Applications of variational theory in certain optimum shape problems in hydrodynamics

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by

Abdelrahman Hussein Essawy

Department of Mathematics,
University of Leicester.

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To

my little son Mohamed and my parents

ABSTRACT

PART I

In a recent paper Wu, T.Y. & Whitney, A.K., the authors studied optimum shape problems in hydrodynamics. These problems are stated in the form of a singular integral equation depending on the unknown shape and an unknown singularity distribution; the shape is then to be determined so that some given performance criterion has to be $\left(\begin{matrix}\text{maximized}\\ \text{minimized}\end{matrix}\right)$. In the optimum problem to be studied in this part we continue to assume that the state equation is a linear integral equation but we generalize the Wu & Whitney theory in two different ways.

This method is applied to functional of quadratic form and a necessary condition for the extremum to be a minimum is derived.

PART II

The purpose of this part is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces $\left(\begin{matrix}\text{maximum lift}\\ \text{minimum drag}\end{matrix}\right)$. The problem is discussed in three cases when there is a $\left(\begin{matrix}\text{full}\\ \text{partial}\\ \text{zero}\end{matrix}\right)$ cavity flow past the hydrofoil.

The liquid flow is assumed to be two-dimensional steady, irrotational and incompressible and a linearized theory is assumed.

Two-dimensional vortex and source distributions are used to simulate the two dimensional $\left(\begin{matrix}\text{full}\\ \text{partial}\\ \text{zero}\end{matrix}\right)$ cavity flow past a thin hydrofoil. This method leads to a system of integral equations and these are solved exactly using the Carleman-Muskhelishvili technique. This method is similar to that used by Davies, T.V.

We use variational calculus techniques to obtain the optimum shape of the hydrofoil in order to $\left(\begin{matrix}\text{maximize}\\ \text{minimize}\end{matrix}\right)$ the $\left(\begin{matrix}\text{lift}\\ \text{drag}\end{matrix}\right)$ coefficient subject to constraints on curvature and given length.

The mathematical problem is that of extremizing a functional depending on $\left(\begin{smallmatrix}\gamma \text{ vortex strength} \\ \mu \text{ source strength}\end{smallmatrix}\right)$ and z (the hydrofoil slope); these three functions are related by singular integral equations.

The analytical solution for the unknown shape z and the unknown singularity distribution γ has branch-type singularities at the two ends of the hydrofoil. Analytical solution by singular integral equations is discussed and the approximate solution by the Rayleigh-Ritz method is derived.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

PART III

The purpose of this work is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces minimum drag. A thin hydrofoil of arbitrary shape is in steady, rectilinear, horizontal motion at a depth h beneath the free surface of a liquid.

The usual assumptions in problems of this kind are taken as a basis, namely, the liquid is non-viscous and moving two-dimensionally, steadily and without vorticity, the only force acting on it is gravity.

With these assumptions together with a linearization assumption we determine the forces, due to the hydrofoil beneath a free surface of the liquid.

We use variational calculus techniques similar to those used in Part II to obtain the optimum shape so that the drag is minimized.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

In this part some general expressions are established concerning the forces acting on a submerged vortex and source element beneath a free surface using Blasius' theorem.

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ABSTRACT

PART I

PART I

VARIATIONAL PROBLEMS DEVELOPING THE THEORY DUE TO WU AND WHITNEY

1.a. INTRODUCTION

In a recent paper Wu, T.Y. and Whitney, A.K. (70) the authors studied optimum shape problem in hydrodynamics.

These problems are stated in the form of a singular integral equation depending on the unknown shape and on an unknown singularity distribution, the shape is then to be determined so that some given performance criterion has to be $\left\{\begin{matrix}\text{maximized}\\ \text{minimized}\end{matrix}\right\}$.

Wu assumes that the problem is stated as follows:

Find the function $u(x)$ (assumed Hölder continuous) defined in $-1<x<1$ when

$$v(x) = \frac{1}{\pi} \int_{-1}^{+1} \frac{u(t)dt}{t-x} \qquad (-1<x<1) , \qquad [1.a.1]$$

(It may be noted that [1.a.1] is an integral equation with a Cauchy singularity), so that the functional

$$J = \int_{-1}^{+1} f_o[u(x),v(x),x]dx \qquad , \qquad [1.a.2]$$

is minimized subject to the isoperimetric constraints

$$\int_{-1}^{+1} f_s[u(x),v(x),x]dx = C_s = \text{const.} \qquad s=1,2,\ldots,r, \qquad [1.a.3]$$

being satisfied.

In addition particular Hölder conditions are satisfied by $u(x)$ near the end points $x=\pm 1$.

Using a variational method, Wu shows that the necessary condition for minimization are

$$\frac{\partial f[u(x),v(x),x]}{\partial u} = \frac{1}{\pi} \oint_{-1}^{+1} \frac{\left(\frac{\partial f[u(t),v(t),t]}{\partial v}\right)dt}{t-x} \qquad , \qquad [1.a.4]$$

$$\frac{\partial^2 f[u(x),v(x),x]}{\partial u^2} + \frac{\partial^2 f[u(x),v(x),x]}{\partial v^2} > 0 \qquad . \qquad [1.a.5]$$

Number in brackets refer to the references at the end of this thesis.

The first condition [1.a.4] is a singular integral equation, where

$$f[u(x),v(x),x] = f_o[u(x),v(x),x] - \sum_{s=1}^{r} \lambda_s[f_s(u(x),v(x),x) - \frac{1}{2} C_s] \ , \quad [1.a.6]$$

and $\lambda_1,\lambda_2,\ldots\lambda_r$ are r Lagrange multipliers and this represents a departure from the classical Euler variational problem.

For the extremal solution [1.a.4] is to be solved together with [1.a.1], as a pair of singular integral equations for $u(x)$ and $v(x)$. The extremal solutions, $u(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ and $v(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ when determined in this manner will involve r multipliers constants $\lambda_1,\lambda_2,\ldots, \lambda_r$, which can be determined, by substituting the extremal solutions $u(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ and $v(x;\lambda_1,\lambda_2,\ldots\lambda_r)$ in the isoperimetric constraints [1.a.3].

In the optimum problem to be studied in this section we continue to assume that the state equation is a linear integral equation but we generalize [1.a.1] in two different ways.

It will be shown that the analysis of the first variation of the functional yields a set of dual, nonlinear, integral equations.

A necessary condition for the extremum to be a minimum is derived from consideration of the second variation.

This method is then applied to a function $f[u(x),v(x),x]$ of quadratic form.

1.b. VARIATIONAL PROBLEMS DEVELOPING THE THEORY DUE TO WU AND WHITNEY

STATEMENT OF THE PROBLEM.

The problem considered here may be stated as follows:

To find the real, extremal function $u(x)$ of a real variable x, required to be Hölder continuous * in the region $a<x<b$, satisfying one of the following three singular integral equations:

Case I

$$v(x) = A \fint_a^b \frac{u(t)dt}{t-x} = D_x[u(t)] \quad (a),$$

Case II

$$v(x) = E(x) + \nu u(x) + \frac{1}{\pi} \fint_a^b \frac{u(t)dt}{t-x} = D_x[u(t)] \quad (b), \qquad (a<x<b); \qquad [1.b.1]$$

Case III

$$v(x) = \fint_a^b \left\{ \frac{1}{t-x} + \alpha(t,x) \right\} u(t)dt = D_x[u(t)] \quad (c),$$

where the integral with symbol C signifies its Cauchy principal value, and in [1.b.1.c] the function $\alpha(t,x)$ is a continuous function of x,t, this will be discussed later; ν and A are constants.

We wish to determine $u(x)$ and $v(x)$ so as to minimize the functional

$$J = \int_a^b f_o[u(x),v(x),x]dx \qquad , \qquad [1.b.2]$$

where $f_o[u(x),v(x),x]$ is a given function of $u(x),v(x)$ and x; subject to the r isoperimetric constraints.

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* A function $u(x)$ is said to satisfy the Hölder $\mathcal{H}$ (μ) - condition on path $L(a<x<b)$ if, for any two points x_1, x_2 of L, $|u(x_2)-u(x_1)| \leqslant A|x_1-x_2|^{\mu}$, where A and μ are positive constants.

A is called the Hölder constant and μ the Hölder index.

$$J_s[u(x),v(x),x] = \int_a^b f_s[u(x),v(x),x]dx = C_s \qquad (s=1,2,\ldots,r) \, , \qquad [1.b.3]$$

where C_s is constant and $f_s[u(x),v(x),x]$ is a given function of $u(x),v(x)$ and x .

The unknown functions $u(x)$ and $v(x)$ are also to satisfy the conditions near the end points (a,b) ,

$$u(x) = u_*(x)/(x-C_k)^{\alpha_k+i\beta_k} \quad , \quad 0\leqslant\alpha_k<1 \quad (k=1,2) \, , \qquad [1.b.4]$$

where

$$C_1=a \quad , \quad C_2=b \quad \text{and} \quad i=\sqrt{-1} \quad , \qquad [1.b.5]$$

α_k and β_k are real constants and $u_*(x)$ satisfies the $\mathcal{H}$-condition near and at C_k .

If $u(x)$ is required to vanish at (a,b) , the end condition

$$u(a) = 0 \quad \text{and/or} \quad u(b) = 0 \quad , \qquad [1.b.6]$$

is a special case of [1.b.4] when [1.b.6]is satisfied and $u_*(x)$ satisfies the $\mathcal{H}(\mu<\alpha_k)$ - condition.

The function $f_o[u(x),v(x),x]$ and the constraint functions $f_s[u(x),v(x),x]$ $(s=1,2,\ldots,r)$ are assumed to be at least twice continuously differentiable with respect to their unknown functions $u(x)$ and $v(x)$ and continuous in x .

The notation $D_x[u(t)]$ for the finite singular integral transformation of $u(x)$, as defined in [1.b.1] , will be used throughout and will be called generalized Hilbert transformation.

It may be remarked here that the solution of a maximum problem can be deduced from this minimum one by changing the sign of the function in [1.b.3] .

In the spirit of the classical calculus of variations, we minimize the new functional

$$I[u] = \int_a^b f[u(x),v(x),x;\lambda_1,\lambda_2,\ldots,\lambda_r]dx \;, \qquad [1.b.7]$$

with

$$f[u(x),v(x),x;\lambda_1,\lambda_2,\ldots,\lambda_r]=f_o[u(x),v(x),x]-\sum_{s=1}^{r}\lambda_s[f_s(u(x),v(x),x)-(\frac{C_s}{b-a})] \qquad [1.b.8]$$

where $u(x),v(x)$ are related by [1.b.1] , and $\lambda_1,\lambda_2,\ldots,\lambda_r$ are Lagrange multipliers.

We define an admissible function that function $u(x)$ which satisfies the Hölder condition $\mathcal{H}(\mu<1)$, the isoperimetric constraints [1.b.3] , the prescribed end conditions [1.b.4]; minimizing the function $I[u(x)]$.

THE NECESSARY CONDITION OF OPTIMALITY

Let $u(x)$ denote the required optimal function.

A function $\xi(x)$ will be called an admissible variation if, for any sufficiently small positive constant ε ,

$$u_1(x) = u(x) + \varepsilon\xi(x) \qquad , \qquad [1.b.9]$$

is an admissible function.

The variation $\eta(x)$ in $v(x)$ which corresponds to the admissible variation $\xi(x)$, such that

$$v_1(x) = v(x) + \varepsilon\eta(x) \qquad , \qquad [1.b.10]$$

is found from [1.b.1] to be

$$\eta(x) = D_x[\xi(t)] \quad , \qquad [1.b.11]$$

where $D_x[\xi(t)]$ is defined by

In Case I

$$\eta(x) = A \oint_a^b \frac{\xi(t)dt}{t-x} = D_x[\xi(t)] \quad \text{(a)}$$

In Case II

$$\eta(x) = \nu\xi(x) + \frac{1}{\pi}\oint_a^b \frac{\xi(t)dt}{t-x} = D_x[\xi(t)] \quad \text{(b)} \qquad (a<x<b). \qquad [1.b.12]$$

In Case III

$$\eta(x) = \oint_a^b \left\{\frac{1}{t-x} + \alpha(t,x)\right\}\xi(t)dt = D_x[\xi(t)] \quad \text{(c)}$$

If $\xi(x)$ is an admissible variation, then $I[u(x) + \varepsilon\xi(x)]$ is a function of ε which has an extreme value when $\varepsilon=0$.

The variation ΔI of the function I due to the variations $\xi(x)$ and $\eta(x)$ is defined by

$$\Delta I = \int_a^b f[u(x)+\varepsilon\xi(x),v(x)+\varepsilon\eta(x),x]dx - \int_a^b f[u(x),v(x),x]dx \quad . \qquad [1.b.13]$$

For sufficiently small ε , expansion of the above integral by Taylor's series yields

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{2!}\delta^2 I + \frac{\varepsilon^3}{3!}\delta^3 I + \dots\dots , \qquad [1.b.14]$$

where the first variation δI and the second variation $\delta^2 I$ are

$$\delta I = \int_a^b [f_u(u(x),v(x),x).\xi(x)+f_v(u(x),v(x),x).\eta(x)]dx , \qquad [1.b.15]$$

$$\delta^2 I = \int_a^b [f_{uu}(u(x),v(x),x).\xi^2(x)+2f_{uv}(u(x),v(x),x)\xi(x).\eta(x)+$$
$$+f_{vv}(u(x),v(x),x)\eta^2(x)]dx , \qquad [1.b.16]$$

in which the sub-indices denote partial differentiations, and $\eta(x)$ is given by [1.b.12] .

The variations δI, $\delta^2 I$, ... depend on $\xi(x)$ and $u(x)$ since $\eta(x)$ and $v(x)$ can be replaced in [1.b.15] and [1.b.16] using [1.b.1] and [1.b.12].

For $I[u(x),\xi(x)]$ to be minimum, we must have for all admissible variations $\xi(x)$,

$$\delta I[u(x),\xi(x)] = 0 \quad , \qquad [1.b.17]$$

and

$$\delta^2 I[u(x),\xi(x)] \gtrless 0 \quad . \qquad [1.b.18]$$

Equations [1.b.17] and [1.b.18] assure that I takes a minimum.

As $\xi(t)$ and $\eta(x)$ are related by [1.b.12], substituting [1.b.12] in [1.b.17] we obtain

$$\left.\begin{aligned}
&\text{In Case I} \\
&\delta I = \int_a^b \left\{ f_u(u(x),v(x),x)\xi(x) + f_v(u(x),v(x),x) \cdot A \oint_a^b \frac{\xi(t)\,dt}{t-x} \right\} dx = 0 \qquad (a) \\
&\text{In Case II} \\
&\delta I = \int_a^b \left\{ f_u(u(x),v(x),x)\,\xi(x) + f_v(u(x),v(x),x) \cdot \left[\nu\xi(x) + \frac{1}{\pi}\int_a^b \frac{\xi(t)\,dt}{t-x} \right] \right\} dx = 0 \qquad (b) \\
&\text{In Case III} \\
&\delta I = \int_a^b \left\{ f_u(u(x),v(x),x)\xi(x) + f_v[u(x),v(x),x] \cdot \left[\int_a^b \left(\frac{1}{t-x} + \alpha(t,x) \right) \xi(t)\,dt \right] \right\} dx = 0 \quad . \qquad (c)
\end{aligned}\right\} \quad [1.b.19]$$

It is permissible to interchange the order of integration in the double integral in [1.b.19] [see, e.g., Hardy, G.H. (35)] and after interchanging the variable t,x we obtain

In Case I

$$\delta I = \int_a^b \left\{ f_u[u(x),v(x),x] - A \oint_a^b \frac{f_v[u(t),v(t),t]}{t-x} dt \right\} \cdot \xi(x)\,dx = 0 \quad , \quad (a)$$

In Case II

$$\delta I = \int_a^b \left\{ f_u[u(x),v(x),x] + \nu f_v[u(x),v(x),x] - \frac{1}{\pi} \int_a^b \frac{f_v[u(t),v(t),t]\,dt}{t-x} \right\} \xi(x)\,dx = 0 \quad , \quad (b)$$

In Case III

$$\delta I = \int_a^b \left\{ f_u[u(x),v(x),x] - \int_a^b f_v[u(t),v(t),t] \cdot \left[\frac{1}{t-x} - \alpha(x,t)\right] dt \right\} \xi(x)\,dx = 0 \quad . \quad (c)$$

[1.b.20]

Since $\xi(x)$ is arbitrary, the factor in brackets of the integrand in [1.b.20] must vanish identically for all x in (a,b), and thus we derive the following singular integral equations.

In Case I

$$f_u[u(x),v(x),x] = A \oint_a^b \frac{f_v[u(t),v(t),t]\,dt}{t-x} = D_x[f_v(u(t),v(t),t)] \qquad (a)$$

In Case II

$$f_u[u(x),v(x),x] = -\nu f_v[u(x),v(x),x] + \frac{1}{\pi} \oint_a^b \frac{f_v[u(t),v(t),t]\,dt}{t-x} \qquad (b)$$

In Case III

$$f_u[u(x),v(x),x] = \oint_a^b f_v[u(t),v(t),t] \left[\frac{1}{t-x} - \alpha(x,t)\right] dt \qquad (c)$$

[1.b.21]

This integral equation is analogous to the Euler differential equation in the classical theory of calculus variations.

Equations [1.b.21] is generally nonlinear in $u(x)$ and $v(x)$ unless $f[u(x),v(x),x]$ is a polynomial of second degree in $u(x)$ and $v(x)$.

The extremal solution is determined by solving the pair of coupled singular integral equations, [1.b.1] and [1.b.21] subject to conditions [1.b.3] and [1.b.4].

We now suppose that [1.b.1] and [1.b.21] can be solved for an extremal function $u(x;C_1,C_2,\ldots,C_r)$ which involve the constant of constraints $C_1,C_2,\ldots\ldots,C_r$ as parameters.

We now enquire under what condition does this extremal solution satisfy the inequality [1.b.18], so that it actually provides a minimum ?

In order to answer this question, we examine the second variation $\delta^2 I$.

Consider the case in which $f_{uu}[u(x),v(x),x]$, $f_{uv}[u(x),v(x),x]$, $f_{vv}[u(x),v(x),x]$ and $\xi(x)$ are the Hölder continuous on (a,b) . The second term on the right-hand side of equation [1.b.16] is as follows:

$$I_2 = 2\int_a^b f_{uv}[u(x),v(x),x]\xi(x)\eta(x)\,dx \qquad [1.b.22]$$

Substituting from [1.b.12] in [1.b.22] we obtain

In Case I

$$I_2 = 2\int_a^b f_{uv}[u(x),v(x),x]\xi(x)\,dx . A \,⨏_a^b \frac{\xi(t)\,dt}{t-x} =$$

$$= 2A\int_a^b ⨏_a^b f_{uv}[u(x),v(x),x]\xi(t)\xi(x)\,\frac{1}{t-x}\,dt\,dx \quad . \qquad [1.b.23]$$

It is permissible to interchange the order of integral [1.b.23] [see, e.g., Hardy, G.H.(35)] and interchange the variable t,x and when we do so we obtain

$$I_2 = -2A\int_a^b \xi(x)dx \,⨍_a^b \frac{f_{uv}[u(t),v(t),t]}{t-x}\xi(t)dt =$$

$$= -2A\int_a^b ⨍_a^b f_{uv}[u(t),v(t),t]\,\frac{1}{t-x}\xi(t)\xi(x)dtdx \quad . \qquad [1.b.24]$$

We take the mean of two preceding equations [1.b.23] and [1.b.24] we obtain

$$I_2 = A\int_a^b ⨍_a^b \left(\frac{f_{uv}[u(x),v(x),x]-f_{uv}[u(t),v(t),t]}{t-x}\right)\xi(t)\xi(x)dtdx \qquad [1.b.25]$$

In Case II

By similar operations in Case I, substituting from [1.b.12.b] in [1.b.22] we can write

$$I_2 = 2\int_a^b f_{uv}[u(x),v(x),x]\xi(x)dx\left[\nu\,\xi(x) + \frac{1}{\pi}⨍_a^b \frac{\xi(t)dt}{t-x}\right]$$

$$= 2\nu\int_a^b f_{uv}[u(x),v(x),x]\xi^2(x)dx + \frac{2}{\pi}\int_a^b f_{uv}[u(x),v(x),x]\xi(x)dx \,⨍_a^b \frac{\xi(t)dt}{t-x} \,. \qquad [1.b.26]$$

It is permissible to interchange the order of integral [1.b.26] and after interchanging the variable x,t we obtain

$$I_2 = 2\nu\int_a^b f_{uv}[u(x),v(x),x]\xi^2(x)dx - \frac{2}{\pi}\int_a^b \xi(x)dx \,⨍_a^b \frac{f_{uv}[u(t),v(t),t]\xi(t)dt}{t-x} \,. \qquad [1.b.27]$$

We take the mean of two preceding equations [1.b,26] and [1.b.27] and we obtain

$$I_2 = 2\nu\int_a^b f_{uv}[u(x),v(x),x]\xi^2(x)dx + \frac{1}{\pi}\int_a^b ⨍_a^b \left[\frac{f_{uv}[u(x),v(x),x]-f_{uv}[u(t),v(t),t)}{t-x}\right]. \xi(t)\xi(x)dtdx \qquad [1.b.28]$$

In Case III

By similar operations in Case I and Case II, substituting from [1.b.12.c] in [1.b.22] we obtain

$$I_2 = 2\int_a^b f_{uv}[u(x),v(x),x]\xi(x)dx \oint_a^b \left[\frac{1}{t-x} + \alpha(t,x)\right]\xi(t)dt \quad . \qquad [1.b.29]$$

It is permissible to interchange the order of integral [1.b.29] and after interchanging the variables x,t and when we do so we obtain

$$I_2 = -2\int_a^b \xi(x)dx \oint_a^b f_{uv}[u(t),v(t),t]\left[\frac{1}{t-x} - \alpha(x,t)\right]\xi(t)dt \quad . \qquad [1.b.30]$$

We take the mean of two preceding equations [1.b.29] and [1.b.30] we obtain

$$I_2 = \int_a^b \oint_a^b \frac{f_{uv}(u(x),v(x),x) - f_{uv}(u(t),v(t),t)}{t-x} . \xi(t)\xi(x)dtdx +$$

$$+ 2\int_a^b\int_a^b f_{uv}[u(x),v(x),x]\alpha(t,x)\xi(t)\xi(x)dtdx \quad , \qquad [1.b.31]$$

The third term on the right-hand side of equation [1.b.16] is as follows:

$$I_3 = \int_a^b f_{vv}[u(x),v(x),x].\eta^2(x)dx \quad . \qquad [1.b.32]$$

Substituting from [1.b.12] in [1.b.32] we obtain

In Case I

$$I_3 = \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(t)]dx.A \oint_a^b \frac{\xi(t)dt}{t-x} \quad . \qquad [1.b.33]$$

It is permissible to interchange the order of integral [1.b.33] and after interchanging the variables x,t and when we do so we obtain

$$I_3 = -A \int_a^b \xi(x)dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]}{t-x} .D_t[\xi(s)]\,dt$$

$$= -A^2 \int_a^b \xi(x)dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{t-x} \oint_a^b \frac{\xi(s)ds}{s-t} \quad . \qquad [1.b.34]$$

Using the Poincaré-Bertrand formula [see, e.g., Muskhelishvili, N.I. (45)]

$$\oint_a^b \frac{dt}{t-x} \oint_a^b \frac{\phi(t,s)ds}{s-t} = -\pi^2\phi(x,x) + \int_a^b ds \oint_a^b \frac{\phi(t,s)dt}{(t-x)(s-t)} \quad , \qquad [1.b.35]$$

hence

$$I_3 = -A^2 \int_a^b \xi(x)dx \left[-\pi^2 f_{vv}(u(x),v(x),x)\xi(x) + \int_a^b \xi(s)ds \oint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{(t-x)(s-t)}\right] . \qquad [1.b.36]$$

Using partial fractions and [1.b.21.a] we can write

$$I_3 = A^2\pi^2 \int_a^b f_{vv}[u(x),v(x),x]\xi^2(x)dx +$$

$$+ A \int_a^b \oint_a^b \frac{D_x[f_{vv}(u(s),v(s),s)] - D_t[f_{vv}(u(s),v(s),s]}{t-x} \xi(x)\xi(t)dtdx$$

[1.b.37]

In Case II

$$I_3 = \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)]dx\left[\nu\xi(x) + \frac{1}{\pi}\int_a^b \frac{\xi(t)dt}{t-x}\right]$$

$$= \nu \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)] \,.\, \xi(x)dx +$$

$$+ \frac{1}{\pi}\int_a^b f_{vv}[u(x),v(x),x].D_x[\xi(s)]dx \oint_a^b \frac{\xi(t)dt}{t-x} \qquad [1.b.38]$$

It is permissible to interchange the order of integral [1.b.38] and after interchanging the variables x,t and when we do so we obtain

$$I_3 = \nu \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)].\xi(x)dx -$$

$$- \frac{1}{\pi}\int_a^b \xi(x)dx \int_a^b \frac{f_{vv}(u(t),v(t),t)D_t[\xi(s)]dt}{t-x}$$

$$= \nu \int_a^b f_{vv}[u(x),v(x),x]\xi(x)dx\left[\nu.\xi(x) + \frac{1}{\pi}\int_a^b \frac{\xi(t)dt}{t-x}\right] -$$

$$- \frac{1}{\pi}\int_a^b \xi(x)dx \fint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{t-x}\left[\nu\xi(t) + \frac{1}{\pi}\fint_a^b \frac{\xi(s)ds}{s-t}\right]$$

$$= \nu^2 \int_a^b f_{vv}[u(x),v(x),x]\xi^2(x)dx +$$

$$+ \frac{\nu}{\pi}\int_a^b \xi(x)dx \fint_a^b \frac{f_{vv}[u(x),v(x),x]-f_{vv}[u(t),v(t),t]}{t-x}\xi(t)dt -$$

$$- \frac{1}{\pi^2}\int_a^b \xi(x)dx \fint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{t-x}\fint_a^b \frac{\xi(s)ds}{s-t} \qquad [1.b.39]$$

By using the Poincaré-Bertrand formulae, [1.b.35] in the final integral of [1.b.39] we obtain

$$I_3 = \nu^2 \int_a^b f_{vv}[u(x),v(x),x]\xi^2(x)dx +$$

$$+ \frac{\nu}{\pi}\int_a^b \xi(x)dx \fint_a^b \frac{f_{vv}[u(x),v(x),x]-f_{vv}[u(t),v(t),t]}{t-x}\xi(t)dt -$$

$$- \frac{1}{\pi^2}\int_a^b \xi(x)dx\left[-\pi^2 f_{vv}(u(x),v(x),x)\xi(x) + \int_a^b \xi(s)ds \fint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{(t-x)(s-t)}\right].$$

$$[1.b.40]$$

Using partial fractions and [1.b.21.b] we can write

$$I_3 = (1+\nu^2)\int_a^b f_{vv}[u(x),v(x),x]\xi^2(x)\,dx -$$

$$-\frac{2\nu}{\pi}\int_a^b \xi(x)\,dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]-f_{vv}[u(x),v(x),x]}{t-x}\xi(t)\,dt -$$

$$-\frac{1}{\pi}\int_a^b \xi(x)\,dx \oint_a^b \frac{D_t[f_{vv}(u(s),v(s),s)]-D_x[f_{vv}(u(s),v(s),s)]}{t-x}\xi(t)\,dt \quad [1.b.41]$$

In Case III

Substituting from [1.b.12.c] in [1.b.32] we obtain

$$I_3' = \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)]\,dx \oint_a^b \left[\frac{1}{t-x} + \alpha(t,x)\right]\xi(t)\,dt$$

$$= \int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)]\,dx \oint_a^b \frac{\xi(t)\,dt}{t-x} +$$

$$+\int_a^b f_{vv}[u(x),v(x),x]D_x[\xi(s)]\,dx \int_a^b \alpha(t,x)\xi(t)\,dt \quad . \qquad [1.b.42]$$

It is permissible to interchange the order of integral [1.b.42] and after interchanging the variables x,t we will have

$$I_3 = -\int_a^b \xi(x)\,dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]D_t[\xi(s)]\,dt}{t-x} +$$

$$+\int_a^b \xi(x)\,dx \int_a^b f_{vv}[u(t),v(t),t]\alpha(x,t)D_t[\xi(s)]\,dt \quad . \qquad [1.b.43]$$

Using [1.b.12] we obtain

$$I_3 = -\int_a^b \xi(x)\,dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]\,dt}{t-x}\oint_a^b \frac{\xi(s)\,ds}{s-t} -$$

$$-\int_a^b \xi(x)\,dx \oint_a^b \frac{f_{vv}[u(t),v(t),t]\,dt}{t-x}\int_a^b \alpha(s,t)\xi(s)\,ds +$$

$$+\int_a^b \xi(x)\,dx \int_a^b f_{vv}[u(t),v(t),t]\alpha(x,t)\,dt \oint_a^b \frac{\xi(s)\,ds}{s-t} +$$

$$+\int_a^b \xi(x)\,dx \int_a^b f_{vv}(u(t),v(t),t)\alpha(x,t)\,dt \int_a^b \alpha(s,t)\xi(s)\,ds \qquad [1.b.44]$$

We now use the Poincaré-Bertrand formula [1.b.35] in the first integral of [1.b.44], in addition we interchange the order of the second, third and fourth integrals in [1.b.44] and we obtain

$$
\begin{aligned}
I_3 = \pi^2\int_a^b \xi^2(x) f_{vv}[u(x),v(x),x]dx &- \int_a^b \xi(x)dx\int_a^b \xi(s)ds \fint_a^b \frac{f_{vv}[u(t),v(t),t]dt}{(t-x)\,(s-t)} - \\
&- \int_a^b \xi(x)dx\int_a^b \xi(s)ds \fint_a^b \frac{f_{vv}[u(t),v(t),t]\alpha(s,t)dt}{t-x} + \\
&+ \int_a^b \xi(x)dx\int_a^b \xi(s)ds \fint_a^b \frac{f_{vv}[u(t),v(t),t]\alpha(x,t)dt}{s-t} + \\
&+ \int_a^b \xi(x)dx\int_a^b \xi(s)ds\int_a^b f_{vv}[u(t),v(t),t]\alpha(x,t)\alpha(s,t)dt \ .
\end{aligned}
\qquad [1.b.45]
$$

Using partial fractions in the second term and interchanging the variables s,x in fourth term we obtain

$$
\begin{aligned}
I_3 = \pi^2\int_a^b \xi^2 f_{vv}[u(x),v(x),x]dx &- \int_a^b \xi(x)dx \fint_a^b \frac{\xi(s)ds}{s-x} \fint_a^b \left[\frac{f_{vv}[u(t),v(t),t]}{t-x} - \right. \\
&\left. - \frac{f_{vv}[u(t),v(t),t]}{t-s}\right] dt - 2\int_a^b \xi(x)dx\int_a^b \xi(s)ds \fint_a^b \frac{f_{vv}[u(t),v(t),t]\alpha(s,t)dt}{t-x} + \\
&+ \int_a^b \xi(x)dx\int_a^b \xi(s)ds\int_a^b f_{vv}[u(t),v(t),t]\alpha(x,t)\alpha(s,t)dt \ ,
\end{aligned}
\qquad [1.b.46]
$$

and this may be written in the form

$$
\begin{aligned}
I_3 = \pi^2\int_a^b \xi^2(x) f_{vv}[u(x),v(x),x]dx &- \int_a^b \fint_a^b \frac{\phi(x)-\phi(t)}{t-x}\,\xi(t)\xi(x)dtdx - \\
&- 2\int_a^b\int_a^b \psi(x,s)\xi(x)\xi(s)dsdx + \int_a^b\int_a^b \gamma(x,s)\xi(x)\xi(s)dsdx \ ,
\end{aligned}
\qquad [1.b.47]
$$

where

$$
\left.
\begin{aligned}
\phi(x) &= \int_a^b \frac{f_{vv}[u(s),v(s),s]ds}{s-x} \ , \\
\psi(x,s) &= \int_a^b \frac{f_{vv}[u(t),v(t),t]\alpha(s,t)dt}{t-x} \ , \\
\gamma(x,s) &= \int_a^b f_{vv}[u(t),v(t),t]\alpha(x,t)\alpha(s,t)dt \ .
\end{aligned}
\right\}
\qquad [1.b.48]
$$

Using the above results, [1.b.16] can be written in the following form:

$$\delta^2 I = \int_a^b g(x)\xi^2(x)\,dx - \frac{1}{\pi}\int_a^b\int_a^b \frac{h(t)-h(x)}{t-x}\xi(t)\xi(x)\,dt\,dx\ , \quad \text{in case I and case II ;} \qquad [1.b.49]$$

$$\delta^2 I = \int_a^b g(x)\xi^2(x)\,dx - \frac{1}{\pi}\int_a^b\int_a^b \left[\frac{h(t)-h(x)}{t-x} + B(t,x)\right]\xi(t)\xi(x)\,dt\,dx\ , \quad \text{in case III ,} \qquad [1.b.50]$$

where

$$B(t,x) = 2\psi(x,t) - \gamma(x,t) - 2f_{uv}[u(x),v(x),x]\cdot\alpha(t,x)\ , \qquad [1.b.51]$$

$$g(x) \equiv \begin{cases} f_{uu}[u(x),v(x),x] + A^2\pi^2 f_{vv}[u(x),v(x),x]\ , & \text{in case I ;} \\ f_{uu}[u(x),v(x),x] + 2\nu f_{uv}[u(x),v(x),x] + (1+\nu^2)\,f_{vv}[u(x),v(x),x]\ , & \text{in case II ;} \\ f_{uu}[u(x),v(x),x] + \pi^2 f_{vv}[u(x),v(x),x]\ , & \text{in case III ;} \end{cases}$$

[1.b.52]

$$h(t) \equiv \begin{cases} \{A\,f_{uv}[u(t),v(t),t] - A\,D_t[f_{vv}(u(s),v(s),s)]\}/\pi\ , & \text{in case I ;} \\ f_{uv}[u(t),v(t),t] - D_t[f_{vv}(u(s),v(s),s)] + 2\nu f_{vv}[u(t),v(t),t]\ , & \text{in case II ;} \\ \{f_{uv}[u(t),v(t),t] - \phi(t)\}/\pi\ , & \text{in case III .} \end{cases}$$

[1.b.53]

In [1.b.49] and [1.b.50] $g(x), h(x), f_{uu}[u(x),v(x),x], f_{uv}[u(x),v(x),x]$ and $f_{vv}[u(x),v(x),x]$ are assumed to be Hölder continuous in (a,b); this implies for the functions $g(x), h(x)$ that for any two points x_1, x_2 in the open interval (a,b) ,

$$\left.\begin{aligned} |g(x_2)-g(x_1)| &\leq A_1|x_2-x_1|^{\mu_1}\ , \quad (o\leq\mu_1<1, A_1>o)\ ; \\ |h(x_2)-h(x_1)| &\leq A_2|x_2-x_1|^{\mu_2}\ , \quad (o\leq\mu_2<1, A_2>o)\ . \end{aligned}\right\} \qquad [1.b.54]$$

Following Wu & Whitney we now consider the special choice of $\xi(x)$,

$$\xi(x) = \beta.U(\theta)\ , \qquad \theta = \frac{x-x_o}{\varepsilon}\ , \qquad [1.b.55]$$

where $U(\theta)$ is Hölder continuous

$$\left.\begin{aligned} o<U(\theta)\leq 1 &\qquad (|\theta|<1,\ |x-x_o|<\varepsilon)\ , \\ U(\theta)=o &\qquad (|\theta|>1,\ |x-x_o|>\varepsilon)\ , \end{aligned}\right\} \qquad [1.b.56]$$

and x_o is any fixed point in the open interval (a,b) and ε is arbitrarily small so that

$$a<x_o\pm\varepsilon<b \ . \qquad [1.b.57]$$

In [1.b.55] β, the upper bound of $\xi(x)$, is either positive or negative, and is chosen so small that $\delta^3 I$, $\delta^4 I$, etc. can be neglected in comparison with $\delta^2 I$.

With this choice of $\xi(x)$, [1.b.49] and [1.b.60] can be written, by adding and subtracting a term, in the form

$$\left.\begin{aligned}
\delta^2 I &= \int_a^b g(x_o)\xi^2(x)dx + \int_a^b [g(x)-g(x_o)]\xi^2(x)dx - \frac{1}{\pi}\int_a^b\int_a^b \frac{h(t)-h(x)}{t-x}\xi(t)\xi(x)dtdx \\
&= g(x_o)\beta^2\int_{x_o-\varepsilon}^{x_o+\varepsilon} U^2(\theta)d\theta + R \\
&= g(x_o)\beta^2\,\varepsilon\int_{-1}^{+1} U^2(\theta)d\theta + R\ , \\
\text{where}\quad & \\
R &= \int_{x_o-\varepsilon}^{x_o+\varepsilon}\left\{[g(x)-g(x_o)]\xi(x) - \frac{1}{\pi}\int_{x_o-\varepsilon}^{x_o+\varepsilon}\frac{h(t)-h(x)}{t-x}\xi(t)dt\right\}\xi(x)dx\ ,
\end{aligned}\right\}$$

in case I and case II [1.b.58]

and

$$\left.\begin{aligned}
\delta^2 I &= g(x_o)\beta^2\,\varepsilon\int_{-1}^{+1} U^2(\theta)d\theta + R\ , \\
\text{where}\quad & \\
R &= \int_{x_o-\varepsilon}^{x_o+\varepsilon}\left\{[g(x)-g(x_o)]\xi(x) - \frac{1}{\pi}\int_{x_o-\varepsilon}^{x_o+\varepsilon}\left[\frac{h(t)-h(x)}{t-x} + \pi B(t,x)\right]\xi(t)dt\right\}\xi(x)dx\ .
\end{aligned}\right\}$$

in case III [1.b.59]

With this choice of the value of $\xi(x)$, assuming that the upper bound of $B(t,x)$ is A_3 and using the inequalities [1.b.54], we can write the upper bound of R as follows:

$$|R| \leq A_1\beta^2\varepsilon^{1+\mu_1}\int_{-1}^{+1}|\theta|^{\mu_1}U^2(\theta)d\theta + A_2\beta^2\varepsilon^{1+\mu_2}\int_{-1}^{+1}\int_{-1}^{+1}|\phi-\theta|^{\mu_2-1}U(\theta)U(\phi)d\theta d\phi$$

$$\leq \left(\frac{2A_1\beta^2}{1+\mu_1}\right)\varepsilon^{1+\mu_1} + \left(\frac{2A_2\beta^2}{\mu_2(1+\mu_2)}\right)(2\varepsilon)^{1+\mu_2} \quad \text{in case I and case II}\ , \qquad [1.b.60]$$

and

$$|R| \leq A_1\beta^2\varepsilon^{1+\mu_1}\int_{-1}^{+1}|\theta|^{\mu_1}U^2(\theta)\,d\theta + A_2\beta^2\varepsilon^{1+\mu_2}\int_{-1}^{+1}\int_{-1}^{+1}|\phi-\theta|^{\mu_2-1}U(\theta)U(\phi)\,d\theta d\phi \; +$$

$$+ \; A_3\beta^2\varepsilon^2\int_{-1}^{+1}\int_{-1}^{+1}U(\theta)U(\phi)\,d\theta d\phi$$

$$\leq \left(\frac{2A_1\beta^2}{1+\mu_1}\right)\varepsilon^{1+\mu_1} + \left(\frac{2A_2\beta^2}{\mu_2(1+\mu_2)}\right)(2\varepsilon)^{1+\mu_2} + 2A_3\beta^2\varepsilon^2 \qquad \text{in case III} \; . \qquad [1.b.61]$$

We then obtain

$$\lim_{\varepsilon\to o}|R| = o \qquad , \qquad [1.b.62]$$

hence from [1.b.58] and [1.b.59] a necessary condition for minimizing, [1.b.18], reduces to

$$g(x_o) > o \; , \qquad [1.b.63]$$

for every $x_o \in (a,b)$, this implies from [1.b.52] that

$$g(x) \equiv \left\{ \begin{array}{l} f_{uu}[u(x),v(x),x] + A^2\pi^2 f_{vv}[u(x),v(x),x] > o \; , \\ \qquad\qquad\qquad\qquad \text{in case I} \; ; \\ f_{uu}[u(x),v(x),x] + 2\nu f_{uv}[u(x),v(x),x] \; + \\ \qquad + \; (1+\nu^2) f_{vv}[u(x),v(x),x] > o \; , \\ \qquad\qquad\qquad\qquad \text{in case II} \; ; \\ f_{uu}[u(x),v(x),x] + \pi^2 f_{vv}[u(x),v(x),x] > o \; , \\ \qquad\qquad\qquad\qquad \text{in case III} \; , \end{array} \right\}$$

[1.b.64]

This condition is analogous to the Legendre condition in the classical theory of the variational calculus.

Equation [1.b.64] is a <u>necessary condition</u> to be satisfied by a minimizing function. It may also be noted, by analogy with other classical variational problems [see, e.g., Courant, R. and Hilbert, D. (9), Chapter IV], that the strict inequality [1.b.64] is not a <u>sufficient condition</u> for a minimum.

To find a sufficient condition we expand $I[u(x)+\varepsilon\xi(x)]$ by Taylor's theorem with a remainder after two terms.

Thus,

$$I[u(x)+\varepsilon\xi(x)]=I[u(x)]+\varepsilon\delta I[u(x),\xi(x)]+\frac{\varepsilon^2}{2!}\delta^2 I[u(x)+\varepsilon\phi\xi(x),\xi(x)] \qquad (0<\phi<1)\,. \qquad [1.b.65]$$

If $u(x)$ is an extremal function, i.e.,

$$\delta I[u(x),\xi(x)] = 0 \quad , \qquad [1.b.66]$$

then

$$I[u(x)+\varepsilon\xi(x)]-I[u(x)]=\frac{\varepsilon^2}{2!}\delta^2 I[u(x)+\varepsilon\phi\xi(x),\xi(x)] \quad (0<\phi<1) \qquad [1.b.67]$$

Now suppose that inequality [1.b.18] holds not just for the extremal $u(x)$, but for all admissible functions.

Then we may set

$$\varepsilon = 1 \qquad [1.b.68]$$

in the above equation, [1.b.67] to give the condition

$$I[u(x)+\xi(x)]-I[u(x)]=\frac{1}{2}\delta^2 I[u(x)+\phi\xi(x),\xi(x)] > 0 \quad (0<\phi<1) \quad [1.b.69]$$

which is sufficient to show that the extremal $u(x)$ actually minimize I .

Based on the foregoing argument [see, e.g., Wu, T.Y and Whitney, A.K. (70)], a sufficient condition for a minimum is that the quadratic form, in $\xi(x)$ and $\eta(x)$ in the integral representation [1.b.16] of $\delta^2 I$ be positive definite for all admissible $u(x)$ and $\xi(x)$ [and hence all admissible v and η by [1.b.1] and [1.b.12]], that is

$$f_{uu}[u(x),v(x),x] > 0 \qquad (a<x<b) \ , \qquad [1.b.70]$$

and

$$f_{uu}[u(x),v(x),x] \cdot f_{vv}[u(x),v(x),x] > (f_{uv}[u(x),v(x),x])^2 \qquad (a<x<b), [1.b.71]$$

for all admissible $u(x)$ and $v(x)$.

This simple but rough <u>sufficient criterion</u> is a more restrictive inequality than [1.b.64].

1.c. <u>INTEGRANDS $f_s[u(x),v(x),x], s=0,1,\ldots,r$ ARE SECOND DEGREE IN u, AND v; THE FREDHOLM INTEGRAL EQUATION.</u>

We now solve the problem of the previous section namely

$$v(x) = E(x)+\nu u(x) + \frac{1}{\pi} \oint_a^b \frac{u(t)dt}{t-x} = D_t[u(t)] \ , \quad a<x<b \ ; \qquad [1.c.1]$$

when $I[u(x),v(x),x]$ in [1.b.7] is a function of second degree, or when [1.b.8] is of second degree in $u(x)$ and $v(x)$.

In this case the integral [1.b.21.b] is linear in $u(x)$ and $v(x)$.

It is instructive to investigate this case first, since the system of singular integral equations [1.c.1] and [1.b.21.b] can then be reduced to a single Fredholm integral equation of the second kind, or, in certain special cases, the method of singular integral equations can be employed to obtain an analytical solution in a closed form.

Following Wu, T.Y. and Whitney, A.K. (70), let the functions $f_o[u(x),v(x),x]$ and $f_s[u(x),v(x),x]$ in [1.b.8] be given by

$$f_o[u(x),v(x),x] = A_o u^2(x)+2B_o u(x)v(x)+C_o v^2(x)+2P_o u(x)+2Q_o v(x) \ , \quad [1.c.2]$$

and

$$f_s[u(x),v(x),x] = A_s u^2(x)+2B_s u(x)v(x)+C_s v^2(x)+2P_s u(x)+2Q_s v(x)$$
$$(s=1,2,\ldots,r); \quad [1.c.3]$$

the coefficients $A_o, B_o, C_o, \ldots, Q_r$ are known functions of x, assumed to be Hölder continuous [with index $\mu, 0<\mu<1$] on (a,b).

Then the function $f[u(x),v(x),x]$ in [1.b.8] becomes omitting the constant term in [1.b.8]

$$f[u(x),v(x),x] = Au^2(x)+2Bu(x)v(x)+Cv^2(x)+2Pu(x)+2Qv(x), \qquad [1.c.4]$$

where

$$A(x) = A_o(x) - \sum_{s=1}^{r} \lambda_s A_s(x) \quad , \text{ etc.} \qquad [1.c.5]$$

The integral equation [1.b.21.b] now reads

$$[A(x)+\nu B(x)]u(x)+[B(x)+\nu C(x)]v(x)+[P(x)+\nu Q(x)] =$$

$$= \frac{1}{\pi}\oint_a^b \frac{B(t)u(t)+C(t)v(t)+Q(t)}{t-x}\,dt \quad (a<x<b) \quad . \qquad [1.c.6]$$

The necessary condition [1.b.64.b] for minimizing, obtained from consideration of the second variation, becomes

$$A(x)+2\nu B(x)+(1+\nu^2)C(x) > 0 \qquad [1.c.7]$$

which can be checked only when $\lambda_1, \lambda_2, \ldots, \lambda_r$ in [1.c.5] are determined. The coupled integral equations [1.b.6] and [1.c.6] can be reduced, under certain assumptions, to a Fredholm integral equation of the second kind, with a regular symmetric kernel.

The required assumptions are that the coefficients $A, B, \ldots, Q$ as well as solution $u(x), v(x)$, are Hölder continuous on (a,b).

In fact, eliminating v between [1.c.6] and [1.c.1] yields

$$[A(x)+\nu B(x)]u(x)+[B(x)+\nu C(x)]\left[E(x)+\nu u(x)+\frac{1}{\pi}\oint_a^b \frac{u(t)\,dt}{t-x}\right] + [P(x)+\nu Q(x)] =$$

$$= \frac{1}{\pi}\oint_a^b \frac{B(t)u(t)\,dt}{t-x} + \frac{1}{\pi}\oint_a^b \frac{C(t)\,dt}{t-x}\left[E(t)+\nu u(t)+\frac{1}{\pi}\oint_a^b \frac{u(y)\,dy}{y-t}\right] + \frac{1}{\pi}\oint_a^b \frac{Q(t)\,dt}{t-x} \qquad [1.c.8]$$

We interchange the order of integration in the second term on the right-hand side of [1.c.8] and then we obtain

$$[A(x)+2\nu B(x)+\nu^2C(x)]u(t) + \frac{1}{\pi}\oint_a^b \frac{[B(x)+\nu C(x)]-[B(t)+\nu C(t)]}{t-x}u(t)dt + [B(x)+\nu C(x)]E(x)+$$

$$+[P(x)+\nu Q(x)] - \frac{1}{\pi}\oint_a^b \frac{E(t)C(t)+Q(t)}{t-x}dt = - C(x)u(x) + \frac{1}{\pi^2}\oint_a^b u(y)dy \oint_a^b \frac{C(t)\,dt}{(t-x)(y-t)}$$

$$(a<x<b), \quad [1.c.9]$$

where, use has been made of the Poincaré-Bertrand formula [1.b.35] [see,e.g., Muskhelishvili,N.I. (45)].

Thus, [1.c.9] reduces to

$$\alpha(x)u(x) + \int_a^b K(t,x)u(t)dt = \psi(x) \quad , \quad a<x<b \; , \qquad [1.c.10]$$

where

$$\alpha(x) = A(x)+2\nu B(x)+(1+\nu^2)C(x) \quad , \qquad [1.c.11]$$

$$K(t,x) = \frac{\beta(t,x)}{t-x} \quad , \qquad [1.c.12]$$

with

$$\beta(t,x) = [B(x)+\nu C(x)]-[B(t)+\nu C(t)] + \oint_a^b \left[\frac{1}{y-t} - \frac{1}{y-x}\right] C(y)dy \; , \qquad [1.c.13]$$

$$\psi(x) = - E(x)[B(x)+\nu C(x)]-[P(x)+\nu Q(x)] + \oint_a^b \frac{E(t)C(t)+Q(t)}{t-x}dt \; . \qquad [1.c.14]$$

This is a Fredholm integral equation of the second kind, with a regular symmetric kernel, for which a well developed theory is available; and $C(x)$ and $u(x)$ are Hölder continuous on (a,b).

The function $\beta(t,x)$ vanish at $t=x$

$$\beta(t,t) = 0 \quad , \qquad [1.c.15]$$

and $\alpha(x)$, $\beta(t,x)$ and $\psi(x)$ will, in general contain unknown Lagrange multipliers.

Ideally, the integral equation can be solved first for arbitrary values of $\lambda_1, \lambda_2, \ldots, \lambda_r$ which can then be determined by the r constraints

[1.b.3]. Finally, condition [1.c.7] should be checked.

ANALYTICAL SOLUTION BY THE METHOD OF SINGULAR INTEGRAL EQUATIONS

In the general case when the coefficients A,B, and C are arbitrary function of x, the solution of the system of singular integral equations [1.c.1], [1.c.6] cannot be found in closed form (for a general discussion, see Muskhelishvili, part IV, (45) for further discussion of special cases, see Peters, A.S. (50) and Gakhov, F.D. (23)).

However, when the coefficients A,B,C satisfy certain conditions, the system of equations [1.c.1] and [1.c.6] can be reduced in succession to a single integral equation of the Carleman type, which can be solved in turn by known methods, yielding the final solution in closed form.

These analytical solutions are of great interest, since in their construction there are definite degrees of freedom for choosing the strength of the singularity of the solution $u(x)$ at the end points $x=a$ and $x=b$.

With these possibilities, the singular behaviour of $u(x)$ and $v(x)$ near $x=a$ and $x=b$ can be explicitly analysed.

The following are several cases of fairly general interest.

Case I A,B,C constants

Multiplying [1.c.1] by n and adding it to [1.c.6], we obtain

$$[A+\nu B-n\nu]u(x)+[B+\nu C+n]\,v(x)=\frac{1}{\pi}\int_a^b \frac{(B+n)u(t)+Cv(t)}{t-x}\,dt+\frac{1}{\pi}\int_a^b \frac{Q(t)dt}{t-x}-$$

$$-\ [P(x)+\nu Q(x)]+nE(x) \qquad (a<x<b) \qquad . \qquad [1.c.16]$$

We now choose n so that

$$\frac{A+\nu B-n_s\nu}{B+n_s}=\frac{B+\nu C+n_s}{C}=k_s \qquad (s=1,2) \qquad [1.c.17]$$

and let $n_s (s=1,2)$ be the two solutions of the quadratic

hence

$$\left.\begin{aligned} n_1 &= -(B+\nu C)+\sqrt{2B\nu C+\nu^2C^2+AC} \qquad &(a)\\ n_2 &= -(B+\nu C)-\sqrt{2B\nu C+\nu^2C^2+AC} \qquad &(b)\end{aligned}\right\} \tag{1.c.18}$$

We define $k_s (s=1,2)$ as follows

$$k_s = \frac{B+\nu C+n_s}{C} = \pm \frac{1}{C}\sqrt{\nu^2C^2+2\nu CB+AC} \qquad (s=1,2) \quad , \tag{1.c.19}$$

hence

$$k_1 = -k_2 = \frac{1}{C}\sqrt{\nu^2C^2+2\nu CB+AC} \quad . \tag{1.c.20}$$

Using [1.c.17] we can write [1.c.16] in the form

$$k_s\{(B+n_s)u(x)+Cv(x)\} = \frac{1}{\pi}\oint_a^b \frac{(B+n_s)u(t)+Cv(t)}{t-x}\,dt + \frac{1}{\pi}\oint_a^b \frac{Q(t)dt}{t-x} -$$

$$-[P(x)+\nu Q(x)]+n_s E(x) \qquad (s=1,2) \quad . \tag{1.c.21}$$

Substituting [1.c.18] and [1.c.20] in [1.c.21] we obtain

$$\left.\begin{aligned} \sqrt{\nu^2C^2+2\nu CB+AC}\;\phi_1(x) &= \frac{C}{\pi}\oint_a^b \frac{\phi_1(t)dt}{t-x} + \theta_1(x) \qquad &(a)\\ \sqrt{\nu^2C^2+2\nu CB+AC}\;\phi_2(x) &= -\frac{C}{\pi}\oint_a^b \frac{\phi_2(t)dt}{t-x} + \theta_2(x) \qquad &(b)\ ,\end{aligned}\right\} \tag{1.c.22}$$

where

$$\left.\begin{aligned} \theta_1(x) &= \frac{C}{\pi}\oint_a^b \frac{Q(t)dt}{t-x} - C[P(x)+\nu Q(x)] - CE(x)(B+\nu C) +\\ &\qquad + CE(x)\sqrt{2B\nu C+\nu^2C^2+AC} \qquad &(a)\\ \theta_2(x) &= -\frac{C}{\pi}\oint_a^b \frac{Q(t)dt}{t-x} + C[P(x)+\nu Q(x)] + CE(x)(B+\nu C) +\\ &\qquad + CE(x)\sqrt{2B\nu C+\nu^2C^2+AC} \qquad &(b)\ ,\end{aligned}\right\} \tag{1.c.23}$$

and

$$\left.\begin{aligned} \phi_1(x) &= [-\nu C+\sqrt{2B\nu C+\nu^2C^2+AC}]u(x)+Cv(x) \quad &(a)\\ \phi_2(x) &= [-\nu C-\sqrt{2B\nu C+\nu^2C^2+AC}]u(x)+Cv(x) \quad &(b)\end{aligned}\right\} . \qquad [1.c.24]$$

Now, equations in [1.c.22] are a singular integral equation of the Carleman type, the general solutions of which can be found.

With $\phi_s(x)$ $(s=1,2)$ so determined, $u(x)$ and $v(x)$ can be solved.

Case II $A=0$, $B=$ const. and $C=$ const.

This is special limit of Case I, and the general solution can be derived by putting $A=0$ in Case I.

Case III $A=$ const., $B=$ const. and $C=0$.

This is another special limit of Case I.

The corresponding solution can be derived from [1.c.16] in the form

$$u(x) = \frac{1}{A+2\nu B}\left\{-BE(x)-[P(x)+\nu Q(x)] + \oint_a^b \frac{Q(t)dt}{t-x}\right\} \qquad [1.c.25]$$

Case IV A,B,C functions of x .

We solve equation [1.c.6] byaspecial choice of $B(x)$

$$\left.\begin{aligned} B_1(x) &= [A(x).C(x)]^{\frac{1}{2}} \quad &(a)\\ B_2(x) &= -[A(x).C(x)]^{\frac{1}{2}} \quad &(b)\end{aligned}\right\} . \qquad [1.c.26]$$

Substituting [1.c.26] in [1.c.6] we obtain

$$\left.\begin{aligned} [A^{\frac{1}{2}}(x)+\nu C^{\frac{1}{2}}(x)]\phi_1(x) &= \frac{1}{\pi}\oint_a^b \frac{C^{\frac{1}{2}}(t)\phi_1(t)dt}{t-x} + \theta(x) \quad &(a)\\ [A^{\frac{1}{2}}(x)-\nu C^{\frac{1}{2}}(x)]\phi_2(x) &= -\frac{1}{\pi}\oint_a^b \frac{C^{\frac{1}{2}}(t)\phi_2(t)dt}{t-x} + \theta(x) \quad &(b)\end{aligned}\right\} , \qquad [1.c.27]$$

where

$$\theta(x) = \frac{1}{\pi}\oint_a^b \frac{Q(t)dt}{t-x} - [P(x)+\nu Q(x)] \qquad , \qquad [1.c.28]$$

and

$$\left.\begin{aligned} \phi_1(x) &= A^{\frac{1}{2}}(x)u(x)+C^{\frac{1}{2}}(x)v(x) \quad &(a)\\ \phi_2(x) &= A^{\frac{1}{2}}(x)u(x)-C^{\frac{1}{2}}(x)v(x) \quad &(b)\end{aligned}\right\} . \qquad [1.c.29]$$

Now, [1.c.27] is a singular integral equation of the Carleman type, the general solution of which can be found.

With $\phi_s(x)$ $(s=1,2)$ so determined, $u(x)$ can be solved in succession by substituting [1.c.1] in [1.c.29], giving

$$\left.\begin{aligned}
&[A^{\frac{1}{2}}(x)+\nu C^{\frac{1}{2}}(x)]u(x) + \frac{C^{\frac{1}{2}}(x)}{\pi}\oint_a^b \frac{u(t)dt}{t-x} = \phi_1 - E(x)C^{\frac{1}{2}}(x) \quad \text{(a)}\\
&[A^{\frac{1}{2}}(x)-\nu C^{\frac{1}{2}}(x)]u(x) - \frac{C^{\frac{1}{2}}(x)}{\pi}\oint_a^b \frac{u(t)dt}{t-x} = \phi_2 + E(x)C^{\frac{1}{2}}(x) \quad \text{(b)}
\end{aligned}\right\} \quad [1.c.30]$$

which is again a singular integral equation of the Carleman type, and the general solution is known.

Case V $A = 0$, B and C functions of x .

This is special limit of Case IV, and the general solution we can derive is by putting $A = 0$ in Case IV .

Case VI A function of x , B = const. and $C = 0$

This is another special limit of Case IV.

The corresponding solution can be deduced from [1.c.10] in the form

$$u(x) = \frac{1}{A(x)+2\nu B}\left\{-BE(x)-[P(x)+\nu Q(x)] + \oint_a^b \frac{Q(t)dt}{t-x}\right\} \quad . \qquad [1.c.31]$$

PART II

THE OPTIMUM SHAPE OF CAVITATING AND NON - CAVITATING HYDROFOIL

INTRODUCTION

There is an extensive literative connected with cavity type flow [see, e.g., Davies,T.V.(13),(14), Geurst,J.A.(24),(25),(26), Gilberg,G.(27),(28),(29), Parkin,B.R.(46),(47),(48),(49), Tulin,M.P.(62),(63), Wu,T.Y.(76),(78)] but two papers which are most relevant to the present Part II are those to Davies,T.V.(13) and Wu,T.Y. & Whitney, A.K.(70).

The purpose of this work is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces $\begin{Bmatrix}\text{maximum lift}\\ \text{minimum drag}\end{Bmatrix}$.

The problem is discussed in three cases when there is a $\begin{Bmatrix}\text{full}\\ \text{partial}\\ \text{zero}\end{Bmatrix}$ cavity flow past a thin hydrofoil.

The liquid flow is assumed to be two-dimensional steady, irrotational, incompressible and a linearized theory is assumed.

A two-dimensional vortex and source distributions are used to simulate the two-dimensional $\begin{Bmatrix}\text{full}\\ \text{partial}\\ \text{zero}\end{Bmatrix}$ cavity flow past hydrofoil.

This method leads to a system of integral equations and these are solved exactly using the Carleman-Muskhelishvili technique.

This method is similar to that used by Davies,T.V.(13),(14).

A singular integral equation formulation of the boundary value problem is obtained and can be solved to yield expressions for the lift and drag as functions of vortex and source strength, and hydrofoil slope.

We use a variational calculus technique to obtain the optimum shape of the hydrofoil in order to $\begin{Bmatrix}\text{maximize}\\ \text{minimize}\end{Bmatrix}$ the $\begin{Bmatrix}\text{lift}\\ \text{drag}\end{Bmatrix}$ coefficient subject to constraints on curvature and given length. The mathematical problem is that of extremizing a functional depending on $\begin{Bmatrix}\gamma \text{ vortex strength}\\ \mu \text{ source strength}\end{Bmatrix}$ and z (the hydrofoil slope) when these two functions are related by a singular integral equation. It will be shown that the first variation of the functional yield a set of dual, non-linear, singular integral equations.

The analytical solution for the unknown shape z and the unknown singularity distribution $\binom{\gamma}{\mu}$ has branch-type singularities at the two ends of the hydrofoil.

The extremal solutions, $\begin{pmatrix}\gamma(x;\lambda_1,\lambda_2)\\ \mu(x;\lambda_1,\lambda_2)\end{pmatrix}$ and $z(x;\lambda_1,\lambda_2)$ when determined will involve two Lagrange multipliers constants λ_1,λ_2 which can be determined by substituting the extremal solution $\begin{pmatrix}\gamma(x;\lambda_1,\lambda_2)\\ \mu(x,\lambda_1,\lambda_2)\end{pmatrix}$ and $z(x;\lambda_1,\lambda_2)$ in the constraints.

Analytical solution by singular integral equations is discussed, and Rayleigh-Ritz methods are discussed.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

II. THE OPTIMUM SHAPE OF HYDROFOIL WITH NO CAVITATION

INTRODUCTION

The purpose of this problem is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces minimum drag.

The hydrofoil as in the accompanying diagram (Fig.1) is placed in a uniform flow of an incompressible non-viscous liquid filling an infinite space. The liquid flow is taken to be two-dimensional irrotational, steady, and a linearized theory is assumed.

A two-dimensional vortex distribution over the hydrofoil is used to simulate the two-dimensional zero cavity flow past the hydrofoil.

This method leads to a system of integral equations and these are solved exactly using the Carleman-Muskhelishvili technique.

This method is similar to that used by Davies,T.V. (13),(14).

We use variational calculus techniques to obtain the optimum shape of the hydrofoil in order to minimize the drag coefficient subject to constraints on curvature and given length. The mathematical problem is that of extremizing a functional depending on γ (the vortex strength) and z (the hydrofoil slope) when these two functions are related by a singular integral equation.

The analytical solution for the unknown shape z and the unknown singularity distribution γ has branch-type singularities at the two ends of the hydrofoil.

The extremal solutions $\gamma(x;\lambda_1,\lambda_2)$ and $z(x;\lambda_1,\lambda_2)$ when determined will involve two Lagrange multipliers constants λ_1,λ_2 which can be determined, by substituting the extremal solutions $\gamma(x;\lambda_1,\lambda_2)$ and $z(x;\lambda_1,\lambda_2)$ in the constraints.

Analytical solution by a singular integral equation and Rayleigh-Ritz method are discussed.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

IIa EXPRESSION OF THE PROBLEM IN INTEGRAL EQUATIONS FORM

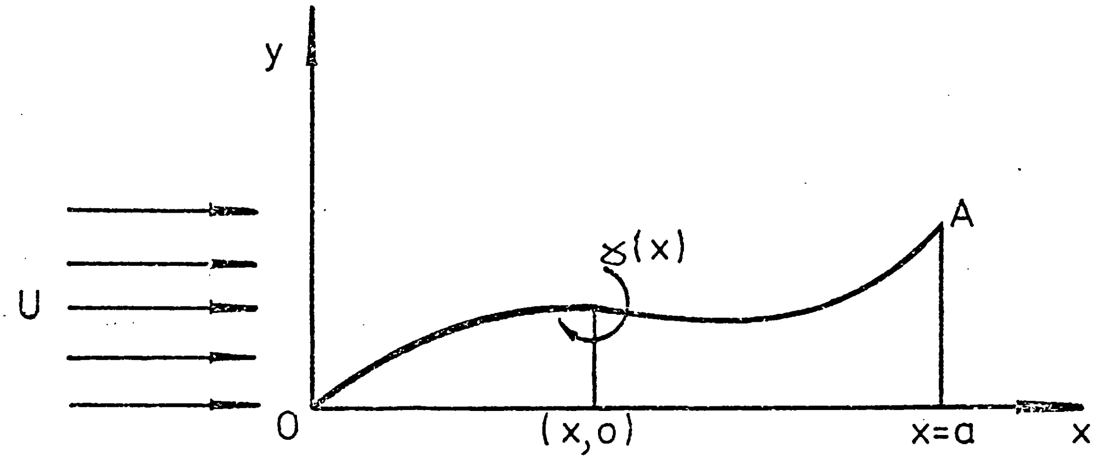

FIG.1.

OA in the Figure represents a hydrofoil of arbitrary shape.

The problem will be solved on the basis of linearized theory and for this purpose we distribute: vortices of strength $\gamma(x)$ per unit length in $o < x < a$ ($\gamma>0$ clockwise) along the x-axis to replace the above physical configuration, and $\gamma(x)$ being an unknown distribution.

The velocity potential due to the distribution of vortices in $0<x<a$ is given by

$$\phi(x,y) = -\frac{1}{2\pi}\int_0^a \gamma(s)\tan^{-1}\left(\frac{y}{x-s}\right) ds \qquad (o<x<a) \quad , \qquad [2.a.1]$$

and the corresponding velocity in y-direction will be

$$v = -\frac{\partial\phi}{\partial y} = \frac{1}{2\pi}\int_0^a \frac{\gamma(s)(x-s)ds}{(x-s)^2+y^2} \qquad [2.a.2]$$

As $y\to0\pm$ we have, for all x

$$\lim_{y\to0\pm} v = \frac{1}{2\pi}\int_0^a \frac{\gamma(s)ds}{x-s} \qquad (o<x<a) \quad . \qquad [2.a.3]$$

The boundary condition on the hydrofoil is

$$z(x) = \frac{v}{U+u} \quad , \quad (z(x)=y'(x)) \quad , \quad (0<x<a) \; , \qquad [2.a.4]$$

where u,v are the components of liquid velocity along x,y axes respectively, U is uniform stream at infinity, parallel to x-axis and $y'(x)$ is the gradient of the hydrofoil at position x .

The equation [2.a.4] is approximated in the usual way to

$$v = Uz(x) \qquad (o<x<a) \quad , \qquad [2.a.5]$$

hence

$$-\frac{1}{2\pi}\int_o^a \frac{\gamma(s)\,ds}{s-x} = Uz(x) \qquad (o<x<a) \quad . \qquad [2.a.6]$$

The linearized form of Bernoulli's equation will be

$$P=P_\infty+\rho U\phi_x \qquad [2.a.7]$$

where P is the pressure, P_∞ the pressure at infinity and ρ is constant density of the liquid.

From [2.a.1] we can write

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\pi}\int_o^a \frac{\gamma(s).y\,ds}{(x-s)^2+y^2} \quad , \qquad [2.a.8]$$

the limiting value of $\frac{\partial\phi}{\partial x}$ as $y\to0\pm$ is

$$\lim_{y\to0\pm}\left(\frac{\partial\phi}{\partial x}\right) = \pm\frac{1}{2}\gamma(x) \qquad (o<x<a) \qquad [2.a.9]$$

IIb DETERMINATING THE GENERAL FORMULA FOR THE LIFT AND DRAG

Let the x- and y- components of the hydrodynamic forces acting on the hydrofoil be denoted by drag D and lift L , then the complex forces acting on a hydrofoil calculated within the linearized theory are given by

$$D+iL = \int_o^a \{P|_{y=o-}-P|_{y=o+}\}idZ \quad . \qquad [2.b.1]$$

Using the results in [2.a.7] and [2.a.9] as $y\to0+$ through positive value we obtain

$$P\big|_{y=o+} = P_\infty + \rho U \lim_{y\to o+} \phi_x$$
$$= P_\infty + \frac{1}{2}\rho U \gamma(x) \qquad , \qquad (o<x<a) \qquad [2.b.2]$$

Using the results in [2.a.7] and [2.a.9] as $y\to 0-$ through negative value we have

$$P\big|_{y=o-} = P_\infty + \rho U \lim_{y\to o-} \phi_x$$
$$= P_\infty - \frac{1}{2}\rho U \gamma(x) \qquad , \qquad (o<x<a) \qquad [2.b.3]$$

It follows that we can write from [2.b.1], [2.b.2] and [2.b.3] the hydrodynamic forces acting on the hydrofoil

$$L = \int_o^a \{P\big|_{y=o-} - P\big|_{y=o+}\}dx$$
$$= -\rho U \int_o^a \gamma(x)dx \qquad (o<x<a) \quad , \qquad [2.b.4]$$

and

$$D = -\int_o^a \{P\big|_{y=o-} - P\big|_{y=o+}\}dy$$
$$= \rho U \int_o^a \gamma(x)y'(x)dx \qquad (o<x<a) \qquad [2.b.5]$$

IIc THE OPTIMUM SHAPE USING VARIATIONAL CALCULUS TECHNIQUES SO THAT THE DRAG IS A MINIMUM

We pose the problem of minimizing the drag coefficient

$$D^* = \frac{D}{\rho U^2} \qquad , \qquad [2.c.1]$$

subject to a constraint on curvature of the form

$$K = \int_o^a z'^2(x)dx \qquad , \qquad [2.c.2]$$

where K is prescribed, together with a constraint on the length of the hydrofoil of the form

$$\ell = \int_0^a \sqrt{1+z^2(x)}\,dx \quad , \qquad [2.c.3]$$

where ℓ is prescribed and $z(x)=y'(x)$ is the gradient of the hydrofoil at position x .

The expression for the drag D is given by

$$D = \rho U \int_0^a \gamma(x).z(x)\,dx \quad . \qquad [2.c.4]$$

STATEMENT OF THE PROBLEM

The general optimum problem considered here may be stated as follows:

To find the real, extremal function $\gamma(x)$ of a real variable, required to be Hölder continuous [see, e.g., Tricomi,F.G.(61] in the region $0<x<a$ together with

$$-\frac{1}{2\pi}\int_0^a \frac{\gamma(s)ds}{s-x} = Uz(x) \qquad (o<x<a) \ , \qquad [2.c.5]$$

so that $\gamma(x)$ and $z(x)$ minimize the new functional

$$I[\gamma(x),z(x),z'(x),x]=D^*+\lambda_1\ell+\lambda_2K = \int_0^a F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2]dx \ , \qquad [2.c.6]$$

with the function $F[\gamma(x),z(x),z'(x),x]$ given by

$$F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2] = \frac{1}{U}z(x)\gamma(x) + \lambda_1\sqrt{1+z^2(x)}+\lambda_2 z'^2(x) \quad , \qquad [2.c.7]$$

where $\gamma(x),z(x)$ are related by [2.c.5] and λ_1,λ_2 are undetermined Lagrange multipliers. We define an admissible function as any function $\gamma(x)$ which satisfies the Hölder condition $\mathcal{H}$ $(\mu<1)$, the constraints [2.c.2] and [2.c.3], and we define the optimal function as an admissible function which minimize the function $I[\gamma,z,z',x]$.

THE NECESSARY CONDITION OF OPTIMALITY

Let $\gamma(x),z(x)$ denote the required optimal vortex distribution function and optimal hydrofoil slope function respectively.

A function $\xi(x)$ will be called admissible variation if, for all sufficiently small positive constant ε

$$\gamma_1(x)=\gamma(x)+\varepsilon\xi(x) \qquad , \qquad [2.c.8]$$

is an admissible function.

The variation in $z(x)$ which corresponds to an admissible variation $\eta(x)$, such that

$$z_1(x)=z(x)+\varepsilon\eta(x) \qquad , \qquad [2.c.9]$$

is found from [2.c.5]

$$\eta(x) = -\frac{1}{2\pi U}\int_0^a \frac{\xi(s)ds}{s-x} \quad , \quad (0<x<a) \quad . \qquad [2.c.10]$$

If $\xi(x)$ is an admissible variation, then $I[\gamma+\varepsilon\xi]$ is a function of ε which has an extreme value when $\varepsilon=0$.

The variation of the function I due to the variation $\xi(x)$ and $\eta(x)$ is

$$\Delta I = \int_0^a F[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x]dx - \int_0^a F[\gamma,z,z',x]dx \quad . \qquad [2.c.11]$$

For sufficiently small ε, expansion of the above integrand in Taylor's series yields

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{2!}\delta^2 I + \ldots \qquad , \qquad [2.c.12]$$

where the first variation δI is defined by

$$\delta I[\gamma,\xi] = \int_0^a [\xi F_\gamma(\gamma,z,z',x)+\eta F_z(\gamma,z,z',x)+\eta' F_{z'}(\gamma,z,z',x)]dx \ , \qquad [2.c.13]$$

in which the sub-indices denote partial differentiations, η is given by [2.c.10] . The variations $\delta I, \delta^2 I, \ldots$ depend on $\xi(x)$ as well as $\gamma(x)$.

We integrate by parts the equation [2.c.13] and it becomes

$$\delta I = \int_0^a [\xi(x)F_\gamma(\gamma,z,z',x)+\eta(x)(F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x))]dx + $$

$$+ [\eta(x)F_{z'}(\gamma,z,z',x)]_0^a \quad . \qquad [2.c.14]$$

Substituting from [2.c.10] in [2.c.14] we obtain

$$\delta I = \int_0^a \left\{ \xi(x) F_\gamma(\gamma,z,z',x) + (F_z(\gamma,z,z',x) - \frac{d}{dx} F_{z'}(\gamma,z,z',x)) \cdot \left(-\frac{1}{2\pi U} \int_0^a \frac{\xi(s)ds}{s-x} \right) \right\} dx +$$

$$+ [\eta(x) . F_{z'}(\gamma,z,z',x)]_0^a \quad . \qquad [2.c.15]$$

It is permissible to interchange the order of the repeated integral [2.c.15] [see, e.g., Hardy,G.H.(35)] and then we obtain

$$\delta I = \int_0^a \xi(x) \left\{ F_\gamma(\gamma,z,z',x) + \frac{1}{2\pi U} \int_0^a \frac{F_z(\gamma,z,z',s) - \frac{d}{ds} F_{z'}(\gamma,z,z',s)}{s-x} ds \right\} dx +$$

$$+ [\eta(x) . F_{z'}(\gamma,z,z',x)]_0^a \quad . \qquad [2.c.16]$$

For $I[\gamma,\xi]$ to be a minimum, we must have for all admissible function $\xi(x)$,

$$\delta I[\gamma,\xi] = 0 \quad . \qquad [2.c.17]$$

Now the stationary condition [2.c.17] must hold for all admissible $\xi(x)$; there are a number of different cases to be considered depending on the end conditions are as follows:

$$\left.\begin{array}{lll} \eta(0)=0 & , \quad \eta(a)=0 & ; \\ \eta(0)=0 & , \quad \eta(a)\neq 0 & ; \\ \eta(0)\neq 0 & , \quad \eta(a)=0 & ; \\ \eta(0)\neq 0 & , \quad \eta(a)\neq 0 & . \end{array}\right\} \qquad [2.c.18]$$

In all cases it is necessary that

$$F_\gamma[\gamma,z,z',x] = -\frac{1}{2\pi U} \int_0^a \frac{F_z[\gamma,z,z',s] - \frac{d}{ds} F_{z'}[\gamma,z,z',s]}{s-x} ds$$

$$(0<x<a) \quad , \qquad [2.c.19]$$

and if $\eta(x)$ does not vanish at an end point then it is necessary that $\frac{\partial F}{\partial z'}$, should vanish at that point; since

$$\frac{\partial F}{\partial z'} = 2\lambda_2 z'(x) \qquad (0<x<a) \quad , \qquad [2.c.20]$$

it then follows that we must have

$$\eta(0)z'(0) = 0 \quad , \quad \eta(a)z'(a) = 0 \quad . \qquad [2.c.21]$$

The boundary conditions for $z(x)$ are called natural boundary conditions [see, e.g., Arthurs,A.M.(3)].

We can write from [2.c.7] the first partial derivatives of the function $F(\gamma,z,z',x;\lambda_1,\lambda_2)$

$$F_\gamma[\gamma,z,z',x] = \frac{1}{U}\, z(x) \quad , \qquad [2.c.22]$$

$$F_z[\gamma,z,z',x] = \frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} + \frac{1}{U}\,\gamma(x) \quad , \qquad [2.c.23]$$

and $\frac{\partial F}{\partial z'}$ is defined by [2.c.20] .

Substituting from [2.c.20],[2.c.22] and [2.c.23] in [2.c.19] we obtain

$$z(x) = -\frac{1}{2\pi}\int_0^a \frac{\left[\frac{\lambda_1 z(s)}{\sqrt{1+z^2(s)}} + \frac{1}{U}\,\gamma(s) - 2\lambda_2 z''(s)\right]ds}{s-x} \quad . \qquad [2.c.24]$$

This equation which is a necessary condition for the existence of an extremal $I[\gamma]$, combines with the integral equation, [2.c.5] to give a pair of singular integral equations, which are to be solved for γ, z subject to appropriate conditions and the constraints, [2.c.2] and [2.c.3].

The term $z(x)$ on the left-hand side of [2.c.24] cancels with the second term on the right-hand side of [2.c.24], from [2.c.5] and [2.c.24] we obtain

$$\int_0^a \frac{\left[\frac{\lambda_1 z(s)}{\sqrt{1+z^2(s)}} - 2\lambda_2 z''(s)\right]ds}{s-x} = 0 \qquad (0<x<a) \quad . \qquad [2.c.25]$$

The general solution of [2.c.25] is

$$2\lambda_2 z''(x) - \frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} = \frac{C}{\sqrt{x(a-x)}} \qquad (0<x<a) \quad , \qquad [2.c.26]$$

where C is an arbitrary constant.

It can easily be shown that λ_2 cannot be zero.

Equation [2.c.26] is a nonlinear differential equation for $z(x)$.

We consider the solution of [2.c.26] for the slope $z(x)$ only in the case of small slope, and we approximate to [2.c.26] as follows:

$$2\lambda_2 z''(x)-\lambda_1 z(x) = \frac{C}{\sqrt{x(a-x)}} \qquad (o<x<a) \ ; \qquad [2.c.27]$$

this is consistent with the linearized assumption on which we base the whole theory.

IId A SUFFICIENT CONDITION FOR THE EXTREMUM TO BE A MINIMUM

A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I .

Since

$$\delta I[\gamma,z,z',x] = 0 \quad , \qquad [2.d.1]$$

the condition for I to be a minimum requires

$$\delta^2 I[\gamma,z,z',x] \geqslant 0 \quad , \qquad [2.d.2]$$

for all admissible variations $\xi(x)$ and $\eta(x)$ consistent with [2.c.10].
The solution of [2.c.10] satisfying the Kutta condition $\xi(a)=o$ is

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a\sqrt{\frac{s}{a-s}}\,\frac{\eta(s)ds}{s-x} \quad . \qquad [2.d.3]$$

Since z has been prescribed at $x=o$ and $x=a$ in [2.c.5] it follows that the variation η will satisfy the conditions

$$\eta(0) = 0 \quad , \qquad \eta(a) = 0 \quad . \qquad [2.d.4]$$

Then, using Taylor's theorem with remainder, we write the increment of the functional $I[\gamma,z,z',x]$ as

$$I[\gamma+\varepsilon\xi,z+\varepsilon\eta,z'+\varepsilon\eta',x]-I[\gamma,z,z',x] = \varepsilon\int_0^a\Big\{\xi(x)F_\gamma(\gamma,z,z',x)+\eta(x)\Big[F_z(\gamma,z,z',x) -$$

$$-\frac{d}{dx}F_{z'}(\gamma,z,z',x)\Big]\Big\}dx+\frac{1}{2}\varepsilon^2\int_0^a\{\xi^2(x)F_{\gamma\gamma}+\eta^2(x)F_{zz}+\eta'^2(x)F_{z'z'}+2\xi(x)\eta(x)F_{\gamma z}+$$

$$+2\xi(x)\eta'(x)F_{\gamma z'}+2\eta(x)\eta'(x)F_{zz'}\}dx + O(\varepsilon^3) \qquad (o<x<a) \quad . \qquad [2.d.5]$$

Denoting the coefficient ε by δI and that ε^2 by $\delta^2 I$, and at a stationary value of I, we have from [2.d.1], [2.d.3] amd [2.d.5]

$$\left.\begin{aligned}\delta I &= \int_0^a \left\{ F_\gamma(\gamma,z,z',x)\xi(x) + [F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)]\eta(x) \right\} dx = 0 \qquad (a)\\ &= \int_0^a \left\{ F_\gamma(\gamma,z,z',x)\,\frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{\eta(s)\,ds}{s-x} + [F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)]\eta(x) \right\} dx \quad (b)\end{aligned}\right\}$$

[2.d.6]

It is permissible to interchange the order of integration in first term in [2.d.6(b)] [see, e.g., Hardy,G.H.(35)] and when we do so we obtain

$$\delta I = \int_0^a \left\{ -\frac{2U}{\pi}\sqrt{\frac{x}{a-x}}\int_0^a \sqrt{\frac{a-s}{s}}\,\frac{F_\gamma(\gamma,z,z',x)\,dx}{s-x} + [F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)] \right\}\eta(x)\,dx = 0 \;.$$

[2.d.7]

Since $\eta(x)$ is arbitrary, the factor in the bracket in [2.d.7] must vanish identically for $o<x<a$, giving the following singular integral equation

$$F_z[\gamma,z,z',x] - \frac{d}{dx}F_{z'}[\gamma,z,z',x] = \frac{2U}{\pi}\sqrt{\frac{x}{a-x}}\int_0^a \sqrt{\frac{a-s}{s}}\;.\;\frac{F_\gamma(\gamma,z,z',s)\,ds}{s-x} \quad (o<x<a)\,. \quad [2.d.8]$$

The second variation defined from [2.d.5] in the form

$$\delta^2 I = \int_0^a \{\xi^2 F_{\gamma\gamma} + \eta^2 F_{zz} + \eta'^2 F_{z'z'} + 2\xi\eta F_{\gamma z} + 2\xi\eta' F_{\gamma z'} + 2\eta\eta' F_{zz'}\}\,dx\;, \qquad [2.d.9]$$

where, by [2.c.7] we can write

$$\left.\begin{aligned} F_{\gamma\gamma}(\gamma,z,z',x) &= 0 \;,\\ F_{zz}(\gamma,z,z',x) &= \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \;,\\ F_{z'z'}[\gamma,z,z',x] &= 2\lambda_2 \;,\\ F_{\gamma z}[\gamma,z,z',x] &= \frac{1}{U} \;,\\ F_{\gamma z'}[\gamma,z,z',x] &= 0 \;,\\ F_{zz'}[\gamma,z,z',x] &= 0 \;. \end{aligned}\right\} \qquad [2.d.10]$$

Substituting from [2.d.10] in [2.d.9] we obtain

$$\delta^2 I = \int_0^a \left\{ \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \eta^2(x) + 2\lambda_2 \eta'^2(x) + \frac{2}{U} \xi(x)\eta(x) \right\} dx \quad . \qquad [2.d.11]$$

Now we calculate the third integral in [2.d.11] which is defined by

$$I_1 = \int_0^a \xi(x)\eta(x)\,dx \quad . \qquad [2.d.12]$$

Using [2.d.3] we obtain

$$I_1 = \frac{2U}{\pi} \int_0^a \sqrt{\frac{a-x}{x}}\, \eta(x)\,dx \int_0^a \sqrt{\frac{s}{a-s}} \cdot \frac{\eta(s)\,ds}{s-x}$$

$$= \frac{2U}{\pi} \int_0^a\int_0^a \sqrt{\frac{a-x}{x}}\sqrt{\frac{s}{a-s}}\; \frac{\eta(s)\eta(x)\,ds\,dx}{s-x} \quad . \qquad [2.d.13]$$

It is permissible to interchange the order of integration on the right-hand side of [2.d.13] and also interchange the variables x and s and when we do so we obtain

$$I_1 = -\frac{2U}{\pi} \int_0^a\int_0^a \sqrt{\frac{a-s}{s}}\sqrt{\frac{x}{a-x}}\; \frac{\eta(s)\eta(x)}{s-x}\,ds\,dx \quad . \qquad [2.d.14]$$

We take the mean of the two preceding equations [2.d.14] and [2.d.13] and then we obtain

$$I_1 = \frac{U}{\pi} \int_0^a\int_0^a \frac{\left[\sqrt{\frac{a-x}{x}}\sqrt{\frac{s}{a-s}} - \sqrt{\frac{a-s}{s}}\sqrt{\frac{x}{a-x}}\right]\eta(s)\eta(x)\,ds\,dx}{s-x}$$

$$= \frac{Ua}{\pi} \int_0^a\int_0^a \frac{\eta(s)\eta(x)\,ds\,dx}{\sqrt{x(a-x)}\quad\sqrt{s(a-s)}}$$

$$= \frac{Ua}{\pi} \left(\int_0^a \frac{\eta(x)\,dx}{\sqrt{x(a-x)}}\right)^2 \quad . \qquad [2.d.15]$$

Substituting from [2.d.15] in [2.d.11] we obtain

$$\delta^2 I = \int_0^a \left\{ \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \eta^2(x) + 2\lambda_2 \eta'^2 + \frac{2}{\pi} \left(\int_0^a \frac{\eta(s)\,ds}{\sqrt{s(a-s)}}\right)^2 \right\} dx \quad . \qquad [2.d.16]$$

As an illustration we consider a special choice of $\eta(x)$ satisfying $\eta(o) = \eta(a) = o$, namely :

$$\eta(x) = \alpha \sin \frac{\pi}{a} x \qquad (o<x<a) \quad , \qquad [2.d.17]$$

then we can write [2.d.16] in the form

$$\delta^2 I = \frac{1}{2}\alpha^2 a\left\{\frac{\lambda_1}{[1+z^2(x)]^{3/2}} + \frac{2\pi^2\lambda_2}{a^2} + \frac{4}{\pi}\left[\int_o^a \frac{\sin\frac{\pi}{a}x\,dx}{\sqrt{x(a-x)}}\right]^2\right\} > o \quad , \qquad [2.d.18]$$

In the case of small slope $z(x)$, we approximate [2.d.18] as follows:

$$\delta^2 I = \frac{1}{2}\alpha^2 a\left\{\lambda_1 + \frac{2\pi^2\lambda_2}{a^2} + \frac{4}{\pi}\left(\int_o^a \frac{\sin\frac{\pi}{a}x\,dx}{\sqrt{x(a-x)}}\right)^2\right\} > o \quad . \qquad [2.d.19]$$

Thus in the case [2.d.17] the sufficient condition for satisfying [2.d.2] is as follows:

$$\lambda_1 + \frac{2\pi^2\lambda_2}{a^2} + \frac{4}{\pi}\left(\int_o^a \frac{\sin(\frac{\pi}{a}x)\,dx}{\sqrt{x(a-x)}}\right)^2 > o \quad . \qquad [2.d.20]$$

IIe ANALYTICAL SOLTUION BY THE RAYLEIGH-RITZ METHOD

We use the Rayleigh-Ritz method [see, e.g., Temple, G. and Bickley, W.G. (58) and Milne, W.E. (44)], to solve equation [2.c.27], namely

$$z''(x) - n\,z(x) = \alpha(x) \quad , \qquad (o < x < a) \quad , \qquad [2.e.1]$$

with

$$\left.\begin{aligned} \alpha(x) &= \frac{E}{\sqrt{x(a-x)}} \quad , \\ n &= \frac{\lambda_1}{2\lambda_2} \quad , \quad E = \frac{C}{2\lambda_2} \quad , \end{aligned}\right\} \qquad [2.e.2]$$

where E is an arbitrary constant and λ_1, λ_2 are Lagrange multipliers, and $z(x)$ is subject to the boundary conditions

$$z(o) = o \quad , \quad z(a) = \beta \, . \qquad [2.e.3]$$

Equation [2.e.1] is the necessary condition for the integral

$$J = \int_o^{\alpha} \{\frac{1}{2} z'^2(x) + \frac{1}{2} n \, z^2(x) + \alpha(x) z(x)\} dx \, , \qquad [2.e.4]$$

to be minimized.

The Rayleigh-Ritz method can be applied to this problem in the following way:

We select a basic set of linearly independent polynomial functions and we assume an expression for $z(x)$ of the form

$$z(x) = \frac{\beta}{a^2} x^2 + a_1 x(a-x) + a_2 x^2(a-x) \quad (o \leq x \leq a) \, , \qquad [2.e.5]$$

which satisfies the end conditions [2.e.3], a_1 and a_2 being arbitrary constants.

The values of $z(x)$ and $z'(x)$ are obtained from equation [2.e.5] and are substituted in [2.e.4] , the result is a quadratic form in a_1 and a_2 :

$$J = \int_o^a \left\{\frac{1}{2} \left[\frac{2\beta}{a^2} x + a_1(a-2x) + a_2(2ax-3x^2)\right]^2 + \frac{1}{2} n \left[\frac{\beta}{a^2} x^2 + a_1 x(a-x) + a_2 x^2 (a-x)\right]^2 + \right.$$

$$\left. + \frac{E}{\sqrt{x(a-x)}} \left[\frac{\beta}{a_2} x^2 + a_1 x(a-x) + a_2 x^2 (a-x)\right] \right\} dx \quad , \qquad [2.e.6]$$

hence

$$J = \frac{\beta^2}{30a} [20+3na^2] + \frac{a^3}{60} a_1^2 [10+na^2] + \frac{a^5}{210} a_2^2 [14+na^2] + \frac{a^4}{60} a_1 a_2 [10+na^2] -$$

$$- \frac{1}{60} \beta a [20-3na^2] a_1 - \frac{1}{30} \beta a^2 [5-na^2] a_2 + E[\frac{3}{8} \beta\pi + \frac{\pi a^2 a_1}{8} + \frac{\pi a^3 a_2}{16}] . \qquad [2.e.7]$$

The necessary conditions for minimizing J , with respect to a_1 and a_2, are

$$\frac{\partial J}{\partial a_1} = \frac{a^3}{30} a_1[10+na^2] + \frac{a^4}{60} a_2[10+na^2] - \frac{1}{60} \beta a[20-3na^2] + \frac{\pi a^2}{8} E = o \; , \qquad [2.e.8]$$

and

$$\frac{\partial J}{\partial a_2} = \frac{a^4 a_1}{60}[10+na^2] + \frac{a^5 a_2}{105} [14+na^2] - \frac{1}{30} \beta a^2 [5-na^2] + \frac{\pi a^3}{16} E = o \; . \qquad [2.e.9]$$

The quantities n and E can now be expressed in terms of a_1 and a_2 but for convenience we introduce

$$\left.\begin{aligned} \xi &= a_1 a^2 \; , \\ \eta &= a_2 a^3 \; , \end{aligned}\right\} \qquad [2.e.10]$$

and then we have

$$n = - \frac{42\,\eta}{[\eta+7.\beta]a^2} \; , \qquad [2.e.11]$$

and

$$E = - \frac{8}{\pi a}\{ \frac{1}{30} .\xi.(10+na^2) + \frac{1}{60} \eta \; (10+na^2) - \frac{1}{60} \beta \; (20-3na^2) \} \; . \qquad [2.e.12]$$

From [2.d.20] it follows that a sufficient condition for the drag to be a minimum is

$$(n + \frac{\pi^2}{a^2}) \;/\; E + \frac{4}{\pi} \left(\int_o^a \frac{\sin \frac{\pi}{a} x \; dx}{\sqrt{x(a-x)}} \right)^2 > o \; . \qquad [2.e.13]$$

Substituting from [2.e.5] in the constraints [2.c.2] and [2.c.3] we obtain

$$\ell = \int_o^a [1 + \frac{1}{2} z^2(x)]dx$$

$$= a + \frac{1}{10} \beta^2 a + \frac{1}{60} a_1^2 a^5 + \frac{1}{210} a^7 a_2^2 + \frac{1}{20} \beta a^3 a_1 + \frac{1}{30} \beta a^4 a_2 + \frac{1}{60} a^6 a_1 a_2 , \qquad [2.e.14]$$

and

$$K = \int_o^a z^2(x)\,dx$$

$$= \frac{4}{3} \beta^2/a + \frac{1}{3} a^3 a_1^2 + \frac{2}{15} a^5 a_2^2 - \frac{2}{3} \beta a_1 a - \frac{1}{3} \beta a_2 a^2 + \frac{1}{3} a_1 a_2 a^4 . \qquad [2.e.15]$$

Equations [2.e.14] and [2.e.15] can be written as follows in terms of ξ and η :

$$\left.\begin{aligned} S_1 &= A_1\xi^2+2H_1\xi\eta+B_1\eta^2+2P_1\xi+2Q_1\eta+C_1 = o \; , \\ S_2 &= A_2\xi^2+2H_2\xi\eta+B_2\eta^2+2P_2\xi+2Q_2\eta+C_2 = o \; , \end{aligned}\right\} \qquad [2.e.16]$$

where

$$\left.\begin{array}{llll} A_1 = \frac{1}{60} & , & A_2 = \frac{1}{3} & , \\ H_1 = \frac{1}{120} & , & H_2 = \frac{1}{6} & , \\ B_1 = \frac{1}{210} & , & B_2 = \frac{2}{15} & , \\ P_1 = \frac{1}{40}\beta & , & P_2 = -\frac{1}{3}\beta & , \\ Q_1 = \frac{1}{60}\beta & , & Q_2 = -\frac{1}{6}\beta & , \\ C_1 = -\left(\frac{\ell-a}{a}\right) + \frac{1}{10}\beta^2 & , & C_2 = -Ka + \frac{4}{3}\beta^2 & . \end{array}\right\} \qquad [2.e.17]$$

We shall consider the special case

$$\left.\begin{array}{ll} \ell = 4.02 \text{ ft} & , \\ a = 4 \text{ ft} & , \\ K = 0.0148 \text{ ft} & , \\ \beta = \tan 12° = 0.21256 & . \end{array}\right\} \qquad [2.e.18]$$

Regarding $S_1 = o$ and $S_2 = o$ as two conics the condition upon λ for the quadratic

$$S_1 + \lambda S_2 = o \qquad [2.e.19]$$

to represent a pair of straight lines is

$$233661\lambda^3 - 27702.7\lambda^2 + 1712.08\lambda + 34.6626 = o \qquad [2.e.20]$$

[see, e.g., THE REV.E.H.ASKWITH, D.D., 1953 "Analytical Geometry of the Conic Sections", Third Ed., Adam & Charles Black] which can be solved to give the following roots

$$\lambda \equiv -0.015718 \ , \ 0.067139 \pm 0.070215i \ . \qquad [2.e.21]$$

Using the real value of λ we can write equation [2.e.19] in the form

$$1.1427\xi^2 + 1.1427\xi\eta + 0.2666\eta^2 + 1.2855\xi + 0.8199\eta - 0.0498 = o \qquad [2.e.22]$$

By factorizing equation [2.e.22], we obtain

$$\xi + 0.3708\eta + 1.1625 = o \quad , \qquad [2.e.23]$$

$$\xi + 0.62917\eta - 0.03752 = o \quad . \qquad [2.e.24]$$

The straight line [2.e.23] when combined with S_1 produces

$$\xi = -1.5576 \mp 1.17002i \; , \; \eta = 1.06558 \pm 3.15518i \; , \qquad [2.e.25]$$

in other words there is no real intersection of this straight line with the conic, while the points of intersection between the straight line [2.e.24] and S_1 are real and are as follows:

$$\left.\begin{array}{lll} \text{(i)} & \xi = -0.06258 \; , & \eta = 0.159102 \; , \\ \text{(ii)} & \xi = 0.308311 \; , & \eta = -0.43039 \; . \end{array}\right\} \qquad [2.e.26]$$

Using [2.e.18] we can write [2.e.13] as follows:

$$(n+0.61685) \,/\, E + 2.799 > o \; . \qquad [2.e.27]$$

Using [2.e.26] and [2.e.18] we can write the values of n and E, [2.e.11] and [2.e.12] in the forms

$$\left.\begin{array}{lll} \text{(i)} & n = -0.25358 \; , & E = 0.07042 \; , \\ \text{(ii)} & n = 1.06837 \; , & E = -0.12408 \; . \end{array}\right\} \qquad [2.e.28]$$

The values in [2.e.28(ii)] do not satisfy the sufficiency condition, [2.e.27] but the values in [2.e.28(i)] satisfy [2.e.27], namely

$$n = -0.25358 \quad , \quad E = 0.07042 \quad , \qquad [2.e.29]$$

and thus the appropriate values of ξ and η, are

$$\xi = 0.308311 \quad , \quad \eta = 0.43039 \quad ; \qquad [2.e.30]$$

using [2.e.30] and [2.e.10] we obtain

$$a_1 = 0.01927 \quad , \quad a_2 = -0.006725 \quad . \qquad [2.e.31]$$

Now we can write the solution $z(x)$, [2.e.5] of the differential equation [2.e.1], using [2.e.31] and [2.e.18] as follows:

$$z(x) = 0.07708x-0.032885x^2+0.006725x^3 = y'(x) \quad (o \leq x \leq 4) \ . \qquad [2.e.32]$$

We integrate [2.e.32] with respect to x and we obtain

$$y(x) = 0.03854x^2-0.010962x^3+0.001681 \ , \qquad (o \leq x \leq 4) \ , \qquad [2.e.33]$$

there being no arbitrary constant since

$$y(o) = o \ . \qquad [2.e.34]$$

The graphs of $y(x)$, $y'(x)$ and $y''(x)$ are shown in Figs. 2,3,4.

IIf THE OPTIMUM SHAPE OF A HYDROFOIL USING VARIATIONAL CALCULUS TECHNIQUES, SO THAT THE LIFT IS A MAXIMUM

We pose the problem of maximizing the lift coefficient

$$L^* = \frac{L}{\rho U^2}$$
$$= -\frac{1}{U}\int_o^a \gamma(x)\,dx \ , \qquad [2.f.1]$$

subject to a constraint on the curvature of the form

$$K = \int_o^a z'^2(x)\,dx \ , \qquad [2.f.2]$$

where K is prescribed, together with a constraint on the length of the hydrofoil of the form

$$\ell = \int_o^a \sqrt{1+z^2(x)}\,dx \ , \qquad [2.f.3]$$

where ℓ is prescribed and $z(x) = y'(x)$ is the gradient of the hydrofoil at position x .

STATEMENT OF THE PROBLEM

The general optimum problem considered here may be stated as follows: To find the real, extremal function $\gamma(x)$ of a real variable, required to be Hölder continuous [see, e.g., Tricomi, F.G. (61)] in the region $(o < x < a)$ together with

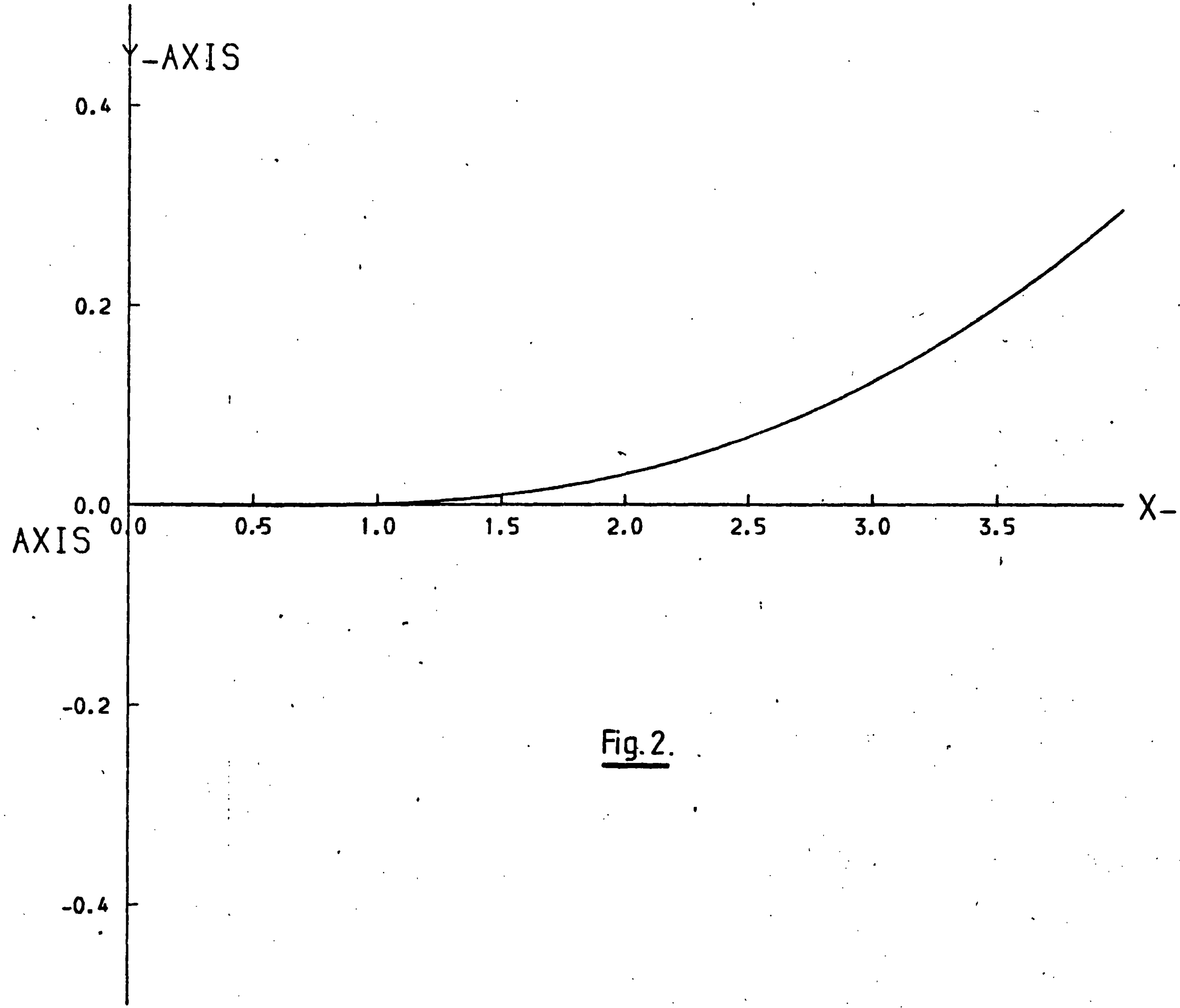

Fig. 2.

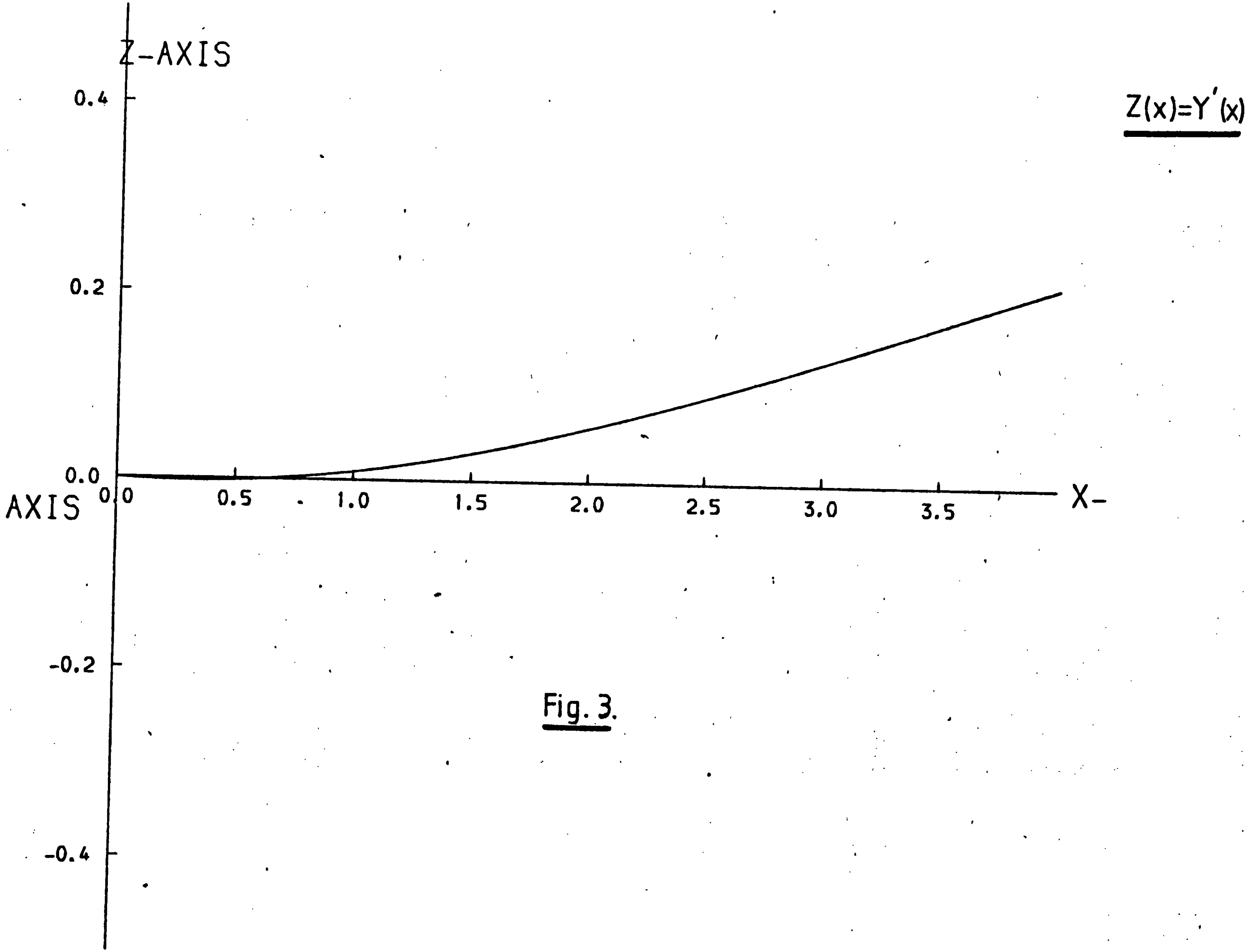

Fig. 3.

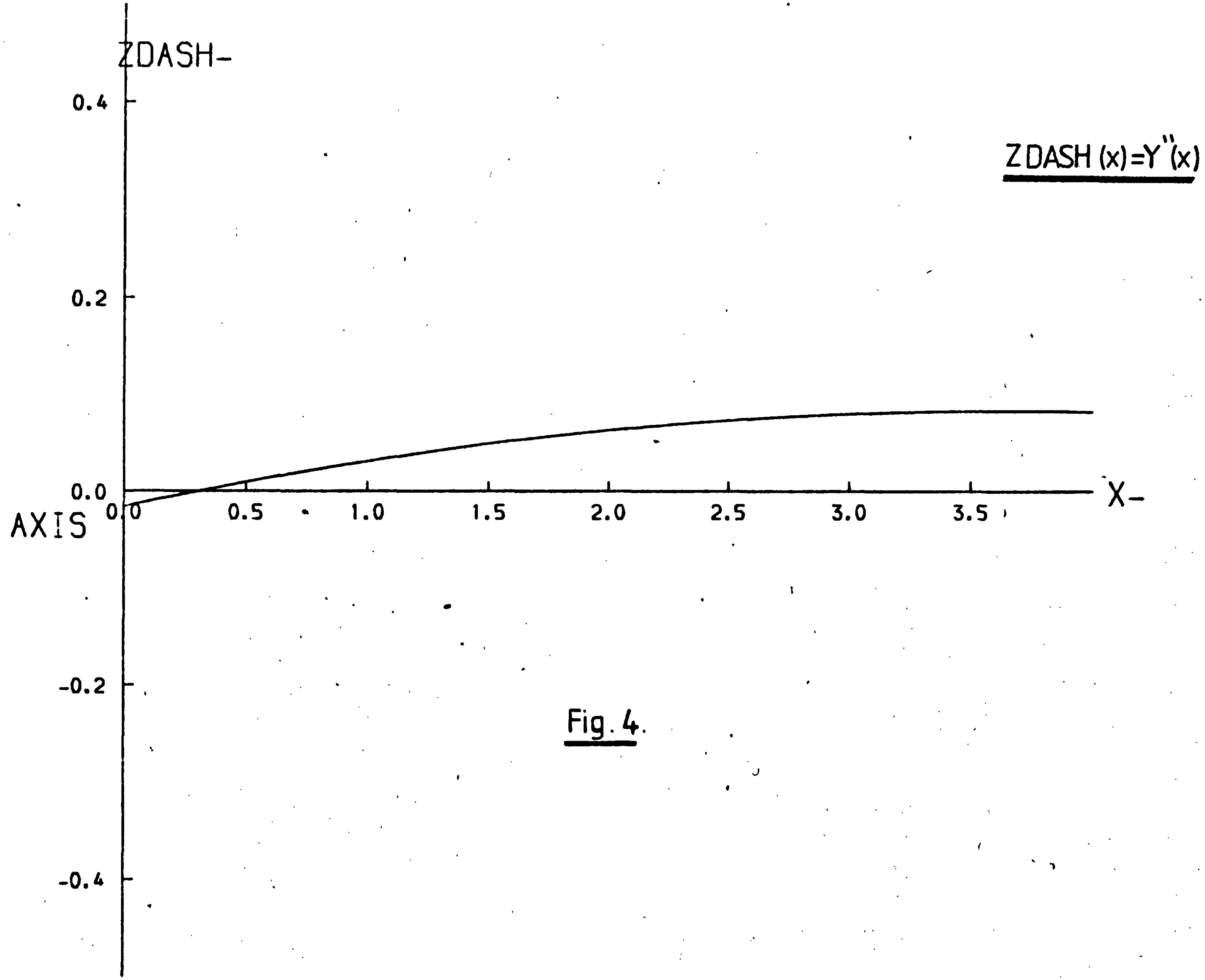

Fig. 4.

$$z(x) = -\frac{1}{2\pi U}\int_o^a \frac{\gamma(s)ds}{s-x} \qquad (o < x < a) \ , \qquad [2.f.4]$$

so that $\gamma(x)$ and $z(x)$ minimize the functional

$$I[\gamma(x),z(x),z'(x),x] = -L^{*}+\lambda_1\ell+\lambda_2 K$$
$$= \int_o^a F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2]dx \ , \qquad [2.f.5]$$

where

$$F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2]=\lambda_1\sqrt{1+z^2(x)}+\lambda_2 z'^2(x) + \frac{1}{U}\gamma(x) \ , \qquad [2.f.6]$$

and $\gamma(x),z(x)$ are related by [2.f.4] and λ_1,λ_2 are Lagrange multipliers.

We define an admissible function as any function $\gamma(x)$ which satisfies the Hölder condition $\mathcal{U}(\mu<1)$ and the constraints [2.f.2] and [2.f.3], and we assume that the optimal function is an admissible function which minimizes the function $I[\gamma,z,z',x]$.

The solution of [2.f.4] satisfying the Kutta condition

$$\gamma(a) = o \qquad [2.f.7]$$

is well known and is given by

$$\gamma(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\oint_o^a \sqrt{\frac{s}{a-s}}\ \frac{z(s)ds}{s-x} \qquad (o<x<a) \qquad [2.f.8]$$

THE NECESSARY CONDITION OF OPTIMALITY

Let $\gamma(x),z(x)$ denote the required optimal vortex distribution function and optimal hydrofoil slope respectively, we write

$$\left.\begin{aligned} \gamma_1(x) &= \gamma(x) + \varepsilon\xi(x) \ , \\ z_1(x) &= z(x) + \varepsilon\eta(x) \ . \end{aligned}\right\} \qquad [2.f.9]$$

We can use [2.f.8] to obtain the following relation between $\xi(x)$ and $\eta(x)$

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\oint_o^a\sqrt{\frac{s}{a-s}}\cdot\frac{\eta(s)ds}{s-x} \ . \qquad [2.f.10]$$

If $\xi(x)$, is an admissible variation, then $I[\gamma(x)+\varepsilon\xi(x),z(x)+\varepsilon\eta(x),z'(x)+\varepsilon\eta'(x),x]$ in [2.f.5] is a function of ε which has an extreme value when $\varepsilon=o$.

For sufficiently small ε , expansion of [2.f.5] in a Taylor series yields

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{2!}\delta^2 I + \ldots \quad . \qquad [2.f.11]$$

We have

$$\Delta I = \int_0^a F[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x]dx - \int_0^a F[\gamma, z, z', x]dx , \qquad [2.f.12]$$

where

$$\delta I = \int_0^a \{\xi(x)F_\gamma(\gamma,z,z',x)+\eta(x)F_z(\gamma,z,z',x)+\eta'(x)F_{z'}(\gamma,z,z',x)\ dx , \qquad [2.f.13]$$

in which the sub-indices denote partial derivatives; it may be noted that ξ and η are related by [2.f.10]. The variations δI, $\delta^2 I,\ldots$ depend on $\xi(x)$ as well as $\gamma(x)$.

We integrate by parts the third term in [2.f.13] and we obtain

$$\delta I=[\eta(x).F_{z'}(\gamma,z,z',x)]_0^a + \int_0^a \{\xi(x)F_\gamma(\gamma,z,z',x)+\eta(x)[F_z(\gamma,z,z',x)-\frac{d}{dx}F_{z'}(\gamma,z,z',x)]\}dx. \qquad [2.f.14]$$

Substituting from [2.f.10] into [2.f.14] we obtain

$$\delta I=[\eta(x).F_{z'}(\gamma,z,z',x)]_0^a + \int_0^a \left\{[F_z(\gamma,z,z',x)-\frac{d}{dx}F_{z'}(\gamma,z,z',x)]\eta(x) + \right.$$
$$\left. + \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\,F_\gamma(\gamma,z,z',x)\oint_0^a\sqrt{\frac{s}{a-s}}\,\frac{\eta(s)ds}{s-x}\right\}dx \quad . \qquad [2.f.15]$$

It is permissible to interchange the order of the double integral on the right-hand side of [2.f.15] [see, e.g., Hardy, G.H. (35)] and interchange the variables x,t and when we do so we obtain

$$\delta I=[\eta(x).F_{z'}(\gamma,z,z',x)]_0^a + \int_0^a \left\{[F_z(\gamma,z,z',x)-\frac{d}{dx}F_{z'}(\gamma,z,z',x)] - \right.$$
$$\left. - \frac{2U}{\pi}\sqrt{\frac{x}{a-x}}\int_0^a\sqrt{\frac{a-s}{s}}\,\frac{F_\gamma(\gamma,z,z',s)ds}{s-x}\right\}\eta(x)dx \quad . \qquad [2.f.16]$$

We have from [2.f.6]

$$\left.\begin{aligned} F_\gamma[\gamma,z,z',x] &= \frac{1}{U} \quad , \\ F_z[\gamma,z,z',x] &= \frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} \quad , \\ F_{z'}[\gamma,z,z',x] &= 2\lambda_2 z'(x) \quad . \end{aligned}\right\} \qquad [2.f.17]$$

Substituting from [2.f.17] in [2.f.16] we obtain

$$\delta I=[2\lambda_2\eta(x)z'(x)]_o^a+\int_o^a\left\{\frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}}-2\lambda_2 z''(x)-\frac{2}{\pi}\sqrt{\frac{x}{a-x}}\oint_o^a\sqrt{\frac{a-s}{s}}\frac{ds}{(s-x)}\right\}\eta(x)\,dx. \quad [2.f.18]$$

For $I[z]$ to be a minimum, we must have for all admissible function, $\eta(x)$

$$\delta I[z,\eta] = o \quad , \qquad [2.f.19]$$

and this implies that the coefficient of $\eta(x)$ in [2.f.18] should vanish that is

$$2\lambda_2 z''(x) - \lambda_1\frac{z(x)}{\sqrt{1+z^2(x)}} = -\frac{2}{\pi}\sqrt{\frac{x}{a-x}}\oint_o^a\sqrt{\frac{a-s}{s}}\frac{ds}{s-x}$$
$$= 2\sqrt{\frac{x}{a-x}} \quad , \qquad [2.f.20]$$

while at the end points it is necessary that

$$\eta(o).z'(o) = o \quad , \quad \eta(a).z'(a) = o \qquad [2.f.21]$$

be satisfied. In all the examples considered subsequently $z(x)$ is postulated at $x=o$ and $x=a$ and this implies that

$$\eta(o) = \eta(a) = o \ . \qquad [2.f.22]$$

Equation [2.f.20] is a nonlinear differential equation for $z(x)$.

We consider the solution of [2.f.20] for the slope $z(x)$ only in the case of small slope, and we approximate to [2.f.20] as follows:

$$z''(x) - n\,z(x) = E\sqrt{\frac{x}{a-x}} \quad , \quad \left(n = \frac{\lambda_1}{2\lambda_2}\ , \ E = \frac{1}{\lambda_2}\ \&\ \lambda_2 \neq o\right),\ (o < x < a). \qquad [2.f.23]$$

It is assumed at this stage that $\frac{\lambda_1}{\lambda_2} < o$ and we show later that $\lambda_1 < o$, $\lambda_2 > o$ are sufficient conditions for a true maximization of L . We write the differential equation in the form

$$z''(x) + m^2 z(x) = E\sqrt{\frac{x}{a-x}} \quad , \quad \left(m^2 = -n = -\frac{\lambda_1}{2\lambda_2}\right) \ , \ (o < x < a) \qquad [2.f.24]$$

To derive the solution of the nonhomogeneous equation, [2.f.24] we apply the usual method of variation of parameters, then we can write $z(x)$ in the form:

$$z(x) = \frac{E}{m}\int_{o}^{x}\sqrt{\frac{\xi}{a-\xi}}\sin m(x-\xi)d\xi + A\sin mx + B\cos mx \quad , \qquad (o < x < a) \tag{2.f.25}$$

where A and B are arbitrary constants.

Using the boundary conditions

$$z(o) = o \ , \quad z(a) = \beta \quad , \tag{2.f.26}$$

we obtain

$$\left.\begin{aligned} A &= -\frac{E}{m\sin ma}\int_{o}^{a}\sqrt{\frac{\xi}{a-\xi}}\ \sin m(a-\xi)d\xi + \beta\ \text{cosec}\ ma \ , \\ B &= o \ . \end{aligned}\right\} \tag{2.f.27}$$

Substituting from [2.f.27] into [2.f.25] we obtain

$$z(x)=y'(x) = \frac{E}{m}\int_{o}^{x}\sqrt{\frac{\xi}{a-\xi}}\sin m(x-\xi)d\xi - \frac{E\sin mx}{m\sin ma}\int_{o}^{a}\sqrt{\frac{\xi}{a-\xi}}\sin m(a-\xi)d\xi + \beta\frac{\sin mx}{\sin ma}$$

$$(o < x < a) \tag{2.f.28}$$

We integrate [2.f.28] with respect to x, and use the boundary condition $y(o)=o$ to obtain

$$y(x) = \frac{E}{m}\int_{o}^{x}d\sigma\int_{o}^{\sigma}\sqrt{\frac{\xi}{a-\xi}}\sin m(x-\xi)d\xi + \frac{E}{m}\frac{(\cos mx-1)}{\sin ma}\int_{o}^{\sigma}\sqrt{\frac{\xi}{a-\xi}}\sin m(a-\xi)d\xi - \beta\frac{(\cos mx-1)}{m\sin ma}$$

$$(o < x < a) \tag{2.f.29}$$

Equation [2.f.29] can be written as follows when the order of integration of the double integral is inverted:

$$y(x) = \frac{E}{m}\int_{o}^{x}d\xi\int_{\xi}^{x}\sqrt{\frac{\xi}{a-\xi}}\sin m(\sigma-\xi)d\sigma + \frac{E(\cos mx-1)}{m^2\sin ma}\int_{o}^{a}\sqrt{\frac{\xi}{a-\xi}}\sin m(a-\xi)d\xi - \beta\frac{(\cos mx-1)}{m\sin ma}$$

$$= -\frac{E}{m^2}\int_{o}^{x}\sqrt{\frac{\xi}{a-\xi}}[\cos m(x-\xi)-1]d\xi + \frac{E(\cos mx-1)}{m^2\sin ma}\int_{o}^{a}\sqrt{\frac{\xi}{a-\xi}}\sin m(a-\xi)d\xi - \frac{(\cos mx-1)}{m\sin ma}.\beta$$

$$(o < x < a) \tag{2.f.30}$$

When we substitute for $z(x)$ and $z'(x)$, using [2.f.28] into the constraints [2.f.2] and [2.f.3] we obtain two equations, in the two unknowns E,m, which have to be evaluated numerically.

We do not complete the solution of this problem using this method since there are two alternative methods of resolving the problem numerically which are discussed in detail in section IIh and Appendix IX.

IIg THE SUFFICIENT CONDITION FOR THE EXTREMUM TO BE A MINIMUM

A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I

Since

$$\delta I[\gamma(x), z(x), z'(x), x] = 0 \qquad [2.g.1]$$

the condition for I to be a minimum requires that

$$\delta^2 I[\gamma(x), z(x), z'(x), x] > 0 \qquad (0<x<a), \qquad [2.g.2]$$

for all admissible variations $\xi(x)$ and $\eta(x)$ consistent with

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}} \oint_0^a \sqrt{\frac{s}{a-s}} \cdot \frac{\eta(s)\,ds}{s-x} \qquad (0<x<a), \qquad [2.g.3]$$

where $\eta(x)$ satisfies the boundary conditions

$$\eta(0) = 0 \quad , \quad \eta(a) = 0 \qquad [2.g.4]$$

Using Taylor's theorem we can write the increment of the functional $I(\gamma, z, z', x)$ in the form

$$\begin{aligned} & I[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x] - I[\gamma, z, z', x] \\ & = \varepsilon \int_0^a \{\xi(x) F_\gamma(\gamma, z, z', x) + \eta(x)[F_z(\gamma, z, z', x) - \frac{d}{dx} F_{z'}(\gamma, z, z', x)]\} dx + \\ & \frac{1}{2}\varepsilon^2 \int_0^a \{\xi^2(x) F_{\gamma\gamma}(\gamma, z, z', x) + \eta^2(x) F_{zz}(\gamma, z, z', x) + \eta'^2(x) F_{z'z'}(\gamma, z, z', x) + \\ & 2\xi(x)\eta(x) F_{\gamma z}(\gamma, z, z', x) + 2\xi(x)\eta'(x) F_{\gamma z'}(\gamma, z, z', x) + 2\eta(x)\eta'(x) F_{zz'}(\gamma, z, z', x)\} dx + 0(\varepsilon^3) \end{aligned}$$

$$(0<x<a) \qquad [2.g.5]$$

Denoting the coefficient ε by δI and that of ε^2 by $\delta^2 I$, at a stationary value of I, we have from [2.g.1], [2.g.3] and [2.g.5]

$$F_z(\gamma,z,z',x) - \frac{d}{dx} F_{z'}(\gamma,z,z',x) = \frac{2U}{\pi}\sqrt{\frac{x}{a-x}} \int_0^a \sqrt{\frac{a-s}{s}} \cdot \frac{F_\gamma(\gamma,z,z',x)\,ds}{s-x} , \quad [2.g.6]$$

and

$$\delta^2 I = \int_0^a \{\xi^2 F_{\gamma\gamma} + \eta^2 F_{zz} + \eta'^2 F_{z'z'} + 2\xi\eta F_{\gamma z} + 2\xi\eta' F_{\gamma z'} + 2\eta\eta' F_{zz'}\}\,dx , \quad [2.g.7]$$

where by [2.f.6] we have

$$\left.\begin{aligned}
F_{\gamma\gamma}[\gamma,z,z',x] &= 0 , \\
F_{zz}[\gamma,z,z',x] &= \frac{\lambda_1}{[1+z^2(x)]^{3/2}} , \\
F_{z'z'}[\gamma,z,z',x] &= 2\lambda_2 , \\
F_{\gamma z}[\gamma,z,z',x] &= 0 , \\
F_{\gamma z'}[\gamma,z,z',x] &= 0 , \\
F_{zz'}[\gamma,z,z',x] &= 0 .
\end{aligned}\right\} \quad [2.g.8]$$

Substituting from [2.g.8] in [2.g.7] we obtain

$$\delta^2 I = \int_0^a \left\{ \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \cdot \eta^2(x) + 2\lambda_2 \eta'^2(x) \right\} dx . \quad [2.g.9]$$

In the case of small slope $z(x)$, we approximate [2.g.9] as follows:

$$\delta^2 I = \int_0^a \{\lambda_2 . \eta^2(x) + 2\lambda_2 . \eta'^2(x)\}\,dx . \quad [2.g.10]$$

We now consider the special choice of $\eta(x)$ satisfying $\eta(o)=\eta(a)=o$, namely:

$$\eta(x) = \alpha \sin \frac{\pi}{a} x , \qquad (o < x < a) , \quad [2.g.11]$$

then we obtain from [2.g.10]:

$$\delta^2 I = \frac{1}{2}\alpha^2 a [\lambda_1 + \frac{2\pi^2}{a^2}\lambda_2] . \quad [2.g.12]$$

A sufficient condition for satisfying [2.g.2] is

$$\lambda_1 + \frac{2\pi^2}{a^2}\lambda_2 > o . \quad [2.g.13]$$

IIh. ANALYTICAL SOLUTION BY THE RAYLEIGH-RITZ METHOD

We use the Rayleigh-Ritz method [see, e.g. Temple, G. and Bickley, W.G. (58) and Milne, W.E. (44)], to solve equation [2.f.25], namely:

$$z''(x) - n\, z(x) = E\sqrt{\frac{x}{a-x}} \quad , \quad (n = \frac{\lambda_1}{2\lambda_2} \;,\; E = \frac{1}{\lambda_2}) \quad , \qquad [2.h.1]$$

where λ_1, λ_2 are Lagrange multipliers. and $z(x)$ is subject to the boundary conditions [2.f.26]. Equation [2.h.1] is the necessary condition for the integral

$$J = \int_o^a \left\{\frac{1}{2} z'^2(x) + \frac{1}{2} n\, z^2(x) + E\sqrt{\frac{x}{a-x}} \cdot z(x)\right\} dx \;, \qquad [2.h.2]$$

to be minimized.

The Rayleigh-Ritz method can be applied to this problem in the following way:

We select a basic set of linearly independent polynomial functions and we

assume an expression for $z(x)$ of the form

$$z(x) = \frac{\beta}{a^2}\, x^2 + a_1 x(a-x) + a_2 x^2 (a-x) \qquad (o \leqslant x \leqslant a) \quad , \qquad [2.h.3]$$

which satisfies the end conditions [2.f.36], a_1 and a_2 being arbitrary constants.

The values of $z(x)$ and $z'(x)$ are obtained from equation [2.h.3] and are substituted in [2.h.2]; the result is a quadratic form in a_1 and a_2, namely

$$J = \frac{2}{3}\frac{\beta^2}{a} + \frac{1}{6} a^3 a_1^2 + \frac{1}{15} a^5 a_2^2 + \frac{1}{6} a_1 a_2 a^4 - \frac{1}{3}\beta a_1 a - \frac{1}{6}\beta a_2 a + \frac{1}{10} n\,\beta^2 + \frac{1}{60} n a_1^2 a^5 +$$

$$+ \frac{1}{210} n a_2^2 a^7 + \frac{1}{20} n a_1 \beta a^3 + \frac{1}{30} n a_2 \beta a^4 + \frac{1}{60} a_1 a_2 n a^6 + \frac{5\pi\beta a}{16} E +$$

$$+ \frac{5\pi a^3 a_1}{16} E + \frac{5\pi a^4 a_2}{128} E = o \;. \qquad [2.h.4]$$

The necessary conditions for minimizing J , with respect to a_1 and a_2 are

$$\frac{\partial J}{\partial a_1} = \frac{a_1 a^3}{30}\,[10+na^2] + \frac{a_2 a^4}{60}\,[10+na^2] - \frac{1}{60}\,\beta a[20-3na^2] + \frac{\pi a^3}{16}\,E = o\ , \qquad [2.h.5]$$

and

$$\frac{\partial J}{\partial a_2} = \frac{a_1 a^4}{60}\,[10+na^2] + \frac{a_2 a^5}{105}\,[14+na^2] - \frac{1}{30}\,\beta\, a^2[5-na^2] + \frac{5\pi a^4}{128}\,E = o\ . \qquad [2.h.6]$$

Using [2.h.5] and [2.h.6] the quantities n and E can now be expressed in terms of a_1 and a_2 , but for convenience we introduce

$$\left.\begin{aligned} \xi &= a_1 a^2 \quad , \\ \eta &= a_2 a^3 \quad , \end{aligned}\right\} \qquad [2.h.7]$$

and we then have

$$n = W/V \quad , \qquad [2.h.8]$$

with

$$\left.\begin{aligned} W &= (\tfrac{5}{24} - \tfrac{1}{6})\xi + (\tfrac{5}{48} - \tfrac{2}{15})\eta - (\tfrac{5}{24} - \tfrac{1}{6})\beta \quad , \\ V &= [(\tfrac{1}{48} - \tfrac{1}{60})\xi + (\tfrac{1}{96} - \tfrac{1}{105})\eta + (\tfrac{1}{32} - \tfrac{1}{30})\beta\ ]a^2 \quad , \end{aligned}\right\} \qquad [2.h.9]$$

and

$$E = -\frac{16}{\pi a^2}\left[\frac{\xi}{30}\,(10+na^2) + \frac{\eta}{60}\,(10+na^2) - \frac{1}{60}\,\beta\,(20-3na^2)\right]\ . \qquad [2.h.10]$$

From [2.g.14] it follows that the sufficient condition for the lift to be a maximum can be expressed in the form:

$$[n + \frac{\pi^2}{a^2}]\ /\ E\ > o \quad , \quad (E = \frac{1}{\lambda_2}\ ,\ n = \frac{\lambda_1}{2\lambda_2})\ . \qquad [2.h.11]$$

Substituting from [2.h.3] in the constraints, [2.f.2] and [2.f.3] we obtain

$$\ell = \int_o^a [1 + \frac{1}{2}\,z^2(x)]dx$$

$$= a + \frac{1}{10}\,\beta^2\, a + \frac{1}{60}\, a_1^2 a^5 + \frac{1}{210}\, a^7 a_2^2 + \frac{1}{20}\,\beta\, a^3 a_1 + \frac{1}{30}\,\beta\, a^4 a_2 + \frac{1}{60}\, a_1 a_2 a^6\ , \qquad [2.h.12]$$

and

$$K = \int_o^a z'^2(x)\,dx$$

$$= \frac{4}{3}\,\frac{\beta^2}{a} + \frac{1}{3}\, a^3 a_1^2 + \frac{2}{15}\, a^5 a_2^2 - \frac{2}{3}\,\beta\, a_1 a - \frac{1}{3}\,\beta\, a_2 a^2 + \frac{1}{3}\, a_1 a_2 a^4\ . \qquad [2.h.13]$$

Equation [2.h.12] and [2.h.13] can be written as follows:

$$\left.\begin{aligned} S_1 &= A_1\xi^2+2H_1\xi\eta+B_1\eta^2+2P_1\xi+2Q_1\eta+C_1 = o \quad , \\ S_2 &= A_2\xi^2+2H_2\xi\eta+B_2\eta^2+2P_2\xi+2Q_2\eta+C_2 = o \quad . \end{aligned}\right\} \qquad [2.h.14]$$

where

$$\left.\begin{aligned} A &= \frac{1}{60} \quad , & A_2 &= \frac{1}{3} \quad , \\ H_1 &= \frac{1}{120} \quad , & H_2 &= \frac{1}{6} \quad , \\ B_1 &= \frac{1}{210} \quad , & B_2 &= \frac{2}{15} \quad , \\ P_1 &= \frac{1}{40}\beta \quad , & P_2 &= -\frac{1}{3}\beta \quad , \\ Q_1 &= \frac{1}{60}\beta \quad , & Q_2 &= -\frac{1}{6}\beta \quad , \\ C_1 &= -\left(\frac{\ell-a}{a}\right)+\frac{1}{10}\beta^2, & C_2 &= -Ka+\frac{4}{3}\beta^2 \quad . \end{aligned}\right\} \qquad [2.h.15]$$

We shall consider the special case

$$\left.\begin{aligned} \ell &= 4.02 \text{ ft} \quad , \\ a &= 4 \text{ ft} \quad , \\ K &= 0.0148 \text{ ft} \quad , \\ \beta &= -\tan 12 = -0.21256 \quad . \end{aligned}\right\} \qquad [2.h.16]$$

Regarding $S_1 = o$ and $S_2 = o$ as two conics the condition upon λ for the quadratic

$$S_1 + \lambda S_2 = o \qquad [2.h.17]$$

to represent a pair of straight lines is

$$233661\lambda^3-27702.7\lambda^2+1712.08\lambda+34.663=o \quad , \qquad [2.h.18]$$

[see, e.g., THE REV. E.H. ASKWITH, D.D., 1953, "Analytical Geometry of the conic sections", third ed. Adam & Charles Black.] which can be solved to give the following roots

$$\lambda \equiv -0.01572,\ 0.067139 \pm 0.070215i \quad . \qquad [2.h.19]$$

Using the real value of λ we can write equation [2.e.17] in the form

$$1.1427\xi^2+1.1427\xi\eta+0.2666\eta^2-1.2855\xi-0.8199\eta-0.04983=o \ . \qquad [2.h.20]$$

By factorizing equation [2.h.20], we obtain

$$\xi + 0.3708\eta-1.1625 = o \quad , \qquad [2.h.21]$$

$$\xi + 0.6292\eta+0.0375 = o \quad . \qquad [2.h.22]$$

The straight line [2.h.21] when combined with $S_1 = o$ produces

$$\xi = 1.5576 \mp 1.17002i,\ \eta=-1.06656 \pm 3.15517i \ , \qquad [2.h.23]$$

in other words there is no real intersection of this straight line with the conic; the points of intersection between the straight line [2.h.22] and S_1 are real and are as follows:

$$\left.\begin{array}{llll} \text{(i)} & \xi = -0.30834 & , \quad \eta = 0.43045 & , \\ \text{(ii)} & \xi = 0.062619 & , \quad \eta = -0.15915 & . \end{array}\right\} \qquad [2.h.24]$$

Using [2.h.16] and [2.h.24] we can write the values of n and E, [2.h.8] and [2.h.10] in the forms

$$\left.\begin{array}{llll} \text{(i)} & n = -2.2599 & , \quad E = -0.17072 & , \\ \text{(ii)} & n = -1.7925 & , \quad E = -0.12293 & . \end{array}\right\} \qquad [2.h.25]$$

We find that both values of n and E in [2.h.25] satisfy the sufficient condition [2.h.11], but the values

$$n = -2.2599 \quad , \quad E = -0.17072 \quad , \qquad [2.h.26]$$

actually provide the maximum value of lift, namely

$$L = 121260 \ \text{lbs} \quad . \qquad [2.h.27]$$

Thus the appropriate values of ξ and η, are

$$\xi = -0.30834 \quad , \quad \eta = 0.43045 \quad ; \qquad [2.h.28]$$

using [2.h.28] and [2.h.7] we obtain

$$a_1 = -0.01927 \quad , \quad a_2 = 0.006726 \quad . \qquad [2.h.29]$$

Now we can write the solution $z(x)$, [2.h.3] of the differential equarion [2.h.1] , using [2.h.9] and [2.h.16] as follows:

$$z(x) = -0.07708x+0.05946x^2-0.006726x^3, \qquad (o \leq x \leq 4) \qquad [2.h.30]$$

We integrate [2.h.30] with respect to x and we obtain

$$y(x) = -0.03854x^2+0.01982x^3-0.001682x^4, \qquad (o \leq x \leq 4) \qquad [2.h.31]$$

there being no arbitrary constant since

$$y(o) = o \quad . \qquad [2.h.32]$$

The graphs of $y(x)$, $y'(x)$ and $y''(x)$ are shown in Figs.5,6 and 7 respectively.

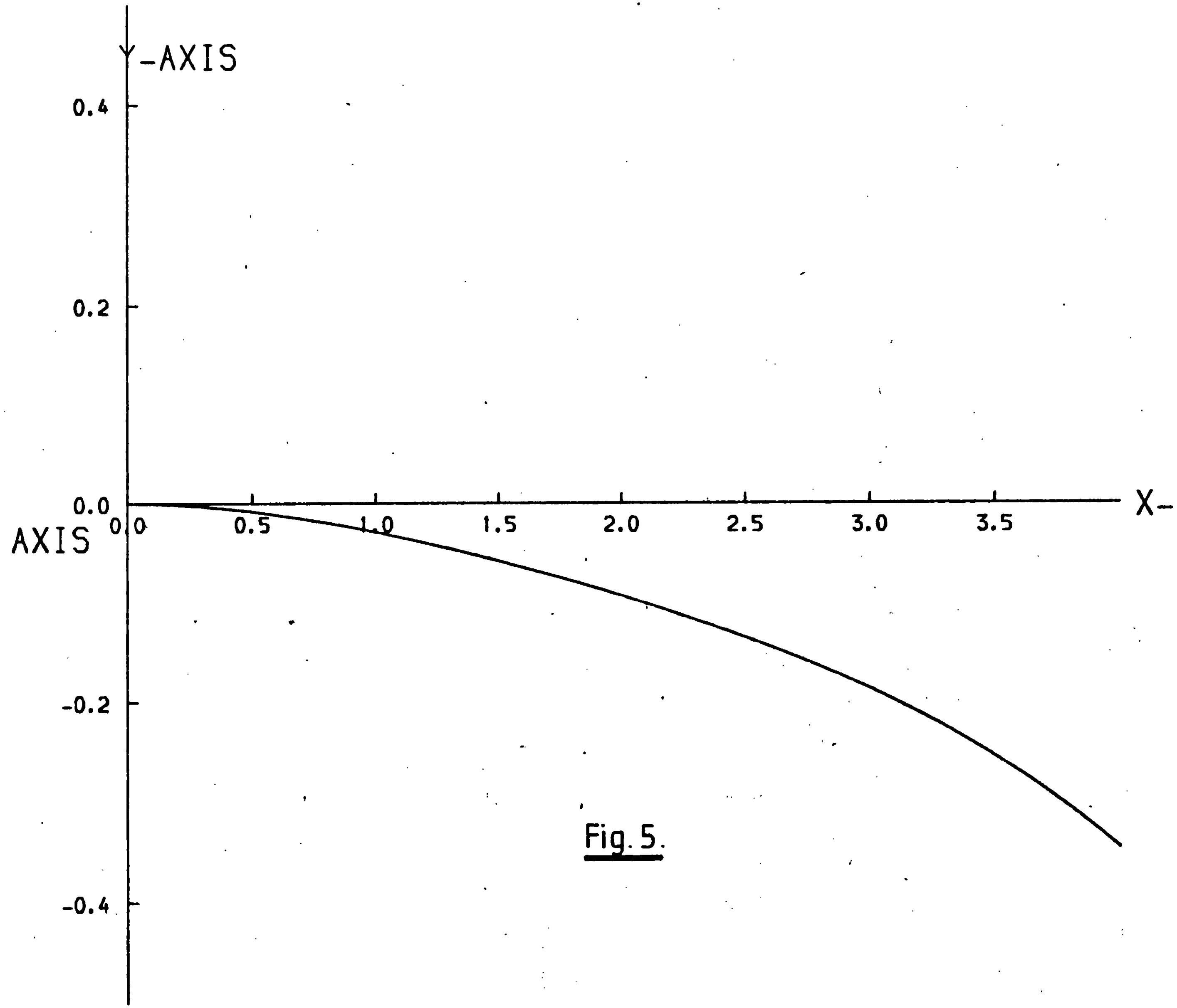

Fig. 5.

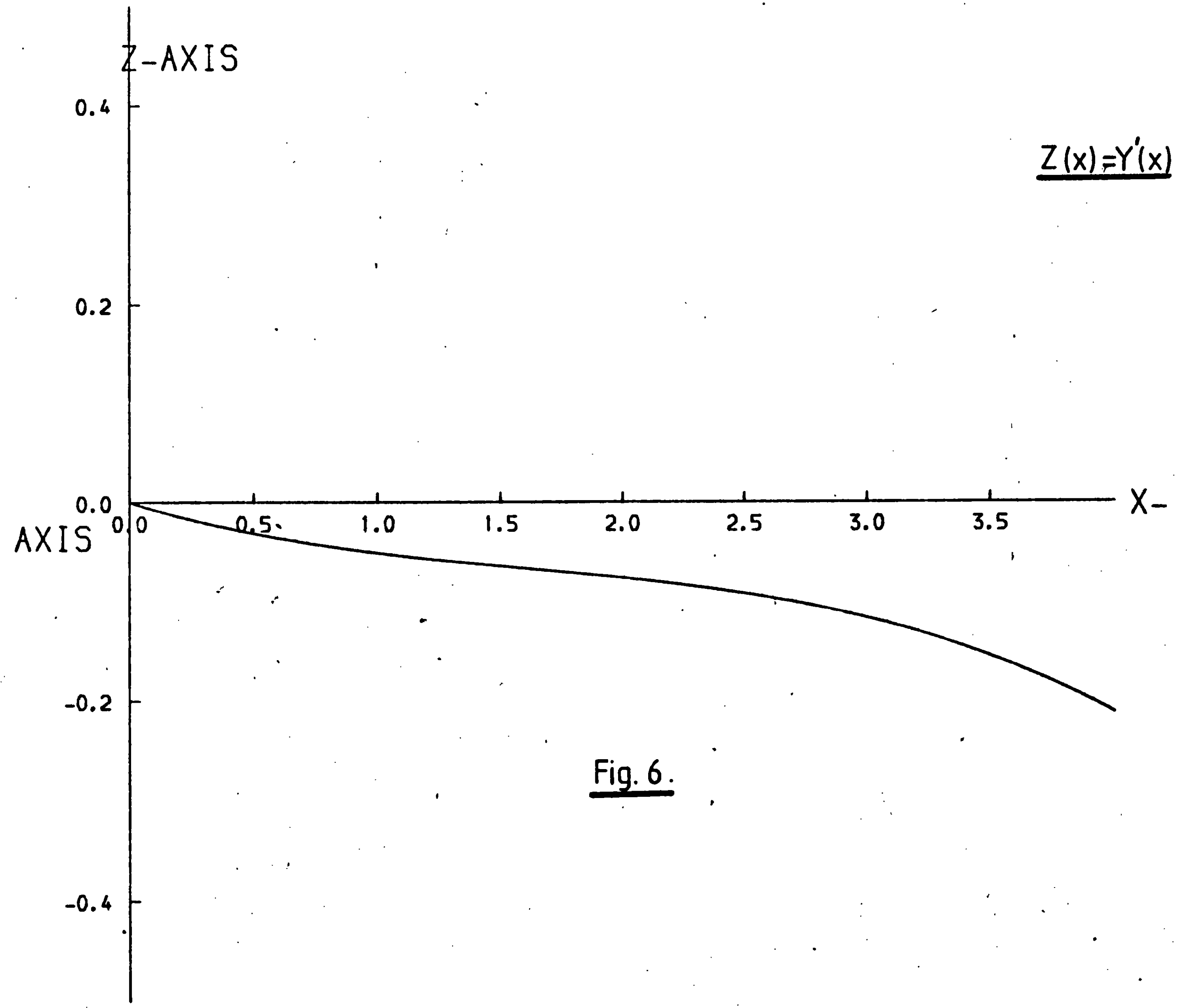

Fig. 6.

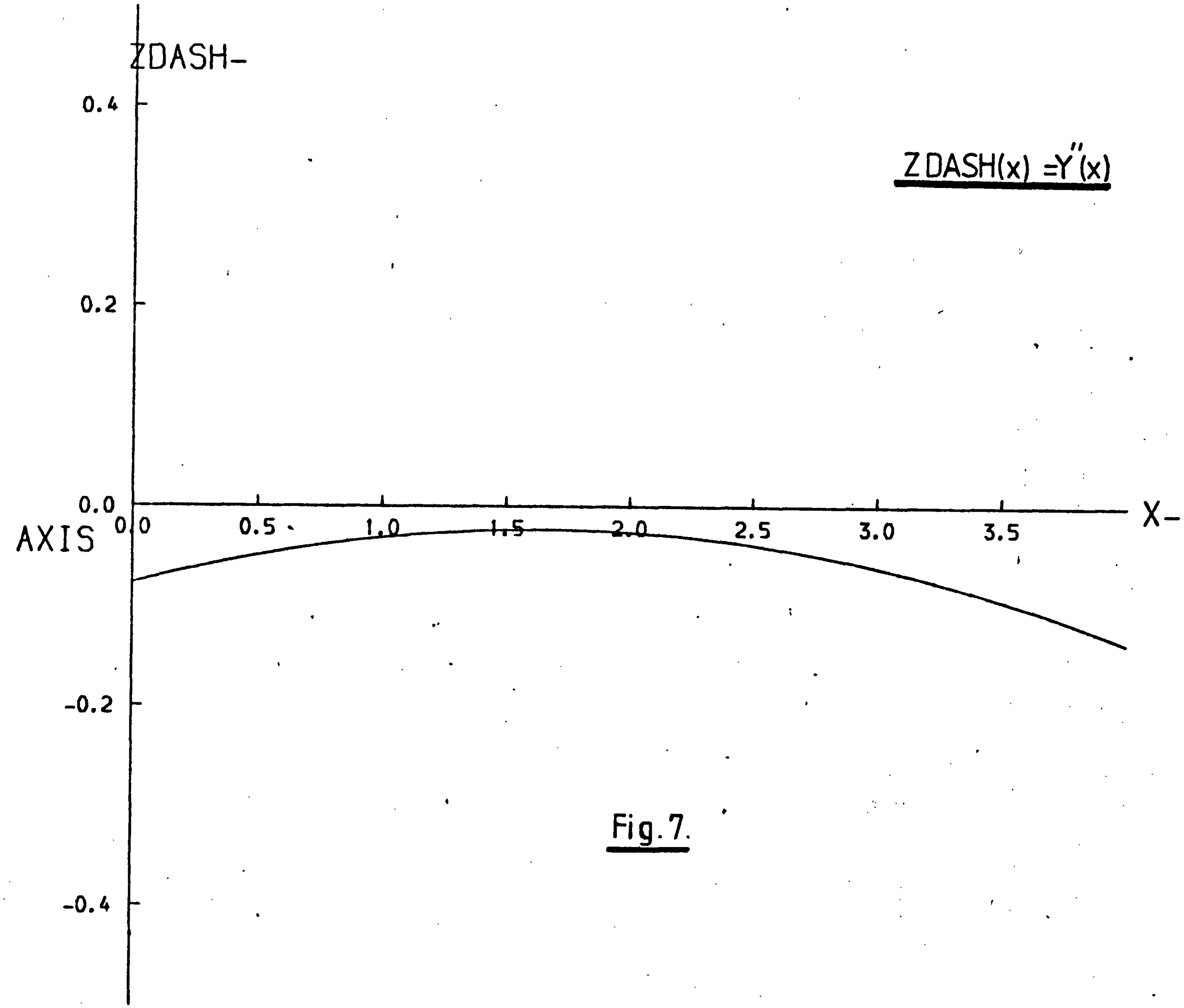

Fig. 7.

IIa THE OPTIMUM SHAPE OF A HYDROFOIL IN STEADY TWO DIMENSIONAL FULL CAVITY FLOW

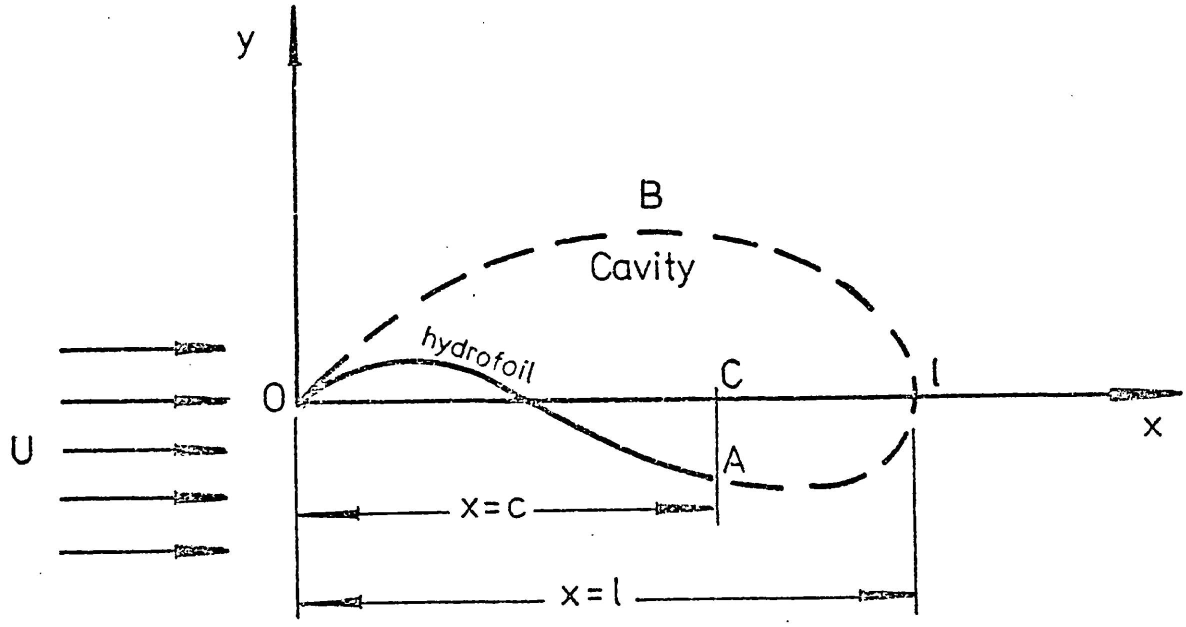

FIG. 8.

INTRODUCTION

In a recent paper Davies,T.V.(13) the author studied the problem of the liquid flow past a sharp edged hydrofoil with a finite cavity using the linearization hypothesis.

We assume that the problem is stated as follows:

Davies assumes that, in a flow of an incompressible liquid past a hydrofoil, a constant pressure cavity develops behind the hydrofoil, it is also assumed that the cavity is of finite length as shown in the diagram (Fig.8).

The method of linearization is used to solve the flow problem.

The hydrofoil is represented by a source-vortex distribution and the problem is resolved by distributing

(a) Sources of strength $m_1(\xi)$ per unit length in $0<\xi<c$,

(b) Sources of strength $m_2(\xi)$ per unit length in $c<\xi<\ell$,

(c) Vortices of strength $\gamma(\xi)$ per unit length in $0<\xi<c$ ($\gamma>0$, clockwise) along the x-axis to replace the above physical configurations.

This problem reduces to three coupled equations as follows:

$$\left.\begin{aligned}
&\int_o^c \frac{\gamma(\xi)d\xi}{\xi-x} = 2\pi U y'(x)+\pi m_1(x) && (o<x<c) \quad ,\\
&-\pi\gamma(x)+\int_o^c \frac{m_1(\xi)d\xi}{\xi-x}+\int_c^\ell \frac{m_2(\xi)d\xi}{\xi-x} = -\pi U\sigma && (o<x<c) \quad ,\\
&\int_o^c \frac{m_1(\xi)d\xi}{\xi-x}+\int_c^\ell \frac{m_2(\xi)d\xi}{\xi-x} = -\pi U\sigma && (c<x<\ell) \quad ,
\end{aligned}\right\} \qquad [3.a.1]$$

where U is the uniform stream at infinity, $\sigma\left(\equiv \frac{P_\infty - P_c}{\frac{1}{2}\rho U^2}\right)$ cavitation number and $y'(x)$ being the gradient of the hydrofoil at position x.

These equations are solved for m_1 to yield

$$\int_o^c \frac{M_1(\xi)d\xi}{\xi-x}\left\{\sqrt{\frac{\xi}{x}}+\sqrt{\frac{\ell-\xi}{\ell-x}}\right\} = G(x) \qquad (o<x<c) \quad , \qquad [3.a.2]$$

where

$$\left.\begin{aligned}
&M_1(\xi) = \frac{m_1(\xi)}{\sqrt{c-\xi}} \quad ,\\
&G(x) = -\frac{\pi U\sigma}{\sqrt{\ell-x}} - \frac{2U}{\sqrt{x}}\int_o^c \sqrt{\frac{\xi}{c-\xi}}\cdot\frac{y'(\xi)d\xi}{\xi-x} \quad .
\end{aligned}\right\} \qquad [3.a.3]$$

Davies has made the substitutions

$$\left.\begin{aligned}
\xi &= \ell\sin^2\theta \quad ,\\
x &= \ell\sin^2\theta_o \quad ,\\
c &= \ell\sin^2\alpha \quad ,
\end{aligned}\right\} \quad (o<\theta_o<\alpha<\tfrac{\pi}{2}) \qquad [3.a.4]$$

to transform the integral [3.a.2] into a standard Cauchy formula, namely

$$\int_o^\alpha \frac{\nu(\theta)d\theta}{\sin(\theta-\theta_o)} = g(\theta_o) \qquad (o<\theta_o<\alpha) \quad , \qquad [3.a.5]$$

with

$$\left.\begin{aligned}
&g(\theta_o) = -\frac{\pi U\sigma\sin\theta_o}{\sqrt{\ell}} - \frac{4U\cos\theta_o}{\sqrt{\ell}}\int_o^\alpha \frac{z(\psi)\sin^2\psi\cos\psi\, d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}\,.\,(\sin^2\psi-\sin^2\theta_o)} \quad ,\\
&\nu(\theta) = \frac{2\mu(\theta)\sin\theta\cos\theta}{\sqrt{\ell}\,.\,\sqrt{\sin^2\alpha-\sin^2\theta}} \qquad [\mu(\theta)=m_1(\ell\sin^2\theta)] \quad ,
\end{aligned}\right\} \qquad [3.a.6]$$

where $z(x) = y'(x)$.

This equation is solved to give

$$\mu(\theta) = \frac{C\sqrt{\ell}\ \sqrt{\sin^2\alpha-\sin^2\theta}}{2\sin\theta\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}} -$$

$$-\frac{1}{2\pi^2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \frac{\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\cos\theta_o\sin(\theta_o-\theta)}\left\{-\pi U\sigma\sin\theta_o-4U\cos\theta_o\int_0^\alpha\frac{\sin^2\psi\cos\psi z(\psi)d\psi}{(\sin^2\psi-\sin^2\theta_o)\sqrt{\sin^2\alpha-\sin^2\psi}}\right\}$$

[3.a.7]

where C can be calculated, using the closure condition [see ref.(13) p.204, Davies, T.V., "Steady two-dimensional cavity flow past an aerofoil using linearized theory"], to be

$$C = \frac{1}{\pi^2}\int_0^\alpha \frac{g(\theta_o)}{\cos\theta_o}\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,d\theta_o\ , \qquad [3.a.8]$$

This can be written as follows:

$$C = -\frac{U\sigma}{\sqrt{\ell}}\left[\cos\frac{1}{2}\alpha - \sqrt{\cos\alpha}\right] - \frac{2U}{\pi\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)d\theta \qquad [3.a.9]$$

[see, Appendix VI].

Substituting from [3.a.9] in [3.a.7] we obtain

$$\mu(\theta) = -\frac{U\sigma}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\left[\cos\frac{1}{2}\alpha - \sqrt{\cos\alpha}\right] -$$

$$-\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\sqrt{\frac{\sin\theta_o}{\sin(\alpha-\theta_o)}}\,z(\theta_o)d\theta_o +$$

$$+\frac{1}{2\pi^2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\frac{\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\cos\theta_o\sin(\theta_o-\theta)}\left\{\pi U\sigma\sin\theta_o +\right.$$

$$\left.+\ 4U\cos\theta_o\int_0^\alpha\frac{\sin^2\psi\cos\psi\ \ z(\psi)d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}.(\sin^2\psi-\sin^2\theta_o)}\right\},\ (o<\theta<\alpha<\frac{\pi}{2})\ . \qquad [3.a.10]$$

Using linearized theory, Davies has derived the hydrodynamic forces which act on the hydrofoil in the form

$$L = -4\rho U^2\ell\int_0^\alpha\frac{\sin^2\theta\cos\theta z(\theta)d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} - 2\rho U\ell\int_0^\alpha\frac{\mu(\theta)\sin^2\theta\cos\theta d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}}\ ,\ [z(\theta)=y(\ell\sin^2\theta)]\ ,$$

[3.a.11]

and

$$D = 4\rho U^2\ell\int_0^\alpha\frac{\sin^2\theta\cos\theta z^2(\theta)d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} + 2\rho U\ell\int_0^\alpha\frac{\mu(\theta)z(\theta)\sin^2\theta\cos\theta d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}}\ , \qquad [3.a.12]$$

where ρ is the constant density of the liquid.

The purpose of this work is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces maximum lift. We use variational calculus techniques to obtain the optimum shape of hydrofoil. The mathematical problem is that of extremizing a functional depending on μ (the sources strength) and z (the hydrofoil slope) when these two functions are related by a singular integral equation.

The extremal solution $\mu(\theta;\lambda_1,\lambda_2)$ and $z(\theta;\lambda_1,\lambda_2)$ when determined will involve two Lagrange multipliers constants λ_1,λ_2, which can be determined, by substituting the extremal solutions $\mu(\theta;\lambda_1,\lambda_2)$ and $z(\theta;\lambda_1,\lambda_2)$ in the two constraints.

The analytical solution for the unknown shape z and the unknown singularity μ has branch-type singularities at the two ends of the hydrofoil.

Analytical solutions by a singular integral equation method and Rayleigh-Ritz method are discussed.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

IIIb THE OPTIMUM SHAPE USING VARIATIONAL CALCULUS TECHNIQUES, SO THAT THE LIFT IS A MAXIMUM.

We pose the problem of maximizing the lift coefficient

$$L^* = \frac{L}{\rho U^2} =$$

$$= -4\ell\int_0^\alpha \frac{\sin^2\theta\cos\theta z(\theta)d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} - \frac{2\ell}{U}\int_0^\alpha \frac{\mu(\theta)\sin^2\theta\cos\theta d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} \quad , \qquad [3.b.1]$$

subject to a constraint on curvature of the form

$$K = \frac{1}{2\ell}\int_0^\alpha z'^2(\theta)\sec\theta \operatorname{cosec}\theta \, d\theta \quad , \qquad [3.b.2]$$

where K is prescribed, together with a constraint on the length S of the hydrofoil of the form

$$S = 2\ell \int_0^\alpha \sqrt{1+z^2(\theta)}.\sin\theta\cos\theta\, d\theta \quad , \qquad [3.b.3]$$

where S is prescribed and $z(\theta)=y'(\ell\sin^2\theta)$ is the gradient of the hydrofoil at the position θ .

STATEMENT OF THE PROBLEM

The general optimum problem considered here may be stated as follows:

To find the real, extremal function $\mu(\theta)$ of a real variable θ , required to be Hölder continuous in the region $o<\theta<\alpha$, which is related to the slope $z(\psi)$ by the equation [3.a.10], namely

$$\begin{aligned}\mu(\theta) = &- \frac{U\sigma}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,[\cos\tfrac{1}{2}\alpha - \sqrt{\cos\alpha}] - \\ &- \frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\, z(\psi)d\psi + \\ &+ \frac{1}{2\pi^2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \frac{\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\cos\theta_o\sin(\theta_o-\theta)}\Big\{\pi U\sigma\sin\theta_o + \\ &+ 4U\cos\theta_o\int_0^\alpha \frac{\sin^2\psi\cos\psi z(\psi)d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}.(\sin^2\psi-\sin^2\theta_o)}\Big\} \quad , \qquad [3.b.4]\end{aligned}$$

so that $\mu(\theta)$ and $z(\theta)$ minimize the functional

$$\begin{aligned}I[\mu(\theta),z(\theta),z'(\theta),\theta] &= - L^* + \lambda_1 S + \lambda_2 K \\ &\equiv \int_0^\alpha F[\mu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2]d\theta \quad , \qquad [3.b.5]\end{aligned}$$

with the function $F[\mu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2]$ given by

$$\begin{aligned}F[\mu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2] = &\lambda_1\sqrt{1+z^2(\theta)}.2\ell\sin\theta\cos\theta + \frac{\lambda_2}{2\ell} z'^2(\theta)\sec\theta\,\mathrm{cosec}\,\theta + \\ &+ 4\ell\,\frac{\sin^2\theta\cos\theta z(\theta)}{\sqrt{\sin^2\alpha-\sin^2\theta}} + \frac{2\ell}{U}\,\frac{\mu(\theta)\sin^2\theta\cos\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} \qquad (o<\theta<\alpha) \; , \qquad [3.b.6]\end{aligned}$$

and λ_1,λ_2 are Lagrange multipliers.

The Necessary Condition of Optimality:

Let $\mu(\theta), z(\theta)$ denote the required optimal source distribution function and optimal hydrofoil slope respectively; we write

$$\left.\begin{aligned} \mu_1(\theta) &= \mu(\theta) + \varepsilon\xi(\theta) \quad , \\ z_1(\theta) &= z(\theta) + \varepsilon\eta(\theta) \quad . \end{aligned}\right\} \qquad [3.b.7]$$

We shall assume that the length of the cavity does not vary in the variation process, this implies that α in [3.a.4] is a constant and we recognize that this is a limitation of the present theory.

We can use [3.b.4] to obtain the following relation between $\xi(\theta)$ and $\eta(\theta)$

$$\xi(\theta) = -\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\eta(\psi)\,d\psi + J(\theta) \ , \quad (o<\theta<\alpha) \ , [3.b.8]$$

where

$$J(\theta) = \frac{2U}{\pi^2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \frac{\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}\int_0^\alpha \frac{\sin^2\psi\ \cos\psi\ \eta(\psi)\,d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}.(\sin^2\psi-\sin^2\theta)} \ . \qquad [3.b.9]$$

Using the Poincaré-Bertrand formula we can write [3.b.9] as follows:

$$J(\theta) = -U\eta(\theta) - \frac{2U\sqrt{\cos\alpha}}{\pi^2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \frac{\eta(\psi)\ \sin^2\psi\ d\psi}{\cos\psi\sqrt{\sin^2\alpha-\sin^2\psi}} .I(\psi,\theta) , \qquad [3.b.10]$$

where

$$I(\psi,\theta) = \int_0^\alpha \frac{\sqrt{\tan\theta_o(\tan\alpha-\tan\theta_o)}\ \sec^2\theta_o\ d\theta_o}{(\tan\theta_o-\tan\theta)(\tan\theta_o-\tan\psi)(\tan\theta_o+\tan\psi)} \qquad \theta,\psi\ \varepsilon\,(o,\alpha)$$

$$= -\frac{\pi}{2\sqrt{\cos\alpha}}\sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\cdot\frac{1}{(\tan\theta+\tan\psi)} \ , \qquad [3.b.11]$$

[see Appendix IX],

hence

$$J(\theta) = -U\eta(\theta) + \frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\ \frac{1}{\sin\theta}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\cdot\frac{\sin\psi\ \eta(\psi)\,d\psi}{\sin(\psi+\theta)} \ , \quad (o<\theta<\alpha<\tfrac{\pi}{2}) \ . \qquad [3.b.12]$$

Substituting from [3.b.12] into [3.b.8] we obtain

$$\xi(\theta) = - U\eta(\theta) + \frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\ \frac{1}{\sin\theta}\int_o^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\ \cdot\ \frac{\sin\psi\eta(\psi)\,d\psi}{\pi\sin(\psi+\theta)} -$$

$$- \frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_o^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\ \eta(\psi)\,d\psi\ , \qquad (o<\theta<\alpha<\tfrac{\pi}{2})\ . \qquad [3.b.13]$$

If $\xi(\theta)$ is an admissible variation, then $I[\mu+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', \theta]$ in [3.b.5] is a function of ε which has an extreme value when $\varepsilon = o$. For sufficiently small ε, expansion of [3.b.7] in a Taylor series yields

$$\Delta I = \varepsilon\xi I + \frac{\varepsilon^2}{2!}\ \xi^2 I + \dots\ \dots \qquad . \qquad [3.b.14]$$

We have

$$\Delta I = \int_o^\alpha F[\mu+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', \theta]d\theta - \int_o^\alpha F(\mu,z,z',\theta)\,d\theta, \qquad [3.b.15]$$

where

$$\xi I = \int_o^\alpha \{\xi(\theta)F_\mu(\mu,z,z',\theta)+\eta(\theta)F_z(\mu,z,z',\theta)+\eta'(\theta)F_{z'}(\mu,z,z',\theta)\}d\theta, \qquad [3.b.16]$$

in which the sub-indices denote partial derivatives; it may be noted that ξ and η are related by [3.b.13]. The variations $\delta I, \delta^2 I, \dots$ depend on $\eta(\theta)$ as well as $z(\theta)$.

We integrate by parts the third term in [3.b.16] and we obtain

$$\delta I = [\eta(\theta).F_{z'}(\mu,z,z',\theta)]_o^\alpha + \int_o^\alpha\{\xi(\theta)F_\mu(\mu,z,z',\theta)+\eta(\theta)[F_z(\mu,z,z',\theta) -$$

$$- \frac{d}{d\theta}\ F_{z'}(\mu,z,z',\theta)]\}d\theta\ . \qquad [3.b.17]$$

Substituting from [3.b.13] into [3.b.17] we can write

$$\delta I = [\eta(\theta)F_{z'}(\mu,z,z',\theta)]_o^\alpha + \int_o^\alpha\left\{\frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\ \frac{F_\mu(\mu,z,z',\theta)}{\sin\theta}\int_o^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\cdot\frac{\sin\psi\eta(\psi)\,d\psi}{\sin(\psi+\theta)} -\right.$$

$$- \frac{U}{\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}F_\mu(\mu,z,z',\theta)\int_o^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\eta(\psi)\,d\psi - UF_\mu(\mu,z,z',\theta)\eta(\theta) +$$

$$\left. + [F_z(\mu,z,z',\theta) - \frac{d}{d\theta}F_{z'}(\mu,z,z',\theta)]\eta(\theta)\right\} d\theta\ . \qquad [3.b.18]$$

It is permissible to interchange the order of the two double integrals in [3.b.18] and interchange the variables θ,ψ and when we do so we obtain

$$\delta I = [\eta(\theta).F_{z'}(\mu,z,z',\theta)]_0^\alpha + \int_0^\alpha \Bigg\{ \frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\sin\theta \int_0^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ \frac{F_\mu(\mu,z,z',\psi)d\psi}{\sin\psi\sin(\psi+\theta)} -$$

$$- \frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}} \int_0^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ \frac{F_\mu(\mu,z,z',\psi)d\psi}{\sin\psi\cos\psi} +$$

$$+ [F_z(\mu,z,z',\theta) - \frac{d}{d\theta}F_{z'}(\mu,z,z',\theta) - UF_\mu(\mu,z,z',\theta)]\Bigg\}\eta(\theta)d\theta\ . \qquad [3.b.19]$$

We have from [3.b.6]

$$\left.\begin{aligned} F_\mu(\mu,z,z',\theta) &= \frac{2\ell}{U}\ \frac{\sin^2\theta\cos\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}}\ , \\ F_z(\mu,z,z',\theta) &= \frac{2\lambda_1\ell z(\theta)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} + 4\ell\ \frac{\sin^2\theta\cos\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}}\ , \\ F_{z'}(\mu,z,z',\theta) &= \frac{\lambda_2}{\ell}\ z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\ . \end{aligned}\right\} \ (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [3.b.20]$$

For $I[z]$ to be a minimum, we must have for all admissible function, $\eta(x)$

$$\delta I[z,\eta] = o\ , \qquad [3.b.21]$$

and this implies that the coefficient of $\eta(x)$ in [3.b.19] should vanish, that is

$$F_z(\mu,z,z',\theta) - \frac{d}{d\theta}F_{z'}(\mu,z,z',\theta) = UF_\mu(\mu,z,z',\theta) -$$

$$\frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\ \sin\theta \int_0^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ \frac{F_\mu(\mu,z,z',\psi)d\psi}{\sin\psi\sin(\psi+\theta)} +$$

$$+ \frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}} \int_0^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ .\ \frac{F_\mu(\mu,z,z',\psi)d\psi}{\sin\psi\cos\psi} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [3.b.22]$$

and if $\eta(\theta)$ does not vanish at an end point then it is necessary that $\eta(\theta)z'(\theta)$ should vanish at that point so that

$$\eta(o).z'(o) = o\ ,\ \eta(\alpha).z'(\alpha) = o\ . \qquad [3.b.23]$$

Substituting from [3.b.20] into [3.b.22] we obtain

$$\frac{\lambda_1 z(\theta)}{\sqrt{1+z^2(\theta)}} . 2\ell\sin\theta\cos\theta - \frac{\lambda_2}{\ell}\frac{d}{d\theta}\left[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right] + \frac{2\ell\sin^2\theta\cos\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} = -$$

$$- \frac{2\ell}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,\tan\theta \int_0^\alpha \frac{\sin\psi d\psi}{\sqrt{\sin\psi\sin(\alpha-\psi)}.(\tan\theta+\tan\psi)} +$$

$$+ \frac{2\ell}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}} \int_0^\alpha \frac{\sin\psi d\psi}{\sqrt{\sin\psi\sin(\alpha-\psi)}}\,, \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad . \qquad [3.b.24]$$

Now we evaluate the integrals I_2 and I_3 which are defined by

$$\left.\begin{aligned} I_2 &= \int_0^\alpha \frac{\sin\psi d\psi}{\sqrt{\sin\psi\sin(\alpha-\psi)}.(\tan\psi+\tan\theta)} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad , \\ I_3 &= \int_0^\alpha \frac{\sin\psi d\psi}{\sqrt{\sin\psi\sin(\alpha-\psi)}} \qquad (o<\alpha<\tfrac{\pi}{2}) \quad . \end{aligned}\right\} \qquad [3.b.25]$$

The integral I_2 is evaluated in Appendix I and is given by

$$I_2 = \pi\cos\theta\left[- \cos\theta\sqrt{\frac{\sin\theta}{\sin(\alpha+\theta)}} + \cos(\tfrac{1}{2}\alpha - \theta)\right] \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad , \qquad [3.b.26]$$

the second integral I_3 becomes

$$I_3 = \pi\sin\frac{1}{2}\alpha \qquad (o<\alpha<\tfrac{\pi}{2}) \quad . \qquad [3.b.27]$$

Substituting from [3.b.27] and [3.b.26] into [3.b.24] we can write

$$\lambda_2\frac{d}{d\theta}\left[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right] - 2\lambda_1\ell^2\frac{z(\theta)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} = -$$

$$- 2\ell^2\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,\sin(\tfrac{1}{2}\alpha - \theta)\cos\theta \quad , \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad . \qquad [3.b.28]$$

This equation is a nonlinear differential equation for $z(\theta)$. We consider the solution of [3.b.28] only in the case when the slope $z(\theta)$ is small and we approximate to [3.b.28] as follows:

$$\lambda_2\frac{d}{d\theta}\left[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right] - 2\lambda_1\ell^2 z(\theta)\sin\theta\cos\theta = - 2\ell^2\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,\sin(\tfrac{1}{2}\alpha - \theta)\cos\theta$$

$$(o<\theta<\alpha<\tfrac{\pi}{2}) \quad . \qquad [3.b.29]$$

This may be written as follows:

$$\frac{1}{2\ell\sin\theta\cos\theta}\frac{d}{d\theta}\left[\frac{1}{2\ell\sin\theta\cos\theta}\frac{dz}{d\theta}\right] - \frac{\lambda_1}{2\lambda_2} z(\theta) = f(\theta) \ , \qquad (o<\theta<\alpha<\tfrac{\pi}{2}), \quad [3.b.30]$$

where

$$f(\theta) = -\frac{1}{2\lambda_2}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\cdot\frac{\sin(\frac{1}{2}\alpha-\theta)}{\sin\theta} \ , \qquad (o<\theta<\alpha<\tfrac{\pi}{2}). \quad [3.b.31]$$

Using the transformation

$$\left.\begin{aligned} x &= \ell\sin^2\theta \ , \\ c &= \ell\sin^2\alpha \ , \end{aligned}\right\} \qquad [3.b.32]$$

we obtain

$$\left.\begin{aligned} & z''(x) - m^2 z(x) = F(x) \ , \quad \left(m^2 = \frac{\lambda_1}{2\lambda_2}\right) \ , \\ & \text{where} \\ & F(x) = -\frac{1}{2\sqrt{2}\,\lambda_2\sqrt[4]{\frac{\ell-c}{\ell}}}\cdot\sqrt{\frac{\sqrt{\frac{x}{\ell-x}}}{\sqrt{\frac{c}{\ell-c}}-\sqrt{\frac{x}{\ell-x}}}}\cdot\left[\sqrt{1-\sqrt{\frac{\ell-c}{\ell}}\sqrt{\frac{\ell-x}{x}}}-\sqrt{1+\sqrt{\frac{\ell-c}{\ell}}}\right] \ , \end{aligned}\right\}$$

$$(o<x<c<\ell). \quad [3.b.33]$$

It is assumed at this stage that $\frac{\lambda_1}{\lambda_2} > o$ and we show later that $\lambda_1 > o$, $\lambda_2 > o$ are sufficient condition for a true maximization of the lift L .

To derive the solution of the nonhomogeneous equation in [3.b.33] we apply the usual method of variation of parameters, then we obtain

$$z(x) = -\frac{1}{m}\int_o^x F(\xi)\sinh m(\xi-x)\,d\xi + C_1\sinh mx + C_2\cosh mx \ , \qquad [3.b.34]$$

where C_1 and C_2 are arbitrary constants.

We shall assume the boundary conditions are given by

$$z(o) = o \ , \ z(c) = \beta \ . \qquad [3.b.35]$$

Using [3.b.35] we obtain

$$\left.\begin{aligned} C_1 &= \frac{1}{m \sinh mc} \int_0^c F(\xi) \sinh m(\xi - c)\, d\xi + \frac{\beta}{\sinh mc} \quad , \\ C_2 &= o \ . \end{aligned}\right\} \qquad [3.b.36]$$

Substituting from [3.b.36] into [3.b.34] we obtain

$$z(x) = y'(x) = -\frac{1}{m}\int_0^x F(\xi)\sinh m(\xi - x)\, d\xi + \frac{\sinh mx}{m \sinh mc}\int_0^c F(\xi)\sinh m(\xi - c)\, d\xi + \beta \frac{\sinh mx}{\sinh mc}$$

$$(o<x<c<\ell) . \qquad [3.b.37]$$

We integrate [3.b.37] with respect to x , and we use the boundary condition

$$y(o) = o \quad , \qquad [3.b.38]$$

to obtain

$$y(x) = -\frac{1}{m}\int_0^x d\sigma \int_0^\sigma F(\xi)\sinh m(\xi - \sigma)\, d\xi + \frac{(\cosh m\,x - 1)}{m^2 \sinh mc}\int_0^c F(\xi)\sinh m(\xi - c)\, d\xi + \beta \frac{(\cosh m\,x - 1)}{m \sinh mc}$$

$$(o<x<c<\ell) . \qquad [3.b.39]$$

Equation [3.b.39] can be written as follows:

$$y(x) = -\frac{1}{m^2}\int_0^x F(\xi)[\cosh m(\xi - x) - 1]\, d\xi + \frac{(\cosh m\,x - 1)}{m^2 \sinh mc}\int_0^c F(\xi)\sinh m(\xi - c)\, d\xi + \beta \frac{(\cosh m\,x - 1)}{m \sinh mc}$$

$$(o<x<c<\ell) . \qquad [3.b.40]$$

The solution [3.b.37] should satisfy the constraints, [3.b.2] and [3.b.3], namely:

$$\left.\begin{aligned} \ell &= \int_0^c \sqrt{1 + z^2(x)}\, dx \approx \int_0^c [1 + \tfrac{1}{2} z^2(x)]\, dx \ , \\ &\text{and} \\ K &= \int_0^c z'^2(x)\, dx \ . \end{aligned}\right\} \qquad [3.b.41]$$

When we substitute for $z(x)$ and $z'(x)$, using [3.b.37] into the constraints [3.b.2] and [3.b.3] we obtain two equations, in the two unknowns E, m, which have to be evaluated numerically. We do not complete the solution of this problem using this method since there are two alternative methods of resolving the problem numerically which are discussed in detail in Section IIId and Appendix IIX.

IIIc. A SUFFICIENT CONDITION FOR THE EXTREMUM TO BE A MINIMUM.

A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I.

Since

$$\delta I[\mu(\theta), z(\theta), z'(\theta), \theta] = o\ , \qquad [3.c.1]$$

the condition for I to be a minimum requires that

$$\delta^2 I[\mu(\theta), z(\theta), z'(\theta), \theta] \geq o \qquad [3.c.2]$$

for all admissible variations $\xi(\theta)$ and $\eta(\theta)$ consistent with

$$\xi(\theta) = -\,U\,\eta(\theta) + \frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,\frac{1}{\sin\theta}\int_o^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\frac{\sin\psi\,\eta(\psi)\,d\psi}{\sin(\psi+\theta)} -$$

$$-\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_o^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\eta(\psi)\,d\psi\ , \quad (o<\theta<\alpha<\tfrac{\pi}{2}). \qquad [3.c.3]$$

where $\eta(\theta)$ satisfies the boundary conditions

$$\eta(o) = o\ ,\ \eta(\alpha) = o\ . \qquad [3.c.4]$$

Using Taylor's theorem we can write the increment of the functional $I[\mu, z, z', \theta]$ in the form

$$I[\mu+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', \theta] - I[\mu, z, z', \theta] = \varepsilon\int_o^\alpha\{\xi(\theta)F_\mu(\mu,z,z',\theta) + \eta(\theta)[F_z(\mu,z,z',\theta) -$$

$$-\frac{d}{d\theta}F_{z'}(\mu,z,z',\theta)]\}d\theta + \frac{1}{2}\varepsilon^2\int_o^\alpha\{\xi^2F_{\mu\mu} + \eta^2F_{zz} + \eta'^2F_{z'z'} + 2\xi\eta F_{\mu z} + 2\xi\eta' F_{\mu z'} + 2\eta\eta' F_{zz'}\}d\theta + O(\varepsilon^3). \qquad [3.c.5]$$

Denoting the coefficient ε by δI and that of ε^2 by $\delta^2 I$, at a stationary value of I , [3.c.1], we have from [3.c.1], [3.c.3] and [3.c.5]:

$$F_z[\mu,z,z',\theta] - \frac{d}{d\theta} F_{z'}[\mu,z,z',\theta] = UF_\mu(\mu,z,z',\theta) -$$

$$- \frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\ \sin\theta \int_o^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ \frac{F_\mu(\mu,z,z',\psi)\,d\psi}{\sin\psi\,\sin(\psi+\theta)} +$$

$$+ \frac{U}{\pi}\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}} \int_o^\alpha \sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}}\ \frac{F_\mu(\mu,z,z',\psi)\,d\psi}{\sin\psi\cos\psi} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad , \qquad [3.c.6]$$

and

$$\delta^2 I = \int_o^\alpha \{\xi^2 F_{\mu\mu} + \eta^2 F_{zz} + \eta'^2 F_{z'z'} + 2\xi\eta F_{\mu z} + 2\xi\eta' F_{\mu z'} + 2\eta\eta' F_{zz'}\}d\theta \quad , \qquad [3.c.7]$$

where, by [3.b.6]

$$\left.\begin{aligned}
F_{\mu\mu}[\mu,z,z',\theta] &= o \quad ,\\
F_{zz}[\mu,z,z',\theta] &= \frac{2\lambda_1 \ell \sin\theta\cos\theta}{[1+z^2(\theta)]^{3/2}} \quad ,\\
F_{z'z'}[\mu,z,z',\theta] &= \frac{\lambda_2}{\ell}\sec\theta\,\mathrm{cosec}\,\theta \quad ,\\
F_{\mu z}[\mu,z,z',\theta] &= o \quad ,\\
F_{\mu z'}[\mu,z,z',\theta] &= o \quad ,\\
F_{zz'}[\mu,z,z',\theta] &= o \quad .
\end{aligned}\right\} \qquad [3.c.8]$$

Substituting from [3.c.8] into [3.c.7] we obtain

$$\delta^2 I = \int_o^\alpha \left\{\frac{2\lambda_1 \ell \sin\theta\cos\theta}{[1+z^2(\theta)]^{3/2}} \cdot \eta^2(\theta) + \frac{\lambda_2}{\ell}\sec\theta\,\mathrm{cosec}\,\theta\,\eta'^2(\theta)\right\} d\theta \ . \qquad [3.c.9]$$

In the case of small slope $z(\theta)$, we approximate [3.c.9] as follows:

$$\delta^2 I = \int_o^\alpha \left\{2\lambda_1 \ell \sin\theta\cos\theta\,\eta^2(\theta) + \frac{\lambda_2}{\ell}\sec\theta\,\mathrm{cosec}\,\theta\,\eta'^2(\theta)\right\} d\theta \ . \qquad [3.c.10]$$

Using the transformation

$$\left.\begin{aligned} x &= \ell \sin^2\theta \quad , \\ c &= \ell \sin^2\alpha \quad , \end{aligned}\right\} \qquad [3.c.11]$$

we obtain

$$\delta^2 I = \int_o^c \{\lambda_1 \phi^2(x) + 2\lambda_2 \phi'^2(x)\}\, dx \ , \qquad [3.c.12]$$

where

$$\left.\begin{aligned} \phi(x) &= \eta(\theta) \quad , \\ \phi'(x) &= \eta'(\theta)/2\ell\sin\theta\cos\theta \ . \end{aligned}\right\} \qquad [3.c.13]$$

Now we consider the special choice of $\phi(x)$ satisfying the conditions [3.c.4]:

$$\phi(x) = \alpha \sin \frac{\pi}{c} x \qquad (o<x<c) \ , \qquad [3.c.14]$$

then we obtain from [3.c.10]:

$$\delta^2 I = \frac{1}{2}\,\alpha^2 c[\lambda_1 + \frac{2\pi^2}{c^2}\lambda_2] \ . \qquad [3.c.15]$$

A sufficient condition for satisfying [3.c.2] is

$$\lambda_1 + \frac{2\pi^2}{c^2}\lambda_2 > o \ . \qquad [3.c.16]$$

IIId. ANALYTICAL SOLUTION BY RAYLEIGH-RITZ METHOD IN THE CASE OF SMALL VALUES OF c/ℓ .

We use the Rayleigh-Ritz method [see, e.g., Temple, G. and Bickley, W.G. (58) and Milne, W.E. (44)] to solve equation [3.b.30], namely

$$\frac{d}{d\theta}[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\,] - m\,z(\theta)\sin\theta\cos\theta = \gamma(\theta) \ , \quad (o<\theta<\alpha<\frac{\pi}{2}), \qquad [3.d.1]$$

with

$$\left.\begin{aligned} \gamma(\theta) &= -\,2\ell^2 E\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\sin(\tfrac{1}{2}\alpha-\theta)\cos\theta \ , \\ m &= 4\ell^2 n \ , \ n = \frac{\lambda_1}{2\lambda_2} \ , \ E = \frac{1}{\lambda_2} \ , \end{aligned}\right\} \qquad [3.d.2]$$

where λ_1, λ_2 are Lagrange multipliers, $z(\theta)$ is subject to the boundary conditions [3.b.37].

The equation [3.d.1] is the necessary condition for the integral

$$J = \int_0^\alpha \{\frac{1}{2} \sec\theta \operatorname{cosec}\theta\, z'^2(\theta) + \frac{1}{2} m \sin\theta \cos\theta\, z^2(\theta) + \gamma(\theta) z(\theta)\}\, d\theta \ , \qquad [3.d.3]$$

to attain a minimum.

We may observe at this point that if α is sufficiently small, that is c/ℓ is sufficiently small, the integral is approximately:

$$J = \int_0^\alpha \{ \frac{1}{2\theta}\, z'^2(\theta) + \frac{1}{2} m\theta\, z^2(\theta) + \gamma(\theta) z(\theta)\}\, d\theta \ , \qquad [3.d.4]$$

where

$$\gamma(\theta) = - 2\ell^2 E \sqrt{\frac{\theta}{\alpha-\theta}}\, (\frac{1}{2}\alpha - \theta) \ , \qquad [3.d.5]$$

where m and E are defined by [3.d.2].

The Rayleigh-Ritz method can be applied to this problem in the following way:

We select a basic set of linearly independent polynomial functions and we assume an expression for $z(\theta)$ of the form

$$z(\theta) = \frac{\beta}{\alpha^2}\theta^2 + a_1\theta^2(\alpha-\theta) + a_2\theta^3(\alpha-\theta) \ , \qquad [3.d.6]$$

which satisfies the end conditions [3.b.37], a_1 and a_2 being arbitrary constants.

The values of z and z' are obtained from equation [3.d.6] and we substituted in [3.d.4], the result is a quadratic a_1 and a_2 ; we have

$$J = \int_0^\alpha \left\{ \frac{1}{2\theta}\left[\frac{2\beta}{\alpha^2}\theta + a_1(2\alpha\theta - 3\theta^2) + a_2(3\alpha\theta^2 - 4\theta^3)\right]^2 + \frac{1}{2} m\theta\left[\frac{\beta}{\alpha^2}\theta^2 + a_1\theta^2(\alpha-\theta) + a_2\theta^3(\alpha-\theta)\right]^2 - \right.$$

$$\left. - 2\ell^2 E\sqrt{\frac{\theta}{\alpha-\theta}}\left[\frac{1}{2}\alpha - \theta\right]\left[\frac{\beta}{\alpha^2}\theta^2 + a_1\theta^2(\alpha-\theta) + a_2\theta^3(\alpha-\theta)\right]\right\} d\theta \qquad [3.d.7]$$

hence

$$J = \frac{\beta^2}{\alpha^2} + \frac{1}{8} a_1^2 \alpha^4 + \frac{7}{120} a_2^2 \alpha^6 + \frac{3}{20} a_1 a_2 \alpha^5 + \frac{1}{12} m \beta^2 \alpha^2 + \frac{1}{336} a_1^2 \alpha^8 m +$$

$$+ \frac{1}{720} a_2^2 \alpha^{10} m + \frac{1}{42} a_1 \beta \alpha^5 m + \frac{1}{56} a_2 \beta \alpha^6 m + \frac{1}{252} a_1 a_2 \alpha^9 m +$$

$$+ 2\ell^2 E\left[\frac{15}{128} \beta \pi \alpha^2 + \frac{\pi}{128} a_1 \alpha^5 + \frac{7}{1024} \pi \alpha^6 a_2\right] . \qquad [3.d.8]$$

The necessary conditions for minimizing J, with respect to a_1 and a_2, are

$$\left.\begin{aligned} \frac{\partial J}{\partial a_1} &= \left(\frac{1}{4} + \frac{1}{168} m\alpha^4\right) a_1 \alpha^4 + \left(\frac{3}{20} + \frac{1}{252} m\alpha^4\right) a_2 \alpha^5 + \frac{1}{42} \beta m \alpha^5 + \frac{\ell^2 \alpha^5 \pi E}{64} = 0 , \\ &\text{and} \\ \frac{\partial J}{\partial a_2} &= \left(\frac{3}{20} + \frac{1}{252} m\alpha^4\right) a_1 \alpha^5 + \left(\frac{7}{60} + \frac{1}{360} m\alpha^4\right) a_2 \alpha^6 + \frac{1}{56} \beta m \alpha^6 + \frac{7\ell^2 \alpha^6 \pi E}{512} = 0 . \end{aligned}\right\} \qquad [3.d.9]$$

The quantities n and E can now be expressed in terms of a_1 and a_2 but for convenience we introduce

$$\left.\begin{aligned} \xi &= a_1 \alpha^3 , \\ \eta &= a_2 \alpha^4 , \end{aligned}\right\} \qquad [3.d.10]$$

and we then have

$$n = W/4\ell^2 \alpha^4 V , \qquad [3.d.11]$$

where

$$\left.\begin{aligned} W &= -\left(\frac{R_2}{4} - \frac{3R_1}{20}\right)\xi - \left(\frac{3R_2}{20} - \frac{7R_1}{60}\right)\eta , \\ V &= \left(\frac{R_2}{168} - \frac{R_1}{252}\right)\xi + \left(\frac{R_2}{252} - \frac{R_1}{360}\right)\eta + \left(\frac{R_2}{42} - \frac{R_1}{56}\right)\beta , \end{aligned}\right\} \qquad [3.d.12]$$

with

$$R_1 = \frac{\pi \ell^2 \alpha^4}{64} , \quad R_2 = \frac{7\pi \ell^2 \alpha^5}{512} , \qquad [3.d.13]$$

and

$$E = -\frac{1}{R_1}\left[\left(\frac{1}{4} + \frac{1}{168} m\alpha^4\right)\xi + \left(\frac{3}{20} + \frac{1}{252} m\alpha^4\right)\eta + \frac{1}{42} m\alpha^4 \beta\right] . \qquad [3.d.14]$$

From [3.c.16] the sufficient condition for the lift to be a maximum is

$$\left[n^2 + \frac{\pi^2}{c^2}\right] / E > o \quad , \quad \left[E = \frac{1}{\lambda_2} \; , \; n = \frac{\lambda_1}{2\lambda_2}\right] . \qquad [3.d.15]$$

Substituting from [3.d.6] in the constraints, [3.b.2] and [3.b.3] we obtain

$$S = 2\ell \int_o^\alpha [1 + \frac{1}{2} z^2(\theta)] \, \theta \, d\theta$$

$$= 2\ell [\frac{1}{2} \alpha^2 + \frac{1}{12} \beta^2\alpha^2 + \frac{1}{336} a_1^2\alpha^8 + \frac{1}{720} a_2^2 \alpha^{10} + \frac{1}{42} a_1 \beta \alpha^5 + \frac{1}{56} a_2 \beta \alpha^6 +$$

$$+ \frac{1}{252} a_1 a_2 \alpha^9] \; , \qquad [3.d.16]$$

and

$$K = \frac{1}{2\ell} \int_o^\alpha \frac{1}{\theta} z'^2(\theta) d\theta$$

$$= \frac{1}{\ell}\left[\frac{\beta^2}{\alpha^2} + \frac{1}{8} a_1^2 \alpha^4 + \frac{7}{120} a_2^2 \alpha^6 + \frac{3}{20} a_1 a_2 \alpha^5\right] . \qquad [3.d.17]$$

Equations [3.d.16] and [3.d.17] can be written as follows:

$$\left.\begin{aligned} S_1 &= A_1\xi^2 + 2H_1\xi\eta + B_1\eta^2 + 2P_1\xi + 2Q_1\eta + C_1 = o \quad , \\ S_2 &= A_2\xi^2 + 2H_2\xi\eta + B_2\eta^2 + 2P_2\xi + 2Q_2\eta + C_2 = o \quad . \end{aligned}\right\} \qquad [3.d.18]$$

where

$$\left.\begin{aligned} A_1 &= \frac{1}{336} \quad , & A_2 &= \frac{1}{8} \quad , \\ B_1 &= \frac{1}{720} \quad , & B_2 &= \frac{7}{120} \quad , \\ H_1 &= \frac{1}{504} \quad , & H_2 &= \frac{3}{40} \quad , \\ P_1 &= \frac{1}{84}\beta \quad , & P_2 &= o \quad , \\ Q_1 &= \frac{1}{112}\beta \quad , & Q_2 &= o \quad , \\ C_1 &= -\left(\frac{S-c}{2c}\right) + \frac{1}{12}\beta^2 , & C_2 &= -Kc + \beta^2 . \end{aligned}\right\} \qquad [3.d.19]$$

We shall consider the special case

$$\left.\begin{array}{l} S = 4.02 \text{ ft} \quad , \\ c = 4 \text{ ft} \quad , \quad \ell = 100 \text{ ft} \quad , \\ K = 0.0148 \text{ ft} \quad , \\ \alpha = 0.2 \quad , \\ \beta = -\tan 12^\circ = -0.21256 \ . \end{array}\right\} \qquad [3.d.20]$$

Regarding $S_1=o$ and $S_2=o$ as two conics the condition upon λ for the quadratic

$$S_1 + \lambda S_2 = o \qquad [3.d.21]$$

to represent a pair of straight lines is

$$233661.8\lambda^3 - 14129.5\lambda^2 + 433.9\lambda + 3.068 = o \quad , \qquad [3.d.22]$$

[see, e.g., THE REV. E.H. ASKWITH, D.D., 1953 "Analytical Geometry of the conic sections" Third Ed., Adam & Charles Black]

which can be solved to give the following roots

$$\lambda \equiv -0.00585,\ 0.03316 \pm 0.033845i \ . \qquad [3.d.23]$$

Using the real value of λ we can write equation [3.d.23] in the form

$$0.22451\xi^2 + 0.30909\xi\eta + 0.10477\eta^2 - 0.50609\xi - 0.37057\eta + 0.1347 = o \ . \qquad [3.d.24]$$

By factorizing equation [3.d.24], we obtain

$$\xi + 0.60355\eta - 1.94587 = o \quad , \qquad [3.d.25]$$

$$\xi + 0.77321\eta - 0.30834 = o \quad . \qquad [3.d.26]$$

The straight line [3.d.25] when combined with $S_1=o$ produces

$$\xi = 1.9068 \mp 3.5418i \quad , \quad \eta = 0.0647 \pm 5.8684i \ , \qquad [3.d.27]$$

in other words, there is no real intersection of this straight line with the conic, while the points of intersection between the straight line [3.d.26] and S_1 are real and are as follows:

$$\left.\begin{array}{lll} \text{(i)} & \xi = -\,0.4013 \;, & \eta = 0.9178 \;, \\ \text{(ii)} & \xi = 0.4137 \;, & \eta = -\,0.1362 \;. \end{array}\right\} \qquad [3.d.28]$$

Using [3.d.28] and [3.d.19] we can write the values of n and E, [3.d.11] and [3.d.14] in the forms

$$\left.\begin{array}{lll} \text{(i)} & n = -\,0.45027 \;, & E = -\,0.18725 \;, \\ \text{(ii)} & n = 1.9293 \;, & E = \;\; 0.38786 \;. \end{array}\right\} \qquad [3.d.29]$$

The values in [3.d.29 . (i)] do not satisfy the sufficiency condition, [3.d.15] but the values in [3.d.29 . (ii)] satisfy [3.d.15], namely

$$n = 1.9293 \quad , \quad E = 0.38786 \quad , \qquad [3.d.30]$$

and thus the appropriate values of ξ and η, are

$$\xi = 0.4137 \quad , \quad \eta = -\,0.1362 \quad , \qquad [3.d.31]$$

using [3.d.31] and [3.d.10] we obtain

$$a_1 = 51.7066 \quad , \quad a_2 = -\,85.1262 \;. \qquad [3.d.32]$$

Now we can write the solution $z(\theta)$, [3.d.6] of the differential equation [3.d.1], using [3.d.32] and [3.d.19] as follows:

$$z(\theta) = 15.655\theta^2 - 68.732\theta^3 + 85.1262\theta^4 \qquad (o \leq \theta \leq 0.2) \quad , \qquad [3.d.33]$$

using the transformation

$$\theta = \sqrt{\frac{x}{\ell}} \quad , \quad (\ell = 100) \quad , \qquad [3.d.34]$$

we obtain

$$z(x) = 0.15655x - 0.06873x^{3/2} + 0.0085x^2 \quad , \quad (o \leq x \leq 4) \quad . \qquad [3.d.35]$$

We integrate [3.d.35] with respect to x and we obtain

$$y(x) = 0.078275x^2 - 0.027496x^{5/2} + 0.00283x^3 \qquad (o \leq x \leq 4) \qquad [3.d.36]$$

there being no arbitrary constant since

$$y(o) = o \;. \qquad [3.d.37]$$

The graphs of $y(x)$, $y'(x)$ and $y''(x)$ are shown in Figures [9,10,11]. The maximum lift corresponding to this solution is 116058 lbs. [see Appendix VIII]

IV THE OPTIMUM SHAPE OF A HYDROFOIL IN STEADY TWO-DIMENSIONAL PARTIAL CAVITY FLOW

INTRODUCTION

A hydrofoil is placed in the uniform flow of an incompressible non-viscous liquid filling an infinite space.

The liquid flow is assumed to be steady and irrotational.

It is assumed the length of the cavity on the upper surface, as shown in Fig.(12) is less than the length of the hydrofoil, and this will be referred to as a partial cavity.

The method of linearization is used to solve the flow problem.

A two-dimensional source and vortex distribution on the x-axis is used to simulate the flow past the hydrofoil and in satisfying the boundary conditions we are led to a system of coupled integral equations; these are solved exactly using the Carleman-Muskhelishvili technique.

This method is similar to that used by Davies (13),(14).

The purpose of this work is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces maximum lift. We use variational calculus techniques to obtain the optimum shape of the hydrofoil. The mathematical problem is that of extremizing a functional depending on ν (the vortex strength) and z (the hydrofoil slope) when the two functions are related by a singular integral equation.

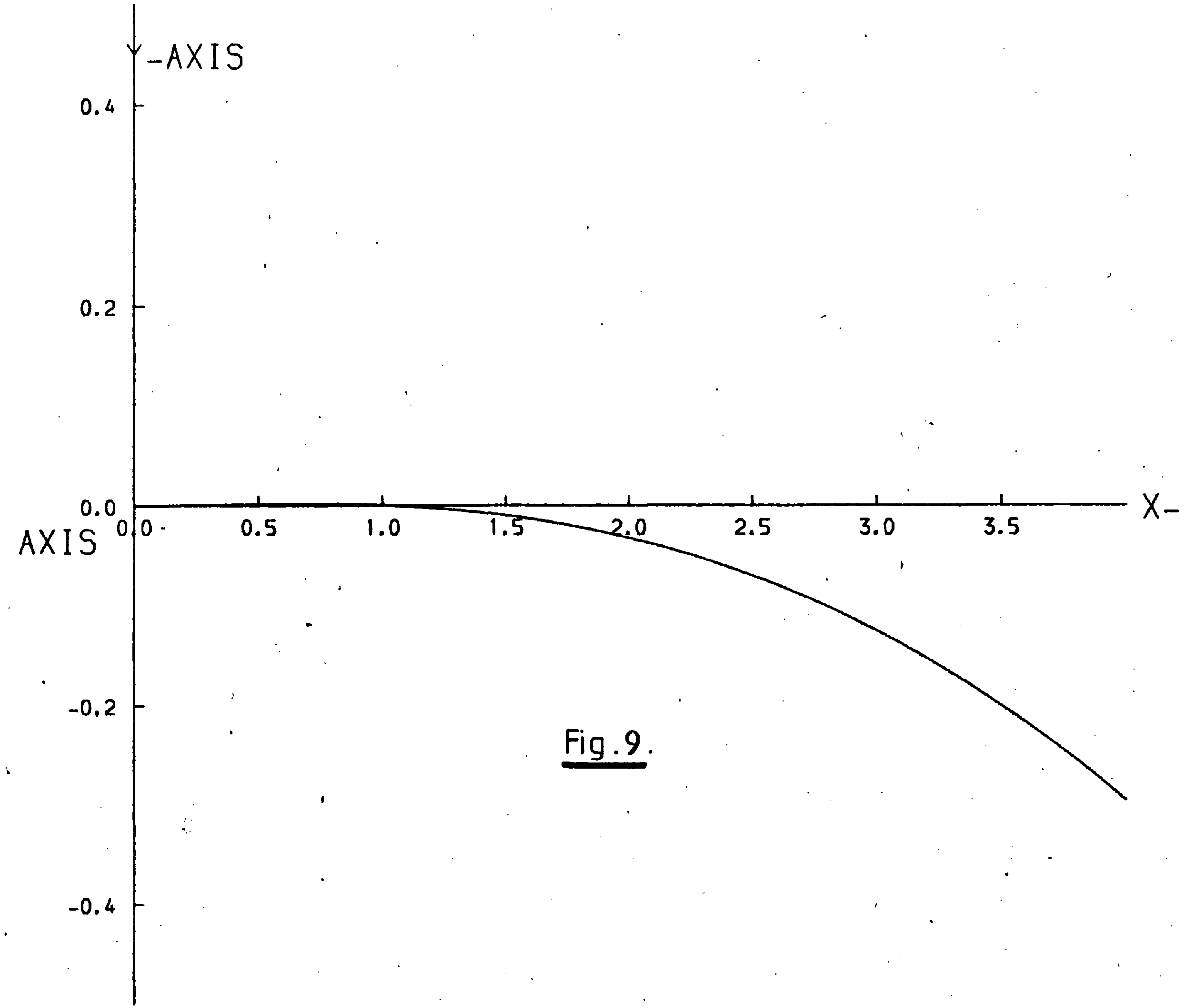

Fig.9.

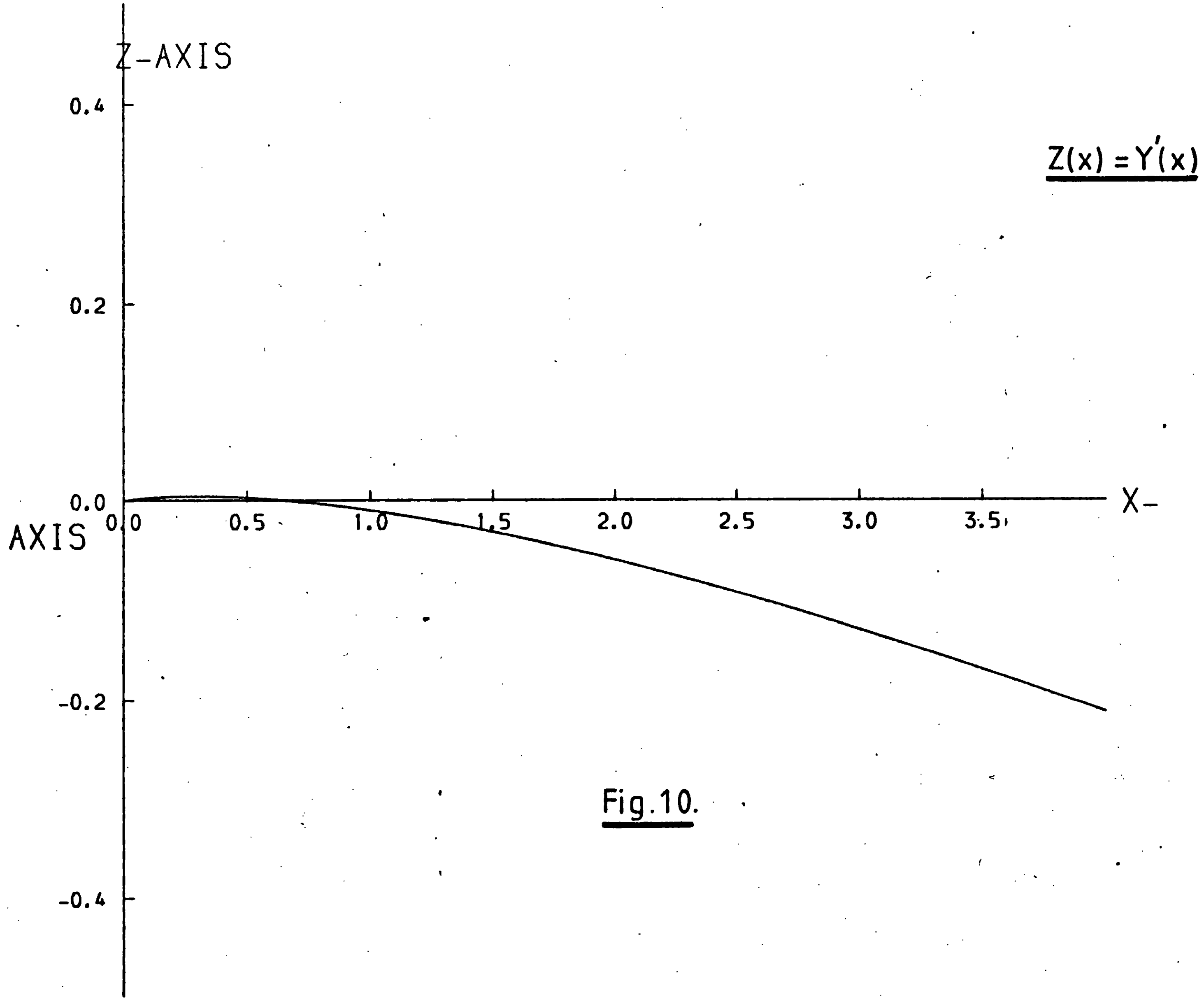

Fig.10.

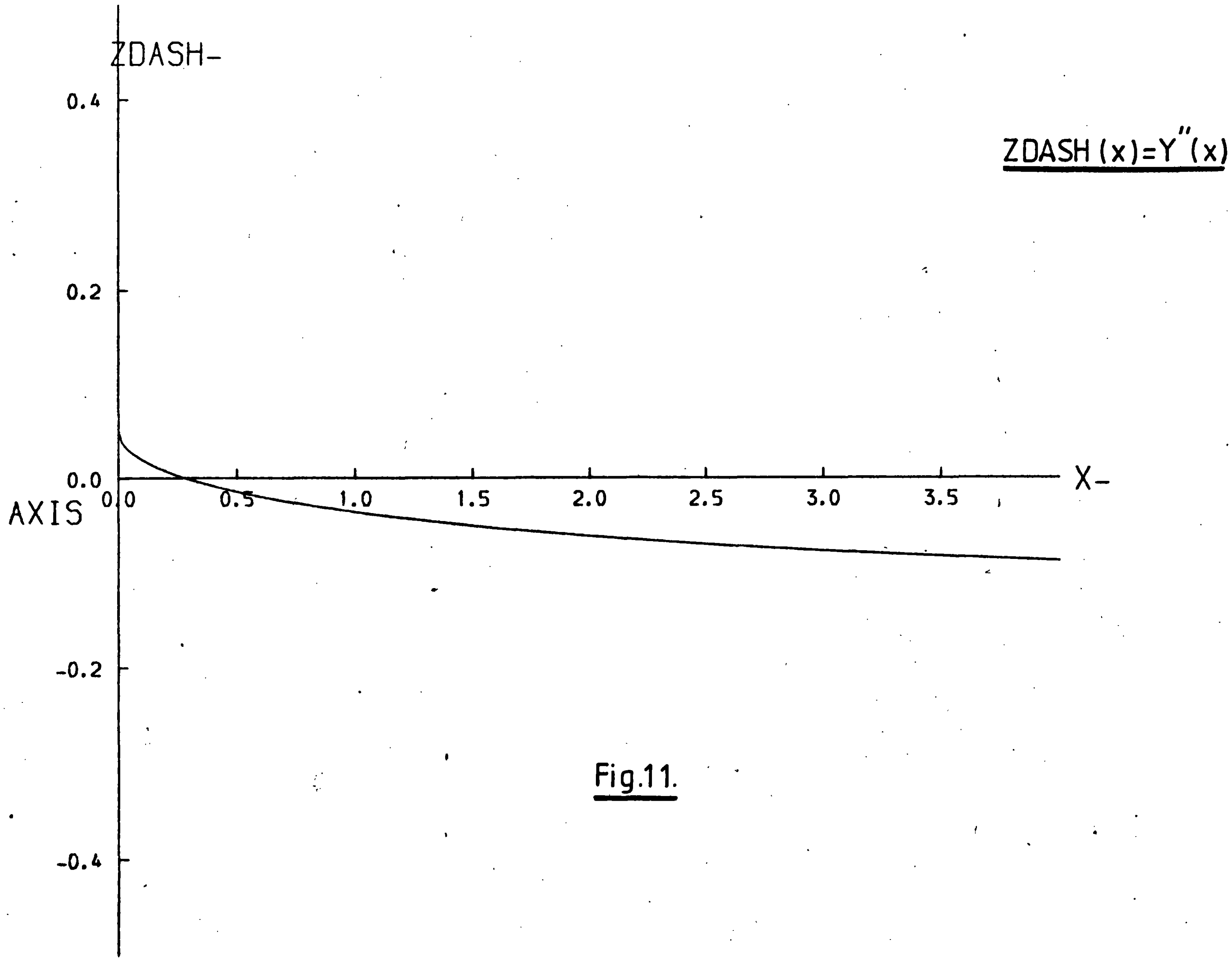

Fig.11.

The extremal solution $\nu(\theta,\lambda_1,\lambda_2)$ and $z(\theta;\lambda_1,\lambda_2)$ when determined will involve two Lagrange multipliers constants λ_1,λ_2 which can be determined, by substituting the extremal solution $\nu(\theta;\lambda_1,\lambda_2)$ and $z(\theta;\lambda_1,\lambda_2)$ in the two constraints.

The analytical solution for the unknown shape z and the unknown singularity ν has branch singularities at the two ends of the hydrofoil.

Analytical solution by a singular integral equation method is discussed. A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

IVa EXPRESSION OF THE PROBLEM IN INTEGRAL EQUATIONS FORM

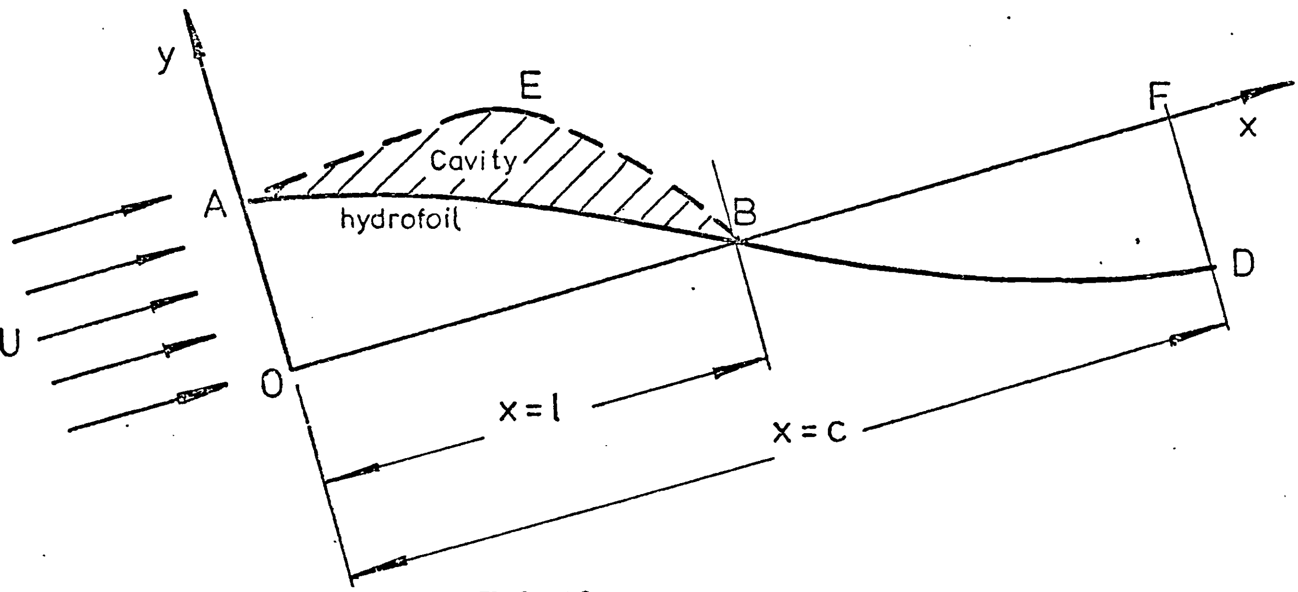

FIG. 12.

ABD in the figure represent the lower surface of hydrofoil and it is assumed that a vapour filled cavity AEB extend along a part AB of the suction side of the hydrofoil as shown in the diagram. (This is called partially cavitated flow.)

We consider the flow of an irrotational and incompressible non-viscous liquid, filling an infinite space.

The velocity of the liquid at infinity (i.e. at a great distance from the hydrofoil) is taken to be U and the pressure there is P_∞ and the pressure inside the cavity is uniform and equal to P_c which is assumed to be less than P_∞ .

We assume the liquid flow to be steady.

The problem will be solved on the basis of linearized theory and for this purpose we introduce the following source and vortex distribution on the x-axis.

(a) Source of strength $m(\xi)$ per unit length in $o<\xi<\ell$,

(b) Vortices of strength $\gamma_1(\xi)$ per unit length in $o<\xi<\ell$,

(c) Vortices of strength $\gamma_2(\xi)$ per unit length in $\ell<\xi<c$,($\gamma>0$ clockwise), to replace the above physical configuration, $\gamma_1(\xi),\gamma_2(\xi)$ and $m(\xi)$ being unknown distributions.

Each of these singular distributions will produce a contribution to the complete velocity potential.

We consider first the source distribution on the hydrofoil, the velocity potential due to the sources will be

$$\phi_1 = -\frac{1}{2\pi}\int_0^\ell m(\xi).\log r.\,d\xi \qquad [r=([x-\xi]^2+y^2)], \qquad [4.a.1]$$

where r denotes the distance from a point (x,y) , in the liquid to (ξ,o) and the velocity component in the y-direction due to these sources will be

$$v_1 = -\frac{\partial\phi_1}{\partial y} = \frac{1}{2\pi}\int_0^\ell \frac{ym(\xi)d\xi}{(x-\xi)^2+y^2} \qquad . \qquad [4.a.2]$$

We require to find the limiting value of v_1 as $y\to 0-$ through negative values (since we shall be concerned with the under surface of the hydrofoil) and by the usual limiting argument we find

$$\lim_{y\to o-} v_1 = -\tfrac{1}{2}m(x) \qquad (o<x<\ell) , \qquad [4.a.3]$$

and

$$\lim_{y\to o-} v_1 = 0 \qquad (\ell<x<c) . \qquad [4.a.4]$$

We now consider the vortex distribution; the velocity potential due to the vortices in $0<x<\ell$ will be

$$\phi_2 = -\frac{1}{2\pi}\int_0^\ell \gamma_1(\xi).\tan^{-1}\left(\frac{y}{x-\xi}\right)d\xi \quad . \qquad [4.a.5]$$

The corresponding component in the y-direction will be

$$v_2 = -\frac{\partial\phi_2}{\partial y} = \frac{1}{2\pi}\int_0^\ell \frac{\gamma_1(\xi).(x-\xi)\,d\xi}{(x-\xi)^2+y^2} \quad . \qquad [4.a.6]$$

As $y\to 0\pm$ we have for all x

$$\lim_{y\to 0\pm} v_2 = \frac{1}{2\pi}\int_0^\ell \frac{\gamma_1(\xi)\,d\xi}{x-\xi} \quad . \qquad [4.a.7]$$

For the vortex distribution in $\ell<x<c$ the velocity potential will be

$$\phi_3 = -\frac{1}{2\pi}\int_\ell^c \gamma_2(\xi)\ \tan^{-1}\left(\frac{y}{x-\xi}\right)d\xi \quad . \qquad [4.a.8]$$

The corresponding velocity component in y-direction will be

$$v_3 = -\frac{\partial\phi_3}{\partial y} = \frac{1}{2\pi}\int_\ell^c \frac{\gamma_2(\xi)\,(x-\xi)\,d\xi}{(x-\xi)^2+y^2} \quad . \qquad [4.a.9]$$

As $y\to 0\pm$ we have for all x

$$\lim_{y\to 0\pm} v_3 = \frac{1}{2\pi}\int_\ell^c \frac{\gamma_2(\xi)\,d\xi}{x-\xi} \quad . \qquad [4.a.10]$$

The complete potential $\phi(x,y)$ will then be given by

$$\phi=\phi_1+\phi_2+\phi_3 \quad . \qquad [4.a.11]$$

It follows now from [4.a.3], [4.a.7] and [4.a.10] that the complete normal velocity in the range $0<x<\ell$ will be

$$-\tfrac{1}{2}m(x) + \frac{1}{2\pi}\int_0^\ell \frac{\gamma_1(\xi)\,d\xi}{x-\xi} + \frac{1}{2\pi}\int_\ell^c \frac{\gamma_2(\xi)\,d\xi}{x-\xi} \qquad (0<x<\ell)\ . \qquad [4.a.12]$$

It follows therefore that the boundary conditions on the hydrofoil will be satisfied if m, γ_1 and γ_2 satisfy

$$\frac{1}{2\pi}\int_0^\ell \frac{\gamma_1(\xi)\,d\xi}{x-\xi} + \frac{1}{2\pi}\int_\ell^c \frac{\gamma_2(\xi)\,d\xi}{x-\xi} - \tfrac{1}{2}m(x) = Uy'(x) \quad (0<x<\ell)\ , \qquad [4.a.13]$$

where $y'(x)$ is the slope of the hydrofoil at position x .

For the range $\ell<x<c$ the normal velocity will be

$$\frac{1}{2\pi}\int_{0}^{\ell}\frac{\gamma_1(\xi)d\xi}{x-\xi}+\frac{1}{2\pi}\int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{x-\xi} \qquad (\ell<x<c) \ . \qquad [4.a.14]$$

It follows therefore that the boundary condition will be satisfied if γ_1 and γ_2 satisfy

$$\frac{1}{2\pi}\int_{0}^{\ell}\frac{\gamma_1(\xi)d\xi}{x-\xi}+\frac{1}{2\pi}\int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{x-\xi}=U\,y'(x) \qquad (\ell<x<c) \ , \qquad [4.a.15]$$

We consider now the condition of constancy of pressure of the surface of the cavity. The general formula for the pressure P is

$$P=P_\infty+\rho U\phi_x \quad , \qquad [4.a.16]$$

and the value of the pressure on the surface of the cavity is

$$P=P_\infty+\rho U\phi_x\Big|_{y=0} \qquad (0<x<\ell), \qquad [4.a.17]$$

and thus on the cavity surface we must have

$$\phi_x\Big|_{y=0}=\frac{P_c-P_\infty}{\rho U} \quad , \qquad [4.a.18]$$

hence from [4.a.18] we have on the cavity surface

$$\phi_x\Big|_{y=0}=-\tfrac{1}{2}U\sigma \quad , \qquad [4.a.19]$$

where

$$\sigma=\frac{P_\infty-P_c}{\frac{1}{2}\rho U^2} \qquad [4.a.20]$$

is the cavitation number.

The limiting value of $\frac{\partial\phi_1}{\partial x}$ as $y\to 0\pm$, can be derived from [4.a.1]

$$\frac{\partial\phi_1}{\partial x}=-\frac{1}{2\pi}\int_{0}^{\ell}\frac{(x-\xi)m(\xi)d\xi}{(x-\xi)^2+y^2} \quad , \qquad [4.a.21]$$

hence

$$\lim_{y\to 0\pm}\left(\frac{\partial\phi_1}{\partial x}\right)=-\frac{1}{2\pi}\int_{0}^{\ell}\frac{m(\xi)d\xi}{x-\xi} \quad , \qquad [4.a.22]$$

for all $x\varepsilon(0,\ell)$.

Likewise from [4.a.5] we have

$$\frac{\partial\phi_2}{\partial x} = \frac{1}{2\pi}\int_0^{\ell} \frac{y\gamma_1(\xi)d\xi}{(x-\xi)^2+y^2} \quad . \qquad [4.a.23]$$

The limiting value of $\frac{\partial\phi_2}{\partial x}$ as $y \to 0\pm$ is

$$\lim_{y\to o\pm}\left(\frac{\partial\phi_2}{\partial x}\right) \equiv \begin{cases} \pm\frac{1}{2}\gamma_1(x) & (o<x<\ell)\ , \\ 0 & (\ell<x<c)\ . \end{cases} \qquad [4.a.24]$$

Likewise from [4.a.8] we obtain

$$\frac{\partial\phi_3}{\partial x} = \frac{1}{2\pi}\int_{\ell}^{c} \frac{\gamma_2(\xi)yd\xi}{(x-\xi)^2+y^2} \quad . \qquad [4.a.25]$$

The limiting value of $\frac{\partial\phi_3}{\partial x}$ as $y \to 0\pm$ is

$$\lim_{y\to o\pm}\left(\frac{\partial\phi_3}{\partial x}\right) \equiv \begin{cases} 0 & (o<x<\ell)\ , \\ \pm\frac{1}{2}\gamma_2(x) & (\ell<x<c)\ . \end{cases} \qquad [4.a.26]$$

Bringing together the various results for $\frac{\partial\phi}{\partial x}$, [4.a.22], [4.a.24] and [4.a.26] and thus we obtain

$$\lim_{y\to o\pm}\left(\frac{\partial\phi}{\partial x}\right) \equiv \begin{cases} -\frac{1}{2\pi}\int_0^{\ell}\frac{m(\xi)d\xi}{x-\xi} \pm \frac{1}{2}\gamma_1(x) & (o<x<\ell)\ , \\ \\ -\frac{1}{2\pi}\int_0^{\ell}\frac{m(\xi)d\xi}{x-\xi} \pm \frac{1}{2}\gamma_2(x) & (\ell<x<c)\ . \end{cases} \qquad [4.a.27]$$

It will then follow from [4.a.19] and [4.a.27] the following must be satisfied as $y\to 0+$ through positive values

$$-\frac{1}{2\pi}\int_0^{\ell}\frac{m(\xi)d\xi}{x-\xi} + \tfrac{1}{2}\gamma_1(x) = \tfrac{1}{2}U\sigma \qquad (o<x<\ell) \quad . \qquad [4.a.28]$$

The three integral equations of the problem are thus [4.a.13], [4.a.15] and [4.a.28] .

IVb SOLUTION OF THE SYSTEM INTEGRAL EQUATIONS

The problem is thus reduced to finding the solution of the singular integral equations

$$\int_0^{\ell}\frac{\gamma_1(\xi)d\xi}{x-\xi} + \int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{x-\xi} - \pi m(x) = 2\pi U\, y'(x) \qquad (o<x<\ell)\ , \qquad [4.b.1]$$

$$\int_0^{\ell} \frac{\gamma_1(\xi)\,d\xi}{x-\xi} + \int_{\ell}^{c} \frac{\gamma_2(\xi)\,d\xi}{x-\xi} = 2\pi U\, y'(x) \qquad (\ell<x<c)\ , \qquad [4.b.2]$$

$$\int_0^{\ell} \frac{m(\xi)\,d\xi}{x-\xi} - \pi\gamma_1(x) = -\ \pi U\sigma \qquad (o<x<\ell)\ , \qquad [4.b.3]$$

for m , γ_1 and γ_2.

We consider first [4.b.3], the inversion of this equation is immediate and we obtain

$$m(x) = \frac{-\ 1}{\pi^2\sqrt{x(\ell-x)}} \int_0^{\ell} \frac{\sqrt{\xi(\ell-\xi)}\,[\pi\gamma_1(x)-\pi U\sigma]\,d\xi}{\xi-x} + \frac{C}{\sqrt{x(\ell-x)}}$$

$$= \frac{-\ 1}{\pi\sqrt{x(\ell-x)}} \int_0^{\ell} \frac{\sqrt{\xi(\ell-\xi)}.\gamma_1(\xi)\,d\xi}{\xi-x} + \frac{U\sigma}{\pi\sqrt{x(\ell-x)}} \int_0^{\ell} \frac{\sqrt{\xi(\ell-\xi)}.\,d\xi}{\xi-x} + \frac{C}{\sqrt{x(\ell-x)}}\ , \qquad [4.b.4]$$

where C is an arbitrary constant.

The second integral in [4.b.4] can be evaluated to give

$$I_1 = \int_0^{\ell} \frac{\sqrt{\xi(\ell-\xi)}\,.\,d\xi}{\xi-x} = \frac{\pi}{2}\ (\ell-2x) \qquad . \qquad [4.b.5]$$

Substituting from [4.b.5] in [4.b.4] we obtain

$$m(x) = \frac{-\ 1}{\pi\sqrt{x(\ell-x)}} \int_0^{\ell} \frac{\gamma_1(\xi)\sqrt{\xi(\ell-\xi)}\,d\xi}{\xi-x} + \frac{C + \frac{1}{2}U\sigma(\ell-2x)}{\sqrt{x(\ell-x)}}\ . \qquad [4.b.6]$$

The arbitrary constant C can be calculated, using the closure condition, namely

$$\int_0^{\ell} m(\xi)\,d\xi = 0 \qquad . \qquad [4.b.7]$$

Consequently, from [4.b.6] we will have

$$\frac{1}{\pi}\int_0^{\ell} \frac{dx}{\sqrt{x(\ell-x)}} \int_0^{\ell} \frac{\gamma_1(\xi)\sqrt{\xi(\ell-\xi)}\,d\xi}{\xi-x} = \int_0^{\ell} \frac{[C + \frac{1}{2}\,U\,\sigma(\ell-2x)]\,dx}{\sqrt{x(\ell-x)}} \qquad . \qquad [4.b.8]$$

It is permissible to interchange the order of integration on the left-hand side of [4.b.8] [see, e.g., Hardy, G.H. (35)] and when we do so we obtain

$$\frac{1}{\pi}\int_0^{\ell}\gamma_1(\xi)\sqrt{\xi(\ell-\xi)}\,d\xi\int_0^{\ell}\frac{dx}{\sqrt{x(\ell-x)}\,.\,(\xi-x)} = \int_0^{\ell}\frac{[C+\frac{1}{2}U\sigma(\ell-2x)]dx}{\sqrt{x(\ell-x)}} \quad . \qquad [4.b.9]$$

In evaluating the inner integral we use the fact that x lies between 0 and ℓ, we have

$$I_2 = \int_0^{\ell}\frac{dx}{\sqrt{x(\ell-x)}\,.\,(\xi-x)} = 0 \quad , \qquad [4.b.10]$$

hence

$$\int_0^{\ell}\frac{[C+\frac{1}{2}U\sigma(\ell-2x)]dx}{\sqrt{x(\ell-x)}} = 0$$

$$= 2\int_0^{\pi/2}[C+\tfrac{1}{2}U\sigma\ell\cos 2\theta]d\theta = 0 \quad , \qquad [4.b.11]$$

and consequently we can write

$$C = 0 \quad . \qquad [4.b.12]$$

Substituting from [4.b.12] in [4.b.7] we obtain

$$m(x) = \frac{-1}{\pi\sqrt{x(\ell-x)}}\int_0^{\ell}\frac{\gamma_1(\xi)\sqrt{\xi(\ell-\xi)}\,d\xi}{\xi-x} + \frac{U\sigma(\ell-2x)}{2\sqrt{x(\ell-x)}} \qquad (o<x<\ell) \quad . \qquad [4.b.13]$$

Now we solve equation [4.b.2] for γ_2 and it is convenient to express this result in the form

$$\int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{\xi-x} = 2\pi U y'(x) - \int_0^{\ell}\frac{\gamma_1(\xi)d\xi}{\xi-x} \qquad (\ell<x<c) \quad . \qquad [4.b.14]$$

The general solution of [4.b.14] is

$$\gamma_2(x) = \frac{-1}{\pi^2\sqrt{(c-x)(x-\ell)}}\oint_{\ell}^{c}\frac{\sqrt{(c-\xi)(\xi-\ell)}}{\xi-x}\left\{2\pi U y'(\xi) - \int_0^{\ell}\frac{\gamma_1(\eta)d\eta}{\eta-\xi}\right\}d\xi + \frac{B}{\pi^2\sqrt{(c-x)(x-\ell)}} , \qquad [4.b.15]$$

B being the arbitrary constant.

Now we consider the boundary condition, at the trailing edge of the hydrofoil where it is postulated the liquid leaves the hydrofoil smoothly along the tangent at the trailing edge this being the Kutta condition.

One of the consequences of this assumption is that the vorticity must vanish at $x=c$ and thus we have

$$\gamma_2(c)=0 \quad , \qquad [4.b.16]$$

hence

$$B = \oint_{\ell}^{c} \frac{\sqrt{(c-\xi)(\xi-\ell)}\,d\xi}{\xi-c} \left\{2\pi U\gamma'(\xi) - \int_{0}^{\ell} \frac{\gamma_1(\eta)\,d\eta}{\eta-\xi}\right\} \quad . \qquad [4.b.17]$$

Substituting from [4.b.17] in [4.b.15] we obtain

$$\gamma_2(x) = -\frac{1}{\pi^2}\sqrt{\frac{c-x}{x-\ell}} \oint_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \left[2\pi U\gamma'(\xi) - \oint_{0}^{\ell} \frac{\gamma_1(\eta)\,d\eta}{\eta-\xi}\right] \frac{d\xi}{\xi-x} \quad , \qquad (\ell<x<c)$$

$$= -\frac{2U}{\pi}\sqrt{\frac{c-x}{x-\ell}} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \frac{\gamma'(\xi)\,d\xi}{\xi-x} + \frac{1}{\pi^2}\sqrt{\frac{c-x}{x-\ell}} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \cdot \frac{d\xi}{\xi-x} \int_{0}^{\ell} \frac{\gamma_1(\eta)\,d\eta}{\eta-\xi} \quad . \qquad [4.b.18]$$

We change the order of integration in the second term on the right-hand side of [4.b.18] and when we do so we obtain

$$\gamma_2(x) = -\frac{2U}{\pi}\sqrt{\frac{c-x}{x-\ell}} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \cdot \frac{\gamma'(\xi)\,d\xi}{\xi-x} + \frac{1}{\pi^2}\sqrt{\frac{c-x}{x-\ell}} \int_{0}^{\ell} \gamma_1(\eta)\,d\eta \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \frac{d\xi}{(\xi-x)(\eta-\xi)}$$

$$= -\frac{2U}{\pi}\sqrt{\frac{c-x}{x-\ell}} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \cdot \frac{\gamma'(\xi)\,d\xi}{\xi-x} + \frac{1}{\pi^2}\sqrt{\frac{c-x}{x-\ell}} \int_{0}^{\ell} \frac{\gamma_1(\eta)\,d\eta}{\eta-x} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \left[\frac{1}{\xi-x} + \frac{1}{\eta-\xi}\right] d\xi \quad ; \qquad [4.b.19]$$

In evaluating the inner integrals we use the fact that x lies between ℓ and c while η lies between o and ℓ, we have

$$\left.\begin{aligned} I_3 &= \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \cdot \frac{d\xi}{\xi-x} = \pi && (\ell<x<c) \ , \\ I_4 &= \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \cdot \frac{d\xi}{\eta-\xi} = -\pi\left[1 - \sqrt{\frac{\ell-\eta}{c-\eta}}\right] && (o<\eta<\ell) \ . \end{aligned}\right\} \qquad [4.b.20]$$

Substituting from [4.b.20] in [4.b.19] we obtain

$$\gamma_2(x) = -\frac{2U}{\pi}\sqrt{\frac{c-x}{x-\ell}} \int_{\ell}^{c} \sqrt{\frac{\xi-\ell}{c-\xi}} \frac{\gamma'(\xi)\,d\xi}{\xi-x} + \frac{1}{\pi}\sqrt{\frac{c-x}{x-\ell}} \int_{0}^{\ell} \sqrt{\frac{\ell-\eta}{c-\eta}} \cdot \frac{\gamma_1(\eta)\,d\eta}{\eta-x} \quad . \qquad [4.b.21]$$

We require now to calculate the integral

$$I_5 = \int_{\ell}^{c} \frac{\gamma_2(\xi)\,d\xi}{\xi-x} \qquad (o<x<\ell) \quad . \qquad [4.b.22]$$

Using [4.b.21] we obtain

$$I_5 = \int_{\ell}^{c} \frac{d\xi}{\xi - x}\left\{-\frac{2U}{\pi}\sqrt{\frac{c-\xi}{\xi-\ell}}\int_{\ell}^{c}\sqrt{\frac{\eta-\ell}{c-\eta}}\cdot\frac{\gamma(\eta)d\eta}{\eta-\xi} + \frac{1}{\pi}\sqrt{\frac{c-\xi}{\xi-\ell}}\int_{0}^{\ell}\sqrt{\frac{\ell-\eta}{c-\eta}}\,\frac{\gamma_1(\eta)d\eta}{\eta-\xi}\right\}$$

$$= -\frac{2U}{\pi}\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\frac{d\xi}{\xi-x}\int_{\ell}^{c}\sqrt{\frac{\eta-\ell}{c-\eta}}\frac{\gamma(\eta)d\eta}{\eta-\xi} + \frac{1}{\pi}\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\frac{d\xi}{\xi-x}\int_{0}^{\ell}\sqrt{\frac{\ell-\eta}{c-\eta}}\frac{\gamma_1(\eta)d\eta}{\eta-\xi} \quad . \qquad [4.b.23]$$

Using the Poincaré-Bertrand formula

$$\int_{C_1}\frac{\phi_1(\xi)d\xi}{\xi-x}\int_{C_2}\frac{\phi_2(\eta)d\eta}{\eta-\xi} = -\pi^2\phi_1(x)\phi_2(x) + \int_{C_2}\phi_2(\eta)d\eta\int_{C_1}\frac{\phi_1(\xi)d\xi}{(\xi-x)(\eta-\xi)}. \qquad [4.b.24]$$

[see, e.g., Muskhelishvili, N.I. (45)] and bearing in mind that $\gamma_1(x)$ is defined in $0<x<\ell$ we obtain

$$I_5 = -\frac{2U}{\pi}\int_{\ell}^{c}\sqrt{\frac{\eta-\ell}{c-\eta}}\gamma(\eta)d\eta\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\frac{d\xi}{(\xi-x)(\eta-\xi)} + \frac{1}{\pi}\int_{0}^{\ell}\sqrt{\frac{\ell-\eta}{c-\eta}}\gamma_1(\eta)d\eta\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\frac{d\xi}{(\xi-x)(\eta-\xi)} + 2\pi U\gamma(x)$$

$$= -\frac{2U}{\pi}\int_{\ell}^{c}\sqrt{\frac{\eta-\ell}{c-\eta}}\frac{\gamma(\eta)d\eta}{\eta-x}\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\left[\frac{1}{\xi-x}+\frac{1}{\eta-\xi}\right]d\xi + \frac{1}{\pi}\int_{0}^{\ell}\sqrt{\frac{\ell-\eta}{c-\eta}}\frac{\gamma_1(\eta)d\eta}{\eta-x}\int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\left[\frac{1}{\xi-x}+\frac{1}{\eta-\xi}\right]d\xi + 2\pi U\gamma(x),$$

$$(o<x<\ell) \quad . \qquad [4.b.25]$$

In evaluating the inner integrals we use the fact that x lies between o and ℓ while η lies between ℓ and c, we can write

$$\left.\begin{aligned} J_1 &= \int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-\ell}}\frac{d\xi}{\xi-x} = \pi\left[\sqrt{\frac{c-x}{\ell-x}} - 1\right] \qquad (o<x<\ell)\ , \\ J_2 &= \int_{\ell}^{c}\sqrt{\frac{c-\xi}{\xi-c}}\frac{d\xi}{\eta-\xi} = \pi \qquad (\ell<\eta<c)\ . \end{aligned}\right\} \qquad [4.b.26]$$

Substituting from [4.b.26] in [4.b.25] we obtain

$$I_5 = \int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{\xi-x} = -2U\sqrt{\frac{c-x}{\ell-x}}\int_{\ell}^{c}\sqrt{\frac{\eta-\ell}{c-\eta}}\frac{\gamma(\eta)d\eta}{\eta-x} + \int_{0}^{\ell}\frac{\gamma_1(\eta)d\eta}{\eta-x}\left[\sqrt{\frac{(\ell-\eta)(c-x)}{(c-\eta)(\ell-x)}} - 1\right] + 2\pi U\gamma(x) \ .$$

$$[4.b.27]$$

If we substitute the expression for $m(x)$ from [4.b.13] in [4.b.1] we can write

$$\int_{0}^{\ell}\frac{\gamma_1(\xi)d\xi}{\xi-x}\left[1 + \sqrt{\frac{\xi(\ell-\xi)}{x(\ell-x)}}\right] + \int_{\ell}^{c}\frac{\gamma_2(\xi)d\xi}{\xi-x} = f_1(x) \qquad , \qquad [4.b.28]$$

where

$$f_1(x) = 2\pi U y'(x) + \frac{\pi U \sigma}{2\sqrt{x(\ell-x)}}(\ell-2x) . \qquad [4.b.29]$$

Substituting from [4.b.27] in [4.b.28] we obtain

$$\int_0^\ell \frac{\gamma_1(\xi)d\xi}{\xi-x}\left[1+\sqrt{\frac{\xi(\ell-\xi)}{x(\ell-x)}}\right]+\int_0^\ell \frac{\gamma_1(\xi)d\xi}{\xi-x}\left[\sqrt{\frac{(\ell-\xi)(c-x)}{(c-\xi)(\ell-x)}}-1\right] \equiv$$

$$\equiv \int_0^\ell \frac{\gamma_1(\xi)d\xi}{\xi-x}\left[\sqrt{\frac{\xi(\ell-\xi)}{x(\ell-x)}}+\sqrt{\frac{(\ell-\xi)(c-x)}{(c-\xi)(\ell-x)}}\right] \equiv g(x) \quad , \quad (0<x<\ell<c) \; , \qquad [4.b.30]$$

where

$$g(x) = \frac{\pi U\sigma}{2\sqrt{x(\ell-x)}}(\ell-2x)+2U\sqrt{\frac{c-x}{\ell-x}}\int_\ell^c \sqrt{\frac{\eta-\ell}{c-\eta}} \cdot \frac{y'(\eta)d\eta}{\eta-x} \quad , \qquad [4.b.31]$$

hence

$$\int_0^\ell \frac{(\ell-\xi)^{\frac{1}{2}}\gamma_1(\xi)d\xi}{\xi-x}\left[\sqrt{\frac{\xi}{x}}+\sqrt{\frac{c-x}{\ell-\xi}}\right] \equiv g(x)\sqrt{\ell-x} \; . \qquad [4.b.32]$$

For convenience we can write

$$\left.\begin{aligned} (\ell-\xi)^{\frac{1}{2}}\gamma_1(\xi) &= \Gamma(\xi) \quad , \\ g(x)(\ell-x)^{\frac{1}{2}} &= G(x) \quad , \end{aligned}\right\} \qquad [4.b.33]$$

and consequently equation [4.b.32] can be expressed in the form

$$\int_0^\ell \frac{\Gamma(\xi)d\xi}{\xi-x}\left[\sqrt{\frac{\xi}{x}}+\sqrt{\frac{c-x}{\ell-\xi}}\right] = G(x) \qquad (0<x<\ell) \; . \qquad [4.b.34]$$

In order to find the solution of equation [4.b.34] we proceed as follows:

We make the transformation

$$\left.\begin{aligned} \xi &= c \sin^2\theta \quad , \\ x &= c \sin^2\theta_0 \quad , \\ \ell &= c \sin^2\alpha \quad , \end{aligned}\right\} \qquad [4.b.35]$$

then we obtain

$$\int_0^\alpha \frac{\Gamma(c\sin^2\theta).\,2\sin\theta\cos\theta d\theta}{[\sin^2\theta - \sin^2\theta_o]}\left[\frac{\sin\theta}{\sin\theta_o} + \frac{\cos\theta_o}{\cos\theta}\right] = G(c\sin^2\theta_o) \quad , \qquad [4.b.36]$$

hence

$$\int_0^\alpha \frac{2\Gamma(c\sin^2\theta).\sin\theta d\theta}{\cos2\theta_o - \cos2\theta}\,[\sin2\theta+\sin2\theta_o] = \sin\theta_o G(c\sin^2\theta_o) \quad .$$

If we introduce the new function

$$\left.\begin{aligned} f(\theta_o) &= \sin\theta_o G(c\sin^2\theta_o) \quad , \\ \phi(\theta) &= 2\Gamma(c\sin^2\theta)\sin\theta \quad , \end{aligned}\right\} \qquad [4.b.37]$$

then we can write [4.b.37] in the form

$$\int_0^\alpha \frac{\phi(\theta)\cos(\theta-\theta_o)d\theta}{\sin(\theta-\theta_o)} = f(\theta_o) \quad . \qquad [4.b.38]$$

We transform the integral equation [4.b.38] into standard Cauchy form [see Appendix II] to give

$$\int_0^b \frac{\Phi(t)dt}{t-t_o} = F(t_o) \quad , \qquad [4.b.39]$$

where

$$\left.\begin{aligned} F(t_o) &= \frac{f(\tan^{-1}t_o)}{t_o^2+1} - \frac{A\,t_o}{1+t_o^2} \qquad (t_o=\tan\theta_o) \quad , \\ \Phi(t) &= \frac{\phi(\tan^{-1}t)}{t^2+1} \qquad (t=\tan\theta) \quad , \\ A &= \int_0^b \frac{\phi(\tan^{-1}t)dt}{t^2+1} = \int_0^\alpha \phi(\theta)\,d\theta \quad (b=\tan\alpha) \quad . \end{aligned}\right\} \qquad [4.b.40]$$

Now we solve equation [4.b.39] and it is convenient to express this result in the form

$$\Phi(t) = \frac{-1}{\pi^2\sqrt{t(b-t)}}\int_0^b \frac{F(t_o)\sqrt{t_o(b-t_o)}dt_o}{t_o-t} + \frac{D}{\sqrt{t(b-t)}}$$

$$= \frac{-1}{\pi^2\sqrt{\tan\theta(\tan\alpha-\tan\theta)}}\int_0^\alpha \frac{F(\tan\theta_o)\sqrt{\tan\theta_o(\tan\alpha-\tan\theta_o)}.\sec^2\theta_o d\theta}{\tan\theta_o-\tan\theta} +$$

$$+ \frac{D}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad . \qquad [4.b.41]$$

Substituting from [4.b.40] in [4.b.41] we obtain

$$\phi(\theta)=\frac{-1}{\pi^2\sqrt{\sin\theta\sin(\alpha-\theta)}}\int_0^\alpha\frac{[f(\theta_o)-A\tan\theta_o]\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}+\frac{D\sqrt{\cos\alpha}\sec\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}, \quad [4.b.42]$$

where

$$\left.\begin{aligned}
&f(\theta_o)=\tfrac{1}{2}\pi U\sigma\sqrt{c}(\sin^2\alpha-2\sin^2\theta_o)+4\sqrt{c}U\sin\theta_o\cos\theta_o\int_\alpha^{\frac{\pi}{2}}\frac{\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi\,d\phi}{\sin^2\phi-\sin\theta_o},\\
&\qquad\qquad [z(\phi)=y'(c\sin^2\phi)]\ ,\\
&\phi(\theta)=2\sqrt{c}\sqrt{\sin^2\alpha-\sin^2\theta}.\nu(\theta)\sin\theta\ , \qquad [\nu(\theta)=\gamma_1(c\sin^2\theta)]\ ,
\end{aligned}\right\}[4.b.43]$$

hence

$$\nu(\theta)=\frac{D\sqrt{\cos\alpha}.\sec\theta\operatorname{cosec}\theta}{2\sqrt{c}\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}-$$

$$-\frac{\operatorname{cosec}\theta}{2\pi^2\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\left\{\tfrac{1}{2}\pi U\sigma\int_0^\alpha\frac{(\sin^2\alpha-2\sin^2\theta_o)\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}+\right.$$

$$+4U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi\,d\phi\int_0^\alpha\frac{\sin\theta_o\cos\theta_o\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{[\sin^2\phi-\sin^2\theta_o].\sin(\theta_o-\theta)}-$$

$$\left.-\frac{A}{\sqrt{c}}\int_0^\alpha\frac{\tan\theta_o\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}\right\} \qquad (0<\theta<\alpha<\phi<\tfrac{\pi}{2}) . \quad [4.b.44]$$

In evaluating the inner integrals we use the fact that θ and θ_o lie between 0 and α, while ϕ lies between α and $\frac{\pi}{2}$, so that

$$\left.\begin{aligned}
I_8&=\int_0^\alpha\frac{\tan\theta_o\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}\\
&=-\frac{\pi\sqrt{\cos\alpha}}{\cos\theta}+\pi\cos(\theta-\tfrac{1}{2}\alpha)\ , \qquad (0<\theta<\alpha<\tfrac{\pi}{2})\ ,\\
I_9&=\int_0^\alpha\frac{\sin\theta_o\cos\theta_o\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)[\sin^2\phi-\sin^2\theta_o]}\\
&=-\frac{\pi\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}[\tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})+\tan\theta(\sqrt{\tan\phi-\tan\alpha}-\\
&\quad-\sqrt{\tan\phi+\tan\alpha})]+\pi\cos(\theta-\tfrac{1}{2}\alpha)\ , \qquad (0<\theta<\alpha<\phi<\tfrac{\pi}{2})\ .
\end{aligned}\right\}$$

[4.b.45]

These integrals are evaluated in Appendix III.

Substituting from [4.b.45] in [4.b.44] we obtain

$$\nu(\theta) = \frac{D\sqrt{\cos\alpha}\sec\theta\operatorname{cosec}\theta}{2\sqrt{c}\sqrt{\sin^2\alpha-\sin^2\theta}\,.\sqrt{\sin\theta\sin(\alpha-\theta)}} -$$

$$- \frac{\operatorname{cosec}\theta}{2\pi^2\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\Bigg\{\tfrac{1}{2}\pi U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)} +$$

$$+ 4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.z(\phi)\sin\phi\, d\phi\Bigg[\Big[\frac{-\pi\sqrt{\tan\phi}\sqrt{\cos\alpha}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +$$

$$+ \tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\pi\cos(\theta-\tfrac{1}{2}\alpha)\Big]\Bigg] + \frac{A}{\sqrt{c}}\left(\frac{\pi\sqrt{\cos\alpha}}{\cos\theta} - \pi\cos(\theta-\tfrac{1}{2}\alpha)\right)\Bigg\},$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \qquad [4.b.46]$$

where D is an arbitrary constant.

The constant A is evaluated in Appendix IV , and can be written in the form

$$A = 2\sqrt{c}\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}\,.\sin\theta\ \nu(\theta)\,d\theta \qquad [\nu(\theta)=\gamma_1(c\sin^2\theta)]$$

$$= \pi D \qquad [4.b.47]$$

Substituting from [4.b.47] in [4.b.46] we obtain

$$\nu(\theta) = \frac{D\ \cos(\theta-\frac{1}{2}\alpha)}{2\sqrt{c}\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta}\ \ \sin\theta\sin(\alpha-\theta)} -$$

$$- \frac{\operatorname{cosec}\theta}{2\pi\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\Bigg\{\tfrac{1}{2}U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)} +$$

$$+ 4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi\, z(\phi)\,d\phi\Bigg[\Big[\frac{-\sqrt{\cos\alpha}\ \sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +$$

$$+ \tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\Big]\Bigg] \qquad ,[\nu(\theta)=\gamma_1(c\sin^2\theta),z(\phi)=\gamma(c\sin^2\phi)]$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \quad . \qquad [4.b.48]$$

Now we calculate the arbitrary constant D , using [4.b.13] and [4.b.35] we can write

$$m(x) = \frac{1}{\pi\sqrt{x(\ell-x)}}\left\{\tfrac{1}{2}\pi U\sigma(\ell-2x) - \int_0^{\ell}\frac{\gamma_1(\xi)\sqrt{\xi(\ell-\xi)}\,.d\xi}{\xi-x}\right\}$$

$$= \frac{1}{\pi\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta_o}}\{\tfrac{1}{2}\pi U\sigma(\sin^2\alpha-2\sin^2\theta_o)- I\} , \qquad [4.b.49]$$

where the integral I is evaluated in Appendix V and can be written in the form

$$I = 2\int_0^{\alpha}\frac{\nu(\theta)\sin^2\alpha\cos\theta\sqrt{\sin^2\alpha-\sin^2\theta}d\theta}{\sin^2\alpha-\sin^2\theta_o}$$

$$= \frac{1}{\sqrt{\cos\alpha}\sqrt{\tan\theta_o}}\left\{\frac{\pi D\cos(\frac{1}{2}\alpha+\theta_o)}{2\sqrt{c}\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} + \tfrac{1}{2}\pi U\sigma\sqrt{\cos\alpha}\sqrt{\tan\theta_o}(\sin^2\alpha-2\sin^2\theta_o) - \right.$$

$$- \frac{\pi U\sigma\sin^2\alpha\sin(\frac{1}{2}\alpha-\theta_o)}{4\cos\theta_o\sqrt{\tan\alpha-\tan\theta_o}} + \frac{\pi U\sigma}{16\cos\theta_o\cos\frac{1}{2}\alpha\sqrt{\tan\alpha-\tan\theta_o}}[\sin\alpha(\cos\theta_o-\cos(\alpha-\theta_o)) +$$

$$+ 8\sin^2\theta_o\cos\tfrac{1}{2}\alpha(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_o))] + 2\pi Uz(\theta_o)\sqrt{\cos\alpha}\sqrt{\tan\theta_o}\sin\theta_o\cos^2\theta_o\sqrt{\sin^2\alpha-\sin^2\theta_o} +$$

$$+ \frac{U}{\cos^2\theta_o}\int_{\alpha}^{\frac{\pi}{2}}\frac{\sqrt{\sin^2\phi-\sin^2\alpha}\sqrt{\tan\phi}\sin\phi z(\phi)d\phi}{(\tan^2\phi-\tan^2\theta_o)}\left(\left(\frac{\tan\phi}{\sqrt{\tan\alpha+\tan\theta_o}}[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]\right.\right.$$

$$\left.\left.- \frac{\tan\theta_o}{\sqrt{\tan\alpha+\tan\theta_o}}[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}]\right)\right) -$$

$$\left.- \frac{2U\cos(\frac{1}{2}\alpha+\theta_o)}{\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} . \int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi\right\} . \qquad [4.b.50]$$

Substituting from [4.b.50] into [4.b.49] we obtain

$$m(c\sin^2\theta_o) = - 2\pi Uz(\theta_o)\cos^2\theta_o - \frac{\mathrm{cosec}\,\theta_o}{\sqrt{\cos\alpha}\sqrt{\tan\theta_o}\sqrt{\sin^2\alpha-\sin^2\theta_o}}\left\{\frac{\pi D\cos(\frac{1}{2}\alpha+\theta_o)}{2\sqrt{c}\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} - \right.$$

$$- \frac{\pi U\sigma\sin^2\alpha\sin(\frac{1}{2}\alpha-\theta_o)}{4\cos\theta_o\sqrt{\tan\alpha-\tan\theta_o}} + \frac{\pi U\sigma}{16\cos\theta_o\cos\frac{1}{2}\alpha\sqrt{\tan\alpha-\tan\theta_o}}[\sin\alpha(\cos\theta_o-\cos(\alpha-\theta_o)) +$$

$$+ 8\sin^2\theta_o\cos\tfrac{1}{2}\alpha(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_o))] +$$

$$+ \frac{U}{\cos^2\theta_o}\int_{\alpha}^{\frac{\pi}{2}}\frac{\sqrt{\sin^2\phi-\sin^2\alpha}\sqrt{\tan\phi}\sin\phi z(\phi)d\phi}{(\tan^2\phi-\tan^2\theta_o)}\left(\frac{\tan\phi}{\sqrt{\tan\alpha+\tan\theta_o}}[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] -\right.$$

$$\left.- \frac{\tan\theta_o}{\sqrt{\tan\alpha+\tan\theta_o}}[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}]\right) -$$

$$\left.- \frac{2U\cos(\frac{1}{2}\alpha+\theta_o)}{\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} . \int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi \right\} . \qquad [4.b.51]$$

The function $m(c\sin^2\theta_o)$ for small value of θ_o is of the form

$$m(c\sin^2\theta_o) = \frac{A_o}{\theta_o^{3/2}} \quad \frac{A_1}{\theta_o^{\frac{1}{2}}} + O(1) + A_2\theta^{\frac{1}{2}} + A_3\theta^{3/2}$$

$$= \frac{A_o}{\left(\frac{x}{c}\right)^{\frac{3}{4}}} + \frac{A_1}{\left(\frac{x}{c}\right)^{\frac{1}{4}}} + O(1) + A_2\left(\frac{x}{c}\right)^{\frac{1}{4}} + A_3\left(\frac{x}{c}\right)^{\frac{3}{4}} , \qquad [4.b.52]$$

where

$$\left.\begin{aligned}
A_o &= -\frac{1}{\sin^{3/2}\alpha}\left\{ \frac{\pi D}{2\sqrt{c}}\cos\tfrac{1}{2}\alpha - \frac{\pi U\sigma}{4}\sin^3\tfrac{1}{2}\alpha(1+2\cos\alpha) + \right. \\
&\quad \left. + U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,\sin\phi z(\phi)d\phi\left\{\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}} - 2\cos\tfrac{1}{2}\alpha\right\}\right\}, \\
A_1 &= -\frac{1}{\sin^{3/2}\alpha}\left\{ -\frac{\pi D\sin\frac{1}{2}\alpha}{2\sqrt{c}} + \frac{\pi U\sigma\sin^2\alpha(2\cos\alpha+1)}{16\ \cos\frac{1}{2}\alpha} + \right. \\
&\quad \left. + U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi d\phi\left\{\frac{1}{\tan\phi}\left[\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}-\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right] + 2\sin\tfrac{1}{2}\alpha\right\}\right\}, \\
A_2 &= -\frac{\pi U\sigma}{2\sin^{3/2}\alpha}\left(\sqrt{\cos\alpha}+\cos\tfrac{1}{2}\alpha\right) , \quad A_3 = -\frac{\pi U\sigma\sin\frac{1}{2}\alpha}{2\sin^{3/2}\alpha} .
\end{aligned}\right\}$$

$$[4.b.53]$$

The behaviour of $m(x)$ for small x has been discussed in earlier papers in cavity theory and the accepted behaviour is $m(x)\ \alpha x^{-\frac{1}{4}}$ [see, e.g., Davies, T.V. (13),(14)] hence we choose A_o to be zero in order to achieve the proper behaviour.

Thus we obtain

$$D = \frac{2\sqrt{c}}{\pi\cos\frac{1}{2}\alpha}\left\{\tfrac{1}{4}\pi U\sigma\sin^3\tfrac{1}{2}\alpha(1+2\cos\alpha) - U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\sin\phi z(\phi)d\phi\left\{\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}} + \right.\right.$$

$$\left.\left. + \sqrt{\frac{\tan\phi+\tan\alpha}{\tan\alpha}} - 2\cos\tfrac{1}{2}\alpha\right\}\right\} . \qquad [4.b.54]$$

Substituting from [4.b.54] in [4.b.48] we obtain

$$
\begin{aligned}
\nu(\theta) = &-\frac{\operatorname{cosec}\theta}{2\pi\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\cdot\Bigg\{\cos(\theta-\tfrac{1}{2}\alpha)\Bigg[B_1 - \\
&-2U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\,\sin\phi\, z(\phi)\,d\phi\Bigg\{\Bigg[B_2\Bigg(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\Bigg)+B_3\Bigg]\Bigg\}\Bigg]+ \\
&+\tfrac{1}{2}U\sigma\int_{0}^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)}+4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\,\sin\phi\, z(\phi)\,d\phi\,\cdot \\
&\cdot\Bigg[\Bigg[-\frac{\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}\Big(\tan\phi\big[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}\big]+\tan\theta\big[\sqrt{\tan\phi-\tan\alpha}\,- \\
&\sqrt{\tan\phi+\tan\alpha}\big]\Big)+\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\Bigg\}\quad ,
\end{aligned}
\qquad [4.b.55]
$$

with

$$
\left.
\begin{aligned}
B_1 &= \frac{\pi U\sigma\sin^3\tfrac{1}{2}\alpha(1+2\cos\alpha)}{4\cos\tfrac{1}{2}\alpha}\ , \\
B_2 &= \frac{1}{\cos\tfrac{1}{2}\alpha}\ , \\
B_3 &= -2\ .
\end{aligned}
\right\}
\qquad [4.b.56]
$$

IVc DETERMINATION THE GENERAL FORMULA FOR THE LIFT AND DRAG

Let the x- and y- components of the hydrodynamic forces acting on the hydrofoil be denoted by drag D and lift L , and these forces are calculated as follows:

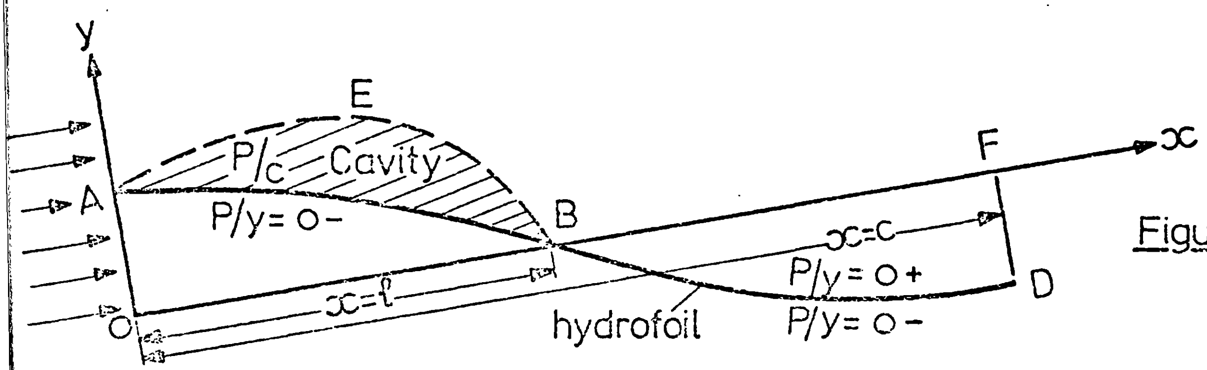

Figure 13.

The value of the pressure on the surface of the cavity is

$$P=P_{\infty}+\rho U\phi_x\Big|_{y=o} \qquad (o<x<c) \quad , \qquad [4.c.1]$$

and thus on the cavity surface we must have

$$\phi_x\Big|_{y=o} = \frac{P_c-P_{\infty}}{\rho U} \qquad (o<x<c) \quad , \qquad [4.c.2]$$

hence from [4.c.2] we have on the cavity surface

$$\phi_x\Big|_{y=o} = -\tfrac{1}{2}U\sigma \qquad \left(\sigma = \frac{P_{\infty}-P_c}{\tfrac{1}{2}\rho U^2}\right) \quad , \qquad [4.c.3]$$

where

$$\phi_x\Big|_{y=o\pm} = \lim_{y\to o\pm}\left(\frac{\partial\phi}{\partial x}\right) = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} \pm \tfrac{1}{2}\gamma_1(x) & (o<x<\ell)\ , \\ -\dfrac{1}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} \pm \tfrac{1}{2}\gamma_2(x) & (\ell<x<c)\ . \end{cases} \qquad [4.c.4]$$

Denoting P on $y=o+$ and $y=o-$ respectively by $P\Big|_{y=o+}$ and $P\Big|_{y=o-}$.

As y tends to zero through positive values we can write from [4.c.1] and [4.c.4]

$$P\Big|_{y=o+} \equiv \begin{cases} P_{\infty} - \dfrac{\rho U}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} + \tfrac{1}{2}\rho U\gamma_1(x) & (o<x<\ell)\ , \\ P_{\infty} - \dfrac{\rho U}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} + \tfrac{1}{2}\rho U\gamma_2(x) & (\ell<x<c)\ . \end{cases} \qquad [4.c.5]$$

As $y\to o-$ through negative values (since we shall be concerned with the under-surface of the hydrofoil) we can write from [4.c.1] and [4.c.4]

$$P\Big|_{y=o-} \equiv \begin{cases} P_{\infty} - \dfrac{\rho U}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} - \tfrac{1}{2}\rho U\gamma_1(x) & (o<x<\ell)\ , \\ P_{\infty} - \dfrac{\rho U}{2\pi}\displaystyle\int_o^{\ell}\frac{m(\xi)d\xi}{x-\xi} - \tfrac{1}{2}\rho U\gamma_2(x) & (\ell<x<c)\ . \end{cases} \qquad [4.c.6]$$

Hence from [4.c.5] and [4.c.6] we have

$$P\Big|_{y=o-} - P\Big|_{y=o+} = -\rho U\gamma_2(x) \qquad (\ell<x<c) \quad . \qquad [4.c.7]$$

On y=o- we eliminate P_∞ between [4.c.1] and [4.c.3] we obtain

$$P\Big|_{y=o-} - P_c = \rho U\phi_x\Big|_{y=o-} + \tfrac{1}{2}\rho U^2\sigma \qquad (o<x<\ell) \quad . \qquad [4.c.8]$$

Substituting from [4.c.4] in [4.c.8] we obtain

$$P\Big|_{y=o-} - P_c = \rho U\left[-\frac{1}{2\pi}\int_o^\ell \frac{m(\xi)d\xi}{x-\xi} - \tfrac{1}{2}\gamma_1(x)\right] + \tfrac{1}{2}\rho U^2\sigma \qquad (o<x<\ell) \quad . \qquad [4.c.9]$$

The complex forces acting on the hydrofoil calculated with the linearized theory is given by

$$D+iL = \int_o^\ell [P\Big|_{y=o-} - P_c]idz + \int_\ell^c [P\Big|_{y=o-} - P\Big|_{y=o+}]idz \quad . \qquad [4.c.10]$$

We can write from [4.c.10] the expression for the lift L in the form

$$L = \int_o^\ell [P\Big|_{y=o-} - P_c]dx + \int_\ell^c [P\Big|_{y=o-} - P\Big|_{y=o+}]dx \quad . \qquad [4.c.11]$$

Substituting from [4.c.9] and [4.c.7] in [4.c.11] we obtain

$$L = \rho U\int_o^\ell \left[-\frac{1}{2\pi}\int_o^\ell \frac{m(\xi)d\xi}{x-\xi} - \tfrac{1}{2}\gamma_1(x)\right]dx + \tfrac{1}{2}\rho U^2\sigma\int_o^\ell dx - \rho U\int_\ell^c \gamma_2(x)dx \quad . \qquad [4.c.12]$$

We have from [4.b.3]

$$\int_o^\ell \frac{m(\xi)d\xi}{x-\xi} = \pi\gamma_1(x) - \pi U\sigma \quad . \qquad [4.c.13]$$

Substituting [4.c.13] in [4.c.12] we can write

$$L = \rho U^2\sigma\ell - \rho U\int_o^\ell \gamma_1(x)dx - \rho U\int_\ell^c \gamma_2(x)dx \quad . \qquad [4.c.14]$$

From [4.b.21] we have relation between γ_1 and γ_2 defined by

$$\gamma_2(x) = -\frac{2U}{\pi}\sqrt{\frac{c-x}{x-\ell}}\int_\ell^c \sqrt{\frac{\xi-\ell}{c-\xi}}\,\frac{y'(\xi)d\xi}{\xi-x} + \frac{1}{\pi}\sqrt{\frac{c-x}{x-\ell}}\int_o^\ell \sqrt{\frac{\ell-\eta}{c-\eta}}\cdot\frac{\gamma_1(\eta)d\eta}{\eta-x} \quad . \qquad [4.c.15]$$

Using the above expression for γ_2 in [4.c.14] we obtain

$$L = \rho U^2\sigma\ell - \rho U\int_o^\ell \gamma_1(x)dx + \frac{2\rho U^2}{\pi}\int_\ell^c \sqrt{\frac{c-x}{x-\ell}}\,dx\int_\ell^c \sqrt{\frac{\xi-\ell}{c-\xi}}\cdot\frac{y'(\xi)d\xi}{\xi-x} - \frac{\rho U}{\pi}\int_\ell^c \sqrt{\frac{c-x}{x-\ell}}\,dx\int_o^\ell \sqrt{\frac{\ell-\eta}{c-\eta}}\cdot\frac{\gamma_1(\eta)d\eta}{\eta-x} \quad .$$

[4.c.16]

It is permissible to interchange the order of the two double integrals in [4.c.16] and then we obtain

$$L=\rho U^2\sigma\ell-\rho U\int_0^\ell \gamma_1(x)\,dx+\frac{2\rho U^2}{\pi}\int_\ell^c\sqrt{\frac{\xi-\ell}{c-\xi}}\,y'(\xi)\,d\xi\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\,\frac{dx}{\xi-x}-\frac{\rho U}{\pi}\int_0^\ell\sqrt{\frac{\ell-\eta}{c-\eta}}\,\gamma_1(\eta)\,d\eta\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\,\frac{dx}{\eta-x}\ . \qquad [4.c.17]$$

Now we evaluate the inner integrals in [4.c.17]

$$\left.\begin{aligned} I_1&=\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\cdot\frac{dx}{\xi-x}=\pi && (\ell<\xi<c)\ ,\\ I_2&=\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\cdot\frac{dx}{\eta-x}=\pi\left(1-\sqrt{\frac{c-\eta}{\ell-\eta}}\right) && (o<\eta<\ell)\ .\end{aligned}\right\} \qquad [4.c.18]$$

Substituting from [4.c.18] in [4.c.17] we obtain

$$L=\rho U^2\sigma\ell+2\rho U^2\int_\ell^c\sqrt{\frac{\xi-\ell}{c-\xi}}\,.y'(\xi)\,d\xi-\rho U\int_0^\ell\sqrt{\frac{\ell-\eta}{c-\eta}}\,\gamma_1(\eta)\,d\eta\ . \qquad [4.c.19]$$

By similar operations we can write expression for drag D

$$D=-\rho U^2\sigma[y(\ell)-y(0)]+\rho U\int_0^\ell\gamma_1(x)\,y'(x)\,dx-\frac{2\rho U^2}{\pi}\int_\ell^c\sqrt{\frac{\xi-\ell}{c-\xi}}\,y'(\xi)\,d\xi\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\cdot\frac{y'(x)\,dx}{\xi-x}\ +$$

$$+\frac{\rho U}{\pi}\int_0^\ell\sqrt{\frac{\ell-\eta}{\eta-x}}\,\gamma_1(\eta)\,d\eta\int_\ell^c\sqrt{\frac{c-x}{x-\ell}}\cdot\frac{y'(x)\,dx}{\eta-x}\quad[0<\eta<\ell<x<c,\ell<\xi<c]\ . \qquad [4.c.20]$$

Using the transformation

$$\left.\begin{aligned}\xi&=c\sin^2\theta\ ,\\ x&=c\sin^2\theta_o\ ,\\ \ell&=c\sin^2\alpha\ ,\end{aligned}\right\} \qquad [4.c.21]$$

we can write [4.c.19] and [4.c.20] in the forms

$$L=\rho U^2\sigma\ell+4\rho U^2\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.z(\phi)\sin\phi\,d\phi-2\rho U\int_0^\alpha\sqrt{\sin^2\alpha-\sin^2\theta}\,\nu(\theta)\sin\theta\,d\theta\ ,$$

$$[\nu(\theta)=\gamma_1(c\sin^2\theta),z(\phi)=y(c\sin^2\phi)]\ , \qquad [4.c.22]$$

and

$$D = -\rho U^2\sigma[y(\ell)-y(0)]+2\rho U\int_0^{\alpha}\nu(\theta)z(\theta)\sin\theta\cos\theta d\theta \; -$$

$$- \frac{8\rho U^2}{\pi}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi\int_{\alpha}^{\frac{\pi}{2}}\frac{z(\psi)\sin^2\psi\cos\psi d\psi}{\sqrt{\sin^2\psi-\sin^2\alpha}.(\sin^2\phi-\sin^2\psi)} +$$

$$+ \frac{4\rho U}{\pi}\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}.\nu(\theta)\sin\theta d\theta\int_{\alpha}^{\frac{\pi}{2}}\frac{z(\phi)\sin^2\phi\cos\phi d\phi}{\sqrt{\sin^2\phi-\sin^2\alpha}.(\sin^2\theta-\sin^2\phi)} \quad . \qquad [4.c.23]$$

IVd THE OPTIMUM SHAPE OF A HYDROFOIL IN PARTIAL CAVITY FLOW USING VARIATIONAL CALCULUS TECHNIQUES, SO THAT THE LIFT IS A MAXIMUM

We pose the problem of maximizing the lift coefficient

$$L^* = \frac{L}{\rho U^2} =$$

$$= \sigma\ell - \frac{2c}{\pi}\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}.\nu(\theta).\sin\theta d\theta+4c\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi z(\phi)d\phi, \qquad [4.d.1]$$

subject to a constraint on curvature of the form

$$K = \frac{1}{2c}\int_0^{\frac{\pi}{2}} z'^2(\theta).\sec\theta\operatorname{cosec}\theta d\theta \; , \qquad [4.d.2]$$

where K is prescribed, together with a constraint on the length of the hydrofoil of the form

$$S = 2c\int_0^{\frac{\pi}{2}}\sqrt{1+z^2(\theta)}.\sin\theta\cos\theta d\theta \; , \qquad [4.d.3]$$

where S is prescribed and $z(\theta)=y'(c\sin^2\theta)$ is the gradient of the hydrofoil at position θ .

We also assume in the above optimum problem that α is kept constant.

STATEMENT OF THE PROBLEM

The general optimum problem considered here may be stated as follows:

To find the real, extremal function $\nu(\theta)$ of a real variable, required to be Hölder continuous [see, e.g., Tricomi,F.G.(61)] in the region $0<\theta<\alpha$ together with

$$\nu(\theta) = \frac{-\operatorname{cosec}\theta}{2\pi\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}} \cdot \Bigg\{ \cos(\theta-\tfrac{1}{2}\alpha) \cdot \Bigg[B_1 -$$

$$- 2U\int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi z(\phi)\,d\phi \left\{\left[B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}} + \sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right) + B_3\right]\right\}\Bigg] +$$

$$+ \tfrac{1}{2}U\sigma\int_0^\alpha \frac{(\sin^2\alpha-2\sin^2\theta_o)\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)} +$$

$$+ 4U\int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi z(\phi)\,d\phi\Bigg[\Bigg[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}\Big(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +$$

$$\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}]\Big) + \cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\Bigg\} , \qquad [4.d.4]$$

with

$$\left.\begin{aligned} B_1 &= \frac{\pi U\sigma\sin^3\frac{1}{2}\alpha(1+2\cos\alpha)}{4\cos\frac{1}{2}\alpha} ,\\ B_2 &= \frac{1}{\cos\frac{1}{2}\alpha} ,\\ B_3 &= -2 . \end{aligned}\right\} \qquad [4.d.5]$$

so that $\nu(\theta)$ and $z(\theta)$ minimize the new functional.

$$I[\nu(\theta),z(\theta),z'(\theta),\theta] = -L^* + \lambda_1 K + \lambda_2 S \equiv$$

$$\equiv \int_0^{\frac{\pi}{2}} F[\nu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2]\,d\theta , \qquad [4.d.6]$$

with the function $F[\nu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2]$ given by

$$F[\nu(\theta),z(\theta),z'(\theta),\theta;\lambda_1,\lambda_2] \equiv \begin{cases} \dfrac{\lambda_1}{2c} z'^2(\theta)\sec\theta\operatorname{cosec}\theta + \lambda_2\sqrt{1+z^2(\theta)}.2c\sin\theta\cos\theta + \\ \quad + \dfrac{2c}{U}\sqrt{\sin^2\alpha-\sin^2\theta}.\sin\theta\nu(\theta), \quad (0<\theta<\alpha) \\ \dfrac{\lambda_1}{2c} z'^2(\theta)\sec\theta\operatorname{cosec}\theta + \lambda_2\sqrt{1+z^2(\theta)}.2c\sin\theta\cos\theta - \\ \quad - 4c\sqrt{\sin^2\alpha-\sin^2\theta}.\sin\theta z(\theta) , \quad (\alpha<\theta<\frac{\pi}{2}) \end{cases} \qquad [4.d.7]$$

where $\nu(\theta), z(\theta)$ are related by [4.d.4] and λ_1, λ_2 are Lagrange multipliers.

We define an admissible function as any function $\nu(\theta)$ which satisfies the Hölder condition $\mathcal{H}(\mu<1)$, the constraints [4.d.2] and [4.d.3], and we assume that the optimal function is an admissible function which minimizes the function $I[\nu,z,z',\theta]$.

THE NECESSARY CONDITION OF OPTIMALITY

Let $\nu(\theta),z(\theta)$ denote the required optimal vortex distribution function and optimal hydrofoil slope respectively; we write

$$\left.\begin{aligned} \nu_1(\theta)&=\nu(\theta)+\varepsilon\xi(\theta) \quad , \\ z_1(\theta)&=z(\theta)+\varepsilon\eta(\theta) \quad . \end{aligned}\right\} \qquad [4.d.8]$$

We can use [4.d.4] to obtain the following relation between $\xi(\theta)$ and $\eta(\theta)$

$$\xi(\theta) = \frac{-2U\operatorname{cosec}\theta}{\pi\sqrt{\sin^2\alpha-\sin^2\theta}\ \sqrt{\sin\theta\sin(\alpha-\theta)}}\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi).\sin\phi d\phi\left\{\left[\tfrac{1}{2}B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\right.\right.\right.$$

$$\left.\left.+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+\tfrac{1}{2}.B_3\right].\cos(\theta-\tfrac{1}{2}\alpha)+\left[\left[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\right.\right.$$

$$\left.\left.\left.+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\right]\right]\right\} ,$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) , \qquad [4.d.9]$$

where B_2 and B_3 defined by [4.d.5] .

If $\xi(\theta)$ is an admissible variation, then $I[\nu(\theta)+\varepsilon\xi(\theta),z(\theta)+\varepsilon\eta(\theta),z'(\theta)+\varepsilon\eta'(\theta),\theta]$ in [4.d.6] is a function of ε which has an extreme value when $\varepsilon=o$.

For sufficiently small ε , expansion of [4.d.6] in a Taylor series yields

$$\Delta I=\varepsilon\delta I+\frac{\varepsilon^2}{2!}\ \delta^2 I+\dots\dots \qquad . \qquad [4.d.10]$$

We have

$$\Delta I = \int_0^{\frac{\pi}{2}} F[\nu+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', \theta]d\theta - \int_0^{\frac{\pi}{2}} F[\nu, z, z', \theta]d\theta \quad , \qquad [4.d.11]$$

where

$$\delta I = \int_0^{\frac{\pi}{2}} \{\xi(\theta)F_\nu(\nu,z,z',\theta)+\eta(\theta)F_z(\nu,z,z',\theta)+\eta'(\theta)F_{z'}(\nu,z,z',\theta)\}d\theta \ , \qquad [4.d.12]$$

in which the sub-indices denote partial derivatives; it may be noted that $\xi(\theta)$ and $\eta(\theta)$ are related by [4.d.9].

The variations $\delta I, \delta^2 I, \ldots$ depend on $\eta(\theta)$ as well as $z(\theta)$.

We integrate the third term in [4.d.12] by parts and we obtain

$$\delta I=[\eta(\theta).F_{z'}(\nu,z,z',\theta)]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\{\xi(\theta)F_\nu(\nu,z,z',\theta)+\eta(\theta)[F_z(\nu,z,z',\theta) - \frac{d}{d\theta}F_{z'}(\nu,z,z',\theta)]\}d\theta \ . \qquad [4.d.13]$$

Substituting from [4.d.9] in [4.d.13] we can write

$$\delta I=[\eta(\theta).F_{z'}(\nu,z,z',\theta)]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\Bigg\{\Bigg\{\frac{-2U\mathrm{cosec}\theta F_\nu(\nu,z,z',\theta)d\theta}{\pi\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\,\eta(\phi)\sin\phi d\phi$$

$$\Bigg\{\left[\frac{B_2}{2}\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+\tfrac{1}{2}B_3\right].\ \cos\left(\theta-\tfrac{1}{2}\alpha\right)\ +$$

$$+\Bigg[\left[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])\ +\right.$$

$$\left.+\cos\left(\theta-\tfrac{1}{2}\alpha\right)\right]\Bigg]\Bigg\}\ +\ \eta(\theta)[F_z(\nu,z,z',\theta)-\frac{d}{d\theta}F_{z'}(\nu,z,z',\theta)]\Bigg\}\Bigg\}\ d\theta \quad . \qquad [4.d.14]$$

It is permissible to interchange the order of integral on the right-hand side of [4.d.14] [see, e.g., Hardy,G.H.(39)] and when we do so we obtain

$$\delta I=[\eta . F_{z'}(\nu,z,z',\theta)]_{o}^{\frac{\pi}{2}} - \frac{2U}{\pi}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi\,\eta(\phi)\,d\phi\int_{o}^{\frac{\pi}{2}}\frac{\operatorname{cosec}\theta F_{\nu}(\nu,z,z',\theta)d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\,.$$

$$\left\{\left[\tfrac{1}{2}B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+\tfrac{1}{2}B_3\right]\cos(\theta-\tfrac{1}{2}\alpha)\,+\right.$$

$$+\left[\left[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])\,+\right.\right.$$

$$\left.\left.\left.+\cos(\theta-\tfrac{1}{2}\alpha)\right]\right]\right\}+\int_{o}^{\frac{\pi}{2}}\eta(\theta)[F_z(\nu,z,z',\theta)-\frac{d}{d\theta}F_{z'}(\nu,z,z',\theta)]d\theta\,. \qquad [4.d.15]$$

We have from [4.d.7]

$$F_{\nu}[\nu,z,z',\theta] \equiv \begin{cases}\frac{2c}{U}(\sqrt{\sin^2\alpha-\sin^2\theta})\sin\theta & (o<\theta<\alpha)\,,\\ 0 & (\alpha<\theta<\frac{\pi}{2})\,,\end{cases}$$

$$F_z[\nu,z,z',\theta] = \begin{cases}\frac{\lambda_2 z(\theta)}{\sqrt{1+z^2(\theta)}}\,.2c\sin\theta\cos\theta & (o<\theta<\alpha)\,,\\ \frac{\lambda_2 z(\theta)}{\sqrt{1+z^2(\theta)}}\,.2c\sin\theta\cos\theta-4c\sqrt{\sin^2\theta-\sin^2\alpha}.\sin\theta & (\alpha<\theta<\frac{\pi}{2})\,,\end{cases} \qquad [4.d.16]$$

$$F_{z'}[\nu,z,z',\theta] = \frac{\lambda_1}{2c}z'(\theta).\sec\theta\operatorname{cosec}\theta \qquad (o<\theta<\frac{\pi}{2})\,.$$

Substituting from [4.d.16] in [4.d.15] we obtain

$$\delta I=[\eta(\theta).\frac{\lambda_1}{2c}z'(\theta)\sec\theta\operatorname{cosec}\theta]_{o}^{\frac{\pi}{2}} - \frac{4c}{\pi}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi\,\eta(\phi)\,d\phi\int_{o}^{\alpha}\frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\,.$$

$$.\left\{\left[\tfrac{1}{2}B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+\tfrac{1}{2}B_3\right]\cos(\theta-\tfrac{1}{2}\alpha)\,+\right.$$

$$+\left[\left[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])\,+\right.\right.$$

$$\left.\left.\left.+\cos(\theta-\tfrac{1}{2}\alpha)\right]\right]\right\}+\int_{o}^{\frac{\pi}{2}}\eta(\theta)\left[\frac{\lambda_2 z(\theta)}{\sqrt{1+z^2(\theta)}}\,.\,2c\sin\theta\cos\theta-\frac{d}{d\theta}\left(\frac{\lambda_1}{c}z'(\theta)\sec\theta\operatorname{cosec}\theta\right)\right]d\theta\,+$$

$$+\int_{\alpha}^{\frac{\pi}{2}}\eta(\theta)\left[\frac{\lambda_2 z(\theta)}{\sqrt{1+z^2(\theta)}}\,.\,2c\sin\theta\cos\theta-4c\sqrt{\sin^2\theta-\sin^2\alpha}.\sin\theta - \frac{d}{d\theta}\left(\frac{\lambda_1}{c}z'(\theta)\sec\theta\operatorname{cosec}\theta\right)\right]d\theta\,.$$

[4.d.17]

Now we evaluate the integrals

$$\left.\begin{aligned}
I_1 &= \int_0^\alpha \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}.(\tan^2\phi-\tan^2\theta)} && (o<\theta<\alpha<\phi<\tfrac{\pi}{2}), \\
I_2 &= \int_0^\alpha \frac{\tan\theta\, d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}.(\tan^2\phi-\tan^2\theta)} && (o<\theta<\alpha<\phi<\tfrac{\pi}{2}), \\
I_3 &= \int_0^\alpha \frac{\cos(\theta-\frac{1}{2}\alpha)\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}} && (o<\theta<\alpha<\tfrac{\pi}{2}), \\
I_4 &= \int_0^\alpha \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}} && (o<\theta<\alpha<\tfrac{\pi}{2}).
\end{aligned}\right\} \quad [4.d.18]$$

The integrals in [4.d.18] are evaluated in Appendix IV and the values of these integrals are

$$\left.\begin{aligned}
I_1 &= \frac{\pi}{2\sqrt{\cos\alpha}\tan\phi\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi+\tan\alpha}} + \frac{1}{\sqrt{\tan\phi-\tan\alpha}}\right], \\
I_2 &= \frac{\pi}{2\sqrt{\cos\alpha}\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi+\tan\alpha}} - \frac{1}{\sqrt{\tan\phi-\tan\alpha}}\right], \\
I_3 &= \pi, \\
I_4 &= \frac{\pi}{\sqrt{\cos\alpha}}.
\end{aligned}\right\} \quad [4.d.19]$$

Substituting from [4.d.19] in [4.d.17] and interchanging the variables ϕ, θ and when we do so we obtain

$$\begin{aligned}
\delta I = {}& \frac{\lambda_1}{2c}\Big[\eta(\theta).z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\Big]_0^{\frac{\pi}{2}} + \int_0^\alpha \eta(\theta)\left\{\frac{2\lambda_2 c z(\theta)}{\sqrt{1+z^2(\theta)}}.\sin\theta\cos\theta - \frac{\lambda_1}{c}\frac{d}{d\theta}[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta]\right\}d\theta \\
&+ \int_\alpha^{\frac{\pi}{2}} \eta(\theta)\left\{\frac{2\lambda_2 c}{\sqrt{1+z^2(\theta)}}.\sin\theta\cos\theta - \frac{\lambda_1}{c}\frac{d}{d\theta}[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta] - 2c\sqrt{\sin^2\theta-\sin^2\alpha}\left[B_2\left(\sqrt{\frac{\tan\theta-\tan\alpha}{\tan\theta}}\right.\right.\right. + \\
&+ \left.\left.\left.\sqrt{\frac{\tan\theta+\tan\alpha}{\tan\theta}}\right)\right].\sin\theta\right\}d\theta .
\end{aligned} \quad [4.d.20]$$

For I[z] to be a minimum, we must have for all admissible function, $\eta(\theta)$

$$\delta I[z,\eta] = o \quad , \qquad [4.d.21]$$

and this implies that the coefficient of $\eta(\theta)$ in [4.d.20] should vanish

$$\left.\begin{aligned}
&\lambda_1 \frac{d}{d\theta}\left[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right] - 2c^2 \frac{\lambda_2\, z(\theta)}{\sqrt{1+z^2(\theta)}} \,.\, \sin\theta\cos\theta = o \quad , \quad (o<\theta<\alpha<\tfrac{\pi}{2}) , \\
&\lambda_1 \frac{d}{d\theta}\left[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right] - 2c^2 \frac{\lambda_2\, z(\theta)}{\sqrt{1+z^2(\theta)}} \,.\, \sin\theta\cos\theta = \\
&= \frac{2c^2\sqrt{\sin^2\theta-\sin^2\alpha}\,.\sin\theta}{\cos\frac{1}{2}\alpha}\left[\sqrt{\frac{\tan\theta-\tan\alpha}{\tan\theta}} + \sqrt{\frac{\tan\theta+\tan\alpha}{\tan\theta}}\right] , \quad (\alpha<\theta<\tfrac{\pi}{2}) ,
\end{aligned}\right\} [4.d.22]$$

while at the end points it is necessary that

$$\eta(o)\,z'(o) = o \quad , \quad \eta\left(\tfrac{\pi}{2}\right) z'\left(\tfrac{\pi}{2}\right) = o \quad , \qquad [4.d.23]$$

be satisfied.

We consider the solution of [4.d.22] for the slope $z(\theta)$ only in the case of small slope, and we approximate to [4.d.22] as follows:

$$\frac{1}{2c\sin\theta\cos\theta}\frac{d}{d\theta}\left[\frac{1}{2c\sin\theta\cos\theta}\, z'(\theta)\right] - \frac{\lambda_2}{2\lambda_1}\; z(\theta) \equiv \begin{cases} o \quad , & (o<\theta<\alpha<\frac{\pi}{2}) ; \\ f(\theta) \quad , & (\alpha<\theta<\frac{\pi}{2}) \quad , \end{cases} \qquad [4.d.24]$$

where

$$f(\theta) = \frac{\sqrt{\tan^2\theta-\tan^2\alpha\cos\alpha}}{2\lambda_1\,\cos\frac{1}{2}\alpha}\left[\sqrt{\frac{\tan\theta-\tan\alpha}{\tan\theta}} + \sqrt{\frac{\tan\theta+\tan\alpha}{\tan\theta}}\right] . \qquad [4.d.25]$$

Using the transformation

$$\left.\begin{aligned} x &= c\,\sin^2\theta \quad , \\ \ell &= c\,\sin^2\alpha \quad , \end{aligned}\right\} \qquad [4.d.26]$$

we obtain

$$z''(x) - m^2\, z(x) \equiv \begin{cases} o \quad , & (o<x<\ell) \quad ; \\ F(x) \quad , & (\ell<x<c) \quad , \end{cases} \qquad [4.d.27]$$

where

$$F(x) = \frac{1}{\sqrt{2}\,\lambda_1\sqrt{1+\sqrt{\frac{c-\ell}{c}}}}\sqrt{\frac{x-\ell}{c-x}}\left(\sqrt{\frac{\sqrt{\frac{x}{c-x}}-\sqrt{\frac{\ell}{c-\ell}}}{\sqrt{\frac{x}{c-x}}}}+\sqrt{\frac{\sqrt{\frac{x}{c-x}}+\sqrt{\frac{\ell}{c-\ell}}}{\sqrt{\frac{x}{c-x}}}}\right), \quad (\ell<x<c),$$

$$m^2 = \frac{\lambda_2}{2\lambda_1} . \qquad [4.d.28]$$

It is assumed at this stage that $\frac{\lambda_1}{\lambda_2} > o$ and we show later that $\lambda_1 > o$, $\lambda_2 > o$ are sufficient conditions for a true maximization of L .

To derive the solution of the nonhomogeneous equation in [4.d.27], we apply the usual method of variation of parameters, then we obtain

$$z(x) \equiv \begin{cases} A_1 \sinh mx + B_1 \cosh mx , & (o<x<\ell) ; \\ -\frac{1}{m}\int_\ell^x F(\xi)\sinh m(\xi-x)\,d\xi + A_2\sinh mx + B_2\cosh mx , & (\ell<x<c) , \end{cases} \qquad [4.d.29]$$

where A_1, A_2, B_1 and B_2 are arbitrary constants.

We shall assume the boundary conditions are given by

$$z(o) = o \quad , \quad z(c) = \beta \quad . \qquad [3.d.30]$$

Since $z(x)$ and $z'(x)$ are continuous at $x=\ell$ and $z(x)$ satisfies the boundary conditions [4.d.30], we obtain

$$z(x) \equiv \begin{cases} \frac{\beta\sinh mx}{\sinh mc} + \frac{\sinh mx}{m\sinh mc}\int_\ell^c F(\xi)\sinh m(\xi-c)\,d\xi , & (o<x<\ell) ; \\ \frac{\beta\sinh mx}{\sinh mc} + \frac{\sinh mx}{m\sinh mc}\int_\ell^c F(\xi)\sinh m(\xi-c)\,d\xi - \\ \qquad -\frac{1}{m}\int_\ell^x F(\xi)\sinh m(\xi-x)\,d\xi , & (\ell<x<c) . \end{cases} \qquad [4.d.31]$$

We integrate [4.d.31] with respect to x , and use the boundary condition

$$y(o) = o \qquad [4.d.32]$$

to obtain

$$y(x) \equiv \begin{cases} \dfrac{\beta(\cosh m x-1)}{m \sinh mc} + \dfrac{(\cosh m x-1)}{m^2 \sinh mc}\displaystyle\int_{\ell}^{c} F(\xi)\sinh m(\xi-c)d\xi \ , & (o<x<\ell) \ ; \\ \dfrac{\beta(\cosh m x-1)}{m \sinh mc} + \dfrac{(\cosh m x-1)}{m^2 \sinh mc}\displaystyle\int_{\ell}^{c} F(\xi)\sinh m(\xi-c)d\xi \ - & \\ \qquad - \dfrac{1}{m^2}\displaystyle\int_{\ell}^{x} F(\xi)\,[\cosh m(\xi-x)-1]d\xi \ , & (\ell<x<c) \ . \end{cases} \qquad [4.d.33]$$

The solution [4.d.32] should satisfy the constraints, [4.d.2] and [4.d.3] and we thus obtain two equations in two unknowns λ_2, m , which have to be evaluated numerically.

We do not complete the solution of this problem using this method since there is an alternative method of resolving the problem numerically which is discussed in detail in Appendix XIII.

IVe A SUFFICIENT CONDITION FOR THE EXTREMUM TO BE A MINIMUM

A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I .

Since

$$\delta I[\nu(\theta), z(\theta), z'(\theta), \theta] = o \quad , \qquad [4.e.1]$$

the condition for I to be a minimum requires

$$\delta^2 I[\nu(\theta), z(\theta), z'(\theta), \theta] > o \quad , \qquad [4.e.2]$$

for all admissible variation $\xi(\theta)$ and $\eta(\theta)$ consistent with

$$\xi(\theta) = \frac{-\,2U \operatorname{cosec}\theta}{\pi\sqrt{\sin^2\alpha-\sin^2\theta}\,.\sqrt{\sin\theta\sin(\alpha-\theta)}}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\eta(\phi)\sin\phi d\phi\Bigg\{\tfrac{1}{2}\cos(\theta-$$
$$-\tfrac{1}{2}\alpha)\left[B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)-2\right]+$$
$$+\Bigg[\Big[-\frac{\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi\,[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta\,[\sqrt{\tan\phi-\tan\alpha}-$$
$$-\sqrt{\tan\phi+\tan\alpha}]) + \cos(\theta-\tfrac{1}{2}\alpha)\ \Big]\Bigg]\Bigg\} \qquad (o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \ , \qquad [4.e.3]$$

where

$$B_2 = \sec\tfrac{1}{2}\alpha \quad . \tag{4.e.4}$$

Since z has been prescribed at $x=o$ and $x=c$ in [4.c.30] it follows that the variation η will satisfy the conditions

$$\eta(o) = o \quad , \quad \eta(\tfrac{\pi}{2}) = o \quad . \tag{4.e.5}$$

Then, using Taylor's theorem, we can write the increment of the functional $I[\nu,z,z',\theta]$ as

$$I[\nu+\varepsilon\xi,z+\varepsilon\eta,z'+\varepsilon\eta',\theta]-I[\nu,z,z',\theta] = \varepsilon\int_o^{\frac{\pi}{2}}\{\xi(\theta)F_\nu(\nu,z,z',\theta)+\eta(\theta)[F_z(\nu,z,z',\theta) - $$

$$- \frac{d}{d\theta}F_{z'}(\nu,z,z',\theta)]\}\,d\theta + \tfrac{1}{2}\varepsilon^2\int_o^{\frac{\pi}{2}}\{\xi^2(\theta)F_{\nu\nu}+\eta^2(\theta)F_{zz}+\eta'^2(\theta)F_{z'z'}+2\xi(\theta)\eta(\theta)F_{\nu z} +$$

$$+ 2\xi(\theta)\eta'(\theta)F_{\nu z'}+2\eta(\theta)\eta'(\theta)F_{zz'}\}\,d\theta + O(\varepsilon^3) \qquad (o<\theta<\tfrac{\pi}{2}) \quad . \tag{4.e.6}$$

Denoting the coefficient ε by δI and that of ε^2 by $\delta^2 I$, at a stationary value of I, we have from [4.e.3], [4.e.5] and [4.e.1]

$$\delta^2 I = \int_o^{\frac{\pi}{2}}\{\xi^2(\theta)F_{\nu\nu}+\eta^2(\theta)F_{zz}+\eta'^2(\theta)F_{z'z'}+2\xi(\theta)\eta(\theta)F_{\nu z}+2\xi(\theta)\eta'(\theta)F_{\nu z'}+2\eta(\theta)\eta'(\theta)F_{zz'}\}\,d\theta \, , \tag{4.e.7}$$

where

$$\left.\begin{aligned}
F_{\nu\nu}[\nu,z,z',\theta] &= o \quad ,\\
F_{zz}[\nu,z,z',\theta] &= \frac{2\lambda_2 c\sin\theta\cos\theta}{[1+z^2(\theta)]^{3/2}} \quad ,\\
F_{z'z'}[\nu,z,z',\theta] &= \frac{\lambda_1}{c}\sec\theta\,\mathrm{cosec}\,\theta ,\\
F_{\nu z}[\nu,z,z',\theta] &= o \quad ,\\
F_{\nu z'}[\nu,z,z',\theta] &= o \quad ,\\
F_{zz'}[\nu,z,z',\theta] &= o \quad .
\end{aligned}\right\} \tag{4.e.8}$$

Substituting from [4.e.8] in [4.e.7] we obtain

$$\delta^2 I = \int_0^{\frac{\pi}{2}} \left\{ \frac{2\lambda_2 c \sin\theta\cos\theta}{[1+z^2(\theta)]^{3/2}} \eta^2(\theta) + \frac{\lambda_1 \eta'^2(\theta)}{c} \sec\theta \operatorname{cosec}\theta \right\} d\theta \quad . \qquad [4.e.9]$$

In the case of small slope $z(x)$, we approximate [4.e.9] as follows:

$$\delta^2 I = \int_0^{\frac{\pi}{2}} \left\{ 2\lambda_2 c \sin\theta\cos\theta\eta^2(\theta) + \frac{\lambda_1 \eta'^2(\theta)}{c} \sec\theta \operatorname{cosec}\theta \right\} d\theta \quad . \qquad [4.e.10]$$

Using the transformation

$$x = c \sin^2\theta \quad , \qquad [4.e.11]$$

we obtain

$$\delta^2 I = \int_0^c \{\lambda_2 \phi^2(x) + 2\lambda_1 \phi'^2(x)\} dx \quad , \qquad [4.e.12]$$

where

$$\left.\begin{aligned} \phi(x) &= \eta(\theta) \quad , \\ \phi'(x) &= \eta'(\theta)/2c\sin\theta\cos\theta \quad . \end{aligned}\right\} \qquad [4.e.13]$$

We consider a special choice of $\phi(x)$ satisfying the boundary conditions

$$\phi(o) = o \quad , \quad \phi(c) = o \quad , \qquad [4.e.14]$$

namely

$$\phi(x) = \alpha \sin \frac{\pi}{c} x \quad , \qquad [4.e.15]$$

and in this case

$$\delta^2 I = \tfrac{1}{2}\alpha^2 \left\{ \lambda_2 + \frac{2\pi^2}{c^2} \lambda_1 \right\} \quad . \qquad [4.e.16]$$

Thus in the case [2.e.15] the sufficient condition for satisfying [4.e.2] is as follows:

$$\lambda_2 + \frac{2\pi^2}{c^2} \lambda_1 > o \quad . \qquad [4.e.17]$$

IVf THE OPTIMUM SHAPE OF A HYDROFOIL IN PARTIAL CAVITY FLOW USING CLASSICAL EULER METHOD [ALTERNATIVE METHOD]

We pose the problem of maximising the lift

$$L=\rho U^2\sigma\ell-2\rho Uc\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}.\nu(\theta)\sin\theta d\theta+4\rho U^2c\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi, \qquad [4.f.1]$$

subject to a constraint on curvature of the form

$$K = \frac{1}{2c}\int_0^{\frac{\pi}{2}} z'^2(\theta)\sec\theta\,\mathrm{cosec}\theta d\theta \ , \qquad [4.f.2]$$

where K is prescribed, together with a constraint on the hydrofoil of the form

$$S = 2c\int_0^{\frac{\pi}{2}}\sqrt{1+z^2(\theta)}.\sin\theta\cos\theta d\theta \quad , \qquad [4.f.3]$$

where S is prescribed, together with a constraint on the length of the hydrofoil at position θ .

The functions ν and z are related by

$$\nu(\theta) = -\frac{\mathrm{cosec}\theta}{2\pi\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\cdot\Bigg\{\cos(\theta-\tfrac{1}{2}\alpha)\Bigg[B_1 -$$

$$-2U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi z(\phi)d\phi\left(\left(B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+B_3\right)\right)\Bigg]+$$

$$+\tfrac{1}{2}U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}d\theta_0}{\sin(\theta_0-\theta)}+$$

$$+4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi d\phi\Bigg[\Bigg[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+$$

$$+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\Bigg\} \quad , \qquad [4.f.4]$$

with

$$\left.\begin{aligned} B_1 &= \frac{\pi U\sigma \sin^3\tfrac{1}{2}\alpha(1+2\cos\alpha)}{4\cos\tfrac{1}{2}\alpha} \;, \\ B_2 &= \frac{1}{\cos\tfrac{1}{2}\alpha} \;, \\ B_3 &= -2 \;. \end{aligned}\right\} \qquad [4.f.5]$$

Substituting from [4.f.4] in [4.f.1] we obtain

$$\begin{aligned} L={}&\rho U^2\sigma\ell+4\rho U^2 c\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\,z(\phi)\sin\phi\, d\phi+\frac{\rho U c}{\pi}\int_0^\alpha\frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\Bigg\{ \\ &\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg[\; B_1-2U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi\,.\,z(\phi)\,d\phi\left[\left[B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+B_3\right]\right]\Bigg]+ \\ &+\tfrac{1}{2}U\sigma\int_0^\alpha\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)}+ \\ &+4U\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi\, d\phi\Bigg[\Bigg[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}\,(\tan\phi\,[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]+ \\ &+\tan\theta\,[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\Bigg\} \;. \end{aligned} \qquad [4.f.6]$$

It is permissible to interchange the order of integral on the right-hand side of [4.f.6] [see, e.g., Hardy,G.H. (35)] and when we do so we obtain

$$\begin{aligned} L={}&\rho U^2\sigma\ell+4\rho U^2 c\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\,z(\phi)\sin\phi\, d\phi+\frac{\rho U c}{\pi}\Bigg\{\; B_1\int_0^\alpha\frac{\cos(\theta-\tfrac{1}{2}\alpha)\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}+ \\ &+\tfrac{1}{2}U\sigma\int_0^\alpha(\sin^2\alpha-2\sin^2\psi)\sqrt{\sin\psi\sin(\alpha-\psi)}\,d\psi\int_0^\alpha\frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\sin(\psi-\theta)}\Bigg\}- \\ &-\frac{4\rho U^2 c}{\pi}\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi\, z(\phi)\,d\phi\int_0^\alpha\frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\Bigg\{\tfrac{1}{2}\left[\left[B_2\left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\right.\right.\right. \\ &\left.\left.\left.+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right)+B_3\right]\right]\,.\,\cos(\theta-\tfrac{1}{2}\alpha)\quad+\quad\Bigg[\Bigg[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}\,(\tan\phi\,[\sqrt{\tan\phi-\tan\alpha}+ \\ &+\sqrt{\tan\phi+\tan\alpha}]+\tan\theta\,[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\Bigg\} \end{aligned}$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \;. \qquad [4.f.7]$$

Now we evaluate the integrals

$$\left.\begin{aligned}
I_1 &= \int_0^{\alpha} \frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\sin(\psi-\theta)} , \\
I_2 &= \int_0^{\alpha} \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}} , \\
I_3 &= \int_0^{\alpha} \frac{\cos(\theta-\frac{1}{2}\alpha)d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}} , \\
I_4 &= \int_0^{\alpha} \frac{\tan\theta d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}(\tan^2\phi-\tan^2\theta)} , \\
I_5 &= \int_0^{\alpha} \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}(\tan^2\phi-\tan^2\theta)} .
\end{aligned}\right\} \quad [4.f.8]$$

The integrals in [4.f.8] are evaluated in Appendix IV and the values of these integrals are

$$\left.\begin{aligned}
I_1 &= 0 , \\
I_2 &= \frac{\pi}{\sqrt{\cos\alpha}} , \\
I_3 &= \pi \\
I_4 &= \frac{\pi}{2\sqrt{\cos\alpha}\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} - \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right] , \\
I_5 &= \frac{\pi}{2\sqrt{\cos\alpha}\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} + \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right] .
\end{aligned}\right\} \quad [4.f.9]$$

Substituting from [4.f.9] in [4.f.7] we obtain

$$L=\rho U^2\sigma\ell + B_1\rho Uc\frac{2\rho U^2c}{\cos\frac{1}{2}\alpha}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi d\phi \;.\; \left(\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right) , \quad [4.f.10]$$

where B_1 is defined by [4.f.5].

We now use the classical Euler method to obtain the optimum shape of cavitating hydrofoil so that the lift is a maximum subject to the constraints [4.f.2] and [4.f.3]. This method can be applied to this problem in the following way. We consider the minimization of the function I defined as follows:

$$I \equiv \begin{cases} \displaystyle\int_0^{\alpha} G(z,z',\theta)\,d\theta & (o<\theta<\alpha) \\ \displaystyle\int_{\alpha}^{\frac{\pi}{2}} \{F(z,\theta)+G(z,z',\theta)\}\,d\theta & (\alpha<\theta<\frac{\pi}{2}) \end{cases} \qquad [4.f.11]$$

where

$$\left.\begin{aligned} F(z,\theta) &= \frac{2\rho U^2 c\sqrt{\sin^2\phi-\sin^2\alpha}}{\cos\frac{1}{2}\alpha}\, z(\phi)\sin\phi \left\{\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}}+\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}}\right\}, \\ G(z,z',\theta) &= \frac{\lambda_1}{2c}\, z'^2(\theta)\sec\theta\,\mathrm{cosec}\,\theta+2\lambda_2 c\sqrt{1+z^2(\theta)}\sin\theta\cos\theta \, . \end{aligned}\right\}$$

[see, e.g., FOX, C. (21)]. [4.f.12]

Using Euler equation

$$\left.\begin{aligned} &\frac{\partial}{\partial z}[\lambda G(z,z',\theta)] - \frac{d}{d\theta}\left[\frac{\partial}{\partial z'}(\lambda G(z,z',\theta))\right] = 0 && (o<\theta<\alpha), \\ &\frac{\partial}{\partial z}[F(z,\theta)+\lambda G(z,z',\theta)] - \frac{d}{d\theta}\left[\frac{\partial}{\partial z'}(F_z(z,\theta)+\lambda G(z,z',\theta))\right] = 0 , && (\alpha<\theta<\frac{\pi}{2}), \end{aligned}\right\}$$

[4.f.13]

we obtain

$$\lambda_1 \frac{d}{d\theta}[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta]-2\lambda_2 c^2\,\frac{z(0)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} = 0 \qquad (o<\theta<\alpha) ,$$

$$\lambda_1 \frac{d}{d\theta}[z'(\theta)\sec\theta\,\mathrm{cosec}\,\theta]-2\lambda_2 c^2\,\frac{z(\theta)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} = \frac{2c^2\sqrt{\sin^2\theta-\sin^2\alpha}}{\cos\frac{1}{2}\alpha}\sin\theta\left\{\sqrt{\frac{\tan\theta-\tan\alpha}{\tan\theta}} + \sqrt{\frac{\tan\theta+\tan\alpha}{\tan\theta}}\right\} \qquad (\alpha<\theta<\frac{\pi}{2}) . \qquad [4.f.14]$$

The equations in [4.f.14] are nonlinear differential equations for $z(\theta)$.

We consider the solution of [4.f.14] for the slope $z(\theta)$ only in the case of small slope, and we approximate to [4.f.14] as follows:

$$\left.\begin{aligned}
&\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\left[\frac{1}{\sin\theta\cos\theta}\frac{dz'}{d\theta}\right]-\frac{2\lambda_2 z(\theta)c^2}{\lambda_1}=0 \quad , \qquad (o<\theta\leq\alpha),\\
&\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\left[\frac{1}{\sin\theta\cos\theta}\frac{dz'}{d\theta}\right]-\frac{2\lambda_2}{\lambda_1}z(\theta)c^2=\frac{2c^2\cos\alpha}{\lambda_1\cos\frac{1}{2}\alpha}\sqrt{\tan^2\theta-\tan^2\alpha}\left(\sqrt{\frac{\tan\theta-\tan\alpha}{\tan\theta}}+\right.\\
&\qquad\left.+\sqrt{\frac{\tan\theta+\tan\alpha}{\tan\theta}}\right) \quad , \qquad (\alpha<\theta<\frac{\pi}{2}) \; .
\end{aligned}\right\}$$

[4.f.15]

These results are exactly the same as the results obtained from employing a variational calculus technique involving a singular integral equation which was introduced earlier in Part I. The sufficient condition for extremum to be a minimum is derived from consideration of the second variation and it is called Legendre test.

$$\left.\begin{aligned}
&F_{zz}[z,z',\theta] > 0 \quad , \\
&F_{z'z'}[z,z',\theta] > 0 \quad , \\
&F_{zz}[z,z',\theta].F_{z'z'}[z,z',\theta] > F^2_{zz'}[z,z',\theta] \quad .
\end{aligned}\right\} \qquad [4.f.16]$$

Hence

$$\left.\begin{aligned}
\lambda_1 &> 0 \; , \\
\lambda_2 &> 0 \; .
\end{aligned}\right\} \qquad [4.f.17]$$

That is the same result as obtained in [4.e.12] .

PART III

THE OPTIMUM SHAPE OF NON - CAVITATING HYDROFOIL BENEATH A FREE SURFACE

INTRODUCTION

There is an extensive literative connected with wave resistance due to a submerged obstacle [see, e.g., Havelock, T.H.(31),(32),(33),(34), Kochin, N.E.(37), Kotchin,N.E.(38), Wehausen, J.V. and Laiton, E.V.(65), Kreisel, G.(40) and Riabouchinsky, D.(54)] but the two papers which are most relevant to present Part III are those to Kochin, N.E., Kibel, I.A. and Roze, N.V.(38) and Wu, T.Y. and Whitney, A.K.(70)].

A hydrofoil of arbitrary shape is in steady, rectilinear, horizontal motion at a depth h beneath the free surface of a liquid.

The usual assumptions in problems of this kind are taken as a basis, namely, the liquid is non-viscous and moving two-dimensionally, steadily and without vorticity, the only force acting on it is gravity.

With these assumptions together with a linearization assumption we determine the forces, due to the hydrofoil beneath a free surface of the liquid.

The hydrodynamic forces due to a vortex of strength γ and source of strength m beneath a free surface are derived using Blasius' theorem in Chapter V.

A singular integral equation formulation of the boundary value problem is obtained and can be solved to yield expressions for the lift and drag as functions of the unknown singularity distribution γ being the vortex strength together with the unknown shape, z (hydrofoil slope), these expressions are given for a hydrofoil of arbitrary shape.

The purpose of this work is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces minimum drag.

We use variational calculus techniques to obtain the optimum shape of the hydrofoil.

The mathematical problem is that of extremizing a function depending on γ (vortex strength) and z (the hydrofoil slope) when the two functions are related by a singular integral equation. The analytical solution for the unknown shape z and the unknown singularity has branch-type singularities at the two ends of the hydrofoil.

The extremal solution $\gamma(x;\lambda_1,\lambda_2)$ and $z(x;\lambda_1,\lambda_2)$ when determined will involve two Lagrange multipliers constants λ_1, λ_2, which can be determined, by substituting the extremal solution $\gamma(x;\lambda_1,\lambda_2)$ and $z(x;\lambda_1,\lambda_2)$ in two constraints.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation.

Va UNIFORM STREAM PAST SUBMERGED VORTEX

The vortex is beneath the free surface of a liquid at a depth h and it is convenient to make use of complex variable techniques.

We assume the liquid is non-viscous and moving two-dimensionally steadily and without vorticity, the only force acting on it is gravity. With these assumptions together with a linearization assumption we solve this problem. We will denote by Ox the horizontal coordinate axis lying on the free surface of a liquid in state of rest, and by Oy the vertical axis directed upward.

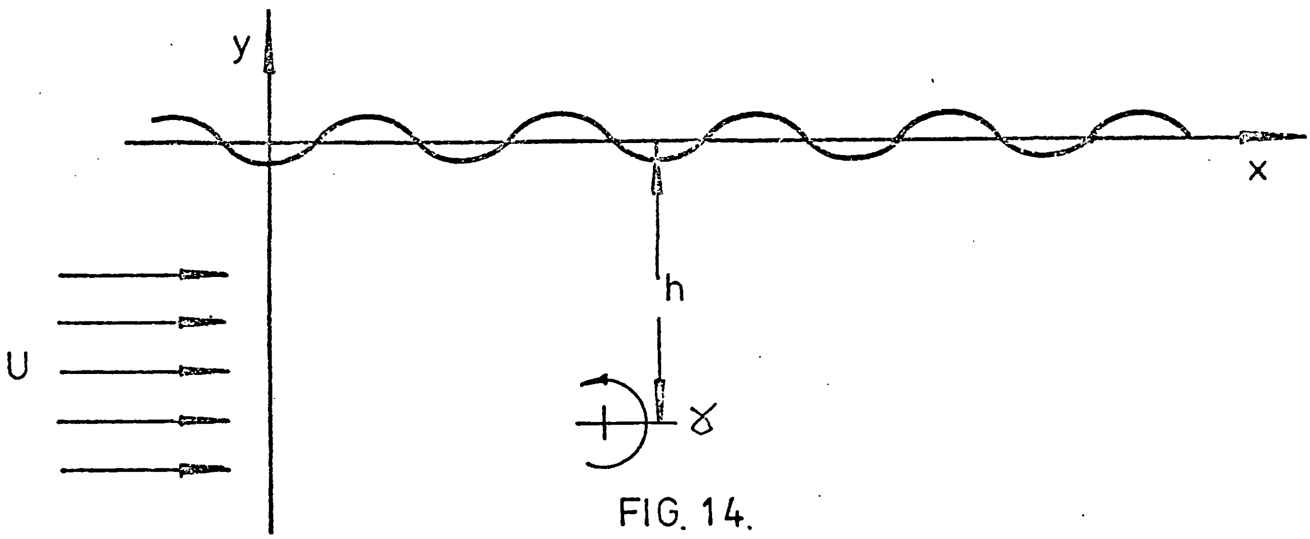

FIG. 14.

In addition, we introduce the complex variable

$$Z = x+iy \qquad , \tag{5.a.1}$$

and the complex velocity potential

$$w = \phi+i\psi \qquad . \tag{5.a.2}$$

First of all, we solve the problem of a vortex of intensity γ located at a depth h below the free surface of the liquid which is flowing uniformly with speed U in the x-positive direction.

Denoting the complex velocity potential of this steady motion by

$$W = \Phi+i\Psi \qquad , \tag{5.a.3}$$

we will have

$$\left.\begin{aligned} &W=w-Uz \quad , \\ &\Phi=\phi-Ux \quad , \\ &\Psi=\psi-Uy \quad . \end{aligned}\right\} \qquad [5.a.4]$$

In this steady motion, the free boundary of the liquid is a streamline and a surface of constant pressure.

By Bernoulli's formula, we have for the pressure P

$$P=C-\tfrac{1}{2}\rho(u^2+v^2)-\rho gy \quad , \qquad [5.a.5]$$

where u,v are the components of liquid velocity along x,y axes respectively.

Substituting here the equations

$$\left.\begin{aligned} u &= -\frac{\partial\Phi}{\partial x} = -\frac{\partial\phi}{\partial x} + U \quad , \\ v &= -\frac{\partial\Phi}{\partial y} = -\frac{\partial\phi}{\partial y} \quad , \end{aligned}\right\} \qquad [5.a.6]$$

we obtain readily

$$P = C - \tfrac{1}{2}\rho U^2 + \rho U \frac{\partial\phi}{\partial x} - \tfrac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right] - \rho gy \quad . \qquad [5.a.7]$$

Since the second degree terms are ignored, we obtain

$$P=C-\tfrac{1}{2}\rho U^2+\rho U \frac{\partial\phi}{\partial x} - \rho gy \quad . \qquad [5.a.8]$$

It can be shown that the boundary condition on the free surface can be written in the form

$$I_{mag}\left[i\,\frac{d^2w}{dz^2} - \nu\,\frac{dw}{dz}\right] = 0 \quad , \ [\nu = \frac{g}{U^2} \ , \ y = 0] \quad , \qquad [5.a.9]$$

[see, e.g., Kochin,N.E., Kibel',I.A. and Roze,N.V.(38)].

The conditions at infinity are as follows:

$$\left.\begin{aligned} &\frac{dW}{dz} \to -U \quad , \quad |z| \to \infty \quad , \\ &\frac{dw}{dz} \to 0 \quad , \quad |z| \to \infty \quad . \end{aligned}\right\} \qquad [5.a.10]$$

In the special case of the motion now under consideration, the function $\frac{dw}{dz}$ must be holomorphic in the entire half-plane $y < o$, except at the point where the vortex is located.

Let this be the point $z = -ih$ (with coordinates $x=o$, $y = -h$).

Near this point, the function $w(z)$ must have the form

$$w(z) = -\frac{\gamma}{2\pi i}\log(z+ih) + g(z) \quad , \qquad [5.a.11]$$

where $g(z)$ is a holomorphic function in the neighbourhood of the point $z=ih$.

For the function $\frac{dw}{dz}$, we have obtained the representation

$$\frac{dw}{dz} = -\frac{\gamma}{2\pi i}\frac{1}{(z+ih)} + g'(z) \quad . \qquad [5.a.12]$$

Now we form the function

$$f(z) = i\frac{d^2w}{dz^2} - \nu\frac{dw}{dz} \quad . \qquad [5.a.13]$$

It is holomorphic in the entire half-plane $y<o$, except at $z=-ih$, near which we have

$$f(z) = \frac{\gamma}{2\pi}\frac{1}{(z+ih)^2} + \frac{\gamma\nu}{2\pi i}\frac{1}{(z+ih)} + f_1(z) \quad , \qquad [5.a.14]$$

where $f_1(z)$ is holomorphic in the neighbourhood of the point $z=ih$.

As a consequence of the condition [5.a.9], the function $f(z)$ assumes a real value on the real axis; $f(z)$ is defined above in [5.a.14] in the half-plane $y<o$, but it can be continued analytically into the upper half-plane $y>o$ by Schwarz's principle of symmetry. In fact, the values of function $f(z)$ at two points symmetric with respect to the x-axis must be complex conjugate, so that one must take

$$f(z) = \overline{f(\bar{z})} \quad . \qquad [5.a.15]$$

We then obtain a function which is defined and is analytic in the entire plane of the complex variable z.

This function has at the point $z=-ih$ the singularity determined by [5.a.14]; in addition, it will have a singularity at point $z=ih$, since it follows from [5.a.14] and [5.a.15] that we will have in the neighbourhood of this point the representation

$$f(z) = \frac{\gamma}{2\pi}\frac{1}{(z-ih)^2} - \frac{\gamma\nu}{2\pi i}\frac{1}{(z-ih)} + \overline{f_1(\bar{z})} \quad , \qquad [5.a.16]$$

which shows that the point $z=ih$ is a second-order pole for the function $f(z)$. The function $f(z)$ has no other singular points in the finite part of the z-plane. Assuming it to be holomorphic in the neighbourhood of the point at infinity and to vanish for $z=\infty$, we introduce a new function $F(z)$ defined by

$$F(z) = f(z) - \frac{\gamma}{2\pi}\frac{1}{(z+ih)^2} - \frac{\gamma\nu}{2\pi i}\frac{1}{(z+ih)} - \frac{\gamma}{2\pi}\frac{1}{(z-ih)^2} + \frac{\gamma\nu}{2\pi i}\frac{1}{(z-ih)} \ . \qquad [5.a.17]$$

This function $F(z)$ has the properties

(i) $F(z)$ has no singularities in z plane,

(ii) As $z \to \infty$, $F(z)$ tends to be constant since $f(z)$ is bounded at infinity, hence, using Liouville's theorem [see, e.g., Copson,E.T.(8)] it follows that $F(z)$ is constant and hence

$$f(z) - \frac{\gamma}{2\pi}\frac{1}{(z+ih)^2} - \frac{\gamma\nu}{2\pi i}\frac{1}{(z+ih)} - \frac{\gamma}{2\pi}\frac{1}{(z-ih)^2} + \frac{\gamma\nu}{2\pi i}\frac{1}{(z-ih)} = C \ , \qquad [5.a.18]$$

where C is constant.

Hence we arrive at the result

$$f(z) = i\frac{d^2w}{dz^2} - \nu\frac{dw}{dz}$$

$$= \frac{\gamma}{2\pi}\left\{\frac{1}{(z+ih)^2} - \frac{i\nu}{(z+ih)} + \frac{1}{(z-ih)^2} + \frac{\nu i}{(z-ih)}\right\} + C \ . \qquad [5.a.19]$$

The general solution of the homogeneous equation

$$i\frac{d^2w}{dz^2} - \nu\frac{dw}{dz} = 0 \qquad [5.a.20]$$

is

$$w(z) = A + Be^{-i\nu z} \ . \qquad [5.a.21]$$

To derive the solution of the nonhomogeneous equation [5.a.19] we apply the usual method of variation of parameters, assuming A and B to be functions of z ; thus we arrive at the equations

$$\left.\begin{aligned} &\frac{dA}{dz} + \frac{dB}{dz} e^{-i\nu z} = 0 , \\ &\nu \frac{dB}{dz} e^{-i\nu z} = \frac{\gamma}{2\pi}\left[\frac{1}{(z+ih)^2} - \frac{\nu i}{(z+ih)} + \frac{1}{(z-ih)^2} + \frac{\nu i}{(z-ih)}\right] . \end{aligned}\right\} \qquad [5.a.22]$$

Assuming the functions A and B tend to zero as the point z moves to infinity in the direction of the positive real axis, i.e. for $z\to\infty$ we obtain easily

$$\left.\begin{aligned} &A = \frac{\gamma}{2\pi\nu}\left\{\frac{1}{z+ih} + \frac{1}{z-ih}\right\} - \frac{\gamma}{2\pi i}\log\frac{z+ih}{z-ih} , \\ &B = \frac{\gamma}{2\pi\nu}\int_{\infty}^{z} e^{i\nu t}\left[\frac{1}{(t+ih)^2} + \frac{1}{(t-ih)^2} - \frac{\nu i}{(t+ih)} + \frac{\nu i}{(t-ih)}\right] dt . \end{aligned}\right\} \qquad [5.a.23]$$

However, integration by parts yields

$$\left.\begin{aligned} &\int_{\infty}^{z} \frac{e^{i\nu t}dt}{(t+ih)^2} = -\frac{e^{i\nu z}}{z+ih} + i\nu\int_{\infty}^{z}\frac{e^{i\nu t}dt}{(t+ih)} , \\ &\int_{\infty}^{z} \frac{e^{i\nu t}dt}{(t-ih)^2} = -\frac{e^{i\nu z}}{z-ih} + i\nu\int_{\infty}^{z}\frac{e^{i\nu t}dt}{(t-ih)} . \end{aligned}\right\} \qquad [5.a.24]$$

Substituting from [5.a.23] and [5.a.24] in [5.a.21] we obtain

$$w(z) = -\frac{\gamma}{2\pi i}\left\{\log\left(\frac{z+ih}{z-ih}\right) + 2e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu t}dt}{(t-ih)}\right\} . \qquad [5.a.25]$$

The path of integration in [5.a.25] is as shown in Fig.15.

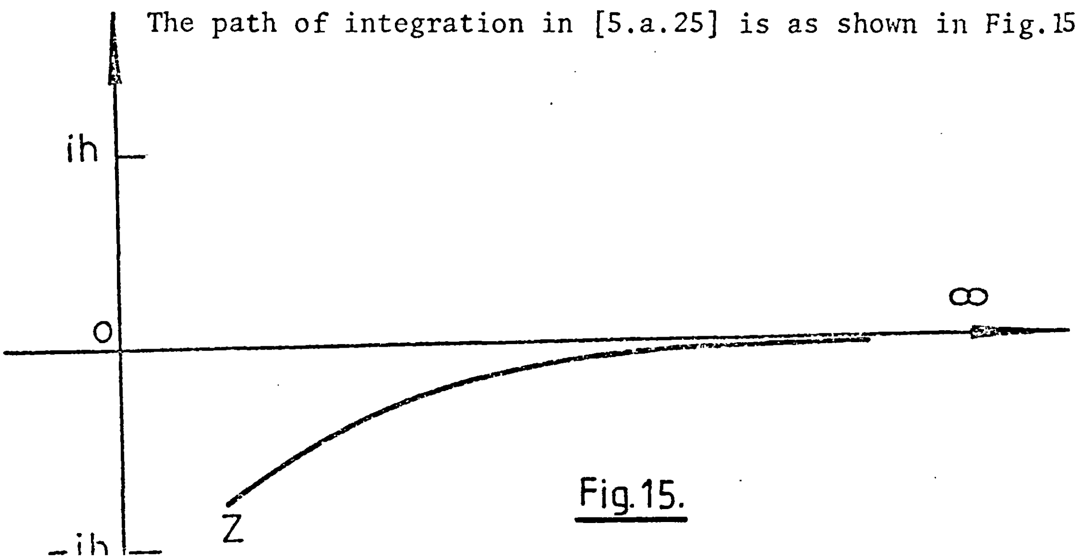

Fig.15.

The formula [5.a.25] is taken as basic in the calculation of the hydrodynamic forces acting on surrounding contour about the vortex.

We compute now the force acting on the vortex using the Blasius theorem and integrating along a contour c enclosing the vortex, the lift and drag can be expressed in the form

$$L+iD = -\tfrac{1}{2}\rho\oint_c \left(\frac{dW}{dz}\right)^2 dz \,, \qquad [5.a.26]$$

where ρ is constant liquid density.

Using [5.a.4] and [5.a.25] we can write

$$\frac{dW}{dz} = -\frac{\gamma}{2\pi i}\,\frac{1}{(z+ih)} + \alpha(z) \,, \qquad [5.a.27]$$

where

$$\alpha(z) = -U - \frac{\gamma}{2\pi i}\,\frac{1}{(z-ih)} + \frac{\gamma\nu}{\pi}\, e^{-i\nu z}\int_\infty^z \frac{e^{i\nu t}dt}{t-ih} \,, \qquad [5.a.28]$$

is a holomorphic function in the lower half-plane.

Therefore the residue of the function

$$\left(\frac{dW}{dz}\right)^2 = -\frac{\gamma^2}{4\pi^2}\,\frac{1}{(z+ih)^2} - \frac{\gamma}{\pi i}\cdot\frac{\alpha(z)}{(z+ih)} + \alpha^2(z) \,, \qquad [5.a.29]$$

at the point $z=-ih$, the location of the vortex, is equal to $[-\gamma.\alpha(-ih)]/\pi i$ and we have, by the residue theorem,

$$\oint_c \left(\frac{dW}{dz}\right)^2 dz = -\,2\gamma.\alpha(-ih) \,. \qquad [5.a.30]$$

Thus

$$\begin{aligned} L+iD &= \rho\gamma.\alpha(-ih) \\ &= -\rho\gamma U + \frac{\rho\gamma^2}{4\pi h} - \frac{\rho\gamma^2\nu}{\pi}\, e^{-\nu h}\int_\infty^{-ih} \frac{e^{i\nu t}.dt}{t-ih} \,. \end{aligned} \qquad [5.a.31]$$

Introducing into the last integral for t the new variable s defined by

$$s=i\nu(t-ih) \,, \qquad [5.a.32]$$

we can reduce it to the form

$$\int_{\infty}^{-ih} \frac{e^{i\nu t}dt}{t-ih} = e^{-\nu h} \int_{i\infty}^{2\nu h} \frac{e^{s}ds}{s} \quad , \qquad [5.a.33]$$

where it must be emphasized that in the t plane the path of integration must lie completely in the lower half-plane; in the s plane, however, it must be in the upper half-plane.

Using the Cauchy theorem we can write

$$\int_{i\infty}^{z} \frac{e^{s}ds}{s} = \int_{-\infty}^{z} \frac{e^{s}ds}{s} = E_i(z) \quad , \qquad [5.a.34]$$

where $E_i(z)$ is the exponential integral [see, e.g., Abramowitz,M .and Stegun,I. (1),5,pp.227-253] .

If z is real positive and the path of integral lies in the upper half plane, the integral can be taken along the real axis, when the point $s=o$ must be bypassed along an infinitesimal semicircle.

Since the residue of the integral at this point is equal to unity, the value of the integral along the infinitesimal semicircle is equal to $-\pi i$. Therefore the imaginary part of $E_i(x)$ is equal to $-\pi$.

Next, we introduce the notation

$$E_{i_1}(x) = R\ell E_i(x) \quad ; \qquad [5.a.35]$$

we then obtain the equation

$$E_i(x) = E_{i_1}(x) - \pi i \quad , \qquad [5.a.36]$$

on the assumption that the path of integration is located in the upper half plane.

Thus we find the equation

$$L+iD = -\rho\gamma U + \frac{\rho\gamma^2}{4\pi h} - \frac{\rho\gamma^2\nu}{\pi} \cdot e^{-2\nu h} \cdot E_{i_1}(2\nu h) + i\rho\gamma^2\nu e^{-2\nu h} \quad ; \qquad [5.a.37]$$

Separating real and imaginary parts and substituting for ν its value g/U^2 , we obtain

$$\left.\begin{aligned} D &= \frac{\rho g \gamma^2}{U^2}\, e^{-2gh/U^2} , \\ L &= -\rho U\gamma + \frac{\rho\gamma^2}{4\pi h} - \frac{\rho g \gamma^2}{\pi U^2} \cdot e^{-2gh/U^2} E_{i_1}\left(\frac{2gh}{U^2}\right) . \end{aligned}\right\} \qquad [5.a.38]$$

Vb UNIFORM STREAM PAST SUBMERGED SOURCE ELEMENT

A source element of intensity m located at a depth h below the free surface of the liquid which moving uniformly with speed U in the x positive direction.

The usual assumptions in this problem is taken as a basis, namely, the liquid is non-viscous and moving two dimensionally steadily and without vorticity, the only force acting on it is gravity.

With these assumptions together with a linearization assumption we can write the complex potential of a submerged source element using the similar operations as used in Va , in the form

$$w(z) = -\frac{m}{2\pi}\left\{\log(z^2+h^2) - 2e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu t}dt}{t-ih}\right\} , \left(\nu = \frac{g}{U^2}\right) . \qquad [5.b.1]$$

Now we compute the forces acting on the source as follows:

If we denote the projections of these forces on the x and y axes by drag D and lift L , respectively, we will have by Blasius' theorem

$$L+iD = -\tfrac{1}{2}\rho\oint_c \left(\frac{dW}{dz}\right)^2 dz , \qquad [5.b.2]$$

where

$$W(z) = w(z) - Uz , \qquad [5.b.3]$$

the integral is taken along any closed contour c surrounding the source element.

Using [5.b.1] and [5.b.3] we can write

$$\frac{dW}{dz} = -\frac{m}{2\pi}\,\frac{1}{(z+ih)} + \alpha(z) , \qquad [5.b.4]$$

where

$$\alpha(z) = -U + \frac{m}{2\pi}\frac{1}{(z-ih)} - \frac{im\nu}{\pi} \cdot e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu t}dt}{t-ih}, \qquad [5.b.5]$$

is a holomorphic function in the lower half plane.

Therefore the residue of the function

$$\left(\frac{dW}{dz}\right)^2 = \frac{m^2}{4\pi^2}\frac{1}{(z+ih)^2} - \frac{m}{\pi}\frac{\alpha(z)}{(z+ih)} + \alpha^2(z) \qquad [5.b.6]$$

at the point $z=-ih$, the location of the source element, is equal to $\frac{-m}{\pi}\,\alpha(-ih)$ and we obtain, by the residue theorem,

$$\oint_c \left(\frac{dW}{dz}\right)^2 dz = -\,2mi\alpha(-ih) \qquad , \qquad [5.b.7]$$

Thus

$$\begin{aligned} L+iD &= \rho mi\alpha(-ih) \\ &= -\rho mUi + \frac{\rho m^2}{4\pi h} - \frac{\rho m^2\nu}{\pi}\, e^{-\nu h}\int_{\infty}^{-ih}\frac{e^{i\nu t}dt}{t-ih} \\ &= -\rho mUi + \frac{\rho m^2}{4\pi h} - \frac{\rho m^2\nu}{\pi}\, e^{-2\nu h}\int_{-\infty}^{2\nu h}\frac{e^s ds}{s}, \quad [s=i\nu(t-ih)]. \end{aligned} \qquad [5.b.8]$$

Separating real and imaginary parts and substituting for ν its value $\frac{g}{U^2}$, we obtain

$$\left.\begin{aligned} D &= -\,\rho mU + \frac{\rho g}{U^2}m^2\, e^{-2gh/U^2}, \\ L &= \frac{\rho m^2}{4\pi h} - \frac{\rho m^2 g}{\pi U^2}\, e^{-2gh/U^2} . E_{i_1}(2gh/U^2), \end{aligned}\right\} \qquad [5.b.9]$$

with

$$E_{i_1}(x) = R\ell E_i(x) \quad , \quad \left[E_i(x) = \int_{-\infty}^{x}\frac{e^s ds}{s}\right] , \qquad [5.b.10]$$

where $E_i(x)$ is the exponential integral [see, e.g., Abramowitz, M. and Stegun, I., (1), 5, pp. 227-253].

Vc THE HYDROFOIL BENEATH A FREE SURFACE

A hydrofoil of arbitrary shape is in steady, rectilinear motion at a depth h beneath the free surface of a uniform liquid flow with speed U in the x-positive direction.

We assume the liquid is non-viscous and moving two-dimensionally and without vorticity, the only force acting on it is gravity.

The problem will be solved on the basis of linearized theory and for this purpose we introduce the following vortex distribution on the x-axis:

Vortices of strength $\gamma(\xi)$ per unit length in $o<\xi<a, y=-h$, $(\gamma>o$, clockwise).

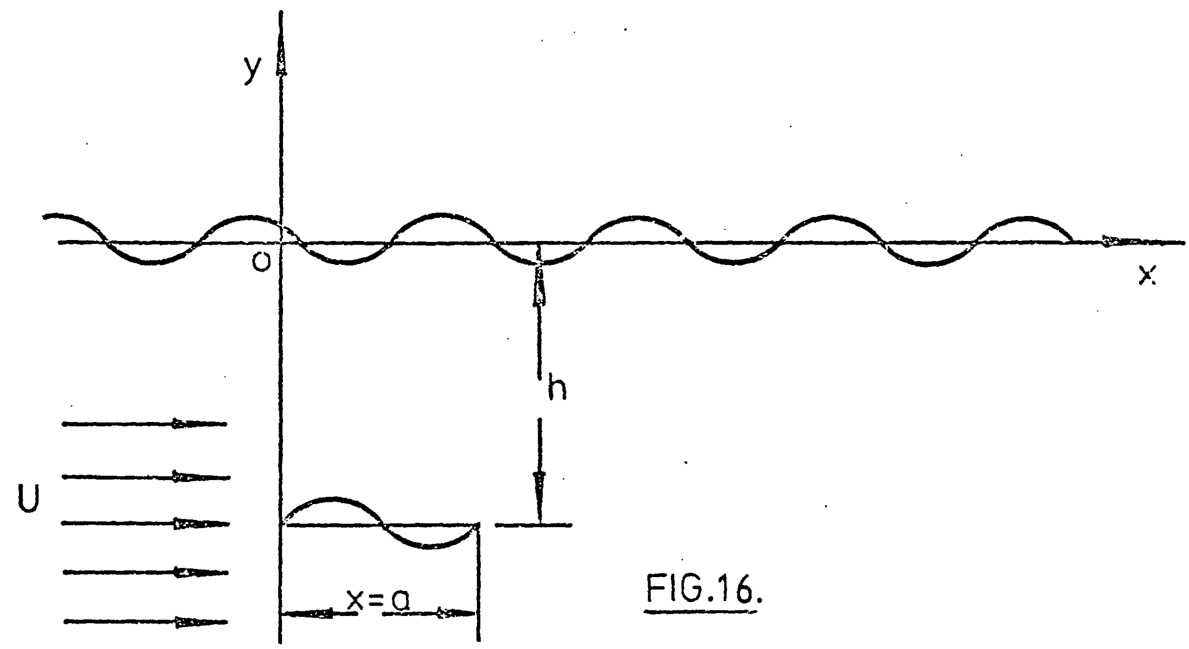

FIG.16.

The complex potential due to a single vortex located at $(\xi,-h)$ as follows:

$$w(z) = -\frac{\gamma(\xi)}{2\pi i}\left\{\log\left(\frac{z-c}{z-\bar{c}}\right) + 2e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu t}dt}{t-\bar{c}}\right\} , \quad [\nu = \frac{g}{U^2} , \; c=\xi-ih] . \qquad [5.c.1]$$

The complex potential due to the complete distribution of vortices as described above will be

$$w(z) = -\int_{o}^{a}\frac{\gamma(\xi)}{2\pi i}\left\{\log\left(\frac{z-c}{z-\bar{c}}\right) + 2e^{-i\nu z}.\int_{\infty}^{z}\frac{e^{i\nu t}dt}{t-\bar{c}}\right\} d\xi \quad . \qquad [5.c.2]$$

Using the identity

$$e^{-i\nu z}\int_{\infty}^{z'} \frac{e^{i\nu t}dt}{t-\bar{c}} = -\int_{0}^{\infty} \frac{e^{i\nu s}ds}{z+s-\bar{c}} \qquad [t=z+s, \bar{c}=\xi+ih] \;, \qquad [5.c.3]$$

we obtain

$$w(z) = -\int_0^a \frac{\gamma(\xi)}{2\pi i}\left\{\log\left(\frac{z-c}{z-\bar{c}}\right) - 2\int_0^\infty \frac{e^{i\nu s}ds}{z+s-\bar{c}}\right\} d\xi \quad . \qquad [5.c.4]$$

Denoting the potential of this steady motion by

$$W=\Phi+i\Psi \quad , \qquad [5.c.5]$$

we will have

$$W=w-Uz \;,\; \Phi=\phi-Ux \;,\; \Psi=\psi-Uy \;. \qquad [5.c.6]$$

Hence

$$\Phi(x,y) = -Ux-\frac{1}{2\pi}\int_0^a \gamma(\xi)\left\{\tan^{-1}\frac{y+h}{x-\xi} - \tan^{-1}\frac{y-h}{x-\xi} - 2\int_0^\infty \frac{[(x-\xi+s)\sin\nu s-(y-h)\cos\nu s]ds}{[(x-\xi+s)^2+4h^2]}\right\} d\xi$$

$$[5.c.7]$$

$$\Psi(x,y) = -Uy-\frac{1}{2\pi}\int_0^a \gamma(\xi)\left\{\log\frac{r}{r'} - 2\int_0^\infty \frac{[(y-h)\cos\nu s+(x-\xi+s)\sin\nu s]ds}{[(x-\xi+s)^2+(y-h)^2]}\right\} d\xi \quad , \qquad [5.c.8]$$

with

$$\left.\begin{aligned} r &= \sqrt{(x-\xi)^2+(y+h)^2} \quad , \\ r' &= \sqrt{(x-\xi)^2+(y-h)^2} \quad . \end{aligned}\right\} \qquad [5.c.9]$$

Using [5.c.4] and [5.c.6] we can write

$$\frac{dW}{dz} = -U + \int_0^a \frac{\gamma(\xi)}{2\pi i}\left\{\frac{1}{(z-c)} - \frac{1}{(z-\bar{c})} + 2\int_0^\infty \frac{e^{i\nu s}ds}{(z+s-\bar{c})^2}\right\} \quad . \qquad [5.c.10]$$

However, integration by parts yields

$$\int_0^\infty \frac{e^{i\nu s}ds}{(z+s-\bar{c})^2} = -\int_0^\infty e^{i\nu s}.d\left(\frac{1}{z+s-\bar{c}}\right) = \frac{1}{z-\bar{c}} + i\nu\int_0^\infty \frac{e^{i\nu s}ds}{z+s-\bar{c}} \quad . \qquad [5.c.11]$$

Substituting from [5.c.11] in [5.c.10] we obtain

$$\frac{dW}{dz} = -U + \frac{1}{2\pi i}\int_0^a \gamma(\xi)\left\{\frac{1}{z-c} + \frac{1}{z-\bar{c}} + 2\nu i\int_0^\infty \frac{e^{i\nu s}ds}{z+s-\bar{c}}\right\} d\xi \;. \qquad [5.c.12]$$

This may be written in the form

$$\frac{dW}{dz} = -u+iv \quad , \qquad [5.c.13]$$

with

$$\left.\begin{aligned} u &= -\frac{\partial\Phi}{\partial x} = -\frac{\partial\phi}{\partial x} + U \quad , \\ v &= -\frac{\partial\Phi}{\partial y} = -\frac{\partial\phi}{\partial y} \quad , \end{aligned}\right\} \qquad [5.c.14]$$

where

$$\phi_x = \frac{1}{2\pi}\int_0^a \gamma(\xi)\left\{\frac{y+h}{(x-\xi)^2+(y+h)^2} + \frac{y-h}{(x-\xi)^2+(y-h)^2} - 2\nu\int_0^\infty \frac{[(x-\xi+s)\cos\nu s+(y-h)\sin\nu s]ds}{[(x-\xi+s)^2+(y-h)^2]}\right\} d\xi \quad , \qquad [5.c.15]$$

$$\phi_y = -\frac{1}{2\pi}\int_0^a \gamma(\xi)\left\{\frac{x-\xi}{(x-\xi)^2+(y+h)^2} + \frac{x-\xi}{(x-\xi)^2+(y-h)^2} - 2\nu\int_0^\infty \frac{[(x-\xi+s)\sin\nu s-(y-h)\cos\nu s]ds}{[(x-\xi+s)^2+(y-h)^2]}\right\} d\xi \quad . \qquad [5.c.16]$$

Here in [5.c.14], u,v are the components of liquid velocity along x,y axes respectively.

Let the x- and y- components of the hydrodynamic forces acting on the hydrofoil be denoted by drag D and lift L, then the complex forces acting on a hydrofoil calculated within the linearized theory are given by

$$D+iL = \int_0^a \{P\big|_{y=o-} - P\big|_{y=o+}\}\, i dz \quad . \qquad [5.c.17]$$

By Bernoulli formula, we have for the pressure P the formula

$$P = C - \tfrac{1}{2}\rho(u^2+v^2) - \rho g y \quad , \qquad [5.c.18]$$

Substituting from [5.c.14] in [5.c.18] we obtain

$$P = C - \tfrac{1}{2}\rho U^2 + \rho U \frac{\partial\phi}{\partial x} - \tfrac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] - \rho g y \quad . \qquad [5.c.19]$$

Since the second degree terms are ignored we obtain

$$P = C - \tfrac{1}{2}\rho U^2 + \rho U \frac{\partial\phi}{\partial x} - \rho g y \quad . \qquad [5.c.20]$$

Using [5.c.20] we can write [5.c.17] in the form

$$D+iL = \rho U\int_0^a \{\phi_x\big|_{y=-h-o} - \phi_x\big|_{y=-h+o}\}\, i dz \quad , \qquad [5.c.21]$$

where

$$\phi_x\Big|_{y=-h\pm o}=\pm\tfrac{1}{2}\gamma(x)-\frac{1}{2\pi}\int_0^a\gamma(\xi)\left\{\frac{2h}{(x-\xi)^2+4h^2}+2\nu\int_0^\infty\frac{[(x-\xi+s)\cos\nu s-2h\sin\nu s]ds}{[(x-\xi+s)^2+4h^2]}\right\}d\xi \quad . \tag{5.c.22}$$

Hence

$$D+iL=-\rho U\int_0^a\gamma(x).idz \quad . \tag{5.c.23}$$

Now we can write expressions for the lift L and drag D in the forms

$$\left.\begin{aligned} L &= -\rho U\int_0^a\gamma(x)dx \quad , \\ D &= \rho U\int_0^a\gamma(x).z(x)dx \qquad (z(x)=y'(x)). \end{aligned}\right\} \tag{5.c.24}$$

The boundary condition on the hydrofoil is

$$z(x)=\frac{v}{U+u} \qquad (o<x<a) \quad . \tag{5.c.25}$$

This equation is approximated in the usual way to

$$\begin{aligned} z(x) &= \frac{1}{U}v \\ &= -\frac{1}{U}\frac{\partial\phi}{\partial y}\Big|_{y=-h} \quad . \end{aligned} \tag{5.c.26}$$

Using [5.c.16] we can write [5.c.26] in the form

$$z(x)=\frac{1}{2\pi U}\int_0^a\gamma(\xi)\left\{\frac{1}{(x-\xi)}+k(x-\xi)\right\}d\xi \quad , \tag{5.c.27}$$

with

$$k(x-\xi)=\frac{x-\xi}{(x-\xi)^2+4h^2}-2\nu\int_0^\infty\frac{[(x-\xi+s)\sin\nu s+2h\cos\nu s]ds}{[(x-\xi+s)^2+4h^2]} \quad . \tag{5.c.28}$$

VIa THE OPTIMUM SHAPE OF A HYDROFOIL OF MINIMUM DRAG BENEATH A FREE SURFACE

We pose the problem of minimizing the drag coefficient

$$D^* = \frac{D}{\rho U^2} \quad , \qquad [6.a.1]$$

subject to a constraint on curvature of the form

$$K = \int_0^a z'^2(x)dx \quad , \qquad [6.a.2]$$

where K is prescribed, together with a constraint on the length of the hydrofoil of the form

$$\ell = \int_0^a \sqrt{1+z^2(x)}\,.dx \quad , \qquad [6.a.3]$$

where ℓ is prescribed and $z(x)$ is the gradient of the hydrofoil at position x .

The expression for the drag D is given by

$$D = \rho U \int_0^a \gamma(x).z(x).dx \quad . \qquad [6.a.4]$$

STATEMENT OF THE PROBLEM

The general optimum problem considered here may be stated as follows:

To find the real, extremal function $\gamma(x)$ of a real variable, required to be Hölder continuous [see, e.g., Tricomi,F.G.(61)] in the region $o<x<a$ together

$$z(x) = \frac{1}{2\pi U}\int_0^a \gamma(s)\left[\frac{1}{x-s} + k(x-s)\right] ds \quad , \qquad [6.a.5]$$

with

$$k(x-s) = \frac{x-s}{(x-s)^2+4h^2} - 2\nu\int_0^\infty \frac{[(x-s+t)\sin\nu t+2h\cos\nu t]dt}{[(x-s+t)^2+4h^2]} \quad , \qquad [6.a.6]$$

so that $\gamma(x)$ and $z(x)$ minimize the functional

$$I[\gamma(x),z(x),z'(x),x] = D^* + \lambda_1\ell + \lambda_2 K$$
$$= \int_o^a F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2]dx \quad , \qquad [6.a.7]$$

with the function $F[\gamma(x),z(x),z'(x),x]$ given by

$$F[\gamma(x),z(x),z'(x),x;\lambda_1,\lambda_2] = \frac{1}{U} z(x)\gamma(x) + \lambda_1\sqrt{1+z^2(x)} + \lambda_2 z'^2(x) \quad , \qquad [6.a.8]$$

where $\gamma(x),z(x)$ are related by [6.a.5] and, λ_1,λ_2 are Lagrange multipliers.

We define an admissible function as any function $\gamma(x)$ which satisfies the Hölder condition $\mathcal{H}$ $(\mu<1)$, the constraints [6.a.2] and [6.a.3], and we define the optimal function as an admissible function which minimize the function $I[\gamma,z,z',x]$.

THE NECESSARY CONDITION OF OPTIMALITY

Let $\gamma(x),z(x)$ denote the required optimal vortex distribution function and optimal hydrofoil slope function respectively.

A function $\xi(x)$ will be called admissible variation if, for all sufficiently small positive constant ε

$$\gamma_1(x) = \gamma(x) + \varepsilon\xi(x) \quad , \qquad [6.a.9]$$

is an admissible function.

The variation in $z(x)$ which corresponds to an admissible variation $\eta(x)$, such that

$$z_1(x) = z(x) + \varepsilon\eta(x) \quad , \qquad [6.a.10]$$

is found from [6.a.5]

$$\eta(x) = \frac{1}{2\pi U}\int_o^a \xi(s)\left[\frac{1}{x-s} + k(x-s)\right]ds \qquad (o<x<a) \quad . \qquad [6.a.11]$$

If $\xi(x)$ is an admissible variation, then $I[\gamma+\varepsilon\xi]$ is a function of ε which has an extreme value when $\varepsilon = o$.

The variation of the function I due to the variation $\xi(x)$ and $\eta(x)$ is

$$\Delta I=\int_0^a F[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x]dx - \int_0^a F[\gamma, z, z', x]dx \quad . \qquad [6.a.12]$$

For sufficiently small ε, expansion of the above integrand in Taylor's series yields

$$\Delta I=\varepsilon\delta I + \frac{\varepsilon^2}{2!}\delta^2 I + \ldots \quad , \qquad [6.a.13]$$

where the first variation δI is defined by

$$\delta I=\int_0^a [\xi F_\gamma(\gamma, z, z', x)+\eta . F_z(\gamma, z, z', x)+\eta' F_{z'}(\gamma, z, z', x)]dx \ , \qquad [6.a.14]$$

in which the sub-indices denote partial differentiations, η is given by [6.a.11].

The variations δI, $\delta^2 I, \ldots$ depend on $\xi(x)$ as well as $\gamma(x)$.

We integrate by parts the equation [6.a.14] and it becomes

$$\delta I=\int_0^a [\xi(x)F_\gamma(\gamma, z, z', x)+\eta(x)\left(F_z(\gamma, z, z', x) - \frac{d}{dx}F_{z'}(\gamma, z, z', x)\right)]dx+[\eta(x).F_{z'}(\gamma, z, z', x)]_0^a \ .$$

[6.a.15]

Substituting from [6.a.11] in [6.a.15] we obtain

$$\delta I=[\eta(x).F_{z'}(\gamma, z, z', x)]_0^a + \int_0^a \Bigg\{\xi(x)F_\gamma(\gamma, z, z', x)+(F_z(\gamma, z, z', x) - $$
$$- \frac{d}{dx}F_{z'}(\gamma, z, z', x))\left(\frac{1}{2\pi U}\int_0^a \xi(s)\left[\frac{1}{x-s} + k(x-s)\right]ds\right)\Bigg\} dx \quad . \qquad [6.a.16]$$

It is permissible to interchange the order of the double integral in [6.a.16] [see, e.g., Hardy, G.H. (35)] and interchange between the variables x, s and when we do we obtain

$$\delta I=[\eta(x).F_{z'}(\gamma, z, z', x)]_0^a + \int_0^a \Bigg\{F_\gamma(\gamma, z, z', x) + \frac{1}{2\pi U}\int_0^a \Bigg[F_z(\gamma, z, z', s) - $$
$$- \frac{d}{ds}F_{z'}(\gamma, z, z', s)\Bigg]\left[\frac{1}{s-x} + k(s-x)\right] ds\Bigg\} \xi(x)dx \quad . \qquad [6.a.17]$$

For $I[\gamma, \xi]$ to be a minimum, we must have for all admissible function $\xi(x)$,

$$\delta I[\gamma,\xi] = 0 \quad . \qquad [6.a.18]$$

Now the stationary condition [6.a.18] must hold for all admissible $\xi(x)$; there are a number of different cases to be considered on the end conditions as follows:

$$\left.\begin{array}{llll} \eta(0) = 0 & , & \eta(a) = 0 & ; \\ \eta(0) = 0 & , & \eta(a) \neq 0 & ; \\ \eta(0) \neq 0 & , & \eta(a) = 0 & ; \\ \eta(0) \neq 0 & , & \eta(a) \neq 0 & . \end{array}\right\} \qquad [6.a.19]$$

In all cases it is necessary that

$$F_\gamma(\gamma,z,z',x) = -\frac{1}{2\pi U}\int_0^a \left[F_z(\gamma,z,z',s) - \frac{d}{ds}F_{z'}(\gamma,z,z',s)\right]\left[\frac{1}{s-x} + k(s-x)\right] ds \quad (o<x<a) \ , \qquad [6.a.20]$$

and if $\eta(x)$ does not vanish at an end point then it is necessary that $\frac{\partial F}{\partial z'}$ should vanish at that point; since $\frac{\partial F}{\partial z'} = 2\lambda_2 z'(x)$ it then follows that, we must have

$$\eta(0).z'(0) = 0 \quad , \quad \eta(a).z'(a) = 0 \quad . \qquad [6.a.21]$$

The boundary conditions for $z(x)$ are called natural boundary conditions [see, e.g., Arthurs (3)].

We can write from [6.a.8] the first partial derivatives of the function $F(\gamma,z,z',x;\lambda_1,\lambda_2)$

$$\left.\begin{array}{ll} F_\gamma[\gamma,z,z',x] = \frac{1}{U}\,z(x) & , \\ F_z[\gamma,z,z',x] = \frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} + \frac{1}{U}\,\gamma(x) & , \\ F_{z'}[\gamma,z,z',x] = 2\lambda_2 z'(x) & . \end{array}\right\} \qquad [6.a.22]$$

Substituting from [6.a.22] in [6.a.20] we obtain

$$z(x) = -\frac{1}{2\pi}\int_0^a \left[\frac{1}{U}\,\gamma(s) + \frac{\lambda_1 z(s)}{\sqrt{1+z^2(s)}} - 2\lambda_2 z''(s)\right]\left[\frac{1}{s-x} + k(s-x)\right] ds \quad (o<x<a) . \ [6.a.23]$$

This equation which is a necessary condition for the existence of an extremal $I[\gamma]$, combines with the integral equation, [6.a.5], to give a pair of singular integral equations which are to be solved for γ, z subject to appropriate conditions and the constraints, [6.a.2] and [6.a.3].

Substituting from [6.a.5] in [6.a.23] we obtain

$$\frac{1}{U}\int_0^a \gamma(s)\,[k(s-x)+k(x-s)]\,ds + \int_0^a\left[\frac{\lambda_1 z(s)}{\sqrt{1+z^2(s)}} - 2\lambda_2 z''(s)\right]\left[\frac{1}{s-x} + k(s-x)\right]ds = 0. \quad [6.a.24]$$

We consider the solution of [6.a.24] for the slope $z(x)$ only in the case of small slope, and we approximate to [6.a.24] as follows:

$$\frac{1}{U}\int_0^a \gamma(s)\,[k(s-x)+k(x-s)]\,ds + \int_0^a [\lambda_1 z(s) - 2\lambda_2 z''(s)]\left[\frac{1}{s-x} + k(s-x)\right]ds = 0. \quad [6.a.25]$$

Now we use the method of iteration to solve equation [6.a.25] as follows:

We introduce function sequences of the form

$$\left.\begin{array}{l} z_0,\ z_1,\ z_2,\ \ldots\ \ldots \quad , \\ \gamma_0,\ \gamma_1, \gamma_2,\ \ldots\ \ldots \quad , \end{array}\right\} \quad [6.a.26]$$

and the stages in the iteration procedure would be as follows:

(a) First we solve

$$\left.\begin{array}{l} \displaystyle\int_0^a [2\lambda_2 z_0''(s) - \lambda_1 z_0(s)]\,\frac{ds}{s-x} = 0 \quad , \\ \displaystyle z_0(x) = -\frac{1}{2\pi U}\int_0^a \gamma_0(s)\frac{ds}{s-x} \quad , \end{array}\right\} \quad (o<x<a)\ , \quad [6.a.27]$$

for γ_0, z_0.

(b) Secondly, we solve

$$\left.\begin{array}{l} \displaystyle\int_0^a [2\lambda_2 z_1''(s) - \lambda_1 z_1(s)]\frac{ds}{s-x} = \frac{1}{U}\int_0^a \gamma_0(s)\,[k(s-x)+k(x-s)]\,ds - \int_0^a [2\lambda_2 z_0''(s) - \lambda_1 z_0(s)]\,k(s-x)\,dx, \\ \displaystyle z_1(x) = -\frac{1}{2\pi U}\int_0^a \frac{\gamma_1(s)\,ds}{s-x} + \frac{1}{2\pi U}\int_0^a \gamma_0(s)\,k(x-s)\,ds \ , \qquad (o<x<a)\ , \end{array}\right\}$$

[6.a.28]

We solve the first equation in [6.a.27] by writing it in the form

$$2\lambda_2^{(o)}\, z_o''(x) - \lambda_1^{(o)}\, z_o(x) = \frac{C_o}{\sqrt{x(a-x)}} \qquad [(o<x<a), z_o(x)=y_o'(x)] \ , \qquad [6.a.29]$$

where C_o is an arbitrary constant.

We can write equation [6.a.29] as follows:

$$z_o''(x) + m_o^2 z_o(x) = \frac{C_o}{2\lambda_2^{(o)}} \cdot \frac{1}{\sqrt{x(a-x)}} \ , \qquad \left[m_o^2 = -\frac{\lambda_1^{(o)}}{2\lambda_2^{(o)}} , \ (o<x<a) \right] , \qquad [6.a.30]$$

and the boundary conditions to be satisfied are

$$z_o(o) = y_o'(o) = o \ , \ z_o(a) = y_o'(a) = \beta \ , \ y_o(o) = o \quad . \qquad [6.a.31]$$

This problem is identical with that solved in Chapter II where it has been verified that $\lambda_1^{(o)} < o$ and $\lambda_2^{(o)} > o$, so that m_o^2 is real. Here we proceed to solve [6.a.30] using Duhamel's method for the determination of the particular integral [see, e.g., Ritger, P.D. and Rose, N.J. (56)].

We integrate [6.a.30] with respect to x to obtain

$$M_x y_o \equiv y_o''(x) + m_o^2 y_o(x) = \frac{C_o}{2\lambda_2^{(o)}} \sin^{-1}\left(\frac{x-\frac{1}{2}a}{\frac{1}{2}a}\right) + G_o \equiv u(x) \ , \qquad [6.a.32]$$

where G_o is an arbitrary constant.

We can write the general solution of [6.a.32] in the form

$$y_o(x) = A_o \cos m_o x + B_o \sin m_o x + \int_o^x \left[\frac{C_o}{2\lambda_2^{(o)}} \sin^{-1}\left(\frac{\tau-\frac{1}{2}a}{\frac{1}{2}a}\right) + G_o \right] \frac{1}{m_o} \sin m_o(x-\tau)\, d\tau \ , \qquad [6.a.33]$$

where A_o and B_o are arbitrary constants and we obtain

$$y_o(x) = D_o + F_o \cos m_o x + B_o \sin m_o x - \frac{C_o}{2\lambda_2^{(o)} m_o^2} \int_o^x \frac{\cos m_o(x-\tau)\, d\tau}{\sqrt{\tau(a-\tau)}} + \frac{C_o}{2\lambda_2^{(o)} m_o^2} \sin^{-1}\left(\frac{x-\frac{1}{2}a}{\frac{1}{2}a}\right) ,$$

$$(o<x<a) \ , \qquad [6.a.34]$$

where

$$\left.\begin{aligned} F_o &= A_o - \frac{G_o}{m_o^2} + \frac{C_o}{4\lambda_2^{(o)} m_o^2} \quad , \\ D_o &= \frac{G_o}{m_o^2} \quad . \end{aligned}\right\} \qquad [6.a.35]$$

Using the first and second boundary conditions in [6.a.31] we obtain

$$y_o(x) = - F_o(1-\cos m_o x) - \frac{C_o}{2\lambda_2^{(o)} m_o^2} \int_o^x \frac{\cos m_o(x-\tau)\,d\tau}{\sqrt{\tau(a-\tau)}} +$$

$$+ \frac{C_o}{2\lambda_2^{(o)} m_o^2} \sin^{-1}\left(\frac{x-\frac{1}{2}a}{\frac{1}{2}a}\right) + \frac{\pi C_o}{4\lambda_2^{(o)} m_o^2} \, , \qquad (o<x<a) \; . \qquad [6.a.36]$$

Differentiating [6.a.36] with respect to x we obtain

$$z_o(x) = - F_o m_o \sin m_o x + \frac{C_o}{2\lambda_2^{(o)} m_o^2} \int_o^x \frac{\sin m_o(x-\tau)\,d\tau}{\sqrt{\tau(a-\tau)}} \, , \quad [z_o(x)=y_o'(x) \, , \; (o<x<a)] \; . \qquad [6.a.37]$$

The function $z_o(x)$ in [6.a.37] should satisfy the constraints, [6.a.2] and [6.a.3], and the boundary condition $z_o(a) = \beta$ at $x = a$; in this way we obtain three equations in three unknowns m_o, F_o and $\frac{C_o}{2\lambda_2^{(o)}}$.

We require now to solve the second equation in [6.a.27] for γ_o and the solution of [6.a.27] satisfying the Kutta condition

$$\gamma_o(a) = o \qquad [6.a.38]$$

is given by

$$\gamma_o(x) = \frac{2U}{\pi} \sqrt{\frac{a-x}{x}} \oint_o^a \sqrt{\frac{s}{a-s}} \cdot \frac{z_o(s)\,ds}{s-x} \quad . \qquad [6.a.39]$$

Substituting from [6.a.37] into [6.a.39] we obtain

$$\gamma_o(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}} \oint_o^a \sqrt{\frac{s}{a-s}} \left[- F_o m_o \sin m_o x + \frac{C_o}{2\lambda_2^{(o)} m_o} \int_o^x \frac{\sin m_o(x-\tau)\,d\tau}{\sqrt{\tau(a-\tau)}} \right] \frac{ds}{s-x} . \qquad [6.a.40]$$

Using the above results, we now consider the equations [6.a.28] for γ_1 and z_1 , which is the second stage of the iteration.

First we write the inversion of the first equation in [6.a.28] as follows:

$$2\lambda_2^{(1)} z_1''(x) - \lambda_1^{(1)} z_1(x) = \frac{C_1}{\sqrt{x(a-x)}} - \frac{1}{\pi^2\sqrt{x(a-x)}} \oint_o^a \frac{\sqrt{s(a-s)}\,\phi(s)\,ds}{s-x} , \qquad [6.a.41]$$

where

$$\phi(s) = \frac{1}{U}\int_o^a \gamma_o(\tau)\,[k(\tau-s)+k(s-\tau)]\,d\tau + \int_o^a [\lambda_1^{(o)} z_o(\tau) - 2\lambda_2^{(o)} z_o''(\tau)]\,k(\tau-s)\,d\tau, \quad [6.a.42]$$

and C_1 is an arbitrary constant.

Substituting from [6.a.29] and [6.a.40] into [6.a.42] we obtain

$$\phi(s) = \frac{2}{\pi}\int_o^a [k(\tau-s)+k(s-\tau)]\sqrt{\frac{a-\tau}{\tau}}\,d\tau \oint_o^a \sqrt{\frac{t}{a-t}}\Big[-F_o m_o \sin m_o t +$$

$$+ \frac{E_o}{m_o}\int_o^t \frac{\sin m_o(t-\xi)\,d\xi}{\sqrt{\cdot(a-\xi)}}\Big]\frac{dt}{t-\tau} - C_o \int_o^a \frac{k(\tau-s)\,d\tau}{\sqrt{\tau(a-\tau)}} , \quad [(o<s<a),\ E_o = \frac{C_o}{2\lambda_2^{(o)}}] \ . [6.a.43]$$

Equation [6.a.41] can be written as follows:

$$z_1''(x) + m_1^2 z_1(x) = F(x) \quad , \quad [m_1^2 = -\frac{\lambda_1^{(1)}}{2\lambda_2^{(1)}} \quad , \quad (o<x<a)], [6.a.44]$$

where

$$F(x) = \frac{1}{\sqrt{x(a-x)}}\left\{E_1 - \frac{D_1}{\pi^2}\oint_o^a \frac{\sqrt{s(a-s)}\,\phi(s)\,ds}{s-x}\right\}, \ [E_1 = \frac{C_1}{2\lambda_2^{(1)}} \ , \ D_1 = \frac{1}{2\lambda_2^{(1)}} \ , \ (o<x<a)] \ . \qquad [6.a.45]$$

It is assumed at this stage $\frac{\lambda_1^{(1)}}{\lambda_2^{(1)}} < o$ and we show later that $\lambda_1^{(1)} < o$, $\lambda_2^{(1)} > o$ are sufficient conditions for a true minimization of the drag D .

The boundary conditions to be satisfied by $z_1(x)$ are

$$\left.\begin{array}{lll} z_1(o) = y_1'(o) = o \ , & z_1(a) = y_1'(a) = \beta \ , & z_1'(o) = o \ , \\ y_1(o) = o & , \ y_1(a) = y_o & , \ [\beta \ , \ y_o \ \text{prescribed}] \ . \end{array}\right\} \qquad [6.a.46]$$

The solution of the non-homogeneous differential equation in [6.a.44] is as follows:

$$z_1(x) = -\frac{1}{m_1}\int_o^x F(\xi)\sin m_1(\xi-x)d\xi + A_1\sin m_1x + B_1\cos m_1x \quad (o<x<a) \quad ,[6.a.47]$$

where A_1 and B_1 are arbitrary constants.

From [6.a.47] , [6.a.45] and [6.a.43] we have six unknowns m_1, E_1, D_1, C_o, A_1 and B_1 and the seventh unknown arises from the integration of $z_1(x) = y_1'(x)$, accordingly we can satisfy five boundary conditions, [6.a.46] and two constraints, [6.a.42] and [6.a.3]. The above choice of the five boundary conditions is a special case which is studied in detail below, but other choices are possible.

Using the boundary conditions, [6.a.46] we can write [6.a.47] in the form

$$z_1(x) = y_1'(x) = -\frac{1}{m_1}\int_o^x F(\xi)\sin m_1(\xi-x)d\xi + \frac{\sin m_1 x}{m_1\sin m_1 a}\int_o^a F(\xi)\sin m_1(\xi-a)d\xi +$$

$$+\beta\frac{\sin m_1x}{\sin m_1a}, \qquad [(o<x<a),\ z_1(x) = y_1'(x)] . \qquad [6.a.48]$$

We integrate [6.a.48] with respect to x and use the boundary condition $y(o) = o$, [6.a.46] to obtain

$$y_1(x) = -\frac{1}{m_1^2}\int_o^x F(\xi).[1-\cos m_1(\xi-x)]d\xi + \frac{(\cos m_1x-1)}{m_1^2\sin m_1 a}\int_o^a F(\xi)\sin m_1(a-\xi)d\xi -$$

$$-\beta\frac{(\cos m_1x-1)}{m_1\sin m_1 a} \qquad (o<x<a). \qquad [6.a.49]$$

The function $z_1(x)$ in [6.a.48] should satisfy the constraints, [6.a.2] and [6.a.3] and the boundary conditions $z_1'(o) = o$ and $y_1(a) = y_o$; in this way we obtain four equations in four unknowns m_1, E_1, D_1 and C_o . No numerical work has been done on the above problem for general values of the depth parameter h , but more detailed consideration is now given to the case when h is large in which numerical results are provided.

The integrand in [6.a.6] can be written in the form

$$\psi(\xi) = \frac{(\xi+t)\sin\nu t+2h\cos\nu t}{(t^2+4h^2)+2\xi t+\xi^2} \quad , \qquad \xi = x-s \quad ; \qquad [6.a.50]$$

when h is large and $|x-s| << h$, $\psi(\xi)$ can be expanded using Taylor's series to yield

$$\psi(\xi) = \frac{(\xi+t)\sin\nu t+2h\cos\nu t}{(t^2+4h^2)}\left\{1 - \frac{2\xi t}{t^2+4h^2} + 0\left(\frac{\xi^2}{t^2+4h^2}\right)\right\}$$

$$= \frac{t\cdot\sin\nu t+2h\cos\nu t}{(t^2+4h^2)} + \xi\,\frac{\sin\nu t}{t^2+4h^2} - 0\left(\frac{\xi}{(t^2+4h^2)^2}\right) . \qquad [6.a.51]$$

Consequently we approximate [6.a.6] by the following equation

$$k(x-s) = \frac{x-s}{4h^2} - 2\nu\int_0^\infty\frac{[t\sin\nu t+2h\cos\nu t]dt}{t^2+4h^2} - 2\nu(x-s)\int_0^\infty\frac{\sin\nu t dt}{t^2+4h^2}$$

$$= \alpha(x-s) - 2\nu.(I_1+2hI_2) , \qquad [6.a.52]$$

with

$$\left.\begin{aligned} I_1 &= \int_0^\infty\frac{t\sin\nu t dt}{t^2+4h^2} = \frac{\pi}{2}\,e^{-2h\nu} ,\\ I_2 &= \int_0^\infty\frac{\cos\nu t dt}{t^2+4h^2} = \frac{\pi}{4h}\,e^{-2h\nu} ,\end{aligned}\right\} \qquad [6.a.53]$$

[see e,g,, Copson, E.T.(8)] ,

and

$$\alpha = \frac{1}{4h^2} - 2\nu\int_0^\infty\frac{\sin\nu t dt}{t^2+4h^2} . \qquad [6.a.54]$$

Substituting from [6.a.53] into [6.a.52] we obtain

$$k(x-s) = \alpha.(x-s) - 2\pi\nu e^{-2\nu h} . \qquad [6.a.55]$$

Using [6.a.55] and [6.a.39] we can write equation [6.a.41] as follows:

$$\int_0^a[2\lambda_2^{(1)} z_1''(s)-\lambda_1^{(1)} z_1(s)]\,\frac{ds}{s-x} = - C_o\int_0^a\frac{1}{\sqrt{s(a-s)}}\,[\alpha(s-x)-2\pi\nu e^{-2\nu h}] -$$

$$- \frac{4\pi\nu}{U}\,.\,e^{-2\nu h}\int_0^a\gamma_o(s)\,ds$$

$$= - C_o[\tfrac{\pi}{2}\alpha.(a-2x)-2\pi^2\nu e^{-2\nu h}] - 8\pi\nu e^{-2\nu h}\int_0^\infty\sqrt{\frac{s}{a-s}}\,z_o(s)\,ds \quad (o<x<a) . \qquad [6.a.56]$$

Equation [6.a.56] can be solved by standard methods to give

$$2\lambda_2^{(1)} z_1''(x) - \lambda_1^{(1)} z_1(x) = \frac{C_1}{\sqrt{x(a-x)}} + \frac{1}{\pi\sqrt{x(a-x)}} \oint_0^a \frac{\sqrt{s(a-s)}\,.\,[A_o s + B_o]\,ds}{s-x}\ , \quad (o<x<a)\ , \tag{6.a.57}$$

where

$$\left.\begin{aligned} A_o &= -\,\alpha C_o \ , \\ B_o &= -\tfrac{1}{2}a.C_o\alpha - 2\,\pi\,\nu.C_o.e^{-2\nu h} + 8\nu e^{-2\nu h}\int_0^a \sqrt{\frac{\tau}{a-\tau}}\; z_o(\tau)\,d\tau\ , \end{aligned}\right\} \tag{6.a.58}$$

and C_1 is an arbitrary constant.

Hence

$$z_1''(x) + m_1^2 z_1(x) = F(x)\ , \qquad \left[m_1^2 = -\frac{\lambda_1^{(1)}}{2\lambda_2^{(1)}}\ ,\ (o<x<a)\right]\ , \tag{6.a.59}$$

where

$$F(x) = \frac{1}{\sqrt{x(a-x)}}\left\{E_1 + D_1\left[C_o\left[\frac{\pi}{8}\,\alpha(a^2-8ax+8x^2) - \pi\nu(a-2x).e^{-2\nu h}\right] + \right.\right.$$

$$\left.\left. + 4\nu(a-2x)e^{-2\nu h}.\int_0^a \sqrt{\frac{s}{a-s}}\; z_o(s)\,ds\right]\right\}\ , \quad \left[E_1 = \frac{C_1}{2\lambda_2^{(1)}}\ ,\ D_1 = \frac{1}{2\lambda_2^{(1)}}\ ,\ (o<x<a)\right]\ . \tag{6.a.60}$$

Equation [6.a.59] has been solved subject to the boundary conditions [6.a.46] to obtain

$$z_1(x) = y_1'(x) = -\frac{1}{m_1}\int_0^x F(\xi)\sin m_1(\xi-x)\,d\xi + \frac{\sin m_1 x}{m_1 \sin m_1 a}\int_0^a F(\xi)\sin m_1(\xi-a)\,d\xi\ +$$

$$+\ \beta\,\frac{\sin m_1 x}{\sin m_1 a} \qquad (o<x<a)\ , \tag{6.a.61}$$

[see, [6.a.44]].

The function $z_1(x)$ in [6.a.61] should satisfy the constraints, [6.a.2] and [6.a.3] and the boundary conditions $z_1'(o) = o$ and $y_1(a) = y_o$; in this way we obtain four equations in four unknowns m_1, E_1, D_o and C_o, which have to be evaluated numerically. This problem is resolved numerically in Appendix XIV.

VIb A SUFFICIENT CONDITION FOR THE EXTREMUM TO BE A MINIMUM

A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I .

Since

$$\delta I[\gamma, z, z', x] = 0 \qquad [6.b.1]$$

the condition for I to be a minimum requires

$$\delta^2 I[\gamma, z, z', x] > 0 \quad , \qquad [6.b.2]$$

with γ , z related by

$$\gamma_1(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,[z_1(s) - \frac{1}{2\pi U}\int_0^a \gamma_0(\tau)k(\tau - s)d\tau]\,\frac{ds}{s-x}$$

$$= A(x) + \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{z_1(s)}{s-x}\,ds \quad , \qquad [6.b.3]$$

for all admissible variations $\xi(x)$ and $\eta(x)$ consistent with

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}} \cdot \frac{\eta(s)ds}{s-x} \qquad [6.b.4]$$

Since z has been prescribed at x=o and x=a in [4.c.3a] it follows that the variation η will satisfy the conditions.

$$\eta(0) = 0 \quad , \quad \eta(a) = 0 \qquad [6.b.5]$$

Then, using Taylor's theorem with remainder, we write the increment of the functional $I[\gamma, z, z', x]$ as

$$I[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x] - I[\gamma, z, z', x] =$$

$$= \varepsilon\int_0^a \{\xi(x)F_\gamma(\gamma,z,z',x) + \eta(x)[F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)]\}dx +$$

$$\tfrac{1}{2}\varepsilon^2\int_0^a \{\xi^2(x)F_{\gamma\gamma} + \eta^2(x)F_{zz} + \eta'^2 F_{z'z'} + 2\xi(x)\eta(x)F_{\gamma z} + 2\xi(x)\eta(x)F_{\gamma z} + 2\eta(x)\eta'(x)F_{zz'}\}dx + O(\varepsilon^3)$$

$$(o<x<a) \quad . \qquad [6.b.6]$$

Denoting the coefficient ε by δI and that ε^2 by $\delta^2 I$, and at a stationary value of I , we have from [6.b.1], [6.b.4] and [6.b.6]

$$\delta I = \int_0^a \{F_\gamma(\gamma,z,z',x).\xi(x) + [F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)][\frac{1}{2\pi U}\int_0^a \xi(s)(\frac{1}{x-s} + k(x-s))ds]\}dx$$

$$= \int_0^a \{F_\gamma(\gamma,z,z',x) + \frac{1}{2\pi U}\int_0^a [F_z(\gamma,z,z',s) - \frac{d}{ds}F_{z'}(\gamma,z,z',s)][\frac{1}{s-x} + k(s-x)]ds\}\xi(x)dx \quad .$$

$$[6.b.7]$$

Since $\eta(x)$ is arbitrary, the factor in the bracket in [6.b.7] must vanish identically for $o<x<a$, giving the following singular integral equation

$$F_\gamma(\gamma,z,z',x) = -\frac{1}{2\pi U}\int_0^a [F_z(\gamma,z,z',s) - \frac{d}{ds}F_{z'}(\gamma,z,z',s)][\frac{1}{s-x} + k(s-x)]ds, \quad (o<x<a) \quad .$$

$$[6.b.8]$$

The second variation defined from [6.b.6] in the form

$$\delta^2 I = \int_0^a \{\xi^2 F_{\gamma\gamma} + \eta^2 F_{zz} + \eta'^2 F_{z'z'} + 2\xi\eta F_{\gamma z} + 2\xi\eta' F_{\gamma z'} + 2\eta\eta' F_{zz'}\}dx \;, \qquad [6.b.9]$$

where, by [6.a.8] we can write

$$\left.\begin{aligned} F_{\gamma\gamma}[\gamma,z,z',x] &= 0 \quad ,\\ F_{zz}[\gamma,z,z',x] &= \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \quad ,\\ F_{z'z'}[\gamma,z,z',x] &= 2\lambda_2 \quad ,\\ F_{\gamma z}[\gamma,z,z',x] &= \frac{1}{U} \quad ,\\ F_{\gamma z}[\gamma,z,z',x] &= 0 \quad ,\\ F_{zz'}[\gamma,z,z',x] &= 0 \quad . \end{aligned}\right\} \qquad [6.b.10]$$

Substituting from [6.b.10] in [6.b.9] we obtain

$$\delta^2 I=\int_0^a \left\{ \frac{\lambda_1}{[1+z^2(x)]^{3/2}} \cdot \eta^2(x)+2\lambda_2\eta'^2(x) + \frac{2}{U}\,\xi(x)\eta(x)\right\} dx \quad . \qquad [6.b.11]$$

Now we calculate the third integral in [6.b.11] which is defined by

$$I_1=\int_0^a \xi(x)\eta(x)\,dx \qquad [6.b.12]$$

Using [6.b.4] we obtain

$$I_1=\frac{2U}{\pi}\int_0^a \sqrt{\frac{a-x}{x}}\,\eta(x)\,dx\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{\eta(s)\,ds}{s-x}$$

$$= \frac{2U}{\pi}\int_0^a\int_0^a \sqrt{\frac{a-x}{x}}\sqrt{\frac{s}{a-s}}\;\frac{\eta(s)\eta(x)\,ds\,dx}{s-x} \quad . \qquad [6.b.13]$$

It is permissible to interchange the order of integration on the right-hand side of [6.b.13] and also interchange the variables x and s and when we do so we obtain

$$I_1 = -\frac{2U}{\pi}\int_0^a\int_0^a \sqrt{\frac{a-s}{s}}\sqrt{\frac{x}{a-x}}\;\frac{\eta(s)\eta(x)}{s-x}\,ds\,dx \quad . \qquad [6.b.14]$$

We take the mean of the two preceding equations [6.b.14] and [6.b.13] and then we obtain

$$I_1=\frac{U}{\pi}\int_0^a\int_0^a \frac{\left[\sqrt{\frac{a-x}{x}}\sqrt{\frac{s}{a-s}} - \sqrt{\frac{a-s}{s}}\sqrt{\frac{x}{a-x}}\right]\eta(s)\eta(x)\;ds\;dx}{s-x}$$

$$= \frac{Ua}{\pi}\int_0^a\int_0^a \frac{\eta(s)\eta(x)\,ds\;dx}{\sqrt{x(a-x)}\,.\sqrt{s(a-s)}}$$

$$= \frac{Ua}{\pi}\left(\int_0^a \frac{\eta(x)\,dx}{\sqrt{x(a-x)}}\right)^2 \quad . \qquad [6.b.15]$$

Substituting from [6.b.15] in [6.b.11] we obtain

$$\delta^2 I=\int_0^a \left\{ \frac{\lambda_1}{[1+z^2(x)]^{3/2}}\,\eta^2(x) + 2\lambda_2\eta'^2(x) + \frac{2}{\pi}\left(\int_0^a \frac{\eta(s)\,ds}{\sqrt{s(a-s)}}\right)^2 \right\} dx \quad . \qquad [6.b.16]$$

In the case of small slope $z(x)$, we approximate [6.b.16] as follows:

$$\delta^2 I = \int_0^a \left\{ \lambda_1 \eta^2(x) + 2\lambda_2 \eta'^2(x) + \frac{2}{\pi} \left(\int_0^a \frac{\eta(s)\,ds}{\sqrt{s(a-s)}} \right)^2 \right\} dx \,. \qquad [6.b.17]$$

We consider a special choice of $\eta(x)$ which satisfies the boundary conditions [6.b.5], namely:

$$\eta(x) = \alpha \sin \frac{\pi}{a} x \quad , \qquad [6.b.18]$$

hence

$$\delta^2 I = \tfrac{1}{2} \alpha^2 a \left\{ \lambda_1 + \frac{2\pi^2}{a^2} \lambda_2 + \frac{4}{\pi} \left(\int_0^a \frac{\sin \frac{\pi}{a} s\,ds}{\sqrt{s(a-s)}} \right)^2 \right\} . \qquad [6.b.19]$$

Thus in the case [6.b.2] the sufficient condition for satisfying [6.b.2] is as follows:

$$\lambda_1 + \frac{2\pi^2}{a^2} \lambda_2 + \frac{4}{\pi} \left(\int_0^a \frac{\sin \frac{\pi}{a} s\,ds}{\sqrt{s(a-s)}} \right)^2 > 0 \,. \qquad [6.b.20]$$

CONCLUSION

We have studied the problems of $\begin{pmatrix}\text{maximum lift } L\\ \text{minimum drag } D\end{pmatrix}$ for a hydrofoil of given length in cavitating and noncavitating flow and subject to an integral constraint upon the curvature.

PART I.

In the new class of variational problems, in which the unknown functions are related by a singular integral equation (depending on the unknown shape and on an unknown singularity distribution) the shape is then to be determined so that some given performance has to be $\begin{pmatrix}\text{maximized}\\ \text{minimized}\end{pmatrix}$. In this part we generalize the Wu & Whitney theory (70) in two different ways in the state equation.

We assume that the problem is stated as follows:

Find the function $u(x)$ (assumed Hölder continuous) defined in $a<x<b$ when

$$\left.\begin{aligned}&\text{Case I}\\ &v(x)=E(x)+\nu u(x)+\frac{1}{\pi}\oint_a^b\frac{u(t)dt}{t-x} \qquad (a)\quad,\\ &\text{Case II}\\ &v(x)=\oint_a^b\left\{\frac{1}{t-x}+\alpha(t,x)\right\}u(t)dt \qquad (b)\quad,\end{aligned}\right\}\ (a<x<b) \qquad [7.a.1]$$

so that the functional

$$J=\int_a^b f_o[u(x),v(x),x]dx \quad, \qquad [7.a.2]$$

is minimized subject to the constraints

$$J_s=\int_a^b f_s[u(x),v(x),x]dx=C_s \quad, \quad s=1,2,\ldots,r, \qquad [7.a.3]$$
$$= \text{const.}$$

being satisfied.

Using a variational method, the necessary conditions for minimization have been derived as follows:

Case I

$$f_u[u(x),v(x),x]=-\nu f_v[u(x),v(x),x]+\frac{1}{\pi}\oint_a^b\frac{f_v[u(t),v(t),t]dt}{t-x} \quad \text{(a)},$$

and

$$f_{uu}[u(x),v(x),x]+2\nu f_{uv}[u(x),v(x),x]+(1+\nu^2)f_{vv}[u(x),v(x),x]>0 \quad \text{(b)},$$

Case II $\qquad$ [7.a.4]

$$f_u[u(x),v(x),x]=\oint_a^b f_v[u(t),v(t),t]\left[\frac{1}{t-x}-\alpha(x,t)\right]dt \quad \text{(a)},$$

and

$$f_{uu}[u(x),v(x),x]+\pi^2 f_{vv}[u(x),v(x),x]>0 \quad \text{(b)}$$

The first condition [7.a.4a] is a singular integral, where

$$f[u(x),v(x),x]=f_o[u(x),v(x),x]-\sum_{s=1}^{r}\lambda_s[f_s(u(x),v(x)-C_s] , \qquad [7.a.5]$$

and $\lambda_1,\lambda_2,\ldots,\lambda_r$ are undetermined Lagrange multipliers.

For the extremal solution the equation [7.a.4a] is to be solved together with equation [7.a.1], as a pair of singular integral equations for $u(x)$ and $v(x)$.

The extremal solutions, $u(x;\lambda_2,\lambda_2,\ldots,\lambda_r$, and $v(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ when determined in this manner will involve r Lagrange multipliers $\lambda_1,\lambda_2,\ldots,\lambda_r$, which can be determined, by substituting the extremal solutions $u(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ and $v(x;\lambda_1,\lambda_2,\ldots,\lambda_r)$ in the constraints [7.a.3].

A necessary condition for the extremum to be a minimum is derived from consideration of the second variation.

It may be remarked here that the solution of a maximum can be deduced from this minimum one by changing the sign of the function in [7.a.3].

Equation [7.a.4a] is generally nonlinear in $u(x)$ and $v(x)$ unless $f[u(x),v(x),x]$ is a polynomial of second degree in $u(x)$ and $v(x)$.

The variational calculus techniques is applied to a function $f[u(x),v(x),x]$ of quadratic form in Case I, [7.a.4].

In this case the integral [7.a.4a] is linear in $u(x)$ and $v(x)$.

It is instructive to investigate this case first, since the system of singular integral equations [7.a.1a] and [7.a.4a] can then be reduced to a single Fredholm integral equation of the second kind, or, in certain special cases, the method of singular integral equations can be employed to obtain an analytical solution in a closed form.

PART II

In this part we evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces $\begin{Bmatrix}\text{maximum lift}\\ \text{minimum drag}\end{Bmatrix}$;

the problem is discussed in three cases when there is a $\begin{Bmatrix}\text{full}\\ \text{partial}\\ \text{zero}\end{Bmatrix}$ cavity flow past a thin hydrofoil.

The liquid flow past a thin hydrofoil of unknown shape is assumed to be two-dimensional irrotational, steady, incompressible and the liquid extends to infinity. A linearized theory is assumed and two-dimensional vortex and source distributions are used to simulate the two-dimensional $\begin{Bmatrix}\text{full}\\ \text{partial}\\ \text{zero}\end{Bmatrix}$ flow past the hydrofoil. This method leads to a system of integral equations and these are solved exactly using the Carleman-Muskhelishvili technique. This is similar to that used by Davies, T.V. (13), (14).

The system of the integral equations can be reduced to a single singular integral equation involving the source strength and hydrofoil slope in Chapt. III and vertex strength and hydrofoil slope in Chapt. II & IV.

In these problems we use variational calculus techniques to obtain the optimum shape of the hydrofoil in order to $\begin{Bmatrix}\text{maximize}\\ \text{minimize}\end{Bmatrix}$ the $\begin{Bmatrix}\text{lift}\\ \text{drag}\end{Bmatrix}$ coefficient subject to constraints on curvature and given length.

The mathematical problem is that of extremizing a functional depending on γ(vortex strength), μ (source strength) and z (the hydrofoil slope) when these three functions are related by singular integral equations. The analytical solution for the unknown shape z and the unknown singularity $\gamma(x)$ and $\mu(x)$ has a branch type singularity at the two ends of the hydrofoil.

Analytical solutions by a singular integral equation method and the Rayleigh-Ritz method are discussed.

A sufficient condition for the extremum to be a minimum is derived from consideration of the second variation and leads to

$$\lambda_1 > 0 \quad , \quad \lambda_2 > 0 \quad . \qquad [7.a.6]$$

The optimum shape using the classical Euler method is discussed in Chapter IV. We assume that the length of the cavity is kept constant in the optimization process and we recognize that this is a limitation in the theory in Part I.

PART III

The usual assumptions in problems of the obstacle beneath a free surface are taken as a basis: namely, the liquid is non-viscous and moving two-dimensionally, steadily and without vorticity, the only force acting on it is gravity. With these assumptions together with a linearization assumption we determine the forces, due to the hydrofoil beneath a free surface of the liquid. We use variational calculus techniques similar to those used in Part II to obtain the optimum shape so that the drag is minimum.

Analytical solutions by a singular integral equation method, Duhamel's method and some approximate methods are discussed for the linearized theory. In this part some general expressions are established for the forces acting on a submerged vortex and source element beneath a free surface using Blasius' theorem.

VII PRINCIPAL NOTATIONS

The following list is intended to show the most frequent conventional meaning with which certain symbols are used in this thesis.

The list is not exhaustive nor does it preclude some symbols being used in other senses (which are always defined).

$m(x)$	Strength sources along the x-axis
$\gamma(x)$	Strength vortices along the x-axis
W	Complex potential $\equiv \phi+i\psi$
ϕ	Velocity potential
ψ	Stream function
L	Lift
D	Drag
L^*	Lift coefficient $\equiv L/\rho U^2$
D^*	Drag coefficient $= D/\rho U^2$
r	Distance
x,y	Cartesian coordinates in physical plane with free stream in position x direction and leading edge at (o,o)
z	Complex variable $\equiv z \equiv x+iy$
ξ,η	Coordinates in ζ plane, admissible variations
U	Uniform stream at infinity, parallel to x-axis
u,v	Components of fluid velocity along x,y axes respectively, admissible variations.
z	$\equiv y'(x)$ hydrofoil slope at position x
P	Pressure
P_∞	Static pressure of stream at infinity
$P_c(<P_\infty)$	Cavity pressure
σ	$\equiv \dfrac{P_\infty-P_c}{\frac{1}{2}\rho U^2}$ cavitation number.
A,B,C,D,α,β,etc	Constants

λ_1, λ_2	Lagrange multipliers
θ, ϕ, α	Angles in general
β	Slope at the end point
ε	Small positive constant
ρ	Constant fluid density
ν	$\equiv g/U^2$, parameters, $\nu(\theta)=\gamma_1(c\sin^2\theta)$ is vortex strength
π	Ratio of circumference of a circle to its number
g	Gravitational force per unit mass
i	Stands for the imaginary number $\equiv \sqrt{-1}$
h	Depth
μ	Strength sources
ℓ, s	Hydrofoil length $\equiv \int \sqrt{1+y'^2(x)}\,dx$
K	Curvature constraint, kernel
λ	Parameter
q	Resultant velocity
m, n	Constants
$E_i(x)$	Exponential integral $\left(E_i(x) \equiv \int_{-\infty}^{x} \frac{e^s ds}{s}\right)$

VII APPENDIX I

As seen from [3.b.42] the integral I_2 has been calculated as follows:

$$I_2=\int_0^{\alpha}\frac{\sin\psi d\psi}{\sqrt{\sin\psi\sin(\alpha-\psi)}.(\tan\psi+\tan\theta)} \qquad (o<\theta<\alpha\frac{\pi}{2})$$

$$=\frac{1}{\sqrt{\cos\alpha}}\int_0^{\alpha}\frac{\tan\psi d\psi}{\sqrt{\tan\psi(\tan\alpha-\tan\psi)}.(\tan\psi+\tan\theta)} \quad . \qquad [7.1.1]$$

We make the transformation

$$\tan\psi=t^2\tan\alpha \quad , \qquad [7.1.2]$$

then we obtain

$$I_2=\frac{2\tan\alpha}{\sqrt{\cos\alpha}}\int_0^1\frac{t^2dt}{\sqrt{1-t^2}(t^2\tan\alpha+\tan\theta)(1+t^4\tan^2\alpha)}$$

$$=\frac{2\cos^2\theta}{\sqrt{\cos\alpha}}\{\ -\tan\theta J_1+J_2\ \} \quad , \qquad [7.1.3]$$

where

$$J_1=\int_0^1\frac{dt}{\sqrt{1-t^2}(t^2\tan\alpha+\tan\theta)} \quad , \qquad [7.1.4]$$

$$J_2=\int_0^1\frac{(t^2\tan\theta\tan\alpha+1)dt}{\sqrt{1-t^2}.(1+t^4\tan^2\alpha)} \quad . \qquad [7.1.5]$$

First, we calculate integral [7.1.4] using the transformation

$$t=\sin x \quad , \qquad [7.1.6]$$

we obtain

$$J_1=\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^2x\tan\alpha+\tan\theta}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2xdx}{(\tan\alpha+\tan\theta)\tan^2x+\tan\theta} \quad . \qquad [7.1.7]$$

If we take

$$y=\tan x \quad , \qquad [7.1.8]$$

then we can write

$$J_1 = \frac{1}{\tan\alpha+\tan\theta}\int_0^\infty \frac{dy}{y^2+y_1^2} \qquad \left[y_1^2 = \frac{\tan\theta}{\tan\alpha+\tan\theta}\right]$$

$$= \frac{\pi}{2\sqrt{\tan\theta(\tan\alpha+\tan\theta)}} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \quad . \qquad [7.1.9]$$

Now we calculate the second integral [7.1.5], using [7.1.6] we obtain

$$J_2=\int_0^{\frac{\pi}{2}} \frac{[\sin^2x\tan\theta\tan\alpha+1]dx}{(1+\sin^4x\tan^2\alpha)}$$

$$=\int_0^{\frac{\pi}{2}} \frac{[\tan^2x\tan\theta\tan\alpha+\sec^2x]\sec^2xdx}{[\sec^4x+\tan^4x\tan^2\alpha]} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \ . \qquad [7.1.10]$$

Using [7.1.8] we obtain

$$J_2=\int_0^\infty \frac{[(\tan\theta\tan\alpha+1)y^2+1]dy}{[\sec^2\alpha y^4+2y^2+1]}$$

$$= \cos^2\alpha\int_0^\infty \frac{[(\tan\theta\tan\alpha+1)y^2+1]dy}{y^4+2\cos^2\alpha y^2+\cos^2\alpha} \quad . \qquad [7.1.11]$$

If we take

$$y=\sqrt{\cos\alpha}.\tau \quad , \qquad [7.1.12]$$

then we obtain

$$J_2=\sqrt{\cos\alpha}\int_0^\infty \frac{[(\tan\theta\tan\alpha+1)\cos\alpha\tau^2+1]d\tau}{\tau^4+2\cos\alpha\tau^2+1} \quad . \qquad [7.1.13]$$

Setting

$$\sin\tfrac{1}{2}\alpha=a \quad , \qquad [7.1.14]$$

hence

$$J_2=\sqrt{\cos\alpha}\int_0^\infty \frac{[(\tan\theta\tan\alpha+1)\cos\alpha\tau^2+1]d\tau}{(\tau^2-2a\tau+1)(\tau^2+2a\tau+1)}$$

$$= \frac{\sqrt{\cos\alpha}}{2}\,[J_3-J_4] \ , \qquad [7.1.15]$$

where

$$J_3=\int_0^\infty \frac{[(\frac{1}{2a}\ \tan\theta\tan\alpha\cos\alpha-a)\tau+1]d\tau}{\tau^2-2a\tau+1} \quad , \qquad [7.1.16]$$

$$J_4=\int_0^{\infty}\frac{[(\frac{1}{2a}\tan\theta\tan\alpha\cos\alpha-a)\tau-1]d\tau}{\tau^2+2a\tau+1} \quad . \qquad [7.1.17]$$

Now we calculate the first integral, [7.1.16], by writing it in the form

$$J_3=\int_0^{\infty}\frac{[(\frac{1}{2a}\tan\theta\tan\alpha\cos\alpha-a)\tau+1]d\tau}{(\tau-a)^2+(1-a^2)} \quad . \qquad [7.1.18]$$

If we take

$$\tau-a=\sqrt{1-a^2}\tan\phi \quad , \qquad [7.1.19]$$

then we obtain

$$J_3=\frac{1}{\sqrt{1-a^2}}\int_{\phi_1}^{\phi_2}\{[\frac{1}{2a}\tan\theta\sin\alpha-a][a+\sqrt{1-a^2}\tan\phi]+1\}d\phi$$

$$=\frac{1}{\sqrt{1-a^2}}\{[\frac{1}{2a}\tan\theta\sin\alpha-a][a\phi-\sqrt{1-a^2}\ln(\cos\phi)]+\phi\}\Big|_{\phi_1}^{\phi_2} \quad . \qquad [7.1.20]$$

The second integral, [7.1.17], can be calculated by taking

$$\tau+a=\sqrt{1-a^2}\tan\psi \quad , \qquad [7.1.21]$$

then we can write

$$J_4=\frac{1}{\sqrt{1-a^2}}\{[\frac{1}{2a}\tan\theta\sin\alpha-a][-a\psi-\sqrt{1-a^2}\ln(\cos\psi)]-\psi\}\Big|_{\psi_1}^{\psi_2} \quad . \qquad [7.1.22]$$

Using [7.1.20] and [7.1.22] we can write the value of integral [7.1.15] in the form

$$J_2=\frac{\sqrt{\cos\alpha}}{2\sqrt{1-a^2}}\lim_{\tau\to\infty}\left\{(\frac{1}{2a}\tan\theta\sin\alpha-a)\left(-\sqrt{1-a^2}\ln\frac{\sqrt{(1-a^2)+(\tau+a)^2}}{\sqrt{(1-a^2)+(\tau-a)^2}}+\right.\right.$$

$$\left.\left.+a\tan^{-1}\frac{\tau-a}{\sqrt{1-a^2}}+a\tan^{-1}\frac{\tau+a}{\sqrt{1-a^2}}\right)+\tan^{-1}\frac{\tau-a}{\sqrt{1-a^2}}+\tan^{-1}\frac{\tau+a}{\sqrt{1-a^2}}\right\}$$

$$=\frac{\pi\sqrt{\cos\alpha}\cos(\frac{1}{2}\alpha-\theta)}{2\cos\theta} \quad . \qquad [7.1.23]$$

Substituting from [7.1.9] and [7.1.23] in [7.1.3] we can write the value of integral [7.1.1] in the form

$$I_2=\pi\cos\theta\left[-\cos\theta\sqrt{\frac{\sin\theta}{\sin(\alpha+\theta)}}+\cos(\tfrac{1}{2}\alpha-\theta)\right] \quad (o<\theta<\alpha<\frac{\pi}{2}) \quad , \qquad [7.1.24]$$

VII APPENDIX II

As seen from [4.b.38] the integral equation

$$\int_o^\alpha \frac{\phi(\theta)\cos(\theta-\theta_o)\,d\theta}{\sin(\theta-\theta_o)} = f(\theta_o) \qquad (o<\theta_o<\alpha) \ , \qquad [7.2.1]$$

can be transformed into standard Cauchy formula as follows

If we use the transformation

$$\left.\begin{array}{ll} t=\tan\theta & , \\ t_o=\tan\theta_o & , \\ b=\tan\alpha & , \end{array}\right\} \qquad [7.2.2]$$

then we can write

$$\frac{\cos(\theta-\theta_o)}{\sin(\theta-\theta_o)} = \frac{1+\tan\theta\tan\theta_o}{\tan\theta - \tan\theta_o} = \frac{1+t\,t_o}{t-t_o} \ , \qquad [7.2.3]$$

hence

$$\frac{\cos(\theta-\theta_o)}{\sin(\theta-\theta_o)} = \frac{1+(t-t_o)t_o+t_o^2}{t-t_o} \ . \qquad [7.2.4]$$

And consequently we can write [7.2.1] in this form

$$\int_o^b \frac{\phi(\tan^{-1}t)\,dt}{t^2+1}\left[\frac{t_o^2+1}{t-t_o} + t_o\right] = f(\tan^{-1}t_o) \ , \qquad [7.2.5]$$

taking

$$\Phi(t) = \frac{\phi(\tan^{-1}t)}{t^2+1} \ , \qquad [7.2.6]$$

we obtain

$$\int_o^b \frac{\Phi(t)\,dt}{t-t_o} = \frac{f(\tan^{-1}t_o)}{t_o^2+1} - \frac{At_o}{1+t_o^2} \ , \qquad [7.2.7]$$

with

$$A=\int_o^b \frac{\phi(\tan^{-1}t)}{t^2+1}\,dt \ . \qquad [7.2.8]$$

It may be verified that the integral equation [7.2. 7] can be written in the form

$$\int_{0}^{b} \frac{\Phi(t)dt}{t-t_o} = F(t_o) \quad , \qquad [7.2.9]$$

where

$$F(t_o) = \frac{f(\tan^{-1} t_o)}{t_o^2+1} - \frac{At_o}{t_o^2+1} \quad . \qquad [7.2.10]$$

VII APPENDIX III

As seen from [4.b.45] the integrals

$$I_8=\int_0^{\alpha}\frac{\tan\theta_0\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)}\qquad(0<\theta<\alpha<\tfrac{\pi}{2})\quad, \tag{7.3.1}$$

$$I_9=\int_0^{\alpha}\frac{\sin\theta_0\cos\theta_0\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)[\sin^2\phi-\sin^2\theta_0]}\,(0<\theta<\alpha<\phi<\tfrac{\pi}{2})\quad, \tag{7.3.2}$$

are calculated as follows:

The integral equation [7.3.1] can be written in the form

$$I_8=\frac{\sqrt{\cos\alpha}}{\cos\theta}\int_0^{\alpha}\frac{\tan\theta_0\sqrt{\tan\theta_0(\tan\alpha-\tan\theta_0)}\,d\theta_0}{\tan\theta_0-\tan\theta}\quad. \tag{7.3.3}$$

We make the transformation

$$\tan\theta_0=t^2\tan\alpha\quad, \tag{7.3.4}$$

we obtain

$$I_8=\frac{2\sqrt{\cos\alpha}\tan^3\alpha}{\cos\theta}\int_0^1\frac{t^4\sqrt{1-t^2}\,dt}{(t^2\tan\alpha-\tan\theta)(1+t^4\tan^2\alpha)}$$

$$=\frac{2\sqrt{\cos\alpha}\tan\alpha}{\cos\theta}\quad[\sin^2\theta J_1+J_2]\quad, \tag{7.3.5}$$

where

$$J_1=\int_0^1\frac{\sqrt{1-t^2}\,dt}{t^2\tan\alpha-\tan\theta} \tag{7.3.6}$$

$$J_2=\int_0^1\frac{\sqrt{1-t^2}[\cos^2\theta\tan\alpha t^2+\sin\theta\cos\theta]\,dt}{1+t^4\tan^2\alpha}\quad. \tag{7.3.7}$$

Now we calculate the first integral, [7.3.6], by taking

$$t=\sin x\quad, \tag{7.3.8}$$

we obtain

$$J_1 = \frac{1}{\tan\alpha-\tan\theta}\int_0^{\frac{\pi}{2}} \frac{dx}{\tan^2 x - \left(\frac{\tan\theta}{\tan\alpha-\tan\theta}\right)}$$

$$= \frac{1}{\tan\alpha-\tan\theta}\int_0^{\infty} \frac{dy}{(y^2-y_1^2)(y^2+1)} \quad [y=\tan x, y_1^2 = \frac{\tan\theta}{\tan\alpha-\tan\theta}]. \quad [7.3.9]$$

This can be written in the form

$$J_1=\frac{1}{\tan\alpha}\ [J_3-J_4] \quad , \qquad [7.3.10]$$

where

$$\left.\begin{aligned} J_3&=\int_0^{\infty}\frac{dy}{y^2-y_1^2} = 0 \quad , \\ J_4&=\int_0^{\infty}\frac{dy}{y^2+1} = \frac{\pi}{2} \quad . \end{aligned}\right\} \qquad [7.3.11]$$

Substituting from [7.3.11] in [7.3.10] we obtain

$$J_1 = -\frac{\pi}{2}\cot\alpha \quad . \qquad [7.3.12]$$

Now we evaluate integral [7.3.7], using [7.3.8] we obtain

$$J_2=\int_0^{\frac{\pi}{2}} \frac{[\cos^2\theta\tan\alpha+\sin\theta\cos\theta]\tan^2 x+\sin\theta\cos\theta}{\sec^2\alpha\tan^4 x+2\tan^2 x+1}\,dx \quad . \qquad [7.3.13]$$

Hence

$$J_2=\int_0^{\infty} \frac{[\cos^2\theta\tan\alpha+\sin\theta\cos\theta]y^2+\sin\theta\cos\theta}{[\sec^2\alpha y^4+2y^2+1][y^2+1]}\,dy \quad [y=\tan x]$$

$$= [-\cos^2\theta J_5+J_6]\cot\alpha \quad , \qquad [7.3.14]$$

where

$$J_5=\int_0^{\infty}\frac{dy}{y^2+1} = \frac{\pi}{2} \qquad [7.3.15]$$

and

$$J_6=\int_0^{\infty} \frac{[\cos^2\theta\sec^2\alpha y^2+(\cos^2\theta+\sin\theta\cos\theta\tan\alpha)]dy}{\sec^2\alpha y^4+2y^2+1}$$

$$= \cos^2\alpha\int_0^{\infty} \frac{[\cos^2\theta\sec^2\alpha y^2+\cos^2\theta+\sin\theta\cos\theta\tan\alpha]dy}{y^4+2\cos^2\alpha y^2+\cos^2\alpha} \quad . \qquad [7.3.16]$$

For convenience of computation, [7.3.16] if we change variables by taking

$$y = \sqrt{\cos\alpha}.\tau \quad , \qquad [7.3.17]$$

then we obtain

$$J_6=\sqrt{\cos\alpha}\int_0^\infty \frac{[\cos^2\theta\sec\alpha\tau^2+\cos^2\theta+\sin\theta\cos\theta\tan\alpha]d\tau}{\tau^4+2\cos\alpha\tau^2+1}$$

$$=\sqrt{\cos\alpha}\int_0^\infty \frac{[\cos^2\theta\sec\alpha\tau^2+\cos^2\theta+\sin\theta\cos\theta\tan\alpha]d\tau}{(\tau^2-2a\tau+1)(\tau^2+2a\tau+1)} \quad [a=\sin\tfrac{1}{2}\alpha] \quad [7.3.18]$$

Hence

$$J_6=\sqrt{\cos\alpha}\int_0^\infty \left[\frac{K(\theta,\alpha)\tau+\lambda(\theta,\alpha)}{\tau^2-2a\tau+1} - \frac{K(\theta,\alpha)\tau-\lambda(\theta,\alpha)}{\tau^2+2a\tau+1}\right] d\tau$$

$$=\sqrt{\cos\alpha}\ [J_7-J_8] \quad , \quad [7.3.19]$$

where

$$\left.\begin{aligned} K(\theta,\alpha) &= \frac{1}{4\sin\frac{1}{2}\alpha}\,[\cos^2\theta\sec\alpha-\cos^2\theta-\sin\theta\cos\theta\tan\alpha] \ , \\ \lambda(\theta,\alpha) &= \tfrac{1}{2}[\cos^2\theta+\sin\theta\cos\theta\tan\alpha] \quad , \end{aligned}\right\} \quad [7.3.20]$$

and

$$J_7=\int_0^\infty \frac{[K(\theta,\alpha)\tau+\lambda(\theta,\alpha)]d\tau}{(\tau-a)^2+(1-a^2)} \quad , \quad [7.3.21]$$

$$J_8=\int_0^\infty \frac{[K(\theta,\alpha)\tau-\lambda(\theta,\alpha)]d\tau}{(\tau+a)^2+(1-a^2)} \quad , \quad [7.3.22]$$

taking

$$\tau-a=\sqrt{1-a^2}\tan\phi \quad , \quad [7.3.23]$$

we obtain

$$J_7=\frac{1}{\sqrt{1-a^2}}\int_{\phi_1}^{\phi_2}\{\sqrt{1-a^2}K(\theta,\alpha)\tan\phi+[aK(\theta,\alpha)+\lambda(\theta,\alpha)]\}d\phi$$

$$=\frac{1}{\sqrt{1-a^2}}\{-\sqrt{1-a^2}K(\theta,\alpha)\ln(\cos\phi)+[aK(\theta,\alpha)+\lambda(\theta,\alpha)]\phi\}_{\phi_1}^{\phi_2} \quad . \quad [7.3.24]$$

Taking

$$\tau+a=\sqrt{1-a^2}\tan\psi \quad , \quad [7.3.25]$$

we can write [7.3.22] in the form

$$J_8=\frac{1}{\sqrt{1-a^2}}\{-\sqrt{1-a^2}K(\theta,\alpha)\ln(\cos\phi)-[\lambda(\theta,\alpha)+aK(\theta,\alpha)]\psi\}_{\psi_1}^{\psi_2} \quad . \quad [7.3.26]$$

Using [7.3.24] and [7.3.26] we can write [7.3.19] in the form

$$J_6=\frac{\sqrt{\cos\alpha}}{\sqrt{1-a^2}}\lim_{\tau\to\infty}\left\{-\sqrt{1-a^2}.K(\theta,\alpha)\ln\frac{\sqrt{(\tau-a)^2+(1-a^2)}}{\sqrt{(\tau+a)^2+(1-a^2)}}+[aK(\theta,\alpha)+\lambda(\theta,\alpha)]\left[\tan^{-1}\frac{\tau-a}{\sqrt{1-a^2}}+\right.\right.$$

$$\left.\left.+\tan^{-1}\frac{\tau+a}{\sqrt{1-a^2}}\right]\right\}=\frac{\pi\sqrt{\cos\alpha}}{4\sqrt{1-a^2}}\;[\cos^2\theta\sec\alpha+\cos^2\theta+\sin\theta\cos\theta\tan\alpha]\qquad(a=\sin\tfrac{1}{2}\alpha]\;.\qquad[7.3.27]$$

Using [7.3.27] and [7.3.15] we can write the value of the integral [7.3.14] in the form

$$J_2=-\frac{\pi}{2}\cos^2\theta\cot\alpha+\frac{\pi\sin\theta\cos(\theta-\frac{1}{2}\alpha)\sqrt{\cos\alpha}}{2\sin\alpha}\;.\qquad[7.3.28]$$

Substituting from [7.3.28] and [7.3.12] in [7.3.5] we obtain

$$I_8=-\frac{\pi\sqrt{\cos\alpha}}{\cos\theta}+\pi\cos(\theta-\tfrac{1}{2}\alpha)\qquad(o<\theta<\alpha\leq\tfrac{\pi}{2})\;.\qquad[7.3.29]$$

Now we calculate integral [7.3.2], by writing it in the form

$$I_9=\frac{\sqrt{\cos\alpha}}{\cos\theta\cos^2\phi}\int_0^\alpha\frac{\tan\theta_o\sqrt{\tan\theta_o(\tan\alpha-\tan\theta_o)}\,d\theta_o}{(\tan\theta_o-\tan\theta)[\tan^2\phi-\tan^2\theta_o]}\;,\qquad(o<\theta_o<\alpha<\phi<\tfrac{\pi}{2})\;.\qquad[7.3.30]$$

Using [7.3.4] we obtain

$$I_9=\frac{2\sqrt{\cos\alpha}\tan^3\alpha}{\cos\theta\cos^2\phi}\int_0^1\frac{t^4\sqrt{1-t^2}\,dt}{(t^2\tan\alpha-\tan\theta)(\tan^2\phi-t^4\tan^2\alpha)(1+t^4\tan^2\alpha)}$$

$$=\frac{2\sqrt{\cos\alpha}\tan\alpha}{\cos\theta}\;[J_9-J_{10}]\;,\qquad[7.3.31]$$

where

$$J_9=\tan^2\phi\int_0^1\frac{\sqrt{1-t^2}\,dt}{(t^2\tan\alpha-\tan\theta)(\tan^2\phi-t^4\tan^2\alpha)}\quad(o<\theta<\alpha<\phi<\tfrac{\pi}{2})\;,\qquad[7.3.32]$$

$$J_{10}=\int_0^1\frac{\sqrt{1-t^2}\,dt}{(t^2\tan\alpha-\tan\theta)(1+t^4\tan^2\alpha)}\qquad(o<\theta<\alpha<\tfrac{\pi}{2})\;.\qquad[7.3.33]$$

First we calculate integral [7.3.32], using partial fractions we obtain

$$J_9=\frac{\tan^2\phi}{\tan^2\phi-\tan^2\theta}\;[J_{11}+J_{12}]\;,\qquad[7.3.34]$$

where

$$J_{11}=\int_0^1\frac{\sqrt{1-t^2}\,dt}{t^2\tan\alpha-\tan\theta}=J_1=-\frac{\pi}{2}\cot\alpha\qquad[7.3.35]$$

$$J_{12}=\int_0^1\frac{\sqrt{1-t^2}\,[t^2\tan\alpha+\tan\theta]\,dt}{\tan^2\phi-t^4\tan^2\alpha}$$

$$=\frac{1}{2\tan\phi}\;[(\tan\phi+\tan\theta)J_{13}-(\tan\phi-\tan\theta)J_{14}]\;,\qquad[7.3.36]$$

with

$$J_{13}=\int_0^1 \frac{\sqrt{1-t^2}dt}{\tan\phi - t^2\tan\alpha} \qquad (o<\alpha<\phi<\frac{\pi}{2})$$

$$= -\frac{\pi}{2\tan\alpha}\left[\sqrt{\frac{\tan\phi-\tan\alpha}{\tan\phi}} - 1\right] \quad , \qquad [7.3.37]$$

and

$$J_{14}=\int_0^1 \frac{\sqrt{1-t^2}dt}{\tan\phi + t^2\tan\alpha}$$

$$= \frac{\pi}{2\tan\alpha}\left[\sqrt{\frac{\tan\phi+\tan\alpha}{\tan\phi}} - 1\right] \quad . \qquad [7.3.38]$$

Hence

$$J_{12}= -\frac{\pi}{4\tan\phi\tan\alpha}\{\sqrt{\tan\phi}[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] + \frac{\tan\theta}{\tan\phi}[\sqrt{\tan\phi-\tan\alpha} -$$

$$-\sqrt{\tan\phi+\tan\alpha}] - 2\tan\phi\} \quad , \qquad (o<\theta<\alpha<\phi<\frac{\pi}{2}) \ . \qquad [7.3.39]$$

Substituting from [7.3.39] and [7.3.35] in [7.3.34] we obtain

$$J_9 = -\frac{\pi\tan\phi}{4\tan\alpha[\tan^2\phi-\tan^2\theta]}\{\sqrt{\tan\phi}[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] + \frac{\tan\theta}{\sqrt{\tan\phi}}[\sqrt{\tan\phi-\tan\alpha} -$$

$$\sqrt{\tan\phi+\tan\alpha}]\} \qquad (o<\theta<\alpha<\phi<\frac{\pi}{2}) \ . \qquad [7.3.40]$$

For convenience of computation, [7.3.33], we can write it in the form

$$J_{10}=\cos^2\theta J_{15}-J_{16} \quad , \qquad [7.3.41]$$

where

$$J_{15}=\int_0^1 \frac{\sqrt{1-t^2}dt}{t^2\tan\alpha-\tan\theta} = J_1 = -\frac{\pi}{2}\cot\alpha \quad ; \qquad [7.3.42]$$

and

$$J_{16}=\int_0^1 \frac{\sqrt{1-t^2}(\cos^2\theta\tan\alpha t^2+\sin\theta\cos\theta)dt}{1+t^4\tan^2\alpha} = J_2$$

$$= -\frac{\pi}{2}\cos^2\theta\cot\alpha + \frac{\pi\sqrt{\cos\alpha}}{2\sin\alpha}\cdot\cos(\theta-\tfrac{1}{2}\alpha)\cos\theta \ . \qquad [7.3.43]$$

Substituting from [7.3.42] and [7.3.43] in [7.3.41] we obtain

$$J_{10} = -\frac{\pi\sqrt{\cos\alpha}\cos\theta}{2\sin\alpha}\cdot\cos(\theta-\tfrac{1}{2}\alpha) \quad . \qquad [7.3.44]$$

Using [7.3.40] and [7.3.44] we can write the value of integral [7.3.31] in the form

$$I_9 = -\frac{\pi\sqrt{\tan\phi}\sqrt{\cos\alpha}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}\{\tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})+\tan\theta(\sqrt{\tan\phi-\tan\alpha}-$$

$$-\sqrt{\tan\phi+\tan\alpha})\} + \pi\cos(\theta-\tfrac{1}{2}\alpha), \quad (0<\theta<\alpha<\phi<\tfrac{\pi}{2}) \; .$$

[7.3.45]

VII APPENDIX IV

As seen from [4.b.47] the constant A can be calculated as follows

$$A=2\sqrt{c}\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}\,.\sin\theta.\nu(\theta).d\theta \quad , \qquad [7.4.1]$$

where

$$\nu(\theta) = \frac{D\sqrt{\cos\alpha}\sec\theta\operatorname{cosec}\theta}{2\sqrt{c}\sqrt{\sin^2\alpha-\sin^2\theta}.\sqrt{\sin\theta\sin(\alpha-\theta)}} -$$

$$-\frac{1}{2\pi^2\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta}.\sqrt{\sin\theta\sin(\alpha-\theta)}}\left\{\tfrac{1}{2}\pi U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)} +\right.$$

$$+4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi).\sin\phi d\phi\left[\left[\frac{-\pi\sqrt{\tan\phi}\sqrt{\cos\alpha}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +\right.\right.$$

$$\left.\left.\left.+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\pi\cos(\theta-\tfrac{1}{2}\alpha)\right]\right] + \frac{A}{\sqrt{c}}\left(\frac{\pi\sqrt{\cos\alpha}}{\cos\theta} - \pi\cos(\theta-\tfrac{1}{2}\alpha)\right)\right\} ,$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \quad , \qquad [7.4.2]$$

where D is an arbitrary constant.

Substituting from [7.4.2] in [7.4.1] we obtain

$$A = -\frac{\sqrt{c}}{\pi}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\left\{\tfrac{1}{2}U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_0-\theta)} +\right.$$

$$+4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi).\sin\phi d\phi\left[\left[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +\right.\right.$$

$$\left.\left.\left.+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\right]\right] + \frac{A}{\sqrt{c}}\left(\frac{\sqrt{\cos\alpha}}{\cos\theta} - \cos(\theta-\tfrac{1}{2}\alpha)\right)\right\} +$$

$$+ D\sqrt{\cos\alpha}\int_0^{\alpha}\frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}} \qquad (o<\theta<\alpha<\phi<\tfrac{\pi}{2}) \quad . \qquad [7.4.3]$$

It is permissible to interchange the order of integral [7.4.3] [see, e.g., Hardy,G.H. (35)] and when we do so we obtain

$$A = \frac{U\sqrt{c\sigma}}{2\pi}\int_0^\alpha (\sin^2\alpha - 2\sin^2\theta_o)\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o\int_0^\alpha \frac{d\theta}{\sin(\theta-\theta_o)\sqrt{\sin\theta\sin(\alpha-\theta)}} -$$

$$- \frac{4\sqrt{c}U}{\pi}\int_\alpha^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\; z(\phi)\sin\phi d\phi\Big\{-$$

$$-\frac{1}{2}\sqrt{\cos\alpha}\sqrt{\tan\phi}\int_0^\alpha \frac{d\theta}{\cos\theta(\tan^2\phi-\tan^2\theta)\sqrt{\sin\theta\sin(\alpha-\theta)}}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +$$

$$+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}]) + \int_0^\alpha \frac{\cos(\theta-\frac{1}{2}\alpha)d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\Big\} - A\int_0^\alpha \frac{[\frac{\sqrt{\cos\alpha}}{\cos\theta} - \cos(\theta-\frac{1}{2}\alpha)]d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}} +$$

$$+ D\sqrt{\cos\alpha}\int_0^\alpha \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}} \qquad (o<\theta<\alpha<\phi<\frac{\pi}{2}) \qquad [7.4.4]$$

In evaluating the inner integrals we use the fact that θ and θ_o lies between o and α , while ϕ lies between α and $\frac{\pi}{2}$, so that

$$I_{10} = \int_0^\alpha \frac{d\theta}{\sin(\theta-\theta_o)\sqrt{\sin\theta\sin(\alpha-\theta)}}$$

$$= -\frac{1}{\sqrt{\cos\alpha}.\cos\theta_o}\int_o^1 \frac{\sec^2\theta d\theta}{(\tan\theta_o-\tan\theta)\sqrt{\tan\theta(\tan\alpha-\tan\theta)}} \qquad [7.4.5]$$

We take the transformation

$$\tan\theta = t^2\tan\alpha \qquad , \qquad [7.4.6]$$

then we obtain

$$I_{10} = -\frac{2}{\sqrt{\cos\alpha}.\cos\theta_o}\int_o^1 \frac{dt}{\sqrt{1-t^2}(\tan\theta_o - t^2\tan\alpha)} \qquad [7.4.7]$$

We make further transformation

$$t = \sin x \qquad [7.4.8]$$

we obtain

$$I_{10} = \frac{2}{\cos\theta_o\sqrt{\cos\alpha}[\tan\alpha-\tan\theta_o]}\int_o^{\frac{\pi}{2}} \frac{\sec^2 x dx}{\tan^2 x - \frac{\tan\theta_o}{\tan\alpha-\tan\theta_o}}$$

$$= \frac{2}{\sqrt{\cos\alpha}\cos\theta_o(\tan\alpha-\tan\theta_o)}\int_o^\infty \frac{dy}{y^2-y_o^2} \quad , \quad \left[y_o^2 = \frac{\tan\theta_o}{\tan\alpha-\tan\theta_o}\right]$$

$$= \; 0 \qquad [7.4.9]$$

Now we calculate the second inner integral I_{11} which is defined by

$$I_{11} = \int_0^\alpha \frac{\cos(\theta-\frac{1}{2}\alpha)\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}} \qquad (o<\theta<\alpha<\tfrac{\pi}{2})$$

$$= \cos\tfrac{1}{2}\alpha\ J_1+\sin\tfrac{1}{2}\alpha J_2 \quad , \tag{7.4.10}$$

where

$$J_1 = \frac{1}{\sqrt{\cos\alpha}}\int_0^\alpha \frac{d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}} \quad , \tag{7.4.11}$$

$$J_2 = \frac{1}{\sqrt{\cos\alpha}}\int_0^\alpha \frac{\tan\theta\,d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}} \quad . \tag{7.4.12}$$

Using [7.4.6] we can write [7.4.11]

$$J_1 = \frac{2}{\sqrt{\cos\alpha}}\int_0^1 \frac{dt}{\sqrt{1-t^2}.(1+t^4\tan^2\alpha)} \tag{7.4.13}$$

Using [7.4.8] we obtain

$$J_1 = \frac{2}{\sqrt{\cos\alpha}}\int_0^{\frac{\pi}{2}} \frac{(1+\tan^2x)\sec^2x dx}{\sec^2\alpha\tan^4x+2\tan^2x+1}$$

$$= \frac{2\cos^2\alpha}{\sqrt{\cos\alpha}}\int_0^\infty \frac{(1+y^2)\,dy}{y^4+2\cos^2\alpha y^2+\cos^2\alpha} \quad , \quad (y=\tan x) \quad , \tag{7.4.14}$$

taking

$$y=\sqrt{\cos\alpha}.\tau \quad , \tag{7.4.15}$$

we obtain

$$J_1 = 2\int_0^\infty \frac{(1+\cos\alpha\tau^2)\,d\tau}{(\tau^2+1)^2 - 4\tau^2\sin^2\frac{\alpha}{2}}$$

$$= 2\int_0^\infty\left[\frac{K\tau+\lambda}{(\tau-a)^2+(1-a^2)} - \frac{K\tau-\lambda}{(\tau+a)^2+(1-a^2)}\right] d\tau, (a=\sin\tfrac{1}{2}\alpha) \quad , \tag{7.4.16}$$

where

$$\left.\begin{aligned} K &= \frac{\cos\alpha-1}{4\sin\frac{1}{2}\alpha} \quad , \\ \lambda &= \tfrac{1}{2} \end{aligned}\right\} \tag{7.4.17}$$

Hence

$$J_1 = \pi \cos\tfrac{1}{2}\alpha \quad . \tag{7.4.18}$$

Using [7.4.6] we can write integral [7.4.12] in the form

$$J_2 = \frac{2\tan\alpha}{\sqrt{\cos\alpha}} \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}.(1+t^4\tan^2\alpha)} \qquad [7.4.19]$$

Using [7.4.8] we obtain

$$J_2 = \frac{2\tan\alpha}{\sqrt{\cos\alpha}} \int_0^1 \frac{\tan^2 x.\sec^2 x dx}{\sec^2\alpha\tan^4 x+2\tan^2 x+1}$$

$$= \frac{2\tan\alpha\cos^2\alpha}{\sqrt{\cos\alpha}} \int_0^\infty \frac{y^2 dy}{y^4+2\cos^2\alpha y^2+\cos^2\alpha} \qquad [7.4.20]$$

Using [7.4.15] we can write

$$J_2 = 2\sin\alpha \int_0^\infty \frac{\tau^2 d\tau}{(\tau^2+1)^2-4\tau^2\sin^2\frac{\alpha}{2}}$$

$$= 2\sin\alpha \int_0^\infty \left[\frac{K\tau+\lambda}{(\tau-a)^2+(1-a^2)} - \frac{K\tau-\lambda}{(\tau+a)^2+(1-a^2)}\right] d\tau$$

$$(a=\sin\tfrac{1}{2}\alpha) , \qquad [7.4.21]$$

where

$$\left.\begin{aligned} K &= \frac{1}{4a} , \\ \lambda &= 0 . \end{aligned}\right\} \qquad [7.4.22]$$

Hence

$$J_2 = \pi \sin\tfrac{1}{2}\alpha . \qquad [7.4.23]$$

Substituting from [7.4.18] and [7.4.23] in [7.4.10] we obtain

$$I_{11} = \pi . \qquad [7.4.24]$$

Now we evaluate the third inner integral in [7.4.4.] which is defined by

$$I_{12} = \int_0^\alpha \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}.(\tan^2\phi-\tan^2\theta)}$$

$$= \frac{1}{\sqrt{\cos\alpha}} \int_0^\alpha \frac{\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}.(\tan^2\phi-\tan^2\theta)}$$

$$(o<\theta<\alpha<\phi<\tfrac{\pi}{2}) . \qquad [7.4.25]$$

Using [7.4.6] we obtain

$$I_{12} = \frac{2}{\tan^2\alpha\sqrt{\cos\alpha}} \int_0^1 \frac{dt}{\sqrt{1-t^2}(t_o^4 - t^4)} \quad , \quad \left(t_o^2 = \frac{\tan\phi}{\tan\alpha}\right)$$

$$= \frac{1}{\sqrt{\cos\alpha}\tan\phi\tan\alpha} [J_5 + J_6] \quad , \qquad [7.4.26]$$

with

$$\left.\begin{aligned} J_5 &= \int_0^1 \frac{dt}{\sqrt{1-t^2}(t_o^2 - t^2)} \quad , \\ J_6 &= \int_0^1 \frac{dt}{\sqrt{1-t^2}(t_o^2 + t^2)} \quad . \end{aligned}\right\} \qquad [7.4.27]$$

Using [7.4.8] we can write the first integral in [7.4.27] in the form

$$J_5 = \int_0^{\frac{\pi}{2}} \frac{dx}{t_o^2 - \sin^2 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x dx}{(t_o^2 - 1)\tan^2 x + t_o^2} \quad . \qquad [7.4.28]$$

Hence

$$J_5 = \frac{1}{t_o^2 - 1} \int_0^{\infty} \frac{dy}{y^2 + y_3^2} \quad , \quad \left[y_3^2 = \frac{t_o^2}{t_o^2 - 1}\right]$$

$$= \frac{\pi}{2} \frac{1}{t_o\sqrt{t_o^2 - 1}}$$

$$= \frac{\pi\tan\alpha}{2\sqrt{\tan\phi}\sqrt{\tan\phi - \tan\alpha}} \quad . \qquad [7.4.29]$$

Using [7.4.8] we can write the second integral in [7.4.27] in the form

$$J_6 = \int_0^{\frac{\pi}{2}} \frac{dx_2}{t_o^2 + \sin x}$$

$$= \frac{1}{t_o^2 + 1} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x dx}{\tan^2 x + \frac{t_o^2}{t_o^2 + 1}} \quad , \left(t_o^2 = \frac{\tan\phi}{\tan\alpha}\right) \quad . \qquad [7.4.30]$$

Hence

Hence

$$J_6 = \frac{1}{t_o^2+1}\int_o^\infty \frac{dy}{y^2+y_4^2} \quad , \qquad \left(y_4^2 = \frac{t_o^2}{t_o^2+1}\right)$$

$$= \frac{\pi}{2t_o\sqrt{t_o^2+1}}$$

$$= \frac{\pi\tan\alpha}{2\sqrt{\tan\phi}\sqrt{\tan\phi+\tan\alpha}} \qquad [7.4.31]$$

Substituting from [7.4.31] and [7.4.29] in [7.4.26] we obtain

$$I_{12} = \frac{\pi}{2\sqrt{\tan\phi}\sqrt{\cos\alpha}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} + \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right]\cdot\frac{1}{\tan\phi} \qquad [7.4.32]$$

Now we calculate the fourth inner integral in [7.4.4] which is defined by

$$I_{13} = \int_o^\alpha \frac{\tan\theta d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}.(\tan^2\phi-\tan^2\theta)} \quad (o<\theta<\alpha<\phi<\tfrac{\pi}{2})$$

$$= \frac{1}{\sqrt{\cos\alpha}}\int_o^\alpha \frac{\tan\theta\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}.(\tan^2\phi-\tan^2\theta)} \ . \qquad [7.4.33]$$

Using [7.4.6] we obtain

$$I_{13} = \frac{2}{\tan\alpha\sqrt{\cos\alpha}}\int_o^1 \frac{t^2dt}{\sqrt{1-t^2}.(t_o^4-t^4)} \quad , \quad \left(t_o^2 = \frac{\tan\phi}{\tan\alpha}\right)$$

$$= \frac{1}{\sqrt{\cos\alpha}\cdot\tan\alpha}\,[J_7-J_8] \quad , \qquad [7.4.34]$$

where

$$J_7 = \int_o^1 \frac{dt}{\sqrt{1-t^2}(t_o^2-t^2)}$$

$$= J_5 = \frac{\pi\tan\alpha}{2\sqrt{\tan\phi}\sqrt{\tan\phi-\tan\alpha}} \quad , \quad (o<\alpha<\phi<\tfrac{\pi}{2}) \quad , \qquad [7.4.35]$$

and

$$J_8 = \int_o^1 \frac{dt}{\sqrt{1-t^2}(t_o^2+t^2)}$$

$$= J_6 = \frac{\pi\tan\alpha}{2\sqrt{\tan\phi}\sqrt{\tan\phi+\tan\alpha}} \quad , \quad (o<\alpha<\phi<\tfrac{\pi}{2}) \ . \qquad [7.4.36]$$

Substituting from [7.4.35] and [7.4.36] in [7.4.34] we can write the value of integral [7.4.33] in the form

$$I_{13} = \frac{\pi}{2\sqrt{\cos\alpha}\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} - \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right] \quad . \qquad [7.4.37]$$

Now we calculate the last inner integral in [7.4.4] which is defined by

$$I_{14} = \int_0^\alpha \frac{d\theta}{\cos\theta\sqrt{\sin\theta\sin(\alpha-\theta)}}$$

$$= \frac{1}{\sqrt{\cos\alpha}}\int_0^\alpha \frac{\sec^2\theta\, d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}} \quad . \qquad [7.4.38]$$

Using [7.4.6] we obtain

$$I_{14} = \frac{2}{\sqrt{\cos\alpha}}\int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{\pi}{\sqrt{\cos\alpha}} \quad . \qquad [7.4.39]$$

Using the above result we can write [7.4.4] in the form

$$A = \pi D \quad . \qquad [7.4.40]$$

VII APPENDIX V

As seen from [4.b.50] the integral I can be calculated as follows:

$$I = 2\int_0^{\alpha} \frac{\nu(\theta)\sin^2\theta\cos\theta\sqrt{\sin^2\alpha-\sin^2\theta}\,.d\theta}{\sin^2\theta-\sin^2\theta_o}, \qquad [7.5.1]$$

where

$$\nu(\theta) = \frac{D\cos(\theta-\tfrac{1}{2}\alpha)}{2\sqrt{c}\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta}\,.\sqrt{\sin\theta\sin(\alpha-\theta)}} -$$

$$-\frac{1}{2\pi\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta}\sqrt{\sin\theta\sin(\alpha-\theta)}}\Bigg\{\tfrac{1}{2}U\sigma\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\sin(\theta_o-\theta)} +$$

$$+4U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi.z(\phi)\Bigg[\Bigg[\frac{-\sqrt{\cos\alpha}\sqrt{\tan\phi}}{2\cos\theta(\tan^2\phi-\tan^2\theta)}(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}] +$$

$$+\tan\theta[\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha}])+\cos(\theta-\tfrac{1}{2}\alpha)\Bigg]\Bigg]\,d\phi, \qquad (o<\theta<\alpha<\phi<\tfrac{\pi}{2}). \qquad [7.5.2]$$

Substituting from [7.5.2] in [7.5.1] we obtain

$$I = \frac{D}{\sqrt{c}}\int_0^{\alpha}\frac{\cos(\theta-\tfrac{1}{2}\alpha)\sin\theta\cos\theta\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.(\sin^2\theta-\sin^2\theta_o)} -$$

$$-\frac{U\sigma}{2\pi}\int_0^{\alpha}\frac{\sin\theta\cos\theta\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.(\sin^2\theta-\sin^2\theta_o)}\int_0^{\alpha}\frac{(\sin^2\alpha-2\sin^2\psi)\sqrt{\sin\psi\sin(\alpha-\psi)}\,d\psi}{\sin(\psi-\theta)} +$$

$$+\frac{U\sqrt{\cos\alpha}}{\pi}\int_0^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.(\sin^2\theta-\sin^2\theta_o)}\int_{\alpha}^{\frac{\pi}{2}}\frac{\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi\sqrt{\tan\phi}\,.z(\phi)\,d\phi}{\tan^2\phi-\tan^2\theta}\Bigg[\Bigg[$$

$$\tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})+\tan\theta(\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha})\Bigg]\Bigg] -$$

$$-\frac{4U}{\pi}\int_0^{\alpha}\frac{\sin\theta\cos\theta\quad\cos(\theta-\tfrac{1}{2}\alpha)\,d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.(\sin^2\theta-\sin^2\theta_o)}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,.\sin\phi z(\phi)\,d\phi, \quad (o<\theta_o<\alpha<\phi<\tfrac{\pi}{2}).$$

(7.5.3]

This may be written in the form

$$I = \frac{D}{\sqrt{c}} \int_0^{\alpha} \frac{\cos(\theta-\tfrac{1}{2}\alpha)\sin\theta\cos\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\sin^2\theta-\sin^2\theta_o)} -$$

$$-\frac{U\sigma}{2\pi\cos^2\theta_o}\int_0^{\alpha}\frac{\tan\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}.(\tan\theta+\tan\theta_o)}\left(\frac{\sec^2\theta\ d\theta}{\tan\theta-\tan\theta_o}\right)\int_0^{\alpha}(\sin^2\alpha\ -$$

$$-2\sin^2\psi)\cos^2\psi\sqrt{\tan\psi(\tan\psi-\tan\psi)}\left(\frac{\sec^2\psi d\psi}{\tan\psi-\tan\theta}\right) +$$

$$+\frac{2U}{\pi}\int_0^{\alpha}\frac{\tan\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}.(\tan\theta+\tan\theta_o)}\left(\frac{\sec^2\theta d\theta}{\tan\theta-\tan\theta_o}\right)\int_{\alpha}^{\frac{\pi}{2}}\frac{\sqrt{\sin^2\phi-\sin^2\alpha}.\sin\phi\cos^2\phi\sqrt{\tan\phi}z(\phi)}{(\tan\phi+\tan\theta)}$$

$$\left(\frac{\sec^2\phi\ d\phi}{\tan\phi-\tan\theta}\right)[[\ \tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})+\tan\theta(\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha})]]\ -$$

$$-\frac{4U}{\pi}\int_0^{\alpha}\frac{\sin\theta\cos\theta\ \ \cos(\theta-\tfrac{1}{2}\alpha)d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\sin^2\theta-\sin^2\theta_o)}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi \quad . \qquad [7.5.4]$$

Using the Poincare-Bertrand formula

$$\int_{L_1}\frac{f_1(\theta)\sec^2\theta d\theta}{\tan\theta-\tan\theta_o}\int_{L_2}\frac{f_2(\psi)\sec^2\psi d\psi}{\tan\psi-\tan\theta} = -\pi^2 f_1(\theta_o)f_2(\theta_o)+\int_{L_2}f_2(\psi)\sec^2\psi d\psi\int_{L_1}\frac{f_1(\theta)\sec^2\theta d\theta}{(\tan\theta-\tan\theta_o)(\tan\psi-\tan\theta)} \qquad [7.5.5]$$

we obtain

$$I = \frac{D}{\sqrt{c}}\int_0^{\alpha}\frac{\cos(\theta-\tfrac{1}{2}\alpha)\sin\theta\cos\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\sin^2\theta-\sin^2\theta_o)} +$$

$$+\tfrac{1}{2}\pi U\sigma(\sin^2\alpha-2\sin^2\theta_o) - \frac{U\sigma}{2\pi\cos^2\theta_o}\int_0^{\alpha}(\sin^2\alpha\ -$$

$$-2\sin^2\psi)\sqrt{\tan\psi(\tan\alpha-\tan\psi)}d\psi\int_0^{\alpha}\frac{\tan\theta\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}(\tan\theta-\tan\theta_o)(\tan\theta+\tan\theta_o)(\tan\psi-\tan\theta)} +$$

$$+2\pi U z(\theta_o)\sin\theta_o\cos^2\theta_o\sqrt{\sin^2\alpha-\sin^2\theta_o}\ +$$

$$+\frac{2U}{\pi}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.\sqrt{\tan\phi}.\sin\phi z(\phi)d\phi\int_0^{\alpha}\frac{\sin\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\sin^2\theta-\sin^2\theta_o)(\tan^2\phi-\tan^2\theta_o)}[[$$

$$\tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})+\tan\theta(\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha})]]\ -$$

$$-\frac{4U}{\pi}\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi\int_0^{\alpha}\frac{\sin\theta\cos\theta\cos(\theta-\tfrac{1}{2}\alpha)d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\sin^2\theta-\sin^2\theta_o)}$$

$$= \frac{D}{\sqrt{c}} I_{16} + \tfrac{1}{2}\pi U\sigma(\sin^2\alpha - 2\sin^2\theta_o) - \frac{U\sigma}{2\pi\cos^2\theta_o}\int_o^\alpha (\sin^2\alpha - 2\sin^2\psi)\sqrt{\tan\psi(\tan\alpha-\tan\psi)}\,.\,I_{17}\cdot d\psi +$$

$$+ 2\pi U z(\theta_o)\sin\theta_o\cos^2\theta_o\sqrt{\sin^2\alpha-\sin^2\theta_o} +$$

$$+ \frac{2U}{\pi}\int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\,.\,\sqrt{\tan\phi}\sin\phi\, z(\phi)\,[\tan\phi(\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha})I_{18} +$$

$$+ (\sqrt{\tan\phi-\tan\alpha}-\sqrt{\tan\phi+\tan\alpha})I_{19}]d\phi - \frac{4U}{\pi}\int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\, z(\phi)\sin\phi d\phi\,.\,I_{16} \quad , \quad [7.5.6]$$

where

$$I_{16} = \int_o^\alpha \frac{\cos(\theta-\frac{1}{2}\alpha)\sin\theta\cos\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.\,(\sin^2\theta-\sin^2\theta_o)} \quad , \quad (o<\theta_o<\alpha<\tfrac{\pi}{2}) \;, \quad [7.5.7]$$

$$I_{17} = \int_o^\alpha \frac{\tan\theta\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}\,.\,(\tan\theta-\tan\theta_o)(\tan\theta+\tan\theta_o)(\tan\psi-\tan\theta)} \quad , \quad [\theta_o,\psi\varepsilon(o,\alpha)] \;, \quad [7.5.8]$$

$$I_{18} = \int_o^\alpha \frac{\sin\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.\,(\sin^2\theta-\sin^2\theta_o)(\tan^2\phi-\tan^2\theta)} \quad , \quad (o<\theta_o<\alpha<\phi<\tfrac{\pi}{2}) \quad [7.5.9]$$

$$I_{19} = \int_o^\alpha \frac{\sin\theta\tan\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}\,.\,(\sin^2\theta-\sin^2\theta_o)(\tan^2\phi-\tan^2\theta)} \quad , \quad (o<\theta_o<\alpha<\phi<\tfrac{\pi}{2}) \quad [7.5.10]$$

First we calculate integral [7.5.7], by writing it in the form

$$I_{16} = \frac{\cos\frac{1}{2}\alpha}{\sqrt{\cos\alpha}\cos^2\theta_o}\int_o^\alpha \frac{\tan\theta[1+\tan\frac{1}{2}\alpha\tan\theta]d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}\,.\,(\tan^2\theta-\tan^2\theta_o)}$$

$$= \frac{\cos\frac{1}{2}\alpha}{\sqrt{\cos\alpha}\cos\theta_o}[J_1+\tan\tfrac{1}{2}\alpha J_2] \quad , \quad [7.5.11]$$

where

$$J_1 = \int_o^\alpha \frac{\tan\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}\,.\,(\tan^2\theta-\tan^2\theta_o)} \quad , \quad [7.5.12]$$

$$J_2 = \int_o^\alpha \frac{\tan^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}\,.\,(\tan^2\theta-\tan^2\theta_o)} \quad . \quad [7.5.13]$$

We make the transformation

$$\tan\theta = t^2\tan\alpha \qquad [7.5.14]$$

we obtain

$$J_2 = \frac{2}{\tan\alpha}\int_0^1 \frac{t^2\,dt}{\sqrt{1-t^2}(t^4-t_o^4)(1+t^4\tan^2\alpha)} \quad , \quad \left(t_o^2 = \frac{\tan\theta_o}{\tan\alpha}\right)$$

$$= \frac{2\cos^2\theta_o}{\tan\alpha}\,[J_3 - J_4\tan^2\alpha] \quad , \qquad [7.5.15]$$

where

$$J_3 = \int_0^1 \frac{t^2dt}{\sqrt{1-t^2}(t^4-t_o^4)}$$

$$= \frac{\pi\tan\alpha}{4\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} \qquad (o<\theta_o<\alpha<\tfrac{\pi}{2}) \quad , \qquad [7.5.16]$$

and

$$J_4 = \int_0^1 \frac{t^2\,dt}{\sqrt{1-t^2}.(1+t^4\tan^2\alpha)}$$

$$= \frac{\pi\sqrt{\cos\alpha}\cos\alpha}{4\cos\frac{1}{2}\alpha} \qquad (o<\alpha<\tfrac{\pi}{2}) \quad . \qquad [7.5.17]$$

Substituting from [7.5.16] and [7.5.17] in [7.5.15] we obtain

$$J_1 = \pi\cos^2\theta_o\left[\frac{1}{2\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} - \frac{\sqrt{\cos\alpha}\sin\alpha}{2\cos\frac{1}{2}\alpha}\right] \quad . \qquad [7.5.18]$$

Now we calculate the second integral [7.5.13], using [7.5.14] we obtain

$$J_2 = 2\int_0^1 \frac{t^4dt}{\sqrt{1-t^2}.(t^4-t_o^4)(1+t^4\tan^2\alpha)} \quad .$$

$$= 2\cos^2\theta_o\left[\frac{\tan^2\theta_o}{\tan^2\alpha}J_5+J_6\right] \quad , \quad \left(t_o^2 = \frac{\tan\theta_o}{\tan\alpha}\right) \quad , \qquad [7.5.19]$$

with

$$J_5 = \int_0^1 \frac{dt}{\sqrt{1-t^2}(t^4-t_o^4)}$$

$$= -\frac{\pi\tan^2\alpha}{4\tan\theta_o\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} \quad , \qquad [7.5.20]$$

and

$$J_6 = \int_0^1 \frac{dt}{\sqrt{1-t^2}.(1+t^4\tan^2\alpha)}$$

$$= \frac{\pi}{2}\sqrt{\cos\alpha}\cos\tfrac{1}{2}\alpha \qquad [7.5.21]$$

Substituting from [7.5.20] and [7.5.21] in [7.5.19] we obtain

$$J_2 = \pi\cos^2\theta_o\left[\frac{-\tan\theta_o}{2\sqrt{\tan\theta_o}(\tan\alpha+\tan\theta)} + \sqrt{\cos\alpha}\cos\tfrac{1}{2}\alpha\right] \qquad [7.5.22]$$

Using [7.5.22] and [7.5.18] we can write the value of integral [7.5.11] in the form

$$I_{16} = \frac{\pi\cos(\tfrac{1}{2}\alpha+\theta_o)}{2\sqrt{\sin\theta_o\sin(\alpha+\theta_o)}} \qquad [7.5.23]$$

Now we calculate integral [7.5.8], using [7.5.14] we obtain

$$I_{17} = -\frac{2}{\tan^2\alpha}\int_0^1 \frac{t^2dt}{\sqrt{1-t^2}.(t^2-t_o^2)(t^2+t_o^2)(t^2-\tau^2)}, \left[t_o^2=\frac{\tan\theta_o}{\tan\alpha}, \tau^2=\frac{\tan\psi}{\tan\alpha}\right]$$

$$= \frac{-2}{\tan\alpha(\tan^2\psi-\tan^2\theta_o)}\{\tan\psi.J_7-\tfrac{1}{2}(\tan\psi+\tan\theta_o)J_8-\tfrac{1}{2}(\tan\psi-\tan\theta_o)J_9\}, \qquad [7.5.24]$$

with

$$J_7 = \int_0^1 \frac{dt}{\sqrt{1-t^2}.(t^2-\tau^2)} \qquad (o<\tau<1)$$

$$= 0 \qquad [7.5.25]$$

$$J_8 = \int_0^1 \frac{dt}{\sqrt{1-t^2}(t^2+t_o^2)} \qquad (o<t_o<1)$$

$$= \frac{\pi}{2\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} \qquad (o<\theta_o<\alpha<\tfrac{\pi}{2}), \qquad [7.5.26]$$

$$J_9 = \int_0^1 \frac{dt}{\sqrt{1-t^2}(t^2-t_o^2)} \qquad (o<t_o<1)$$

$$= 0 \qquad [7.5.27]$$

Using the above results we can write the value of integral [7.5.24] in the form

$$I_{17} = \frac{\pi}{2(\tan\psi-\tan\theta_o)\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} \qquad (o<\theta_o<\alpha<\tfrac{\pi}{2}) \quad . \qquad [7.5.28]$$

Now we calculate integral [7.5.9] , using [7.5.14] we obtain

$$I_{18} = \frac{1}{\sqrt{\cos\alpha}\cos^2\theta_o}\int_o^\alpha \frac{\tan\theta\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta)}(\tan^2\theta-\tan^2\theta_o)(\tan^2\phi-\tan^2\theta)}$$

$$= \frac{2}{\sqrt{\cos\alpha}\cos^2\theta_o\tan^3\alpha}\int_o^1 \frac{t^2dt}{\sqrt{1-t^2}(t^4-t_o^4)(\tau^4-t^4)} , \left[t_o^2 = \frac{\tan\theta_o}{\tan\alpha} , \tau^2 = \frac{\tan\phi}{\tan\alpha}\right] . [7.5.29]$$

Hence

$$I_{18} = \frac{2}{\sqrt{\cos\alpha}\cos^2\theta_o\tan\alpha(\tan^2\phi-\tan^2\theta_o)} \cdot [J_{10}+J_{11}] \quad , \qquad [7.5.30]$$

with

$$J_{10} = \int_o^1 \frac{t^2dt}{\sqrt{1-t^2}(t^4-t_o^4)}$$

$$= \tfrac{1}{2}\left\{\int_o^1 \frac{dt}{\sqrt{1-t^2}(t^2-t_o^2)} + \int_o^1 \frac{dt}{\sqrt{1-t^2}(t^2+t_o^2)}\right\} \qquad (o<t<1)$$

$$= \frac{\pi\tan\alpha}{4\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} \quad , \qquad [7.5.31]$$

and

$$J_{11} = \int_o^1 \frac{t^2dt}{\sqrt{1-t^2}(\tau^4-t_o^4)}$$

$$= \tfrac{1}{2}\left\{\int_o^1 \frac{dt}{\sqrt{1-t^2}(\tau^2-t^2)} - \int_o^1 \frac{dt}{\sqrt{1-t^2}(\tau^2+t^2)}\right\} \qquad (\tau>1)$$

$$= \frac{\pi\tan\alpha}{4\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} - \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right] \quad . \qquad [7.5.32]$$

Substituting from [7.5.32] and [7.5.31] in [7.5.30] we can write the value of integral I_{18}

$$I_{18}= \frac{\pi}{2\sqrt{\cos\alpha}\cos^2\theta_o(\tan^2\phi-\tan^2\theta_o)}\left[\frac{1}{\sqrt{\tan\theta_o}\sqrt{\tan\alpha+\tan\theta_o}} + \frac{1}{\sqrt{\tan\phi}}\left(\frac{1}{\sqrt{\tan\phi-\tan\alpha}} - \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right)\right]$$

$$(o<\theta_o<\alpha<\phi<\tfrac{\pi}{2}) \quad . \qquad [7.5.33]$$

Now we calculate integral [7.5.10], by writing it in the form

$$I_{19} = \frac{1}{\sqrt{\cos\alpha}\cos^2\theta_o}\int_o^\alpha \frac{\tan^2\theta\sec^2\theta d\theta}{\sqrt{\tan\theta(\tan\alpha-\tan\theta).(\tan^2\theta-\tan^2\theta_o)(\tan^2\phi-\tan^2\theta)}} \quad . \qquad [7.5.34]$$

Using [7.5.14] we obtain

$$I_{19} = \frac{2}{\sqrt{\cos\alpha}\cos^2\theta_o\tan^2\alpha}\int_o^1 \frac{t^4 dt}{\sqrt{1-t^2}(t^4-t_o^4)(\tau^4-t^4)} \ , \quad \left(\tau^2 = \frac{\tan\phi}{\tan\alpha}\ ,\ t_o^2 = \frac{\tan\theta_o}{\tan\alpha}\right)$$

$$= \frac{1}{\sqrt{\cos\alpha}\cos^2\theta_o\tan^2\alpha(\tan^2\phi-\tan^2\theta_o)}\left[-\tan^2\theta_o J_{12}+\tan^2\phi J_{13}\right] \ , \qquad [7.5.35]$$

with

$$J_{12} = \int_o^1 \frac{dt}{\sqrt{1-t^2}(t^4-t_o^4)} \qquad (o<t_o<1)$$

$$= \frac{\pi\tan^2\alpha}{4\tan\theta_o\sqrt{\tan\theta_o}(\tan\alpha+\tan\theta_o)} \qquad (o<\theta_o<\alpha<\tfrac{\pi}{2}) \ , \qquad [7.5.36]$$

and

$$J_{13} = \int_o^1 \frac{dt}{\sqrt{1-t^2}(\tau^4-t^2)} \qquad (o<t<1<\tau)$$

$$= \frac{\pi\tan^2\alpha}{4\tan\phi\sqrt{\tan\phi}}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} + \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right] \quad (o<\alpha<\phi<\tfrac{\pi}{2}) \ . \qquad [7.5.37]$$

Using [7.5.37] and [7.5.36] we can write the value of integral [7.5.35] in the form

$$I_{19} = \frac{\pi\sec^2\theta_o}{2\sqrt{\cos\alpha}(\tan^2\phi-\tan^2\theta_o)}\left\{-\frac{\tan\theta_o}{\sqrt{\tan\theta_o}(\tan\alpha+\tan\theta_o)} + \sqrt{\tan\phi}\left[\frac{1}{\sqrt{\tan\phi-\tan\alpha}} + \frac{1}{\sqrt{\tan\phi+\tan\alpha}}\right]\right\} \ , \quad (o<\theta_o<\alpha<\phi<\tfrac{\pi}{2}) \ . \qquad [7.5.38]$$

Substituting the above results in [7.5.6] we obtain

$$I = \frac{1}{\sqrt{\cos\alpha}\sqrt{\tan\theta_o}} \left\{ \frac{\pi D \cos(\frac{1}{2}\alpha+\theta_o)}{2\sqrt{c}\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} + \right.$$

$$\tfrac{1}{2}\pi U\sigma\sqrt{\cos\alpha}\sqrt{\tan\theta_o}(\sin^2\alpha-2\sin^2\theta_o) - \frac{U\sigma\sqrt{\cos\alpha}}{4\cos^2\theta_o\sqrt{\tan\alpha+\tan\theta_o}} I_{20}+$$

$$+ 2\pi Uz(\theta_o)\sqrt{\cos\alpha}\sqrt{\tan\theta_o}\sin\theta_o\cos^2\theta_o\sqrt{\sin^2\alpha-\sin^2\theta_o} +$$

$$+ \frac{U}{\cos^2\theta_o}\int_\alpha^{\frac{\pi}{2}} \frac{\sqrt{\sin^2\phi-\sin^2\alpha}\sqrt{\tan\phi}\sin\phi z(\phi)d\phi}{(\tan^2\phi-\tan^2\theta_o)} \left(\!\!\left(\tan\phi[\sqrt{\tan\phi-\tan\alpha}+ \right.\right.$$

$$\left.\left. + \sqrt{\tan\phi+\tan\alpha}] \frac{1}{\sqrt{\tan\alpha+\tan\theta_o}} - (\sqrt{\tan\phi-\tan\alpha} - \sqrt{\tan\phi+\tan\alpha})\cdot\frac{\tan\theta_o}{\sqrt{\tan\alpha+\tan\theta_o}}\right)\!\!\right) -$$

$$\left. - \frac{U\cos(\frac{1}{2}\alpha+\theta_o)}{\cos\theta_o\sqrt{\tan\alpha+\tan\theta_o}} \int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}.z(\phi)\sin\phi d\phi \right\} , \qquad [7.5.39]$$

where

$$I_{20} = \int_0^\alpha \frac{(\sin^2\alpha-2\sin^2\psi)\sqrt{\tan\psi(\tan\alpha-\tan\psi)}d\psi}{\tan\psi-\tan\theta_o}$$

$$= \sin^2\alpha\ J_1 - 2J_2 , \qquad [7.5.40]$$

with

$$J_1 = \int_0^\alpha \frac{\sqrt{\tan\psi(\tan\alpha-\tan\psi)}d\psi}{\tan\psi-\tan\theta_o} = \frac{\pi\cos\theta_o\sin(\frac{1}{2}\alpha-\theta_o)}{\sqrt{\cos\alpha}} , \qquad [7.5.41]$$

and

$$J_2 = \int_0^\alpha \frac{\sin^2\psi\sqrt{\tan\psi(\tan\alpha-\tan\psi)}d\psi}{\tan\psi-\tan\theta_o} =$$

$$= 2\{\sin^3\theta_o\cos\theta_o J_3-\cos\theta_o\tan\alpha J_4+\cos\theta_o\tan\alpha J_5\} , \qquad [7.5.42]$$

where

$$\left.\begin{aligned} J_3 &= \int_0^1 \frac{\sqrt{1-t^2}dt}{t^2-t_o^2} = 0 , \qquad \left(t^2 = \frac{\tan\psi}{\tan\alpha} , \ t_o^2 = \frac{\tan\theta_o}{\tan\alpha}\right), \\ J_4 &= \int_0^1 \frac{\sqrt{1-t^2}dt[\sin^3\theta_o\tan\alpha t^2-\cos\theta_o(1+\sin^2\theta_o)]}{1+t^4\tan^2\alpha} , \\ J_5 &= \int_0^1 \frac{\sqrt{1-t^2}dt[\sin\theta_o\tan\alpha t^2-\cos\theta_o]}{(1+t^4\tan^2\alpha)^2} . \end{aligned}\right\} \qquad [7.5.43]$$

The last two integrals in [7.5.43] can be written in the forms

$$\left.\begin{aligned} J_4 &= \sin^3\theta_o \tan\alpha J_6 - (1+\sin^2\theta_o)\cos\theta_o J_7 , \\ J_5 &= \sin\theta_o \tan\alpha J_8 - \cos\theta_o J_9 , \end{aligned}\right\} \qquad [7.5.44]$$

with

$$\left.\begin{aligned} J_6 &= \int_0^1 \frac{t^2\sqrt{1-t^2}\,dt}{1+t^4\tan^2\alpha} = \frac{\pi}{2\tan^2\alpha}\left[\frac{\cos\frac{1}{2}\alpha}{\sqrt{\cos\alpha}} - 1\right] , \\ J_7 &= \int_0^1 \frac{\sqrt{1-t^2}\,dt}{1+t^4\tan^2\alpha} = \frac{\pi\sqrt{\cos\alpha}}{4\cos\frac{1}{2}\alpha} , \\ J_8 &= \int_0^1 \frac{t^2\sqrt{1-t^2}\,dt}{(1+t^4\tan^2\alpha)^2} = \frac{\pi\cos\alpha\sin\frac{1}{2}\alpha}{8\sqrt{\cos\alpha}\tan\alpha} , \\ J_9 &= \int_0^1 \frac{\sqrt{1-t^2}\,dt}{(1+t^4\tan^2\alpha)^2} = \frac{\pi\sqrt{\cos\alpha}}{8\cos\frac{1}{2}\alpha}\,[2-\sin^2\tfrac{1}{2}\alpha] . \end{aligned}\right\} \qquad [7.5.45]$$

Substituting from [7.5.45] in [7.5.44] we obtain

$$\left.\begin{aligned} J_4 &= \frac{-\pi}{4\cos\frac{1}{2}\alpha\sqrt{\cos\alpha}\tan\alpha}\,[2\sin^2\theta_o\cos(\tfrac{1}{2}\alpha)\cdot(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_o))+\sin\alpha\cos\theta_o] , \\ J_5 &= -\frac{\pi\cos\alpha}{16\sqrt{\cos\alpha}\cos\frac{1}{2}\alpha}\,[\cos(\alpha-\theta_o)+3\cos\theta_o] . \end{aligned}\right\} \qquad [7.5.46]$$

Using [7.5.46] we can write the value of integral [7.5.42] in the form

$$J_2 = \frac{\pi\cos\theta_o}{8\cos\frac{1}{2}\alpha\sqrt{\cos\alpha}}\,\{\sin\alpha[\cos\theta_o-\cos(\alpha-\theta_o)]+8\sin^2\theta_o\cos\tfrac{1}{2}\alpha(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_o))\} \cdot \qquad [7.5.47]$$

Now we can write the value of integral [7.5.40], using [7.5.47] and [7.5.41] we obtain

$$I_{20} = \frac{\pi\cos\theta_o\sin(\frac{1}{2}\alpha-\theta_o)\sin^2\alpha}{\sqrt{\cos\alpha}} - \frac{\pi\cos\theta_o}{4\cos\frac{1}{2}\alpha\sqrt{\cos\alpha}}\,\{\sin\alpha[\cos\theta_o-\cos(\alpha-\theta_o)] +$$

$$+ 8\sin^2\theta_o\cos\tfrac{1}{2}\alpha(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_o))\} . \qquad [7.5.48]$$

Substituting from [7.5.48] in [7.5.39] we can write the value of integral I in the form

$$
\begin{aligned}
I = \frac{1}{\sqrt{\cos\alpha}\sqrt{\tan\theta_0}} \Bigg\{ & \frac{\pi D\cos(\frac{1}{2}\alpha+\theta_0)}{2\sqrt{c}\cos\theta_0\sqrt{\tan\alpha+\tan\theta_0}} + \tfrac{1}{2}\pi U\sigma\sqrt{\cos\alpha}\sqrt{\tan\theta_0}(\sin^2\alpha-2\sin^2\theta_0) - \\
& - \frac{\pi U\sigma\sin^2\alpha\sin(\frac{1}{2}\alpha-\theta_0)}{4\cos\theta_0\sqrt{\tan\alpha-\tan\theta_0}} + \frac{\pi U\sigma}{16\cos\theta_0\cos\frac{1}{2}\alpha\sqrt{\tan\alpha-\tan\theta_0}}\,[\sin\alpha(\cos\theta_0-\cos(\alpha-\theta_0)) + \\
& + 8\sin^2\theta_0\cos\alpha(\sqrt{\cos\alpha}+\cos(\tfrac{1}{2}\alpha-\theta_0))] + 2\pi Uz(\theta_0)\sqrt{\cos\alpha}\sqrt{\tan\theta_0}\sin\theta_0\cos^2\theta_0\sqrt{\sin^2\alpha-\sin^2\theta_0} + \\
& + \frac{U}{\cos^2\theta_0}\int_{\alpha}^{\frac{\pi}{2}} \frac{\sqrt{\sin^2\phi-\sin^2\alpha}.\sqrt{\tan\phi}\sin\phi\, z(\phi)\,d\phi}{(\tan^2\phi-\tan^2\theta_0)} \left(\!\!\left(\frac{\tan\phi}{\sqrt{\tan\alpha+\tan\theta_0}}.[\sqrt{\tan\phi-\tan\alpha} + \right.\right.\\
& \left.\left. + \sqrt{\tan\phi+\tan\alpha}] - \frac{\tan\theta_0}{\sqrt{\tan\alpha+\tan\theta_0}}.[\sqrt{\tan\phi-\tan\alpha}+\sqrt{\tan\phi+\tan\alpha}]\right)\!\!\right) - \\
& - \frac{2U\cos(\frac{1}{2}\alpha+\theta_0)}{\cos\theta_0.\sqrt{\tan\alpha+\tan\theta_0}} \cdot \int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}.\,z(\phi)\sin\phi\, d\phi \Bigg\} \qquad [7.5.49]
\end{aligned}
$$

VII APPENDIX VI A formula for C in terms of the hydrofoil slope.

As seen from [3.a.8] the constant C can be calculated as follows:

$$C = \frac{1}{\pi^2}\int_0^\alpha \frac{g(\theta_o)}{\cos\theta_o}\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,d\theta_o \quad , \qquad [7.6.1]$$

with

$$g(\theta_o) = -\frac{\pi U\sigma\sin\theta_o}{\sqrt{\ell}} - \frac{4U\cos\theta_o}{\sqrt{\ell}}\int_0^\alpha \frac{z(\psi)\sin^2\psi\,\cos\psi\,d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}\,.\,(\sin^2\psi-\sin^2\theta_o)} \,. \qquad [7.6.2]$$

Substituting from [7.6.2] into [7.6.1] we obtain

$$C = -\frac{U'\sigma}{\pi\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,\tan\theta_o\,d\theta_o -$$

$$-\frac{4U\cos\theta_o}{\pi^2\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,d\theta_o\int_0^\alpha\frac{z(\psi)\sin^2\psi\,\cos\psi\,d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}\,.\,(\sin^2\psi-\sin^2\theta_o)} \,. \qquad [7.6.3]$$

It is permissible to interchange the order of the double integral in [7.6.3] and then we obtain

$$C = -\frac{U\sigma}{\pi\sqrt{\ell}}I_1 - \frac{4U\cos\theta_o}{\pi^2\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,d\theta\cdot I_2 \quad , \qquad [7.6.4]$$

where

$$I_1 = \int_0^\alpha\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,\tan\theta_o\,d\theta_o$$

$$= \pi\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] \quad , \qquad [7.6.5]$$

and

$$I_2 = \int_0^\alpha\sqrt{\frac{\sin(\alpha-\theta_o)}{\sin\theta_o}}\,\frac{d\theta_o}{(\sin^2\psi-\sin^2\theta_o)}$$

$$= \frac{\pi}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}} \quad , \qquad [7.6.6]$$

hence

$$C = -\frac{U\sigma}{\sqrt{\ell}}\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] - \frac{2U}{\pi\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\,d\theta \quad . \qquad [7.6.7]$$

APPENDIX VII A formula for the lift L .

As seen from [2.b.4] the values of the lift L can be calculated as follows:

$$L = -\rho U \int_0^a \gamma(x)\,dx \quad , \qquad [7.7.1]$$

with

$$\gamma(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}} \oint_0^a \sqrt{\frac{s}{a-s}} \cdot \frac{z(s)\,ds}{s-x} \qquad (o<x<a) \ . \qquad [7.7.2]$$

Substituting from [7.7.2] into [7.7.1] we obtain

$$L = -\frac{2\rho U^2}{\pi} \int_0^a \sqrt{\frac{a-x}{x}}\,dx \int_0^a \sqrt{\frac{s}{a-s}} \frac{z(s)\,ds}{s-x} \quad . \qquad [7.7.3]$$

It is permissible to interchange the order of integration on the right-hand side of [7.7.3] [see, e.g., Hardy, G.H. (35)] and when we do so we obtain

$$L = \frac{2\rho U^2}{\pi} \int_0^a \sqrt{\frac{s}{a-s}}\ z(s)\,ds \int_0^a \sqrt{\frac{a-x}{x}} \frac{dx}{x-s} \quad . \qquad [7.7.4]$$

We evaluate the inner integral I which is defined by

$$I = \int_0^a \sqrt{\frac{a-x}{x}} \frac{dx}{x-s}$$

$$= -\ \pi \ . \qquad [7.7.5]$$

Substituting from [7.7.5] in [7.7.6] we obtain

$$L = -2\rho U^2 \int_0^a \sqrt{\frac{s}{a-s}}\ z(s)\,ds \quad . \qquad [7.7.6]$$

APPENDIX VIII A formula for the lift L .

As seen from [3.a.9] the value of the lift L can be calculated as follows:

$$L = -4\rho U^2 \ell \int_0^\alpha \frac{\sin^2\theta\cos\theta z(\theta)d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} -$$

$$-2\rho U\ell \int_0^\alpha \frac{\mu(\theta)\sin^2\theta\cos\theta d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} \quad , \qquad [z(\theta)=y'(\ell\sin^2\theta), \mu(\theta)=m_1(\ell\sin^2\theta)] \quad ,[7.8.1]$$

where

$$\mu(\theta) = -\frac{U\sigma}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] -$$

$$-\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\sqrt{\frac{\sin\theta_o}{\sin(\alpha-\theta_o)}}\; z(\theta_o)d\theta_o +$$

$$+\frac{U\sigma}{2\pi\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\frac{\tan\theta_o\sqrt{\sin\alpha\,\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)} +$$

$$+\frac{2U}{\pi^2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\frac{\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}}{\sin(\theta_o-\theta)}\,d\theta_o\int_0^\alpha\frac{\sin^2\psi\,\cos\psi\; z(\psi)d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}.(\sin^2\psi-\sin^2\theta_o)}$$

$$(o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [7.8.2]$$

This equation can be written as follows

$$\mu(\theta) = -\frac{U\sigma}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] -$$

$$-\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha\sqrt{\frac{\sin\theta_o}{\sin(\alpha-\theta_o)}}\; z(\theta_o)d\theta_o +$$

$$+\frac{U\sigma}{2\pi\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\;.\;J_1(\theta)+J_2(\theta) \quad , \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [7.8.3]$$

where

$$J_1(\theta) = \int_0^\alpha\frac{\tan\theta_o\sqrt{\sin\theta_o\sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)}$$

$$= -\frac{\pi\sqrt{\cos\alpha}}{\cos\theta}+\pi\cos(\theta-\tfrac{1}{2}\alpha) \qquad (o<\theta<\alpha\leq\tfrac{\pi}{2}) \qquad [7.8.4]$$

[see, Appendix III]

and

$$J_2(\theta) = \frac{2U}{\pi^2 \sin}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}} \int_0^\alpha \frac{\sqrt{\sin\theta_o \sin(\alpha-\theta_o)}\,d\theta_o}{\sin(\theta_o-\theta)} \int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\frac{z(\psi)\sin\psi d\psi}{\sin(\psi+\theta)}$$

$$= -Uz(\theta) + \frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,\frac{1}{\sin\theta}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\frac{z(\psi)\sin\psi d\psi}{\sin(\psi+\theta)} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [7.8.5]$$

where, in the last step, use has been made of the POINCARÉ-BERTRAND formula [7.5.5].

Substituting from [7.8.5] and [7.8.4] into [7.8.3] we obtain

$$\mu(\theta) = -\frac{U\sigma}{2\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] -$$

$$-\frac{U}{\pi\sin\theta\cos\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\int_0^\alpha \sqrt{\frac{\sin\theta_o}{\sin(\alpha-\theta_o)}}\,z(\theta_o)\,d\theta_o -$$

$$-\frac{U\sigma}{2\sin\theta}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\left[\frac{\sqrt{\cos\alpha}}{\cos\theta} - \cos(\theta-\tfrac{1}{2}\alpha)\right] -$$

$$-Uz(\theta) + \frac{U}{\pi}\sqrt{\frac{\sin(\alpha+\theta)}{\sin\theta}}\,\frac{1}{\sin\theta}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,\frac{z(\psi)\sin\psi d\psi}{\sin(\psi+\theta)} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [7.8.6]$$

Using the above result we can write equation [7.8.1] as follows

$$L = -2\rho U^2 \ell \int_0^\alpha \frac{\sin^2\theta\cos\theta z(\theta)\,d\theta}{\sqrt{\sin^2\alpha-\sin^2\theta}} + \rho U^2\sigma\ell[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}].I_1 +$$

$$+ \rho U^2\sigma\ell[\sqrt{\cos\alpha}I_1 - I_2] + \frac{2\rho U^2\ell}{\pi}\int_0^\alpha \sqrt{\frac{\sin\theta_o}{\sin(\alpha-\theta_o)}}\,z(\theta_o)\,d\theta_o.I_1 -$$

$$-\frac{2\rho U^2\ell}{\ell}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\,z(\psi)\tan\psi d\psi.\,I_3(\psi) \quad , \qquad [7.8.7]$$

where

$$I_1 = \int_0^\alpha \sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,d\theta \quad , \qquad (o<\alpha<\tfrac{\pi}{2}) \qquad [7.8.8]$$

$$I_2 = \int_0^\alpha \sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,\cos\theta\cos(\theta-\tfrac{1}{2}\alpha)\,d\theta \quad , \qquad [7.8.9]$$

$$I_3 = \int_0^\alpha \sqrt{\frac{\sin}{\sin(\alpha-\theta)}}\,\frac{d\theta}{(\tan\theta+\tan\psi)} = \int_0^\alpha \frac{\sin\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}.(\tan\theta+\tan\psi)} \,. \qquad [7.8.10]$$

The last integral has been calculated as shown in Appendix I , to give the following result

$$I_3 = \pi\cos\psi[-\cos\psi\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}} + \cos(\tfrac{1}{2}\alpha-\psi)] \qquad (o<\psi<\alpha<\tfrac{\pi}{2}) \qquad [7.8.11]$$

Now we calculate the first integral I , [7.8.8] as follows:

$$I_1 = \frac{1}{\sqrt{\cos\alpha}}\int_o^\alpha \sqrt{\frac{\tan\theta}{\tan\alpha-\tan\theta}}\,d\theta \quad . \qquad [7.8.12]$$

Using the transformation

$$\tan\theta = t^2\tan\alpha \quad , \qquad [7.8.13]$$

we obtain

$$I_1 = \frac{2\tan\alpha}{\sqrt{\cos\alpha}}\int_o^1 \frac{t^2dt}{\sqrt{1-t^2}(1+t^4\tan^2\alpha)}$$

$$= \pi\sin\tfrac{1}{2}\alpha \qquad [7.8.14]$$

The second integral, [7.8.9] can be written as follows:

$$I_2 = \frac{\cos\frac{1}{2}\alpha}{\sqrt{\cos\alpha}}\int_o^\alpha \sqrt{\frac{\tan\theta}{\tan\alpha-\tan\theta}}\,[1+\tan\tfrac{1}{2}\alpha\tan\theta]\cos^4\theta\sec^2\theta d\theta \quad . \qquad [7.8.15]$$

Using the transformation [7.8.13] we can write

$$I_2 = \frac{2\cos\frac{1}{2}\alpha\tan\alpha}{\sqrt{\cos\alpha}}[J_1+\tan\alpha\tan\tfrac{1}{2}\alpha J_2] \quad , \qquad [7.8.16]$$

with

$$\left.\begin{aligned} J_1 &= \int_o^1 \frac{t^2dt}{\sqrt{1-t^2}(1+t^4\tan^2\alpha)^2}, \\ &= \frac{\pi\cos^2\alpha}{16\sqrt{\cos\alpha}\cos^2\frac{1}{2}\alpha}[4\cos\tfrac{1}{2}\alpha\cos^2\alpha+\sin\alpha\sin\tfrac{3}{2}\alpha] \\ \text{and} \quad J_2 &= \int_o^1 \frac{t^4dt}{\sqrt{1-t^2}(1+t^4\tan^2\alpha)^2} \\ &= \frac{\pi\sqrt{\cos\alpha}\cos^2\alpha\sin\frac{3}{2}\alpha}{8\sin\alpha} \quad . \end{aligned}\right\} \qquad [7.8.17]$$

Substituting from [7.8.17] into [7.8.16] we obtain

$$I_2 = \pi \sin\tfrac{1}{2}\alpha\cos^3\tfrac{1}{2}\alpha \qquad . \tag{7.8.18}$$

Using [7.8.11], [7.8.14] and [7.8.18] we can write [7.8.8] in the form

$$L = \rho U^2\sigma\ell\pi\sin^3\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha + 2\rho U^2\ell\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\, z(\theta)\sin(\tfrac{1}{2}\alpha-\theta)\cos\theta d\theta \ , \tag{7.8.19}$$

where the constant σ is defined by

$$\frac{1}{\pi^2}\int_0^\alpha \frac{g(\theta_0)}{\cos\theta_0}\sqrt{\frac{\sin(\alpha-\theta_0)}{\sin\theta_0}}\, d\theta_0 = C = \frac{U\sigma\cos^2\alpha}{2\sqrt{\ell}\cos\frac{1}{2}\alpha} +$$

$$+ \frac{1}{\pi^3\cos\frac{1}{2}\alpha}\int_0^\alpha\frac{\cos^2\theta d\theta}{\sqrt{\sin\theta\sin(\alpha-\theta)}}\int_0^\alpha\frac{g(\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\cos\theta_0\sin(\theta_0-\theta)} \ , \tag{7.8.20}$$

with

$$g(\theta_0) = -\frac{\pi U\sigma\sin\theta_0}{\sqrt{\ell}} - \frac{4U\cos\theta_0}{\sqrt{\ell}}\int_0^\alpha\frac{z(\psi)\sin^2\psi\cos\psi d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}.(\sin^2\psi-\sin^2\theta_0)} \ . \tag{7.8.21}$$

[see, e,g , T.V.Davies "Steady two-dimensional cavity flow past an aerofoil using linearized theory" 1970, Quart.Journ.Mech. and Applied Math., Vol.XXIII, Pt.1]

The left-hand side of [7.8.20] has been calculated in Appendix VI to give

$$C = -\frac{U\sigma}{\sqrt{\ell}}\,[\cos\tfrac{1}{2}\alpha-\sqrt{\cos\alpha}] - \frac{2U}{\pi\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\, z(\theta)d\theta \qquad . \tag{7.8.22}$$

Now we evaluate the right-hand side of equation [7.8.20] by interchanging the order of its two double integral

$$C = \frac{U\sigma\ \cos^2\alpha}{2\sqrt{\ell}\cos\frac{1}{2}\alpha} - \frac{1}{\pi^3\cos\frac{1}{2}\alpha}\int_0^\alpha\frac{g(\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\cos\theta_0}\, I(\theta_0) \ , \tag{7.8.23}$$

where

$$I(\theta_0) = \int_0^\alpha\frac{\cos^2\theta\ d\theta}{\sin(\theta-\theta_0)\sqrt{\sin\theta\sin(\alpha-\theta)}} = -\pi\sin(\tfrac{1}{2}\alpha+\theta_0) \qquad (o<\theta_0<\alpha<\tfrac{\pi}{2}) \tag{7.8.24}$$

hence

$$C = \frac{U\sigma\cos^2\alpha}{2\sqrt{\ell}\cos\frac{1}{2}\alpha} + \frac{1}{\pi^2\cos\frac{1}{2}\alpha}\int_0^\alpha \frac{g(\theta_0)\sin(\frac{1}{2}\alpha+\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{\cos\theta_0} \qquad [7.8.25]$$

Substituting from [7.8.21] into [7.8.25] we obtain

$$C = \frac{U\sigma\cos^2\alpha}{2\sqrt{\ell}\cos\frac{1}{2}\alpha} - \frac{U\sigma\ I_3}{\pi\sqrt{\ell}\cos\frac{1}{2}\alpha} + \frac{4U}{\pi^2\sqrt{\ell}\cos\frac{1}{2}\alpha}\int_0^\alpha \frac{z(\psi)\sin^2\psi\cos\psi d\psi}{\sqrt{\sin^2\alpha-\sin^2\psi}}\ .\ J(\psi) \quad , \qquad [7.8.26]$$

$$\begin{aligned} I_3 &= \int_0^\alpha \tan\theta_0\sin(\tfrac{1}{2}\alpha+\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0 \\ &\doteqdot \frac{\pi}{4}\,[2+\cos\alpha+\cos^2\alpha-4\sqrt{\cos\alpha}\cos\tfrac{1}{2}\alpha] \ , \end{aligned} \qquad [7.8.27]$$

and

$$\begin{aligned} J(\psi) &= \int_0^\alpha \frac{\sin(\frac{1}{2}\alpha+\theta_0)\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,d\theta_0}{(\sin^2\theta_0-\sin^2\psi)} \\ &= \frac{\pi}{2}\left[\sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}} - 2\cos\tfrac{1}{2}\alpha\right] . \end{aligned} \qquad [7.8.28]$$

Substituting from [7.8.28] and [7.8.27] into [7.8.26] we obtain

$$\begin{aligned} C = {} & \frac{U\sigma}{4\sqrt{\ell}\cos\frac{1}{2}\alpha}\,[4\sqrt{\cos\alpha}\cos\tfrac{1}{2}\alpha+\cos^2\alpha-\cos\alpha-2] \ + \\ & + \frac{2U}{\pi\sqrt{\ell}\cos\frac{1}{2}\alpha}\int_0^\alpha \sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}\ z(\psi)\sin\psi\sin(\tfrac{1}{2}\alpha-\psi)\,d\psi \quad . \end{aligned} \qquad [7.8.29]$$

Now we can write from [7.8.29] and [7.8.22] the value of the cavitation number

$$\begin{aligned} \sigma &= -\frac{2\sqrt{\ell}}{U\cos\alpha\cos\frac{1}{2}\alpha}\left\{\frac{2U}{\pi\sqrt{\ell}\cos\frac{1}{2}\alpha}\int_0^\alpha\sqrt{\frac{\sin\psi}{\sin(\alpha-\psi)}}z(\psi)\sin\psi\sin(\tfrac{1}{2}\alpha-\psi)\,d\psi \ + \right. \\ &\qquad\qquad \left. + \frac{2U}{\pi\sqrt{\ell}}\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\,d\theta\right\} \\ &= -\frac{U}{\pi\cos\alpha\cos^2\frac{1}{2}\alpha}\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\cos\theta\cos(\tfrac{1}{2}\alpha-\theta)\,d\theta \quad . \end{aligned} \qquad [7.8.30]$$

Substituting from [7.8.30] into [7.8.19] we obtain

$$\begin{aligned} L &= -2\rho U^2\ell\tan\alpha\tan^2\tfrac{1}{2}\alpha\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\cos\theta\cos(\tfrac{1}{2}\alpha-\theta)\,d\theta \ + \\ &\qquad\qquad + 2\rho U^2\ell\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\sin(\tfrac{1}{2}\alpha-\theta)\cos\theta d\theta \\ &= 2\rho U^2\ell\int_0^\alpha\sqrt{\frac{\sin\theta}{\sin(\alpha-\theta)}}\,z(\theta)\cos\theta\,[-\tan\alpha\tan^2\tfrac{1}{2}\alpha\cos(\tfrac{1}{2}\alpha-\theta)+\sin(\tfrac{1}{2}\alpha-\theta)]\,d\theta \ . \end{aligned} \qquad [7.8.31]$$

VII APPENDIX IX

As seen from [3.b.11] the integral

$$I = \int_0^{\alpha} \frac{\sqrt{\tan\theta_0(\tan\alpha-\tan\theta_0)}\sec^2\theta_0 d\theta_0}{(\tan\theta_0-\tan\theta)(\tan\theta_0-\tan\psi)(\tan\theta_0+\tan\psi)} \qquad \theta_0\ ,\ \psi\ \varepsilon\ (o,\alpha) \qquad [7.9.1]$$

is calculated as follows:

We make the transformation

$$\tan\theta_0 = t^2\tan\alpha \quad , \qquad [7.9.2]$$

and obtain

$$I = 2\tan^2\alpha \int_0^1 \frac{t^2\sqrt{1-t^2}\ dt}{(t^2\tan\alpha-\tan\theta)(t^2\tan\alpha-\tan\psi)(t^2\tan\alpha+\tan\psi)} \ . \qquad [7.9.3]$$

Using partial fractions we obtain

$$I = \frac{\tan\alpha}{(\tan^2\theta-\tan^2\psi)} \{2\tan\theta J_1 - (\tan\theta+\tan\psi)J_2 - (\tan\theta-\tan\psi)J_3\} \ , \qquad [7.9.4]$$

where

$$J_1 = \int_0^1 \frac{\sqrt{1-t^2}\ dt}{t^2\tan\alpha-\tan\theta} \qquad (o<\theta<\alpha<\tfrac{\pi}{2}) \qquad [7.9.5]$$

$$J_2 = \int_0^1 \frac{\sqrt{1-t^2}\ dt}{t^2\tan\alpha-\tan\psi} \qquad (o<\psi<\alpha<\tfrac{\pi}{2}) \qquad [7.9.6]$$

$$J_3 = \int_0^1 \frac{\sqrt{1-t^2}\ dt}{t^2\tan\alpha+\tan\psi} \qquad (o<\psi<\alpha<\tfrac{\pi}{2}) \qquad [7.9.7]$$

First we calculate the first integral [7.9.5] by taking

$$t = \sin x \quad , \qquad [7.9.8]$$

then we obtain

$$J_1 = \int_0^{\frac{\pi}{2}} \frac{dx}{(\tan\alpha-\tan\theta)\tan^2 x-\tan\theta}$$

$$= -\frac{\pi}{2\tan\alpha} \quad . \qquad [7.9.9]$$

By similar operations we can write the value of integral J_2 , [7.9.6] as follows

$$J_2 = -\frac{\pi}{2\tan\alpha} \quad . \qquad [7.9.10]$$

Now we calculate the third integral [7.9.7], using [7.9.8] and obtain

$$J_3 = \int_0^{\frac{\pi}{2}} \frac{dx}{(\tan\alpha+\tan\psi)\tan^2 x+\tan\psi}$$

$$= -\frac{\pi}{2\tan\alpha}\left[1 - \sqrt{\frac{\tan\alpha+\tan\psi}{\tan\psi}}\right] \quad , \qquad (o < \psi < \alpha < \frac{\pi}{2}) \qquad [7.9.11]$$

Substituting from [7.9.9], [7.9.10] and [7.9.11] into [7.9.4] we obtain

$$I = -\frac{\pi}{2\sqrt{\cos\alpha}}\sqrt{\frac{\sin(\alpha+\psi)}{\sin\psi}} \cdot \frac{1}{(\tan\theta+\tan\psi)} \qquad \theta , \psi \in (o , \alpha) \qquad [7.9.12]$$

NUMERICAL SOLUTIONS

APPENDIX X The Numerical Solution of the Non-Cavity Problem (Minimum Drag), of Chapter II.

We consider here the problem

$$z''(x) - n\frac{z(x)}{\sqrt{1+z^2(x)}} = \frac{E}{\sqrt{x(a-x)}}, \qquad [z(o)=o,\ z(a)=\tan 12°, (o<x<a=4)] \qquad [7.10.1]$$

subject to the constraints

$$\left.\begin{aligned} \ell = 4.02 &= \int_o^a \sqrt{1+z^2(x)}\,dx \,, \\ K = 0.0148 &= \int_o^a z'^2(x)\,dx \,. \end{aligned}\right\} \qquad [7.10.2]$$

We transform [7.10.1] into the simultaneous first order differential equations

$$\left.\begin{aligned} \frac{dW_1(x)}{dx} &= W_2(x) \,, \qquad [W_1(x)=z(x), W_2(x)=z'(x)] \,, \\ \frac{dW_2(x)}{dx} &= \frac{n\,W_1(x)}{\sqrt{1+W_1^2(x)}} + \Psi(x) \,, \end{aligned}\right\} \qquad [7.10.3]$$

where

$$\Psi(x) = \frac{E}{\sqrt{x(a-x)}} \,. \qquad [7.10.4]$$

We consider first the end points $x=o$ and $x=a$ at which the right-hand side of [7.10.1] possesses singularities. First we guess initial values for n and E :

$$\left.\begin{aligned} n_o &= -0.25 \,, \\ E_o &= 0.08 \,. \end{aligned}\right\} \qquad [7.10.5]$$

We now estimate $z(0.01)$ and $z(3.99)$ using Taylor's theorem. We write equation [7.10.1] in the form

(i) for x sufficiently small:

$$\begin{aligned} z''(x) - n\,z(x) &= \frac{E}{\sqrt{a\cdot x}}\left[1 - \frac{x}{a}\right]^{-\frac{1}{2}} \\ &= \frac{E}{a}\left[\left(\frac{x}{a}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{x}{a}\right)^{\frac{1}{2}} + \frac{3}{8}\left(\frac{x}{a}\right)^{3/2} + \frac{5}{16}\left(\frac{x}{a}\right)^{5/2} + \frac{35}{128}\left(\frac{x}{a}\right)^{7/2} + \ldots\right] \,, \end{aligned} \qquad [7.10.6]$$

(ii) for (a-x) sufficiently small:

$$z'(x) - n\,z(x) = \frac{E}{\sqrt{a(a-x)}}\left[1 - \left(\frac{a-x}{a}\right)\right]^{-\frac{1}{2}}$$

$$= \frac{E}{a}\left[\left(\frac{a-x}{x}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{a-x}{a}\right)^{\frac{1}{2}} + \frac{3}{8}\left(\frac{a-x}{a}\right)^{3/2} + \frac{5}{16}\left(\frac{a-x}{a}\right)^{5/2} + \frac{35}{128}\left(\frac{a-x}{a}\right)^{7/2} + \ldots\right] \quad [7.10.7]$$

The solutions of [7.10.6] and [7.10.7] near x=o and x=a respectively, satisfying the end conditions stated in [7.10.1], are as follows:

(i) for x sufficiently small:

$$z(x) = Ea\left\{\frac{4}{3}\left(\frac{x}{a}\right)^{3/2} + \frac{2}{15}\left(\frac{x}{a}\right)^{5/2} + \left(\frac{16a^2}{105}n + \frac{3}{70}\right)\left(\frac{x}{a}\right)^{7/2} + \left(\frac{8a^2}{945}n + \frac{5}{252}\right)\left(\frac{x}{a}\right)^{9/2} + \ldots\right\}, \quad [7.10.8]$$

(ii) for (a-x) sufficiently small:

$$z(x) = \tan 12 - Ea\left\{\frac{4}{3}\left(\frac{a-x}{a}\right)^{3/2} + \frac{2}{15}\left(\frac{a-x}{a}\right)^{5/2} + \left(\frac{16a^2}{105}n + \frac{3}{70}\right)\left(\frac{a-x}{a}\right)^{7/2} + \left(\frac{8a^2}{945}n + \frac{5}{252}\right)\left(\frac{a-x}{a}\right)^{9/2} + \ldots\right\}.$$

[7.10.9]

Using [7.10.1], [7.10.5] and [7.10.8], we can write the first estimate of z(0.01) as follows:

$$z(0.01) \simeq 0.000053347\,. \quad [7.10.10]$$

Using [7.10.1], [7.10.5] and [7.10.9], we can write the first estimate of z(3.99) as follows:

$$z(3.99) \simeq 0.212503215\,. \quad [7.10.11]$$

Then, equation [7.10.4] can be solved numerically subject to the boundary conditions [7.10.10] and [7.10.11] using the NAG library routine D_{02}ADF [see, e.g., HASELGROVE, C.B. (82)], which solves a two-point boundary value problem for a system of two ordinary differential equations:

$$\frac{dW_i}{dx} = f_i(x, W_1, W_2) \quad , \quad i=1,2 \quad (0.01<x<3.99) \,. \quad [7.10.12]$$

In this method two boundary values of the variable W_1 must be specified, and estimates of the remaining two boundary values of W_2 should be supplied, and the subroutine corrects them by a form of Newton iteration. Starting from the known and estimated values of W_i at $x=0.01$, the subroutine integrates the equations forward to a matching point $R(\equiv 2)$ (using Merson's method, D02ABF), similarly starting from $x=3.99$ integrates backward to R. The difference between the forward and backward values of W_i at R should be zero for the true solution. (These differences are called matching functions below.) The subroutine used a generalized Newton method to reduce the matching functions to zero, by calculating corrections to the estimated boundary values. This process is repeated iteratively until an exact matching is obtained or until the number of iterations exceeds 12.

Using the values of z and z' worked out by this routine, $f_1(n_o,E_o)$ and $f_2(n_o,E_o)$ are now estimated as follows:

$$\left.\begin{aligned} f_1(n_o,E_o) &= \int_0^a \sqrt{1+z^2(x)}\,dx \quad , \\ f_2(n_o,E_o) &= \int_0^a z'^2(x)\,dx \quad . \end{aligned}\right\} \qquad [7.10.13]$$

The values of $f_1(n_o,E_o)$ and $f_2(n_o,E_o)$ will normally differ from the true values of ℓ and K, therefore it is necessary to keep improving the n and E values until the true values of $f_1(n_r,E_r) = \ell$ and $f_2(n_r,E_r) = K$ are reached. This can be achieved by considering the following function of n and E:

$$F(n,E) = (\ell-f_1(n,E))^2 + (K-f_2(n,E))^2 \quad . \qquad [7.10.14]$$

It can be seen that $F(n,E)$ is a nonlinear function of n and E, however $F(n,E)$ can be approximated by a quadratic function of the form

$$F(\underline{V}) = \frac{1}{2}\,(\underline{V}-\underline{V}^{(r)})^T.B^{(r)}.(\underline{V}-\underline{V}^{(r)}) + (\underline{V}-\underline{V}^{(r)})^T.\underline{g}^{(r)} + F(\underline{V}^{(r)}), \qquad [7.10.15]$$

where

$$\underline{V} \equiv \begin{bmatrix} n \\ E \end{bmatrix} , \qquad [7.10.16]$$

and

$$\underline{V}^{(r)} \equiv \begin{bmatrix} n^{(r)} \\ E^{(r)} \end{bmatrix} , \qquad [7.10.17]$$

is the rth stage in iteration process.

[see, e.g. GILL, P.E. and MURRAY, W., (81)2.

In [7.10.15] $g^{(r)}$ and $B^{(r)}$ are matrices involving the first and second derivatives of $F(n,E)$ at $\underline{V}^{(r)}$, namely:

$$\underline{g}^{(r)} \equiv \begin{bmatrix} \left.\dfrac{\partial F(n,E)}{\partial n}\right|_{n=n_r} \\ \left.\dfrac{\partial F(n,E)}{\partial E}\right|_{E=E_r} \end{bmatrix} , \qquad [7.10.18]$$

and

$$B^{(r)} \equiv \begin{bmatrix} \left.\dfrac{\partial^2 F(n,E)}{\partial n^2}\right|_{n=n_r} & \left.\dfrac{\partial^2 F(n,E)}{\partial n \partial E}\right|_{n=n_r, E=E_r} \\ \left.\dfrac{\partial^2 F(n,E)}{\partial E \partial n}\right|_{n=n_r, E=E_r} & \left.\dfrac{\partial^2 F(n,E)}{\partial E^2}\right|_{E=E_r} \end{bmatrix} . \qquad [7.10.19]$$

For the stated function $F(n,E)$, the analytic partial derivatives are not available and therefore the partial derivatives have to be computed by finite-difference approximations.

Two possible choice of finite-difference formula for the approximate derivatives $\bar{g}_i^{(r)}$ are

$$\left.\begin{aligned} h_i^{(r)}\, \bar{g}_i^{(r)} &= F(V^{(r)} + h_i^{(r)} e_i) - F(V^{(r)}) , \\ &\text{and} \\ 2h_i^{(r)}\, \bar{g}_i^{(r)} &= F(V^{(r)} + h_i^{(r)} e_i) - F(V^{(r)} - h_i^{(r)} e_i) , \end{aligned}\right\} \qquad [7.10.20]$$

where e_i denotes the ith column of the identity matrix, $h_i^{(r)}$ remains constant throughout the iterations and the size of each $h_i^{(r)}$ depends upon the scaling of the variables.

Now, suppose that the function $F(\underline{V})$ of equation [7.10.15] takes its minimum values at

$$V = V^{(r+1)} \quad , \tag{7.10.21}$$

then

$$\nabla F(\underline{V}^{(r+1)}) = B^{(r)}(\underline{V}^{(r+1)}-\underline{V}^{(r)})+\underline{g}^{(r)} = o \quad , \tag{7.10.22}$$

hence

$$\underline{V}^{(r+1)} = \underline{V}^{(r)} - (B^{(r)})^{-1}.\underline{g}^{(r)} \; . \tag{7.10.23}$$

Taking

$$\underline{P}^{(r)} = \underline{V}^{(r+1)} - \underline{V}^{(r)} \quad , \tag{7.10.24}$$

then

$$B^{(r)}\underline{P}^{(r)} = - \underline{g}^{(r)} \tag{7.10.25}$$

The matrix $B^{(r)}$ is expressed in the following form

$$L^{(r)}D^{(r)}L^{(r)T}\underline{P}^{(r)} = - \underline{g}^{(r)} \quad , \tag{7.10.26}$$

where $L^{(r)}$ is a unit-lower triangular matrix, and $D^{(r)}$ a diagonal matrix. The vector $P^{(r)}$ can be found by solving successively

$$\left.\begin{aligned} & L^{(r)}\underline{V} = - \underline{g}^{(r)} \quad , \\ \text{and} \quad & \\ & L^{(r)T}\underline{P}^{(r)} = (D^{(r)})^{-1}\underline{V} \, , \end{aligned}\right\} \tag{7.10.27}$$

using a forward and backward substitution. This is a summary of the NAG library routine E_{04}CEF for minimising the function $F(n,E)$, [7.10.13] of the two independent variables n and E , using the quasi-Newton method due to GILL, MURRAY and PITFIELD.

In this way the following optimum computed values of n and E are obtained:

$$\left.\begin{aligned} n &= - 0.256414 \quad , \\ E &= + 0.07101039 \; , \end{aligned}\right\} \tag{7.10.28}$$

then, on substituting into [7.10.8] and [7.10.9] we obtain

$$\left.\begin{aligned} z(0.01) &\simeq 0.0000473521 \quad , \\ z(3.99) &\simeq 0.212509209 \quad . \end{aligned}\right\} \qquad [7.10.29]$$

Equation [7.10.29] represents the improved values of $z(0.01)$ and $z(3.99)$ and knowing these improved values, the whole method is repeated to obtain a further improvement until the optimum values of $z(0.01)$ and $z(3.99)$ are reached.

The graphs of $y(x)$, $y'(x)$ $(\equiv z(x))$ and $y''(x)$ are shown in Figs.17, 18 and 19 respectively.

There is reasonable agreement between the values of n and E obtained using the numerical method and the Rayleigh-Ritz method.

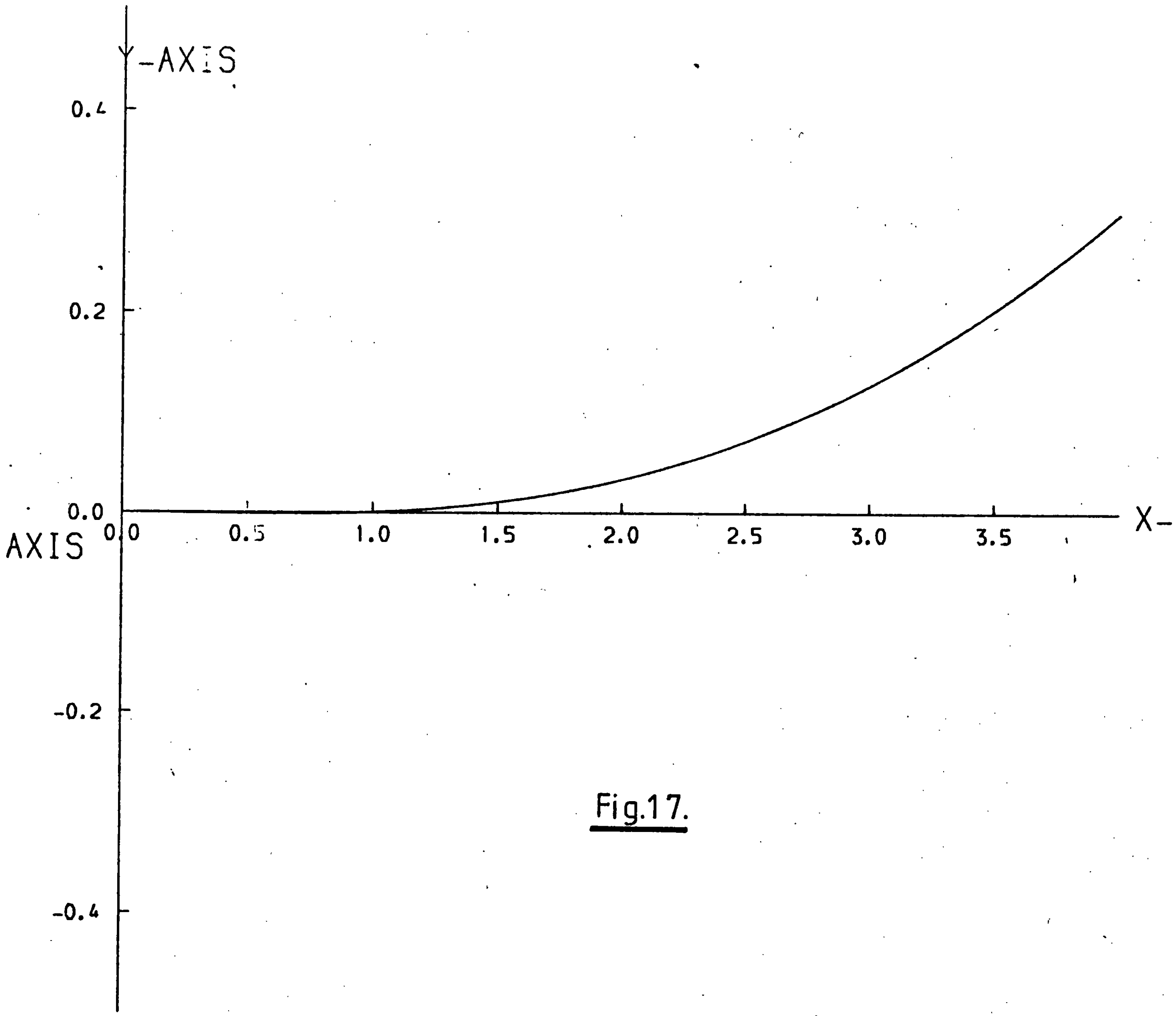

Fig.17.

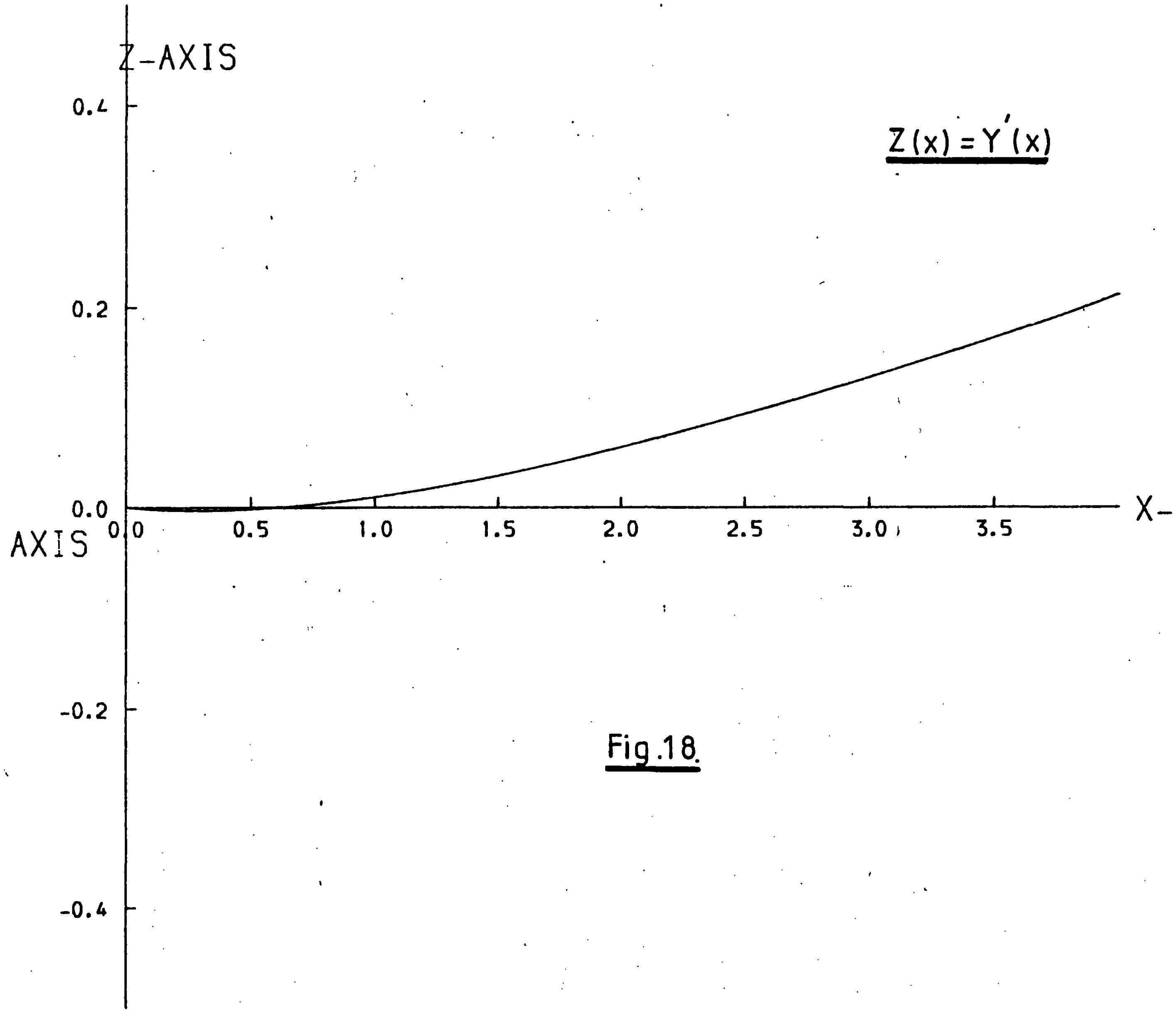

Fig.18.

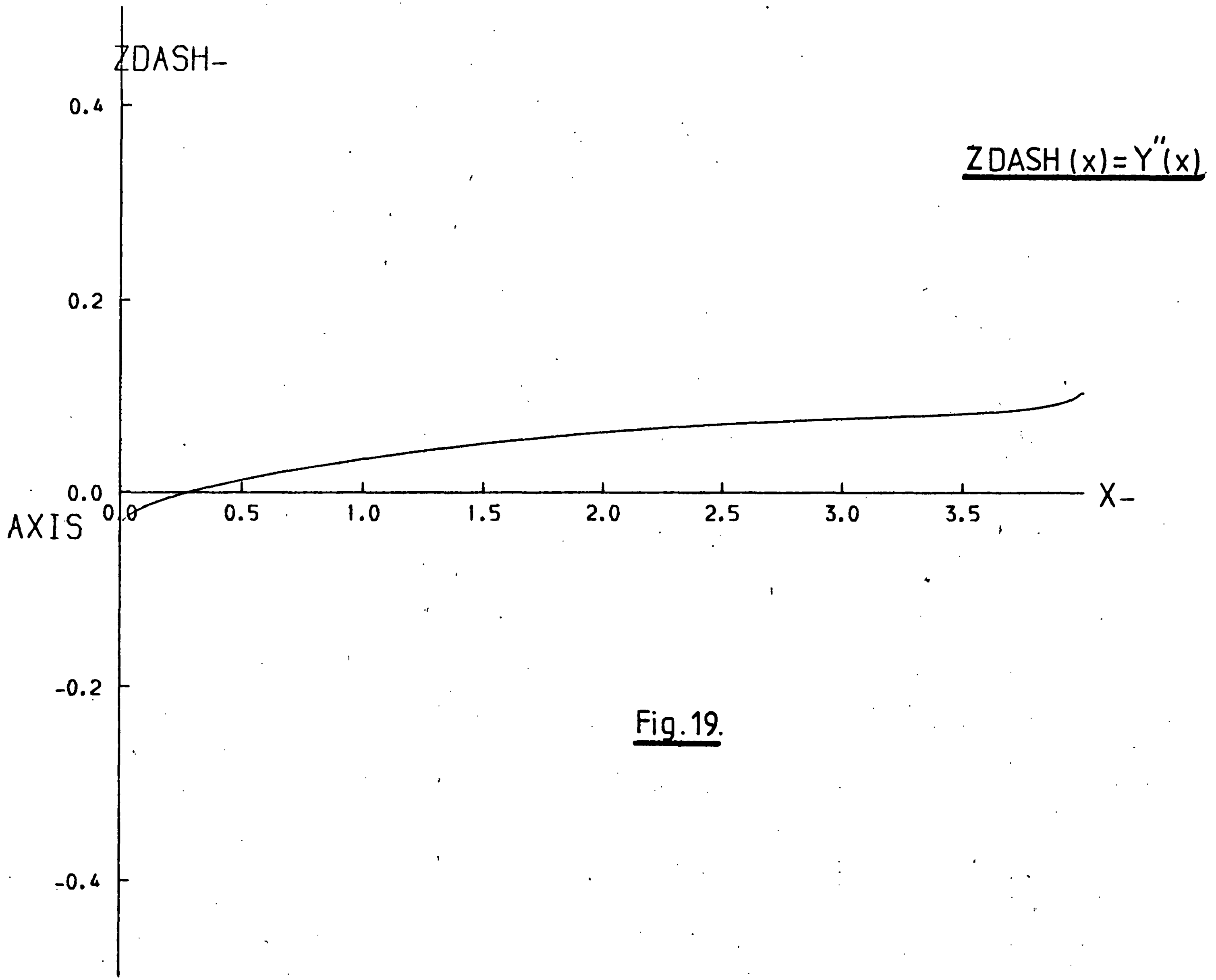

Fig. 19.

APPENDIX XI The Numerical Solution of the Non-Cavity Problem (Maximum Lift), of Chapter II.

We consider here the problem

$$z''(x) - n \frac{z(x)}{\sqrt{1+z^2(x)}} = E\sqrt{\frac{x}{a-x}} \quad , \; [n = \frac{\lambda_1}{2\lambda_2} \; , \; E = \frac{1}{\lambda_2} \; , \; z(o)=o \; , z(a)=-\tan 12^\circ, (o<x<a=4)] \qquad [7.11.1]$$

subject to the constraints

$$\left.\begin{aligned} \ell &= 4.02 = \int_o^a \sqrt{1+z^2(x)}\,dx \quad , \\ K &= 0.0148 = \int_o^a z'^2(x)\,dx \quad . \end{aligned}\right\} \qquad [7.11.2]$$

We transform [7.11.1] into the simultaneous first order differential equations

$$\left.\begin{aligned} \frac{dW_1(x)}{dx} &= W_2(x) \quad , \qquad [W_1(x) = z(x) \; , \; W_2(x) = z'(x)] \; , \\ \frac{dW_2(x)}{dx} &= \frac{n\,W_1(x)}{\sqrt{1+W_1{}^2(x)}} + \Psi(x) \quad , \end{aligned}\right\} \qquad [7.11.3]$$

where

$$\Psi(x) = E\sqrt{\frac{x}{a-x}} \quad . \qquad [7.11.4]$$

Equation [7.11.3] has the same structure as [7.10.3], therefore we solve equation [7.11.3] by the same method stated in Appendix X. The graphs of this numerical method for $y(x)$, $y'(x)$ ($\equiv z(x)$) and $y''(x)$ are shown in Figs. 20, 21 and 22 respectively. There is reasonable agreement between the values of n and E obtained using the numerical solution and the Rayleigh-Ritz method, namely:

$$\left.\begin{aligned} n &= -\,1.7852 \quad , \\ E &= -\,0.13796 \quad . \end{aligned}\right\} \qquad [7.11.5]$$

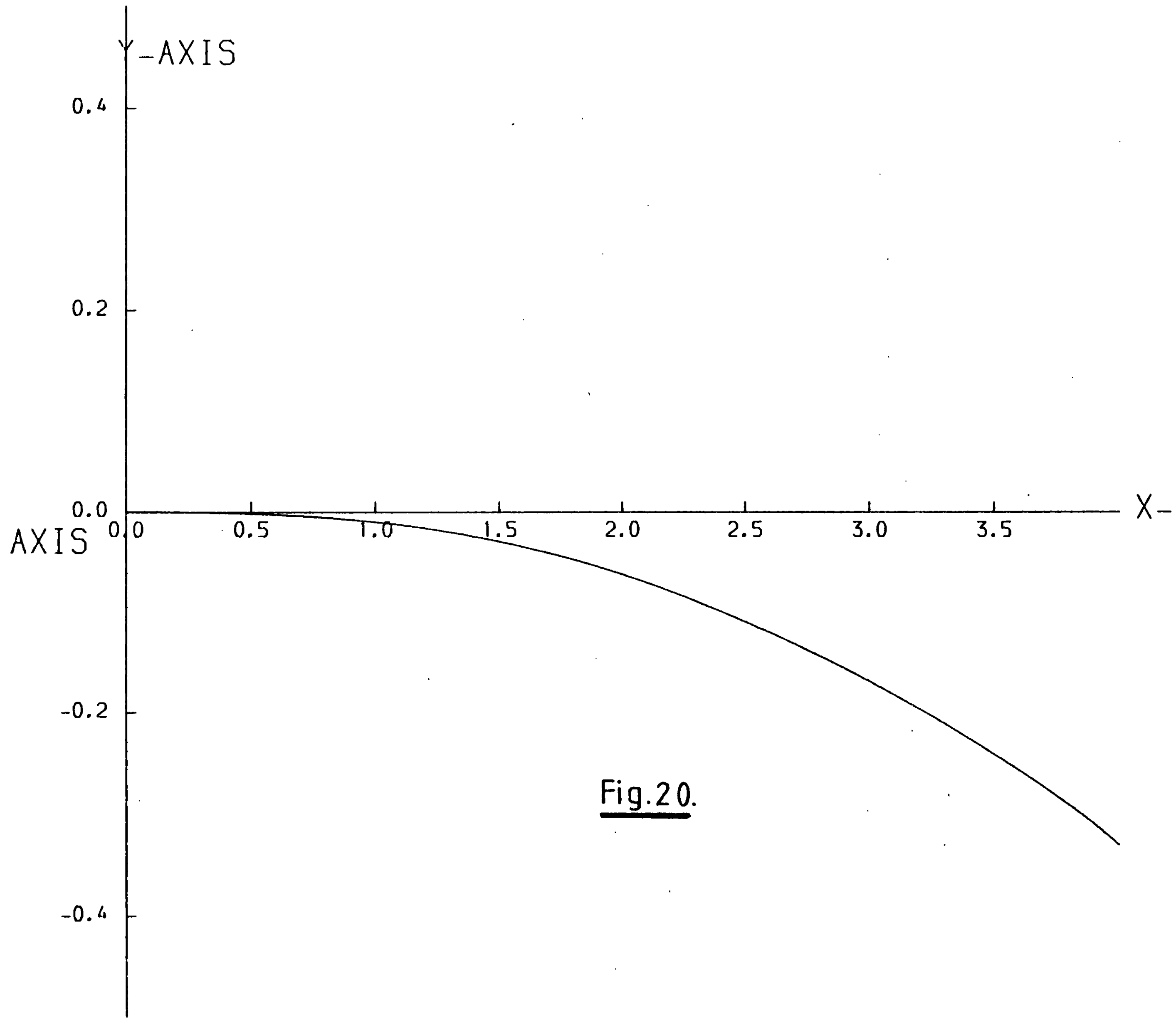

Fig.20.

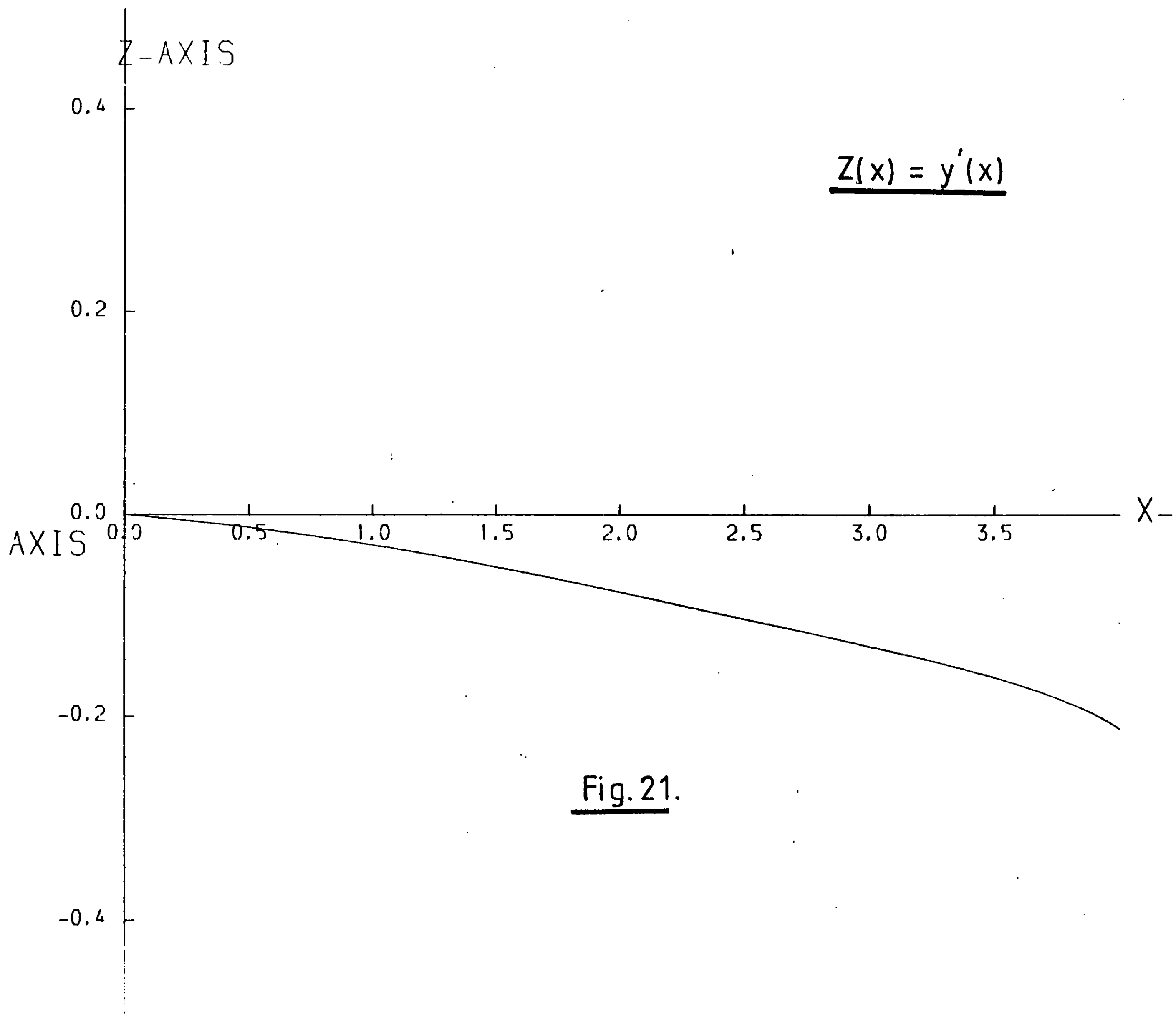

Fig. 21.

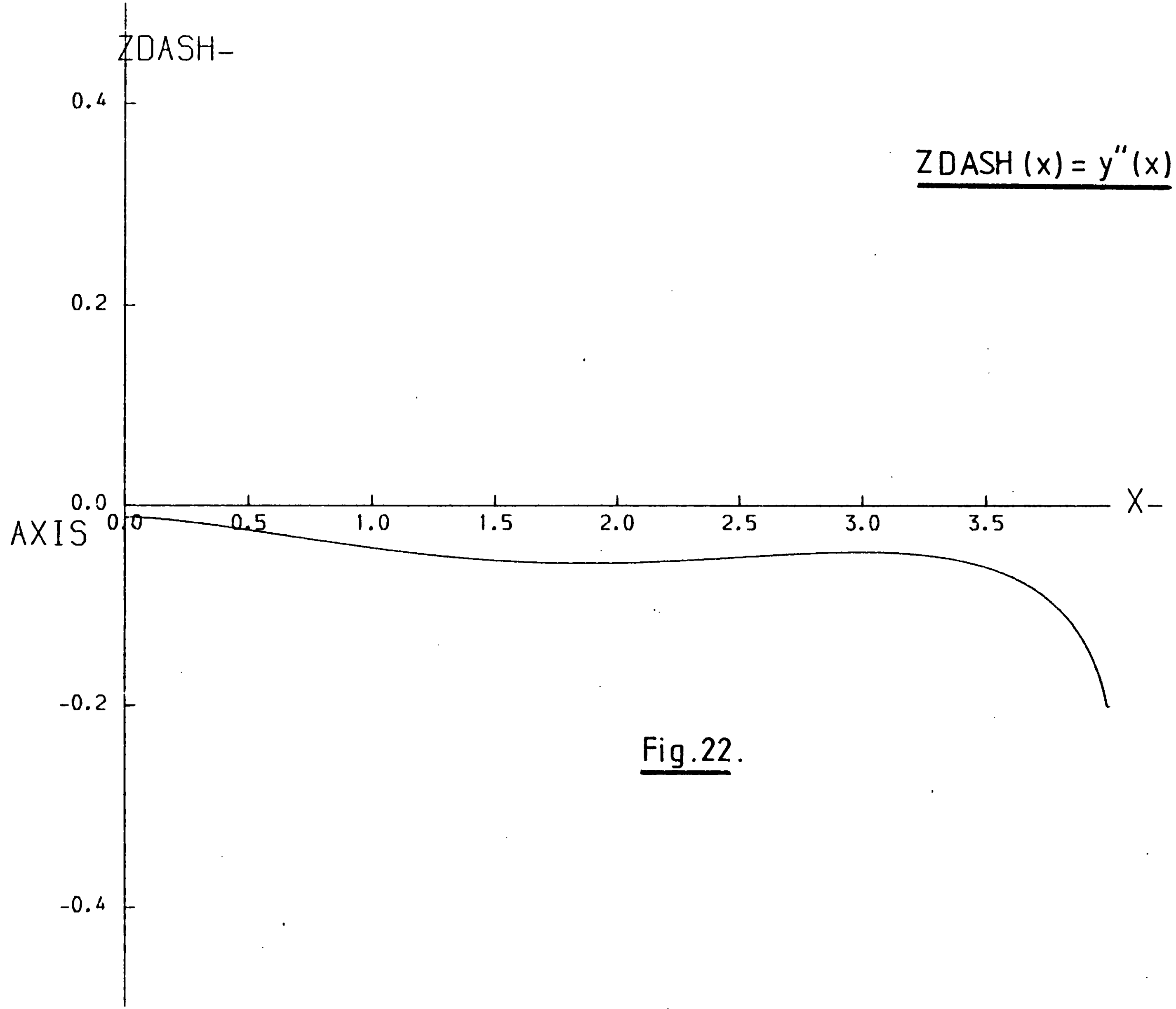

Fig.22.

APPENDIX XII The Numerical Solution of the Full-Cavity Problem (Maximum Lift), of Chapter III.

We consider here the problem

$$z''(x) = \frac{n\,z(x)}{\sqrt{1+z^2(x)}} = -\frac{E}{2\sqrt{2}}\sqrt[4]{\frac{\ell}{\ell-c}}\cdot\sqrt{\frac{\sqrt{\frac{x}{\ell-x}}}{\sqrt{\frac{c}{\ell-c}}-\sqrt{\frac{x}{\ell-x}}}}\cdot\left[\sqrt{1-\sqrt{\frac{\ell-c}{\ell}}\sqrt{\frac{\ell-x}{x}}}-\sqrt{1+\sqrt{\frac{\ell-c}{\ell}}}\right]$$

$$\left[n=\frac{\lambda_1}{2\lambda_2}\;,\;E=\frac{1}{\lambda_2}\;,\;z(o)=o,\;z(c)=-\tan 12^{\circ},o<x<c=4<\ell)\right] \quad [7.12.1]$$

subject to the constraints

$$\left.\begin{aligned} S &= 4.02 = \int_o^c \sqrt{1+z^2(x)}\,dx \quad , \\ K &= 0.0148 = \int_o^c z'^2(x)\,dx \quad . \end{aligned}\right\} . \qquad [7.12.2]$$

We transform [7.12.1] into the simultaneous first order differential equations

$$\left.\begin{aligned} \frac{dW_1(x)}{dx} &= W_2(x)\;, \qquad [W_1(x)=z(x),\;W_2(x)=z'(x)]\;, \\ \frac{dW_2(x)}{dx} &= \frac{n\,W_1(x)}{\sqrt{1+W_1^2(x)}} + \Psi(x)\quad , \end{aligned}\right\} (o<x<c<\ell)\;, \qquad [7.12.3]$$

where

$$\Psi(x) = -\frac{E}{2\sqrt{2}}\sqrt[4]{\frac{\ell}{\ell-c}}\cdot\sqrt{\frac{\sqrt{\frac{x}{\ell-x}}}{\sqrt{\frac{c}{\ell-c}}-\sqrt{\frac{x}{\ell-x}}}}\cdot\left[\sqrt{1-\sqrt{\frac{\ell-c}{\ell}}\sqrt{\frac{\ell-x}{x}}}-\sqrt{1+\sqrt{\frac{\ell-c}{\ell}}}\right] . \quad [7.12.4]$$

This problem has been solved using the same method as in Appendix X for the two different values $\ell=7$ and $\ell=100$. The values of n and E obtained for the two different values of ℓ are:

$n = 1.3$, $E = 0.3$ where $\ell = 7$,

$n = 1.93$, $E = 0.388$ where $\ell = 100$.

For $\ell=7$ and $\ell=100$, $y(x)$, $y'(x)$ ($\equiv z(x)$) and $y''(x)$ are plotted in Figs. 23, 24, 25, 26, 27 and 28 respectively.

In Chapter III the problem with $\ell=100$ has been solved using the Rayleigh-Ritz method. There is reasonable agreement between the values of n and E obtained using the numerical method and the Rayleigh-Ritz method.

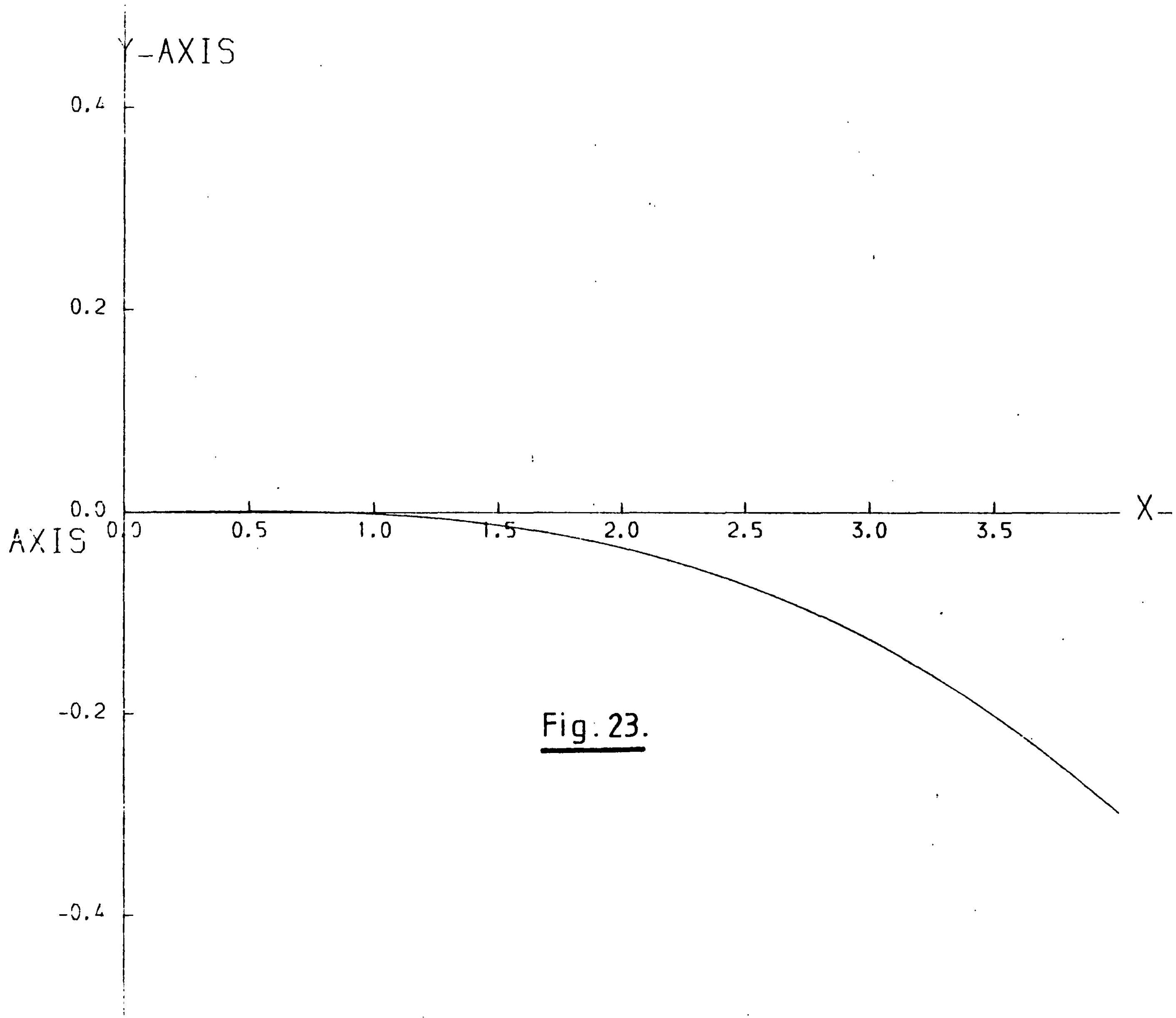

Fig. 23.

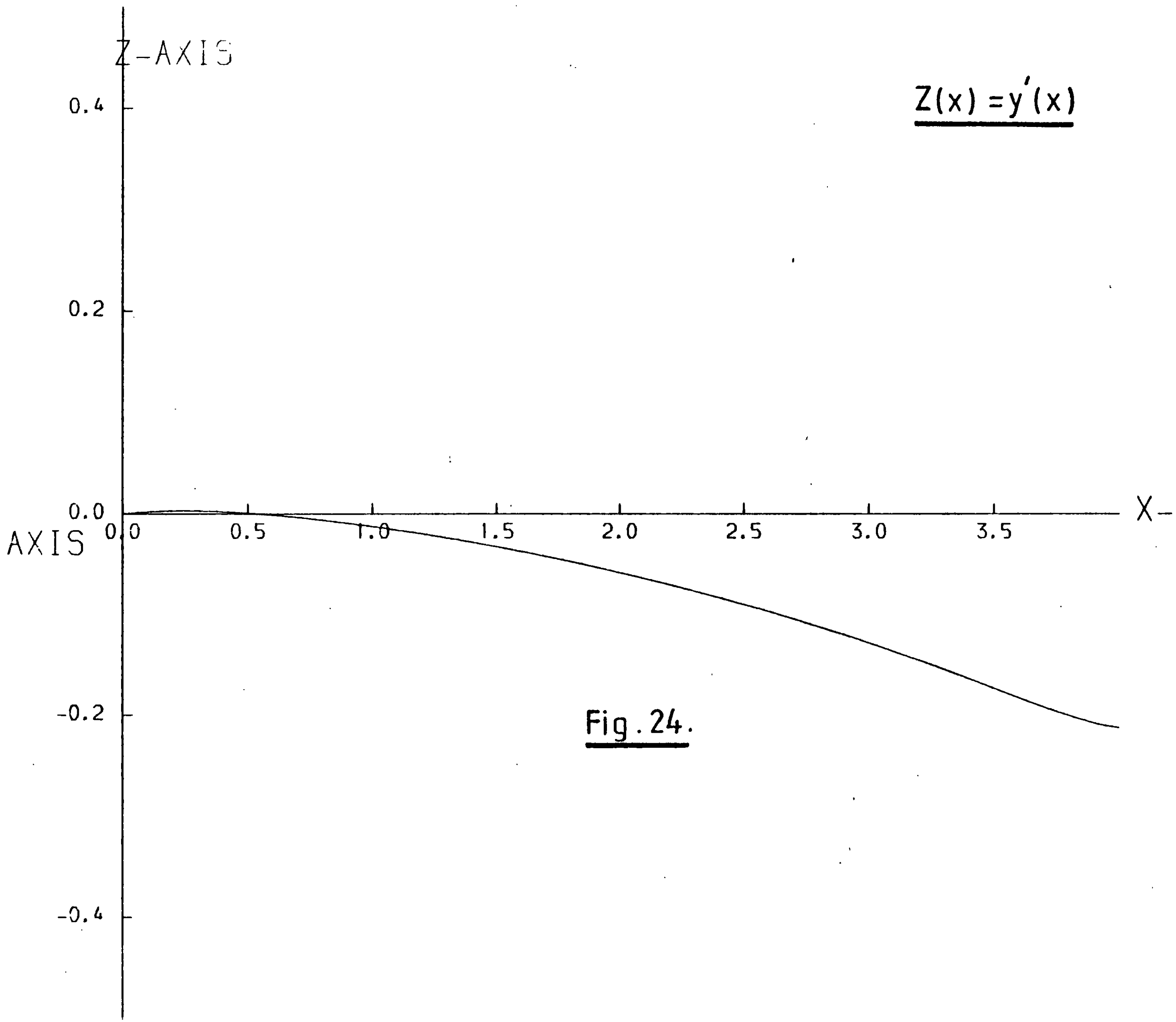

Fig. 24.

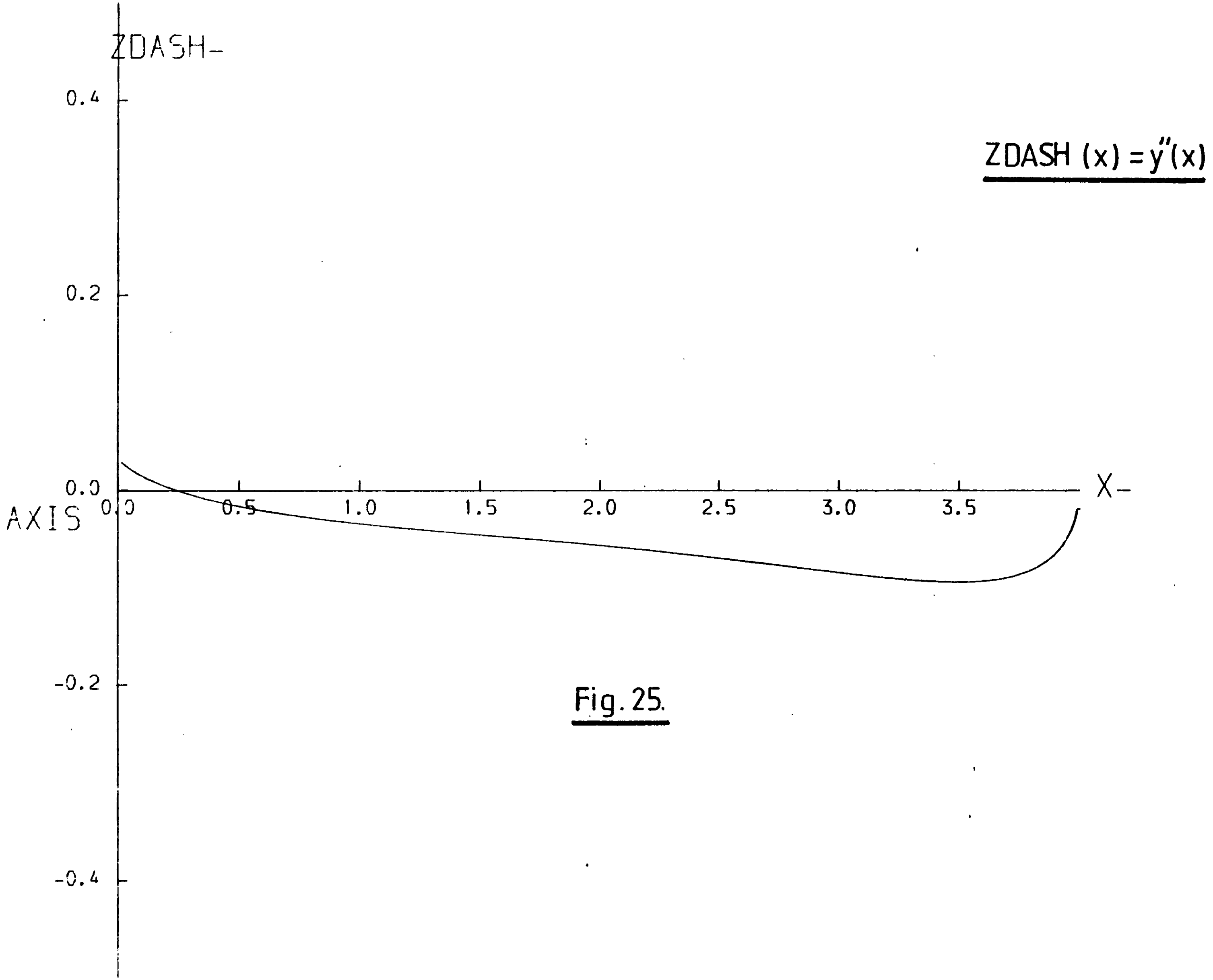

Fig. 25.

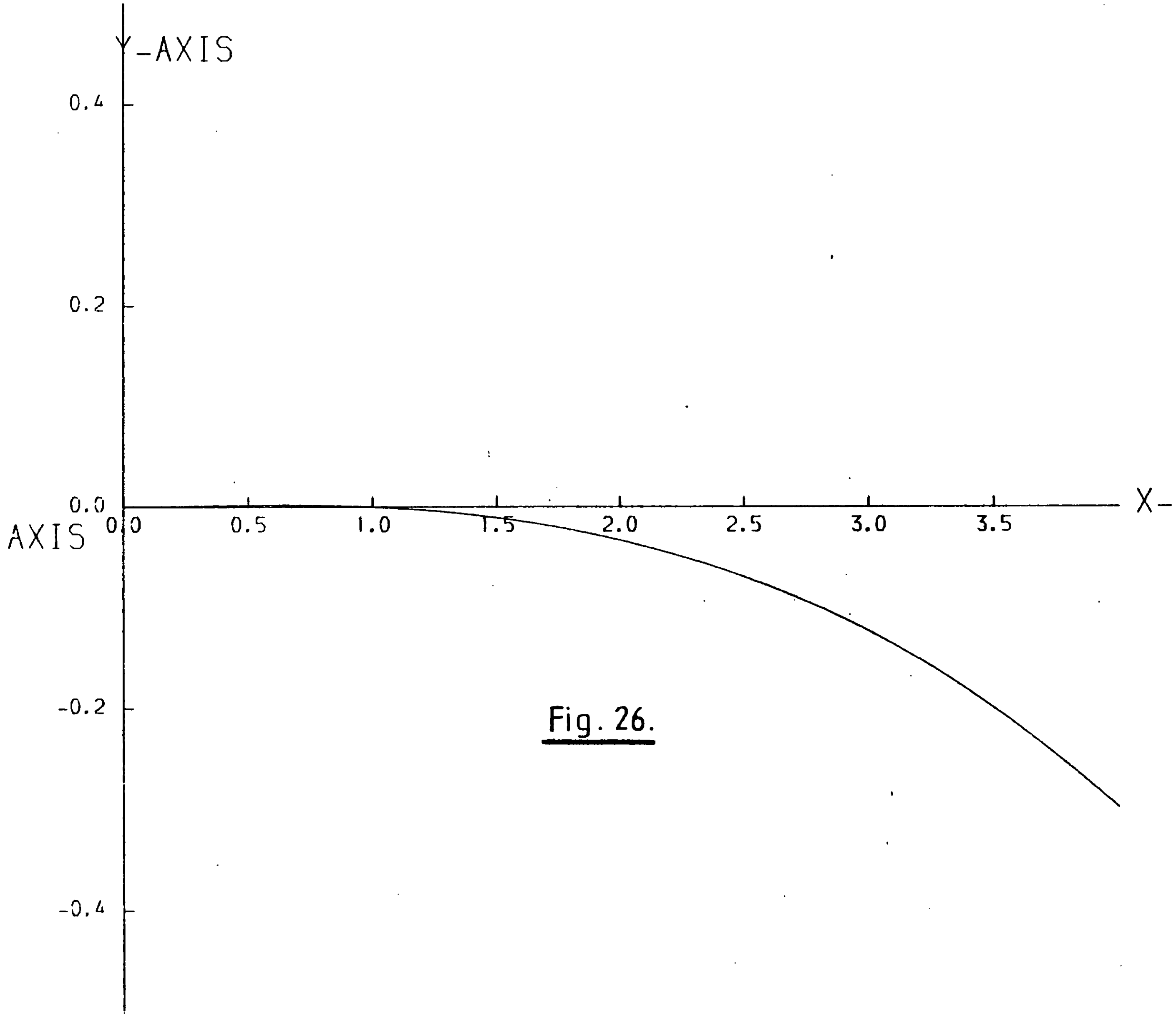

Fig. 26.

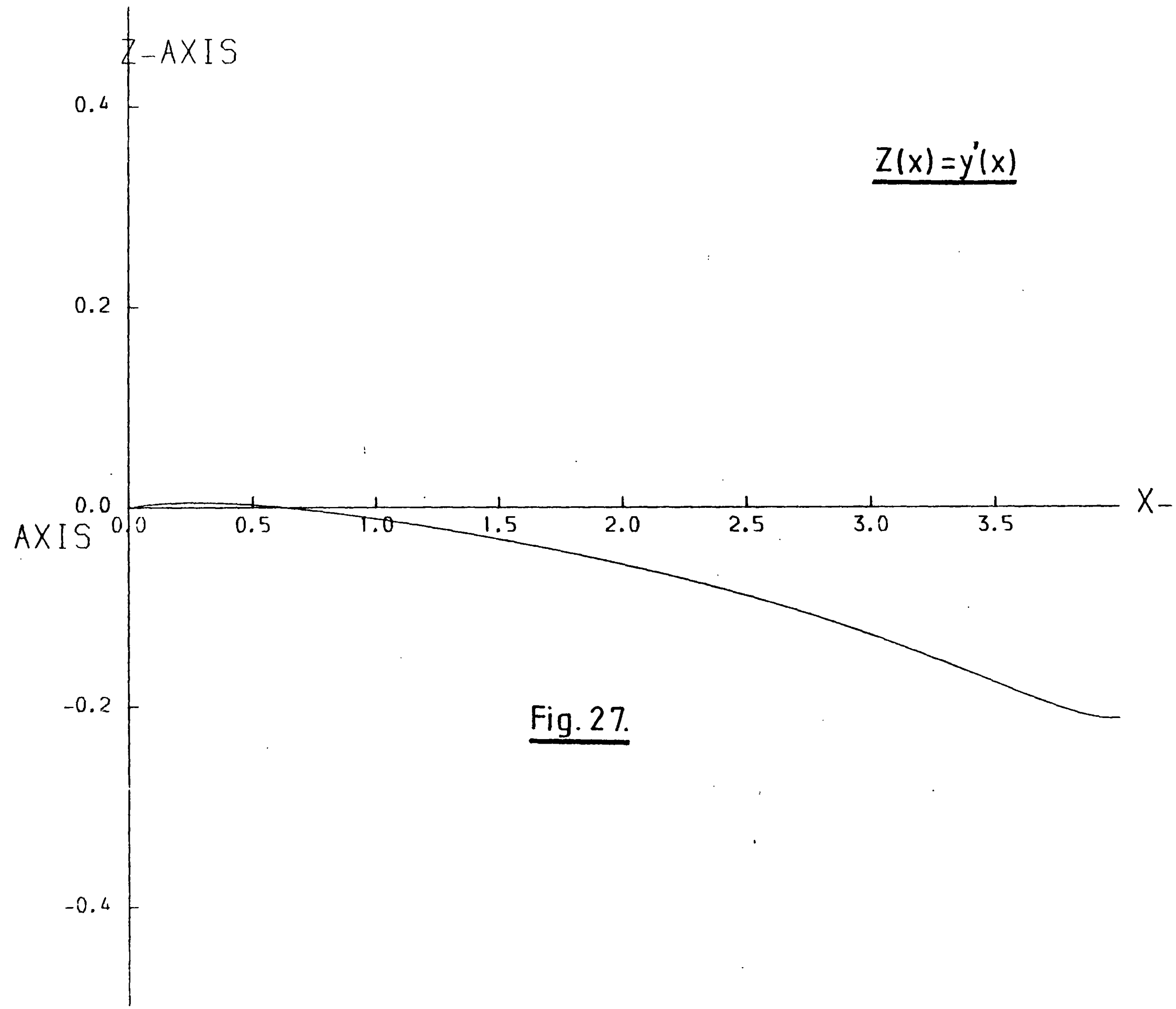

Fig. 27.

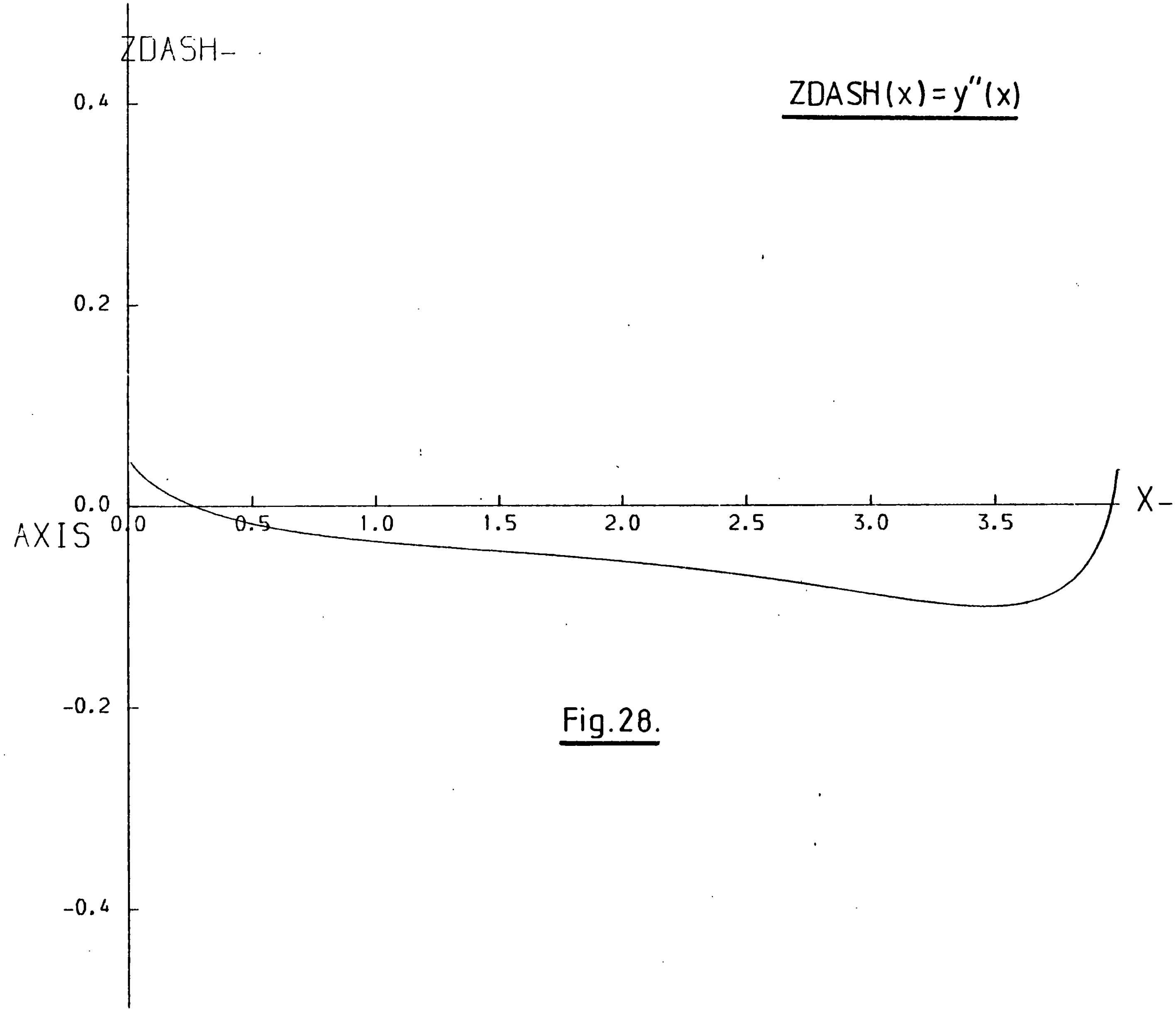

Fig.28.

APPENDIX XIII The Numerical Solution of the Partial-Cavity (Maximum Lift), of Chapter IV.

We consider here the problem

$$z''(x) - n\, z(x) \equiv \begin{cases} 0 & (0 < x < \ell = 4) \\ \dfrac{E}{\sqrt{2}\sqrt{1+\sqrt{\dfrac{c-\ell}{c}}}} \cdot \sqrt{\dfrac{x-\ell}{c-x}} \left[\sqrt{\dfrac{\sqrt{\dfrac{x}{c-x}} - \sqrt{\dfrac{\ell}{c-\ell}}}{\sqrt{\dfrac{x}{c-x}}}} + \sqrt{\dfrac{\sqrt{\dfrac{x}{c-x}} + \sqrt{\dfrac{\ell}{c-\ell}}}{\sqrt{\dfrac{x}{c-x}}}} \right] & (\ell = 4 < x < c = 7)\ , \end{cases}$$

$$\left[n = \frac{\lambda_2}{2\lambda_1}\ ,\ E = \frac{1}{\lambda_1}\ ,\ z(0) = 0\ ,\ z(c) = -\tan 12^\circ\right] \qquad [7.13.1]$$

subject to the constraints

$$\left.\begin{aligned} S = 7.02 &= \int_0^c \sqrt{1+z^2(x)}\,dx\ , \\ K = 0.0148 &= \int_0^c z'^2(x)\,dx\ . \end{aligned}\right\} \qquad [7.13.2]$$

We transform [7.13.1] into the simultaneous first order differential equations

$$\left.\begin{aligned} \frac{dW_1(x)}{dx} &= W_2(x)\ , \qquad [W_1(x) = z(x),\ W_2(x) = z'(x)] \\ \frac{dW_2(x)}{dx} &= \frac{n\, W_1(x)}{\sqrt{1+W_1{}^2(x)}} + \Psi(x)\ , \end{aligned}\right\} \qquad [7.13.3]$$

where

$$\left.\Psi(x) \equiv \begin{cases} 0 & (0 < x < \ell)\ , \\ \dfrac{E}{\sqrt{2}\sqrt{1+\sqrt{\dfrac{c-\ell}{\ell}}}} \cdot \sqrt{\dfrac{x-\ell}{c-x}} \left[\sqrt{\dfrac{\sqrt{\dfrac{x}{c-x}} - \sqrt{\dfrac{\ell}{c-\ell}}}{\sqrt{\dfrac{x}{c-x}}}} + \sqrt{\dfrac{\sqrt{\dfrac{x}{c-x}} + \sqrt{\dfrac{\ell}{c-\ell}}}{\sqrt{\dfrac{x}{c-x}}}} \right], & (\ell < x < c). \end{cases}\right\} \qquad [7.13.4]$$

Equation [7.13.3] has the same structure as [7.10.3], therefore we solve equation [7.13.3] by the same method stated in Appendix X.

The values of n and E are

$$\left.\begin{aligned} n &= 1.5 \quad , \\ E &= 0.09 \quad . \end{aligned}\right\} \qquad [7.13.5]$$

The graphs of this numerical method for $y(x)$, $y'(x)$ ($\equiv z(x)$) and $y''(x)$ are shown in Figs. 29, 30 and 31 respectively.

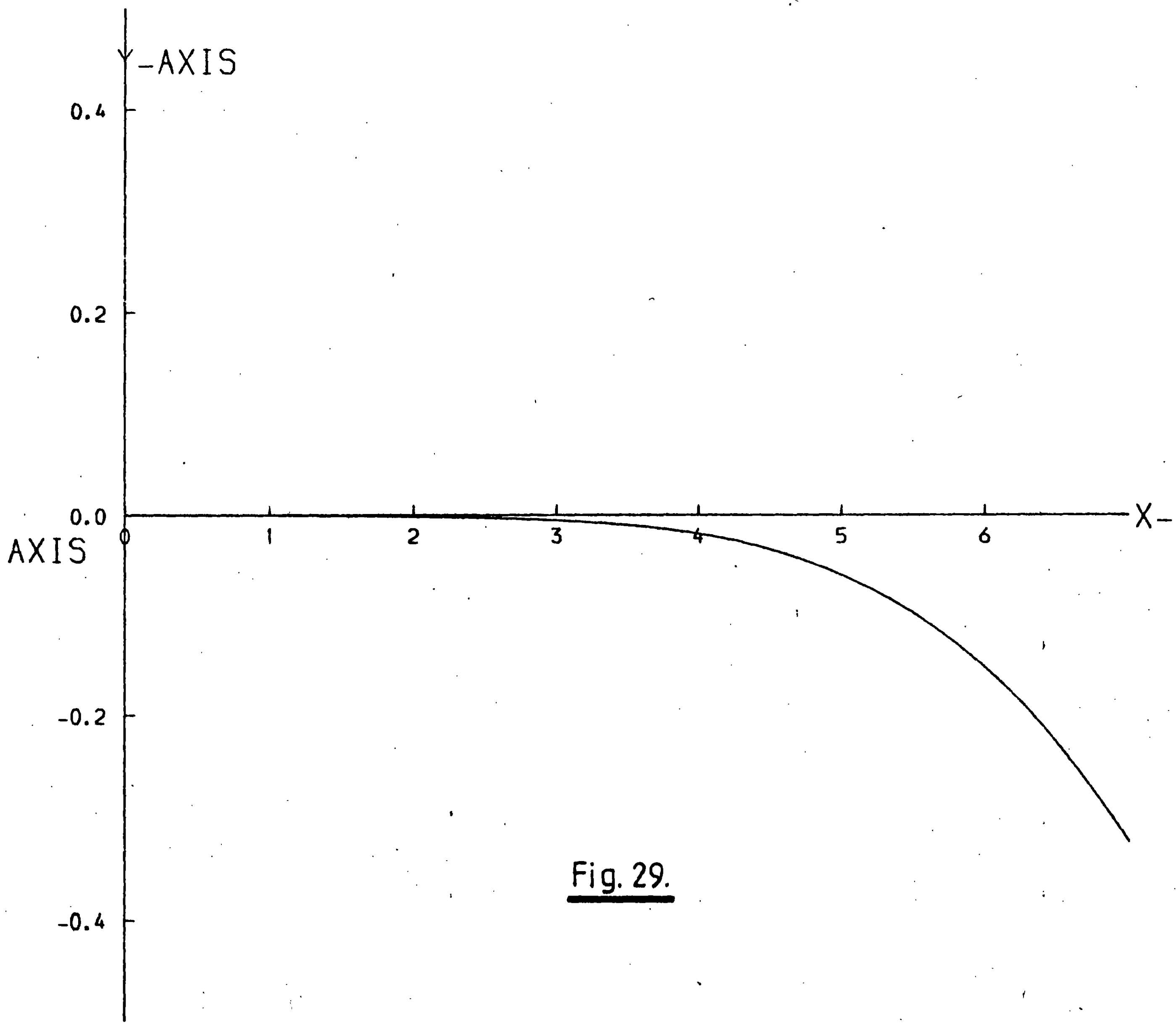

Fig. 29.

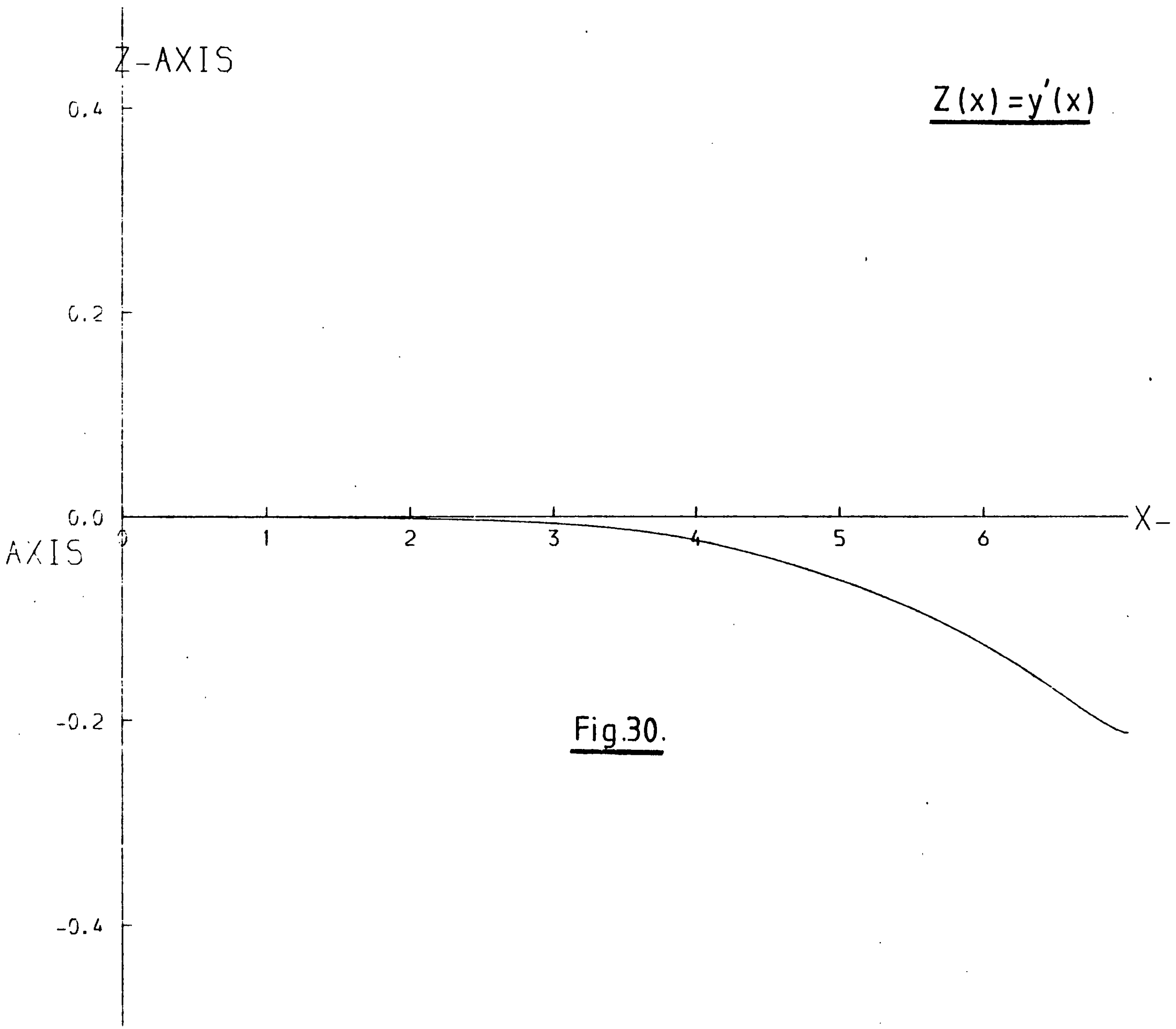

Fig.30.

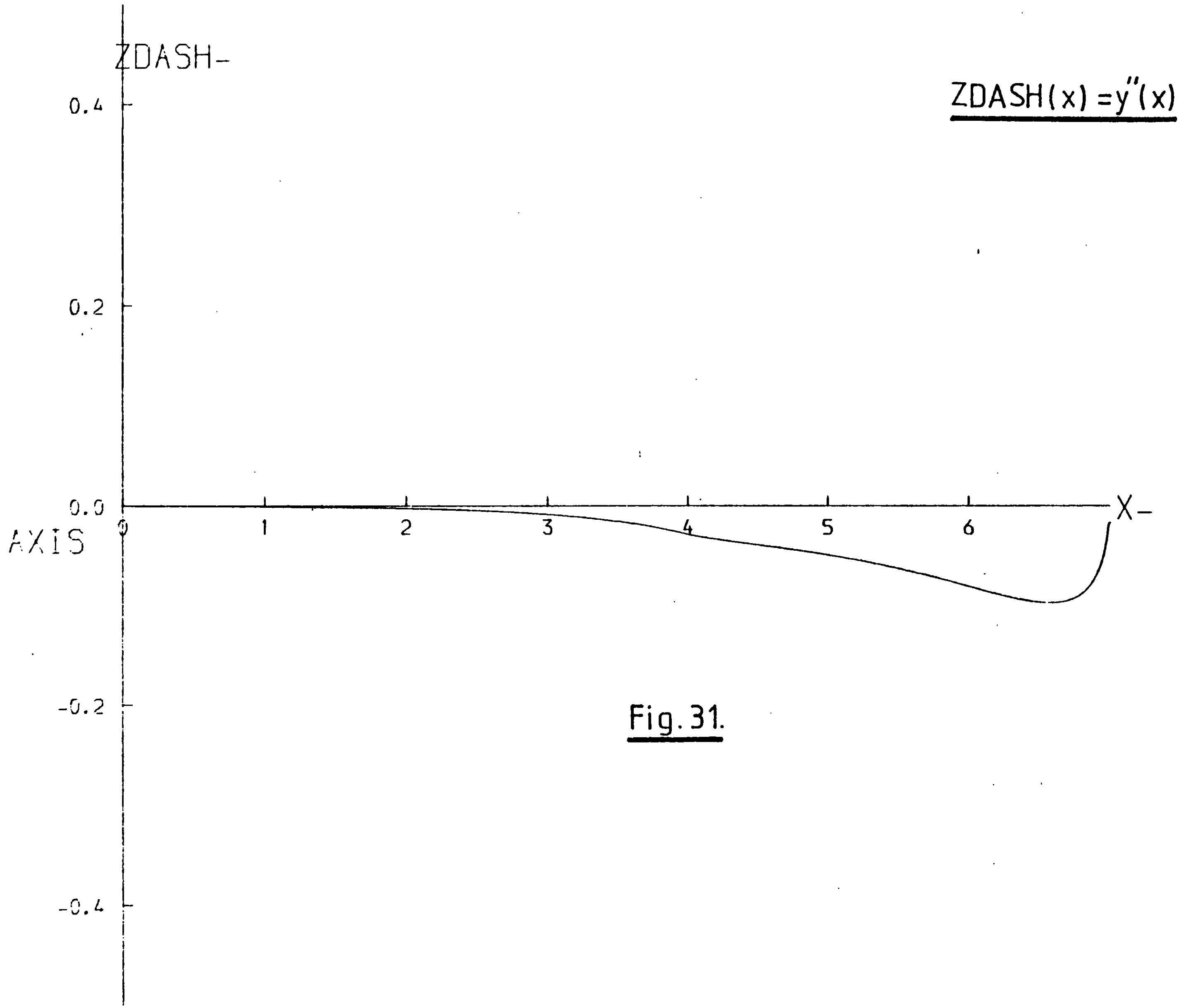

Fig. 31.

APPENDIX XIV The Numerical Solution of the Non-Cavity Problem Beneath a Free Surface (Minimum Drag) of Chapter VI.

We consider here the problem

$$z_1''(x) - n_1 \frac{z_1(x)}{\sqrt{1+z_1^2(x)}} = \frac{1}{\sqrt{x(a-x)}}\{E_1+D_1(C_o[\tfrac{\pi}{8}\alpha.(a^2-8ax+8x^2)-\pi\nu(a-2x)e^{-2\nu h}] +$$

$$+ 4\nu A_o.(a-2x)e^{-2\nu h}\} ,$$

$$[z_1(o)=o, z_1(a)=\tan 12^\circ, z_1'(o)=o, n_1=\frac{\lambda_1^{(1)}}{2\lambda_2^{(1)}}, E=\frac{C_1}{2\lambda_2^{(1)}}, D=\frac{1}{2\lambda_2^{(1)}}, (o<x<a)] , \qquad [7.14.1]$$

with

$$\left.\begin{aligned} A_o &= \int_o^a \sqrt{\frac{s}{a-s}}\; z_o(s)\,ds \\ &= 0.81244 , \\ \alpha &= \frac{1}{4h^2} - 2\nu\int_o^\infty \frac{\sin\nu t\,dt}{t^2+4h^2} \\ &= 0.0007071 , \\ \nu &= \frac{g}{U^2} = 0.0092975\ ft^{-1} , \\ a &= 4\ ft,\ h = 16\ ft. \end{aligned}\right\} \qquad [7.14.2]$$

In [7.14.2] $z_o(s)$ is the solution of the differential equation

$$z_o''(x) - n_o \frac{z_o(x)}{\sqrt{1+z_o^2(x)}} = \frac{E_o}{\sqrt{x(a-x)}} ,$$

$$[n_o = \frac{\lambda_1^{(o)}}{2\lambda_2^{(o)}}, E_o = \frac{C_o}{2\lambda_2^{(o)}}, z_o(o)=o, z_o(a)=\tan 12^\circ, (o<x<a)] , \qquad [7.14.3]$$

which has been solved in Appendix X.

Equation [7.14.1] can be written as follows:

$$z_1''(x) - n_1 \frac{z_1(x)}{\sqrt{1+z_1^2(x)}} = \frac{1}{\sqrt{x(a-x)}} \{B_o+B_1x+B_2x^2\} ,$$

$$[z_1(o)=o,\ z_1(a)=\tan 12^\circ;\ z_1'(o)=o,\ (o < x < a)] , \qquad [7.14.4]$$

with

$$\left.\begin{aligned} B_o &= E_1+D_1[(\tfrac{\pi}{8}\, a^2\alpha-\pi\nu ae^{-2\nu h})C_o+4\nu ae^{-2\nu h}A_o] , \\ B_1 &= D_1[(-\pi\alpha a+2\pi\nu e^{-2\nu h})C_o-8\nu A_oe^{-2\nu h}] , \\ B_2 &= \pi\alpha C_oD_1 . \end{aligned}\right\} \qquad [7.14.5]$$

Equation [7.14.4] will be solves subject to the stated boundary conditions together with the following constraints:

$$\left.\begin{aligned} \ell &= 4.02 = \int_o^a \sqrt{1+z_1'^2(x)}\,dx , \\ K &= 0.0148 = \int_o^a z_1'^2(x)\,dx , \\ y_o &= 0.3 = \int_o^a z_1(x)\,dx . \end{aligned}\right\} \qquad [7.14.6]$$

We transform [7.14.4] into the simultaneous first order differential equations

$$\left.\begin{aligned} \frac{dW_1(x)}{dx} &= W_2(x) , \quad [W_1(x)=z_1(x), W_2(x)=z_1'(x)] , \\ \frac{dW_2(x)}{dx} &= \frac{n_1W_1(x)}{\sqrt{1+W_1^2(x)}} + \Psi(x) , \end{aligned}\right\} \qquad [7.14.7]$$

where

$$\Psi(x) = \frac{1}{\sqrt{x(a-x)}} [B_o+B_1x+B_2x^2] . \qquad [7.14.8]$$

We consider first the end points $x=o$ and $x=a$ at which the right-hand side of [7.14.4] possesses singularities. First we guess initial values for n_1, E_1, D_1 and C_o :

$$\left.\begin{aligned} n_1^{(o)} &= -\ 0.25 \quad , \\ E_1^{(o)} &= 0.07 \quad , \\ D_1^{(o)} &= 0.001 \quad , \\ C_o^{(o)} &= .0.01 \quad . \end{aligned}\right\} \qquad [7.14.9]$$

We now estimate $z_1(0.01)$, $z_1'(0.01)$ and $z_1(3.99)$ using Taylor's theorem. We write equation [7.14.4] in the form

(i) for x sufficiently small:

$$z_1''(x) - n_1 z_1(x) = \frac{1}{\sqrt{a,x}} \left[1 - \frac{x}{a}\right]^{-\frac{1}{2}} \{B_o + B_1 a(\tfrac{x}{a}) + B_2 a^2 (\tfrac{x}{a})^2\}$$

$$= \frac{1}{a} \{B_o (\tfrac{x}{a})^{-\frac{1}{2}} + [\tfrac{1}{2}B_o + B_1 a](\tfrac{x}{a})^{\frac{1}{2}} + [\tfrac{3}{8}B_o + \tfrac{1}{2}B_1 a + B_2 a^2](\tfrac{x}{a})^{3/2} +$$

$$+ [\tfrac{1}{2}B_2 a^2 + \tfrac{3}{8}B_1 a + \tfrac{5}{16} B_o](\tfrac{x}{a})^{5/2} + \ldots \} , \qquad [7.14.10]$$

where

$$\left.\begin{aligned} B_o &= 0.07009 \quad , \\ B_1 &= -\ 0.0000447 \quad , \\ B_2 &= 0.15707 \times 10^{-10} \quad , \end{aligned}\right\} \qquad [7.14.11]$$

(ii) for $(a-x)$ sufficiently small

$$z_1''(x) - n_1 z_1(x) = \frac{1}{\sqrt{a(a-x)}} \left[1 - (\tfrac{a-x}{a})\right]^{-\frac{1}{2}} \{\gamma_o + \gamma_1 (\tfrac{a-x}{a}) + \gamma_2 (\tfrac{a-x}{a})^2\}$$

$$= \frac{1}{a} \{\gamma_o (\tfrac{a-x}{a})^{-\frac{1}{2}} + [\tfrac{1}{2}\gamma_o + \gamma_1 a](\tfrac{a-x}{a})^{\frac{1}{2}} + [\tfrac{3}{8}\gamma_o + \tfrac{1}{2}\gamma_1 a + \gamma_2 a^2](\tfrac{a-x}{a})^{3/2} +$$

$$+ [\tfrac{1}{2}\gamma_2 a^2 + \tfrac{3}{8}\gamma_1 a + \tfrac{5}{16}\gamma_o](\tfrac{a-x}{a})^{5/2} + \ldots\ldots\} , \qquad [7.14.12]$$

where

$$\left.\begin{aligned}\gamma_o &= B_o+B_1a+B_2a \\ &= 0.06991 \quad , \\ \gamma_1 &= -B_1a-2a^2B_2 \\ &= 0.0001787 \quad , \\ \gamma_2 &= B_2a^2 \\ &= 0.25 \times 10^{-9} \quad .\end{aligned}\right\} \qquad [7.14.13]$$

The solutions [7.14.10] and [7.14.12] near $x=o$ and $x=a$ respectively, satisfying the end conditions stated in [7.14.4], are as follows

(i) for x sufficiently small:

$$z_1(x) = a\{\frac{4}{3}B_o(\frac{x}{a})^{3/2} + \frac{4}{15}(\frac{1}{2}B_o+B_1a)(\frac{x}{a})^{5/2} + \frac{4}{35}(\frac{4}{3}B_on+\frac{3}{8}B_o+\frac{1}{2}B_1a+B_2a^2)(\frac{x}{a})^{7/2} +$$

$$+\frac{4}{63}[\frac{4}{15}(\frac{1}{2}B_o+B_1a)n+\frac{1}{2}B_2a^2+\frac{3}{8}B_1a^2](\frac{x}{a})^{9/2} + \ldots\} \quad , \qquad [7.14.14]$$

(ii) for $(a-x)$ sufficiently small:

$$z_1(x) = \tan 12 - a\{\frac{4}{3}\gamma_o(\frac{a-x}{a})^{3/2} + \frac{4}{15}(\frac{1}{2}\gamma_o+\gamma_1a)(\frac{a-x}{a})^{5/2} + \frac{4}{35}(\frac{4}{3}\gamma_on+\frac{3}{8}\gamma_o+\frac{1}{2}\gamma_1a+\gamma_2a^2)(\frac{a-x}{a})^{7/2} +$$

$$+\frac{4}{63}[\frac{4}{15}(\frac{1}{2}\gamma_o+\gamma_1a)n+\frac{1}{2}\gamma_2a^2+\frac{3}{8}\gamma_1a^2](\frac{a-x}{a})^{9/2} + \ldots\ldots\} \quad . \qquad [7.14.15]$$

Using [7.14.9] and [7.14.14], we can write the first estimate of $z_1(0.01)$ and $z_1'(0.01)$ as follows:

$$\left.\begin{aligned}z_1(0.01) &\simeq 0.00004672 \quad , \\ z_1'(0.01) &\simeq 0.006991 \quad .\end{aligned}\right\} \qquad [7.14.16]$$

Using [7.14.9] and [7.14.15], we can write the first estimate of $z(3.99]$ as follows:

$$z_1(3.99) \simeq 0.21251 \quad . \qquad [7.14.17]$$

Equation [7.14.7] can then be solved numerically subject to the boundary conditions [7.14.16] and [7.14.17], using the NAG library routine D_{02}ADF [see, e.g., HASELGROVE, C.B. (82)], which solves the two-point boundary value problem for a system of two ordinary differential equations.

Using the values of z_1 and z_1' worked out by this routine, $f_1(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$, $f_2(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$ and $f_3(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$ are now estimated as follows:

$$\left.\begin{aligned} f_1(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)}) &= \int_0^a \sqrt{1+z_1'^2(x)}\,dx \quad , \\ f_2(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)}) &= \int_0^a z_1'^2(x)\,dx \quad , \\ f_3(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)}) &= \int_0^a z_1(x)\,dx \quad . \end{aligned}\right\} \qquad [7.14.18]$$

The values of $f_1(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$, $f_2(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$ and $f_3(n_1^{(o)},E_1^{(o)},D_1^{(o)},C_o^{(o)})$ will normally differ from the desired true values of ℓ, K and y_o in [7.14.6]. Therefore it is necessary to keep improving n_1,E_1,D_1 and C_o until the exact values are reached.

This can be achieved by considering the following funciton of n_1,E_1,D_1 and C_o :

$$\begin{aligned} F(n_1,E_1,D_1,C_o) = {} & [\ell-f_1(n_1,E_1,D_1,C_o)]^2 + [K-f_2(n_1,E_1,D_1,C_o)]^2 + \\ & + [y_o-f_3(n_1,E_1,D_1,C_o)]^2 \; . \end{aligned} \qquad [7.14.19]$$

Using the NAG library routine $E_{04}CEF$ to minimize $F(n_1,E_1,D_1,C_o)$, [see, e.g. GILL, P.E. and MURRAY, W., (81)] we obtain

$$\left.\begin{aligned} n_1 &= -\,0.134 \quad , \\ E_1 &= 0.03145 \quad , \\ D_1 &= 0.1397 \quad , \\ C_o &= -\,0.03856 \quad , \end{aligned}\right\} \qquad [7.14.20]$$

then on substituting into [7.14.14] and [7.14.15] we obtain

$$\left.\begin{aligned} z_1(0.01) &= 0.00002964 \quad , \\ z_1'(0.01) &= 0.004445 \quad , \\ z_1(3.99) &= 0.2125439 \quad . \end{aligned}\right\} \qquad [7.14.21]$$

Equation [7.14.20] represents the improved values of $z_1(0.01)$, $z_1'(0.01)$ and $z_1'(3.99)$ and knowing these improved values, the whole method is repeated to get a further improvement until optimum values of $z_1(0.01)$, $z_1'(0.01)$ and $z_1(3.99)$ are reached. The graphs of this numerical solution for $y_1(x)$, $y_1'(x)$ ($\equiv z_1(x)$) and $y_1''(x)$ are shown in Figs. 32,33 and 34 respectively.

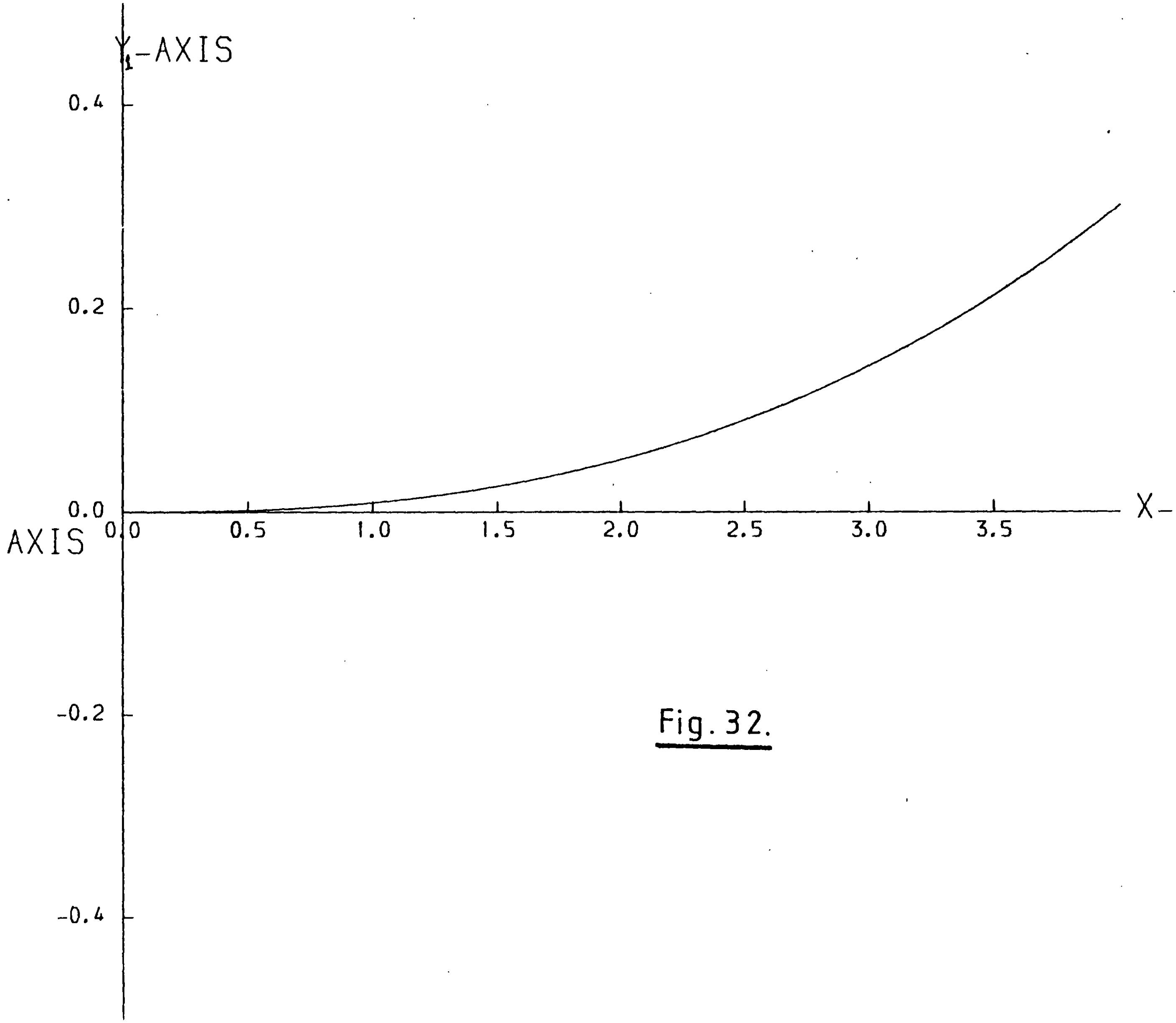

Fig. 32.

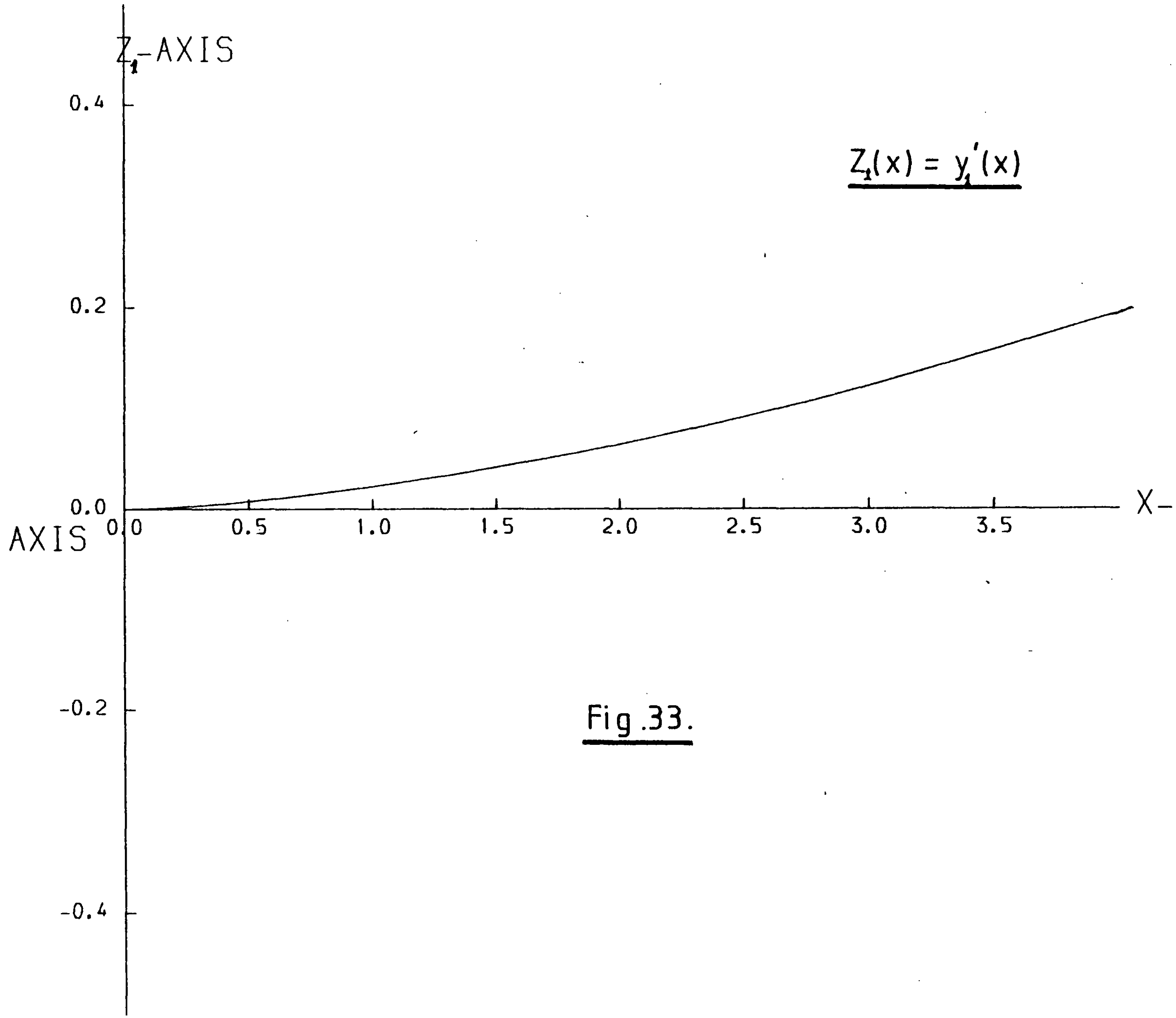

Fig.33.

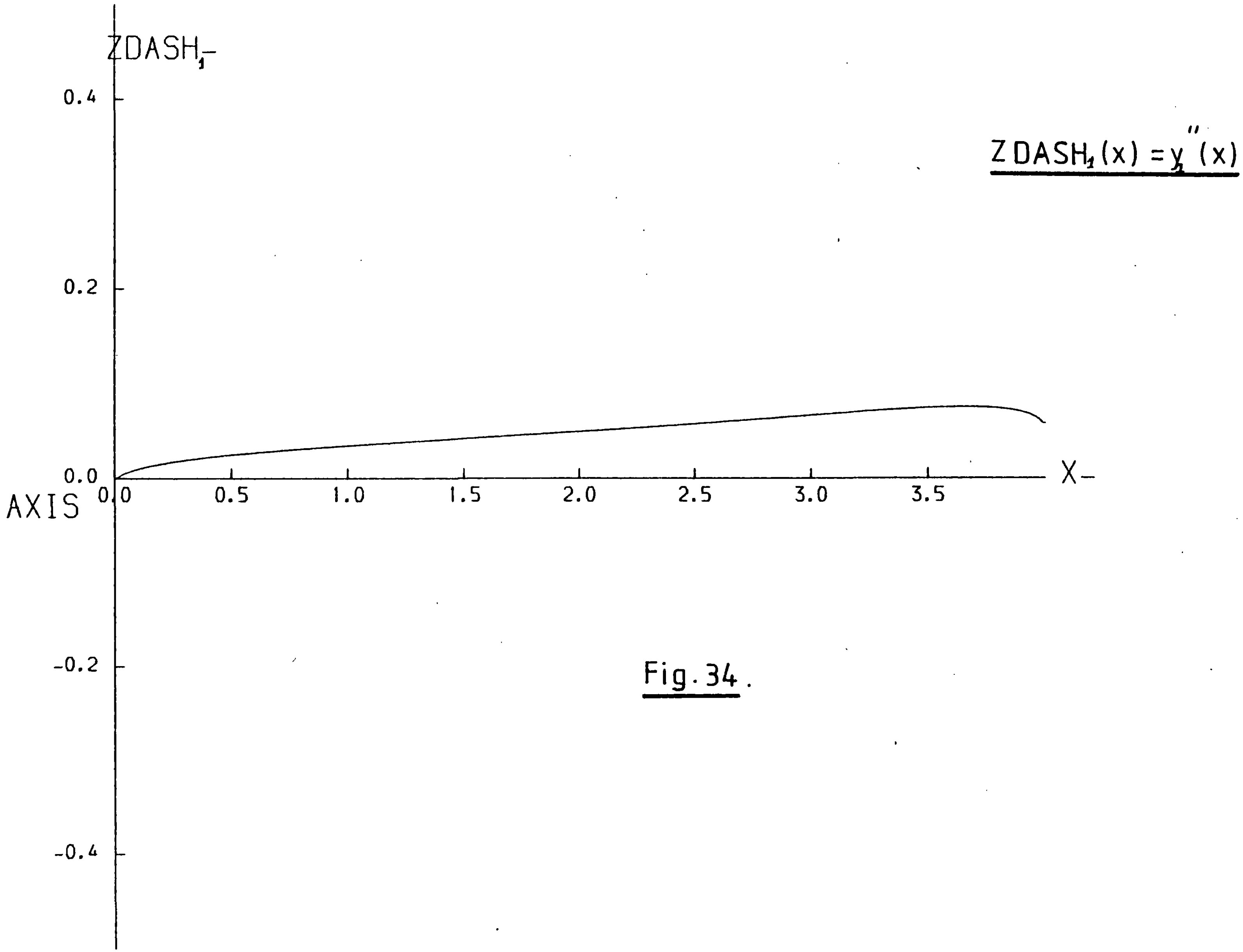

Fig. 34.

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These notes appear on the examiners' report form and are here reproduced for the benefit of internal examiners.

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   (a) a reasoned assessment of the candidate's performance, and
   (b) a recommendation of conferment or non-conferment of the degree, or re-examination of the candidate.

   If an agreed report cannot be submitted, each examiner should report separately. The candidate's work will then be referred to the adjudication of a second external examiner.

   (As a gloss on this section, examiners are requested to include in their report a brief description of the main problems under study, so that their reasoned assessment of the candidate's performance might be understood by all members of the Faculty Board or relevant Board of Studies.)

2. Where the candidate offers both written papers and a dissertation the examiners should report on each part of the examination separately, though basing their recommendation on the candidate's performance as a whole, including the oral examination. The examiners may at their discretion fail or refer without an oral examination a candidate whose written performance they consider inadequate.

3. The written papers, dissertation or thesis should comply with the requirements (including those relating to length, presentation, relevance, and style) laid down in the notes issued for the guidance of candidates. Examiners should state that these requirements have been met, or indicate any departure from them. Examiners should note that no change should be made in the title of a dissertation or thesis once it has been submitted for examination.

4. Where the examiners recommend the award of a Ph.D. degree, they must certify that the thesis contains work worthy of publication.

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