = 1000kms. The subscript denotes the power of ten by which the entry should be multiplied.

γh	L = 250kms		L = 500kms		L = 750kms		L = 1000kms	
	c^+_R	c^+_I	c^+_R	c^+_I	c^+_R	c^+_I	c^+_R	c^+_I
0.0	800.0	0.0	800.0	0.0	800.0	0.0	800.0	0.0
1.0	811.8	2.0^{-4}	811.8	2.9^{-4}	811.8	3.5^{-4}	811.8	4.0^{-4}
2.5	856.2	5.5^{-4}	856.2	7.7^{-4}	856.2	9.5^{-4}	856.2	1.1^{-3}
3.0	872.0	5.5^{-4}	872.0	7.8^{-4}	872.0	9.5^{-4}	872.0	1.1^{-3}
5.0	922.0	3.4^{-4}	922.0	4.8^{-4}	922.0	5.9^{-4}	922.0	6.8^{-4}
10.0	972.4	6.1^{-5}	972.4	8.8^{-5}	972.4	1.0^{-4}	972.4	1.3^{-4}

Table 2. Variation of c^+ cm/s ($c^+ = c^+_R + ic^+_I$) with γh and L. The subscript denotes the power of ten by which the entry should be multiplied.

CHAPTER 3.5

AN APPROXIMATE SOLUTION OF THE STABILITY PROBLEM FOLLOWING MEKSYN

We now aim to find criteria for stability following methods of Meskyn (26). From Chaper 3.2, (3.53) and (3.54), the problem is to find a solution of

$$\frac{\varepsilon}{\rho_1}\frac{d^4P}{dz^4} - i\,(U(z)-c)\left[\frac{d^2P}{dz^2} - \gamma^2 P\right] = 0$$

subject to the boundary conditions,

$$\frac{\varepsilon}{\rho_1}\frac{d^3P}{dz^3} - i\left[(U(z)-c)\frac{dP}{dz} - U'(z)P\right] = 0 \quad \text{at } z=0,h\,,$$

$$\frac{d^2P}{dz^2} - H\frac{dP}{dz} = 0 \qquad \text{at } z=0\,,$$

$$\frac{d^2P}{dz^2} = 0 \qquad \text{at } z=h\,.$$

THE SOLUTION FOLLOWING MEKSYN

We introduce a new parameter Z defined by

$$U(z) - c = U_1(Z - Z_c)\,, \quad 0 \leqslant Z \leqslant 1\,, \tag{3.128}$$

where Z is a complex constant. The above equations can now be rewritten in a nondimensional form, namely

$$\frac{d^4P}{dZ^4} - i\lambda(Z-Z_c)\left[\frac{d^2P}{dZ^2} - \gamma_1^2 P\right] = 0 \tag{3.129}$$

subject to

$$\left.\begin{aligned} &\frac{d^3P}{dZ^3} - i\lambda\left[(Z-Z_c)\frac{dP}{dZ} - P\right] = 0 \quad \text{at } Z=0,1,\\ &\frac{d^2P}{dZ^2} - hH\frac{dP}{dZ} = 0 \qquad \text{at } Z=0\,,\\ &\frac{d^2P}{dZ^2} = 0 \qquad \text{at } Z=1\,. \end{aligned}\right\} \tag{3.130}$$

The nondimensional quantities, λ and γ_1 are defined by

$$\lambda = U_1 h^2 \rho_1/\varepsilon \quad , \quad \gamma_1 = \gamma h\,.$$

It should be noted that λ is a large quantity and plays the part of a Reynolds Number (31).

Following Heisenberg and others (20) we can look for solutions of P(Z) of the form

$$P(Z) = \psi^0(Z) + \lambda^{-1}\psi^1(Z) + \lambda^{-2}\psi^2(Z) + \cdots \tag{3.131}$$

so that $\psi^0(Z)$ will satisfy the differential equation

$$\frac{d^2\psi^0}{dZ^2} - \gamma_1^2\psi = 0 \; ; \tag{3.132}$$

clearly the two solutions for $\psi^0(Z)$ are

$$\psi^0(Z) = e^{\pm\gamma_1 Z}.$$

It is convenient to have the first two solutions for P in a slightly different form from $\psi^0(Z)$ and we shall write them as $P_1(Z)$ and $P_2(Z)$, where

$$\left.\begin{aligned} P_1(Z) &= e^{\gamma_1(Z-Z_c)}, \\ P_2(Z) &= e^{-\gamma_1(Z-Z_c)}. \end{aligned}\right\} \tag{3.133}$$

It can easily be seen that a boundary layer will exist in the neighbourhood of $Z = Z_c$. The solutions $P_1(Z)$ and $P_2(Z)$ are good approximations of the solution to (3.129) away from the boundary layer.

We now introduce yet another new parameter, η defined by

$$\eta = \lambda^{1/3}(Z - Z_c). \tag{3.134}$$

The system of equation, (3.129) and (3.130) become

$$\frac{d^4P}{d\eta^4} - i\eta\left[\frac{d^2P}{d\eta^2} - R^2P\right] = 0 \tag{3.135}$$

subject to

$$\left.\begin{aligned} &\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 && \text{at } \eta = \eta_0, \eta_h, \\ &\frac{d^2P}{d\eta^2} - M\frac{dP}{d\eta} = 0 && \text{at } \eta = \eta_0, \\ &\frac{d^2P}{d\eta^2} = 0 && \text{at } \eta = \eta_h. \end{aligned}\right\} \tag{3.136}$$

where the constants R, M, η_0 and η_h are defined by,

$$\left.\begin{aligned} R^2 &= \lambda^{-2/3}\gamma_1^2 \\ M &= h\lambda^{1/3}H \\ \eta_0 &= -\lambda^{1/3}Z_c \\ \eta_h &= \lambda^{1/3}(1-Z_c) \end{aligned}\right\} \tag{3.137}$$

Later it will be necessary to know where Z_c lies in the complex plane. As in Part 2, we use heuristic arguments to assume that

$$Z_c = a - ib \tag{3.138}$$

where $a >> b$ and $0 \leqslant a, b, \leqslant 1$.

Meksyn (26) has used the Laplace Integral method to resolve equations of the form (3.135) and so we will look for a solution for $P(\eta)$ in the form

$$P(\eta) = \int_c \chi(t)\exp(-t\eta e^{-i\pi/6})\,dt \tag{3.139}$$

where c is some curve in the complex t plane; this will be a solution of (3.135) provided that

$$\int_c \left(t^4 e^{-\frac{2\pi i}{3}} - i\eta t^2 e^{-i\pi/3} + i\eta R^2\right)\chi e^{\phi}\,dt = 0 \tag{3.140}$$

where

$$\phi = -t\eta e^{-i\pi/6}.$$

We integrate by parts the last two terms in (3.140) to obtain

$$\left[ie^{i\pi/6}(t^2e^{-i\pi/3} - R^2)\chi e^{\phi}\right]_c + \int_c \left(t^4 e^{-\frac{2i\pi}{3}}\chi - \frac{d}{dt}\left(e^{2i\pi/3}(t^2e^{-i\pi/3} - R^2)\chi\right)\right)e^{\phi}\,dt = 0.$$

Accordingly a solution of the form (3.138) is possible provided

$$\frac{d}{dt}\left[e^{2\pi i/3}\chi(t)\left(t^2e^{-i\pi/3} - R^2\right)\right] - t^4e^{-2i\pi/3}\chi(t) = 0 \tag{3.141}$$

and

$$\left[(t^2e^{-i\pi/3} - R^2)\chi(t)e^{\phi}\right]_c = 0\,. \tag{3.142}$$

We can rewrite (3.141) as

$$(t^2 - R^2e^{i\pi/3})\frac{d\chi}{dt} + (t^4 + 2t)\chi = 0.$$

The variables can now be separated to give

$$\frac{d\chi}{\chi} + dt\left[t^2 + R^2 e^{i\pi/3} + \frac{R^4 e^{2i\pi/3}}{(t^2 - R^2 e^{i\pi/3})} + \frac{2t}{t^2 - R^2 e^{i\pi/3}} \right] = 0 \qquad (3.143)$$

which can be integrated to obtain the solution for χ, namely

$$\chi(t) = (t - Re^{i\pi/6})^{-1-iR^3/2}(t + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3}\right]. \qquad (3.144)$$

Thus a solution for $P(\eta)$ exists in the form

$$P(\eta) = \int_c (t - Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta e^{-i\pi/6}\right] dt \qquad (3.145)$$

where the curve c is chosen so that, from (3.142)

$$\left[(t - Re^{i\pi/6})^{-\frac{iR^3}{2}}(t + Re^{i\pi/6})^{\frac{iR^3}{3}} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta e^{-i\pi/6}\right] \right]_c = 0. \qquad (3.146)$$

From (3.146) we can see that $t = Re^{i\pi/6}$ and $t = -Re^{i\pi/6}$ are branch points. If we write $t = re^{i\theta}$ then

$$-t^3 = -r^3(\cos 3\theta + i\sin 3\theta).$$

Accordingly $\exp(-\frac{1}{3}t^3) \to 0$ as $r \to \infty$ provided that $\cos 3\theta > 0$; the following cases are therefore possible

$$\left.\begin{array}{lll} -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} & \text{i.e.} & -\frac{\pi}{6} < \theta < \frac{\pi}{6}, \\ \frac{3\pi}{2} < 3\theta < \frac{5\pi}{2} & \text{i.e.} & \frac{\pi}{2} < \theta < \frac{5\pi}{6}, \\ \frac{7\pi}{2} < 3\theta < \frac{9\pi}{2} & \text{i.e.} & \frac{7\pi}{6} < \theta < \frac{3\pi}{2}, \end{array}\right\} \qquad (3.147)$$

and the three choices for c are shown in Fig. 1. As in the case discussed by Meksyn only two of the curves, $\alpha_1\beta_1$, A_3B_3 and A_4B_4 are independent. The three paths, when described consecutively in the same direction correspond to a closed path round the origin. Provided the curves lie within the branch points $\pm Re^{i\pi/6}$, the integrand is regular inside the closed contour and the integral vanishes.

We will choose the curves A_3B_3 and A_4B_4 for the two independent solutions, $P_3(\eta)$ and $P_4(\eta)$ respectively so that,

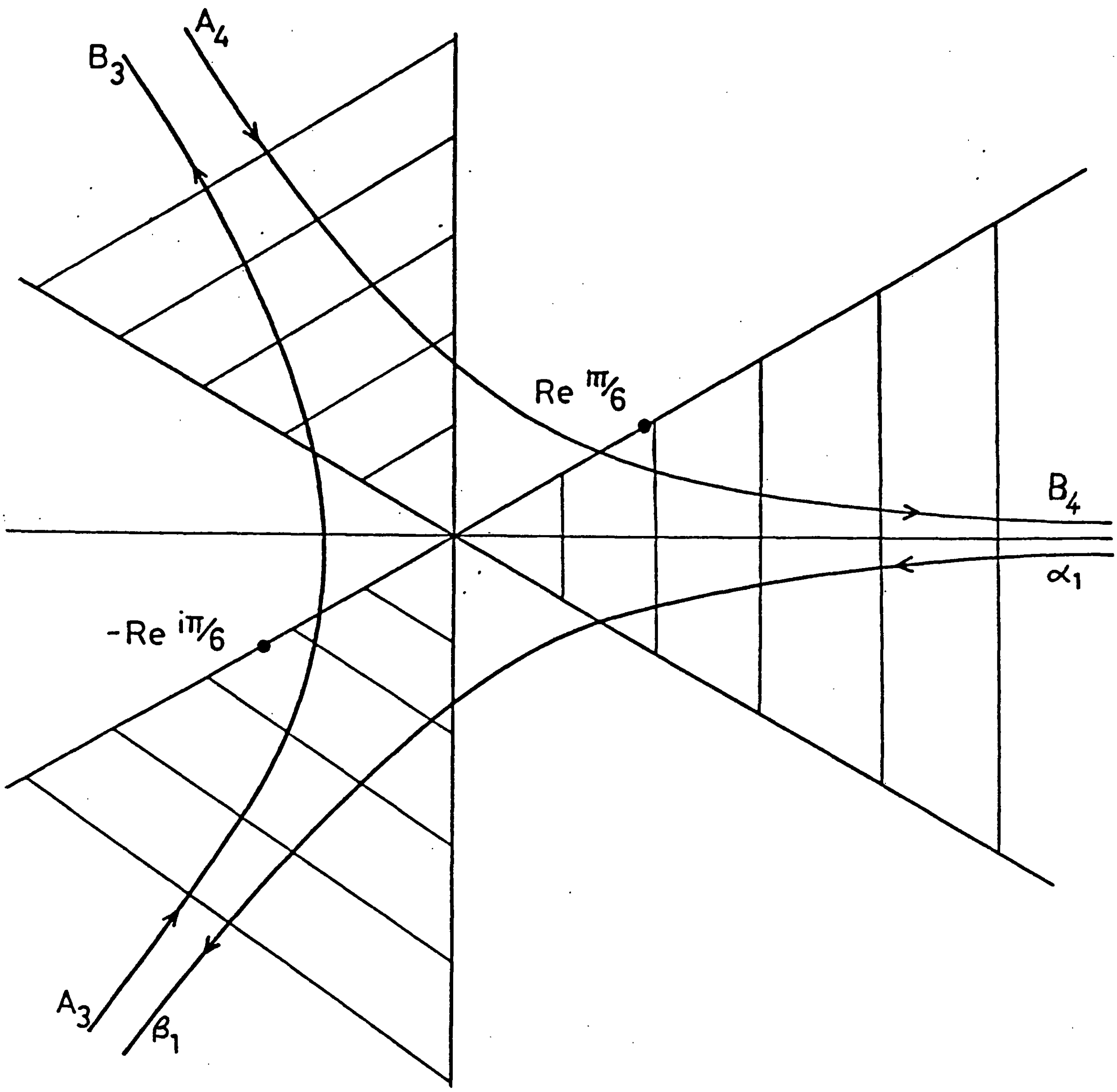


Fig. 1: Three possible paths of integration for P(η)

$$P_s(\eta)\int_{A_sB_s}(t-Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}}\exp\left[-\tfrac{1}{3}t^3-tR^2e^{i\pi/3}-t\eta e^{-i\pi/6}\right]d\eta\,,\quad s=3,4. \tag{3.148}$$

The two solutions $P_1(Z)$ and $P_2(Z)$ can be rewritten as

$$\begin{aligned} P_1(Z) &= P_1(\eta) = e^{\eta R},\\ P_2(Z) &= P_2(\eta) = e^{-\eta R}. \end{aligned} \tag{3.149}$$

The general solution for $P(\eta)$ is therefore

$$P(\eta) = AP_1(\eta) + BP_2(\eta) + CP_3(\eta) + DP_4(\eta) \tag{3.150}$$

where A, B, C and D are arbitary constants.

<u>FORMULATION OF THE EQUATION OF CONSISTENCY</u>

We substitute (3.150) into the homogeneous boundary conditions (3.136) to obtain four equations for the arbitary constants A, B, C and D,

$$\left.\begin{aligned} &A(L+i)P_1(\eta_0) + B(L+i)P_2(\eta_0) + C(L+i)P_3(\eta_0) + D(L+i)P_4(\eta_0) = 0,\\ &A(L+i)P_1(\eta_h) + B(L+i)P_2(\eta_h) + C(L+i)P_3(\eta_h) + D(L+i)P_4(\eta_h) = 0,\\ &AJP_1(\eta_0) + BJP_2(\eta_0) + CJP_3(\eta_0) + DJP_4(\eta_0) = 0,\\ &AP_1''(\eta_h) + BP_2''(\eta_h) + CP_3''(\eta_h) + DP_4''(\eta_h) = 0. \end{aligned}\right\} \tag{3.151}$$

The operators L and J in (3.151) are defined by

$$L \equiv \frac{d^3}{d\eta^3} - i\eta\frac{d}{d\eta}\,,$$

$$J \equiv \frac{d^2}{d\eta^2} - M\frac{d}{d\eta}\,.$$

The four equations of (3.151) lead to the equation of consistency, namely

$$
\begin{vmatrix}
(L+i)P_1(\eta_0) & (L+i)P_2(\eta_0) & (L+i)P_3(\eta_0) & (L+i)P_4(\eta_0) \\
(L+i)P_1(\eta_h) & (L+i)P_2(\eta_h) & (L+i)P_3(\eta_h) & (L+i)P_4(\eta_h) \\
JP_1(\eta_0) & JP_2(\eta_0) & JP_3(\eta_0) & JP_4(\eta_0) \\
P_1''(\eta_h) & P_2''(\eta_h) & P_3''(\eta_h) & P_4''(\eta_h)
\end{vmatrix} = 0. \qquad (3.152)
$$

The terms in the last two columns of the determinant in (3.152) are integrals which are evaluated by the method of stationary points and the details of the method are presented in Appendix E. It is found that $P_3(\eta_0)$ and $P_4(\eta_h)$ are negligibly small and so their contributions to the boundary conditions at η_0 and η_h respectively are taken to be zero. When we have substituted into (3.152) the functions $P_1(\eta)$ and $P_2(\eta)$ from (3.149) and using (E.43), (E.44), (E.70) and (E.71) from Appendix E, the equation of consistency becomes

$$
\begin{vmatrix}
(R^3-i(\eta_0 R-1))e^{\eta_0 R} & (-R^3+i(\eta_0 R+1))e^{-\eta_0 R} & 0 & \frac{iR^2}{R+Be^{-i\pi/6}}\left[\eta_0+\frac{1}{(R+Be^{-i\pi/6})}\right] \\
(R^3-i(\eta_h R-1))e^{\eta_h R} & (-R^3+i(\eta_h R+1))e^{-\eta_h R} & \frac{iR^2}{R+\alpha e^{i\pi/6}}\left[-\eta_h+\frac{1}{R+\alpha e^{i\pi/6}}\right] & 0 \\
(R^2-RM)e^{\eta_0 R} & (R^2+RM)e^{-\eta_0 R} & 0 & (R+Be^{-i\pi/6})(R+Be^{-i\pi/6}+M) \\
R^2 e^{\eta_h R} & R^2 e^{-\eta_h R} & (R+\alpha e^{i\pi/6})^2 & 0
\end{vmatrix} = 0.
$$

(3.153)

APPROXIMATE CRITERIA FOR STABILITY

We will look at the order of magnitude of the various terms in (3.153), From (3.137) and Appendix E we find that

$$\alpha = O(\lambda^{1/6}),\ B = O(\lambda^{1/6}),\ M = O(\lambda^{5/6}),\ R = O(\lambda^{-1/3}),\ \eta = O(\lambda^{1/3}). \qquad (3.154)$$

After dividing the third row of the determinant in (3.153) by $\lambda^{1/2}$ and the fourth row of the determinant in (3.153) by $\lambda^{-2/3}$, we can write (3.153) in the general form,

$$\begin{vmatrix} \lambda^{-1}A_1 + a_1 & \lambda^{-1}B_1 + b_1 & 0 & \alpha_1\lambda^{-1/2} \\ \lambda^{-1}A_2 + a_2 & \lambda^{-1}B_2 + b_2 & \alpha_2\lambda^{-1/2} & 0 \\ \lambda^{-7/6}A_3 + a_3 & \lambda^{-7/6}B_3 + b_3 & 0 & \alpha_3\lambda^{1/2} \\ a_4 & b_4 & \alpha_4\lambda & 0 \end{vmatrix} = 0 \tag{3.155}$$

where A_i, a_i, B_i, b_i and α_i $(i = 1, 2, 3, 4)$ are independent of λ and in the last two columns we have retained the largest terms only, namely

$$\alpha_1\lambda^{1/2} = \frac{iR^2\eta_0}{Be^{-i\pi/6}}, \quad \alpha_2\lambda^{1/2} = \frac{-iR^2\eta_h}{\alpha e^{i\pi/6}}, \quad \alpha_3\lambda^{1/2} = MBe^{-i\pi/6}, \quad \alpha_4\lambda = \alpha^2 e^{i\pi/3}.$$

We expand the determinant by the last two columns to give

$$\alpha_1\alpha_2\lambda^{-1}\begin{vmatrix} \lambda^{-7/6}A_3+a_3 & \lambda^{-7/6}B_3+b_3 \\ a_4 & b_4 \end{vmatrix} + \alpha_4\alpha_1\lambda^{1/2}\begin{vmatrix} \lambda^{-1}A_2+a_2 & \lambda^{-1}B_2+b_2 \\ \lambda^{-7/6}A_3+a_3 & \lambda^{-7/6}B_3+b_3 \end{vmatrix}$$

$$- \alpha_3\alpha_4\begin{vmatrix} \lambda^{-1}A_1+a_1 & \lambda^{-1}B_1+b_1 \\ a_4 & b_4 \end{vmatrix} + \alpha_3\alpha_4\lambda^{3/2}\begin{vmatrix} \lambda^{-1}A_1+a_1 & \lambda^{-1}B_1+b_1 \\ \lambda^{-1}A_2+a_2 & \lambda^{-1}B_2+b_2 \end{vmatrix} = 0. \tag{3.156}$$

We will approximate the equation (3.156) by retaining the terms of order $\lambda^{3/2}$ and $\lambda^{1/2}$, namely,

$$\lambda^{3/2}\alpha_3\alpha_4\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \lambda^{1/2}\alpha_3\alpha_4\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} + \lambda^{1/2}\alpha_1\alpha_4\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \tag{3.157}$$

which is

$$B^2e^{-i\pi/3}M\left[\begin{vmatrix} (1-\eta_0R)e^{\eta_0R} & (1+\eta_0R)e^{-\eta_0R} \\ (1-\eta_hR)e^{\eta_hR} & (1+\eta_hR)e^{-\eta_hR} \end{vmatrix} + R^6\begin{vmatrix} e^{\eta_0R} & e^{-\eta_0R} \\ e^{\eta_hR} & e^{-\eta_hR} \end{vmatrix}\right]$$

$$+ R^3\eta_0M\lambda^{-1/2}\begin{vmatrix} (1-\eta_hR)e^{\eta_hR} & (1+\eta_hR)e^{-\eta_hR} \\ -e^{\eta_0R} & e^{-\eta_0R} \end{vmatrix} = 0. \tag{3.158}$$

At this stage we will return to the original parameters and re-introduce U(z), the thermal wind, remembering that

$$\eta_0 = -\lambda^{1/3} Z_c \quad , \quad \eta_h = \lambda^{1/3}(1 - Z_c),$$

$$U(z) - c = U_1(Z - Z_c),$$

$$U(Z) = U_0 + U_1(Z - 1/2), \qquad 0 \leq Z \leq 1.$$

Using the above formulae, with $c = c_0 + ic_1$ (3.158) becomes

$$\frac{1}{2}\left(U_0 + \frac{U_1}{2} - c_0 - c_1\right)\left[-\frac{\gamma_1^2}{U_1^2}(U_0 - c_0 - ic_1)^2 + \frac{\gamma_1^2}{4} - \gamma_1 \coth\gamma_1 + 1\right]$$

$$-\lambda^{-1}\gamma_1^3 e^{i\pi/3}(U_0 - c_0 - ic_1 + U_1/2)\left[\frac{\gamma_1}{U_1}\left(U_0 - c_0 - ic_1 - \frac{U_1}{2}\right) + \coth\gamma_1\right]$$

$$+\lambda^{-1}\frac{\gamma_1^3}{2}\left[U_0 + \frac{U_1}{2} - c_0 - c_1\right] = 0. \qquad (3.159)$$

Thus c can be written as a series of the form

$$c = A_0 + iA_1 + \lambda^{-1}(B_0 + iB_1) + O(\lambda^{-3/2}) \qquad (3.160)$$

Note that the term of order $\lambda^{-3/2}$ in (3.160) will arise by retaining the term

$$\alpha_3\alpha_2 \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix}$$

from (3.156) in the equation of consistency, (3.157). It is also important to note that the series (3.160) is not the same series expansion found for c in Part 2, which was a series in powers of $\lambda^{-1/2}$.

The first term, $(A_0 + iA_1)$ in the series (3.160) for c, satisfies the equation

$$(U_0 - A_0 - iA_1)^2 = \frac{U_1^2}{\gamma_1^2}\left[1 + \frac{\gamma_1^2}{4} - \gamma_1 \coth\gamma_1\right]. \qquad (3.161)$$

This is, of course, the Eady result found in Chapter 3.3. If we write

$$D(\gamma_1) = \left[1 + \gamma_1^2/4 - \gamma_1 \coth\gamma_1\right]^{1/2},$$

$$d(\gamma_1) = \left[\,\left|1 + \gamma_1^2/4 - \gamma_1 \coth\gamma_1\right|\,\right]^{1/2},$$

then we find as before,

$$\left.\begin{aligned} &D(\gamma_1) = id\,, \quad A_0 + iA_1 = U_0 \pm \frac{iU_1 d}{\gamma_1}\,, \qquad 0 < \gamma_1 < 2.4, \\ &D(\gamma_1) = d\,, \quad A_0 + iA_1 = U_0 \pm \frac{U_1 d}{\gamma_1}\,, \qquad 2.4 < \gamma_1\,. \end{aligned}\right\} \qquad (3.162)$$

To find the second term, (B_0+iB_1) in the series (3.160) we substitute the first two terms of the series in (3.160) into (3.159) and retain terms of order λ^{-1} to obtain the following equation

$$B_0+iB_1 = \frac{e^{i\pi/3}\left[\frac{\gamma_1^3}{U_1}\left((U_0-A_0-iA_1)^2-U_1^2/4\right)+\gamma_1^2(U_0-A_0-iA_1+U_1/2)\coth\gamma_1\right]-\gamma_1^2/2\left[U_0+U_1/2-A_0-A_1\right]}{\gamma_1/U_1^2\left[U_0-A_0-iA_1\right]\left[U_0+U_1/2-A_0-A_1\right]}\,.$$

We substitute for (A_0+iA_1) from (3.161) and find that

$$B_0+iB_1 = U_1\gamma_1^2\,\frac{\left[e^{i\pi/3}(1-\gamma_1/2\coth\gamma_1 \pm D\coth\gamma_1)-\gamma_1/4\pm d/2\right]}{\pm D(\gamma_1/2\pm d)}\,.$$

As for (A_0+iA_1) there are two values for (B_0+iB_1) for the ranges $0<\gamma_1<2.4$ and $2.4<\gamma_1$ and we find that

$$\underset{0<\gamma_1<2.4}{B_0+iB_1} = \frac{U_1\gamma_1^2}{\pm d(\gamma_1/2\pm d)}\left\{\left[\frac{\sqrt{3}}{2}\left(1-\frac{\gamma_1}{2}\coth\gamma_1\right)\pm\frac{d}{2}\coth\gamma_1\right]\right.$$
$$\left.+i\left[-\frac{1}{2}+\frac{\gamma_1}{4}\mp\frac{d}{2}+\frac{\gamma_1}{4}\coth\gamma_1 \pm d\frac{\sqrt{3}}{2}\coth\gamma_1\right]\right\}, \qquad (3.163)$$

$$\underset{2.4<\gamma_1}{B_0+iB_1} = \frac{U_1\gamma_1^2}{\pm d(\gamma_1/2\pm d)}\left\{\left[\frac{1}{2}-\frac{\gamma_1}{4}\coth\gamma_1\mp\frac{d}{2}\coth\gamma_1-\frac{\gamma_1}{4}\mp\frac{d}{2}\right]\right.$$
$$\left.+i\left[\frac{\sqrt{3}}{2}\left(1-\frac{\gamma_1}{2}\coth\gamma_1 \pm d\coth\gamma_1\right)\right]\right\}. \qquad (3.164)$$

The formulae for (B_0+iB_1) in (3.164) and(3.165) are complicated and so we will look at the regions γ_1 small, $\gamma_1\sim 2.4$ and γ_1 large in more depth.

(1) Small Values of γ_1

For small γ_1 we can approximate $\coth\gamma_1$ to $\left(\frac{1}{\gamma_1}+\frac{\gamma_1^2}{3}+\ldots\ldots..\right)$ and $D(\gamma_1)$ becomes approximately $\pm i\gamma_1/\sqrt{12}$. Thus we find that

$$\left.\begin{aligned} D(\gamma_1)=+\frac{i\gamma_1}{\sqrt{12}}:\quad A_0+iA_1 &\sim U_0+\frac{iU_1}{\sqrt{12}}\,,\quad B_0+iB_1\sim U_1\left[2-\frac{i}{2}\right],\\ D(\gamma_1)=-\frac{i\gamma_1}{\sqrt{12}}:\quad A_0+iA_1 &\sim U_0-\frac{iU_1}{\sqrt{12}}\,,\quad B_0+iB_1\sim -5U_1\left[1-i\right]. \end{aligned}\right\} \qquad (3.165)$$

(2) $\gamma_1 \sim 2.4$

To approximate (B_0+iB_1) near $\gamma_1 = 2.4$, we write $\gamma_1 = 2.4 + \varepsilon$, where ε is assumed to be small. We substitute for γ_1 in (3.159) to obtain

$$\frac{1}{2}\left[U_0+\frac{U_1}{2}-c_0-c_1\right]\left[-\frac{\gamma_1^2}{U_1^2}(U_0-c_0-ic_1)^2+\mu\varepsilon\right]$$
$$-\lambda^{-1}\gamma_1^3 e^{-i\pi/3}(U_0-c_0-ic_1+U_1/2)\left[\gamma_1/U_1(U_0-c_0-ic_1-U_1/2)+\coth\gamma_1\right]$$
$$+\lambda^{-1}\frac{\gamma_1^3}{2}(U_0+U_1/2-c_0-c_1) = 0$$

where $\mu = \gamma_1\coth^2\gamma_1 - \coth\gamma_1 - \gamma_1/2$. When ε is small compared with λ^{-1} we have

$$(U_0-c_0-ic_1)^2 = \gamma_1\lambda^{-1}U_1^2\,\frac{\left[-2e^{-i\pi/3}(U_0-c_0-ic_1+\frac{U_1}{2})\left(\frac{\gamma_1}{U_1}(U_0-c_0-ic_1-U_1/2)+\coth\gamma_1\right)+(U_0+U_1/2-c_0-c_1)\right]}{(U_0+U_1/2-c_0-c_1)}.$$

Thus, by comparing orders of magnitude we have that,

$$c_0 \sim U_0+\mu_1\lambda^{-1}U_1\ ,\quad c_1\sim\mu_2\lambda^{-1}U_1\quad(\mu_1,\mu_2 \text{ finite constants}) \tag{3.166}$$

Since from (3.160) we have that

$$c_0 = A_0+\lambda^{-1}B_0+\cdots\ ,$$
$$c_1 = A_1+\lambda^{-1}B_1+\cdots\ ,$$

we can see that B_0 and B_1 remain finite in this region.

(3) Large Values of γ_1

For large γ_1 we can approximate $\coth\gamma_1$ to 1 and $D(\gamma_1)$ becomes approximately $\pm(1-\gamma_1/2)$ and we find that,

$$\left.\begin{aligned} D(\gamma_1)&\sim(1-\gamma_1/2):\ A_0+iA_1\sim U_0-\frac{U_1}{2},\ B_0+iB_1\sim(\lambda_1\gamma_1+i\lambda_2\gamma_1^2),\\ D(\gamma_1)&\sim-(1-\gamma_1/2):\ A_0+iA_1\sim U_0+\frac{U_1}{2},\ B_0+iB_1\sim(-\lambda_3\gamma_1-i\lambda_4\gamma_1^2).\end{aligned}\right\} \tag{3.167}$$

where λ_i (i = 1-4) are positive constants.

DISCUSSION

We shall return to the formulation of the perturbation pressure as defined in (3.30),

$$p'(x,z,t) = P(z)e^{ik(x-ct)}$$

and notice that when c has a positive imaginary term unstable waves can exist. Unlike the Eady model we now have instability for all values of γ_1.

For the range $0 < \gamma_1 < 2.4$ we have shown that the second term in the expansion of c, (B_0+iB_1) remains finite and it is therefore the Eady term (A_0+iA_1) which dominates. However, for the region $2.4 < \gamma_1$ where instability does not exist in the Eady model the second order term (B_0+iB_1) is important. As is seen above B_1 can be positive and is of order γ_1^2 and thus for large γ_1 the term $i\lambda^{-1}B_1$ can be large and produce unstable growth. These unstable waves will travel at great speeds as $\lambda^{-1}B_0$ will also be large for large γ_1.

For the term $\lambda^{-1}B_1$ to be of the same order of magnitude as the Eady instability term $(U_1\frac{d}{\gamma_1})$, of order U_1, we find that

$$\gamma_1 \sim U_1 h(\rho_1/\varepsilon)^{1/2}.$$

Using the values in (3.55) and (3.57) we deduce the following :

$$\gamma_1 \sim 10^9 \qquad \text{(for molecular conductivity)},$$

$$\gamma_1 \sim 10^7 \qquad \text{(for eddy conductivity)}.$$

For these values of γ_1, $\lambda^{-1}B_0$ is small and so the speed of the unstable wave is approximately $A_0 = U_0 - U_1/2$.

CHAPTER 3.6

AN APPROXIMATE SOLUTION OF THE STABILITY PROBLEM FOLLOWING MEKSYN, RETAINING THE β TERM

FORMULATION OF THE MODEL

In Chapter 3.2, we formulated a fourth order differential equation for P(z), (3.35). At that stage, we used the approximation, $\beta = 0$. Using the methods introduced in Chapter 3.5 it is now possible to retain the β term and find a solution for P(z). The differential equation for P(z), from (3.35) takes the form

$$\frac{k_1}{c_1\rho_1}\frac{d^4P}{dz^4} - ik(U(z)-c)\left[\frac{d^2P}{dz^2} - \frac{g\alpha Ak^2}{\rho_1 L^2}P\right] - \frac{ik\beta g\alpha A}{\rho_1 L^2}P = 0. \qquad (3.168)$$

We will denote

$$\varepsilon = \frac{k_1}{c_1 k}, \quad \gamma^2 = \frac{g\alpha Ak^2}{\rho_1 L^2}, \quad m = \frac{\beta g\alpha A}{\rho_1 L^2},$$

and thus (3.168) simplifies to

$$\frac{\varepsilon}{\rho_1}\frac{d^4P}{dz^4} - i(U(z)-c)\left[\frac{d^2P}{dz^2} - \gamma^2 P\right] - imP = 0. \qquad (3.169)$$

We now use the change of variable as in Chapter 3.5, namely

$$U(z) - c = U_1(Z - Z_c), \quad 0 \leq Z \leq 1;$$

the equation (3.169) reduces to

$$\frac{d^4P}{dZ^4} - i\lambda(Z-Z_c)\left[\frac{d^2P}{dZ^2} - \gamma_1^2 P\right] - i\lambda M'P = 0 \qquad (3.170)$$

where

$$\lambda = U_1 h^2\rho_1/\varepsilon, \quad \gamma_1 = \gamma h \quad \text{and} \quad M' = h^2 m/U_1.$$

Finally we introduce the variable η,

$$\eta = \lambda^{1/3}(Z - Z_c) \qquad (3.171)$$

and the resulting equation for P is

$$\frac{d^4P}{d\eta^4} - i\eta\left[\frac{d^2P}{d\eta^2} - R^2P\right] - iM_1P = 0 \qquad (3.172)$$

where

$$R^2 = \lambda^{-2/3}\gamma_1^2 \quad \text{and} \quad M_1 = \lambda^{-1/3}M'.$$

The boundary equations are not changed by retaining the β term, from (3.136), the boundary conditions are

$$\left.\begin{aligned} &\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_0, \eta_h, \\ &\frac{d^2P}{d\eta^2} - M\frac{dP}{d\eta} = 0 \quad \text{at } \eta = \eta_0, \\ &\frac{d^2P}{d\eta^2} = 0 \quad \text{at } \eta = \eta_h. \end{aligned}\right\} \qquad (3.173)$$

As in Chapter 3.5 we use heuristic arguments to assume that

$$Z_c = a - ib$$

where

$$0 \leq a, b \leq 1, \quad a >> b.$$

THE SOLUTION FOLLOWING MEKSYN

Following the method of Meksyn (26) described in Chapter 3.5 we will look for a solution for $P(\eta)$ in the form

$$P(\eta) = \int_c \chi(t) \exp\left[-t\eta e^{-i\pi/6}\right] dt. \qquad (3.174)$$

We substitute $P(\eta)$ from (3.174) into (3.172) and obtain

$$\int_c \left(t^4 e^{-2\pi i/3} - i\eta t^2 e^{-i\pi/3} + i\eta R^2 - iM_1\right)\chi e^{\phi} dt = 0 \qquad (3.175)$$

where

$$\phi(\eta) = -t\eta e^{-i\pi/6}.$$

We integrate by parts the middle two terms in (3.175) and it then becomes

$$\int_c \left(t^4 e^{-\frac{2\pi i}{3}}\chi - iM_1\chi - \frac{d}{dt}\left(e^{\frac{2i\pi}{3}}(t^2 e^{-i\pi/3} - R^2)\chi\right)\right)e^{\phi} dt$$
$$+ \left[ie^{i\pi/6}(t^2 e^{-i\pi/3} - R^2)\chi e^{\phi}\right]_c = 0. \qquad (3.176)$$

A solution of the form (3.174) does therefore exist provided

$$\frac{d}{dt}\left[e^{2\pi i/3}(t^2 e^{i\pi/3} - R^2)\chi\right] + \left[-t^4 e^{-2\pi i/3} + iM_1\right]\chi = 0 \qquad (3.177)$$

and

$$\left[(t^2 e^{-i\pi/3} - R^2)\chi e^{\phi}\right]_c = 0. \qquad (3.178)$$

Expanding (3.177) results in the following differential equation for $\chi(t)$:

$$(t^2 - R^2 e^{i\pi/3})\frac{d\chi}{dt} + (t^4 + 2t - iM_1 e^{2\pi i/3})\chi = 0.$$

We can separate the variables in the above equation and the solution for

$\chi(t)$ takes the form

$$\chi(t) = (t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3}\right] \tag{3.179}$$

where

$$M_2 = \frac{M_1}{2R} + \frac{iR^3}{2}. \tag{3.180}$$

Thus M_2 is a complex quantity with $\mathrm{Re}\{M_2\} = \frac{M'}{2\gamma_1} > 0$ and $\mathrm{Re}\{M_2\} >> \mathrm{Im}\{M_2\}$.

The solution for $P(\eta)$ is therefore

$$P(\eta) = \int_c (t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt \tag{3.181}$$

where c must be chosen so that

$$\left[(t - Re^{i\pi/6})^{-M_2}(t + Re^{i\pi/6})^{+M_2}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]\right]_c = 0 .$$

If we write $t = re^{i\theta}$ then

$$-t^3 = -r^3(\cos 3\theta + i\sin 3\theta)$$

and $\exp\{-\tfrac{1}{3}t^3\} \to 0$ as $r \to \infty$ provided that $\cos 3\theta > 0$. As in Chapter 3.5, (3.148) we have three possible cases for θ,

$$\left.\begin{array}{l} -\frac{\pi}{6} < \theta < \pi/6 \;, \\ \frac{\pi}{2} < \theta < 5\pi/6 \;, \\ 7\pi/6 < \theta < 3\pi/2 \;. \end{array}\right\} \tag{3.182}$$

As in Chapter 3.5, three possible contours for c are shown in Fig. 1, namely $\alpha_1\beta_1$, A_3B_3 and A_4B_4 . It has been shown that only two of these are independent and again we will choose A_3B_3 and A_4B_4 for two independent solutions $P_3(\eta)$ and $P_4(\eta)$ so that

$$P_s(\eta) = \int_{A_sB_s} (t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt, \quad s = 3, 4.$$

There are however a group of three more contours, $\alpha_2\beta_2$, A_1B_1 and A_2B_2 as shown in Fig. 2 which satisfy one of the conditions in (3.182). The three curves circle the singularity $-Re^{i\pi/6}$ and tend to infinity in the acceptable shaded areas in which one of the conditions of (3.182) hold. The singularity at $Re^{i\pi/6}$ is to be avoided since $P(\eta) \to \infty$ at this point. Again, the three curves are not independent. Since the contribution to the integral of the singularity $-Re^{i\pi/6}$ is zero ($\mathrm{Re}\{M_2\} > 0$) the integral inside the closed contour of the three paths $\alpha_2\beta_2$, A_1B_1 and A_2B_2 is zero. We will choose A_1B_1 and A_2B_2 for two independent solutions $P_1(\eta)$ and $P_2(\eta)$ so that

$$P_s(\eta) = \int_{A_sB_s} (t - Re^{i\pi/6})^{-1-M_2} (t - Re^{i\pi/6})^{-1+M_2} \exp[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}]\, dt, \quad s = 1,2,3,4. \tag{3.183}$$

The four solutions are independent since A_1B_1 and A_2B_2 cannot be deformed into A_3B_3 and A_4B_4 without leaving the shaded areas. The general solution for $P(\eta)$ is therefore

$$P(\eta) = \alpha P_1(\eta) + \beta P_2(\eta) + \gamma P_3(\eta) + \delta P_4(\eta). \tag{3.184}$$

FORMULATION OF THE EQUATION OF CONSISTENCY

The equation of consistency arising from the four homogeneous boundary conditions, (3.173) leads to an equation of the form

$$\begin{vmatrix} (L+i)P_1(\eta_0) & (L+i)P_2(\eta_0) & (L+i)P_3(\eta_0) & (L+i)P_4(\eta_0) \\ (L+i)P_1(\eta_h) & (L+i)P_2(\eta_h) & (L+i)P_3(\eta_h) & (L+i)P_4(\eta_h) \\ JP_1(\eta_0) & JP_2(\eta_0) & JP_3(\eta_0) & JP_4(\eta_0) \\ P_1''(\eta_h) & P_2''(\eta_h) & P_3''(\eta_h) & P_4''(\eta_h) \end{vmatrix} = 0 \tag{3.185}$$

where the operators L and J are defined as

$$L \equiv \frac{d^3}{d\eta^3} - i\eta\frac{d}{d\eta}, \quad J \equiv \frac{d^2}{d\eta^2} - M\frac{d}{d\eta}.$$

The terms in (3.185) are calculated in Appendix F; the determinant in (3.185) is simplified by approximations made in the Appendix and (3.185) becomes

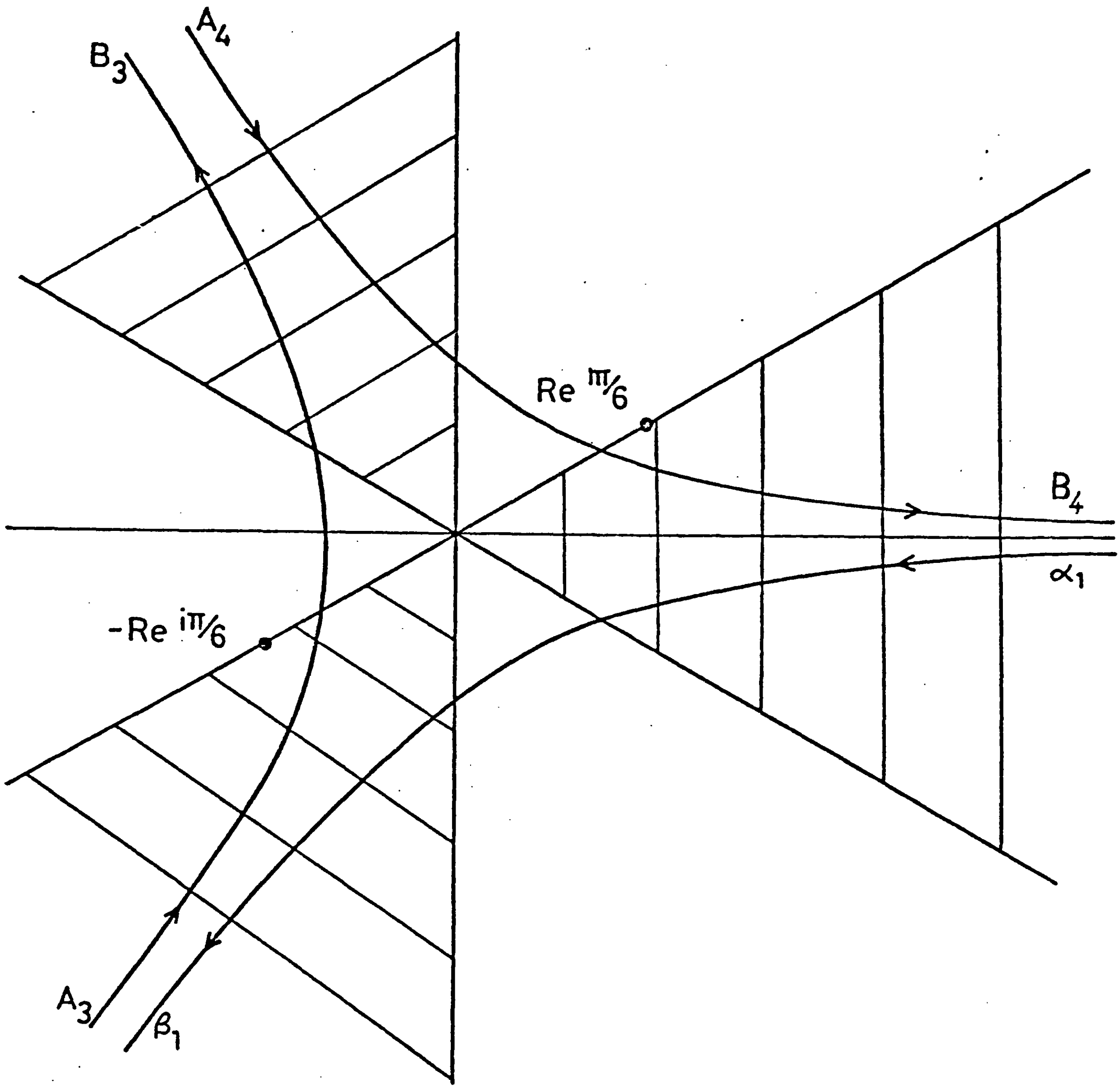


Fig. 1: Three possible paths of integration for P(η)

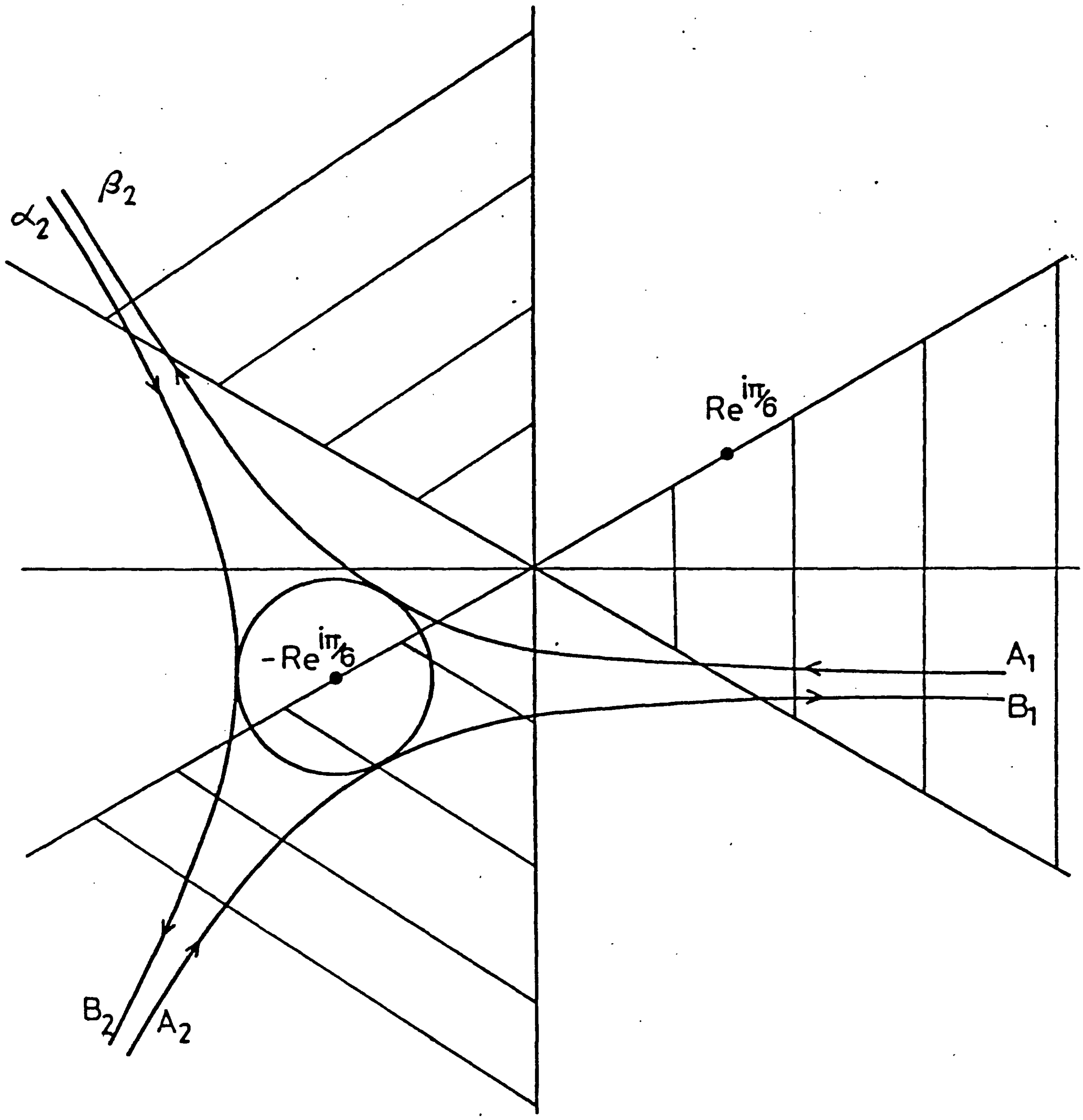


Fig. 2: Three more possible paths of integration for $P(\eta)$

$$\begin{vmatrix} C_1\alpha_1 & D_2'C & 0 & \alpha_1 \\ D_1'A & C_2\beta_1 & \beta_1 & 0 \\ C_1\alpha_2 & D_2'D & 0 & \alpha_2 \\ D_1'B & C_2\beta_2 & \beta_2 & 0 \end{vmatrix} = 0$$

which when expanded, reduces to

$$\frac{AC}{\alpha_1\beta_1} - \frac{CB}{\alpha_1\beta_2} - \frac{AD}{\beta_1\alpha_2} + \frac{DB}{\alpha_2\beta_2} = 0 \tag{3.186}$$

where

$$\left.\begin{aligned} A &= \frac{1}{\eta_h^3}\Gamma(3+M_2)(-\tau_h)^{3/2}W_{-3/2-M_2,1}(-\tau_h) + i\Gamma(1+M_2)(-\tau_h)^{1/2}W_{-1/2-M_2,0}(-\tau_h) + i\Gamma(M_2)W_{-M_2,-1/2}(-\tau_h), \\ B &= \frac{1}{\eta_h^2}\Gamma(2+M_2)(-\tau_h)W_{-1-M_2,1/2}(-\tau_h), \\ C &= \frac{-1}{\eta_0^3}\Gamma(3+M_2)(-\tau_0)^{\frac{3}{2}}W_{-3/2-M_2,1}(-\tau_0) + i\Gamma(1+M_2)(-\tau_0)^{1/2}W_{-1/2-M_2,0}(-\tau_0) + i\Gamma(M_2)W_{-M_2,-1/2}(-\tau_0), \\ D &= \frac{1}{\eta_0^2}\Gamma(2+M_2)(-\tau_0)W_{-1-M_2,1/2}(-\tau_0) + \frac{M}{\eta_0}\Gamma(1+M_2)(-\tau_0)^{1/2}W_{-1/2-M_2,0}(-\tau_0), \end{aligned}\right\} \tag{3.187}$$

$$\left.\begin{aligned} \alpha_1 &= i\left[\frac{\eta_0R^2-M_1}{Be^{-i\pi/6}-R} + \frac{R^2}{(Be^{-i\pi/6}-R)^2}\right], \\ \alpha_2 &= (Be^{-i\pi/6}-R)(Be^{-i\pi/6}-R+M), \\ \beta_1 &= i\left[\frac{-R^2\eta_h+M_1}{\alpha e^{i\pi/6}+R} + \frac{R^2}{(\alpha e^{i\pi/6}+R)^2}\right], \\ \beta_2 &= (\alpha e^{i\pi/6}+R)^2. \end{aligned}\right\} \tag{3.188}$$

The function $W_{k,m}(\tau)$ in the expression, (3.187) is the Whittaker function (32) defined by

$$W_{k,m}(\tau) = \frac{e^{-\frac{1}{2}\tau}\tau^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k-1/2+m}\left(1+\frac{t}{\tau}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt.$$

When we look at the orders of magnitude of the various terms in (3.186) we find that

$$\alpha_1 = O(\lambda^{-1/2}), \quad \alpha_2 = O(\lambda), \quad \beta_1 = O(\lambda^{-1/2}), \quad \beta_2 = O(\lambda^{1/3}),$$

and if we assume from the recurrence regions (33) that the Whittaker functions in (3.187) are all of the same order then

$$B/A = O(\lambda^{-2/3}), \quad C/A = O(\lambda^0), \quad D/A = O(\lambda^{1/2}).$$

We can therefore approximate (3.186) to

$$AC = 0 \tag{3.189}$$

where the variables τ_h, τ_o and Z_C are defined as

$$\tau_o = 2\gamma_1 Z_c \;, \quad \tau_h = -2\gamma_1(1 - Z_c),$$

$$Z_c = a - ib, \quad 0 < a, b < 1 \;, \quad a >> b \,.$$

The nondimensional quantity γ_1 is of finite magnitude and in the models of Eady and of Chapter 3.5 the important range for γ_1 is $0 < \gamma_1 < 5$. Within this range of γ_1, τ_o and τ_h are of finite magnitude and thus the use of asymptotic expansions for the Whittaker function is, we note, of limited value. Since these functions are not tabulated the solution of (3.189) would best be found using a computer. This numerical problem is suggested as one possible extension to the author's work.

CHAPTER 3.7

SUMMARY

The work in Part 3 is based on the one layer model of Eady. In his model, Eady superposed a wave perturbation on a sheared flow and solved the linearised perturbation equations, subject to homogeneous boundary conditions, to obtain criteria for stability. In this present work, by using difference methods the author was able to find identical criteria for stability without having to solve the perturbation equations.

It was of primary interest in Part 3 to extend the work of Eady to a two layer model and to include thermal conductivity in the heat transfer process. Consequently, a second layer was introduced, the thermal conductivity term was retained in the heat transfer equation and the boundary conditions of continuity of heat transfer and temperature between the two layers were added to complete the model.

Both the method of direct solution of the perturbation equations and of difference techniques were used to find the criteria for stability in this extended model. Unfortunately, unlike the Eady model, the two methods did not yield the same results:† as expected, the method of direct solution of the perturbation equations produced the stability criteria of Eady with first order correction terms. By contrast, this was not the case for the criteria found by using difference methods. Although more weight must be placed on the results found by solving the perturbation equations directly, the difference technique must not be disregarded. This method is extremely valuable in finding stability criteria of models where the complexity of the perturbation equations prohibits a direct solution.

The model was developed further by including a variable coriolis parameter. The perturbation equations were solved, subject to the same

† see next page

homogeneous boundary conditions of the authors model and a stability equation was found. Clearly, the application of the difference method to this model would yield interesting results.

From the work of Namias* it was assumed that two separate stable regimes might exists. It is of interest to note that the stability criteria of a two layer model of Davies*, in which heat transfer was not considered, reproduced two regions of stability. However, neither the Eady results nor those of the author's model retain this pattern. Further study in this field is therefore necessary to fully understand stability of one and two layer models in which heat transfer is included.

† It is possible that the Meksyn method and the difference method are both valid and are reflections of the two different unstable régimes in the work of Davis. Further work using a slightly modified approach to the difference method might reconcile the apparent disagreement.

* See General Introduction

APPENDIX B

A COMPARISON OF LIQUID AND FLUID MODELS ON A ROTATING SPHERE

In the liquid model considered in Chapter 3.2 the equations of motion were approximated to (3.2), (3.3) and (3.5). For completeness we will retain $w\frac{\partial u}{\partial z}$ and $w\frac{\partial v}{\partial z}$ in (3.2) and (3.3) and we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - lv = -\frac{1}{\rho_1}\frac{\partial p}{\partial x}, \tag{B.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + lu = -\frac{1}{\rho_1}\frac{\partial p}{\partial y}, \tag{B.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{B.3}$$

The equation of continuity is approximated to (3.6), namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{B.4}$$

In the heat equation of (3.7) only the vertical component of thermal conductivity is retained, resulting in

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k_1}{c_1\rho_1}\frac{\partial^2 T}{\partial z^2}. \tag{B.5}$$

To compare the above system of equations with that of a fluid we will introduce a new co-ordinate system and change from the (x, y, z, t) indepen- -dent variables to (x, y, p, t) where p is the pressure function. We take the isobaric surface (39), at which the pressure is p, to be

$$z = H(x, y, p(x, y, z, t), t). \tag{B.6}$$

It is assumed that p always satisfies the hydrostatic equation, (B.3) and thus if we differentiate (B.6) partially with respect to x, y, z and t respectively we find

$$0 = H_x + H_p p_x, \tag{B.7}$$

$$0 = H_y + H_p P_y \,, \tag{B.8}$$

$$1 = \quad H_p P_z \,, \tag{B.9}$$

$$0 = H_t + H_p P_t \,. \tag{B.10}$$

When we combine (B.9) and (B.3) we have a new form for the hydrostatic equation

$$g\frac{\partial H}{\partial p} = -\frac{1}{\rho}\,, \tag{B.11}$$

which when substituted into (B.7) and (B.8) yields

$$gH_x = \frac{1}{\rho} P_x \,, \tag{B.12}$$

$$gH_y = \frac{1}{\rho} P_y \,. \tag{B.13}$$

The quantity gH is often refered to as the geopotential.

We will take the complete transformation from (x, y, z, t) to (x, y, p, t) as follows:

$$\left.\begin{aligned} x &= X\,, \\ y &= Y\,, \\ z &= H(X, Y, p, \tau)\,, \\ t &= \tau\,. \end{aligned}\right\} \tag{B.14}$$

Thus we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial X} &= \frac{\partial}{\partial x} + H_X \frac{\partial}{\partial z}\,, \\ \frac{\partial}{\partial Y} &= \frac{\partial}{\partial y} + H_Y \frac{\partial}{\partial z}\,, \\ \frac{\partial}{\partial p} &= \qquad\quad H_p \frac{\partial}{\partial z}\,, \\ \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} + H_\tau \frac{\partial}{\partial z}\,, \end{aligned}\right\} \tag{B.15}$$

with the inverse formulae

$$\left.\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial X} - \frac{H_x}{H_p}\frac{\partial}{\partial p}, \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial Y} - \frac{H_y}{H_p}\frac{\partial}{\partial p}, \\
\frac{\partial}{\partial z} &= \qquad \frac{1}{H_p}\frac{\partial}{\partial p}, \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} - \frac{H_\tau}{H_p}\frac{\partial}{\partial p}.
\end{aligned}\right\} \tag{B.16}$$

If we take

$$W = \frac{1}{H_p}\left[w - H_\tau - uH_x - vH_Y\right] \tag{B.17}$$

we can write the first two equations of motion in the form

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} + W\frac{\partial u}{\partial p} - lv = -g\frac{\partial H}{\partial x}, \tag{B.18}$$

$$\frac{\partial v}{\partial \tau} + u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y} + W\frac{\partial v}{\partial p} + lu = -g\frac{\partial H}{\partial Y}. \tag{B.19}$$

It can be seen that W is taking the place of the vertical velocity in the new co-ordinate system and it can easily be shown from (B.16) and (B.17) that

$$W \equiv \frac{dp}{dt} \tag{B.20}$$

i.e. that W is the rate of change of pressure following the motion in the (x, y, z, t) co-ordinate system. It is also interesting to note that w can be written as

$$w = H_\tau + uH_X + vH_Y + WH_p \tag{B.21}$$

so that w is the rate of change of $H(x, y, p, \tau)$ following the motion.

We transform the equation of continuity for a fluid,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) + \frac{\partial}{\partial y}(v\rho) + \frac{\partial}{\partial z}(\rho w) = 0,$$

using the same process, to obtain

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial W}{\partial p} = 0, \tag{B.22}$$

which now has the same structure as (B.4).

Finally we consider the heat equation in the form

$$\rho \frac{dQ}{dt} = k_1 \nabla^2 T \tag{B.23}$$

where Q is the heat content per unit mass and can be written as

$$\delta Q = c_1 \delta T - \frac{1}{\rho} \delta p = c_1 T \frac{\delta \theta}{\theta} \tag{B.24}$$

where θ is the potential temperature. For a liquid therefore dQ can be approximated to $c_1 \delta T$. For a fluid, retaining only the vertical thermal conductivity term in (B.23) we have

$$\rho \frac{T}{\theta} c_1 \frac{d\theta}{dt} = k_1 \frac{\partial^2 T}{\partial z^2} \, . \tag{B.25}$$

In the new system of co-ordinates (B.25) is transformed to

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + W \frac{\partial \theta}{\partial p} = \frac{k_1 R}{c_1 P_0} \frac{\theta^{\frac{1}{1-\gamma}}}{T^{\frac{\gamma}{1-\gamma}} H_p^2} \left[\frac{\partial^2 T}{\partial p^2} - \frac{H_{pp}}{H_p} \frac{\partial T}{\partial p} \right] . \tag{B.26}$$

If thermal conductivity is ignored then the heat equation is preserved with T being replaced by θ . It is clear that the use of (x, y, p, t) as a new co-ordinate system simplifies the structure of the equations of motion, continuity and heat transfer for a fluid. Comparing the equations for a liquid and a fluid we have:

LIQUID	FLUID
$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - lv = -\frac{1}{\rho_1}\frac{\partial p}{\partial x}$	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial p} - lv = -g\frac{\partial H}{\partial x}$
$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + lu = -\frac{1}{\rho_1}\frac{\partial p}{\partial y}$	$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial p} + lu = -g\frac{\partial H}{\partial y}$
$0 = -\frac{\partial p}{\partial z} - g\alpha T$	$0 = gp^{\frac{1}{\gamma}}\frac{\partial H}{\partial p} + \frac{R\theta}{P_0^{1-\frac{1}{\gamma}}}$
$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial p} = 0$
$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k_1}{c_1\rho_1}\frac{\partial^2 T}{\partial z^2}$	$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + W\frac{\partial \theta}{\partial p} = \frac{k_1 R}{c_1 P_0}\frac{\theta^{\frac{1}{1-\gamma}}}{T^{\frac{\gamma}{1-\gamma}} H_p^2}\left[\frac{\partial^2 T}{\partial p^2} - \frac{H_{pp}}{H_p}\frac{\partial T}{\partial p}\right]$

where for the liquid we have taken $\rho = \alpha T$.

The boundary conditions for the model must also be considered when changing from a liquid into a fluid. The conditions that $w = 0$ at $z = 0$ and $z = h$ are transformed, using (B.14) and B.21) to

$$H_\tau + uH_x + vH_y + WH_p = 0 \quad \text{at} \quad H = 0, h. \tag{B.27}$$

This is a far more complicated expression for the two boundary conditions. The formulation of the remaining boundary conditions, (3.44) - (3.47) remains the same so that comparing the conditions, with T* as the temperature of the ocean we have:

LIQUID	FLUID
$W = 0$ at $z = 0, h$	$H_\tau + uH_x + vH_y + WH_p = 0$ at $H = 0, h$
$T = T^*$ at $z = 0$	$T = T^*$ at $H = 0$
$k_1 \frac{\partial T}{\partial z} = k_2 \frac{\partial T^*}{\partial z}$ at $z = 0$	$k_1 \frac{\partial T}{\partial p} = k_2 \frac{\partial T^*}{\partial p}$ at $H = 0$
$k_2 \frac{\partial T^*}{\partial z} = 0$ at $z = -h_2$	$\frac{k_2 \partial T^*}{H_p \partial p} = 0$ at $H = -h_2$
$k_1 \frac{\partial T}{\partial z} = 0$ at $z = h$	$\frac{k_1}{H_p} \frac{\partial T}{\partial p} = 0$ at $H = h$

The corresponding equations for the fluid are now very similar to those of the liquid with the main differences in the hydrostatic equation, the thermal conductivity term in the heat equation and the boundary con--dition, $w = 0$ at $z = 0, h$. Thus we can infer that some of the basic results for a liquid will also be found, possibly in a more complicated form, in a fluid model.

APPENDIX C

APPROXIMATE SOLUTIONS FOR C FOR THE CASES n = 4 AND n = 5.

In the Chapter 3.4 we present the use of the difference method to formulate an equation of consistency, (3.97). For small n, the determinant in (3.97) can be expanded and solutions for c are found. We present the cases of n = 4 and n = 5 in detail below.

(1) The Case n = 4

The equation of consistency, (3.97) for n = 4 is

$$|A_4| = \begin{vmatrix} 1+hH/4 & -2-hH/4 & 1 & 0 & 0 \\ -E+im(U_0-c-U_1/4) & 3E+im(-U_0+c+U_1/2) & -3E & E & 0 \\ E & -4E-im(U_0-c) & 6E+im\delta(U_0-c) & -4E-im(U_0-c) & E \\ 0 & -E & -3E & -3E+im(U_0-c+U_1/2) & E+im(-U_0+c-U_1/4) \\ 0 & 0 & 1 & -2 & 1 \end{vmatrix} = 0 \tag{C.1}$$

where

$$m = h^2/16, \quad E = \varepsilon/\rho_1, \quad \delta = 2+\gamma^2h^2/16 = 2+\beta .$$

The constant β is nondimensional. The determinant is expanded and written in powers of E,

$$\begin{aligned} |A_4| = \; & E^2 im\left[-(2+\beta hH/4)(U_0-c)+2W\right] \\ & + Em^2\left[(2\beta+\beta hH/2)(U_0-c)^2+(Hh/2+\beta hH/4)(U_0-c)W-\frac{hH}{2}W^2\right] \\ & + im^3\left[\beta(U_0-c)^3+(hH/4+\beta hH/4)(U_0-c)^2W+hH/4\,(U_0-c)W^2\right] \end{aligned} \tag{C.2}$$

where $W = U_1/4$. We substitute for H from (3.60),

$$H = \varepsilon^{-1/2}c^{1/2}\lambda(1-i) ,$$

to obtain

$$\begin{aligned} |A_4| = \; & \left(\frac{\varepsilon}{\rho_0}\right)^2 im\left[2W-2(U_0-c)-\varepsilon^{-1/2}c^{1/2}\frac{\lambda\beta h}{4}(1-i)(U_0-c)\right] \\ & + \frac{\varepsilon}{\rho_0}m^2\left[2\beta(U_0-c)^2+\varepsilon^{-1/2}c^{1/2}\lambda(1-i)\left(\frac{\beta h}{2}(U_0-c)^2+\left(\frac{h}{2}+\frac{\beta h}{4}\right)(U_0-c)W-\frac{h}{2}W^2\right)\right] \\ & + im^3\left[\beta(U_0-c)^3+\varepsilon^{-1/2}c^{1/2}\lambda(1-i)\left(\left(\frac{h}{4}+\frac{\beta h}{4}\right)(U_0-c)^2W-\frac{h}{4}(U_0-c)W^2\right)\right] = 0. \end{aligned} \tag{C.3}$$

We assume that c can be expanded in powers of ε in the following two ways

$$c = c_0 + \varepsilon^{1/2} c_1 + \varepsilon c_2 + \cdots\cdots \tag{C.4}$$

$$c = \frac{a_0}{\varepsilon} + \frac{a_1}{\varepsilon^{1/2}} + a_2 + \cdots\cdots\cdot \tag{C.5}$$

where the coefficients c_0 , c_1 , ... and a_0, a_1 ...... are independent of ε . By comparing coefficients of $\varepsilon^{-1/2}$ in (C.3) we find three possible values for c_0 in (C.4), namely

$$\left.\begin{aligned} c_0^1 &= 0 , \\ c_0^2 &= U_0 , \\ c_0^3 &= U_0 - W/(1+\beta) . \end{aligned}\right\} \tag{C.6}$$

The corresponding values of c_1 to the values of c_0 in (C.6) are obtained by comparing coefficients of ε in (C.3) and are found to be

$$\left.\begin{aligned} c_1^1 &= 0 , \\ c_1^2 &= 0 , \\ c_1^3 &= \frac{(1-i)\beta W}{h\lambda(1+\beta)^3 (U_0 - W/(1+\beta))^{1/2}} . \end{aligned}\right\} \tag{C.7}$$

The second series, (C.5) for c produces one more root for c. By comparing the powers of ε^{-3} in (C.3) we find

$$a_0 = -\frac{i\lambda^2 W^2 (1+\beta)^2 h^2}{8\beta^2} . \tag{C.8}$$

The second term in the expansion, a_1, found by comparing coefficients of $\varepsilon^{-5/2}$, is zero. The four values for c are therefore

$$c_1 = O(\varepsilon) , \tag{C.9}$$

$$c_2 = U_0 + O(\varepsilon), \tag{C.10}$$

$$c_3 = U_0 - \frac{W}{1+\beta} + \varepsilon^{1/2} \frac{(1-i)\beta W}{h\lambda(1+\beta)^3 (U_0 - \frac{W}{1+\beta})^{1/2}} + O(\varepsilon), \tag{C.11}$$

$$c_4 = -\varepsilon^{-1} \frac{i}{2} \left[\frac{\lambda h W (1+\beta)}{2\beta} \right]^2 + O(\varepsilon^0). \tag{C.12}$$

Higher order terms for the expansion of c can be found in the same way.

(2) The Case n = 5

The equation of consistency, (3.97) for n = 5 reduces to

$$|A_5| = \begin{vmatrix} 1+hH/5 & -2-hH/5 & 1 & 0 & 0 & 0 \\ -E+im(V-3W) & 3E+im(-V+5W) & -3E & E & 0 & 0 \\ E & -4E+im(-V+W) & 6E+im\delta(V-W) & -4E+im(-V+W) & E & 0 \\ 0 & E & -4E+im(V+W) & 6E+im\delta(V+W) & -4E+im(V+W) & E \\ 0 & 0 & -E & 3E & -3E+im(V+5W) & E+im(-V-3W) \\ 0 & 0 & 0 & 1 & -2 & 1 \end{vmatrix} = 0$$

(C.13)

where

$$m = h^2/25\,,\quad E = \epsilon/\rho_1\,,\quad \delta = 2+\frac{\sigma^2h^2}{25} = 2+\beta\,,\quad V = U_0-c\,,\quad W = U_1/10\,.$$

As before we expand the determinant in powers of ϵ, namely

$$\begin{aligned} |A_5| = {} & \left(\frac{\epsilon}{\rho_1}\right)^3 im\Big[(2+\beta)W + \epsilon^{-1/2}c^{1/2}\alpha(1-i)\big[-2\beta(U_0-c)+(-8+\beta)W\big]\Big] \\ & +\left(\frac{\epsilon}{\rho_1}\right)^2 im^2\Big[(-1+6\beta+\beta^2)(U_0-c)^2-(28+14\beta)(U_0-c)W+(6\beta-\beta^2)W^2 \\ & \qquad +\epsilon^{-1/2}c^{1/2}\alpha(1-i)\big[(-2+9\beta+2\beta^2)(U_0-c)^2+(94+54\beta)(U_0-c)W+(-20+13\beta-24\beta^2)W^2\big]\Big] \\ & +\frac{\epsilon}{\rho_1}im^3\Big[(-1+8\beta+2\beta^2)(U_0-c)^3-(1+\beta)(U_0-c)^2W+(6+6\beta-2\beta^2)(U_0-c)W^2 \\ & \qquad +\epsilon^{-1/2}c^{1/2}\alpha(1-i)\big[(3\beta+2\beta^2)(U_0-c)^3+(6+21\beta+4\beta^2)(U_0-c)^2W+(-24+13\beta-2\beta^2)(U_0-c)W^2+(19\beta-4\beta^2)W^4\big]\Big] \\ & +m^4\Big[(-2\beta-\beta^2)(U_0-c)^4+2\beta^2(U_0-c)^2W^2+(2\beta-\beta^2)W^4 \\ & \qquad +\epsilon^{-1/2}c^{1/2}\alpha(1-i)\big[(-2-6\beta-\beta^2)(U_0-c)^3W+(6-2\beta-2\beta^2)(U_0-c)^2W^2+(2+6\beta+2\beta^2)(U_0-c)W^3 \\ & \qquad +(-6+2\beta+2\beta^2)W^4\big]\Big] \end{aligned}$$

(C.14)

where we have substituted for H from (3.60) and α is defined as

$$\alpha = h\lambda/5\,.$$

As in the previous section we find c as series expansions in ϵ and obtain the five roots;

$$c_1 = O(\epsilon),$$ (C.15)

$$C_2 = U_0 + W + O(\varepsilon), \tag{C.16}$$

$$C_3 = U_0 - W + O(\varepsilon), \tag{C.17}$$

$$C_4 = U_0 - \frac{(6-2\beta-2\beta^2)W}{(2+6\beta+2\beta^2)} + c_1^+ \varepsilon^{1/2} + O(\varepsilon), \tag{C.18}$$

$$C_5 = -2i\left[\frac{\alpha W(2+6\beta+2\beta^2)}{(2\beta+\beta^2)}\right]^2 \varepsilon^{-1} + O(\varepsilon^0), \tag{C.19}$$

where c_1^+ is defined as

$$c_1^+ = \frac{(1+i)\left[(-2\beta-\beta^2)(U_0-c_0^+)^4 + 2\beta^2(U_0-c_0^+)^2W^2 + (2\beta-\beta^2)W^4\right]}{2(c_0^+)^{1/2}\alpha W\left(3(-2-6\beta-2\beta^2)(U_0-c_0^+)^2 + 2(6-2\beta-2\beta^2)(U_0-c_0^+)W + (2+6\beta+2\beta^2)W^2\right)}\,. \tag{C.20}$$

APPENDIX D

COMPUTATIONAL METHODS TO SOLVE $|A_n(c)| = 0$

The problem to be solved is

$$|A_n(c)| = 0 \cdot \tag{D.1}$$

where $|A_n(c)|$ is an (n + 1) x (n + 1) determinant with complex coefficients, dependent on a variable c which has to be determined. The problem therefore breaks down into two distinct parts, the first being to multiply out the determinant and the second to determine values of c such that (D.1) holds. The first part is resolved by using Crout's method and the second by using Muller's method and discussions of both are presented below.

CROUT'S METHOD

Before describing Crout's method, let us consider the simplest method of evaluating determinants which is attributed to Gauss(34). We shall consider the example

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad . \tag{D.2}$$

We use a_{11} to eliminate a_{21} and a_{31} by the row operations $r_2 - (a_{21}/a_{11})r_1$ and $r_3 - (a_{31}/a_{11})\, r_1$ where r_1, r_2 and r_3 are the first, second and third rows. We then use the new element in the second row and second column, $(a_{22} - a_{21}\, a_{12}/a_{11})$ to eliminate the term in the second column and third row thus producing an upper triangular determinant,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{vmatrix} \tag{D.3}$$

and so the value of the determinant is $(a_{11}\, U_{22}\, U_{33})$. In this elimination process the row used to eliminate terms is called the pivotal row. The element in the pivotal row which eliminates the rest of the column is called the pivot. In the above example a_{11} is the pivot for the first two operations. It is clear that this method breaks down when ever a pivot is zero. A unique process to avoid this difficulty is to choose as pivot the largest element in the relevant column of the determinant, taking the columns successively so that the other elements are eliminated in natural order. This method is called partial pivoting. The determinant thus formed may no longer be an upper triangular form, but can easily be transformed by interchanging certain rows and the value of the determinant will be $(-1)^p$ times the diagonal terms where p is the number of interchanges. An added advantage to this method is that by choosing the largest element as pivot, the accuracy of the results increases when calculations are rounded.

However, the above method forces us to compute the determinant after each row operation, producing a lengthy computation. The more sophisticated method of Crout (35, 36) has been devised so that the final form is found without calculating and recording the elements of the intermediate determinants. The determinant, $|A|$ say, is found in the form of a product of determinants, $|L||U|$ where $|L|$ is a lower triangular and $|U|$ an upper triangular determinant. Crout's method ensures that $|U|$ is a unit triangular determinant. This can be made clearer if we consider an example. If $|A|$ is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad (D.4)$$

then the computation is performed by means of the following equations. For the first row,

$$\left.\begin{aligned} U_{11} &= a_{11}\,, \\ V_{1s} &= a_{1s}/U_{11} \quad s>1\,, \end{aligned}\right\} \qquad (D.5)$$

for the second row,

$$\left.\begin{aligned} n_{12} &= a_{21}\,, \\ u_{22} &= a_{22} - n_{21}v_{12}\,, \\ v_{2s} &= (a_{2s} - n_{21}v_{1s})/u_{22}\,,\quad s>2\,, \end{aligned}\right\} \tag{D.6}$$

for the third row,

$$\left.\begin{aligned} n_{31} &= a_{31}\,, \\ n_{32} &= a_{32} - n_{31}v_{12}\,, \\ u_{33} &= a_{33} - n_{31}v_{13} - n_{32}v_{23}\,, \\ v_{3s} &= (a_{3s} - n_{31}v_{1s} - n_{32}v_{2s})/u_{33}\,,\quad s>3\,, \end{aligned}\right\} \tag{D.7}$$

and finally for the fourth row,

$$\left.\begin{aligned} n_{41} &= a_{41}\,, \\ n_{42} &= a_{42} - n_{41}v_{12}\,, \\ n_{43} &= a_{43} - n_{41}v_{13} - n_{32}v_{23}\,, \\ u_{44} &= a_{44} - n_{41}v_{14} - n_{42}v_{24} - n_{43}v_{34}\,, \end{aligned}\right\} \tag{D.8}$$

so that the decomposition of $|A|$ by Crout's method is

$$|A| = \begin{vmatrix} u_{11} & 0 & 0 & 0 \\ n_{21} & u_{22} & 0 & 0 \\ n_{31} & n_{32} & u_{33} & 0 \\ n_{41} & n_{42} & n_{43} & u_{44} \end{vmatrix} \begin{vmatrix} 1 & v_{12} & v_{13} & v_{14} \\ 0 & 1 & v_{23} & v_{24} \\ 0 & 0 & 1 & v_{34} \\ 0 & 0 & 0 & 1 \end{vmatrix} . \tag{D.9}$$

We note that the first determinant on the right hand side of (D.9), is a lower triangular determinant and the second, $|U|$, is a unit upper triangular determinant. The method is similar to that of Gauss but more complicated operations are needed to find the determinant $|U|$.

Again a breakdown of the method can occur when elements in $|A|$ are zero, and yet $|A|$ is not zero. Clearly, a method similar to partial pivoting for the Gauss elimination mustbeincorporated into Crout's method. The pivots are chosen so that the off-diagonal elements in $|L|$ will not exceed unity and the diagonal elements of $|U|$ are as large as possible.

at each stage. In the first stage the possible values of U_{11} are the elements in the first column of $|A|$ and the largest element is chosen, interchanging rows if necessary. The first column of $|L|$ is now fixed and the values for the elements in the first row of $|U|$ are found but not fixed. The term U_{22} is now chosen to be the largest and again it may be necessary to interchange rows. Thus the process continues and again the value of $|A|$ is $(-1)^p$ times the product of diagonal terms of $|L|$ where p is the number of interchanges of rows.

For the methods described here we have taken real values for the elements of $|A|$. When the elements are complex the same procedure is followed and no further discussion is needed for this case.

MULLER'S METHOD

We now have $|A_n(c)|$ in the form,

$$|A_n(c)| = f(c) + ig(c) \tag{D.10}$$

where will in general be complex, $c = c_R + ic_I$. Thus $|A_n(c)|$ has roots when

$$\left.\begin{aligned} f(c_R + ic_I) &= 0, \\ g(c_R + ic_I) &= 0, \end{aligned}\right\}$$

or when the modulus of $(f(c) + ig(c))$ is zero, i.e.

$$f^2(c_R + ic_I) + g^2(c_R + ic_I) = 0. \tag{D.11}$$

A method by Muller (37,38) is used which finds any prescribed number of zeros, real or complex of an arbitary function. The method is iterative and converges almost quadratically in the vicinity of a root. Another advantage is that it does not require the evaluation of the derivative of the function. To explain briefly the method we will take a general example of finding the roots of $f(x) = 0$. Let x_i, x_{i-1}, x_{i-2} be three distinct approximations to a root and let $f_i = f(x_i)$, $f_{i-1} = f(x_{i-1})$, $f_{i-2} = f(x_{i-2})$ be the corresponding values of $f(x)$. Three points determine a unique quadratic in one variable. We wish to determine the polynomial which passes through (x_i, f_i), (x_{i-1}, f_{i-1}) and (x_{i-2}, f_{i-2}). Assuming the polynomial can be expressed in the form

$$p(x) = a_2x^2 + a_1x + a_0 \tag{D.12}$$

we have that

$$\left.\begin{aligned} a_2x_i^2 + a_1x_i + a_0 &= f_i \\ a_2x_{i-1}^2 + a_1x_{i-1} + a_0 &= f_{i-1} \\ a_2x_{i-2}^2 + a_1x_{i-2} + a_0 &= f_{i-2} \end{aligned}\right\} \tag{D.13}$$

and a_2, a_1 and a_0 can be determined. The roots of this polynomial are determined from the quadratic formula,

$$x = \frac{2a_0}{-a_1 \pm (a_1^2 - 4a_0a_2)^{1/2}} \; . \tag{D.14}$$

The sign before the radical is chosen so that the denominator is largest in magnitude and the corresponding root is taken as the next approximation x_{i+1}. The process is repeated using x_{i+1}, x_i and x_{i-1} as the new approximations to the root.

For accuracy and convenience, the sequence of steps proposed by Muller, alter slightly from above and are described below

1) Choose x_i, x_{i-1}, x_{i-2} as approximations to a zero of $f(x)$.
   Compute f_i, f_{i-1}, f_{i-2}.

2) Compute

$$h = x_i - x_{i-1} \; ,$$

$$\lambda_i = \frac{h}{x_{i-1} - x_{i-2}} \; ,$$

$$\delta = 1 + \lambda_i \; .$$

3) Compute

$$g_i = f_{i-2}\lambda_i^2 - f_{i-1}\delta_i^2 + f_i(\lambda_i+\delta_i).$$

4) Compute

$$\lambda_{i+1} = \frac{-2f_i\,i}{-g_i \pm (g_i^2 - 4f_i\delta_i\lambda_i(f_{i-2}\lambda_i - f_{i-1}\delta_i + f_i))^{1/2}}$$

choosing the sign so the denominator is largest.

5) $x_{i+1} = x_i + h\lambda_{i+1}$ is the next approximation.

6) Compute $f(x_{i+1}) = f_{i+1}$.

7) Repeat 1) to 6) until

$|f_{i+1}(x_{i+1})| < \eta$ for a prescribed η.

The above iterative alogorithm was performed using the Leicester University Cyber 72 computer. Each root was obtained to an accuracy of four significant figures.

APPENDIX E

CONTRIBUTIONS OF $P_3(\eta)$ AND $P_4(\eta)$ TO THE BOUNDARY CONDITIONS

To solve the equation of consistency in Chapter 3.5, (3.152) the individual contributions to the boundary conditions in (3.136) from $P_1(\eta)$, $P_2(\eta)$, $P_3(\eta)$ and $P_4(\eta)$ have to be evaluated. For reference purposes the four conditions are:

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_0 , \tag{1}$$

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_h , \tag{2}$$

$$\frac{d^2P}{d\eta^2} - M\frac{dP}{d\eta} = 0 \quad \text{at } \eta = \eta_0 , \tag{3}$$

$$\frac{d^2P}{d\eta^2} = 0 \quad \text{at } \eta = \eta_h . \tag{4}$$

The contributions made by $P_1(\eta)$ and $P_2(\eta)$ are easily determined and we present below the details of the evaluation of the contributions made by $P_3(\eta)$ and $P_4(\eta)$ only, using the method of stationary points(26)throughout. The function $P_3(\eta)$ and $P_4(\eta)$ are defined in (3.148) to be

$$P_s(\eta) = \int_{A_sB_s} (t - Re^{i\pi/6})^{-1-\frac{iR^3}{2}} (t + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right] dt , \quad s = 3, 4, \tag{5}$$

and the contours A_sB_s are defined in Fig. 1 of Chapter 3.5.

(1) $P_4(\eta)$ at η_h , (η large, $\arg\eta = 0$)

We will aim to show that $P_4(\eta_h)$ can be approximated to zero. From (5) we have

$$P_4(\eta_h) = \int_{A_4B_4} (t - Re^{i\pi/6})^{-1-\frac{iR^3}{2}} (t + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp f(t, \eta_h)\, dt \tag{E.1}$$

where

$$f(t,\eta) = -\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}. \tag{E.2}$$

We can use the method of stationary points to find $P_4(\eta_h)$. The stationary points of $f(t,\eta_h)$ are given by $f'(t) = 0$, i.e.

$$t^2 = -\eta_h e^{-i\pi/6} - R^2 e^{i\pi/3} \tag{E.3}$$

and since $|\eta_h|$ is sufficiently large compared to R to dominate the right hand side of (E.3), there will be two stationary points $\pm t_0$ where t_0 may be approximated to

$$t_0 = \eta_h^{1/2} e^{\frac{5i\pi}{12}} . \tag{E.4}$$

The path of integration A_4B_4, in the complex plane must be chosen to contain the stationary points. From the definition of A_4B_4 in Fig.1 of Chapter 3.5 it is clear that $-t_0$ cannot lie on the curve. Thus we choose A_4B_4 to pass through t_0 and disregard $-t_0$. In the neighbourhood of t_0 we write

$$t = t_0 + \tau$$

and so we have, retaining terms up to τ^2,

$$f(t) = f(t_0) + \tfrac{1}{2}\tau^2 f''(t_0)$$

where we can take the approximate values of $f(t_0)$ and $f''(t_0)$ to be

$$f(t_0) = -\frac{2}{3}\eta_h^{3/2} e^{i\pi/4} , \quad f''(t_0) = -2\eta_h^{1/2} e^{5i\pi/12} .$$

We shift the origin to t_0 and obtain an approximate solution for $P_4(\eta_h)$ as

$$P_4(\eta_h) \sim \int_{-\omega}^{\omega} (t_0 - Re^{i\pi/6})^{-1-\frac{iR^3}{2}} (t_0 + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp\left[f(t_0) + \tfrac{1}{2}\tau^2 f''(t_0)\right] d\tau$$

where ω is small and generally complex. The greatest part of the integral comes from small values of τ ($\mathrm{Re}[f''(t_0)] < 0$) and so we can extend the limits of integration to $\pm\infty$ resulting in

$$P_4(\eta_h) \sim t_0^{-2} \exp f(t_0) \int_{-\infty}^{\infty} \exp\left[-\tau^2 \eta_h^{1/2} e^{\frac{5i\pi}{12}}\right] d\tau . \tag{E.5}$$

We write

$$\tau = \eta_h^{-1/4} e^{-\frac{5i\pi}{24}} \varepsilon$$

noting that the direction $\arg\tau = -\frac{5i\pi}{24}$ conforms with the general direction of A_4B_4, so from (E.5) we obtain

$$P_4(\eta_h) \sim t_0^{-2} \exp f(t_0) \left[\eta_h^{1/4} e^{\frac{5i\pi}{24}}\right]^{-1} \int_{-\infty}^{\infty} e^{-\varepsilon^2} d\varepsilon$$

and hence

$$P_4(\eta_h) \sim \pi^{1/2} \eta_h^{-5/4} e^{-\frac{25i\pi}{24}} \exp\left[-\tfrac{2}{3}\eta_h^{3/2} e^{i\pi/4}\right] .$$

Thus $P_4(\eta_h) \sim 0$. (E.6)

(2) $P_3(\eta)$ at η_0, (η large, $\arg\eta = \pi$)

Exactly the same argument can be used to show that $P_3(\eta_0) \sim 0$. From (5) we have

$$P_3(\eta_0) = \int_{A_3B_3} (t - Re^{i\pi/6})^{-1-\frac{iR^3}{2}} (t + Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp f(t,\eta_0)\, dt \tag{E.7}$$

with $f(t,\eta)$ defined in (E.2). The two stationary points of $f(t,\eta_0)$ are $\pm t_1$, where

$$t_1 = |\eta_0|^{1/2} e^{\frac{11i\pi}{12}}.$$

Again, we disregard $-t_1$, as it does not lie on the path of integration, A_3B_3. In the neighbourhood of t_1 we write

$$t = t_1 + \tau$$

and so for η_0 we have

$$f(t,\eta_0) = f(t_1) + \tfrac{1}{2}\tau^2 f''(t_1)$$

where

$$f(t_1) = \frac{2}{3}|\eta_0|^{3/2} e^{3i\pi/4}, \quad f''(t_1) = -2|\eta_0|^{1/2} e^{\frac{11i\pi}{12}}.$$

We shift the origin to t_1 and obtain the approximate solution for $P_3(\eta_0)$,

$$P_3(\eta_0) \sim t_1^{-2} \exp f(t_1) \int_{-\omega}^{\omega} \exp\left[-\tau^2 |\eta_0|^{1/2} e^{\frac{11i\pi}{12}}\right] d\tau . \tag{E.8}$$

As before we can extend the limits of integration to $\pm\infty$ and writing

$$\tau = \varepsilon e^{13i\pi/24} |\eta_0|^{-1/4}$$

we note that the direction $\arg\tau$ confirms with the general direction of A_3B_3. We finally have

$$P_3(\eta_0) \sim t_1^{-2} \exp f(t_1) \left[|\eta_0|^{1/4} e^{\frac{11i\pi}{24}}\right]^{-1} \int_{-\infty}^{\infty} e^{-\varepsilon^2}\, d\varepsilon$$

and hence

$$P_3(\eta_0) \sim \pi^{1/2} |\eta_0|^{-5/4} e^{\frac{-31i\pi}{24}} \exp\left[\tfrac{2}{3}|\eta_0|^{3/2} e^{\frac{3i\pi}{4}}\right].$$

Thus $P_3(\eta_0) \sim 0$. (E.9)

(3) Boundary Conditions (2) and (4) for $P_4(\eta)$

The contribution to (2) from $P_4(\eta)$ will be

$$I_2^4 = \frac{d^3P_4}{d\eta^3} - i\left[\eta\frac{dP_4}{d\eta} - P_4\right]\Bigg|_{\eta=\eta_h} \equiv (L+i)\,P_4\Big|_{\eta=\eta_h} \tag{E.10}$$

and using (E.1)

$$I_2^4 = (L+i)\int_{A_4B_4}(t-Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt\,\Bigg|_{\eta=\eta_h}.$$

We can take the operator (L+i) inside the integral sign and therefore obtain

$$I_2^4 = \int_{A_4B_4}(-t^3e^{-\frac{i\pi}{2}} + i\eta_h te^{-i\pi/6} + i)(t-Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta_h e^{-i\pi/6}\right]dt. \tag{E.11}$$

The integral I_2^4 therefore is essentially the same type of integral as (E.1) and so with no further discussion we can infer that

$$I_2^4 = 0. \tag{E.12}$$

The contribution to (4) from $P_4(\eta)$ will be

$$I_4^4 = \frac{d^2P_4}{d\eta^2}\Bigg|_{\eta=\eta_h}. \tag{E.13}$$

Again we can take the operator $\frac{d^2}{d\eta^2}$ inside the integral sign of (E.1) to obtain

$$I_4^4 = \int_{A_4B_4} t^2e^{-i\pi/3}(t-Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta_h e^{-i\pi/6}\right]dt$$

and we find again that

$$I_4^4 = 0. \tag{E.14}$$

(4) Boundary Conditions (1) and (3) for $P_3(\eta)$

The contribution to (1) from $P_3(\eta)$ will be

$$I_3^1 = \frac{d^3P_3}{d\eta^3} - i\left[\frac{dP_3}{d\eta} - P_3\right]\Bigg|_{\eta=\eta_0} = (L+i)P_3\Big|_{\eta=\eta_0}. \tag{E.15}$$

We substitute for $P_3(\eta)$ from (5) and take the operator (L+i) inside the integral to obtain

$$I_3^1 = \int_{A_3B_3}(-t^3 + i\eta_0 te^{-i\pi/6} + i)(t-Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta_0 e^{-i\pi/6}\right]dt$$

and using the same method as in the previous section we find

$$I_3^1 = 0. \tag{E.16}$$

Similarly the contribution to (3) from $P_3(\eta)$,

$$I_3^3 = \frac{d^2P_3}{d\eta^2} - M\frac{dP}{d\eta}\Big|_{\eta=\eta_0} \tag{E.17}$$

results in the integral

$$I_3^3 = \int_{A_3B_3} (t^2 e^{-i\pi/3} + Mt e^{-i\pi/6})(t - Re^{i\pi/6})^{-1-\frac{iR^3}{2}}(t+Re^{i\pi/6})^{-1+\frac{iR^3}{2}} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_0 e^{-i\pi/6}\right] dt$$

and thus

$$I_3^3 = 0. \tag{E.18}$$

(5) Boundary Condition (1) for $P_4(\eta)$

For the remaining boundary conditions we will approximate $P_4(\eta)$ to

$$P_4(\eta) = \int_{A_4B_4} (t - Re^{i\pi/6})^{-1}(t + Re^{i\pi/6})^{-1} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta e^{-i\pi/6}\right] dt \tag{E.19}$$

where we have ignored the terms $\frac{iR^3}{2}$ in the powers of $(t - Re^{i\pi/6})$ and $(t + Re^{i\pi/6})$ as R is a small quantity, of order $\lambda^{-1/3}$.

The boundary condition, (1) is

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_0$$

We can take the differential equation (3.135) and write it as

$$\frac{d}{d\eta}\left[\frac{d^3P}{d\eta^3} - i\eta\frac{dP}{d\eta} + iP\right] + i\eta R^2 P = 0.$$

Therefore an alternative form for the boundary condition (1) is

$$-iR^2\int_{\infty}^{\eta_0} \eta P\, d\eta = 0 \tag{E.20}$$

The contribution of P_4 to the boundary contribution can be written as

$$I_1^4 = -iR^2\int_{\infty}^{\eta_0} \eta \int_{A_4B_4} (t - Re^{i\pi/6})^{-1}(t + Re^{i\pi/6})^{-1} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta e^{-i\pi/6}\right] dt\, d\eta$$

We change the order of integration and using the result that

$$\int_{\infty}^{\eta_0} \eta \exp\left[-t\eta e^{-i\pi/6}\right] d\eta = -\left[\frac{\eta_0 e^{i\pi/6}}{t} + \frac{e^{i\pi/3}}{t^2}\right] \exp\left[-t\eta_0 e^{-i\pi/6}\right]$$

we find I_1^4 takes the form

$$I_1^4 = iR^2 \int_{A_4B_4} \left[\frac{\eta_0 e^{i\pi/6}}{t} + \frac{e^{i\pi/3}}{t^2}\right](t - Re^{i\pi/6})^{-1}(t + Re^{i\pi/6})^{-1} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_0 e^{-i\pi/6}\right] dt. \tag{E.21}$$

We will choose the path A_4B_4 in the following way:

$$A_4B_4 = A_4L + LM + MB_4$$

where

$$A_4L : \; t = Re^{i\pi/6} + re^{i2\pi/3} \quad , \quad \infty \leq r \leq \varepsilon , \tag{E.22}$$

$$LM : \; t = Re^{i\pi/6} - \varepsilon e^{i\theta} \quad , \quad 2\pi/3 \leq \theta \leq 0 , \tag{E.23}$$

$$MB_4 : \; t = Re^{i\pi/6} + r \quad , \quad \varepsilon \leq r \leq \infty . \tag{E.24}$$

We now look at the integral I_1^4 along the three parts of A_4B_4. The integral along A_4L is of the form

$$I_1 = -iR^2 \int_{\varepsilon}^{\infty} \left[\frac{\eta_0}{R+ir} + \frac{1}{(R+ir)^2} \right] \frac{1}{r} \left(2Re^{i\pi/6} + re^{2\pi i/3} \right)^{-1} \exp \phi_1(r) \, dr \tag{E.25}$$

where we approximate the exponential term to the largest coefficient of each power of r (neglecting R compared to η_0), to $\phi_1(r)$, namely,

$$\phi_1(r) = -\tfrac{1}{3}r^3 + iRr^2 - i\eta_0 r - R\eta_0 . \tag{E.26}$$

The integral along LM becomes

$$I_2 = iR^2 \int_0^{2\pi/3} \left[\frac{\eta_0}{R} + \frac{1}{R^2} \right] \frac{e^{-\eta_0 R}}{2Re^{i\pi/6}} i\,d\theta = -\frac{\pi R}{3} \left(\frac{\eta_0}{R} + \frac{1}{R^2} \right) e^{-\eta_0 R - i\pi/6} \tag{E.27}$$

and the integral along MB_4 reduces to

$$I_3 = iR^2 \int_{\varepsilon}^{\infty} \left[\frac{\eta_0 e^{i\pi/6}}{Re^{i\pi/6}+r} + \frac{e^{i\pi/3}}{(Re^{i\pi/6}+r)^2} \right] \frac{1}{r} \left(2Re^{i\pi/6} + r \right)^{-1} \exp \phi_2(r) \, dr \tag{E.28}$$

where we approximate the exponential term to $\phi_2(r)$ defined as

$$\phi_2(r) = -\tfrac{1}{3}r^3 - Re^{i\pi/6}r^2 - r\eta_0 e^{-i\pi/6} - \eta_0 R . \tag{E.29}$$

Therefore

$$I_1^4 = I_1 + I_2 + I_3$$

and we can now take the limit, $\varepsilon \to 0$ in I_1 and I_3. We will use the method of stationary points to evaluate I_1 and I_3.

For I_1 the function $\phi_1(r)$ is, from (E.26)

$$\phi_1(r) = -\tfrac{1}{3}r^3 + iRr^2 - i\eta_0 r - R\eta_0 .$$

The last term is a constant which can be taken outside the integral. The term iRr^2 is small as R is of the order $\lambda^{-1/3}$, and is therefore not as important as $(-\tfrac{1}{3}r^3 - i\eta_0 r)$. From the definition of η and Z_c in (3.134) and

(3.138) we have

$$-i\eta_0 = \lambda^{1/3}(b+ia).$$

The dominant terms of $\phi_1(r)$ are therefore

$$-\tfrac{1}{3}r^3 + \lambda^{1/3}br.$$

We will write

$$I_1 = -iR^2e^{-\eta_0 R}\int_0^\infty f_1(r)\exp\left[-\tfrac{1}{3}r^3+\lambda^{1/3}br\right]dr \tag{E.30}$$

where

$$f_1(r) = \frac{1}{r}\left[\frac{\eta_0}{R+ir}+\frac{1}{(R+ir)^2}\right](2Re^{i\pi/6}+re^{2\pi i/3})^{-1}\exp\left[iRr^2+ia\lambda^{1/3}r\right]. \tag{E.31}$$

The stationary points of $(-\tfrac{1}{3}r^3+\lambda^{1/3}br)$ are $\pm A$ where

$$A = \lambda^{1/6}b^{1/2}. \tag{E.32}$$

We disregard the negative value as -A is not along the line of integration.

In the neighbourhood of A we write

$$r = A + \tau$$

and so retaining terms up to τ^2,

$$I_1 = -iR^2e^{-\eta_0 R}\int_{-\omega}^{\omega} f_1(A)\exp\left[\tfrac{2}{3}A^3 - A\tau^2\right]d\tau$$

where ω is small. As before we can extend the limits of integration to $\pm\infty$

and so we finally have

$$I_1 = -iR^2e^{-\eta_0 R}f_1(A)\frac{\pi^{1/2}}{A^{1/2}}\exp\left[\tfrac{2}{3}A^3\right]. \tag{E.33}$$

We now use the same method to evaluate I_3. From (3.134) and (3.138)

we have

$$-\eta_0 e^{-i\pi/6} = \frac{\lambda^{1/3}}{2}\left[(a-b)+i(-a-b\sqrt{3})\right]$$

The dominant terms of $\phi_2(r)$ are

$$-\tfrac{1}{3}r^3 + \frac{\lambda^{1/3}}{2}(a-b)r.$$

We therefore write I_3 in the form

$$I_3 = iR^2e^{-\eta_0 R}\int_0^\infty f_2(r)\exp\left[-\tfrac{1}{3}r^3+\frac{\lambda^{1/3}}{2}(a-b)r\right]dr \tag{E.34}$$

where

$$f_2(r) = \frac{1}{r}\left[\frac{\eta_0}{Re^{i\pi/6}+r}+\frac{1}{(Re^{i\pi/6}+r)^2}\right](2Re^{i\pi/6}+r)^{-1}\exp\left[-Re^{i\pi/6}r^2-\frac{\lambda^{1/3}}{2}i(a+b\sqrt{3})r\right]. \tag{E.35}$$

The stationary points of (E.34) are $\pm$B where

$$B = \left[\lambda^{1/3}(a-b)/2\right]^{1/2} \tag{E.36}$$

and again we disregard the negative value of B as it is not along the line of integration. In the neighbourhood of B we write

$$r = B + \tau$$

and retaining terms up to τ^2 we have for I_3,

$$I_3 = iR^2 e^{-\eta_0 R} \int_{-\omega}^{\omega} f_2(B) \exp[\tfrac{2}{3}B^3 - B\tau^2] \, d\tau$$

where ω is small. We can now extend the limits of integration to $\pm\infty$ to obtain

$$I_3 = iR^2 e^{-\eta_0 R} f_2(B) \frac{\pi^{1/2}}{B^{1/2}} \exp[\tfrac{2}{3}B^3]. \tag{E.37}$$

The complete contribution therefore of this boundary condition from $P_4(\eta)$ is therefore

$$I_1^4 = iR^2 e^{-\eta_0 R} \left[-\frac{\pi^{1/2}}{A^{1/2}} f_1(A) e^{\frac{2}{3}A^3} + \frac{\pi i e^{-i\pi/6}}{3R^2} \left[\eta_0 + \frac{1}{R}\right] + \frac{\pi^{1/2}}{B^{1/2}} f_2(B) e^{\frac{2}{3}B^3} \right]. \tag{E.38}$$

(6) Boundary Condition (3) for $P_4(\eta)$

The contribution from the boundary condition (3),

$$\frac{d^2P}{d\eta^2} - M\frac{dP}{d\eta} = 0 \quad \text{at } \eta = \eta_0,$$

due to P_4 would produce the following integral,

$$I_3^4 = \int_{A_4B_4} (t^2 e^{-i\pi/3} + Mte^{-i\pi/6})(t - Re^{i\pi/6})^{-1}(t + Re^{i\pi/6})^{-1} \exp[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_0 e^{-i\pi/6}] \, dt. \tag{E.39}$$

This is exactly the same form of integral as in section (5) and the path for A_4B_4 will be chosen in the same way. The two stationary points will be A and B, defined in (E.32) and (E.36) as before. The difference is in the functions $f_1(r)$ and $f_2(r)$. For I_1, along A_4L we have

$$I_1 = -\int_{\varepsilon}^{\infty} (R + re^{i\pi/2})(R + re^{i\pi/2} + M)\frac{1}{r}(2Re^{i\pi/6} + re^{\frac{2\pi i}{3}})^{-1} \exp \phi_1(r) \, dr.$$

Along LM we obtain

$$I_2 = \frac{\pi i}{3}(R + M)e^{-\eta_0 R - \frac{i\pi}{6}}$$

and along MB,

$$I_3 = \int_{\varepsilon}^{\infty} (R + re^{-i\pi/6})(R + re^{-i\pi/6} + M)\frac{1}{r}(2Re^{i\pi/6} + r)^{-1} \exp \phi_2(r) \, dr.$$

Thus we find that the contribution to the boundary condition from P_4 is

$$I_3^4 = e^{-\eta_0 R}\left[-\frac{\pi^{1/2}}{A^{1/2}}\, g_1(A)e^{2/3 A^3} + \frac{\pi i}{3}(R+M)e^{-i\pi/6} + \frac{\pi^{1/2}}{B^{1/2}}\, g_2(B)e^{2/3 B^3}\right] \tag{E.40}$$

where

$$g_1(A) = (R+iA)(R+iA+M)\frac{1}{A}(2Re^{i\pi/6}+Ae^{2\pi i/3})^{-1}\exp\left[iRA^2 + i\lambda^{1/3}aA\right], \tag{E.41}$$

$$g_2(B) = (R+Be^{-i\pi/6})(R+Be^{-i\pi/6}+M)\frac{1}{B}(2Re^{i\pi/6}+B)^{-1}\exp\left[-RB^2e^{i\pi/6} - iB(a+b\sqrt{3})\right]. \tag{E.42}$$

The stationary point B is much larger than A and the term involving B in I_3^4 and I_1^4 is dominant and so we will ignore the other two terms. The functions $f_2(B)$ and $g_2(B)$ contain many common factors and when we divide the fourth column of the determinant in (3.152) by these common terms we have remaining

$$I_1^4 = iR^2\left[\frac{\eta_0}{(R+Be^{-i\pi/6})} + \frac{1}{(R+Be^{i\pi/6})^2}\right], \tag{E.43}$$

$$I_3^4 = (R+Be^{-i\pi/6})(R+Be^{-i\pi/6}+M). \tag{E.44}$$

(7) <u>The Boundary Condition (2) for $P_3(\eta)$</u>

As for $P_4(\eta)$, we now approximate $P_3(\eta)$ to

$$P_3(\eta) = \int_{A_3B_3}(t-Re^{i\pi/6})^{-1}(t+Re^{i\pi/6})^{-1}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt \tag{E.45}$$

ignoring $\frac{iR^3}{2}$ in the powers of $(t-Re^{i\pi/6})$ and $(t+Re^{i\pi/6})$. We can take the boundary condition (2) in an integral form, namely

$$-iR^2\int_{-\infty}^{\eta_h}\eta P\,d\eta = 0$$

and so the contribution of P_3 to the boundary condition (2) is

$$I_2^3 = -iR^2\int_{-\infty}^{\eta_h}\eta\int_{A_3B_3}(t-Re^{i\pi/6})^{-1}(t+Re^{i\pi/6})^{-1}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt\,d\eta.$$

As before we can change the order of integration and the integral takes the form

$$I_2^3 = -iR^2 \int_{A_3B_3} \left[\frac{\eta_h e^{i\pi/6}}{t} + \frac{e^{i\pi/3}}{t^2} \right] (t - Re^{i\pi/6})^{-1} (t + Re^{i\pi/6})^{-1} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_h e^{-i\pi/6} \right] dt. \tag{E.46}$$

The path A_3B_3 is chosen as follows:

$$A_3B_3 = A_3L + LM + MB_3$$

where

$$A_3L: \quad t = -Re^{i\pi/6} + re^{\frac{4\pi i}{3}}, \quad \infty \leq r \leq \varepsilon, \tag{E.47}$$

$$LM: \quad t = -Re^{i\pi/6} - \varepsilon e^{i\theta}, \quad 4\pi/3 \leq \theta \leq 2\pi/3, \tag{E.48}$$

$$MB_3: \quad t = -Re^{i\pi/6} + re^{2\pi i/3}, \quad \varepsilon \leq r \leq \infty. \tag{E.49}$$

Along A_3L the integral becomes

$$I_1 = -iR^2 \int_{\varepsilon}^{\infty} \left[\frac{\eta_h}{-R + re^{\frac{7\pi i}{6}}} + \frac{1}{(-R + re^{\frac{7\pi i}{6}})^2} \right] \frac{1}{r} (-2Re^{i\pi/6} + re^{4\pi i/3})^{-1} \exp \phi_3(r) \tag{E.50}$$

where $\phi_3(r)$ is approximated to

$$\phi_3(r) = -\tfrac{1}{3}r^3 + Re^{5i\pi/6} r^2 + \eta_h e^{i\pi/6} r + \eta_h R. \tag{E.51}$$

The integral along LM takes the form

$$I_2 = -iR^2 \int_{2\pi/3}^{4\pi/3} \left[-\frac{\eta_h}{R} + \frac{1}{R^2} \right] \frac{e^{-\eta_h R}}{2Re^{i\pi/6}} \, i d\theta = \frac{\pi}{3} e^{\eta_h R - i\pi/6} \left[-\eta_h + \frac{1}{R} \right] \tag{E.52}$$

and finally the integral along MB_3 will be

$$I_3 = iR^2 \int_{\varepsilon}^{\infty} \left[\frac{\eta_h}{-R + ir} + \frac{1}{(-R + ir)^2} \right] \frac{1}{r} (-2Re^{i\pi/6} + re^{2\pi i/3})^{-1} \exp \phi_4(r) \, dr \tag{E.53}$$

where $\phi_4(r)$ is approximated to

$$\phi_4(r) = -\tfrac{1}{3}r^3 - iRr^2 - i\eta_h r + \eta_h R. \tag{E.54}$$

Therefore we have that

$$I_2^3 = I_1 + I_2 + I_3$$

and since I_2 has been evaluated we can take the limit, $\varepsilon \to 0$ in I_1 and I_3 and use the method of stationary points to evaluate the integrals.

In I_1, the term $\eta_h e^{i\pi/6}$ can be written as,

$$\eta_h e^{i\pi/6} = \frac{\lambda^{1/3}}{2}\left[(\sqrt{3}(1-a)-b) + i(1-a+b\sqrt{3})\right]$$
$$= d_1 + id_2 . \qquad (E.55)$$

Thus, the important terms in $\phi_3(r)$ are

$$-\tfrac{1}{3}r^3 + d_1 r$$

and so we write I_1 in the form

$$I_1 = -iR^2 e^{\eta_h R}\int_0^\infty f_3(r)\exp\left[-\tfrac{1}{3}r^3 + d_1 r\right] dr \qquad (E.56)$$

where

$$f_3(r) = \left[\frac{\eta_h}{-R+re^{7\pi i/6}} + \frac{1}{(-R+re^{7\pi i/6})^2}\right]\frac{1}{r}(-2Re^{i\pi/6}+re^{4\pi i/3})^{-1}\exp\left[Re^{5\pi i/6}r^2 + id_2 r\right]. \qquad (E.57)$$

The stationary points of $(-\tfrac{1}{3}r^3 + d_1 r)$ are $\pm\alpha$ where

$$\alpha = \sqrt{d_1} = \left[\lambda^{1/3}(\sqrt{3}(1-a)-b)/2\right]^{1/2} \qquad (E.58)$$

and we neglect $-\alpha$ as it is not along the line of integration. In the neighbourhood of α we take

$$r = \alpha + \tau$$

and so retaining terms up to τ^2 we have

$$I_1 = -iR^2 e^{\eta_h R}\int_{-\omega}^{\omega} f_3(\alpha)\exp\left[\tfrac{2}{3}\alpha^3 - \alpha\tau^2\right] d\tau .$$

We extend the limits of integration to $\pm\infty$ and hence

$$I_1 = -iR^2 e^{\eta_h R} f_3(\alpha)\frac{\pi^{1/2}}{\alpha^{1/2}} e^{\frac{2}{3}\alpha^3}. \qquad (E.59)$$

In the integral I_3, the term $-i\eta_h$ in $\phi_4(r)$ is

$$-i\eta_h = \lambda^{1/3}\left[b - i(1-a)\right] \qquad (E.60)$$

and so the important terms in $\phi_4(r)$ are $(-\tfrac{1}{3}r^3 + \lambda^{1/3}br)$ and we write I_3 in the form

$$I_3 = iR^2 e^{\eta_h R}\int_0^\infty f_4(r)\exp\left[-\tfrac{1}{3}r^3 + \lambda^{1/3}br\right] dr \qquad (E.61)$$

where

$$f_4(r) = \left[\frac{\eta_h}{-R+ir} + \frac{1}{(-R+ir)^2}\right]\frac{1}{r}(-2Re^{i\pi/6}+re^{2\pi i/3})^{-1}\exp\left[-iRr^2 - i\lambda^{1/3}(1-a)r\right]. \qquad (E.62)$$

The stationary points of $(-\frac{1}{3}r^3+\lambda^{1/3}br)$ are $\pm\beta$ where

$$\beta = [\lambda^{1/3}b]^{1/2}. \tag{E.63}$$

As before $-\beta$ is neglected and in the neighbourhood of β we write

$$r=\beta+\tau$$

so that

$$I_3 = iR^2e^{\eta_h R}\int_{-\omega}^{\omega} f_4(\beta)\exp\left[\tfrac{2}{3}\beta^3-\beta\tau^2\right]d\tau .$$

We extend the limits of integration to $\pm\infty$ and hence

$$I_3 = iR^2e^{\eta_h R}f_4(\beta)\frac{\pi^{1/2}}{\beta^{1/2}}e^{2/3\beta^3}. \tag{E.64}$$

Thus the integral I_2^3 will be

$$I_2^3 = iR^2e^{\eta_h R}\left[-\frac{\pi^{1/2}}{\alpha^{1/2}}f_3(\alpha)e^{2/3\alpha^3}-\frac{i\pi e^{-i\pi/6}}{3R^2}\left[-\eta_h+\frac{1}{R}\right]+\frac{\pi^{1/2}}{\beta^{1/2}}f_4(\beta)e^{2/3\beta^3}\right]. \tag{E.65}$$

(8) <u>The Boundary Condition (4) for $P_3(\eta)$</u>

As in section (4) the contribution of $P_3(\eta)$ to the boundary condition (4) will take the form

$$I_4^3 = \int_{A_3B_3} t^2e^{-i\pi/3}(t-Re^{i\pi/6})^{-1}(t+Re^{i\pi/6})^{-1}\exp\left[-\tfrac{1}{3}t^3-tR^2e^{i\pi/3}-t\eta_h e^{-i\pi/6}\right]dt . \tag{E.66}$$

This is exactly the same type of integral as in the previous section. Using the same path for A_3B_3 the integral splits up into three parts as before. Along A_3L

$$I_1 = -\int_{\epsilon}^{\infty}(R+re^{i\pi/6})^2\frac{1}{r}\left[-2Re^{i\pi/6}-re^{4i\pi/3}\right]^{-1}\exp\phi_3(r)\,dr,$$

along LM we have

$$I_2 = \int_{2\pi/3}^{4\pi/3}-\frac{R^2e^{\eta_h R}}{2Re^{i\pi/6}}\,id\theta = \frac{i\pi R}{3}e^{\eta_h R-i\pi/6}$$

and finally along MB

$$I_3 = \int_{\epsilon}^{\infty}(-R+ir)^2\frac{1}{r}(-2Re^{i\pi/6}+re^{2\pi i/3})^{-1}\exp\phi_4(r)\,dr .$$

Following the method of the previous section we can write

$$I_4^3 = e^{\eta_h R}\left[-\frac{\pi^{1/2}}{\alpha^{1/2}}g_3(\alpha)e^{2/3\alpha^3}-\frac{i\pi R}{3}e^{-i\pi/6}+\frac{\pi^{1/2}}{\beta^{1/2}}g_4(\beta)e^{2/3\beta^3}\right] \tag{E.67}$$

where

$$g_3(\alpha) = (R+\alpha e^{i\pi/6})^2 \frac{1}{\alpha} \left[-2Re^{i\pi/6}+\alpha e^{4\pi i/3}\right]^{-1} \exp\left[R\alpha^2 e^{5\pi i/6}+id_2\alpha\right] , \tag{E.68}$$

$$g_4(\beta) = (-R+i\beta)^2 \frac{1}{\beta} \left[-2Re^{i\pi/6}+\beta e^{2\pi i/3}\right]^{-1} \exp\left[-iR\beta^2-i\lambda^{2/3}(1-a)\beta\right] . \tag{E.69}$$

The stationary point α is much larger than β and so we can approximate the terms I_2^3 and I_4^3 by retaining the term involving α . The functions $f_3(\alpha)$ and $g_3(\alpha)$ have terms in common as before and if we divide the third column of the determinant in (3.152) by these factors we have remaining

$$I_2^3 = iR^2\left[\frac{-\eta_h}{(R+\alpha e^{i\pi/6})} + \frac{1}{(R+\alpha e^{i\pi/6})^2}\right] , \tag{E.70}$$

$$I_4^3 = (R+\alpha e^{i\pi/6})^2 . \tag{E.71}$$

APPENDIX F

CONTRIBUTIONS OF $P_1(\eta)$, $P_2(\eta)$, $P_3(\eta)$ AND $P_4(\eta)$ TO THE BOUNDARY CONDITIONS OF THE MODEL RETAINING THE β TERM

To solve the equation of consistency in Chapter 3.6, (3.184) the individual contributions of $P_1(\eta)$, $P_2(\eta)$, $P_3(\eta)$ and $P_4(\eta)$ to the boundary conditions in (3.173) need to be evaluated. The conditions are presented below for reference.

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_0, \tag{1}$$

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \quad \text{at } \eta = \eta_h, \tag{2}$$

$$\frac{d^2P}{d\eta^2} - M\frac{dP}{d\eta} = 0 \qquad \text{at } \eta = \eta_0, \tag{3}$$

$$\frac{d^2P}{d\eta^2} = 0 \qquad \text{at } \eta = \eta_h. \tag{4}$$

The functions $P_1(\eta)$, $P_2(\eta)$, $P_3(\eta)$ and $P_4(\eta)$ are defined in (3.183) as follows

$$P_s(\eta) = \int_{A_sB_s} (t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3 - tR^2e^{i\pi/3} - t\eta e^{-i\pi/6}\right]dt \tag{5}$$

where the contours A_sB_s are defined in Figs. 1 and 2 of Chapter 3.6.

(1) $P_4(\eta)$ at η_h and $P_3(\eta)$ at η_0

The functions $P_4(\eta)$ and $P_3(\eta)$ are almost identical to the corresponding functions $P_4(\eta)$ and $P_3(\eta)$ in Appendix E, the only difference being that $\frac{iR^3}{2}$ is replaced by M_2. The behaviour of $P_4(\eta_h)$ and $P_3(\eta_0)$ is not altered by this change and so with no further discussion we take

$$P_4(\eta_h) = 0, \quad P_3(\eta_0) = 0, \tag{F.1}$$

and therefore

$$(L+i)P_4(\eta_h) = 0, \quad \frac{d^2P_4(\eta_h)}{d\eta^2} = 0 \tag{F.2}$$

and

$$(L+i)P_3(\eta_0)=0\ , \quad \left[\frac{d^2}{d\eta^2}-M\frac{d}{d\eta}\right]P_3(\eta_0')=0\ , \tag{F.3}$$

where the operator L is defined as

$$L \equiv \frac{d^3}{d\eta^3}-i\eta\frac{d}{d\eta}\ .$$

(2) Boundary Condition (1) for $P_4(\eta)$

The differential equation for $P_4(\eta)$ in (3.172) can be written as

$$\frac{d}{d\eta}\left[\frac{d^3P}{d\eta^3}-i\left(\eta\frac{dP}{d\eta}-P\right)\right]+iR^2\eta P-iM_1P=0\ . \tag{F.4}$$

Thus the boundary condition (1) can be written as

$$i\int_{\infty}^{\eta_0}(M_1-\eta R^2)\,P\,d\eta=0\ . \tag{F.5}$$

The contribution of $P_4(\eta)$ to (1) can take the form

$$I_4'=i\int_{\infty}^{\eta_0}(M_1-\eta R^2)P_4\,d\eta\ ;$$

hence, substituting for $P_4(\eta)$ from (5)

$$I_4'=i\int_{\infty}^{\eta_0}(M_1-\eta R^2)\int_{A_4B_4}(t-Re^{i\pi/6})^{-1-M_2}(t+Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3-tR^2e^{i\pi/3}-t\eta e^{-i\pi/6}\right]dt\,d\eta\ . \tag{F.6}$$

The order of integration may be changed, thus (F.6) becomes

$$I_4'=i\int_{A_4B_4}\left[(R^2\eta_0-M_1)\frac{e^{i\pi/6}}{t}+\frac{R^2e^{i\pi/3}}{t^2}\right](t-Re^{i\pi/6})^{-1-M_2}(t-Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3-tR^2e^{i\pi/3}-t\eta_0e^{-\pi/6}\right]dt. \tag{F.7}$$

We must choose a slightly different path for A_4B_4 than in the previous chapter as $t=Re^{i\pi/6}$ is a singularity which must be avoided; however the singularity at $t=-Re^{i\pi/6}$ is well behaved since $Re\{M_2\}>0$ and so we take

$$A_4B_4=A_4K+KN+NB_4$$

where

$$A_4K: \quad t = -Re^{i\pi/6} + re^{2\pi i/3}, \qquad \infty \leq r \leq \varepsilon, \tag{F.8}$$

$$KN: \quad t = -Re^{i\pi/6} - \varepsilon e^{i\theta}, \qquad 2\pi/3 \leq \theta \leq 0, \tag{F.9}$$

$$NB_4: \quad t = -Re^{i\pi/6} + r, \qquad \varepsilon \leq r \leq \infty. \tag{F.10}$$

Along A_4K, using (C.8) I_4' becomes

$$I_1 = -e^{\frac{2i\pi M_2}{3}} \int_{\varepsilon}^{\infty} \left[\frac{\eta_0 R^2 - M_1}{R+ir} + \frac{1}{(R+ir)^2} \right] (re^{2\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \phi_1(r)\, dr \tag{F.11}$$

where $\phi_1(r)$ is approximated to

$$\phi_1(r) = -\tfrac{1}{3}r^3 - iRr^2 - i\eta_0 r + \eta_0 R. \tag{F.12}$$

On KN, the integral I_4' reduces to

$$I_2 = i \int_0^{2\pi/3} \left[\frac{R^2\eta_0 - M_1}{-R} + 1 \right] (-2Re^{i\pi/6})^{-1-M_2} e^{\eta_0 R} \varepsilon^{M_2} i\, d\theta \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \tag{F.13}$$

and along NB_4 we have

$$I_3 = i \int_{\varepsilon}^{\infty} \left[\frac{\eta_0 R^2 - M_1}{re^{-i\pi/6} - R} + \frac{R^2}{(re^{-i\pi/6} - R)^2} \right] (r - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \phi_2(r)\, dr \tag{F.14}$$

where $\phi_2(r)$ is taken as

$$\phi_2(r) = -\tfrac{1}{3}r^3 + r^2 Re^{i\pi/6} - \eta_0 re^{-i\pi/6} + \eta_0 R. \tag{F.15}$$

Thus $I_4' = I_1 + I_2 + I_3$ and we can now let $\varepsilon \rightarrow 0$, and use the method of stationary points to evaluate I_1 and I_3. It is useful to note that the integral I_1 and I_3 are very similar to the integrals of Appendix E, (E.25) and (E.28). The value of the stationary points for I_1 and I_3 will be exactly the same, namely

$$\text{for } I_1: \; r = A = \lambda^{1/6} b^{1/2}, \tag{F.16}$$

$$\text{for } I_3: \; r = B = [\lambda^{1/3}(a-b)/2]^{1/2}. \tag{F.17}$$

Thus we can follow the same method as Appendix E which yields

$$I_4^1 = e^{R\eta_0}\left[-\frac{i\pi^{1/2}}{A^{1/2}} f_1'(A) e^{2/3 A^3} + \frac{i\pi^{1/2}}{B^{1/2}} f_2'(B) e^{2/3 B^3}\right], \tag{F.18}$$

where

$$f_1'(A) = \left[\frac{\eta_0 R^2 - M_1}{Ai + R} + \frac{R^2}{(Ai+R)^2}\right](Ae^{2\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} A^{-1+M_2} \exp\left[-iRA^2 + i\lambda^{1/3} aA + \frac{2\pi i M_2}{3}\right], \tag{F.19}$$

$$f_2'(B) = \left[\frac{\eta_0 R^2 - M_1}{Be^{-i\pi/6} - R} + \frac{R^2}{(Be^{-i\pi/6} - R)^2}\right](B - 2Re^{i\pi/6})^{-1-M_2} B^{-1+M_2} \exp\left[B^2 Re^{i\pi/6} - \frac{iB}{2}\lambda^{1/3}(a + b\sqrt{3})\right]. \tag{F.20}$$

(3) <u>The Boundary Condition (3) for $P_4(\eta)$</u>

The contribution of $P_4(\eta)$ to boundary condition (3) will take the form

$$I_4^3 = \int_{A_4 B_4} (t^2 e^{-i\pi/3} + Mte^{-i\pi/6})(t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_0 e^{-i\pi/6}\right] dt. \tag{F.21}$$

Using the same path as in the previous section I_4^3 splits up into the following integrals, along A_4K, KN and NB_4:

$$I_1 = e^{\frac{-2\pi i M_2}{3}} \int_\varepsilon^\infty (ir - R)(ir - R + M)(re^{2\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \phi_1(r)\, dr,$$

$$I_2 = \int_0^{2\pi/3} -R(-R+M)(-2Re^{i\pi/6})^{-1-M_2} e^{\eta_0 R} \varepsilon^{M_2} i d\theta \to 0 \text{ as } \varepsilon \to 0,$$

$$I_3 = \int_\varepsilon^\infty (re^{-i\pi/6} - R)(re^{-i\pi/6} - R + M)(r - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \phi_2(r)\, dr,$$

where the functions $\phi_1(r)$ and $\phi_2(r)$ are defined in (F.12) and (F.15). We can now let $\varepsilon \to 0$ and using the method of stationary points the resulting value of I_4^3 will be

$$I_4^3 = e^{\eta_0 R}\left[-\frac{\pi^{1/2}}{A^{1/2}} g_1'(A) e^{2/3 A^3} + \frac{\pi^{1/2}}{B^{1/2}} g_2'(B) e^{2/3 B^3}\right] \tag{F.22}$$

where

$$g_1'(A) = (iA - R)(iA - R + M)(Ae^{2\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} A^{-1+M_2} \exp\left[-iRA^2 + i\lambda^{1/3} aA\right], \tag{F.23}$$

$$g_2'(B) = (Be^{-i\pi/6} - R)(Be^{-i\pi/6} - R + M)(B - 2Re^{i\pi/6})^{-1-M_2} B^{-1+M_2} \exp\left[B^2 Re^{i\pi/6} - \frac{iB}{2}\lambda^{1/3}(a+b\sqrt{3})\right]. \tag{F.24}$$

As before in Appendix E we note that $B >> A$ and so I_4^1 and I_4^3 are approximated to the terms from the stationary point of B. When substituting into the fourth column of the determinant in (3.184) we may divide the column by the common factors of I_4^1 and I_4^3 and the remaining expressions will be

$$I_4^1 = i\left[\frac{\eta_0 R^2 - M_1}{Be^{-i\pi/6} - R} + \frac{R^2}{(Be^{-i\pi/6} - R)^2}\right], \tag{F.25}$$

$$I_4^3 = (Be^{-i\pi/6} - R)(Be^{-i\pi/6} - R + M). \tag{F.26}$$

(4) The Boundary Condition (2) for $P_3(\eta)$

Taking the differential equation for $P(\eta)$ in the form of (3.172) we can write the boundary equation (2),

$$\frac{d^3P}{d\eta^3} - i\left[\eta\frac{dP}{d\eta} - P\right] = 0 \qquad \text{at } \eta = \eta_h$$

in the form

$$i\int_{-\infty}^{\eta_h} (M_1 - \eta R^2) P \, d\eta = 0. \tag{F.27}$$

The contribution of $P_3(\eta)$ to the boundary condition takes the form

$$I_3^2 = i\int_{-\infty}^{\eta_h} (M_1 - \eta R^2) P_3(\eta)\, d\eta ;$$

substituting for $P_3(\eta)$ from (5) we have

$$I_3^2 = i\int_{-\infty}^{\eta_h} (M_1 - \eta R^2) \int_{A_3B_3} (t - Re^{i\pi/6})^{-1-M_2} (t + Re^{i\pi/6})^{-1+M_2} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta e^{-i\pi/6}\right] dt$$

and changing the order of integration I_3^2 becomes

$$I_3^2 = i\int_{A_3B_3} \left[\frac{(R^2\eta_h - M_1)e^{i\pi/6}}{t} + \frac{R^2 e^{i\pi/3}}{t^2}\right](t - Re^{i\pi/6})^{-1-M_2}(t + Re^{i\pi/6})^{-1+M_2} \exp\left[-\tfrac{1}{3}t^3 - tR^2 e^{i\pi/3} - t\eta_h e^{-i\pi/6}\right] dt. \tag{F.28}$$

We can choose the path A_3B_3 as in Appendix E,

$$A_3B_3 = A_3L + LM + MB_3$$

where

$$A_3L: \ t = -Re^{i\pi/6} + re^{4i\pi/3}, \qquad \infty \leq r \leq \varepsilon, \tag{F.29}$$

$$LM: \ t = -Re^{i\pi/6} - \varepsilon e^{i\theta}, \qquad 4\pi/3 \leq \theta \leq 2\pi/3, \tag{F.30}$$

$$MB_3: \ t = -Re^{i\pi/6} + re^{2\pi i/3}, \qquad \varepsilon \leq r \leq \infty. \tag{F.31}$$

Using the method of stationary points we can evaluate I_3^2 along A_3B_3. Along A_3L, I_3^2 becomes

$$I_1 = ie^{\frac{4i\pi M_2}{3}} \int_{\varepsilon}^{\infty} \left[\frac{R^2\eta_h - M_1}{(re^{i\pi/6}+R)} - \frac{R^2}{(re^{i\pi/6}+R)^2}\right] (re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+m_2} \exp\phi_3(r)\, dr \tag{F.32}$$

where

$$\phi_3(r) = -\tfrac{1}{3}r^3 + Rr^2e^{5i\pi/6} + \eta_h re^{i\pi/6} + \eta_h R. \tag{F.33}$$

For LM, the integral reduces to

$$I_2 = i\int_{2\pi/3}^{4\pi/3} (\eta_h R^2 - M_1 - R)(-R)^{-1}(-2Re^{i\pi/6})^{-1-M_2} e^{\eta_h R} \varepsilon^{M_2}\, i\, d\theta \to 0 \text{ as } \varepsilon \to 0. \tag{F.34}$$

Finally for MB_3

$$I_3 = ie^{\frac{2\pi i M_2}{3}} \int_{\varepsilon}^{\infty} \left[\frac{R^2\eta_h - M_1}{ir - R} + \frac{R^2}{(ir-R)^2}\right] (re^{2\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+m_2} \exp\phi_4(r)\, dr \tag{F.35}$$

where $\phi_4(r) = -\tfrac{1}{3}r^3 - iRr^2 - i\eta_h r + \eta_h R.$ (F.36)

Thus $I_3^2 = I_1 + I_2 + I_3$ and we can take the limit, $\varepsilon \to 0$.

The stationary points for I_1 and I_3 are as in (E.58) and (E.63),

$$\alpha = \left[\sqrt{3}\,\lambda^{1/3}(1-a)/2\right]^{1/2}, \tag{F.37}$$

$$\beta = \left[\lambda^{1/3} b\right]^{1/2}. \tag{F.38}$$

Therefore we have for I_3^2,

$$I_3^2 = ie^{\eta_h R}\left[\frac{\pi^{1/2}}{\alpha^{1/2}} f_3'(\alpha) e^{2/3\alpha^3} + \frac{\pi^{1/2}}{\beta^{1/2}} f_4'(\beta) e^{2/3\beta^3}\right] \tag{F.39}$$

where

$$f_3'(\alpha) = \left[\frac{R^2\eta_h - M_1}{(\alpha e^{i\pi/6}+R)} - \frac{R^2}{(\alpha e^{i\pi/6}+R)^2}\right](\alpha e^{\frac{4\pi i}{3}} - 2Re^{i\pi/6})^{-1-M_2} \alpha^{-1+M_2} \exp\left[Re^{\frac{5i\pi}{6}}\alpha^2 + \frac{\lambda^{1/3}}{2} i\alpha + \frac{4\pi i M_2}{3}\right], \tag{F.40}$$

$$f_4'(\beta)=\left[\frac{R^2\eta_h-M_1}{i\beta-R}+\frac{R^2}{(i\beta-R)^2}\right](\beta e^{\frac{2\pi i}{3}}-2Re^{i\pi/6})^{-1-M_2}\beta^{-1+M_2}\exp\left[-iR\beta^2-i\lambda^{1/3}\beta+\frac{2\pi iM_2}{3}\right].$$

(F.41)

(5) <u>The Boundary Condition (4) for $P_3(\eta)$</u>

The contribution of $P_3(\eta)$ to the boundary condition (4) will be

$$I_3^4=\int_{A_3B_3}t^2e^{-i\pi/3}(t-Re^{i\pi/6})^{-1-M_2}(t+Re^{i\pi/6})^{-1+M_2}\exp\left[-\tfrac{1}{3}t^3-tR^2e^{i\pi/3}-t\eta_he^{-i\pi/6}\right]dt.$$

(F.42)

Using the same path as above, the integral breaks into three parts, along A_3L, LM and MB_3:

$$I_1=-e^{\frac{4\pi iM_2}{3}}\int_\varepsilon^\infty(re^{i\pi/6}+R)^2(re^{4\pi i/3}-2Re^{i\pi/6})^{-1-M_2}r^{-1+M_2}\exp\phi_3(r)\,dr,$$

$$I_2=-\int_{2\pi/3}^{4\pi/3}R(-2Re^{i\pi/6})^{-1-M_2}e^{\eta_hR}\varepsilon^{M_2}i\,d\theta\to0\text{ as }\varepsilon\to0,$$

$$I_3=e^{\frac{2\pi iM_2}{3}}\int_\varepsilon^\infty(ir-R)^2(re^{2\pi i/3}-2Re^{i\pi/6})^{-1-M_2}r^{-1+M_2}\exp\phi_4(r)\,dr,$$

where $\phi_3(r)$ and $\phi_4(r)$ are defined in (F.33) and (F.36). Using methods as above we find

$$I_3^4=e^{\eta_hR}\left[\frac{\pi^{1/2}}{\alpha^{1/2}}g_3'(\alpha)e^{2/3\alpha^3}+\frac{\pi^{1/2}}{\beta^{1/2}}g_4'(\beta)e^{2/3\beta^3}\right]$$

(F.43)

where

$$g_3'(\alpha)=(\alpha e^{i\pi/6}+R)^2(\alpha e^{4\pi i/3}-2Re^{i\pi/6})^{-1-M_2}\alpha^{-1+M_2}\exp\left[R\alpha^2e^{5\pi i/6}+\frac{\lambda^{1/3}}{2}i\alpha+\frac{4\pi iM_2}{3}\right],$$

(F.44)

$$g_4'(\beta)=(i\beta-R)^2(\beta e^{\frac{2\pi i}{3}}-2Re^{i\pi/6})^{-1-M_2}\beta^{-1+M_2}\exp\left[-iR\beta^2-\lambda^{1/3}\beta i+\frac{2\pi iM_2}{3}\right].$$

(F.45)

As before as $\alpha\gg\beta$ we can approximate I_3^2 and I_3^4 to the terms involving α only. The common factors of $f_3'(\alpha)$ and $g_3'(\alpha)$ can be cancelled through the third column of the determinant in (3.184). The remaining terms are therefore

$$I_3^2=i\left[\frac{-R^2\eta_h+M_1}{\alpha e^{i\pi/6}+R}+\frac{R^2}{(\alpha e^{i\pi/6}+R)^2}\right],$$

(F.46)

$$I_3^4=(\alpha e^{i\pi/6}+R)^2.$$

(F.47)

(6) <u>The Boundary Conditions (1) and (3) $P_1(\eta)$</u>

The function $P_1(\eta)$ in (5) takes the path A_1B_1 defined in Fig. 2 of Chapter 3.6 which can be split up into three section

$$A_1B_1 = A_1L + LM + MB_1$$

where

$$A_1L : t = -Re^{i\pi/6} + r \,, \quad \infty \leqslant r \leqslant \varepsilon \,, \tag{F.48}$$

$$LM : t = -Re^{i\pi/6} + \varepsilon e^{i\theta}, \quad 0 \leqslant \theta \leqslant 2\pi \,, \tag{F.49}$$

$$MB_1 : t = -Re^{i\pi/6} + re^{2\pi i}, \quad \varepsilon \leqslant r \leqslant \infty \,. \tag{F.50}$$

Along A_1L we have

$$I_1 = -\int_{\varepsilon}^{\infty} (r - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \mathcal{Z}_1(r)\, dr, \tag{F.51}$$

where $\mathcal{Z}_1(r) = -\frac{1}{3}r^3 + Re^{i\pi/6}r^2 - \eta re^{-i\pi/6} + \eta R.$ (F.52)

The term from LM is

$$I_2 = \int_0^{2\pi} (-2Re^{i\pi/6})^{-1-M_2} \varepsilon^{M_2} e^{\eta R}\, i d\theta \to 0 \text{ as } \varepsilon \to 0 \tag{F.53}$$

and the integral from MB_1 becomes

$$I_3 = e^{2\pi i M_2} \int_{\varepsilon}^{\infty} (re^{2\pi i} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \mathcal{Z}_1(r)\, dr \,. \tag{F.54}$$

We can see that $I_3 = -e^{\frac{2\pi i M_2}{3}} I_1$ and as we can now take the limit, $\varepsilon \to 0$ we have for $P_1(\eta)$,

$$P_1(\eta) = \left[e^{2\pi i M_2} - 1\right] \int_0^{\infty} (r - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \mathcal{Z}_1(r)\, dr. \tag{F.55}$$

The integral in (F.55) is in fact exactly the same integral as the integral for $P_4(\eta)$ along NB_4.

We take the formulation as in (F.5) for the boundary condition (1) and so the contribution of $P_1(\eta)$ is of the form

$$I_1' = (e^{2\pi i M_2} - 1) \int_{\infty}^{\eta_0} i(M_1 - \eta R^2) \int_0^{\infty} (r - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp \mathcal{Z}_1(r)\, dr \tag{F.56}$$

and by changing the order of integration we have

$$I_1^1 = i\left(e^{2\pi i M_2}-1\right)\int_0^{\infty}\left[\frac{\eta_0 R^2 - M_1}{re^{-i\pi/6}-R}+\frac{R^2}{(re^{-i\pi/6}-R)^2}\right](r-2Re^{i\pi/6})^{-1-M_2}r^{-1+M_2}\exp\phi_2(r)\,dr$$

where $\phi_2(r)$ is exactly the same function as in (F.15), namely

$$\phi_2(r) = -\tfrac{1}{3}r^3 + r^2Re^{i\pi/6} - \eta_0 re^{-i\pi/6} + \eta_0 R.$$

Thus we can use the results from section (2) to obtain

$$I_1^1 = \left(e^{2\pi i M_2}-1\right)e^{R\eta_0}\,\frac{i\pi^{1/2}}{B^{1/2}}\,f_2^1(B)\,e^{2/3 B^3} \qquad \text{(F.57)}$$

where $f_2^1(B)$ is defined in (F.20).

For later work it is useful to write here,

$$I_1^1 = iC_1\left[\frac{\eta_0 R - M_1}{Be^{-i\pi/6}-R}+\frac{R^2}{(Be^{-i\pi/6}-R)^2}\right] \qquad \text{(F.58)}$$

where

$$C_1 = \left(e^{2\pi i M_2}-1\right)\frac{\pi^{1/2}}{B^{1/2}}B^{-1+M_2}(B-2Re^{i\pi/6})^{-1-M_2}\exp\left[B^2R^2e^{i\pi/3}-\frac{iB}{2}\lambda^{1/3}(a+b\sqrt{3})+R\eta_0+\tfrac{2}{3}B^3\right]. \qquad \text{(F.59)}$$

Similarly the contribution of $P_1(\eta)$ to boundary condition (3) will be

$$I_1^3 = \left(e^{2\pi i M_2}-1\right)\int_0^{\infty}(re^{-i\pi/6}-R)(re^{-i\pi/6}-R+M)(r-2Re^{i\pi/6})^{-1-M_2}r^{-1+M_2}\exp\phi_2(r) \qquad \text{(F.60)}$$

and so, using the results from section (3) we have

$$I_1^3 = \left(e^{2\pi i M_2}-1\right)e^{R\eta_0}\,\frac{i\pi^{1/2}}{B^{1/2}}\,g_2^1(B)\,e^{2/3 B^3} \qquad \text{(F.61)}$$

where $g_2^1(B)$ is defined in (F.24), and we will write I_1^3 in the form

$$I_1^3 = C_1(Be^{-i\pi/6}-R)(Be^{-i\pi/6}-R+M). \qquad \text{(F.62)}$$

(7) The Boundary Conditions (2) and (4) for $P_2(\eta)$

The function $P_2(\eta)$ in (5) takes the path A_2B_2 defined in Fig. 2 in Chapter 3.6 which is separated into three regions,

$$A_2B_2 = A_2L + LM + MB_2$$

where

$$A_2L:\ t = -Re^{i\pi/6} + re^{4\pi i/3}, \quad \infty \le r \le \varepsilon, \qquad \text{(F.63)}$$

$$LM:\ t = -Re^{i\pi/6} + \varepsilon e^{i\theta}, \quad 0 \le \theta \le 2\pi, \qquad \text{(F.64)}$$

$$MB_2:\ t = -Re^{i\pi/6} + re^{4\pi i/3 + 2\pi i}, \quad \varepsilon \le r \le \infty. \qquad \text{(F.65)}$$

Along A_2L we have

$$I_1 = -e^{\frac{4\pi i M_2}{3}}\int_{\varepsilon}^{\infty}(re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2}\exp \psi_2(r)\, dr. \tag{F.66}$$

The term along LM will be

$$I_2 = \int_0^{2\pi}(-2Re^{i\pi/6})^{-1-M_2}\varepsilon^{M_2}e^{\eta R}\, i\, d\theta \to 0 \text{ as } \varepsilon \to 0 \tag{F.67}$$

and the integral along MB_2 is

$$I_3 = \exp\left[\frac{4\pi i M_2}{3} + 2\pi i M_2\right]\int_{\varepsilon}^{\infty}(re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2}\exp \psi_2(r)\, dr \tag{F.68}$$

where $\psi_2(r) = -\tfrac{1}{3}r^3 + Re^{5\pi i/6}r^2 + \eta re^{-i\pi/6} + \eta R.$ (F.69)

We can see that $I_3 = -e^{2\pi i M_2} I_1$, and as we can now take the limit, $\varepsilon \to 0$

we have for $P_2(\eta)$

$$P_2(\eta) = [e^{2\pi i M_2} - 1]e^{\frac{4\pi i M_2}{3}}\int_0^{\infty}(re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2}\exp \psi_2(r)\, dr. \tag{F.70}$$

The integral in (F.70) is the same integral of $P_3(\eta)$ along A_3L.

We take the boundary condition (2) in the form of (F.27) and so the contribution of $P_2(\eta)$ is

$$I_2^2 = (e^{2\pi i M_2} - 1)e^{\frac{4\pi i M_2}{3}}\int_{\infty}^{\eta_h} i(M_1 - \eta R^2)\int_0^{\infty}(re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2}\exp \psi_2(r)\, dr$$

and by changing the order of integration we have

$$I_2^2 = (e^{2\pi i M_2} - 1)e^{\frac{4\pi i M_2}{3}}\int_0^{\infty}\left[\frac{R^2\eta_h - M_1}{(re^{i\pi/6} + R)} - \frac{R^2}{(re^{i\pi/6} + R)^2}\right](re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2} r^{-1+M_2}\exp \phi_3(r)\, dr \tag{F.71}$$

where $\phi_3(r)$ is defined in (F.33),

$$\phi_3(r) = -\tfrac{1}{3}r^3 + Rr^2e^{5\pi i/6} + \eta_h re^{-i\pi/6} + \eta_h R\,.$$

Thus we can use the results from section (4) to obtain

$$I_2^2 = i(e^{2\pi i M_2} - 1)\, e^{\eta_h R}\frac{\pi^{1/2}}{\alpha^{1/2}} f_3'(\alpha)e^{\frac{2}{3}\alpha^3} \tag{F.72}$$

where $f_3'(\alpha)$ is defined in (F.40). We will rewrite I_2^2 in the form

$$I_2^2 = C_2 i\left[\frac{-R^2\eta_h + M_1}{(\alpha e^{i\pi/6} + R)} + \frac{R^2}{(\alpha e^{i\pi/6} + R)^2}\right] \tag{F.73}$$

where

$$C_2 = -\frac{\pi^{1/2}}{\alpha^{1/2}}(e^{2\pi i M_2} - 1)(\alpha e^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2}\alpha^{-1+M_2}\exp\left[\eta_h R + \tfrac{2}{3}\alpha^3 + \alpha^2 Re^{5\pi i/6} + \frac{\lambda^{1/3} i\alpha}{2}\right]. \tag{F.74}$$

Similarly the contribution of $P_2(\eta)$ to boundary condition (4) will be

$$I_2^4 = (e^{2\pi i M_2}-1)\, e^{\frac{4\pi i M_2}{3}} \int_0^\infty (re^{i\pi/6}+R)^2 (re^{4\pi i/3} - 2Re^{i\pi/6})^{-1-M_2}\, r^{-1+M_2} \exp\phi_3(r)\, dr$$

and so using the results from section (5) we have

$$I_2^4 = (e^{2\pi i M_2}-1)\, e^{\eta_h R}\, \frac{\pi^{1/2}}{\alpha^{1/2}}\, g_3'(\alpha)\, e^{2/3\,\alpha^3} \tag{F.75}$$

where $g_3'(\alpha)$ is defined in (F.44) and thus

$$I_2^4 = C_2\,(\alpha e^{i\pi/6} + R)^2. \tag{F.76}$$

(8) Boundary Conditions (2) and (4) for $P_1(\eta)$

Using the original form of boundary condition (2), these two conditions (2) and (4) produce the following two integrals as the contributions of $P_1(\eta)$:

$$I_1^2 = e^{\eta_h R}[e^{2\pi i M_2}-1] \int_0^\infty (-r^3 e^{-i\pi/2} + i\eta_h r e^{-i\pi/6} + i)(r - 2Re^{i\pi/6})^{-1-M_2}\, r^{-1+M_2} \exp Q_1(r)\, dr, \tag{F.77}$$

$$I_1^4 = e^{\eta_h R}[e^{2\pi i M_2}-1] \int_0^\infty r^2 e^{-i\pi/3} (r - 2Re^{i\pi/6})^{-1-M_2}\, r^{-1+M_2} \exp Q_1(r)\, dr, \tag{F.78}$$

where $$Q_1(r) = -\tfrac{1}{3}r^3 + r^2 R e^{i\pi/6} - \eta_h r e^{-i\pi/6}. \tag{F.79}$$

We have that

$$\eta_h = \lambda^{1/3}[(1-a)+ib]$$

and so $\mathrm{Re}\{\eta_h e^{-i\pi/6}\} > 0$, and therefore because η_h is large we can approximate $Q_1(r)$ to $-\eta_h r e^{-i\pi/6}$. Substituting a new variable, u where

$$u = \eta_h r e^{-i\pi/6} \tag{F.80}$$

the integrals I_1^2 and I_1^4 become

$$I_1^2 = D_1 \int_0^\infty \left[\frac{u^3}{\eta_h^3} + iu + i\right] \left(1 + \frac{u}{-2R\eta_h}\right)^{-1-M_2} u^{-1+M_2}\, e^{-u}\, du \tag{F.81}$$

$$I_1^4 = D_1 \int_0^\infty \frac{u^2}{\eta_h^2} \left(1 + \frac{u}{-2R\eta_h}\right)^{-1-M_2} u^{-1+M_2}\, e^{-u}\, du \tag{F.82}$$

where $$D_1 = \eta_h (e^{2\pi i M_2}-1)(-2Re^{i\pi/6})^{-1-M_2} \exp(\eta_h R - i\pi/6). \tag{F.83}$$

The integrals are therefore summations of Whittaker functions. In general we have (32)

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty t^{-k-\frac{1}{2}+m} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t} dt. \tag{F.84}$$

Thus we have

$$I_1^2 = D_1' \left[\frac{1}{\eta_h^3} \Gamma(3+M_2)(-\tau_h)^{3/2} W_{-\frac{3}{2}-M_2,1}(-\tau_h) + i\Gamma(1+M_2)(\tau_h)^{\frac{1}{2}} W_{-\frac{1}{2}-M_2,0}(-\tau_h) + i\Gamma(M_2) W_{-M_2,-\frac{1}{2}}(-\tau_h) \right] \tag{F.85}$$

$$I_1^4 = D_1' \left[\frac{1}{\eta_h^2} \Gamma(2+M_2) W_{-1-M_2,\frac{1}{2}}(-\tau_h) \right] \tag{F.86}$$

where

$$\tau_h = 2R\eta_h, \tag{F.87}$$

$$D_1' = D_1 e^{-\frac{1}{2}\tau_h} (-\tau_h)^{M_2}. \tag{F.88}$$

(9) Boundary Conditions (1) and (3) for $P_2(\eta)$

Using the original form of the boundary condition (1) the contributions of $P_2(\eta)$ to (1) and (3) are

$$I_2^1 = (e^{2\pi i M_2}-1) e^{\eta_0 R} e^{\frac{4\pi i M_2}{3}} \int_0^\infty (r^3 e^{\pi i/2} - i\eta_0 r e^{i\pi/6} + i)(r e^{4\pi i/3} - 2R e^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp Q_2(r) dr, \tag{F.89}$$

$$I_2^3 = (e^{2\pi i M_2}-1) e^{\eta_0 R} e^{\frac{4\pi i M_2}{3}} \int_0^\infty (r^2 e^{i\pi/3} - M r e^{i\pi/6})(r e^{4\pi i/3} - 2R e^{i\pi/6})^{-1-M_2} r^{-1+M_2} \exp Q_2(r) dr, \tag{F.90}$$

where $Q_2(r) = -\frac{1}{3}r^3 + Rr^2 e^{5\pi i/6} + \eta_0 r e^{i\pi/6}$. (F.91)

Noting that $\eta_0 = -\lambda^{1/3}[a-ib]$ we can approximate $Q_2(r)$ to $\eta_0 r e^{i\pi/6}$ since η_0 is large. Thus using a new variable, v where

$$v = -\eta_0 r e^{i\pi/6} \tag{F.92}$$

we have

$$I_2^1 = D_2 \int_0^\infty \left[\frac{-v^3}{\eta_0^3} + iv + i \right] \left(1 + \frac{v}{-2R\eta_0}\right)^{-1-M_2} v^{-1+M_2} e^{-v} dv, \tag{F.93}$$

$$I_2^3 = D_2 \int_0^\infty \left[\frac{v^2}{\eta_0^2} + \frac{Mv}{\eta_0} \right] \left(1 + \frac{v}{-2R\eta_0}\right)^{-1-M_2} v^{-1+M_2} e^{-v} dv, \tag{F.94}$$

where

$$D_2 = [1 - e^{2\pi i M_2}] \eta_0 (-2R\eta_0)^{-1-M_2} \exp\left(\eta_0 R + \frac{4\pi i M_2}{3} + \left(\frac{i\pi}{6}\right)^{-2M_2-1}\right). \tag{F.95}$$

Again the integrals are summations of Whittaker functions,

$$I_2^1 = D_2' \Big[-\frac{1}{\eta_0^3} \Gamma(3+M_2)(-\tau_0)^{3/2} W_{-3/2-M_2,1}(-\tau_0) + i\Gamma(1+M_2)(-\tau_0)^{1/2} W_{-1/2-M_2,0}(-\tau_0)$$
$$+ i\Gamma(M_2) W_{-M_2,-1/2}(-\tau_0) \Big], \tag{F.96}$$

$$I_2^3 = D_2' \Big[\frac{1}{\eta_0^2} \Gamma(2+M_2)(-\tau_0) W_{-1-M_2,1/2}(-\tau_0) + \frac{M}{\eta_0} \Gamma(1+M_2)(-\tau_0)^{1/2} W_{-1/2-M_2,0}(-\tau_0) \Big], \tag{F.97}$$

where $D_2' = D_2 \, e^{-\frac{1}{2}\tau_0} (-\tau_0)^{M_2}$ (F.98)

and $\tau_0 = 2R\eta_0$. (F.99)

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THE EFFECTS OF HEAT TRANSFER ON OCEAN/ATMOSPHERE GENERAL CIRCULATION MODELS

BY S.K.L. JONES

ABSTRACT

We investigate in three problems some effects of heat transfer in linked ocean/atmosphere models. In all the problems the term involving vertical thermal conduction is retained in the heat transfer equation and both molecular and eddy values for the conductivity are considered.

In Part 1 we look at a two layer model, ignoring all macroscopic motion; the governing equation for both layers is therefore the heat transfer equation. With suitable boundary conditions the 'phase lag' between a heat source in the upper layer and the temperature at the inteface of the layers (the sea surface) is studied.

In Part 2 we consider a one layer model. A perturbation model due to Blinova is extended to include the heat transfer equation. One boundary condition introduces a time dependent heat source at the bottom of the layer, simulating a heating at the sea surface. The stream function is obtained at the bottom of the layer.

Finally, in Part 3, the stability of a two layer liquid model is examined. Macroscopic motion in the lower layer is ignored. The perturbation equations for the two layers are solved and homogeneous boundary equations yield an equation of consistency for the system which leads to criteria for stability. These criteria are found using difference methods and, following Meksyn we produce first order correction terms to Eady's well known stability results. Using Meksyn's methods once more, the model is extended to include a variable coriolis parameter and a stability equation is found.