PARAMETER REDUCTION IN DEFINITION BY MULTI-SUCCESSOR RECURSION

by

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This thesis is submitted for the
degree of Doctor of Philosophy
of the University of Leicester

1973

ABSTRACT

It is well known that in primitive recursive arithmetic with a single successor the number of parameters in a definition by recursion may be successively reduced. In this thesis I examine the possibility of effecting a similar reduction in the number of parameters in a definition by recursion in a multi-successor arithmetic.

The reduction process involves the discovery in multi-successor arithmetic of analogues of pairing functions and of functions which select the elements of an ordered pair. One of the difficulties in finding such functions is the construction within multi-successor arithmetic of suitable product and square foot functions and establishing the properties of these functions, and the pairing functions, within a formalisation of multi-successor arithmetic. The reduction process involves of course an examination of what functions, if any, need to be adjoined to the initial functions to secure the generality of the reduction.

ACKNOWLEDGEMENTS

May I express my gratitude for the assistance and encouragement given by my supervisor, Professor R. L. Goodstein.

Further I would like to thank the Science Research Council for providing financial assistance from October 1968 to October 1971.

CONTENTS

CHAPTER I

INTRODUCTION

The subject of multiple successor arithmetic was first introduced in a paper by Vladeta Vučković in Mathematica Scandinavica in 1959 [5]. In single successor arithmetic every numeral has only one immediate successor. However in the paper mentioned above Vučković considered the possibility that every numeral has 'n' immediate successors. The successor of x in single successor arithmetic is written Sx, and for the successors in multiple successor arithmetic we shall write S_uX where u will range from 1 to n. In Vučković's paper the suffixes ranged from 0 to $n-1$, though in this thesis we have found it more convenient for the suffixes to range from 1 to n. Rather than make an analysis of single successor arithmetic, we shall suppose that the reader has a working knowledge of this theory. For references see [1], [2] and [3].

Both single and multiple successor arithmetics are based on the theory of Recursive Functions, that is functions defined by a schema, which defines the function at a starting value and then at the successor of X in terms of the value of that function at X. In the arithmetics that are considered in this thesis the starting value will be 0. By substitution the function is then defined successively for all values of X.

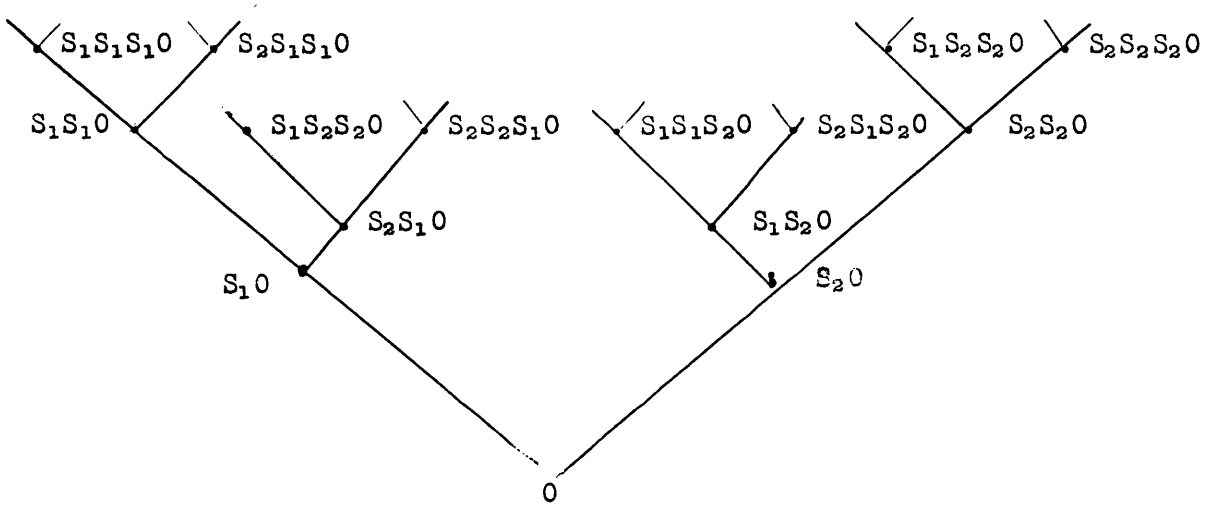
From hereon in this thesis lower case letters will be used for variables and function symbols in single successor arithmetic, and higher case letters for variables and function symbols in multiple successor arithmetic. It is worth noting that we are in fact considering an infinite though countable set of arithmetics, that is for all values of $n \geqslant 1$. The theory that we shall consider is the commutative theory.

The numerals of single successor arithmetic are as follows:

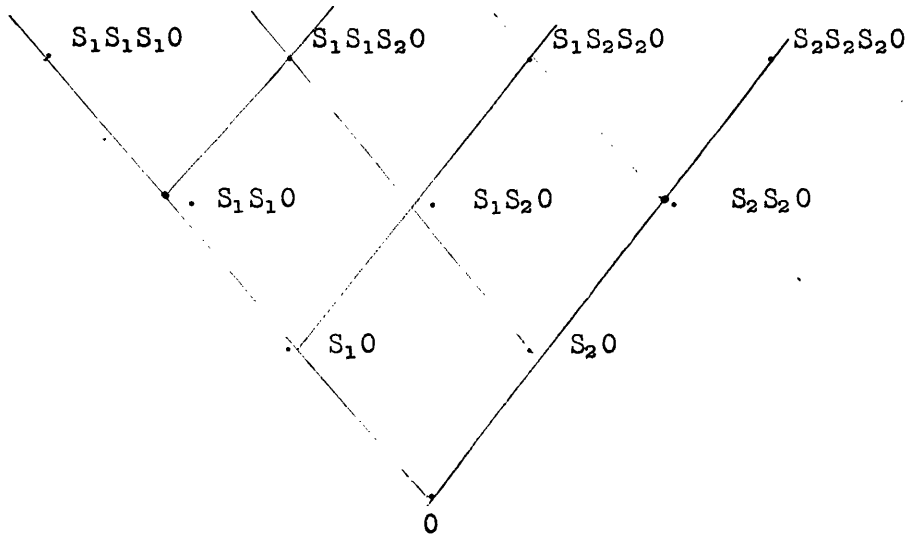
0, S0, SS0, SSS0, SSSS0, SSSSS0,..... and clearly these form a complete ordering. The condition that $x \geqslant y$ is given by the condition $y \dot{-} x = 0$. This is a total ordering since for all x and y either $x \dot{-} y = 0$ or $y \dot{-} x = 0$, and if both are equal to zero then $x = y$. Now in multiple successor arithmetic the numerals are as follows:

$0,\ S_1 0,\ S_2 0, \ldots S_n 0,\ S_1 S_1 0,\ S_1 S_2 0,\ \ldots S_n S_n 0,\ S_1 S_1 S_1 0, \ldots S_n S_n S_n 0, \ldots\ldots$

These numerals have a partial ordering which can be illustrated in '2' successor arithmetic with the aid of a lattice diagram.



In commutative successor arithmetic the diagram becomes:



The theory of single successor arithmetic is built up from three initial functions, namely the Identity, Zero and Successor functions given by

$I_i^k(x_1,x_2,\ldots,x_k)$ written x_i

$Z(x)$ written 0

$S(x)$ written Sx .

However in multiple successor arithmetic we obviously require the Identity and Zero functions though, in order to obtain 'n' successors for X we require 'n' successor functions amongst the initial functions. The initial set of functions is therefore as follows:

$I_i^k(X_1,X_2,\ldots,X_k)$ written X_i

$Z(X)$ written 0

$S_u(X)$ written S_uX for $u=1,\ldots,n$.

In order to define further functions in single successor arithmetic use is made of a defining schema called 'Definition by Primitive Recursion':-

$f(x,y)$ is said to be defined by primitive recursion from $g(y),h(x,y,z)$ if

$$f(0,y) = g(y)$$

$$f(Sx,y) = h(x,y,f(x,y))$$

where $g(..)$ and $h(....)$ are initial functions or previously defined functions. Primitive recursive functions can also be defined by substitution from functions which have been previously defined by primitive recursion.

The above schema obviously defines $f(x,y)$ for all values of the variable x. That is, the function is defined at a starting value, namely 0. Then from the second equation subsequent values of the function can be found by substituting $x = 0, S0, SS0, SSS0, SSSS0,\ldots..$ in turn. In other words in multiple successor arithmetic with 'n'

successors, the basic defining schema for a primitive recursive function would need to define the function at a starting value, namely 0, and then have n further equations to evaluate the function for each different successor of X. Hence this schema will consist of 'n+1' equations as follows:-

A function F(X,Y) is defined by primitive recursion in multiple successor arithmetic by

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,Y,F(X,Y)) \qquad u = 1,\ldots,n$$

where the functions A(..) and B_u(.....) are initial functions or have been previously defined. A primitive recursive function can be defined by substitution as in single successor arithmetic.

In this thesis the successors of our arithmetic are commutative. Clearly this condition is not implicit in the equations for the definition of primitive recursion. It is therefore necessary to impose a restriction on the functions B_u(....) $u = 1,\ldots,n$ in order that the commutativity of the successors is maintained. We require that $F(S_uS_vX,Y) = F(S_vS_uX,Y)$ for all u,v. Since

$$F(S_uS_vX,Y) = B_u(S_vX,Y,F(S_vX,Y))$$

$$= B_u(S_vX,Y,B_v(X,Y,F(X,Y)))$$

and

$$F(S_vS_uX,Y) = B_v(S_uX,Y,F(S_uX,Y))$$

$$= B_v(S_uX,Y,B_u(X,Y,F(X,Y)))$$

the condition is therefore that

$$B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(S_uX,Y,B_u(X,Y,Z)) \qquad \text{for all } u,v.$$

For an illustration of this consider in '2' successor arithmetic the function defined by

$$F(0,Y) = Y$$
$$F(S_1X,Y) = F(X,Y)$$
$$F(S_2X,Y) = S_2F(X,Y)$$

F(X,Y) is a primitive recursive function in commutative multiple successor arithmetic; since

$$B_1(S_2X,Y,B_2(X,Y,Z)) = B_1(S_2X,Y,S_2Z)$$
$$= S_2Z$$

and

$$B_2(S_1X,Y,B_1(X,Y,Z)) = B_2(S_1X,Y,Z)$$
$$= S_2Z$$
$$= B_1(S_2X,Y,B_2(X,Y,Z))$$

the commutativity condition is satisfied.

Now consider the function defined by

$$F(0,Y) = Y$$
$$F(S_1X,Y) = S_1F(X,Y)$$
$$F(S_2X,Y) = 0$$

F(X,Y) is not a primitive recursive function in commutative recursive successor arithmetic since

$$B_1(S_2X,Y,B_2(X,Y,Z)) = B_1(S_2X,Y,0)$$
$$= S_10$$
$$B_2(S_1X,Y,B_1(X,Y,Z)) = B_2(S_1X,Y,S_1Z)$$
$$= 0$$
$$\neq S_10$$

the commutativity condition is not satisfied.

The uniqueness rule of inference for a primitive recursive function in single successor arithmetic is given by

$$\begin{array}{l} f(0) = g(0) \\ f(Sx) = h(x,f(x)) \\ g(Sx) = h(x,g(x)) \\ \hline f(x) = g(x) \end{array}$$

That is if two functions have the same defining equations in the definition by primitive recursion, then these two functions are identical, maintaining the uniqueness of definition by primitive recursion. Clearly in 'n' successor arithmetic we require the uniqueness rule of inference to be as follows:

$$\begin{array}{ll} F(0) = G(0) & \\ F(S_uX) = B_u(X,F(X)) & u = 1,\ldots,n \\ G(S_uX) = B_u(X,G(X)) & u = 1,\ldots,n \\ \hline F(X) = G(X) & \end{array}$$

Parameters have been omitted from the above statements though these are implicit in the rules.

Let M_n be the set of numerals in 'n' successor arithmetic, then

$$M_1 = 0,\ S0,\ SS0,\ SSS0,\ SSSS0, \ldots\ldots$$

Hence for any primitive recursive function the domain of the arguments cover all numerals that can be generated by repeated application of the successor functions to the zero function. That is letting $D(X)$ represent the domain of X, for a function $F(X)$ we have

$$D(F(X)) = M_n \text{ in 'n' successor arithmetic,}$$

and letting $R(X)$ represent the range of X, for a function $F(X)$ we have

$$R(F(X)) \subseteq M_n \text{ in 'n' successor arithmetic.}$$

For an example of this consider the identity function $I(X)$, in this instance,

$$D(I(X)) = M_n$$

$$R(I(X)) = M_n$$

and for the function $2.X$ (supposing a function such that $F(X) = X + X$ exists in our arithmetic)

$$D(2.X) = M_n$$

$$R(2.X) \subset M_n \quad (\text{'}\subset\text{' meaning 'is a proper subset of'})$$

and in single successor arithmetic we would have

$$R(2.X) = 0,\ SS0,\ SSSS0,\ SSSSSS0,\ SSSSSSSS0, \ldots\ldots\ldots$$

Also consider the functions $S_u X \quad u = 1,\ldots,n$. For this we have

$$D(S_u X) = M_n$$

$$R(S_u X) \subset M_n \quad \text{since there is no } X \text{ such that } S_u X = 0 .$$

Equality in M_n is defined by means of an axiom due to Professor R. L. Goodstein, which is designed to avoid irregular models, given by

$$S_a S_b S_c \ldots\ldots\ldots S_q 0 = S_{a'} S_{b'} S_{c'} \ldots\ldots\ldots S_{q'} 0$$

with $a \leqslant b \leqslant c \leqslant \ldots\ldots\ldots \leqslant q$ and $a' \leqslant b' \leqslant c' \leqslant \ldots\ldots\ldots \leqslant q'$ if and only if

$$a = a',\ \ b = b',\ \ c = c', \ldots\ldots\ldots,\ q = q' .$$

However it has been shown by M. T. Partis in [4] that this axiom can be replaced by the condition $S_1 0 \neq S_2 0$.

We shall introduce several basic primitive recursive functions in 'n' successor arithmetic, and consider the properties of these functions.

First we shall define the function $Y \sigma_v X,\ v = 0,\ldots,n-1$ (the case $v = n$ will be considered as the case $v = 0$), as defined in Vučković's original paper.

The defining schema is

$$Y \sigma_v 0 = Y$$

$$Y \sigma_v S_u X = S_{u+v}(Y \sigma_v X) \qquad u = 1,\ldots,n .$$

The suffix $u+v$ is maintained within the range $[1,n]$ by taking the excess over 'n'. The commutativity condition for this function is given by

$$\begin{aligned} B_u(S_w X, Y, B_w(X,Y,Z)) &= B_u(S_w X, Y, S_{w+v} Z) \\ &= S_{u+v} S_{w+v} Z \\ &= S_{w+v} S_{u+v} Z \\ &= B_w(S_u X, Y, S_{u+v} Z) \\ &= B_w(S_u X, Y, B_u(X,Y,Z)) \ . \end{aligned}$$

In order to prove the properties of this function we require the uniqueness rule

$$\begin{array}{ll} F(0) = G(0) & \\ F(S_u X) = B_u(X, F(X)) & u = 1,\ldots,n \\ G(S_u X) = B_u(X, G(X)) & u = 1,\ldots,n \\ \hline F(X) = G(X) & \end{array}$$

A complete formal development of the arithmetic will be left to a later chapter; here we shall just prove simply certain properties of several linear functions in the arithmetic.

First we consider Vučković's function $Y \sigma_v X \quad v = 0,\ldots,n-1$.

LEMMA $\quad (S_u Y)\, \sigma_v X = S_u(Y\, \sigma_v X) \quad u = 1,\ldots,n$

<u>Proof</u> Denoting the right-hand side of the equation by $R(X)$ and the left-hand side by $L(X)$ we have that

$$\begin{aligned} L(0) &= (S_u Y)\, \sigma_v 0 \\ &= S_u Y && \text{Def'n} \\ R(0) &= S_u(Y\, \sigma_v 0) \\ &= S_u Y && \text{Def'n} \end{aligned}$$

$$L(S_w X) = (S_u Y)\ \sigma_v S_w X$$

$$= S_{w+v}(S_u Y)\ \sigma_v X \qquad \text{Defn}$$

$$= S_{w+v} L(X)$$

$$R(S_w X) = S_u(Y\ \sigma_v S_w X)$$

$$= S_u S_{w+v}(Y\ \sigma_v X) \qquad \text{Defn}$$

$$= S_{w+v} S_u(Y\ \sigma_v X) \qquad \text{Condition of commutativity on successors}$$

$$= S_{w+v} R(X)\ .$$

Hence by the Uniqueness rule

$$L(X) = R(X)$$

End of proof.

In general $X\ \sigma_v(Y\ \sigma_v Z) \neq (X\ \sigma_v Y)\ \sigma_v Z$.

A simple counter-example serves to illustrate this; consider

$$Y = S_1 0, \quad Z = S_2 0$$

Then

$$X\ \sigma_v(Y\ \sigma_v Z) = X\ \sigma_v(S_1 0\ \sigma_v S_2 0)$$

$$= X\ \sigma_v S_{2+v}(S_1 0\ \sigma_v 0) \qquad \text{Defn}$$

$$= X\ \sigma_v S_{2+v} S_1 0 \qquad \text{Defn}$$

$$= S_{2+v+v}(X\ \sigma_v S_1 0) \qquad \text{Defn}$$

$$= S_{2+v+v} S_{1+v}(X\ \sigma_v 0) \qquad \text{Defn}$$

$$= S_{2+v+v} S_{1+v} X \qquad \text{Defn}$$

$$(X\ \sigma_v Y)\ \sigma_v Z = (X\ \sigma_v S_1 0)\ \sigma_v S_2 0$$

$$= S_{2+v}((X\ \sigma_v S_1 0)\ \sigma_v 0) \qquad \text{Defn}$$

$$= S_{2+v}(X\ \sigma_v S_1 0) \qquad \text{Defn}$$

$$= S_{2+v} S_{1+v}(X\ \sigma_v 0) \qquad \text{Defn}$$

$$= S_{2+v} S_{1+v} X \qquad \text{Defn}$$

and $S_{2+v+v}S_{1+v}X$ is equal to $S_{2+v}S_{1+v}X$ only when $v = 0$.

Also we find that in general

$$X\,\sigma_v Y \neq Y\,\sigma_v X\ ,$$

In order to prove this consider the particular instance where $Y = 0$:

$$X\,\sigma_v 0 \neq 0\,\sigma_v X.$$

If $X = S_2 0$ then

$$\begin{aligned} X\,\sigma_v 0 &= S_2 0\,\sigma_v 0 \\ &= S_2(0\,\sigma_v 0) && \text{Previous Lemma} \\ &= S_2 0 && \text{Defn} \end{aligned}$$

and

$$\begin{aligned} 0\,\sigma_v X &= 0\,\sigma_v S_2 0 \\ &= S_{2+v}(0\,\sigma_v 0) && \text{Defn} \\ &= S_{2+v}0 && \text{Defn} \end{aligned}$$

and $S_{2+v}0$ is only equal to $S_2 0$ when $v = 0$

Now we consider if this function is distributive over itself, that is if

$$X\,\sigma_v(Y\sigma_u Z) = (X\,\sigma_v Y)\,\sigma_u(X\,\sigma_v Z) \qquad (x \neq 0)$$

This condition is never true; put $Y = S_1 0$, $Z = S_2 0$

$$\begin{aligned} X\,\sigma_v(Y\,\sigma_u Z) &= X\,\sigma_v(S_1 0\,\sigma_u S_2 0) \\ &= X\,\sigma_v S_1 S_{2+u}0 && \text{Defn} \\ &= S_{1+v}S_{2+u+v}X && \text{Defn} \end{aligned}$$

$$\begin{aligned} (X\,\sigma_v Y)\,\sigma_u(X\,\sigma_v Z) &= (X\,\sigma_v S_1 0)\,\sigma_u(X\,\sigma_v S_2 0) \\ &= (S_{1+v}X)\,\sigma_u(S_{2+v}X) && \text{Defn} \\ &= S_{1+v}S_{2+v+u}(X\,\sigma_u X) && \text{Defn} \end{aligned}$$

and $X \neq X\,\sigma_u X$ for any value of u $(x \neq 0)$.

Therefore we have proved that Vučković's function $Y\,\sigma_v X$ is neither associative, nor commutative, except possibly in the case $v = 0$; this particular case is that of addition, that is the function $Y + X$ defined by the usual schema

$$Y + 0 = Y$$

$$Y + S_u X = S_u(Y + X) \qquad u = 1,\ldots,n$$

PROPOSITION $\quad Y\,\sigma_0 X = Y + X$

<u>Proof</u>

$$\begin{aligned}
L(0) &= Y\,\sigma_0 0 \\
&= Y && \text{Defn} \\
R(0) &= Y + 0 \\
&= Y && \text{Defn} \\
L(S_u X) &= Y\,\sigma_0 S_u X \\
&= S_{u+0}(Y\,\sigma_0 X) && \text{Defn} \\
&= S_u(Y\,\sigma_0 X) && \text{Normal arithmetical rules of addition} \\
&= S_u L(X) && u = 1,\ldots,n \\
R(S_u X) &= Y + S_u X \\
&= S_u(Y + X) && \text{Defn} \\
&= S_u R(X) && u = 1,\ldots,n
\end{aligned}$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \quad \text{as required .}$$

It will be shown that addition is commutative and associative by a series of Lemmas.

LEMMA $0 + X = X$

<u>Proof</u>

$$L(0) = 0 + 0$$

$$= 0 \qquad \text{Def'n}$$

$$R(0) = 0$$

$$L(S_u X) = 0 + S_u X$$

$$= S_u(0 + X) \qquad \text{Def'n}$$

$$= S_u L(X)$$

$$R(S_u X) = S_u X$$

$$= S_u R(X)$$

Hence by the uniqueness rule of inference

$L(X) = R(X)$ as required.

LEMMA $S_u Y + X = S_u(Y + X)$

<u>Proof</u>

$$L(0) = S_u Y + 0$$

$$= S_u Y \qquad \text{Def'n}$$

$$R(0) = S_u(Y + 0)$$

$$= S_u Y \qquad \text{Def'n}$$

$$L(S_v X) = S_u Y + S_v X$$

$$= S_v(S_u Y + X) \qquad \text{Def'n}$$

$$= S_v L(X)$$

$$R(S_v X) = S_u(Y + S_v X)$$

$$= S_u S_v(Y + X) \qquad \text{Def'n}$$

$$= S_v S_u(Y + X) \qquad \text{Commutativity of successor}$$

$$= S_v R(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \quad \text{as required}$$

PROPOSITION $\quad Y + X = X + Y \qquad$ Commutativity

<u>Proof</u>

$$L(0) = Y + 0$$

$$= Y \qquad \text{Defn}$$

$$R(0) = 0 + Y$$

$$= Y \qquad \text{Previous Lemma}$$

$$L(S_u X) = Y + S_u X$$

$$= S_u(Y + X) \qquad \text{Defn}$$

$$= S_u L(X)$$

$$R(S_u X) = S_u X + Y$$

$$= S_u(Y + X) \qquad \text{Previous Lemma}$$

$$= S_u R(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \quad \text{as required .}$$

PROPOSITION $\quad (X + Y) + Z = X + (Y + Z) \qquad$ Associativity

<u>Proof</u>

$$L(0) = (0 + Y) + Z$$

$$= Y + Z \qquad \text{Previous Lemma}$$

$$R(0) = 0 + (Y + Z)$$

$$= Y + Z \qquad \text{Previous Lemma}$$

$$L(S_u X) = (S_u X + Y) + Z$$

$$= S_u(X + Y) + Z \qquad \text{Previous Lemma}$$

$$= S_u((X + Y) + Z) \qquad \text{Previous Lemma}$$

$$= S_u L(X)$$

$$R(S_uX) = S_uX + (Y + Z)$$

$$= S_u(X + (Y + Z)) \qquad \text{Previous Lemma}$$

$$= S_uR(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) .$$

Having considered addition functions in our arithmetic, we shall now proceed to consider difference functions.

The basic functions for the addition functions are the successor functions. For the difference functions the basic functions are the predecessor functions P_vX $v = 1,\ldots,n$, defined by the schema

$$P_v0 = 0$$

$$P_vS_uX = \begin{cases} X, & u = v \\ S_uP_vX, & u \neq v \end{cases}$$

the commutativity condition is

$$B_u(S_wX,Y,B_w(X,Y,Z)) = \begin{cases} S_wX & u=v \\ S_uB_w(X,Y,Z) & u\neq v \end{cases}$$

$$= \begin{cases} S_wX & u=v \\ S_uX & u\neq v \ \& \ w=v \\ S_uS_wZ & u\neq v \ \& \ w\neq v \end{cases}$$

$$= \begin{cases} S_uX & u=v \ \& \ w=v \\ S_wX & u=v \ \& \ w\neq v \\ S_uX & u\neq v \ \& \ w=v \\ S_wS_uZ & u\neq v \ \& \ w\neq v \end{cases} \qquad \text{Commutativity of successors}$$

$$= \begin{cases} S_uX & w=v \\ S_wX & w\neq v \ \& \ u=v \\ S_wS_uZ & w\neq v \ \& \ u\neq v \end{cases}$$

$$= \begin{cases} S_u X & w=v \\ S_w B_u(X,Y,Z) & w \neq v \end{cases}$$

$$= B_w(S_u X, Y, B_u(X,Y,Z)) \quad \text{as required}$$

We are now able to introduce the difference function $Y \dot{-} X$, defined by the schema.

$$Y \dot{-} 0 = Y$$

$$Y \dot{-} S_u X = P_u(Y \dot{-} X) \qquad u = 1,\ldots,n$$

In order to prove the commutativity condition for this function we first need to prove the following Lemma:

LEMMA $\quad P_u P_v X = P_v P_u X$

<u>Proof</u>

$$L(0) = P_u P_v 0$$

$$= P_u 0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$R(0) = P_v P_u 0$$

$$= P_v 0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$L(S_w X) = P_u P_v S_w X$$

$$= \begin{cases} P_u X & v=w & \text{Def'n} \\ P_u S_w P_v X & v \neq w & \text{Def'n} \end{cases}$$

$$= \begin{cases} P_u X & v=w & \\ P_v X & v \neq w \ \& \ u=w & \text{Def'n} \\ S_w P_u P_v X & v \neq w \ \& \ u \neq w & \text{Def'n} \end{cases}$$

$$= \begin{cases} P_u X & v=w \\ P_v X & v \neq w \ \& \ u=w \\ S_w L(X) & v \neq w \ \& \ u \neq w \end{cases}$$

$$R(S_wX) = P_vP_uS_wX$$

$$= \begin{cases} P_vX & u=w \quad \text{Defn} \\ P_vS_w\,P_uX & u\neq w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} P_vX & u=w \\ P_uX & u\neq w \ \&\ v=w \quad \text{Defn} \\ S_wP_vP_uX & u\neq w \ \&\ v\neq w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} P_uX & v=w \\ P_vX & v\neq w \ \&\ u=w \\ S_wP_vP_uX & v\neq w \ \&\ u\neq w \end{cases}$$

$$= \begin{cases} P_uX & v=w \\ P_vX & v\neq w \ \&\ u=w \\ S_wR(X) & v\neq w \ \&\ u\neq w \end{cases}$$

Now letting

$$B_w(X,Y,Z) = \begin{cases} P_uX & w=v \\ P_vX & w=u \\ S_wZ & w\neq v \ \&\ w\neq u \end{cases}$$

then $$L(S_wX) = B_w(X,Y,L(X))$$

$$R(S_wX) = B_w(X,Y,R(X)) \ .$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \quad \text{as required}$$

The commutativity condition for $Y \dot{-} X$ follows by

$$B_u(S_vX,Y,B_v(X,Y,Z)) = P_uB_v(X,Y,Z)$$

$$= P_uP_vZ$$

$$= P_vP_uZ \qquad \text{Previous Lemma}$$

$$= P_vB_u(X,Y,Z)$$

$$= B_v(S_uX,Y,B_u(X,Y,Z)) \quad \text{as required}$$

Corresponding to the Vučković's functions σ_v for addition functions, for difference functions, we define functions τ_v by the sehema

$$Y \tau_v 0 = Y$$

$$Y \tau_v S_u X = P_{u+v}(Y \tau_v X) \qquad v=0,\ldots,n-1,$$

again taking the excess over n for $u+v$. The case of $v=n$ is considered as $v=0$.

For the commutativity condition we have

$$\begin{aligned} B_u(S_w X, Y, B_w(X,Y,Z)) &= P_{u+v} B(X,Y,Z) \\ &= P_{u+v} P_{w+v} Z \\ &= P_{w+v} P_{u+v} Z && \text{Previous Lemma} \\ &= P_{w+v} B_u(X,Y,Z) \\ &= B_w(S_u X, Y, B_u(X,Y,Z)) && \text{as required} \end{aligned}$$

As for Vučković's functions σ_v, the particular case $v=0$ being addition, so for τ_v, with $v=0$ we have the normal difference function $Y \dot{-} X$.

PROPOSITION $\quad Y \tau_0 X = Y \dot{-} X$

<u>Proof</u>

$$\begin{aligned} L(0) &= Y \tau_0 0 \\ &= Y && \text{Defn} \\ R(0) &= Y \dot{-} 0 \\ &= Y && \text{Defn} \\ L(S_u X) &= Y \tau_0 S_u X \\ &= P_{u+0}(Y \tau_0 X) && \text{Defn} \\ &= P_u L(X) \end{aligned}$$

$$R(S_uX) = Y \dot{-} S_uX$$

$$= P_u(Y \dot{-} X) \qquad \text{Defn}$$

$$= P_uR(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X)$$

The common properties of the addition function and the difference function in single successor arithmetic are

$$y \dot{-} x = Sy \dot{-} Sx$$

$$(y + x) \dot{-} x = y$$

$$x + (y \dot{-} x) = y + (x \dot{-} y)$$

For the first two of these we shall prove in multiple successor arithmetic, that is

$$Y \dot{-} X = S_vY \dot{-} S_vX \qquad v = 1,\ldots,n$$

$$(Y + X) \dot{-} X = Y$$

The third equation $X + (Y \dot{-} X) = Y + (X \dot{-} Y)$, known as the Key Equation, will not be proved in this thesis. The proof in primitive recursive multiple successor arithmetic is due to Professor R.L.Goodstein, and can be found in [9].

PROPOSITION $\quad Y \dot{-} X = S_vY \dot{-} S_vX$

<u>Proof</u>

$$L(0) = Y \dot{-} 0$$

$$= Y \qquad \text{Defn}$$

$$R(0) = S_vY \dot{-} S_v0$$

$$= P_v(S_vY \dot{-} 0) \qquad \text{Defn}$$

$$= P_vS_vY \qquad \text{Defn}$$

$$= Y \qquad \text{Defn}$$

$$L(S_uX) = Y \dot{-} S_uX$$

$$= P_u(Y \dot{-} X) \qquad \text{Def'n}$$

$$= P_uL(X)$$

$$R(S_uX) = S_vY \dot{-} S_vS_uX$$

$$= S_vY \dot{-} S_uS_vX \qquad \text{Commutativity of successors}$$

$$= P_u(S_vY \dot{-} S_vX) \qquad \text{Def'n}$$

$$= P_uR(X)$$

Hence by the uniqueness rule of inference

$$(L(X) = R(X) \quad \text{as required}$$

PROPOSITION $\quad (Y + X) \dot{-} X = Y$

<u>Proof</u>

$$L(0) = (Y + 0) \dot{-} 0$$

$$= Y + 0 \qquad \text{Def'n}$$

$$= Y \qquad \text{Def'n}$$

$$R(0) = Y \qquad \text{Def'n}$$

$$L(S_uX) = (Y + S_uX) \dot{-} S_uX$$

$$= S_u(Y + X) \dot{-} S_uX \qquad \text{Def'n}$$

$$= (Y + X) \dot{-} X \qquad \text{Previous Proposition}$$

$$= L(X)$$

$$R(S_uX) = Y$$

$$= R(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \quad \text{as required.}$$

Let us now consider the following three equations, relating σ_v and τ_v.

1. $$Y\,\tau_v X = S_u Y\,\tau_v S_u X$$

2. $$(Y\,\sigma_v X)\,\tau_v X = Y$$

3. $$X\,\sigma_v(Y\,\tau_v X) = Y\,\sigma_v(X\,\tau_v Y)$$

It would surely be remarkable if all three equations were true, especially 3., which would be a generalisation of the Key Equation, but unfortunately this equation is not true. Equation 1. needs a slight adjustment, and equation 2. is true. Equation 1. we adjust to

$$Y\,\tau_v X = S_{u+v} Y\,\tau_v S_u X$$

PROPOSITION $\quad Y\,\tau_v X = S_{u+v} Y\,\tau_v S_u X$

<u>Proof</u>

$$\begin{aligned}
L(0) &= Y\,\tau_v 0 \\
&= Y && \text{Defn} \\
R(0) &= S_{u+v} Y\,\tau_v S_u 0 \\
&= P_{u+v}(S_{u+v} Y\,\tau_v 0) && \text{Defn} \\
&= P_{u+v} S_{u+v} Y && \text{Defn} \\
&= Y && \text{Defn} \\
L(S_w X) &= Y\,\tau_v S_w X \\
&= P_{w+v}(Y\,\tau_v X) && \text{Defn} \\
&= P_{w+v} L(X)
\end{aligned}$$

$$R(S_w X) = S_{u+v} Y \, \tau_v S_u S_w X$$

$$= S_{u+v} Y \, \tau_v S_w S_u X \qquad \text{Commutativity of successors}$$

$$= P_{w+v}(S_{u+v} Y \, \tau_v S_u X) \qquad \text{Defn}$$

$$= P_{w+v} R(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X) \text{ as required}$$

PROPOSITION $\quad (Y \, \sigma_v X) \, \tau_v X = Y$

<u>Proof</u>

$$L(0) = (Y \, \sigma_v 0) \, \tau_v 0$$

$$= Y \, \sigma_v 0 \qquad \text{Defn}$$

$$= Y \qquad \text{Defn}$$

$$R(0) = Y$$

$$L(S_u X) = (Y \, \sigma_v S_u X) \, \tau_v S_u X$$

$$= S_{u+v}(Y \, \sigma_v X) \, \tau_v S_u X \qquad \text{Defn}$$

$$= (Y \, \sigma_v X) \, \tau_v X \qquad \text{Previous Proposition}$$

$$= L(X)$$

$$R(S_u X) = Y$$

$$= R(X)$$

Hence by the uniqueness rule of inference

$$L(X) = R(X)$$

Further, the equation $Z \dot{-} (Y + X) = (Z \dot{-} Y) \dot{-} X$ is provable in both single and multiple successor arithmetic; using the σ_v and τ_v functions we are able to make the following generalisation of this equation:

$$Z\ \tau_u(Y\ \sigma_v X) = (Z\ \tau_u Y)\ \tau_{u+v} X$$

PROPOSITION $\quad Z\ \tau_u(Y\ \sigma_v X) = (Z\ \tau_u Y)\ \tau_{u+v} X$

<u>Proof</u>

$$\begin{aligned}
L(0) &= Z\ \tau_u(Y\ \sigma_v 0) \\
&= Z\ \tau_u Y && \text{Defn} \\
R(0) &= (Z\ \tau_u Y)\ \tau_{u+v} 0 \\
&= Z\ \tau_u Y && \text{Defn} \\
L(S_w X) &= Z\ \tau_u(Y\ \sigma_v S_w X) \\
&= Z\ \tau_u S_{w+v}(Y\ \sigma_v X) && \text{Defn} \\
&= P_{w+v+u}(Z\ \tau_u(Y\ \sigma_v X)) && \text{Defn} \\
&= P_{w+u+v} L(X) \\
R(S_w X) &= (Z\ \tau_u Y)\ \tau_{u+v} S_w X \\
&= P_{w+u+v}((Z\ \tau_u Y)\ \tau_{u+v} X) && \text{Defn} \\
&= P_{w+u+v} R(X)
\end{aligned}$$

Hence by the uniqueness rule of inference

$$L(X) = R(X)$$

PROPOSITION $\quad Y\ \sigma_v(X\ \tau_v Y) \neq X\ \sigma_v(Y\ \tau_v X)$

<u>Proof</u> A simple counter-example will serve to illustrate this result.

Put $Y = S_1 0, \quad X = S_2 0, \quad v = 1$

$$\begin{aligned}
L(X) &= S_1 0\ \sigma_1(S_2 0\ \tau_1 S_1 0) \\
&= S_1 0\ \sigma_1 P_2(S_2 0\ \tau_1 0) && \text{Defn} \\
&= S_1 0\ \sigma_1 P_2 S_2 0 && \text{Defn} \\
&= S_1 0\ \sigma_1 0 && \text{Defn} \\
&= S_1 0 && \text{Defn}
\end{aligned}$$

$$
\begin{aligned}
R(X) &= S_2 0\ \sigma_1(S_1 0\ \tau_1 S_2 0) \\
&= S_2 0\ \sigma_1 P_3(S_1 0\ \tau_1 0) \\
&= S_2 0\ \sigma_1 P_3 S_1 0 \\
&= S_2 0\ \sigma_1 S_1 0 \\
&= S_2(S_2 0\ \sigma_1 0) \\
&= S_2 S_2 0 \\
&\neq L(X)
\end{aligned}
$$

The multiplication functions in this arithmetic will be left until a later chapter.

A function particular to multiple successor arithmetic, which we have not yet considered, is the component function. This is a most valuable function which allows us to apply and utilise many of the results and principles of single successor arithmetic. The component function $C_v X$ is defined by the schema

$$C_v 0 = 0$$

$$C_v S_u X = \begin{cases} C_v X & u \neq v \\ S_u C_v X & u = v \end{cases}$$

the commutativity condition is

$$
\begin{aligned}
B_u(S_w X, Y, B_w(X,Y,Z)) &= \begin{cases} B_w(X,Y,Z) & u \neq v \\ S_u B_w(X,Y,Z) & u = v \end{cases} \\
&= \begin{cases} Z & u \neq v \ \& \ w \neq v \\ S_w Z & u \neq w \ \& \ w = v \\ S_u Z & u = v \ \& \ w \neq v \\ S_u S_w Z & u = v \ \& \ w = v \end{cases}
\end{aligned}
$$

$$= \begin{cases} Z & w \neq v \ \& \ u \neq v \\ S_u Z & w \neq v \ \& \ u = v \\ S_w Z & w = v \ \& \ u \neq v \\ S_w S_u Z & w = v \ \& \ u = v \end{cases}$$

Commutativity of successors

$$= \begin{cases} B_u(X,Y,Z) & w \neq v \\ S_w B_u(X,Y,Z) & w = v \end{cases}$$

$$= B_w(S_u X, Y, B_u(X,Y,Z)) \text{ as required}$$

Clearly $C_v X$ only allows use of the successors S_v in X, and thus reduces X to one type of successor. The reason the word component is used for this function is due to M.T.Partis in [4], where it is proved that variables in multiple successor arithmetic can be regarded as 'n'tuples with only one successor. The function $C_v X$ reduces X to only the 'v'th component, that is if

$$X = (X_1, X_2, \ldots\ldots\ldots, X_n)$$

$$C_v X = (0, 0, \ldots, 0, X_v, 0, \ldots, 0)$$

Further properties of the component function will be considered in the formal development of the arithmetic in chapter III.

PROBLEMS ATTEMPTED

The problems that have been attempted in this thesis are:

1. Reduction of the number of parameters in definition by primitive recursion.

An 'm+1'ary function $F(X, Y_1, Y_2, \ldots. Y_m)$ is said to be defined by primitive recursion from A and B if

$$F(0, Y_1, Y_2, \ldots, Y_m) = A(Y_1, Y_2, \ldots, Y_m)$$

$$F(S_u X, Y_1, Y_2, \ldots, Y_m) = B_u(X, Y_1, Y_2, \ldots, Y_m, F(X, Y_1, Y_2, \ldots, Y_m))$$

$$u = 1, \ldots, n$$

where $A(Y_1,Y_2,\ldots,Y_m)$ and $B_u(X,Y_1,Y_2,\ldots,Y_m,Z)$ $u=1,\ldots,n$ are initial functions or previously defined functions, and further, in order to preserve commutativity of the successors

$$B_u(S_vX,Y_1,Y_2,\ldots,Y_m,B_v(X,Y_1,Y_2,\ldots Y_m,Z)) = B_v(S_uX,Y_1,Y_2,\ldots,Y_m,B_u(X,Y_1,Y_2,\ldots,Y_m,Z))$$

for all u,v.

The problem is to reduce this definition to the defining schemata.

(i) Define $F(X,Y)$ by the schema, called R_1

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,Y,F(X,Y)) \qquad u=1,\ldots,n$$

where $B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(S_uX,Y,B_u(X,Y,Z))$ for all u,v.

(ii) Define $F(X,Y)$ by the schema, called $R_1{}^*$

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,F(X,Y)) \qquad u=1,\ldots,n$$

where $B_u(S_vX,B_v(X,Z) = B_v(S_uX,B_u(X,Z))$ for all u,v.

(iii) Define $F(X,Y)$ by the schema, called $R_1{}^{**}$

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(Y,F(X,Y)) \qquad u=1,\ldots,n$$

where $B_u(Y,B_v(Y,Z)) = B_v(Y,B_u(Y,Z))$ for all u,v.

(iv) Define $F(X,Y)$ by the schema called $R_1{}^{***}$

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(F(X,Y)) \qquad u=1,\ldots,n$$

where $B_u(B_v(Z)) = B_v(B_u(Z))$ for all u,v.

We consider further what functions, if any, are required to be added to the initial set of functions in order to produce the set of all primitive recursive functions, from each of these definitions.

In single successor arithmetic, the reduction of parameters in the definition of primitive recursion is to be found in [1] and [3].

2. A complete analysis of X.

In single successor arithmetic a numeral x can be compared relatively simply to any other numeral with the aid of the difference function $y \dot{-} x$. Using both $y \dot{-} x$ and $x \dot{-} y$ a complete comparison is made between x and y. In multiple successor arithmetic, however, X is not so easily analysed. The questions which we seek to answer are the following, in order to obtain a complete analysis of X.

(i) How many successor symbols in X?

(ii) How many different successor symbols?

(iii) What these different successor symbols are.

(iv) How many of each successor symbol?

With these four questions answered the analysis of X is obviously complete and a comparison between X and Y can be effected.

CHAPTER II

REDUCTION OF PARAMETERS IN THE DEFINITION BY PRIMITIVE RECURSION

In this chapter we will consider the problem of the reduction of parameters in the definition of primitive recursion in Commutative Multiple Successor Arithmetics.

The first task will be to consider the reduction, given that we have already in our arithmetic a set of functions with special properties. Different functions will then be considered for this set. In chapter III the arithmetic will be formalised and a proof of these properties will be given.

In single successor arithmetic the reduction is solved with the aid of three functions $L(x)$, $K(x)$ and $J(u,v)$, such that

$$L(J(u,v)) = u$$

$$K(J(u,v)) = v$$

We shall suppose first of all that in commutative multiple successor arithmetic we have three functions $L(X)$, $K(X)$ & $J(U,V)$ with the above properties. With these functions we are able to reduce the number of parameters in the definition of primitive recursion.

Consider therefore the definition of primitive recursion with 'm' parameters (m finite); this schema we will call R_m:-

A function $F(X,Y_1,Y_2,\ldots,Y_m)$ is said to be defined by primitive recursion from A and B if

$$F(0,Y_1,Y_2,\ldots,Y_m) = A(Y_1,Y_2,\ldots,Y_m)$$

$$F(S_uX,Y_1,Y_2,\ldots,Y_m) = B_u(X,Y_1,Y_2,\ldots,Y_m,F(X,Y_1,Y_2,\ldots,Y_m)) \quad u=1,\ldots,n$$

where $B_u(S_vX,Y_1,Y_2,\ldots,Y_m,B_v(X,Y_1,Y_2,\ldots,Y_m)) = B_v(S_uX,Y_1,Y_2,\ldots,Y_m,B_u(X,Y_1,Y_2,\ldots Y_m))$ all u,v

and $A(\ldots.)$ and $B_u(\ldots.)$ $u=1,\ldots,n$ are previously defined functions.

Now let $W = J(Y_1, Y_2)$

and let $G(X,W,Y_3,Y_4,\ldots,Y_m) = F(X,L(W),K(W),Y_3,Y_4,\ldots,Y_m)$

$$= F(X,Y_1,Y_2,\ldots,Y_m) .$$

Then $G(X,W,Y_3,Y_4,\ldots,Y_m)$ is defined by

$$G(0,W,Y_3,Y_4,\ldots,Y_m) = A(L(W),K(W),Y_3,\ldots,Y_m)$$

$$= A'(W,Y_3,Y_4,\ldots,Y_m)$$

$$G(S_uX,W,Y_3,Y_4,\ldots,Y_m) = B_u(X,L(W),K(W),Y_3,\ldots,Y_m,G(X,W,Y_3,\ldots,Y_m)) \qquad u=1,\ldots,n$$

$$= B'_u(X,W,Y_3,\ldots,Y_m,G(X,W,Y_3,\ldots,Y_m)) \qquad u=1,\ldots,n$$

and the commutativity condition reduces to

$$B_u(S_vX,L(W),K(W),Y_3,\ldots,Y_m,\ B_v(X,L(W),K(W),Y_3,\ldots,Y_m)) =$$

$$B_v(S_uX,L(W),K(W),Y_3,\ldots,Y_m,\ B_u(X,L(W),K(W),Y_3,\ldots,Y_m,Z)) \qquad \text{all } u,v$$

and therefore

$$B'_u(S_vX,W,Y_3,\ldots,Y_m,B'_v(X,W,Y_3,\ldots,Y_m,Z)) = B'_v(S_uX,W,Y_3,\ldots,Y_m,B'_u(X,W,Y_3,\ldots,Y_m,Z)) \qquad \text{all } u,v.$$

Hence the definition by primitive recursion is now reduced to a schema R_{m-1} containing $m-1$ parameters, that is, $F(X,Y_1,Y_2,\ldots,Y_{m-1})$ is defined by

$$F(0,Y_1,Y_2,\ldots,Y_{m-1}) = A(Y_1,Y_2,\ldots,Y_{m-1})$$

$$F(S_uX,Y_1,Y_2,\ldots,Y_{m-1}) = B_u(X,Y_1,\ldots,Y_{m-1},F(X,Y_1,\ldots,Y_{m-1})) \quad u=1,\ldots,n$$

where

$$B_u(S_vX,Y_1,\ldots,Y_{m-1},B_v(X,Y_1,\ldots,Y_{m-1},Z)) = B_v(S_uX,Y_1,\ldots,Y_{m-1},B_u(X,Y_1,\ldots,Y_{m-1},Z)) \qquad \text{all } u,v.$$

By repeated use of this method we can reduce the definition to a definition with only one parameter R_1, that is, $F(X,Y)$ is defined from A and B by:

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,Y,F(X,Y)) \qquad u=1,\ldots,n$$

where $\quad B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(S_uX,Y,B_u(X,Y,Z)) \quad$ all u,v .

The prof just given reduces R_m to R_1 in m−1 steps. A similar technique can be used to complete the proof in a single step. We proceed as follows:

Let $W = J(Y_1,J(Y_2,J(Y_3,\ldots,J(Y_{m-1},Y_m))) \ldots)$

Let $K^2(W) = K(K(W))$, and $K^r(W) = \overbrace{K(K(\ldots K}^{r}(W))\ldots)$

We therefore have that

$$L(W) = Y_1$$

$$L(K(W)) = Y_2$$

$$L(K^2(W)) = Y_3$$

$$\vdots$$

$$L(K^{m-2}(W)) = Y_{m-1}$$

$$K^{m-1}(W) = Y_m$$

and let $\quad G(X,W) = F(X,L(W),L(K(W)),\ldots,L(K^{m-2}(W)),K^{m-1}(W))$.

We therefore define $G(X,W)$ by

$$G(0,W) = A(L(W),L(K(W)),\ldots,L(K^{m-2}(W)),K^{m-1}(W))$$

$$= A'(W), \text{ say.}$$

$$G(S_uX,W) = B_u(X,L(W),L(K(W)),\ldots,L(K^{m-2}(W)),K^{m-1}(W),$$
$$F(X,L(W),L(K(W)),\ldots L(K^{m-1}(W)),K^{m-1}(W)))$$
$$u=1,\ldots,n$$

$$= B'_u(X,W,G(X,W)), \text{ say} \qquad u=1,\ldots,n$$

and the commutativity condition reduces to

$$B_u(S_vX,L(W),L(K(W)),\ldots,L(K^{m-2}(W)),K^{m-1}(W),B_v(X,L(W),L(K(W)),\ldots,$$
$$\ldots,L(K^{m-2}(W)),K^{m-1}(W))) =$$

$$B_v(S_uX,L(W),L(K(W)),\ldots,L(K^{m-2}(W)),K^{m-1}(W),B_u(X,L(W),L(K(W)),\ldots$$
$$\ldots,L(K^{m-2}(W)),K^{m-1}(W)))$$

$$\therefore\ B'_u(S_vX,W,B'_v(X,W,Z)) = B'_v(S_uX,W,B'_u(X,W,Z)) \qquad \text{all } u,v.$$

Thus R_m is reduced to R_1 .

We now require:

(i) to reduce R_1 to $R_1{}^*$ where $F(X,Y)$ is defined by the schema

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,F(X,Y)) \qquad u=1,\ldots,n$$

where $\quad B_u(S_vX,B_v(X,Z)) = B_v(S_uX,B_u(X,Z)) \quad$ all u,v

(ii) to reduce R_1 to $R_1{}^{**}$ where $F(X,Y)$ is defined by the schema

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(Y,F(X,Y)) \qquad u = 1,\ldots,n$$

where $\quad B_u(Y,B_v(Y,Z)) = B_v(Y,B_u(Y,Z)) \qquad$ all u,v.

(iii) to reduce R_1 to $R_1{}^{***}$ where $F(X,Y)$ is defined by the schema

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(F(X,Y)) \qquad u = 1,\ldots,n$$

where $\quad B_u(B_v(Z)) = B_v(B_u(Z)) \qquad$ all u,v.

Reduction of R_1 to R_1^*:

Let $G(X,Y) = J(Y,F(X,Y))$.

Then $G(X,Y)$ is defined by the schema

$$\begin{aligned} G(0,Y) &= J(Y,A(Y)) \\ &= A'(Y), \text{ say} \\ G(S_uX,Y) &= J(Y,F(S_uX,Y)) \\ &= J(Y,B_u(X,Y,F(X,Y))) \\ &= J(L(G(X,Y)),B_u(X,L(G(X,Y)),K(G(X,Y)))) \\ &= B'_u(X,G(X,Y)), \text{ say} \qquad u=1,\ldots,n \end{aligned}$$

and for the commutativity condition we let

$$Z' = J(Y,Z) .$$

$$\begin{aligned} \text{Now } B'_u(S_vX,B'_v(X,Z')) &= B'_u(S_vX,J(L(Z'),B_v(X,L(Z'),K(Z')))) \\ &= B'_u(S_vX,J(Y,B_v(X,Y,Z))) \\ &= J(L(J(Y,B_v(X,Y,Z))),B_u(S_vX,L(J(Y,B_v(X,Y,Z))), \\ &\qquad K(J(Y,B_v(X,Y,Z))))) \\ &= J(Y,B_u(S_vX,Y,B_v(X,Y,Z))) \\ &= J(Y,B_v(S_uX,Y,B_u(X,Y,Z))) \\ &= J(L(J(Y,B_u(X,Y,Z))),B_v(S_uX,L(J(Y,B_u(X,Y,Z))), \\ &\qquad K(J(Y,B_u(X,Y,Z))))) \\ &= B'_v(S_uX,J(Y,B_u(X,Y,Z))) \\ &= B'_v(S_uX,J(L(Z'),B_u(X,L(Z'),K(Z')))) \\ &= B'_v(S_uX,B'_u(X,Z')) \qquad \text{all } u,v, \end{aligned}$$

which is the required commutativity condition.

Thus $F(X,Y) = K(G(X,Y))$.

Reduction of R_1 to R_1^{**}:

Let $G(X,Y) = J(X,F(X,Y))$.

Then $G(X,Y)$ is defined by the schema

$$G(0,Y) = J(0,A(Y))$$

$$= A'(Y), \text{ say}$$

$$G(S_uX,Y) = J(S_uX,F(S_uX,Y)) \qquad u=1,\ldots,n$$

$$= J(S_uX,B_u(X,Y,F(X,Y)))$$

$$= J(S_uL(G(X,Y)),B_u(L(G(X,Y)),Y,K(G(X,Y))))$$

$$= B'_u(Y,G(X,Y)), \text{ say} \qquad u=1,\ldots,n$$

and for the commutativity condition we let

$$Z' = J(X,Z) .$$

Thus

$$B'_u(Y,B'_v(Y,Z')) = B'_u(J(S_vL(Z'),B_v(L(Z'),Y,K(Z'))))$$

$$= B'_u(Y,J(S_vX,B_v(X,Y,Z)))$$

$$= J(S_uL(J(S_vX,B_v(X,Y,Z))),B_u(L(J(S_vX,B_v(X,Y,Z))),$$
$$Y,K(J(S_vX,B_v(X,Y,Z)))))$$

$$= J(S_uS_vX,B_u(S_vX,Y,B_v(X,Y,Z)))$$

$$= J(S_vS_uX,B_v(S_uX,Y,B_u(X,Y,Z)))$$ (commutativity condition for R_1 and commutativity of successors)

$$= J(S_vL(J(S_uX,B_u(X,Y,Z))),B_v(L(J(S_uX,B_u(X,Y,Z))),$$
$$Y,K(J(S_uX,B_u(X,Y,Z)))))$$

$$= B'_v(Y,J(S_uX,B_u(X,Y,Z)))$$

$$= B'_v(J(S_uL(Z'),B_uL(Z'),Y,K(Z'))))$$

$$= B'_v(Y,B'_u(Y,Z')) \qquad \text{all } u,v ,$$

which is the required commutativity condition.

Thus $F(X,Y) = K(G(X,Y))$.

Reduction of R_1 to R_1^{***}:

the reduction of R_1 to R_1^{***} can be achieved in three ways, that is

(1) R_1 to R_1^* then R_1^* to R_1^{***}

(2) R_1 to R_1^{**} then R_1^{**} to R_1^{***}

(3) R_1 direct to R_1^{***} .

Both the reductions R_1^* to R_1^{***} and R_1^{**} to R_1^{***} are similar to the reductions R_1 to R_1^{**} and R_1 to R_1^* respectively.

We shall consider the reductions R_1^* to R_1^{***} and R_1 to R_1^{***}.

Reduction R_1^* to R_1^{***}:

R_1^* is the schema

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,F(X,Y)) \qquad u=1,\ldots,n$$

where $B_u(S_vX,B_v(X,Z)) = B_v(S_uX,B_u(X,Z))$ all u,v.

First let $G(X,Y) = J(X,F(X,Y)$.

Then $G(X,Y)$ is defined by the schema

$$\begin{aligned} G(0,Y) &= J(0,A(Y)) \\ &= A'(Y), \text{ say} \\ G(S_uX,Y) &= J(S_uX,F(S_uX,Y)) \\ &= J(S_uX,B_u(X,F(X,Y))) \qquad u=1,\ldots,n \\ &= J(S_uL(G(X,Y)),B_u(L(G(X,Y)),K(G(X,Y)))) \\ &= B_u'(G(X,Y)), \text{ say}, \qquad u=1,\ldots,n. \end{aligned}$$

and for the commutativity condition we let

$$Z' = J(X,Z) .$$

Then

$$\begin{aligned}
B'_u(B'_v(Z')) &= B'_u(J(S_vL(Z'),B_v(L(Z'),K(Z')))) \\
&= B'_u(J(S_vX,B_v(X,Z))) \\
&= J(S_uL(J(S_vX,B_v(X,Z))),B_u(L(J(S_vX,B_v(X,Z))),K(J(S_vX,B_v(X,Z))))) \\
&= J(S_uS_vX,B_u(S_vX,B_v(X,Z))) \\
&= J(S_vS_uX,B_v(S_uX,B_u(X,Z))) \quad \text{(commutativity condition of } R_1{}^* \text{ and commutativity of successors)} \\
&= J(S_vL(J(S_uX,B_u(X,Z))),B_v(L(J(S_uX,B_u(X,Z))),K(J(S_uX,B_u(X,Z))))) \\
&= B'_v(J(S_uX,B_u(X,Z))) \\
&= B'_v(J(S_uL(Z'),B_u(L(Z'),K(Z')))) \\
&= B'_v(B'_u(Z')) \quad \text{all } u,v,
\end{aligned}$$

which is the required commutativity condition.

Thus $F(X,Y) = K(G(X,Y))$.

Reduction of R_1 to $R_1{}^{***}$:

R_1 is the schema

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,Y,F(X,Y)) \quad u = 1,\ldots,n$$

where $B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(S_uX,Y,B_u(X,Y,Z))$ all u,v.

First let $G(X,Y) = J(X,J(Y,F(X,Y)))$.

Then $G(X,Y)$ is defined by the schema

$$\begin{aligned}
G(0,Y) &= J(0,J(Y,F(0,Y))) \\
&= J(0,J(Y,A(Y))) \\
&= A'(Y), \text{ say}
\end{aligned}$$

$$G(S_uX,Y) = J(S_uX,J(F(S_uX,Y))) \qquad u=1,\ldots,n$$

$$= J(S_uX,J(Y,B_u(X,Y,F(X,Y))))$$

$$= J(S_uL(G(X,Y)),J(L(K(G(X,Y))),B_u(L(G(X,Y)),L(K(G(X,Y))),$$
$$K(K(G(X,Y))))))$$

$$= B'_u(G(X,Y)), \text{ say} \qquad u=1,\ldots,n$$

and for the commutativity condition we let

$$Z' = J(X,J(Y,Z)).$$

Then

$$B'_u(B'_v(Z')) = B'_u(J(S_vL(Z'),J(L(K(Z')),B_v(L(Z'),L(K(Z')),K(K(Z'))))))$$

$$= B'_u(J(S_vX,J(Y,B_v(X,Y,Z))))$$

$$\left[\begin{array}{ll} \text{Put} \quad W_v = J(S_vX,J(Y,B_v(X,Y,Z))) & \text{for suffix } v \text{ or } u \\ \therefore \quad L(W_v) = S_vX, \quad L(K(W_v)) = Y, & K(K(W_v)) = B_v(X,Y,Z) \end{array}\right]$$

$$= J(S_uL(W_v),J(L(K(W_v)),B_u(L(W_v),L(K(W_v)),K(K(W_v)))))$$

$$= J(S_uS_vX,J(Y(B_u(S_vX,Y,B_v(X,Y,Z))))$$

$$= J(S_vS_uX,J(Y,B_v(S_uX,Y,B_u(X,Y,Z)))) \quad \text{(commutativity condition of } R_1 \text{ and commutativity of successors)}$$

$$= J(S_vL(W_u),J(L(K(W_u)),B_v(L(W_u)),L(K(W_u)),K(K(W_u)))))$$

$$= B'_v(J(S_uX,J(Y,B_u(X,Y,Z))))$$

$$= B'_v(J(S_uL(Z'),J(L(K(Z')),B_u(L(Z'),L(K(Z')),K(K(Z'))))))$$

$$= B'_v(B'_u(Z')) \qquad \text{all } u,v$$

which is the required commutativity condition.

Thus $F(X,Y) = K(K(G(X,Y)))$.

In the previous sections the problem of reduction of parameters in the definition of primitive recursion with 'm' parameters has been successfully achieved with the aid of the set of functions $L(X)$, $K(X)$ & $J(U,V)$. Now we must consider what functions have the required properties

$$L(J(U,V)) = U$$
$$K(J(U,V)) = V \quad .$$

In order to discover these functions, we first examine the functions used in the reduction in single successor arithmetic. To do this we need to define the following.

<u>Defn.</u> $y + x$ Addition

$y + 0 = y$

$y + Sx = S(y + x)$

<u>Defn.</u> Px Predecessor

$P0 = 0$

$PSx = x$

<u>Defn.</u> $y \dot{-} x$ Difference

$y \dot{-} 0 = y$

$y \dot{-} Sx = P(y \dot{-} x)$

<u>Defn.</u> $y.x$ Multiplication

$y.0 = 0$

$y.Sx = y.x + y$

<u>Defn.</u> $x.x = x^2$ Notational definition (explicit)

<u>Defn.</u> Rx Square root

$R0 = 0$

$RSx = Rx + (S0 \dot{-} ((SRx)^2 \dot{-} Sx)))$

Clearly, from the definition, Rx is the integer part of the square root of x, or the square root of the greatest perfect square in x.

Now with the aid of these functions we can define $L(x)$, $K(x)$ & $J(u,v)$ as follows:

$$J(u,v) = (u + v)^2 + u$$

$$L(X) = x \dot{-} (Rx)^2$$

$$K(x) = Rx \dot{-} L(x)$$

Obviously, informally,

$$L(J(u,v)) = ((u + v)^2 + u) \dot{-} (R((u + v)^2 + u))^2$$

$$= ((u + v)^2 + u) \dot{-} (u + v)^2$$

$$= u$$

$$K(J(u,v)) = (u + v) \dot{-} u$$

$$= v \quad \text{as required.}$$

In multiple successor arithmetic however the corresponding functions are not quite so straightforward.

As can be clearly seen the functions $L(x)$, $K(x)$ & $J(u,v)$ are based on the relations

$$R(u^2) = u$$

and $$R((u + v)^2 + u) = u + v \quad .$$

In other words, we seek to define a suitable multiplication function with a suitable square root function.

Let us consider the functions we have for multiplication, their properties, and possible square roots.

1. Vučković's multiplication function defined by

$$Y*0 = 0$$

$$Y*S_uX = Y*X\ \sigma_uY \qquad u=1,\ldots,n$$

The commutativity condition requires the equality of

$$B_u(S_vX,Y,B_v(X,Y,Z)) = (Z\ \sigma_vY)\ \sigma_uY$$

$$B_v(S_uX,Y,B_u(X,Y,Z)) = (Z\ \sigma_uY)\ \sigma_vY\ .$$

We shall prove these equal by the following lemma.

LEMMA $\qquad (Y\ \sigma_vX)\sigma_uX = (Y\ \sigma_uX)\ \sigma_vX$

<u>Proof</u>

$$L(0,Y) = (Y\sigma_v0)\ \sigma_u0$$

$$= (Y\ \sigma_v0) \qquad \text{Defn}$$

$$= Y \qquad \text{Defn}$$

$$R(0,Y) = (Y\ \sigma_u0)\ \sigma_v0$$

$$= (Y\ \sigma_u0) \qquad \text{Defn}$$

$$= Y \qquad \text{Defn}$$

$$L(S_wX,Y) = (Y\ \sigma_vS_wX)\ \sigma_uS_wX$$

$$= S_{w+u}(Y\ \sigma_vS_wX)\ \sigma_uX \qquad \text{Defn}$$

$$= S_{w+u}S_{w+v}(Y\ \sigma_vX)\ \sigma_uX \qquad \text{Defn}$$

$$R(S_wX,Y) = (Y\ \sigma_uS_wX)\ \sigma_vS_wX$$

$$= S_{w+v}(Y\ \sigma_uS_wX)\ \sigma_vX \qquad \text{Defn}$$

$$= S_{w+v}S_{w+u}(Y\ \sigma_uX)\ \sigma_vX \qquad \text{Defn}$$

$$= S_{w+u}S_{w+v}(Y\ \sigma_uX)\ \sigma_vX \qquad \text{(commutativity of successors)}$$

$\therefore \qquad L(X,Y) = R(X,Y)$

$\therefore \qquad B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(S_uX,Y,B_u(X,Y,Z))$

The properties of this function include commutativity, associativity, and distribution over σ_u functions for all u; the case of $u = 0$ is addition.

Commutativity is proved by the following lemmas:

LEMMA $\qquad 0*X = 0$

<u>Proof</u>

$$
\begin{aligned}
L(0) &= 0*0 \\
&= 0 && \text{Def'n} \\
R(0) &= 0 \\
L(S_uX) &= 0*S_uX \\
&= 0*X\ \sigma_u 0 && \text{Def'n} \\
&= 0*X && \text{Def'n} \\
&= L(X) \\
R(S_uX) &= 0 \\
&= R(X)
\end{aligned}
$$

$\therefore \qquad L(X) = R(X)$ by uniqueness

LEMMA $\qquad (Z\ \sigma_v Y)\ \sigma_u X = (Z\ \sigma_u X)\ \sigma_v Y$

<u>Proof</u>

$$
\begin{aligned}
L(0,Y,Z) &= (Z\ \sigma_v Y)\ \sigma_u 0 \\
&= (Z\ \sigma_v Y) && \text{Def'n} \\
R(0,Y,Z) &= (Z\ \sigma_u 0)\ \sigma_v Y \\
&= Z\ \sigma_v Y && \text{Def'n}
\end{aligned}
$$

$$L(S_w X,Y,Z) = (Z\ \sigma_v Y)\ \sigma_u S_w X$$

$$= S_{w+u}(Z\ \sigma_v Y)\ \sigma_u X \qquad \text{Def'n}$$

$$= S_{w+u} L(X,Y,Z)$$

$$R(S_w X,Y,Z) = (Z\ \sigma_u S_w X)\ \sigma_v Y$$

$$= (S_{w+u}(Z\ \sigma_u X))\ \sigma_v Y \qquad \text{Def'n}$$

$$= S_{w+u}(Z\ \sigma_u X)\ \sigma_v Y \qquad \text{Previous Lemma}$$

$$= S_{w+u}(R(X,Y,Z))$$

$$\therefore \qquad L(X,Y,Z) = R(X,Y,Z) \qquad \text{by uniqueness}$$

LEMMA $\qquad S_u Y^* X = Y^* X\ \sigma_u X$

<u>Proof</u>

$$L(0,Y) = S_u Y^* 0$$

$$= 0 \qquad \text{Def'n}$$

$$R(0,Y) = Y^* 0\ \sigma_u 0$$

$$= Y^* 0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$L(S_v X,Y) = S_u Y^* S_v X$$

$$= S_u Y^* X\ \sigma_v S_u Y \qquad \text{Def'n}$$

$$= S_{u+v}(S_u Y^* X)\ \sigma_v Y \qquad \text{Def'n}$$

$$= S_{u+v}(L(X,Y))\ \sigma_v Y$$

$$R(S_v X,Y) = Y^* S_v X\ \sigma_u S_v X$$

$$= (Y^* X\ \sigma_v Y)\ \sigma_u S_v X \qquad \text{Def'n}$$

$$= S_{v+u}(Y^* X\ \sigma_v Y)\ \sigma_u X \qquad \text{Def'n}$$

$$= S_{u+v}(Y^* X\ \sigma_u X)\ \sigma_v Y \qquad \text{Previous Lemma}$$

$$= S_{u+v}(R(X,Y))\ \sigma_v Y$$

$\therefore \qquad L(X,Y) = R(X,Y)$ by uniqueness

PROPOSITION $\quad Y*X = X*Y$ Commutativity

<u>Proof</u>

$$L(0,Y) = Y*0$$

$$= 0 \qquad \text{Defn}$$

$$R(0,Y) = 0*Y$$

$$= 0 \qquad \text{Previous Lemma}$$

$$L(S_uX,Y) = Y*S_uX$$

$$= Y*X\ \sigma_uY \qquad \text{Defn}$$

$$= L(X,Y)\ \sigma_uY$$

$$R(S_uX,Y) = S_uX*Y$$

$$= X*Y\ \sigma_uY \qquad \text{Defn}$$

$$= R(X,Y)\ \sigma_uY$$

$\therefore \qquad L(X,Y) = R(X,Y)$ by uniqueness

For the distributive property we need the following Lemmas:

LEMMA $\quad Z\ \sigma_u(Y\ \sigma_vX) = (Z\sigma_uY)\ \sigma_{v+u}X$

<u>Proof</u>

$$L(0,Y,Z) = Z\ \sigma_u(Y\ \sigma_v0)$$

$$= Z\ \sigma_uY \qquad \text{Defn}$$

$$R(0,Y,Z) = (Z\ \sigma_uY)\ \sigma_{v+u}0$$

$$= Z\ \sigma_uY \qquad \text{Defn}$$

$$L(S_wX,Y,Z) = Z\ \sigma_u(Y\ \sigma_vS_wX)$$

$$= Z\ \sigma_uS_{w+v}(Y\ \sigma_vX) \qquad \text{Defn}$$

$$= S_{w+v+u}Z\ \sigma_u(Y\ \sigma_vX) \qquad \text{Defn}$$

$$= S_{w+v+u} L(X,Y,Z)$$

$$R(S_w X,Y,Z) = (Z\ \sigma_u Y)\ \sigma_{v+u} S_w X$$

$$= S_{w+v+u}(Z\ \sigma_u Y)\ \sigma_{v+u} X \qquad \text{Defn}$$

$$= S_{w+v+u} R(X,Y,Z)$$

$\therefore \qquad L(X,Y,Z) = R(X,Y,Z) \qquad$ by uniqueness

PROPOSITION $\quad Z^*(Y\ \sigma_u X) = Z^*Y\ \sigma_u Z^*X \qquad$ Distributive

<u>Proof</u> $\quad L(0,Y,Z) = Z^*(Y\ \sigma_u 0)$

$$= Z^*Y \qquad \text{Defn}$$

$$R(0,Y,Z) = Z^*Y\ \sigma_u Z^*0$$

$$= Z^*Y\ \sigma_u 0 \qquad \text{Defn}$$

$$= Z^*Y \qquad \text{Defn}$$

$$L(S_v X,Y,Z) = Z^*(Y\ \sigma_u S_v X)$$

$$= Z^* S_{v+u}(Y\ \sigma_u X) \qquad \text{Defn}$$

$$= Z^*(Y\ \sigma_u X)\ \sigma_{v+u} Z \qquad \text{Defn}$$

$$= L(X,Y,Z)\ \sigma_{v+u} Z$$

$$R(S_v X,Y,Z) = Z^*Y\ \sigma_u Z^* S_v X$$

$$= Z^*Y\ \sigma_u (Z^*X\ \sigma_v Z) \qquad \text{Defn}$$

$$= (Z^*Y\ \sigma_u Z^*X)\ \sigma_{v+u} Z \qquad \text{Previous Lemma}$$

$$= R(X,Y,Z)\ \sigma_{v+u} Z$$

$\therefore \qquad L(X,Y,Z) = R(X,Y,Z) \qquad$ by uniqueness

PROPOSITION $(Z*Y)*X = Z*(Y*X)$ Associative

<u>Proof</u>

$$L(0,Y,Z) = (Z*Y)*0$$

$$= 0 \qquad \text{Def'n}$$

$$R(0,Y,Z) = Z*(Y*0)$$

$$= Z*0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$L(S_uX,Y,Z) = (Z*Y)*S_uX$$

$$= (Z*Y)*X\ \sigma_u(Z*Y) \qquad \text{Def'n}$$

$$= L(X,Y,Z)\ \sigma_u(Z*Y)$$

$$R(S_uX,Y,Z) = Z*(Y*S_uX)$$

$$= Z*(Y*X\ \sigma_uY) \qquad \text{Def'n}$$

$$= Z*(Y*X)\ \sigma_uZ*Y \qquad \text{Previous Proposition}$$

$$= R(X,Y,Z)\ \sigma_u(Z*Y)$$

$\therefore$ $L(X,Y,Z) = R(X,Y,Z)$ by uniqueness

In general however $Z*(Y\ \tau_uX) \neq Z*Y\ \tau_uZ*X$.

Consider the counter-example

$$u=0,\quad X=S_20,\quad Y=S_10,\quad Z=S_2S_10$$

$$S_2S_10*(S_10\ \tau_0S_20) = S_2S_10*P_2(S_10\ \tau_0\ 0)$$

$$= S_2S_10*P_2S_10 \qquad \text{Def'n}$$

$$= S_2S_10*S_10 \qquad \text{Def'n}$$

$$= S_2S_10*0\ \sigma_1\ S_2S_10 \qquad \text{Def'n}$$

$$= 0\ \sigma_1\ S_2S_10 \qquad \text{Def'n}$$

$$= S_3S_20 \qquad \text{Def'n}$$

$$S_2S_1 0^*S_1 0 \; \tau_0 \; S_2S_1 0^*S_2 0 = (S_2S_1 0^*0 \; \sigma_1 \; S_2S_1 0) \; \tau_0 \; (S_2S_1 0^*0 \; \sigma_2 \; S_2S_1 0) \qquad \text{Def'n}$$

$$= S_3S_2 0 \; \tau_0 \; S_4S_3 0 \qquad \text{Def'n}$$

$$= P_4P_3S_3S_2 0 \qquad \text{Def'n}$$

$$= S_2 0 \qquad \text{Def'n}$$

Thus in general $Z^*(Y \tau_u X) \neq Z^*Y \tau_u Z^*X$.

2. We shall now consider the multiplicative function defined by

$$Y^X 0 = 0$$

$$Y^X S_u X = Y^X X + Y \qquad u = 1,\ldots,n$$

and the commutativity condition

$$B_u(S_v X, Y, B_v(X,Y,Z)) = B_v(X,Y,Z) + Y$$

$$= (Z + Y) + Y$$

$$= B_u(X,Y,Z) + Y$$

$$= B_v(S_u X, Y, B_u(X,Y,Z)) \; .$$

Clearly the function does not distinguish between successors, but unfortunately when $n > 1$ the function is not commutative. We shall prove this by a simple counter-example.

Put $Y = S_1 0$, $X = S_2 0$

$$Y^X X = S_1 0^X S_2 0$$

$$= S_1 0^X 0 + S_1 0 \qquad \text{Def'n}$$

$$= 0 + S_1 0 \qquad \text{Def'n}$$

$$= S_1 0 \qquad \text{Def'n}$$

$$X^XY = S_20^XS_10$$

$$= S_20^X0 + S_20 \qquad \text{Defn}$$

$$= 0 + S_20 \qquad \text{Defn}$$

$$= S_20 \qquad \text{Previous Lemma}$$

PROPOSITION $\quad Z^X(Y+X) = Z^XY + Z^XX \qquad$ Distributive over addition

<u>Proof</u> $\quad L(0,Y,Z) = Z^X(Y + 0)$

$$= Z^XY \qquad \text{Defn}$$

$$R(0,Y,Z) = Z^XY + Z^X0$$

$$= Z^XY + 0 \qquad \text{Defn}$$

$$= Z^XY \qquad \text{Defn}$$

$$L(S_uX,Y,Z) = Z^X(Y + S_uX)$$

$$= Z^XS_u(Y + X) \qquad \text{Defn}$$

$$= Z^X(Y + X) + Z \qquad \text{Defn}$$

$$= L(X,Y,Z) + Z$$

$$R(S_uX,Y,Z) = Z^XY + Z^XS_uX$$

$$= Z^XY + (Z^XX + Z) \qquad \text{Defn}$$

$$= Z^XY + Z^XX + Z \qquad \text{Associative rule for addition}$$

$$= R(X,Y,Z) + Z$$

$\therefore \quad L(X,Y,Z) = R(X,Y,Z) \qquad$ by uniqueness

PROPOSITION $\quad (Z^XY)^XX = Z^X(Y^XX) \qquad$ Associative

<u>Proof</u> $\quad L(0,Y,Z) = (Z^XY)^X0$

$$= 0 \qquad \text{Defn}$$

$$R(0,Y,Z) = Z^X(Y^X 0)$$

$$= Z^X 0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$L(S_u X,Y,Z) = (Z^X Y)^X S_u X$$

$$= (Z^X Y)^X X + (Z^X Y) \qquad \text{Def'n}$$

$$= L(X,Y,Z) + (Z^X Y)$$

$$R(S_u X,Y,Z) = Z^X(Y^X S_u X)$$

$$= Z^X(Y^X X + Y) \qquad \text{Def'n}$$

$$= Z^X(Y^X X) + Z^X Y \qquad \text{Previous Lemma}$$

$$= R(X,Y,Z) + (Z^X Y)$$

$$\therefore \qquad L(X,Y,Z) = R(X,Y,Z) \qquad \text{by uniqueness}$$

However, although this function is distributive over addition, it is not distributive over the σ_u functions.

That is $Z^X(Y\,\sigma_u X) \neq Z^X Y\,\sigma_u Z^X X$

Put $Z = S_2 0$, $Y = S_1 0$, $X = S_2 0$

$$S_2 0^X(S_1 0\,\sigma_u S_2 0) = S_2 0^X(S_{2+u} S_1 0\,\sigma_u 0) \qquad \text{Def'n}$$

$$= S_2 0^X S_{2+u} S_1 0 \qquad \text{Def'n}$$

$$= S_2 0^X S_1 0 + S_2 0 \qquad \text{Def'n}$$

$$= S_2 0^X 0 + S_2 0 + S_2 0 \qquad \text{Def'n}$$

$$= S_2 S_2 0$$

$$S_2 0^X S_1 0\,\sigma_u S_2 0^X S_2 0 = (S_2 0^X 0 + S_2 0)\,\sigma_u(S_2 0^X 0 + S_2 0) \qquad \text{Def'n}$$

$$= S_2 0\,\sigma_u S_2 0 \qquad \text{Def'n}$$

$$= S_{2+u} S_2 0 \qquad \text{Def'n}$$

and $S_{2+u} S_2 0$ is only equal to $S_2 S_2 0$ when $u = 0$.

Also we find that in general

$$Z^X(Y\,\tau_u X) \neq Z^X Y\,\tau_u Z^X X \quad .$$

Put $u = 0$, $Z = S_1 0$, $Y = S_2 S_1 0$, $X = S_1 S_1 0$

$$\begin{aligned}
S_1 0^X\,(S_2 S_1 0\,\tau_u S_1 S_1 0) &= S_1 0^X P_1 P_1 S_2 S_1 0 && \text{Defn}\\
&= S_1 0^X S_2 0 && \text{Defn}\\
&= S_1 0 && \text{Defn}\\
S_1 0^X S_2 S_1 0\,\tau_u S_1 0^X S_1 S_1 0 &= (S_1 0^X S_1 0 + S_1 0)\,\tau_u (S_1 0^X S_1 0 + S_1 0) && \text{Defn}\\
&= (S_1 0^X 0 + S_1 0 + S_1 0)\,\tau_u (S_1 0^X 0 + S_1 0 + S_1 0) && \text{Defn}\\
&= S_1 S_1 0\,\tau_u S_1 S_1 0 && \text{Defn}\\
&= P_1 P_1 S_1 S_1 0 && \text{Defn}\\
&= 0 && \text{Defn}
\end{aligned}$$

3. For the last multiplication function we consider the function defined by

$$Y.0 = 0$$

$$Y.S_u X = Y.X + C_u Y \qquad u = 1,\ldots,n$$

and the commutativity condition is

$$\begin{aligned}
B_u(S_v X, Y, B_v(X,Y,Z)) &= B_v(X,Y,Z) + C_u Y\\
&= (Z + C_v Y) + C_u Y\\
&= (Z + C_u Y) + C_v Y && \text{Associative rule of addition}\\
&= B_u(X,Y,Z) + C_v Y\\
&= B_v(S_u X, Y, B_u(X,Y,Z)) \ .
\end{aligned}$$

This function is commutative, proved by the following Lemmas.

LEMMA $0.X = 0$

Proof

$$L(0) = 0.0$$

$$= 0 \qquad \text{Defn}$$

$$R(0) = 0$$

$$L(S_uX) = 0.S_uX$$

$$= 0.X + C_u0 \qquad \text{Defn}$$

$$= 0.X + 0 \qquad \text{Defn}$$

$$= 0.X \qquad \text{Defn}$$

$$= L(X)$$

$$R(S_uX) = 0$$

$$= R(X)$$

$\therefore$ $L(X) = R(X)$ by uniqueness

LEMMA $S_uY.X = Y.X + C_uX$

Proof

$$L(0,Y) = S_uY.0$$

$$= 0 \qquad \text{Defn}$$

$$R(0,Y) = Y.0 + C_u0$$

$$= 0 + 0 \qquad \text{Defn}$$

$$= 0 \qquad \text{Defn}$$

$$L(S_vX,Y) = S_uY.S_vX$$

$$= S_uY.X + C_vS_uY \qquad \text{Defn}$$

$$= \begin{cases} S_uY.X + S_vC_vY & u=v \quad \text{Defn} \\ S_uY.X + C_vY & u\neq v \quad \text{Defn} \end{cases}$$

$$= \begin{cases} S_v(S_uY.X + C_vY & u=v \\ L(X,Y) + C_vY & u\neq v \end{cases} \qquad \text{Defn}$$

$$= \begin{cases} S_v(L(X,Y) + C_vY) & u=v \\ L(X,Y) + C_vY & u\neq v \end{cases}$$

$$R(S_vX,Y) = Y.S_vX + C_uS_vX$$

$$= Y.X + C_vY + C_uS_vX \qquad \text{Defn}$$

$$= \begin{cases} Y.X + C_vY + S_vC_uX & u=v \\ Y.X + C_vY + C_uX & u\neq v \end{cases} \qquad \begin{matrix} \text{Defn} \\ \text{Defn} \end{matrix}$$

$$= \begin{cases} S_v((Y.X + C_uX) + C_vY) & u=v \\ (Y.X + C_uX) + C_vY & u\neq v \end{cases} \qquad \begin{matrix} \text{Defn \& Associative law of addition} \\ \text{Associative law of addition} \end{matrix}$$

$$= \begin{cases} S_v(R(X,Y) + C_vY) & u=v \\ R(X,Y) + C_vY & u\neq v \end{cases}$$

$\therefore \quad L(X,Y) = R(X,Y)$ by uniqueness.

PROPOSITION $\quad Y.X = X.Y$ Commutative

<u>Proof</u> $\quad L(0,Y) = Y.0$

$$= 0 \qquad \text{Defn}$$

$$R(0,Y) = 0.Y$$

$$= 0 \qquad \text{Previous Lemma}$$

$$L(S_uX,Y) = Y.S_uX$$

$$= Y.X + C_uY \qquad \text{Defn}$$

$$= L(X,Y) + C_uY$$

$$R(S_uX,Y) = S_uX,Y$$

$$= X.Y + C_uY \qquad \text{Previous Lemma}$$

$$= R(X,Y) + C_u Y$$

$\therefore$ $L(X,Y) = R(X,Y)$ by uniqueness

Consider the distributive property of this function over addition.

PROPOSITION $Z.(Y + X) = Z.Y + Z.X$ Distribution over addition

<u>Proof</u>

$$L(0,Y,Z) = Z.(Y + 0)$$

$$= Z.Y \qquad \text{Def'n}$$

$$R(0,Y,Z) = Z.Y + Z.0$$

$$= Z.Y + 0 \qquad \text{Def'n}$$

$$= Z.Y \qquad \text{Def'n}$$

$$L(S_u X,Y,Z) = Z.(Y + S_u X)$$

$$= Z.S_u(Y + X) \qquad \text{Def'n}$$

$$= Z.(Y + X) + C_u Z \qquad \text{Def'n}$$

$$= L(X,Y,Z) + C_u Z$$

$$R(S_u X,Y,Z) = Z.Y + Z.S_u X$$

$$= Z.Y + Z.X + C_u Z \qquad \text{Def'n \& Associative law of addition}$$

$$= R(X,Y,Z) + C_u Z$$

$\therefore$ $L(X,Y,Z) = R(X,Y,Z)$ by uniqueness.

To prove associativity we require to prove the following Lemmas.

<u>LEMMA</u> $C_u C_v X = C_v C_u X$

<u>Proof</u>

$$L(0) = C_u C_v 0$$

$$= C_u 0 \qquad \text{Def'n}$$

$$= 0 \qquad \text{Def'n}$$

$$R(0) = C_v C_u 0$$

$$= C_v 0 \qquad \text{Defn}$$

$$= 0 \qquad \text{Defn}$$

$$L(S_w X) = C_u C_v S_w X$$

$$= \begin{cases} C_u C_v X & v \neq w \quad \text{Defn} \\ C_u S_w C_v X & v = w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} C_u C_v X & v \neq w \\ C_u C_v X & v = w \ \& \ u \neq w \quad \text{Defn} \\ S_w C_u C_v X & u = v = w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} C_u C_v X & v \neq w \text{ or } u \neq w \\ S_w C_u C_v X & u = v = w \end{cases}$$

$$= \begin{cases} L(X) & v \neq w \text{ or } u \neq w \\ S_w L(X) & u = v = w \end{cases}$$

$$R(S_w X) = C_v C_u S_w X$$

$$= \begin{cases} C_v C_u X & u \neq w \quad \text{Defn} \\ C_v S_w C_u X & u = w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} C_v C_u X & u \neq w \\ C_v C_u X & u = w \ \& \ v \neq w \quad \text{Defn} \\ S_w C_v C_u X & u = v = w \quad \text{Defn} \end{cases}$$

$$= \begin{cases} C_v C_u X & u \neq w \text{ or } v \neq w \\ S_w C_v C_u X & u = v = w \end{cases}$$

$$\begin{cases} R(X) & u \neq w \text{ or } v \neq w \\ S_w R(X) & u = v = w \end{cases}$$

$\therefore \quad L(X) = R(X)$ by uniqueness.

LEMMA $\quad C_u(Y + X) = C_uY + C_uX$

<u>Proof</u>

$$L(0,Y) = C_u(Y + 0)$$

$$= C_uY \qquad \text{Def'n}$$

$$R(0,Y) = C_uY + C_u0$$

$$= C_uY + 0 \qquad \text{Def'n}$$

$$= C_uY \qquad \text{Def'n}$$

$$L(S_vX,Y) = C_u(Y + S_vX)$$

$$= C_uS_v(Y + X) \qquad \text{Def'n}$$

$$= \begin{cases} S_vC_u(Y + X) & u=v \quad \text{Def'n} \\ C_u(Y + X) & u\neq v \quad \text{Def'n} \end{cases}$$

$$= \begin{cases} S_vL(X,Y) & u=v \\ L(X,Y) & u\neq v \end{cases}$$

$$R(S_vX,Y) = C_uY + C_uS_vX$$

$$= \begin{cases} C_uY + S_vC_uX & u=v \quad \text{Def'n} \\ C_uY + C_uX & u\neq v \quad \text{Def'n} \end{cases}$$

$$= \begin{cases} S_v(C_uY + C_uX) & u=v \quad \text{Def'n} \\ C_uY + C_uX & u\neq v \end{cases}$$

$$= \begin{cases} S_vR(X,Y) & u=v \\ R(X,Y) & u\neq v \end{cases}$$

$\therefore \quad L(X,Y) = R(X,Y) \qquad$ by uniqueness.

LEMMA $\quad (C_uY).X = C_u(Y,X)$

<u>Proof</u> $L(0,Y) = (C_uY).0$

$= 0$ Defn

$R(0,Y) = C_u(Y.0)$

$= C_u0$ Defn

$= 0$ Defn

$L(S_vX,Y) = (C_uY).S_vX$

$= (C_uY).X + C_v(C_uY)$ Defn

$= L(X,Y) + C_vC_uY$

$R(S_vX,Y) = C_u(Y.S_vX)$

$= C_u(Y.X + C_vY)$ Defn

$= C_u(Y.X) + C_uC_vY$ Defn

$= R(X,Y) + C_vC_uY$ Previous Lemma

$\therefore$ $L(X,Y) = R(X,Y)$ by uniqueness

and therefore by the commutativity property we can say $(C_uY).X = X.(C_uY)$.

PROPOSITION $Z.(Y.X) = (Z.Y).X$ Associative

<u>Proof</u> $L(0,Y,Z) = Z.(Y.0)$

$= Z.0$ Defn

$= 0$ Defn

$R(0,Y,Z) = (Z.Y).0$

$= 0$ Defn

$L(S_uX,Y,Z) = Z.(Y.S_uX)$

$= Z.(Y.X + C_uY)$ Defn

$= Z.(Y.X) + Z.(C_uY)$ Previous Lemma

$$= L(X,Y,Z) + C_u(Z.Y)$$

$$R(S_uX,Y,Z) = (Z.Y).S_uX$$

$$= (Z.Y).X + C_u(Z.Y) \qquad \text{Defn}$$

$$= R(X,Y,Z) + C_u(Z.Y)$$

$$\therefore \qquad L(X,Y,Z) = R(X,Y,Z)$$

This function is not, however, distributive over σ_u functions, nor the τ_u functions, for all u. Though the distributive property is true in both cases with $u = 0$, the first we have proved, the second we shall state, and leave the complete proof until chapter III.

$$Z.(Y \dot{-} X) = Z.Y \dot{-} Z.X \qquad \text{Distributive over difference}$$

Having established the properties of these three multiplication functions, we are now able to consider their squares, and possible square roots.

1. Vučković's multiplication function.

$$Y*0 = 0$$

$$Y*S_uX = Y*X\ \sigma_uY \qquad u=1,\ldots,n \qquad (\sigma_n = \sigma_0)$$

In 4 successor arithmetic the squares of numerals will be as follows.

Numeral	Square	Numeral	Square
0	0	S_1S_10	$S_2S_2S_2S_20$
S_10	S_20	S_1S_20	$S_2S_3S_3S_40$
S_20	S_40	S_1S_30	$S_2S_4S_4S_20$
S_30	S_20	S_1S_40	$S_2S_1S_1S_40$
S_40	S_40	S_2S_20	$S_4S_4S_4S_40$
		S_2S_30	$S_4S_1S_1S_20$

From the above table it can be seen that obviously there cannot be a useful square root function for this multiplication function, as both $S_1 0$ and $S_3 0$ have the same square.

2. The function defined by

$$Y^X 0 = 0$$

$$Y^X S_u X = Y^X X + Y \qquad u=1,\ldots,n$$

In 4 successor arithmetic the squares of numerals will be as follows.

Numeral	Square	Numeral	Square
0	0	$S_1 S_1 0$	$S_1 S_1 S_1 S_1 0$
$S_1 0$	$S_1 0$	$S_1 S_2 0$	$S_1 S_2 S_1 S_2 0$
$S_2 0$	$S_2 0$	$S_2 S_3 0$	$S_2 S_3 S_2 S_3 0$
$S_3 0$	$S_3 0$	$S_1 S_4 0$	$S_1 S_4 S_1 S_4 0$
$S_4 0$	$S_4 0$	$S_1 S_3 S_4 0$	$S_1 S_3 S_4 S_1 S_3 S_4 S_1 S_3 S_4 0$

This function does give a unique square, and further if a square root function could be found then a unique answer would be obtained for square roots of 'perfect squares' under this multiplication rule. It can be seen that the square of a numeral is that numeral repeated the same number of times as there are successor symbols in that numeral. Therefore in order to obtain the square root of a numeral we would first have to analyse the number of successor symbols in that numeral, take the square root of that number, then further either analyse the particular successor symbols or 'divide' the numeral by the square root of the number of successor symbols. The questions of the number of successor symbols in a numeral, and the particular successor symbols which make up that numeral have been fully analysed in chapter IV.

It can be seen from the above that a square root function of this multiplication function would not be a 'convenient' function and therefore we shall leave this function and consider the third multiplication function.

3. The function defined by

$$Y.0 = 0$$

$$Y.S_uX = Y.X + C_uY \qquad u=1,\ldots,n$$

Consider the squares of numerals in 4 successor arithmetic.

Numeral	Square	Numeral	Square
0	0	S_1S_20	S_1S_20
S_10	S_10	S_2S_30	S_2S_30
S_20	S_20	S_2S_20	$S_2S_2S_2S_20$
S_30	S_30	$S_1S_1S_40$	$S_1S_1S_1S_1S_40$
S_40	S_40	$S_4S_4S_40$	$S_4S_4S_4S_4S_4S_4S_4S_4S_40$

This appears to have a very reasonable square, which is the numeral split into components, and each component squared. To obtain the square root therefore we would first have to take components and then take the square root as in single successor arithmetic. Thus we would have functions $Rt_u(X)$ defined by:-

$$Rt_u(0) = 0$$

$$Rt_u(S_vX) = \begin{cases} Rt_u(C_uX) + C_u(S_v0 \dot{-} ((S_vRt_u(C_uX))^2 \dot{-} S_vC_uX)) & u=v \\ Rt_u(C_uX) & u\neq v \end{cases}$$

That is $Rt_u(X)$ is the square root of the 'S_u' successor symbols in X, the complete square root of X would then be given by $Rt(X)$ defined by

$$Rt(X) = Rt_1(X) + Rt_2(X) + \ldots + Rt_n(X) .$$

However we can avoid the $Rt_u(X)$ functions by the definition

$$Rt(0) = 0$$

$$Rt(S_uX) = Rt(X) + (S_u0 \dot{-} ((S_uRt(X))^2 \dot{-} S_uX)) \qquad u=1,\ldots,n$$

The components are taken by the property that

$$C_u(S_u0 \dot{-} A) = (S_u0 \dot{-} A) \ .$$

For this function to be the one required the following properties need to be satisfied:

$$Rt(X.X) = X$$

$$Rt(X.X + X) = X$$

$$Rt((Y + X)^2 + X) = Y + X \qquad (\ (Y + X)^2 = (Y + X) \ . \ (Y + X) \ )$$

Further if $Y > X$ that is $X \dot{-} Y = 0$, then $Rt(Y) > Rt(X)$, that is $Rt(X) \dot{-} Rt(Y) = 0$.

All these properties will be proved formally in chapter III.

Further, we need to prove the properties

$$L(J(U,V)) = U$$

$$K(J(U,V)) = V \ .$$

CHAPTER III

FORMAL DEVELOPMENT OF COMMUTATIVE MULTIPLE SUCCESSOR ARITHMETIC

In this chapter we will develop the arithmetic formally and prove the properties of some of the functions already introduced in previous chapters. The notation used will be to number all axioms, rules of inference, definitions, lemmas, propositions, and theorems. The steps in a proof will then by illustrated by a number corresponding to the statement used on the right-hand side of the page. For the substitution rule of inference and equality rules numbering will not be used.

The functional notation $L(\ldots)$ and $R(\ldots)$ will be used to denote the left- and right-hand sides of equations. Where a letter is quoted in the parenthesis of $L(\ldots)$ or $R(\ldots)$ and does not appear on the corresponding side of the equation, then the zero function $Z(\ldots)$ of that letter is supposed to have been taken.

e.g. $R(X) = 0$ should be taken as $R(X) = Z(X)$, which will not affect the proof.

A proof will conclude with the statement $L(\ldots) = R(\ldots)$. Statements 1 to 6 give the axiomatic statement of the arithmetic.

1. Initial functions

$Z(X)$ written 0

$S_u(X)$ written S_uX $u=1,\ldots,n$

$I_i^k(X_1,X_2,\ldots,X_k)$ written X_i

2. Commutative condition

$S_uS_vX = S_vS_uX \quad u,v=1,\ldots,n$

Definition of functions by explicit and primitive recursive definitions, the latter given by:-

If $A(Y_1,Y_2,\ldots,Y_m)$ and $B_u(X,Y_1,Y_2,\ldots,Y_m,Z)$ $u=1,\ldots,n$, are designated functions (initial functions or previously defined) then $F(X,Y_1,Y_2,\ldots,Y_m)$ is designated if

$$F(0,Y_1,Y_2,\ldots,Y_m) = A(Y_1,Y_2,\ldots,Y_m)$$

$$F(S_uX,Y_1,Y_2,\ldots,Y_m) = B_u(X,Y_1,Y_2,\ldots,Y_m,F(X,Y_1,Y_2,\ldots,Y_m)) \quad u=1,\ldots,n$$

and

$$B_u(S_vX,Y_1,Y_2,\ldots,Y_m,B_v(X,Y_1,Y_2,\ldots,Y_m,Z)) = B_v(S_uX,Y_1,Y_2,\ldots,Y_m,B_u(X,Y_1,Y_2,\ldots,Y_m,Z)) \quad \text{all } u,v,$$

in order to preserve the commutativity of the successors.

3. Rule of inference (equality rule)

$$\frac{\begin{array}{l} A = B \\ A = C \end{array}}{B = C}$$

This statement reads: 'If A is equal to B and A is equal to C then B is equal to C'.

4. Rule of inference (substitution)

$$\frac{F(Y) = G(Y)}{F(A) = G(A)}$$

5. Rule of inference (substitution)

$$\frac{A = B}{F(A) = F(B)}$$

6. Rule of inference (uniqueness rule)

$$\frac{\begin{array}{l} F(0) = G(0) \\ F(S_uX) = H_u(X,F(X)) \\ G(S_uX) = H_u(X,G(X)) \quad u=1,\ldots,n \end{array}}{F(X) = G(X)}$$

7. Definition of $Y + X$

$$Y + 0 = Y$$

$$Y + S_u X = S_u(Y + X) \quad u=1,\ldots,n$$

8. Proposition $A = A$

Proof	$Y + 0 = Y$	7.
	$Y + 0 = Y$	7.
	$Y = Y$	3.
	$A = A$	4.

9. Rule of inference

$$\frac{A = B}{B = A}$$

Proof	$A = B$	Premises
	$A = A$	8,
	$B = A$	3.

10. Rule of inference

$$\frac{\begin{array}{c} A = B \\ B = C \end{array}}{A = C}$$

Proof	$A = B$	Premises
	$B = A$	9.
	$B = C$	Premises
	$A = C$	3.

11. Rule of inference

$$\frac{\begin{array}{l} B = A \\ C = A \end{array}}{B = C}$$

Proof	$B = A$	Premises
	$A = B$	9.
	$C = A$	Premises
	$A = C$	9.
	$B = C$	3.

12. Rule of inference

$$\frac{\begin{array}{l} A = B \\ A = C \\ B = D \end{array}}{C = D}$$

Proof	$A = B$	Premises
	$A = C$	Premises
	$B = C$	3.
	$B = D$	Premises
	$C = D$	3.

13. Proposition

$$0 + X = X$$

Proof	$L(0) = 0 + 0$	
	$= 0$	7.
	$R(0) = 0$	
	$L(S_u X) = 0 + S_u X$	
	$= S_u(0 + X)$	7.
	$= S_u L(X)$	

$$R(S_uX) = S_uX$$
$$= S_uR(X)$$
$$L(X) = R(X) \qquad 6.$$

14. Proposition

$$S_uY + X = S_u(Y + X)$$

Proof

$$L(0,Y) = S_uY + 0$$
$$= S_uY \qquad 7.$$
$$R(0,Y) = S_u(Y + 0)$$
$$= S_uY \qquad 7.$$
$$L(S_vX,Y) = S_uY + S_vX$$
$$= S_v(S_uY + X) \qquad 7.$$
$$= S_vL(X,Y)$$
$$R(S_vX,Y) = S_u(Y + S_vX)$$
$$= S_uS_v(Y + X) \qquad 7.$$
$$= S_vS_u(Y + X) \qquad 2.$$
$$= S_vR(X,Y)$$
$$L(X,Y) = R(X,Y) \qquad 6.$$

15. Proposition

$$X + Y = Y + X$$

Proof

$$L(0,Y) = 0 + Y$$
$$= Y \qquad 13.$$
$$R(0,Y) = Y + 0$$
$$= Y \qquad 7.$$
$$L(S_uX,Y) = S_uX + Y$$
$$= S_u(X + Y) \qquad 14.$$
$$= S_uL(X,Y)$$

$$R(S_uX,Y) = Y + S_uX$$

$$= S_u(Y + X) \qquad 7.$$

$$= S_uR(X,Y)$$

$$L(X,Y) = R(X,Y) \qquad 6.$$

16. Proposition

$$(A + B) + C = A + (B + C)$$

Proof

$$L(0,B,C) = (0 + B) + C$$

$$= B + C \qquad 13.$$

$$R(0,B,C) = 0 + (B + C)$$

$$= B + C \qquad 13.$$

$$L(S_uA,B,C) = (S_uA + B) + C$$

$$= S_u(A + B) + C \qquad 14.$$

$$= S_u((A + B) + C) \qquad 14.$$

$$= S_uL(A,B,C)$$

$$R(S_uA,B,C) = S_uA + (B + C)$$

$$= S_u(A + (B + C)) \qquad 14.$$

$$= S_uR(A,B,C)$$

$$L(A,B,C) = R(A,B,C) \qquad 6.$$

17. Definition of P_uX

$$P_u0 = 0$$

$$P_uS_vX = \begin{cases} X & u=v \\ S_vP_uX & u\neq v \end{cases}$$

18. Definition of $Y \dot{-} X$

$$Y \dot{-} 0 = Y$$

$$Y \dot{-} S_uX = P_u(Y \dot{-} X)$$

19. Proposition

$$0 \dot{-} X = 0$$

Proof

$$L(0) = 0 \dot{-} 0$$

$$= 0 \qquad 18.$$

$$R(0) = 0$$

$$L(S_u X) = 0 \dot{-} S_u X$$

$$= P_u(0 \dot{-} X) \qquad 18.$$

$$= P_u L(X)$$

$$R(S_u X) = 0$$

$$= P_u R(X) \qquad 17.$$

$$L(X) = R(X) \qquad 6.$$

20. Proposition

$$(C \dot{-} B) \dot{-} A = C \dot{-} (B + A)$$

Proof

$$L(0,B,C) = (C \dot{-} B) \dot{-} 0$$

$$= C \dot{-} B \qquad 18.$$

$$R(0,B,C) = C \dot{-} (B + 0)$$

$$= C \dot{-} B \qquad 7.$$

$$L(S_u A,B,C) = (C \dot{-} B) \dot{-} S_u A$$

$$= P_u((C \dot{-} B) \dot{-} A) \qquad 18.$$

$$= P_u L(A,B,C)$$

$$R(S_u A,B,C) = C \dot{-} (B + S_u A)$$

$$= C \dot{-} S_u(B + A) \qquad 7.$$

$$= P_u(C \dot{-} (B + A)) \qquad 18.$$

$$= P_u R(A,B,C)$$

$$L(A,B,C) = R(A,B,C) \qquad 6.$$

21. Proposition

$$S_uB \dot{-} S_uA = B \dot{-} A$$

Proof

$$\begin{aligned}
L(0,B) &= S_uB \dot{-} S_u0 \\
&= P_u(S_uB \dot{-} 0) && 18. \\
&= P_uS_uB && 18. \\
&= B && 17. \\
R(0,B) &= B \dot{-} 0 \\
&= B && 18. \\
L(S_vA,B) &= S_uB \dot{-} S_uS_vA \\
&= S_uB \dot{-} S_vS_uA && 2. \\
&= P_v(S_uB \dot{-} S_uA) && 18. \\
&= P_vL(A,B) \\
R(S_vA,B) &= B \dot{-} S_vA \\
&= P_v(B \dot{-} A) && 18. \\
&= P_vR(A,B) \\
L(A,B) &= R(A,B) && 6.
\end{aligned}$$

22. $$A \dot{-} A = 0$$

Proof

$$\begin{aligned}
L(0) &= 0 \dot{-} 0 \\
&= 0 && 18. \\
R(0) &= 0 \\
L(S_uA) &= S_uA \dot{-} S_uA \\
&= A \dot{-} A && 21. \\
&= L(A) \\
R(S_uA) &= 0 \\
&= R(A) \\
L(A) &= R(A) && 6.
\end{aligned}$$

23. $(B + A) \dot{-} A = B$

Proof

$$L(0,B) = (B + 0) \dot{-} 0$$
$$= B + 0 \qquad 18.$$
$$= B \qquad 7.$$
$$R(0,B) = B$$
$$L(S_uA,B) = (B + S_uA) \dot{-} S_uA$$
$$= S_u(B + A) \dot{-} S_uA \qquad 7.$$
$$= (B + A) \dot{-} A \qquad 21.$$
$$= L(A,B)$$
$$R(S_uA,B) = B$$
$$= R(A,B)$$
$$L(A,B) = R(A,B) \qquad 6.$$

24. $(B + A) \dot{-} (C + A) = B \dot{-} C$

Proof

$$L(0,B,C) = (B + 0) \dot{-} (C + 0)$$
$$= B \dot{-} C \qquad 7.$$
$$R(0,B,C) = B \dot{-} C$$
$$L(S_uA,B,C) = (B + S_uA) \dot{-} (C + S_uA)$$
$$= S_u(B + A) \dot{-} S_u(C + A) \qquad 7.$$
$$= (B + A) \dot{-} (C + A) \qquad 21.$$
$$= L(A,B,C)$$
$$R(S_uA,B,C) = B \dot{-} C$$
$$= R(A,B,C)$$
$$L(A,B,C) = R(A,B,C) \qquad 6.$$

25. $B \dot{-} (B + A) = 0$

Proof

$$L(0,B) = B \dot{-} (B + 0)$$
$$= B \dot{-} B \qquad 7.$$
$$= 0 \qquad 22.$$

$$R(0,B) = 0$$

$$L(S_uA,B) = B \dot{-} (B + S_uA)$$

$$= B \dot{-} S_u(B + A) \qquad 7.$$

$$= P_u(B \dot{-} (B + A)) \qquad 18.$$

$$= P_uL(A,B)$$

$$R(S_uA,B) = 0$$

$$= P_u0 \qquad 17.$$

$$= P_uR(A,B)$$

$$L(A,B) = R(A,B) \qquad 6.$$

26. Rule of inference

$$F(0) = G(0)$$

$$F(S_uX) = G(S_uX) \qquad u=1,2,\ldots,r$$

$$F(S_uX) = H_u(X,F(X)) \qquad u=r+1,\ldots,n$$

$$G(S_uX) = H_u(X,G(X)) \qquad u=r+1,\ldots,n$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$F(X) = G(X)$$

Proof Define $M_u(X,Y)$ by

$$M_u(X,Y) = \begin{cases} F(S_uX) & u=1,2,\ldots,r \\ H_u(X,Y) & u=r+1,\ldots,n \end{cases}$$

Now $F(S_uX) = G(S_uX) \qquad u=1,2,\ldots,r$

Hence $M_u(X,Y) = \begin{cases} G(S_uX) & u=1,2,\ldots,r \\ H_u(X,Y) & u=r+1,\ldots,n \end{cases}$

We now have $F(0) = G(0)$

$$F(S_uX) = M_u(X,Y) \qquad u=1,\ldots,n$$

$$G(S_uX) = M_u(X,Y) \qquad u=1,\ldots,n$$

Hence $F(X) = G(X) \qquad 6.$

27. Definition C_uX

$$C_u0 = 0$$

$$C_uS_vX = \begin{cases} S_vC_uX & u=v \\ C_uX & u\neq v \end{cases}$$

28.

$$C_uC_vA = \begin{cases} 0 & u\neq v \\ C_uA & u=v \end{cases}$$

Proof

$$L(0) = C_uC_v0$$

$$= C_u0 \qquad 27.$$

$$= 0 \qquad 27.$$

$$R(0) = 0$$

$$L(S_wA) = C_uC_vS_wA$$

$$= \begin{cases} C_uS_wC_vA & v=w & 27. \\ C_uC_vA & v\neq w & 27. \end{cases}$$

$$= \begin{cases} S_wC_uC_vA & u=v=w & 27. \\ C_uC_vA & v=w \ \& \ u\neq w & 27. \\ C_uC_vA & v\neq w \end{cases}$$

$$= \begin{cases} S_wL(A) & u=v=w \\ L(A) & u\neq w \text{ or } v\neq w \end{cases}$$

$$R(S_wA) = \begin{cases} 0 & u\neq v \\ C_uS_wA & u=v \end{cases}$$

$$= \begin{cases} 0 & u\neq v \\ S_wC_uA & u=v=w & 27. \\ C_uA & u=v \ \& \ u\neq w & 27. \end{cases}$$

$$\begin{cases} S_wC_uA & u=v=w \\ 0 & u\neq v \\ C_uA & u=v \end{cases} \quad u\neq w \text{ or } v\neq w$$

$$= \begin{cases} S_w R(A) & u=v=w \\ R(A) & u\neq w \text{ or } v\neq w \end{cases}$$

$$L(A) = R(A) \qquad 6.$$

29. $$C_u C_v A = C_v C_u A$$

Proof

$$C_u C_v A = \begin{cases} 0 & u\neq v \qquad 27. \\ C_u A & u=v \qquad 27. \end{cases}$$

$$= \begin{cases} 0 & u\neq v \\ C_v A & u=v \end{cases}$$

$$= C_v C_u A \qquad 27.$$

30. $$C_u(B + A) = C_u B + C_u A$$

Proof

$$L(0,B) = C_u(B + 0)$$

$$= C_u B \qquad 7.$$

$$R(0,B) = C_u B + C_u 0$$

$$= C_u B + 0 \qquad 27.$$

$$= C_u B \qquad 7.$$

$$L(S_v A,B) = C_u(B + S_v A)$$

$$= C_u S_v(B + A) \qquad 7.$$

$$= \begin{cases} S_v C_u(B + A) & u=v \qquad 27. \\ C_u(B + A) & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} S_v L(A,B) & u=v \\ L(A,B) & u\neq v \end{cases}$$

$$R(S_v A,B) = C_u B + C_u S_v A$$

$$= \begin{cases} C_u B + S_v C_u A & u=v \qquad 27. \\ C_u B + C_u A & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} S_v(C_uB + C_uA) & u=v \\ C_uB + C_uA & u\neq v \end{cases} \qquad 7.$$

$$= \begin{cases} S_vR(A,B) & u=v \\ R(A,B) & u\neq v \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

31. $$C_uP_vA = \begin{cases} P_vC_uA & u=v \\ C_uA & u\neq v \end{cases}$$

Proof

$$L(0) = C_uP_v0$$

$$= C_u0 \qquad 17.$$

$$= 0 \qquad 27.$$

$$R(0) = \begin{cases} P_vC_u0 & u=v \\ C_u0 & u\neq v \end{cases}$$

$$= \begin{cases} 0 & u=v & 27.\ 17. \\ 0 & u\neq v & 27. \end{cases}$$

$$= 0$$

$$L(S_wA) = C_uP_vS_wA$$

$$= \begin{cases} C_uS_wP_vA & v\neq w & 17. \\ C_uA & v=w & 17. \end{cases}$$

$$= \begin{cases} S_wC_uP_vA & v\neq w \ \&\ u=w & 27. \\ C_uP_vA & v\neq w \ \&\ u\neq w & 27. \\ C_uA & v=w \end{cases}$$

$$= \begin{cases} S_wL(A) & v\neq w \ \&\ u=w \\ L(A) & v\neq w \ \&\ u\neq w \\ C_uA & v=w \end{cases}$$

$$R(S_wA) = \begin{cases} P_vC_uS_wA & u=v \\ C_uS_wA & u\neq v \end{cases}$$

$$= \begin{cases} P_v S_w C_u A & u=v \ \& \ u=w \\ P_v C_u A & u=v \ \& \ u\neq w \\ S_w C_u A & u\neq v \ \& \ u=w \\ C_u A & u\neq v \ \& \ u\neq w \end{cases} \qquad \begin{matrix} 27. \\ 27. \\ 27. \\ 27. \end{matrix}$$

$$= \begin{cases} C_u A & u=v \ \& \ u=w \\ P_v C_u A & u=v \ \& \ u\neq w \\ S_w C_u A & u\neq v \ \& \ u=w \\ C_u A & u\neq v \ \& \ u\neq w \end{cases} \qquad 17.$$

$$= \begin{cases} C_u A & u=v \ \& \ v=w \ \& \ u=w \\ P_v C_u A & u=v \ \& \ v\neq w \ \& \ u\neq w \\ S_w C_u A & u\neq v \ \& \ v=w \ \& \ u\neq w \\ C_u A & u\neq v \ \& \ v\neq w \ \& \ u\neq w \\ C_u A & u\neq v \ \& \ v\neq w \ \& \ u=w \end{cases}$$

$$= \begin{cases} S_w C_u A & v\neq w \ \& \ u=w \\ P_v C_u A & v\neq w \ \& \ u\neq w \ \& \ u=v \\ C_u A & v\neq w \ \& \ u\neq w \ \& \ u\neq v \\ C_u A & v=w \end{cases}$$

$$= \begin{cases} S_w R(A) & v\neq w \ \& \ u=w \ (u\neq v) \\ R(A) & v\neq w \ \& \ u\neq w \\ C_u A & v=w \end{cases}$$

$$L(A) = R(A) \qquad 26.$$

32. $$C_u(B \dot{-} A) = C_u B \dot{-} C_u A$$

Proof $$L(0,B) = C_u(B \dot{-} 0)$$

$$= C_u B \qquad 18.$$

$$R(0,B) = C_u B \dot{-} C_u 0$$

$$= C_u B \dot{-} 0 \qquad 27.$$

$$= C_u B \qquad 18.$$

$$L(S_vA,B) = C_u(B \dot{-} S_vA)$$

$$= C_uP_v(B \dot{-} A) \qquad 18.$$

$$= \begin{cases} P_vC_u(B \dot{-} A) & u=v \qquad 31. \\ C_u(B \dot{-} A) & u\neq v \qquad 31. \end{cases}$$

$$= \begin{cases} P_vL(A,B) & u=v \\ L(A,B) & u\neq v \end{cases}$$

$$R(S_vA,B) = C_uB \dot{-} C_uS_vA$$

$$= \begin{cases} C_uB \dot{-} S_vC_uA & u=v \qquad 27. \\ C_uB \dot{-} C_uA & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} P_v(C_uB \dot{-} C_uA) & u=v \qquad 18. \\ C_uB \dot{-} C_uA & u\neq v \end{cases}$$

$$= \begin{cases} P_vR(A,B) & u=v \\ R(A,B) & u\neq v \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

33. $C_1A + C_2A + \ldots\ldots + C_nA = A$

Proof

$$L(0) = C_10 + C_20 + \ldots\ldots + C_n0$$

$$= 0 + 0 + \ldots\ldots + 0 \qquad 27.$$

$$= 0 \qquad 7.$$

$$R(0) = 0$$

$$L(S_uA) = C_1S_uA + C_2S_uA + \ldots\ldots + C_nS_uA$$

$$= C_1A + C_2A + \ldots\ldots + S_uC_uA + \ldots\ldots + C_nA \qquad 27.$$

$$= S_u(C_1A + C_2A + \ldots\ldots + C_nA) \qquad 7.$$

$$= S_uL(A)$$

$$R(S_uA) = S_uA$$

$$= S_uR(A)$$

$$L(A) = R(A) \qquad 6.$$

34. $S_vB \dot{-} C_uA = S_v(B \dot{-} C_uA) \quad u \neq v$

Proof

$$L(0,B) = S_vB \dot{-} C_u0$$

$$= S_vB \dot{-} 0 \qquad 27.$$

$$= S_vB \qquad 18.$$

$$R(0,B) = S_v(B \dot{-} C_u0)$$

$$= S_v(B \dot{-} 0) \qquad 27.$$

$$= S_vB \qquad 18.$$

$$L(S_wA,B) = S_vB \dot{-} C_uS_wA$$

$$= \begin{cases} S_vB \dot{-} S_wC_uA & u=w \qquad 27. \\ S_vB \dot{-} C_uA & u \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} P_w(S_vB \dot{-} C_uA) & u=w \qquad 18. \\ S_vB \dot{-} C_uA & u \neq w \end{cases}$$

$$= \begin{cases} P_wL(A,B) & u=w \\ L(A,B) & u \neq w \end{cases}$$

$$R(S_wA,B) = S_v(B \dot{-} C_uS_wA)$$

$$= \begin{cases} S_v(B \dot{-} S_wC_uA) & u=w \qquad 27. \\ S_v(B \dot{-} C_uA) & u \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} S_vP_w(B \dot{-} C_uA) & u=w \qquad 18. \\ S_v(B \dot{-} C_uA) & u \neq w \end{cases}$$

$$= \begin{cases} P_wS_v(B \dot{-} C_uA) & u=w \ (u \neq v \text{ and } \therefore v \neq w) \qquad 17. \\ S_v(B \dot{-} C_uA) & u \neq w \end{cases}$$

$$= \begin{cases} P_wR(A,B) & u=w \\ R(A,B) & u \neq w \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

35. $P_vB + C_uA = P_v(B + C_uA) \qquad u \neq v$

Proof

$$L(0,B) = P_vB + C_u0$$

$$= P_vB + 0 \qquad 27.$$

$$= P_vB \qquad 7.$$

$$R(0,B) = P_v(B + C_u0)$$

$$= P_v(B + 0) \qquad 27.$$

$$= P_vB \qquad 7.$$

$$L(S_wA,B) = P_vB + C_uS_wA$$

$$= \begin{cases} P_vB + S_wC_uA & u=w \qquad 27. \\ P_vB + C_uA & u \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} S_w(P_vB + C_uA) & u=w \qquad 7. \\ P_vB + C_uA & u \neq w \end{cases}$$

$$= \begin{cases} S_wL(A,B) & u=w \\ L(A,B) & u \neq w \end{cases}$$

$$R(S_wA,B) = P_v(B + C_uS_wA)$$

$$= \begin{cases} P_v(B + S_wC_uA) & u=w \qquad 27. \\ P_v(B + C_uA) & u \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} P_vS_w(B + C_uA) & u=w \qquad 7. \\ P_v(B + C_uA) & u \neq w \end{cases}$$

$$= \begin{cases} S_wP_v(B + C_uA) & u=w \ (u \neq v \text{ and } \therefore v \neq w) \qquad 17. \\ P_v(B + C_uA) & u \neq w \end{cases}$$

$$= \begin{cases} S_wR(A,B) & u=w \\ R(A,B) & u \neq w \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

36. $(B + C_uA) \dot{-} C_vA = (B \dot{-} C_vA) + C_uA \quad u \neq v$

Proof

$$L(0,B) = (B + C_u0) \dot{-} C_v0$$

$$= (B + 0) \dot{-} 0 \qquad 27.$$

$$= B \qquad 25.$$

$$R(0,B) = (B \dot{-} C_v0) + C_u0$$

$$= (B \dot{-} 0) + 0 \qquad 27.$$

$$= B \qquad 7.\ 18.$$

$$L(S_wA,B) = (B + C_uS_wA) \dot{-} C_vS_wA$$

$$= \begin{cases} (B + C_uS_wA) \dot{-} S_wC_vA & v=w \qquad 27. \\ (B + C_uS_wA) \dot{-} C_vA & v \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} (B + C_uA \dot{-} S_wC_vA & v=w\ (u \neq v \text{ and } \therefore u \neq w) \qquad 27. \\ (B + S_wC_uA) \dot{-} C_vA & v \neq w\ \&\ u=w \qquad 27. \\ (B + C_uA) \dot{-} C_vA & v \neq w\ \&\ u \neq w \qquad 27. \end{cases}$$

$$= \begin{cases} P_w((B + C_uA) \dot{-} C_vA) & v=w \qquad 18. \\ S_w(B + C_uA) \dot{-} C_vA & v \neq w\ \&\ u=w \qquad 7. \\ (B + C_uA) \dot{-} C_vA & v \neq w\ \&\ u \neq w \end{cases}$$

$$= \begin{cases} P_w((B + C_uA) \dot{-} C_vA) & v=w \\ S_w((B + C_uA) \dot{-} C_vA) & v \neq w\ \&\ u=w \qquad 34. \\ (B + C_uA) \dot{-} C_vA & v \neq w\ \&\ u \neq w \end{cases}$$

$$= \begin{cases} P_wL(A,B) & v=w \\ S_wL(A,B) & v \neq w\ \&\ u=w \\ L(A,B) & v \neq w\ \&\ u \neq w \end{cases}$$

$$R(S_wA,B) = (B \dot{-} C_vS_wA) + C_uS_wA$$

$$= \begin{cases} (B \dot{-} S_w C_v A) + C_u S_w A & v=w \quad 27. \\ (B \dot{-} C_v A) + C_u S_w A & v \neq w \quad 27. \end{cases}$$

$$= \begin{cases} (B \dot{-} S_w C_v A) + C_u A & v=w \ (u \neq v \text{ and } \therefore u \neq w) \quad 27. \\ (B \dot{-} C_v A) + S_w C_u A & v \neq w \ \& \ u=w \quad 27. \\ (B \dot{-} C_v A) + C_u A & v \neq w \ \& \ u \neq w \quad 27. \end{cases}$$

$$= \begin{cases} P_w(B \dot{-} C_v A) + C_u A & v=w \quad 18. \\ S_w((B \dot{-} C_v A) + C_u A) & v \neq w \ \& \ u=w \quad 7. \\ (B \dot{-} C_v A) + C_u A & v \neq w \ \& \ u \neq w \end{cases}$$

$$= \begin{cases} P_w((B \dot{-} C_v A) + C_u A) & v=w(u \neq v \text{ and } \therefore u \neq w) \quad 35. \\ S_w((B \dot{-} C_v A) + C_u A) & v \neq w \ \& \ u=w \\ (B \dot{-} C_v A) + C_u A & v \neq w \ \& \ u \neq w \end{cases}$$

$$= \begin{cases} P_w R(A,B) & v=w \\ S_w R(A,B) & v \neq w \ \& \ u=w \\ R(A,B) & v \neq w \ \& \ u \neq w \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

37. $C_u B \dot{-} C_v A = C_u B \qquad u \neq v$

Proof

$$L(0,B) = C_u B \dot{-} C_v 0$$

$$= C_u B \dot{-} 0 \qquad 27.$$

$$= C_u B \qquad 18.$$

$$R(0,B) = C_u B$$

$$L(S_w A,B) = C_u B \dot{-} C_v S_w A$$

$$= \begin{cases} C_u B \dot{-} S_w C_v A & v=w \quad 27. \\ C_u B \dot{-} C_v A & v \neq w \quad 27. \end{cases}$$

$$= \begin{cases} P_w(C_u B \dot{-} C_v A) & v=w \quad 18. \\ C_u B \dot{-} C_v A & v \neq w \end{cases}$$

$$= \begin{cases} P_w L(A,B) & v=w \\ L(A,B) & v\neq w \end{cases}$$

$$R(S_w A,B) = C_u B$$

$$= \begin{cases} C_u B + 0 & v=w \\ C_u B & v\neq w \end{cases} \qquad 7.$$

$$= \begin{cases} C_u B + P_w 0 & v=w \\ C_u B & v\neq w \end{cases} \qquad 17.$$

$$= \begin{cases} P_w(C_u B + 0) & v=w \ (u\neq v \text{ and } \therefore u\neq w) \\ C_u B & \end{cases} \qquad 35.\ 15.$$

$$= \begin{cases} P_w C_u B & v=w \\ C_u B & v\neq w \end{cases} \qquad 7.$$

$$= \begin{cases} P_w R(A,B) & v=w \\ R(A,B) & v\neq w \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

38. $P_u P_v A = P_v P_u A$

Proof $L(0) = P_u P_v 0$

$$= P_u 0 \qquad 17.$$

$$= 0 \qquad 17.$$

$$R(0) = P_v P_u 0$$

$$= P_v 0 \qquad 17.$$

$$= 0 \qquad 17.$$

$$L(S_w A) = P_u P_v S_w A$$

$$= \begin{cases} P_u A & v=w \\ P_u S_w P_v A & v\neq w \end{cases} \qquad \begin{matrix} 17. \\ 17. \end{matrix}$$

$$= \begin{cases} P_u A & v=w \\ P_v A & v\neq w \ \& \ u=w \\ S_w P_u P_v A & v\neq w \ \& \ u\neq w \end{cases} \qquad \begin{matrix} \\ 17. \\ 17. \end{matrix}$$

$$= \begin{cases} P_u A & v=w \\ P_v A & u=w \\ S_w P_u P_v A & v\neq w \ \& \ u\neq w \end{cases}$$

$$= \begin{cases} P_u A & v=w \text{ or } u=w \\ S_w L(A,B) & v\neq w \ \& \ u\neq w \end{cases}$$

$$R(S_w A) = P_v P_u S_w A$$

$$= \begin{cases} P_v A & u=w \\ P_v S_w P_u A & u\neq w \end{cases} \qquad \begin{matrix} 17. \\ 17. \end{matrix}$$

$$= \begin{cases} P_v A & u=w \\ P_u A & u\neq w \ \& \ v=w \\ S_w P_v P_u A & u\neq w \ \& \ v\neq w \end{cases} \qquad \begin{matrix} \\ 17. \\ 17. \end{matrix}$$

$$= \begin{cases} P_u A & u=w \text{ or } v=w \\ S_w P_v P_u A & u\neq w \ \& \ v\neq w \end{cases}$$

$$= \begin{cases} P_u A & u=w \text{ or } v=w \\ S_w R(A,B) & u\neq w \ \& \ v\neq w \end{cases}$$

$$L(A,B) = R(A,B) \qquad 26.$$

39. $$P_u(B \dot{-} A) = P_u B \dot{-} A$$

Proof $$L(0,B) = P_u(B \dot{-} 0)$$

$$= P_u B \qquad 18.$$

$$R(0,B) = P_u B \dot{-} 0$$

$$= P_u B \qquad 18.$$

$$L(S_v A,B) = P_u(B \dot{-} S_v A)$$

$$= P_u P_v(B \dot{-} A) \qquad 18.$$

$$= P_v P_u(B \dot{-} A) \qquad 38.$$

$$= P_v L(A,B)$$

$$R(S_v A,B) = P_u B \dot{-} S_v A$$

$$= P_v(P_u B \dot{-} A) \qquad 18.$$

$$= P_v R(A,B)$$

$$L(A,B) = R(A,B) \qquad 6.$$

40. $$P_v C_u A = \begin{cases} C_u P_v A & u=v \\ C_u A & u\neq v \end{cases}$$

Proof $$L(0) = P_v C_u 0$$

$$= P_v 0 \qquad 27.$$

$$= 0 \qquad 17.$$

$$R(0) = \begin{cases} C_u P_v 0 & u=v \\ C_u 0 & u\neq v \end{cases}$$

$$= \begin{cases} C_u 0 & u=v \qquad 17. \\ 0 & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} 0 & u=v \qquad 27. \\ 0 & u\neq v \end{cases}$$

$$= 0$$

$$L(S_w A) = P_v C_u S_w A$$

$$= \begin{cases} P_v S_w C_u A & u=w \qquad 27. \\ P_v C_u A & u\neq w \qquad 27. \end{cases}$$

$$= \begin{cases} C_uA & u=w \ \& \ v=w \\ S_wP_vC_uA & u=w \ \& \ v\neq w \\ P_vC_uA & u\neq w \end{cases} \qquad \begin{matrix} 17. \\ 17. \\ \\ \end{matrix}$$

$$= \begin{cases} C_uA & u=v=w \\ S_wL(A) & u=w \ \& \ v\neq w \\ L(A) & u\neq w \end{cases}$$

$$R(S_wA) = \begin{cases} C_uP_vS_wA & u=v \\ C_uS_wA & u\neq v \end{cases}$$

$$= \begin{cases} C_uA & u=v=w \\ C_uS_wP_vA & u=v \ \& \ v\neq w \\ S_wC_uA & u\neq v \ \& \ u=w \\ C_uA & u\neq v \ \& \ u\neq w \end{cases} \qquad \begin{matrix} 17. \\ 17. \\ 27. \\ 27. \end{matrix}$$

$$= \begin{cases} C_uA & u=v=w \\ C_uP_vA & u=v \ \& \ v\neq w \ (u\neq w) \\ S_wC_uA & u\neq v \ \& \ u=w \\ C_uA & u\neq v \ \& \ u\neq w \end{cases} \qquad \begin{matrix} \\ 27. \\ \\ \\ \end{matrix}$$

$$= \begin{cases} C_uA & u=v=w \\ S_wC_uA & u=w \ \& \ v\neq w \ (u\neq v) \\ C_uP_vA & u\neq w \ \& \ u=v \\ C_uA & u\neq w \ \& \ u\neq v \end{cases}$$

$$= \begin{cases} C_uA & u=v=w \\ S_wR(A) & u=w \ \& \ v\neq w \\ R(A) & u\neq w \end{cases}$$

$$L(A) = R(A) \qquad 26.$$

41. $C_uB \dot{-} A = C_uB \dot{-} C_uA$

Proof

$$L(0,B) = C_uB \dot{-} 0$$

$$= C_uB \qquad 18.$$

$$R(0,B) = C_uB \dot{-} C_u0$$

$$= C_uB \dot{-} 0 \qquad 27.$$

$$= C_uB \qquad 18.$$

$$L(S_vA,B) = C_uB \dot{-} S_vA$$

$$= P_v(C_uB \dot{-} A) \qquad 18.$$

$$= \begin{cases} P_v(C_uB \dot{-} A) & u=v \\ P_vC_uB \dot{-} A & u\neq v \end{cases} \qquad 39.$$

$$= \begin{cases} P_v(C_uB \dot{-} A) & u=v \\ C_uB \dot{-} A & u\neq v \end{cases} \qquad 40.$$

$$= \begin{cases} P_vL(A,B) & u=v \\ L(A,B) & u\neq v \end{cases}$$

$$R(S_vA,B) = C_uB \dot{-} C_uS_vA$$

$$= \begin{cases} C_uB \dot{-} S_vC_uA & u=v \qquad 27. \\ C_uB \dot{-} C_uA & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} P_v(C_uB \dot{-} C_uA) & u=v \qquad 18. \\ C_uB \dot{-} C_uA & u\neq v \end{cases}$$

$$= \begin{cases} P_vR(A,B) & u=v \\ R(A,B) & u\neq v \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

42. $$S_u0 \dot{-} A = S_u0 \dot{-} C_uA$$

Proof

$$C_uS_u0 = S_uC_u0 \qquad 27.$$

$$= S_u0 \qquad 27.$$

$$S_u 0 \dot{-} A = C_u S_u 0 \dot{-} A \qquad 27.$$

$$= C_u S_u 0 \dot{-} C_u A \qquad 41.$$

$$= S_u 0 \dot{-} C_u A$$

43. Definition of $Y.X$

$$Y.0 = 0$$

$$Y.S_u X = Y.X + C_u Y \qquad u=1,\ldots,n$$

44. $$0.X = 0$$

Proof

$$L(0) = 0.0$$

$$= 0 \qquad 43.$$

$$R(0) = 0$$

$$L(S_u X) = 0.S_u X$$

$$= 0.X + C_u 0 \qquad 43.$$

$$= 0.X + 0 \qquad 27.$$

$$= 0.X \qquad 7.$$

$$= L(X)$$

$$R(S_u X) = 0$$

$$= R(X)$$

$$L(X) = R(X) \qquad 6.$$

45. $$S_u Y.X = Y.X + C_u X$$

Proof

$$L(0,Y) = S_u Y.0$$

$$= 0 \qquad 43.$$

$$R(0,Y) = Y.0 + C_u 0$$

$$= 0 + 0 \qquad 27.$$

$$= 0$$

$$L(S_vX,Y) = S_uY.S_vX$$

$$= S_uY.X + C_vS_uY$$

$$= \begin{cases} S_uY.X + S_uC_vY & u=v \\ S_uY.X + C_vY & u\neq v \end{cases} \qquad \begin{matrix} 27. \\ 27. \end{matrix}$$

$$= \begin{cases} S_u(S_uY.X) + C_vY & u=v \\ S_uY.X + C_vY & u\neq v \end{cases} \qquad 7.$$

$$= \begin{cases} S_v(L(X,Y) + C_vY) & u=v \\ L(X,Y) + C_vY & u\neq v \end{cases}$$

$$R(S_vX,Y) = Y.S_vX + C_uS_vX$$

$$= \begin{cases} Y.X + C_vY + S_vC_uX & u=v \\ Y.X + C_vY + C_uX & u\neq v \end{cases} \qquad \begin{matrix} 43.\ 27. \\ 43.27. \end{matrix}$$

$$= \begin{cases} S_v(Y.X + C_vY + C_uX) & u=v \\ Y.X + C_vY + C_uX & u\neq v \end{cases} \qquad 7.$$

$$= \begin{cases} S_v(Y.X + C_uX + C_vY) & u=v \\ Y.X + C_uX + C_vY & u\neq v \end{cases} \qquad 15.$$

$$= \begin{cases} S_v(R(X,Y) + C_vY) & u=v \\ R(X,Y) + C_vY & u\neq v \end{cases}$$

$$L(X,Y) = R(X,Y) \qquad 6.$$

46. $$X.Y = Y.X$$

Proof $$L(0,Y) = 0.Y$$

$$= 0 \qquad 44.$$

$$R(0,Y) = Y.0$$

$$= 0 \qquad 43.$$

$$L(S_uX,Y) = S_uX.Y$$

$$= X.Y + C_uY \qquad 45.$$

$$= L(X,Y) + C_uY$$

$$R(S_uX,Y) = Y.S_uX$$

$$= Y.X + C_uY \qquad 43.$$

$$= R(X,Y) + C_uY$$

$$L(X,Y) = R(X,Y) \qquad 6.$$

47. $$C_u(B.A) = C_uB.C_uA$$

Proof

$$L(0,B) = C_u(B.0)$$

$$= C_u0 \qquad 43.$$

$$= 0 \qquad 27.$$

$$R(0,B) = C_uB.C_u0$$

$$= C_uB.0 \qquad 27.$$

$$= 0 \qquad 43.$$

$$L(S_vA,B) = C_u(B.S_vA)$$

$$= C_u(B.A) + C_uC_vB \qquad 30,43.$$

$$= \begin{cases} C_u(B.A) + C_uB & u=v \qquad 28. \\ C_u(B.A) + 0 & u\neq v \qquad 28. \end{cases}$$

$$= \begin{cases} L(A,B) + C_uB & u=v \\ L(A,B) & u\neq v \qquad 7. \end{cases}$$

$$R(S_vA,B) = C_uB.C_uS_vA$$

$$= \begin{cases} C_uB.S_vC_uA & u=v \qquad 27. \\ C_uB.C_uA & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} C_uB.C_uA + C_vC_uB & u=v \qquad 43. \\ C_uB.C_uA & u\neq v \end{cases}$$

$$= \begin{cases} C_uB.C_uA + C_uB & u=v \\ C_uB.C_uA & u\neq v \end{cases} \qquad 28.$$

$$= \begin{cases} R(A,B) + C_uB & u=v \\ R(A,B) & u\neq v \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

48. $$C_uB.A = C_uB.C_uA$$

Proof

$$L(0,B) = C_uB.0$$

$$= 0 \qquad 43.$$

$$R(0,B) = C_uB.C_u0$$

$$= C_uB.0 \qquad 27.$$

$$= 0 \qquad 43.$$

$$L(S_vA,B) = C_uB.S_vA$$

$$= C_uB.A + C_vC_uB \qquad 43.$$

$$= \begin{cases} C_uB.A + C_uB & u=v \qquad 28. \\ C_uB.A + 0 & u\neq v \qquad 28. \end{cases}$$

$$= \begin{cases} L(A,B) + C_uB & u=v \\ L(A,B) & u\neq v \end{cases}$$

$$R(S_vA,B) = C_uB.C_uS_vA$$

$$= \begin{cases} C_uB.S_vC_uA & u=v \qquad 27. \\ C_uB.C_uA & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} C_uB.C_uA + C_vC_uB & u=v \qquad 43. \\ C_uB.C_uA & u\neq v \end{cases}$$

$$= \begin{cases} C_uB.C_uA + C_uB & u=v \qquad 28. \\ C_uB.C_uA & u\neq v \end{cases}$$

$$= \begin{cases} R(A,B) + C_u B & u=v \\ R(A,B) & u \neq v \end{cases}$$

$$L(A,B) = R(A,B) \qquad 6.$$

49. $$C_u A.B = A.C_u B$$

Proof

$$C_u A.B = C_u A.C_u B \qquad 48.$$

$$= C_u B.C_u A \qquad 46.$$

$$= C_u B.A \qquad 48.$$

$$= A.C_u B \qquad 46.$$

50. $$C.(B + A) = C.B + C.A$$

Proof

$$L(0,B,C) = C.(B + 0)$$

$$= C.B \qquad 7.$$

$$R(0,B,C) = C.B + C.0$$

$$= C.B + 0 \qquad 43.$$

$$= C.B \qquad 7.$$

$$L(S_u A,B,C) = C.(B + S_u A)$$

$$= C.S_u(B + A) \qquad 7.$$

$$= C.(B + A) + C_u C \qquad 43.$$

$$= L(A,B,C) + C_u C$$

$$R(S_u A,B,C) = C.B + C.S_u A$$

$$= C.B + C.A + C_u C \qquad 43.$$

$$= R(A,B,C) + C_u C$$

$$L(A,B,C) = R(A,B,C) \qquad 6.$$

51. $(C.B).A = C.(B.A)$

Proof

$$L(0,B,C) = (C.B).0$$

$$= 0 \qquad 43.$$

$$R(0,B,C) = C.(B.0)$$

$$= C.0 \qquad 43.$$

$$= 0 \qquad 43.$$

$$L(S_uA,B,C) = (C.B).S_uA$$

$$= (C.B).A + C_u(C.B) \qquad 43.$$

$$= L(A,B,C) + C_u(C.B)$$

$$R(S_uA,B,C) = C.(B.S_uA)$$

$$= C.(B.A + C_uB) \qquad 43.$$

$$= C.(B.A) + C.C_uB \qquad 50.$$

$$= C.(B.A) + C_u(C.B) \qquad 47.48.49.$$

$$= R(A,B,C) + C_u(C.B)$$

$$L(A,B,C) = R(A,B,C) \qquad 6.$$

52. $B.P_uA = B.A \dot{-} C_uB$

Proof

$$L(0,B) = B.P_u0$$

$$= B.0 \qquad 17.$$

$$= 0 \qquad 43.$$

$$R(0,B) = B.0 \dot{-} C_uB$$

$$= 0 \dot{-} C_uB \qquad 43.$$

$$= 0 \qquad 19.$$

$$L(S_vA,B) = B.P_uS_vA$$

$$= \begin{cases} B.A & u=v \\ B.S_vP_uA & u\neq v \end{cases} \qquad \begin{matrix} 17. \\ 17. \end{matrix}$$

$$= \begin{cases} B.A & u=v \\ B.P_uA + C_vB & u\neq v \end{cases} \qquad 43.$$

$$= \begin{cases} B.A & u=v \\ L(A,B) + C_vB & u\neq v \end{cases}$$

$$R(S_vA,B) = B.S_vA \dot{-} C_uB$$

$$= (B.A + C_vB) \dot{-} C_uB \qquad 43.$$

$$= \begin{cases} (B.A + C_uB) \dot{-} C_uB & u=v \\ (B.A \dot{-} C_uB) + C_vB & u\neq v \end{cases} \qquad 36.$$

$$= \begin{cases} B.A & u=v \\ R(A,B) + C_vB & u\neq v \end{cases} \qquad 23.$$

$$L(A,B) = R(A,B) \qquad 26.$$

53. $C.(B \dot{-} A) = C.B \dot{-} C.A$

Proof $L(0,B,C) = C.(B \dot{-} 0)$

$$= C.B \qquad 18.$$

$$R(0,B,C) = C.B \dot{-} C.0$$

$$= C.B \dot{-} 0 \qquad 43$$

$$= C.B \qquad 18$$

$$L(S_uA,B,C) = C.(B \dot{-} S_uA)$$

$$= C.P_u(B \dot{-} A) \qquad 18.$$

$$= C.(B \dot{-} A) \dot{-} C_uC \qquad 52.$$

$$= L(A,B,C) \dot{-} C_uC$$

$$R(S_uA,B,C) = C.B \dot{-} C.S_uA$$

$$= C.B \dot{-} (C.A + C_uC) \qquad 43.$$

$$= (C.B \dot{-} C.A) \dot{-} C_uC \qquad 20.$$

$$= R(A,B,C) \dot{-} C_uC$$

$$L(A,B,C) = R(A,B,C) \qquad 6.$$

54. Definition of $|A,B|$

$$|A,B| = (A \dot{-} B) + (B \dot{-} A)$$

55. Rule of inference

$$\frac{A + B = 0}{\begin{array}{c} A = 0 \\ B = 0 \end{array}}$$

Proof

$A + B = 0$	Premises
$A = (A + B) \dot{-} B$	23.
$= 0 \dot{-} B$	Premises
$= 0$	19.
$B = (A + B) \dot{-} A$	23. 15.
$= 0 \dot{-} A$	Premises
$= 0$	

56. Rule of inference

$$\frac{A = B}{\begin{array}{c} A \dot{-} B = 0 \\ B \dot{-} A = 0 \end{array}}$$

Proof	$A = B$	Premises
	$A \dot{-} B = B \dot{-} B$	5.
	$= 0$	22.
	$B \dot{-} A = B \dot{-} B$	Premises. 5.
	$= 0$	22.

57. Rule of inference

$$\frac{F(S_u X) = F(X)}{F(X) = F(0)}$$

Proof	$L(0) = F(0)$	
	$R(0) = F(0)$	
	$L(S_u X) = F(S_u X)$	
	$= F(X)$	Premises
	$= L(X)$	
	$R(S_u X) = F(0)$	
	$= R(X)$	
	$L(X) = R(X)$	6.

58. Rule of inference

$$F(0,Y) = 0$$

$$F(X,0) = 0$$

$$\frac{F(C_u S_u X, C_u S_u Y) = F(C_u X, C_u Y)}{F(C_u X, C_u Y) = 0} \quad \text{for one value of } u. \quad \text{for that value of } u.$$

Proof

Consider first

(a) $F(C_u S_u X, C_u Y) = F(C_u X, C_u (Y \dot{-} S_u 0))$

Proof $L(X,0) = F(C_u S_u X, C_u 0)$

$$= F(C_u S_u X, 0) \qquad 27.$$

$$= 0 \qquad \text{Premises}$$

$$R(X,0) = F(C_u X, C_u(0 \dot{-} S_u 0))$$

$$= F(C_u X, C_u 0) \qquad 19.$$

$$= F(C_u X, 0) \qquad 27.$$

$$= 0 \qquad \text{Premises}$$

$$L(X, S_v Y) = F(C_u S_u X, C_u S_v Y)$$

$$= \begin{cases} F(C_u S_u X, C_u S_u Y) & u=v \qquad 27. \\ F(C_u S_u X, C_u Y) & u \neq v \qquad 27. \end{cases}$$

$$= \begin{cases} F(C_u X, C_u Y) & u=v \qquad \text{Premises} \\ L(X,Y) & u \neq v \end{cases}$$

$$R(X, S_v Y) = F(C_u X, C_u(S_v Y \dot{-} S_u 0))$$

$$= \begin{cases} F(C_u X, C_u(S_u Y \dot{-} S_u 0)) & u=v \\ F(C_u X, C_u S_v Y \dot{-} C_u S_u 0) & u \neq v \qquad 32. \end{cases}$$

$$= \begin{cases} F(C_u X, C_u(Y \dot{-} 0)) & u=v \qquad 21. \\ F(C_u X, C_u Y \dot{-} C_u S_u 0) & u \neq v \qquad 27. \end{cases}$$

$$= \begin{cases} F(C_u X, C_u Y) & u=v \\ F(C_u X, C_u(Y \dot{-} S_u 0)) & u \neq v \qquad 32. \end{cases}$$

$$= \begin{cases} F(C_u X, C_u Y) & u=v \\ R(X,Y) & u \neq v \end{cases}$$

$$L(X,Y) = R(X,Y) \text{ for (a)} \qquad 26.$$

Now consider

(b) $\quad F(C_u X, C_u Y) = F(C_u(X \dot{-} S_u 0), C_u(Y \dot{-} S_u 0))$

Proof

$$L(0,Y) = F(C_uX,C_u0)$$

$$= F(C_uX,0) \qquad 27.$$

$$= 0 \qquad \text{Premises}$$

$$R(0,Y) = F(C_u(X \dot{-} S_u0),C_u(0 \dot{-} S_u0))$$

$$= F(C_u(X \dot{-} S_u0),C_u0) \qquad 19.$$

$$= F(C_u(X \dot{-} S_u0),0) \qquad 27.$$

$$= 0 \qquad \text{Premises}$$

$$L(S_vX,Y) = F(C_uS_vX,C_uY)$$

$$= \begin{cases} F(C_uS_uX,C_uY) & u=v \qquad 27. \\ F(C_uX,C_uY) & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} F(C_uX,C_u(Y \dot{-} S_u0)) & u=v \qquad (a) \\ L(X,Y) & u\neq v \end{cases}$$

$$R(S_vX,Y) = F(C_u(S_vX \dot{-} S_u0),C_u(Y \dot{-} S_u0))$$

$$= \begin{cases} F(C_u(S_uX \dot{-} S_u0),C_u(Y \dot{-} S_u0)) & u=v \\ F(C_uS_vX \dot{-} C_uS_u0,C_u(Y \dot{-} S_u0)) & u\neq v \qquad 32. \end{cases}$$

$$= \begin{cases} F(C_u(X \dot{-} 0),6_u(Y \dot{-} S_u0)) & u=v \qquad 21. \\ F(C_uX \dot{-} C_uS_u0,C_u(Y \dot{-} S_u0)) & u\neq v \qquad 27. \end{cases}$$

$$= \begin{cases} F(C_uX,C_u(Y \dot{-} S_u0)) & u=v \qquad 18. \\ F(C_u(X \dot{-} S_u0),C_u(Y \dot{-} S_u0)) & u\neq v \qquad 32. \end{cases}$$

$$= \begin{cases} F(C_uX,C_u(Y \dot{-} S_u0) & u=v \\ R(X,Y) & u\neq v \end{cases}$$

$$L(X,Y) = R(X,Y) \text{ for (b)} \qquad 26.$$

Now let $\qquad G(N) = F(C_u(X \dot{-} N),C_u(Y \dot{-} N))$

Hence $G(S_vN) = F(C_uX \dot{-} S_vN), C_u(Y \dot{-} S_vN))$

$= F(C_uP_v(X \dot{-} N), C_uP_v(Y \dot{-} N))$ 18.

$= \begin{cases} F(C_u((X \dot{-} N) \dot{-} S_u0), C_u((Y \dot{-} N) \dot{-} S_u0) & u=v \quad 17.18. \\ F(C_u(X \dot{-} N), C_u(Y \dot{-} N)) & u \neq v \quad 40. \end{cases}$

$= \begin{cases} F(C_u(X \dot{-} N), C_u(Y \dot{-} N)) & u=v \quad (b) \\ F(C_u(X \dot{-} N), C_u(Y \dot{-} N)) & u \neq v \end{cases}$

$= G(N)$

$G(N) = G(0)$ 57.

$F(C_u(X \dot{-} N), C_u(Y \dot{-} N)) = F(C_u(X \dot{-} 0), C_u(Y \dot{-} 0))$

$= F(C_uX, C_uY)$ 18.

In the particular instance where $N = X$ we have:-

$F(C_uX, C_uY) = F(C_u(X \dot{-} X), C_u(Y \dot{-} C))$

$= F(C_u0, C_u(Y \dot{-} X))$ 22.

$= F(0, C_u(Y \dot{-} X))$ 27.

$= 0$ Premises

59. $C_v(S_u0 \dot{-} A) = \begin{cases} S_u0 \dot{-} A & u=v \\ 0 & u \neq v \end{cases}$

Proof $C_v(S_u0 \dot{-} A) = C_vS_u0 \dot{-} C_vA$

$= \begin{cases} S_u0 \dot{-} C_uA & u=v \quad 27. \\ 0 \dot{-} A & u \neq v \quad 27. \end{cases}$

$= \begin{cases} S_u0 \dot{-} A & u=v \quad 42. \\ 0 & u \neq v \quad 19. \end{cases}$

60. $A.S_u0 = C_uA$

Proof

$$\begin{aligned} A.S_u0 &= A.0 + C_uA && 43. \\ &= 0 + C_uA && 43. \\ &= C_uA && 13. \end{aligned}$$

61. $$S_u0 \dot{-} S_vA = \begin{cases} S_u0 \dot{-} A & u \neq v \\ 0 & u = v \end{cases}$$

Proof

$$\begin{aligned} S_u0 \dot{-} S_vA &= S_u0 \dot{-} C_uS_vA \\ &= \begin{cases} S_u0 \dot{-} C_uA & u \neq v & 27. \\ S_u0 \dot{-} S_uC_uA & u = v & 27. \end{cases} \\ &= \begin{cases} S_u0 \dot{-} A & u \neq v & 42. \\ 0 \dot{-} C_uA & u = v & 21. \end{cases} \\ &= \begin{cases} S_u0 \dot{-} A & u \neq v \\ 0 & u = v & 19. \end{cases} \end{aligned}$$

62. $A.(S_u0 \dot{-} A) = 0$

Proof

$$\begin{aligned} L(0) &= 0.(S_u0 \dot{-} 0) \\ &= 0 && 44. \\ R(0) &= 0 \\ L(S_vA) &= S_vA.(S_u0 \dot{-} S_vA) \\ &= \begin{cases} S_vA.0 & u = v & 61. \\ S_vA.(S_u0 \dot{-} A & u \neq v & 61. \end{cases} \\ &= \begin{cases} 0 & u = v & 43. \\ A.(S_u0 \dot{-} A) + C_v(S_u0 \dot{-} A) & u \neq v & 43. \end{cases} \\ &= \begin{cases} 0 & u = v \\ L(A) + 0 & u \neq v & 59. \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & u=v \\ L(A) & u\neq v \end{cases} \qquad 7.$$

$$R(S_vA) = 0$$

$$= \begin{cases} 0 & u=v \\ 0 & u\neq v \end{cases}$$

$$= \begin{cases} 0 & u=v \\ R(A) & u\neq v \end{cases}$$

$$L(A) = R(A) \qquad 26$$

63. Rule of inference

$$\frac{\begin{array}{c} F(0) = 0 \\ (S_u0 \dot{-} F(X)).F(S_vX) = 0 \end{array}}{F(X) = 0}$$

Proof

Consider first

(a) $(S_u0 \dot{-} F(X)).(S_u0 \dot{-} F(S_vX)) = S_u0 \dot{-} F(X)$

Proof

$$(S_u0 \dot{-} F(X)).(S_u0 \dot{-} F(S_vX)) = (S_u0 \dot{-} F(X)).S_u0 \dot{-} (S_u0 \dot{-} F(X)).F(S_vX) \qquad 53.$$

$$= (S_u0 \dot{-} F(X)).S_u0 \dot{-} 0 \qquad \text{Premises}$$

$$= (S_u0 \dot{-} F(X)).S_u0 \qquad 18.$$

$$= C_u(S_u0 \dot{-} F(X)) \qquad 60.$$

$$= S_u0 \dot{-} F(X) \text{ for (a)} \qquad 59.$$

Now define $G(X)$ by

$$G(0) = S_u0$$

$$G(S_vX) = G(X).(S_u0 \dot{-} F(X))$$

Let	$H(X) = G(S_v X)$	
	$H(0) = G(S_v 0)$	
	$= G(0).(S_u 0 \dot{-} F(0))$	Defn.
	$= G(0).(S_u 0 \dot{-} 0)$	Premises
	$= S_u 0.S_u 0$	18.
	$= C_u S_u 0$	60.
	$= S_u 0$	27.
	$= G(0)$	Defn.
	$H(S_w X) = G(S_v S_w X)$	
	$= G(S_w X).(S_u 0 \dot{-} F(S_w X))$	Defn.
	$= G(X).(S_u 0 \dot{-} F(X)).(S_u 0 \dot{-} F(S_w X))$	Defn.
	$= G(X).(S_u 0 \dot{-} F(X))$	(a)
	$= G(S_w X)$	Defn.
	$H(X) = G(X)$	
	$G(S_v X) = G(X)$	
	$G(X) = G(0)$	57.
	$= S_u 0$	Defn.
	$S_u 0 = S_u 0.(S_u 0 \dot{-} F(X))$	Defn.
	$= C_u(S_u 0 \dot{-} F(X))$	60.
	$= S_u 0 \dot{-} F(X)$	59.
	$F(X).S_u 0 = F(X).(S_u 0 \dot{-} F(X))$	5.
	$C_u F(X) = 0$	62.

As no specification has been made on u we can therefore say

$$F(X) = 0 \qquad \text{by } 53.$$

64. Definition (explicit)

$$(X)^2 = X^2 = X.X$$

65. Definition of Rt(X)

$$Rt(0) = 0$$

$$Rt(S_uX) = Rt(X) + (S_u0 \dot{-} ((S_uRt(X))^2 \dot{-} S_uX)) \quad u=1,\ldots,n$$

66. $(S_u0 \dot{-} ((S_u(A + (S_v0 \dot{-} B)))^2 \dot{-} S_uS_vC)) = (S_u0 \dot{-} ((S_uA)^2 \dot{-} S_uC)) \quad u \neq v$

Proof

$$\begin{aligned}
(S_u0 \dot{-} ((S_u(A+ (S_v0 \dot{-} B)))^2 \dot{-} S_uS_vC)) &= (S_u0 \dot{-} C_u((S_u(A+ (S_v0 \dot{-} B)))^2 \dot{-} S_uS_vC)) && 42.\\
&= (S_u0 \dot{-} (C_u(S_u(A+ (S_v0 \dot{-} B)))^2 \dot{-} C_uS_uS_vC)) && 32.\\
&= (S_u0 \dot{-} ((C_uS_u(A+ (S_v0 \dot{-} B)))^2 \dot{-} S_uC_uS_vC)) && 47.27.\\
&= (S_u0 \dot{-} ((S_u(C_uA+ C_u(S_v0 \dot{-} B)))^2 \dot{-} S_uC_uC)) && 30.27.\\
&= (S_u0 \dot{-} ((S_u(C_uA+ 0))^2 \dot{-} C_uS_uC)) && 59.27.\\
&= (S_u0 \dot{-} ((S_uC_uA)^2 \dot{-} C_uS_uC)) && 7.\\
&= (S_u0 \dot{-} ((C_uS_uA)^2 \dot{-} C_uS_uC)) && 27.\\
&= (S_u0 \dot{-} (C_u(S_uA)^2 \dot{-} C_uS_uC)) && 47.\\
&= (S_u0 \dot{-} C_u((S_uA)^2 \dot{-} S_uC)) && 32.\\
&= (S_u0 \dot{-} ((S_uA)^2 \dot{-} S_uC)) && 42.
\end{aligned}$$

67. The commutativity condition for R(X)

If $u \neq v$ then

$$B_u(S_vX,B_v(X,Z)) = Z + (S_v0 \dot{-} ((S_vZ)^2 \dot{-} S_vX)) +$$

$$(S_u0 \dot{-} ((S_u(Z + (S_v0 \dot{-} ((S_vZ)^2 \dot{-} S_vX))))^2 \dot{-} S_uS_vX)) \quad \text{Defn.}$$

$$= Z + (S_v0 \dot{-} ((S_v(Z + (S_u0 \dot{-} ((S_uZ)^2 \dot{-} S_uX))))^2 \dot{-} S_vS_uX)) + \quad 66.$$

$$(S_u0 \dot{-} ((S_uZ)^2 \dot{-} S_uX)) \quad 66.$$

$$= Z + (S_u0 \dot{-} ((S_uZ)^2 \dot{-} S_uX)) +$$

$$(S_v0 \dot{-} ((S_v(Z+ (S_u0 \dot{-} ((S_uZ)^2 \dot{-} S_uX))))^2 \dot{-} S_vS_uX)) \quad 15.$$

$$= B_v(S_uX,B_u(Y,Z)) \quad \text{for } u \neq v \quad \text{Defn.}$$

If u=v then

$$B_u(S_vX,B_v(X,Z)) = B_v(S_uX,B_u(X,Z) \quad u=v$$

Hence

$$B_u(S_vX,B_v(X,Z)) = B_v(S_uX,B_u(X,Z)) \quad \text{all } u,v.$$

68. $(S_u0 \dot{-} A).(S_u0 \dot{-} (S_u0 \dot{-} A)) = 0$

Proof $\quad L(0) = (S_u0 \dot{-} 0).(S_u0 \dot{-} (S_u0 \dot{-} 0))$

$$= S_u0.(S_u0 \dot{-} S_u0) \quad 18.$$

$$= S_u0.0 \quad 22.$$

$$= 0 \quad 43.$$

$$R(0) = 0$$

$$L(S_vA) = (S_u0 \dot{-} S_vA).(S_u0 \dot{-} (S_u0 \dot{-} S_vA))$$

$$= \begin{cases} (0 \dot{-} A).(S_u0 \dot{-} (0 \dot{-} A)) & u=v \quad 61. \\ (S_u0 \dot{-} A).(S_u0 \dot{-} (S_u0 \dot{-} A)) & u \neq v \quad 61. \end{cases}$$

$$= \begin{cases} 0.(S_u0 \dot{-} 0) & u=v \quad 19. \\ L(A) & u \neq v \end{cases}$$

$$= \begin{cases} 0 & u=v \quad 44. \\ L(A) & u \neq v \end{cases}$$

$$R(S_vA) = 0$$

$$= \begin{cases} 0 & u=v \\ 0 & u\neq v \end{cases}$$

$$= \begin{cases} 0 & u=v \\ R(A) & u\neq v \end{cases}$$

$$L(A) = R(A) \qquad 26.$$

69. $(S_u0 \dot{-} (S_uA \dot{-} B)).(S_u0 \dot{-} (B \dot{-} A)) = 0$

Proof

$$L(A,0) = (S_u0 \dot{-} (S_uA \dot{-} 0)).(S_u0 \dot{-} (0 \dot{-} A))$$

$$= (S_u0 \dot{-} S_uA).(S_u0 \dot{-} 0) \qquad 18.19.$$

$$= 0.S_u0 \qquad 61.$$

$$= 0 \qquad 44.$$

$$L(0,B) = (S_u0 \dot{-} (S_u0 \dot{-} B)).(S_u0 \dot{-} (B \dot{-} 0))$$

$$= (S_u0 \dot{-} (S_u0 \dot{-} B)).(S_u0 \dot{-} B) \qquad 18.$$

$$= 0 \qquad 68.$$

$$L(C_uS_uA,C_uS_uB) = (S_u0 \dot{-} (S_uC_uS_uA \dot{-} C_uS_uB)).(S_u0 \dot{-} (C_uS_uB \dot{-} C_uS_uA))$$

$$= (S_u0 \dot{-} (S_uS_uC_uA \dot{-} S_uC_uB)).(S_u0 \dot{-} (S_uC_uB \dot{-} S_uC_uA)) \qquad 27.$$

$$= (S_u0 \dot{-} (S_uC_uA \dot{-} C_uB)).(S_u0 \dot{-} (C_uB \dot{-} C_uA)) \qquad 22.$$

$$= L(C_uA,C_uB)$$

$$L(C_uA,C_uB) = 0 \qquad 58.$$

$$(S_u0 \dot{-} (S_uA \dot{-} B)).(S_u0 \dot{-} (B \dot{-} A)) = (S_u0 \dot{-} C_u(S_uA \dot{-} B)).(S_u0 \dot{-} C_u(B \dot{-} A)) \qquad 42.$$

$$= (S_u0 \dot{-} (C_uS_uA \dot{-} C_uB)).(S_u0 \dot{-} (C_uB \dot{-} C_uA)) \qquad 32.$$

$$= (S_u0 \dot{-} (S_uC_uA \dot{-} C_uB)).(S_u0 \dot{-} (C_uB \dot{-} C_uA)) \qquad 27.$$

$$= L(C_uA,C_uB)$$

$$= 0 \qquad \text{From above.}$$

70. Rule of inference

$$\frac{A.B = A.C}{A.|B,C| = 0}$$

Proof

$A.\|B,C\|$	$= A.((B \dot{-} C) + (C \dot{-} B))$	54.
	$= A.(B \dot{-} C) + A.(C \dot{-} B)$	50.
	$= (A.B \dot{-} A.C) + (A.C \dot{-} A.B)$	53.
	$= 0 + 0$	Premises. 56.
	$= 0$	7.

It is now necessary to quote the Key Equation. This is proved formally using primitive recursion under the same set of axioms in [4].

Thus

71. Key Equation

$$A + (B \dot{+} A) = B + (A \dot{-} B)$$

72. $(S_u 0 \dot{-} X).F(C_u(X + Y)) = (S_u 0 \dot{-} X).F(C_u Y)$

Proof

$L(0,Y)$	$= (S_u 0 \dot{-} 0).F(C_u(0 + Y))$	
	$= S_u 0.F(C_u Y)$	18.13.
	$= C_u F(C_u Y)$	60.
$R(0,Y)$	$= (S_u 0 \dot{-} 0).F(C_u Y)$	
	$= S_u 0.F(C_u Y)$	18.
	$= C_u F(C_u Y)$	60.
$L(S_v X,Y)$	$= (S_u 0 \dot{-} S_v X).F(C_u(S_v X + Y))$	
	$= \begin{cases} 0.F(C_u(S_v X + Y) & u=v \\ (S_u 0 \dot{-} X).F(C_u S_v(X + Y)) & u \neq v \end{cases}$	61. 61.

$$= \begin{cases} 0 & u=v \\ (S_u 0 \dot{-} X).F(C_u(X + Y)) & u \neq v \end{cases} \qquad 44.$$

$$= \begin{cases} 0 & u=v \\ L(X,Y) & u \neq v \end{cases}$$

$$R(S_v X,Y) = (S_u 0 \dot{-} S_v X).F(C_u Y)$$

$$= \begin{cases} 0.F(C_u Y) & u=v \qquad 61. \\ (S_u 0 \dot{-} X).F(C_u Y) & u \neq v \qquad 61. \end{cases}$$

$$= \begin{cases} 0 & u=v \qquad 44. \\ R(X,Y) & u \neq v \end{cases}$$

$$L(X,Y) = R(X,Y) \qquad 26.$$

In fact the equation

$$(S_u 0 \dot{-} X).F(C_u X + Y) = (S_u 0 \dot{-} X).F(Y)$$

is proved by the above schema, though the equation proved is sufficient for subsequent proofs.

73. Rule of inference

$$\frac{|A,B| = 0}{A = B}$$

Proof $\quad |A,B| = 0 \qquad$ Premises.

$(A \dot{-} B) + (B \dot{-} A) = 0 \qquad$ 54.

$A \dot{-} B = 0 \quad \ldots\ldots \quad (1) \qquad$ 55.

$B \dot{-} A = 0 \quad \ldots \quad (2) \qquad$ 55.

$A = A + 0 \qquad$ 7.

$= A + (B \dot{-} A) \qquad$ From above (2)

$= B + (A \dot{-} B) \qquad$ Key Equation

$= B + 0 \qquad$ From above (1)

$= B \qquad$ 7.

74. Rule of inference

$$F(0,Y) = G(0,Y)$$

$$F(X,0) = G(X,0)$$

$$F(C_uS_uX,C_uS_uY) = F(C_uX,C_uY) \qquad \text{for one value of } u$$

$$G(C_uS_uX,C_uS_uY) = G(C_uX,C_uY) \qquad \text{for the same value of } u.$$

$$F(C_uX,C_uY) = G(C_uX,C_uY) \qquad \text{for that value of } u.$$

Proof

Let $H(X,Y) = |F(X,Y),G(X,Y)|$

Then $H(0,Y) = |F(0,Y),G(0,Y)|$

$= 0$ Premises 56. 7.

$H(X,0) = |F(X,0),G(X,0)|$

$= 0$ Premises 56. 7.

$H(C_uS_vX,C_uS_vY) = |F(C_uS_vX,C_uS_vY),G(C_uS_vX,C_uS_vY)|$

$= |F(C_uX,C_uY),G(C_uX,C_uY)|$ $u=v$ Premises

$|F(C_uX,C_uY),G(C_uX,C_uY)|$ $u\neq v$ 27.

$= H(C_uX,C_uY)$

$H(C_uX,C_uY) = 0$ 58.

$F(C_uX,C_uY) = G(C_uX,C_uY)$ 73.

75. $|S_uX,S_uY| = |X,Y|$

Proof $|S_uX,S_uY| = (S_uX \dot{-} S_uY) + (S_uY \dot{-} S_uX)$ 54.

$= (X \dot{-} Y) + (Y \dot{-} X)$ 21.

$= |X,Y|$ 54.

76. $|C_uX, C_uY| = C_u|X,Y|$

Proof

$$
\begin{aligned}
|C_uX, C_uY| &= (C_uX \dot{-} C_uY) + (C_uY \dot{-} C_uX) && 54. \\
&= C_u(X \dot{-} Y) + C_u(Y \dot{-} X) && 32. \\
&= C_u((X \dot{-} Y) + (Y \dot{-} X)) && 30. \\
&= C_u|X,Y| && 54.
\end{aligned}
$$

77. $|X,Y| = |Y,X|$

Proof

$$
\begin{aligned}
|X,Y| &= (X \dot{-} Y) + (Y \dot{-} X) && 54. \\
&= (Y \dot{-} X) + (X \dot{-} Y) && 15. \\
&= |Y,X| && 54.
\end{aligned}
$$

78. $|0,Y| = |Y,0| = Y$

Proof

$$
\begin{aligned}
|0,Y| &= |Y,0| && 77. \\
&= (Y \dot{-} 0) + (0 \dot{-} Y) && 54. \\
&= Y + 0 && 18.19. \\
&= Y && 7.
\end{aligned}
$$

79. $(S_u0 \dot{-} |X,Y|).(S_u0 \dot{-} (X \dot{-} Y)) = (S_u0 \dot{-} |X,Y|)$

Proof

$$
\begin{aligned}
L(0,Y) &= (S_u0 \dot{-} |0,Y|).(S_u0 \dot{-} (0 \dot{-} Y)) \\
&= (S_u0 \dot{-} Y).(S_u0 \dot{-} 0) && 81.19. \\
&= (S_u0 \dot{-} Y).S_u0 && 18. \\
&= C_u(S_u0 \dot{-} Y) && 60. \\
&= (S_u0 \dot{-} Y) && 59.
\end{aligned}
$$

$$R(0,Y) = (S_u 0 \dot{-} |0,Y|)$$

$$= (S_u 0 \dot{-} Y) \qquad 78.$$

$$L(X,0) = (S_u 0 \dot{-} |X,0|).(S_u 0 \dot{-} (X \dot{-} 0))$$

$$= (S_u 0 \dot{-} X).(S_u 0 \dot{-} X) \qquad 78.18.$$

$$= S_u 0.(S_u 0 \dot{-} X) \dot{-} X.(S_u 0 \dot{-} X) \qquad 53.46.$$

$$= C_u(S_u 0 \dot{-} X) \dot{-} 0 \qquad 60.62.$$

$$= C_u(S_u 0 \dot{-} X) \qquad 18.$$

$$= (S_u 0 \dot{-} X) \qquad 59.$$

$$R(X,0) = (S_u 0 \dot{-} |X,0|)$$

$$= (S_u 0 \dot{-} X) \qquad 78,$$

$$L(C_u S_u X, C_u S_u Y) = (S_u 0 \dot{-} |C_u S_u X, C_u S_u Y|).(S_u 0 \dot{-} (C_u S_u X \dot{-} C_u S_u Y))$$

$$= (S_u 0 \dot{-} |S_u C_u X, S_u C_u Y|).(S_u 0 \dot{-} (S_u C_u X \dot{-} S_u C_u Y)) \qquad 27.$$

$$= (S_u 0 \dot{-} |C_u X, C_u Y|).(S_u 0 \dot{-} (C_u X \dot{-} C_u Y)) \qquad 75.21.$$

$$= L(C_u X, C_u Y)$$

$$R(C_u S_u X, C_u S_u Y) = (S_u 0 \dot{-} |C_u S_u X, C_u S_u Y|)$$

$$= (S_u 0 \dot{-} |S_u C_u X, S_u C_u Y|) \qquad 27.$$

$$= (S_u 0 \dot{-} |C_u X, C_u Y|) \qquad 75.$$

$$= R(C_u X, C_u Y)$$

$$L(C_u X, C_u Y) = R(C_u X, C_u Y) \qquad 74.$$

Hence

$$(S_u 0 \dot{-} |C_u X, C_u Y|).(S_u 0 \dot{-} (C_u X \dot{-} C_u Y)) = (S_u 0 \dot{-} |C_u X, C_u Y|)$$

$$(S_u 0 \dot{-} C_u|X,Y|).(S_u 0 \dot{-} C_u(X \dot{-} Y)) = (S_u 0 \dot{-} C_u|X,Y|) \qquad 76.32$$

$$(S_u 0 \dot{-} |X,Y|).(S_u 0 \dot{-} (X \dot{-} Y)) = (S_u 0 \dot{-} |X,Y|) \text{ as required.} \qquad 42.$$

80. $(S_u0 \dot{-} |X,Y|).(S_u0 \dot{-} F(C_uX)).F(C_uY) = 0$

Proof $(S_u0 \dot{-} X).F(C_u(X + Y)) = (S_u0 \dot{-} X).F(C_uY)$ 72.

Put $X \dot{-} Y$ for X, and we have

$$(S_u0 \dot{-} (X \dot{-} Y)),F(C_u((X \dot{-} Y) + Y)) = (S_u0 \dot{-} (X \dot{-} Y)).F(C_uY).$$

Multiply both sides of the equation by $(S_u0 \dot{-} |X,Y|)$ and therefore

$$(S_u0 \dot{-} |X,Y|).F(C_u((X \dot{-} Y) + Y)) = (S_u0 \dot{-} |X,Y|).F(C_uY) \qquad 70.79.$$

Similarly one can prove the equation

$$(S_u0 \dot{-} |Y,X|).F(C_u((Y \dot{-} X) + X)) = (S_u0 \dot{-} |Y,X|).F(C_uX)$$

Hence from the above two equations, the Key Equation and 77., we have

$$(S_u0 \dot{-} |X,Y|).F(C_uY) = (S_u0 \dot{-} |X,Y|).F(C_uX) .$$

Multiply both sides of the equation by $(S_u0 \dot{-} F(C_uX))$; the right-hand side then becomes zero by 62. and therefore we have

$$(S_u0 \dot{-} |X,Y|).(S_u0 \dot{-} F(C_uX)).F(C_uY) = 0 \quad \text{as required.}$$

81. $(S_u0 \dot{-} (S_u0 \dot{-} A)) + (S_u0 \dot{-} A) = S_u0$

Proof $L(0) = (S_u0 \dot{-} (S_u0 \dot{-} 0)) + (S_u0 \dot{-} 0)$

$= (S_u0 \dot{-} S_u0) + S_u0$ 18.

$= 0 + S_u0$ 22.

$= S_u0$ 13.

$R(0) = S_u0$

$L(S_vA) = (S_u0 \dot{-} (S_u0 \dot{-} S_vA)) + (S_u0 \dot{-} S_vA)$

$$= \begin{cases} (S_u 0 \dot{-} 0) + 0 & u=v \\ (S_u 0 \dot{-} (S_u 0 \dot{-} A)) + (S_u 0 \dot{-} A) & u \neq v \end{cases} \qquad \begin{matrix} 61. \\ 61. \end{matrix}$$

$$= \begin{cases} S_u 0 + 0 & u=v \\ L(A) & u \neq v \end{cases} \qquad 18.$$

$$= \begin{cases} S_u 0 & u=v \\ L(A) & u \neq v \end{cases} \qquad 7.$$

$$R(S_v A) = S_u 0$$

$$= \begin{cases} S_u 0 & u=v \\ S_u 0 & u \neq v \end{cases}$$

$$= \begin{cases} S_u 0 & u=v \\ R(A) & u \neq v \end{cases}$$

$$L(A) = R(A) \qquad 26.$$

82. $(S_u 0 \dot{-} (S_u A \dot{-} B)) + (S_u 0 \dot{-} |A,B|) = (S_u 0 \dot{-} (A \dot{-} B))$

Proof

$$L(A,0) = (S_u 0 \dot{-} (S_u A \dot{-} 0)) + (S_u 0 \dot{-} |A,0|)$$

$$= (S_u 0 \dot{-} S_u A) + (S_u 0 \dot{-} A) \qquad 18.78.$$

$$= 0 + (S_u 0 \dot{-} A) \qquad 61.$$

$$= S_u 0 \dot{-} A \qquad 13.$$

$$R(A,0) = (S_u 0 \dot{-} (A \dot{-} 0))$$

$$= S_u 0 \dot{-} A \qquad 18.$$

$$L(0,B) = (S_u 0 \dot{-} (S_u 0 \dot{-} B) + (S_u 0 \dot{-} |0,B|)$$

$$= (S_u 0 \dot{-} (S_u 0 \dot{-} B)) + (S_u 0 \dot{-} B) \qquad 61.$$

$$= S_u 0 \qquad 81.$$

$$R(0,B) = (S_u0 \dot{-} (0 \dot{-} B))$$

$$= (S_u0 \dot{-} 0) \qquad 19.$$

$$= S_u0 \qquad 18.$$

$$L(C_uS_uA,C_uS_uB) = (S_u0 \dot{-} (S_uC_uS_uA \dot{-} C_uS_uB)) + (S_u0 \dot{-} |C_uS_uA,C_uS_uB|)$$

$$= (S_u0 \dot{-} (S_uS_uC_uA \dot{-} S_uC_uB)) + (S_u0 \dot{-} |S_uC_uA,S_uC_uB|) \qquad 27.$$

$$= (S_u0 \dot{-} (S_uC_uA \dot{-} C_uB)) + (S_u0 \dot{-} |C_uA,C_uB|) \qquad 21.75.$$

$$= L(C_uA,C_uB)$$

$$R(C_uS_uA,C_uS_uB) = (S_u0 \dot{-} (C_uS_uA \dot{-} C_uS_uB))$$

$$= (S_u0 \dot{-} (S_uC_uA \dot{-} S_uC_uB)) \qquad 27.$$

$$= (S_u0 \dot{-} (C_uA \dot{-} C_uB)) \qquad 21.$$

$$= R(C_uA,C_uB)$$

$$L(C_uA,C_uB) = R(C_uA,C_uB) \qquad 74.$$

Hence

$$(S_u0 \dot{-} (S_uC_uA \dot{-} C_uB)) + (S_u0 \dot{-} |C_uA,C_uB|) = (S_u0 \dot{-} (C_uA \dot{-} C_uB))$$

$$(S_u0 \dot{-} (C_uS_uA \dot{-} C_uB)) + (S_u0 \dot{-} C_u|A,B|) = (S_u0 \dot{-} C_u(A \dot{-} B)) \qquad 27.76.32.$$

$$(S_u0 \dot{-} C_u(S_uA \dot{-} B)) + (S_u0 \dot{-} C_u|A,B|) = (S_u0 \dot{-} C_u(A \dot{-} B)) \qquad 32.$$

$$(S_u0 \dot{-} (S_uA \dot{-} B)) + (S_u0 \dot{-} |A,B|) = (S_u0 \dot{-} (A \dot{-} B)) \qquad 42.$$

83. Rule of inference

$$(S_u0 \dot{-} (S_uA \dot{-} B)).C = 0$$

$$\underline{(S_u0 \dot{-} |A,B|).C = 0}$$

$$(S_u0 \dot{-} (A \dot{-} B)).C = 0$$

Proof $\quad (S_u0 \dot{-} (A \dot{-} B)).C = ((S_u0 \dot{-} (S_uA \dot{-} B)) + (S_u0 \dot{-} |A,B|)).C \qquad 82.$

$$= (S_u0 \dot{-} (S_uA \dot{-} B)).C + (S_u0 \dot{-} |A,B|).C \qquad 50.46.$$

$$= 0 + 0 \qquad \text{Premises}$$

$$= 0 \qquad 7.$$

84. $A + (S_u 0 \dot{-} A) = S_u 0 + P_u A$

Proof

$$L(0) = 0 + (S_u 0 \dot{-} 0)$$

$$= S_u 0 \qquad 13.18.$$

$$R(0) = S_u 0 + P_u 0$$

$$= S_u 0 + 0 \qquad 17.$$

$$= S_u 0 \qquad 7.$$

$$L(S_v A) = S_v A + (S_u 0 \dot{-} S_v A)$$

$$= \begin{cases} S_u A + 0 & u=v \quad 61. \\ S_v A + (S_u 0 \dot{-} A) & u \neq v \quad 61. \end{cases}$$

$$= \begin{cases} S_u A & u=v \quad 7. \\ S_v(A + (S_u 0 \dot{-} A)) & u \neq v \quad 14. \end{cases}$$

$$= \begin{cases} S_u A & u=v \\ S_v L(A) & u \neq v \end{cases}$$

$$R(S_v A) = S_u 0 + P_u S_v A$$

$$= \begin{cases} S_u 0 + A & u=v \quad 17. \\ S_u 0 + S_v P_u A & u \neq v \quad 17. \end{cases}$$

$$= \begin{cases} S_u (0 + A) & u=v \quad 14. \\ S_v (S_u 0 + P_u A) & u \neq v \quad 7. \end{cases}$$

$$= \begin{cases} S_u A & u=v \\ S_v R(A) & u \neq v \end{cases}$$

$$L(A) = R(A)$$

85. Rule of inference

$$\frac{\begin{array}{r} C_u B.A = 0 \\ (S_u 0 \dot{-} A).C = 0 \end{array}}{C_u B.C = 0}$$

Proof	$0 = (S_u 0 \dot{-} A).C$	Premises
	$= (S_u 0 \dot{-} C_u A).C$	42.
	$= (S_u 0 \dot{-} C_u A).C.B$	44.73.
	$= S_u 0.C.B \dot{-} C_u A.C.B$	53.46.
	$= C_u(C.B) \dot{-} C.C_u A.B$	60.46.51.
	$= C.C_u B \dot{-} C.C_u B.A$	47.48.49.
	$= C_u B.C \dot{-} C.0$	46.Premises.
	$= C_u B.C \dot{-} 0$	27.
	$= C_u B.C.$	18.

86. Rule of inference

$$\frac{(A \dot{-} B).C = 0}{(A \dot{-} S_u B).C = 0}$$

Proof	$(A \dot{-} S_u B).C = (A \dot{-} (B + S_u 0)).C$	7.
	$= ((A \dot{-} B) \dot{-} S_u 0).C$	20.
	$= (A \dot{-} B).C \dot{-} S_u 0.C$	53.46.
	$= 0 \dot{-} S_u 0.C$	Premises
	$= 0$	19.

87. $(S_u 0 \dot{-} A).(S_u 0 \dot{-} A) = S_u 0 \dot{-} A$

Proof	$(S_u 0 \dot{-} A).(S_u 0 \dot{-} A) = (S_u 0 \dot{-} A).S_u 0 \dot{-} (S_u 0 \dot{-} A).A$	53.46.
	$= C_u(S_u 0 \dot{-} A) \dot{-} 0$	62.63.
	$= C_u(S_u 0 \dot{-} A)$	18.
	$= S_u 0 \dot{-} A$	59.

88. $Rt(C_uX) = C_uRt(X)$

Proof

$$
\begin{aligned}
L(0) &= Rt(C_u0) \\
&= Rt(0) && 27. \\
&= 0 && 65. \\
R(0) &= C_uRt(0) \\
&= C_u0 && 65. \\
&= 0 && 27. \\
L(S_vX) &= Rt(C_uS_vX) \\
&= \begin{cases} Rt(S_uC_uX) & u=v \\ Rt(C_uX) & u\neq v \end{cases} && 27. \\
&= \begin{cases} Rt(C_uX) + (S_u0 \dot{-} ((S_uRt(C_uX))^2 \dot{-} S_uC_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 65. \\
&= \begin{cases} L(X) + (S_u0 \dot{-} C_u((S_uRt(C_uX))^2 \dot{-} C_uS_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 42. \\
&= \begin{cases} L(X) + (S_u0 \dot{-} (C_u(S_uRt(C_uX))^2 \dot{-} C_uC_uS_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 32. \\
&= \begin{cases} L(X) + (S_u0 \dot{-} (C_u(S_uRt(C_uX))^2 \dot{-} C_uS_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 28. \\
&= \begin{cases} L(X) + (S_u0 \dot{-} C_u((S_uRt(C_uX))^2 \dot{-} S_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 32. \\
&= \begin{cases} L(X) + (S_u0 \dot{-} ((S_uL(X))^2 \dot{-} S_uX)) & u=v \\ L(X) & u\neq v \end{cases} && 42.
\end{aligned}
$$

$$R(S_vX) = C_uRt(S_vX)$$

$$= C_u(Rt(X) + (S_v0 \dot{-} ((S_vRt(X))^2 \dot{-} S_vX)) \qquad 65.$$

$$= C_uRt(X) + C_u(S_v0 \dot{-} ((S_vRt(X))^2 \dot{-} S_vX)) \qquad 30.$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} C_u((S_uRt(X))^2 \dot{-} S_uX)) & u=v \qquad 59.42. \\ R(X) + 0 & u\neq v \qquad 59. \end{cases}$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} (C_u(S_uRt(X))^2 \dot{-} C_uS_uX)) & u=v \qquad 32. \\ R(X) & u\neq v \end{cases}$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} (C_uC_u(S_uRt(X))^2 \dot{-} C_uS_uX)) & u=v \qquad 28. \\ R(X) & u\neq v \end{cases}$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} C_u((C_uS_uRt(X))^2 \dot{-} S_uX)) & u=v \qquad 32.47.64. \\ R(X) & u\neq v \end{cases}$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} ((S_uC_uRt(X))^2 \dot{-} S_uX)) & u=v \qquad 27.42. \\ R(X) & u\neq v \end{cases}$$

$$= \begin{cases} R(X) + (S_u0 \dot{-} ((S_uR(X))^2 \dot{-} S_uX)) & u=v \\ R(X) & u\neq v \end{cases}$$

$$L(X) = R(X) \qquad 6.$$

89. $C_uRt(S_vX) = C_uRt(X) \qquad u\neq v$

Proof $C_uRt(S_vX) = Rt(C_uS_vX) \qquad 88.$

$$= Rt(C_uX) \qquad u\neq v \qquad 27.$$

$$= C_uRt(X) \qquad 88.$$

90. $(S_u0 \dot{-} |A,B|) = (S_u0 \dot{-} (A \dot{-} B)) \dot{-} (B \dot{-} A)$

Proof $(S_u0 \dot{-} |A,B|) = (S_u0 \dot{-} ((A \dot{-} B) + (B \dot{-} A))) \qquad 54.$

$$= (S_u0 \dot{-} (A \dot{-} B)) \dot{-} (B \dot{-} A) \qquad 20.$$

91. Rule of inference

$$\frac{\begin{array}{l} A.C.D = 0 \\ (B \dot{-} A).C = 0 \end{array}}{B.C.D = 0}$$

Proof	$0 = (B \dot{-} A).C$	Premises
	$= (B \dot{-} A).C.D$	70.44.
	$= B.C.D. \dot{-} A.C.D$	53.46.
	$= B.C.D \dot{-} 0$	Premises
	$= B.C.D.$	18.

92. Rule of inference

$$\frac{C_u A = 0}{A = 0} \qquad u=1,\ldots,n$$

Proof	$A = C_1 A + C_2 A + \ldots\ldots + C_n A$	33.
	$= 0 + 0 + \ldots\ldots + 0$	Premises
	$= 0$	7.

93. Rule of inference

$$\frac{\begin{array}{l} A.B = 0 \\ A.(S_u 0 \dot{-} B) = 0 \end{array}}{A = 0} \qquad \text{all } u \text{ OR if one value } u \text{ then } C_u A = 0$$

Proof	$0 = A.(S_u 0 \dot{-} B)$	Premises
	$= A.S_u 0 \dot{-} A.B$	53.
	$= C_u A \dot{-} 0$	Premises
	$= C_u A$ for that value of u.	18.
	$A = 0$ if for all u.	92.

94. $C_u(B \dot{-} A).(S_uA \dot{-} B) = 0$

Proof

$$
\begin{aligned}
L(A,0) &= C_u(0 \dot{-} A).(S_uA \dot{-} 0) \\
&= C_u0.S_uA && 19.18. \\
&= 0.S_uA && 27. \\
&= 0 && 44. \\
L(0,B) &= C_u(B \dot{-} 0).(S_u0 \dot{-} B) \\
&= C_uB.(S_u0 \dot{-} B) && 18. \\
&= C_u0 && 62. \\
&= 0 && 27. \\
L(C_uS_uA,C_uS_uB) &= C_u(C_uS_uB \dot{-} C_uS_uA).(S_uC_uS_uA \dot{-} C_uS_uB) \\
&= C_u(S_uC_uB \dot{-} S_uC_uA).(S_uS_uC_uA \dot{-} S_uC_uB) && 27. \\
&= C_u(C_uB \dot{-} C_uA).(S_uC_uA \dot{-} C_uB) && 21. \\
&= L(C_uA,C_uB) \\
L(C_uA,C_uB) &= 0 && 58. \\
C_u(B \dot{-} A).(S_uA \dot{-} B) &= C_uC_u(B \dot{-} A).(S_uA \dot{-} B) && 28. \\
&= C_u(C_uB \dot{-} C_uA).(C_uS_uA \dot{-} C_uB) && 47.32. \\
&= C_u(C_uB \dot{-} C_uA).(S_uC_uA \dot{-} C_uB) && 27. \\
&= L(C_uA,C_uB) \\
&= 0
\end{aligned}
$$

We now have sufficient results to prove the two conditions

$$S_uX \dot{-} (S_uRt(X))^2 = 0$$

$$(Rt(X))^2 \dot{-} X = 0$$

which will lead to the result

$$Rt(X^2) = X \ .$$

From here on $Rt(X)$ will be referred to as RX.

95. $\quad S_uX \dot{-} (S_uRX)^2 = 0$

Proof In this proof we shall dispense with the notation for small steps and only use the notation for major formulae. A labelling method with the number in parenthesis will be used for statements for use later in the proof.

Definition of RX

$$RO = 0$$

$$RS_uX = RX + (S_uO \dot{-} ((S_uRX)^2 \dot{-} S_uX)) \qquad u=1,\ldots,n$$

$$(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).RS_uX = (S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).RX +$$
$$(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).(S_uO \dot{-} ((S_uRX)^2 \dot{-} S_uX))$$

the last term being equal to zero from 69. with $A = S_uX$, $B = (S_uRX)^2$.

$$(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).RS_uX = (S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).RX.$$

By 70. we have

$$(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).|RS_uX,RX| = 0 \qquad u=1,\ldots,n \qquad (1)$$

Now by 72. we have

$$(S_uO \dot{-} |RX,RS_uX|).(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRC_uX)^2)).(S_uS_uX \dot{-} (S_uRS_uC_uX)^2) = 0$$

and obviously

$$(S_uO \dot{-} |RX,RS_uX|).(S_uO \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).(S_uS_uX \dot{-} (S_uRS_uX)^2) = 0 \qquad (2)$$

Multiplying (1) by $(S_uS_uX \dot{-} (S_uRS_uX)^2)$ and using 93. we find

$$(S_u0 \dot{-} (S_uS_uX \dot{-} (S_uRX)^2)).(S_uS_uX \dot{-} (S_uRS_uX)^2) = 0 \qquad (3)$$

Multiply the defining equation for RS_uX by $(S_u0 \dot{-} |(S_uRX)^2, S_uX|)$ and using 71. we find

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).RS_uX = (S_u0 \dot{-} |(S_uRX)^2, S_uX|).RX + (S_u0 \dot{-} |(S_uRX)^2, S_uX|)$$

Hence

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).RS_uX = (S_u0 \dot{-} |(S_uRX)^2, S_uX|).S_uRX$$

and so

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).|RS_uX, S_uRX| = 0 \qquad (4)$$

by 72.

$$(S_u0 \dot{-} |RS_uX, S_uRX|).(S_u0 \dot{-} |(S_uRX)^2, S_uX|).|(RS_uX)^2, S_uX| = 0 \qquad (5)$$

and so, multiplying (4) by $|(RS_uX)^2, S_uX|$, adding (5), taking the common factor out, then using 93. we have

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).|(RS_uX)^2, S_uX| = 0 \qquad (6)$$

It follows therefore by 55. that

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).(S_uX \dot{-} (RS_uX)^2) = 0 .$$

Hence

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).(S_uS_uX \dot{-} S_u(RS_uX)^2) = 0$$

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).(S_uS_uX \dot{-} S_u(RS_uX)^2) \dot{-} (S_u0 \dot{-} |(S_uRX)^2, S_uX|).(C_uRS_uX + C_uRS_uX) = 0$$

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).(S_uS_uX \dot{-} (S_u(RS_uX)^2 + C_uRS_uX + C_uRS_uX)) = 0$$

$$(S_u0 \dot{-} |(S_uRX)^2, S_uX|).(S_uS_uX \dot{-} (S_uRS_uX)^2) = 0 \qquad (7)$$

From (3) and (7) we have by 82.

$$(S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).(S_u S_u X \dot{-} (S_u RS_u X)^2) = 0 \qquad (8)$$

Now

$$(S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).(S_u S_v X \dot{-} (S_u RS_v X)^2) \qquad u \neq v$$

$$= (S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).C_u(S_u S_v X \dot{-} (S_u RS_v X)^2)$$

$$= (S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).(C_u S_u S_v X \dot{-} C_u(S_u RS_v X)^2)$$

$$= (S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).(C_u S_u X \dot{-} C_u(S_u RX)^2)$$

$$= (S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).C_u(S_u X \dot{-} (S_u RX)^2)$$

$$= 0$$

We can therefore say

$(S_u 0 \dot{-} (S_u X \dot{-} (S_u RX)^2)).(S_u S_v X \dot{-} (S_u RS_v X)^2) = 0$ for all u,v,

and $S_u 0 \dot{-} (S_u R0)^2 = 0$.

Hence by rule of inference 63.

$S_u X \dot{-} (S_u RX)^2 = 0$ as required.

96. $(RX)^2 \dot{-} X = 0$

Proof From (4) in 95. we have

$$(S_u 0 \dot{-} |(S_u RX)^2, S_u X|).|RS_u X, S_u RX| = 0 \qquad (1)$$

and from the defining equation for $RS_u X$

$$((S_u RX)^2 \dot{-} S_u X).RS_u X = ((S_u RX)^2 \dot{-} S_u X).RX$$

$$((S_u RX)^2 \dot{-} S_u X).|RS_u X, RX| = 0$$

by 72.

$$(S_u 0 \dot{-} |RX, RS_u X|).(S_u 0 \dot{-} ((RX)^2 \dot{-} X)).((RS_u X)^2 \dot{-} X) = 0$$

[by virtue of the fact that

$$F(C_uX) = C_uF(X) \text{ for } F(X) = (RX)^2 \dot{-} X$$]

Hence by 93.

$$((S_uRX)^2 \dot{-} S_uX).(S_u0 \dot{-} ((RX)^2 \dot{-} X)).((RS_uX)^2 \dot{-} X) = 0$$

and therefore by 86.

$$((S_uRX)^2 \dot{-} S_uX).(S_u0 \dot{-} ((RX)^2 \dot{-} X)).((RS_uX)^2 \dot{-} S_uX) = 0 \qquad (2)$$

From (1) and (2), by 91., splitting (1), using 90. and 55.,in 91. substitute

$$A = ((S_uRX)^2 \dot{-} S_uX)$$

$$B = S_u0 \dot{-} (S_uX \dot{-} (S_uRX)^2)$$

$$C = (RS_uX)^2 \dot{-} S_uX$$

$$D = S_u0 \dot{-} ((RX)^2 \dot{-} X)$$

$$(S_u0 \dot{-} (S_uX \dot{-} (S_uRX)^2)).(S_u0 \dot{-} ((RX)^2 \dot{-} X)).((RS_uX)^2 \dot{-} S_uX) = 0$$

by 95.

$$(S_u0 \dot{-} ((RX)^2 \dot{-} X)).((RS_uX)^2 \dot{-} S_uX) = 0 \qquad (3)$$

Now

$$\begin{aligned}
(S_u0 \dot{-} ((RX)^2 \dot{-} X)).(((RS_vX)^2 \dot{-} S_vX) &= (S_u0 \dot{-} ((RX)^2 \dot{-} X)).C_u((RS_vX)^2 \dot{-} S_vX) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u \neq v \\
&= (S_u0 \dot{-} ((RX)^2 \dot{-} X)).(C_u(RS_vX)^2 \dot{-} C_uS_vX) \\
&= (S_u0 \dot{-} ((RX)^2 \dot{-} X)).(C_u(RX)^2 \dot{-} C_uX) \\
&= (S_u0 \dot{-} ((RX)^2 \dot{-} X)).((RX)^2 \dot{-} X) \\
&= 0
\end{aligned}$$

We can therefore say,

$(S_u 0 \dot{-} ((RX)^2 \dot{-} X)).((RS_v X)^2 \dot{-} S_v X = 0$ for all u,v,

and $(R0)^2 \dot{-} 0 = 0$.

Hence by rule of inference 63.

$$(RX)^2 \dot{-} X = 0$$

97. Rule of inference

$$\frac{\begin{array}{c} A \dot{-} B = 0 \\ B \dot{-} A = 0 \end{array}}{A = B}$$

Proof	$A + (B \dot{-} A) = B + (A \dot{-} B)$	Key equation
	$A + 0 = B + 0$	Premises
	$A = B$	7.

98. Rule of inference

$$\frac{A^2 \dot{-} B^2 = 0}{A \dot{-} B = 0}$$

Proof Consider first the result

(a) $(S_u 0 \dot{-} (X \dot{-} Y).X).(X \dot{-} Y) = 0$

Now obviously

$(S_u 0 \dot{-} (X \dot{-} Y).X).(X \dot{-} Y).X = 0$ 44.

and further,

$$(S_u0 \dot{-} (X \dot{-} Y).X).(X \dot{-} Y).(S_v0 \dot{-} X) = (S_u0 \dot{-} (X \dot{-} Y).X).((S_v0 \dot{-} X).X \dot{-} (S_v0 \dot{-} X).Y) \quad 53.$$

$$= (S_u0 \dot{-} (X \dot{-} Y).X).(0 \dot{-} (S_v0 \dot{-} X).Y) \quad 62.$$

$$= (S_u0 \dot{-} (X \dot{-} Y).X).0 \quad 19.$$

$$= 0 \quad 43.$$

Hence by rule of inference 93.

$$(S_u0 \dot{-} (X \dot{-} Y).X).(X \dot{-} Y) = 0 \qquad \text{for (a)}$$

Put $X = A + B$

$Y = B + B$

$$(S_u0 \dot{-} ((A + B) \dot{-} (B + B)).(A + B)).((A + B) \dot{-} (B + B)) = 0$$

$$(S_u0 \dot{-} (A \dot{-} B).(A + B)).(A \dot{-} B) = 0 \quad 24.$$

$$(S_u0 \dot{-} (A+B).(A \dot{-} B)).(A \dot{-} B) = 0 \quad 46.$$

$$(S_u0 \dot{-} ((A^2 + B.A) \dot{-} (A.B + B^2)).(A \dot{-} B) = 0 \quad 53.$$

$$(S_u0 \dot{-} (A^2 \dot{-} B^2)).(A \dot{-} B) = 0 \quad 46.24.$$

Hence

$$(S_u0 \dot{-} 0).(A \dot{-} B) = 0 \qquad \text{Premises}$$

$$S_u0.(A \dot{-} B) = 0 \qquad 18.$$

$$C_u(A \dot{-} B) = 0 \qquad \text{for } u=1,\ldots,n \qquad 60.$$

$$A \dot{-} B = 0 \qquad \text{as required} \qquad 92.$$

99. Rule of inference

$$\frac{S_uA^2 \dot{-} B^2 = 0}{C_u(S_uA \dot{-} B) = 0}$$

Proof $\quad C_u(B \dot{-} A).(S_uA \dot{-} B) = 0 \qquad$ from 94.

Hence,

$$C_u(B + A).(B \dot{-} A).(S_uA \dot{-} B) = 0 \qquad 5.44.$$

$$C_u(B^2 \dot{-} A^2).(S_uA \dot{-} B) = 0 \quad \text{for all } u. \qquad 53.46.24.$$

Now

$$(S_u0 \dot{-} (B^2 \dot{-} A^2)).(S_u0 \dot{-} (S_uA^2 \dot{-} B^2)) = 0 \quad \text{for all } u, \qquad 69.$$

and therefore

$$(S_u0 \dot{-} (S_uA^2 \dot{-} B^2)).C_u(S_uA \dot{-} B) = 0 \qquad 49.93.$$

$$(S_u0 \dot{-} 0).(S_uA \dot{-} B) = 0 \qquad 49.59.\text{Premises}$$

$$S_u0.(S_uA \dot{-} B) = 0 \qquad 18.$$

$$C_u(S_uA \dot{-} B) = 0 \quad \text{for } u=1,\ldots,n \qquad 60.$$

100. $$RA^2 = A$$

From 95.96., we have

$$S_uX \dot{-} (S_uRX)^2 = 0$$

$$(RX)^2 \dot{-} X = 0$$

Put $$X = A^2$$

$$S_uA^2 \dot{-} (S_uRA^2)^2 = 0$$

$$(RA^2)^2 \dot{-} A^2 = 0$$

$$C_u(S_uA \dot{-} S_uRA^2) = 0 \qquad 99.$$

$$RA^2 \dot{-} A = 0 \qquad 98.$$

$$A \dot{-} RA^2 = 0 \qquad 21.92.$$

Hence $$RA^2 = A \qquad 97.$$

Having proved the basic property of the square root function, we now have to prove the properties

$$X > Y \Rightarrow RX > RY$$

$$R(X^2 + X + X) = R(X^2)$$

and $$R((X + Y)^2 + X) = X + Y$$

101. $Rt(X) \dot{-} Rt(X + Y) = 0$ (equivalent to $X > Y \Rightarrow RX > RY$)

Proof

$$\begin{aligned}
L(X,0) &= Rt(X) \dot{-} Rt(X + 0) \\
&= Rt(X) \dot{-} Rt(X) && 7. \\
&= 0 && 22. \\
R(X,0) &= 0 \\
L(X,S_uY) &= Rt(X) \dot{-} Rt(X + S_uY) \\
&= Rt(X) \dot{-} Rt(S_u(X + Y)) && 7. \\
&= Rt(X) \dot{-} (Rt(X + Y) + (S_u0 \dot{-} ((S_uR(X+ Y))^2 \dot{-} S_u(X+ Y)))) && 65. \\
&= (Rt(X) \dot{-} Rt(X+ Y)) \dot{-} (S_u0 \dot{-} ((S_uRt(X+ Y))^2 \dot{-} S_u(X+ Y))) && 20. \\
&= L(X,Y) \dot{-} (S_u0 \dot{-} ((S_uRt(X+ Y))^2 \dot{-} S_u(X+ Y))) \\
R(X,S_uY) &= 0 \\
&= 0 \dot{-} (S_u0 \dot{-} ((S_uRt(X+ Y))^2 \dot{-} S_u(X+ Y))) && 19. \\
&= R(X,Y) \dot{-} (S_u0 \dot{-} ((S_uRt(X+ Y))^2 \dot{-} S_u(X+ Y))) \\
L(X,Y) &= R(X,Y) && 6.
\end{aligned}$$

102. $Rt(A^2 + A + A) = A$

Proof $$S_uX \dot{-} (S_uRt(X))^2 = 0$$

Put $X = C_u(A^2 + A + A)$

$$S_uC_u(A^2 + A + A) \dot{-} (S_uRt(C_u(A^2 + A + A)))^2 = 0$$

$$(S_uC_uA)^2 \dot{-} (S_uC_uRt(A^2 + A + A))^2 = 0 \qquad 30.47.$$

$$S_uC_uA \dot{-} S_uC_uRt(A^2 + A + A) = 0 \qquad 98.$$

$$C_uA \dot{-} C_uRt(A^2 + A + A) = 0 \qquad (1) \qquad 21.$$

$$(Rt(X))^2 \dot{-} X = 0$$

$$S_u(Rt(X))^2 \dot{-} S_uX = 0 \qquad 21.$$

$$S_u(Rt(C_u(A^2 + A + A)))^2 \dot{-} S_uC_u(A^2 + A + A) = 0$$

$$S_u(C_uRt(A^2 + A + A))^2 \dot{-} (S_uC_uA)^2 = 0 \qquad 30.47.$$

$$C_u(S_uC_uRt(A^2 + A + A) \dot{-} S_uC_uA) = 0 \qquad 99.$$

$$C_uRt(A^2 + A + A) \dot{-} C_uA = 0 \qquad (2) \qquad 21.32.$$

Therefore from 97 we have

$$C_uRt(A^2 + A + A) = C_uA$$

Hence we can say $Rt(A^2 + A + A) = A$ 92.

103. Rule of inference

$$\frac{\begin{array}{c} A \dot{-} B = 0 \\ B \dot{-} C = 0 \end{array}}{A \dot{-} C = 0}$$

Proof

$B + (A \dot{-} B) = A + (B \dot{-} A)$	Key equation
$B = A + (B \dot{-} A)$	Premises, 7.
$B \dot{-} (B \dot{-} A) = (A + (B \dot{-} A)) \dot{-} (B \dot{-} A)$	5.
$= A$	23.
$C + (B \dot{-} C) = B + (C \dot{-} B)$	Key equation
$C = B + (C \dot{-} B)$	Premises, 7.

Hence by substitution

$$A \dot{-} C = (B \dot{-} (B \dot{-} A)) \dot{-} (B + (C \dot{-} B)) \qquad \text{from above}$$

$$= B \dot{-} ((B \dot{-} A) + B + (C \dot{-} B)) \qquad 20.$$

$$= 0 \dot{-} ((B \dot{-} A) + (C \dot{-} B)) \qquad 23.$$

$$= 0 \quad \text{as required.} \qquad 19.$$

104. $Rt((U + V)^2 + U) = U + V$

Proof

$$Rt((U + V)^2 + U) \dot{-} Rt((U + V)^2 + U + V + U + V) = 0 \qquad 101.$$

$$Rt((U + V)^2 + U + V + U + V) \dot{-} (U + V) = 0 \qquad 102.$$

$$Rt((U + V)^2 + U) \dot{-} (U + V) = 0 \quad (1) \qquad 103.$$

$$Rt((U + V)^2) \dot{-} Rt((U + V)^2 + U) = 0 \qquad 101.$$

$$(U + V) \dot{-} Rt((U + V)^2 + U) = 0 \quad (2) \qquad 100.$$

The results (1) and (2) together using 97. give

$$Rt((U + V)^2 + U) = U + V$$

105. $L(J(U,V)) = U$

Proof

$$L(J(U,V) = ((U + V)^2 + U) \dot{-} (Rt((U + V)^2 + U))^2$$

$$= ((U + V)^2 + U) \dot{-} (U + V)^2 \qquad 104.$$

$$= U \qquad 23.$$

106. $K(J(U,V)) = V$

Proof

$$K(J(U,V)) = Rt((U + V)^2 + U) \dot{-} L(J(U,V))$$

$$= (U + V) \dot{-} U \qquad 105.$$

$$= V \qquad 23.$$

We have therefore proved that a set of functions with the required properties to achieve the reduction in parameters in the definition of primitive recursion exists in commutative multiple successor arithmetic.

CHAPTER IV

INITIAL FUNCTIONS REQUIRED FOR THE REDUCED DEFINITIONS BY PRIMITIVE RECURSIONS

In this chapter we will consider the four defining schema which were derived in Chapter II, and examine these to see if we need to adjoin further functions to the initial set of all primitive recursive functions.

The functions used to reduce the parameters in the definitions of primitive recursive functions were $K(X)$, $L(X)$, & $J(U,V)$, defined as follows:

$$J(U,V) = (U + V)^2 + U$$

$$L(X) = X \dot{-} (RX)^2$$

$$K(X) = RX \dot{-} L(X)$$

In order therefore to define all primitive recursive functions in the reduced defining schemas, we may need to adjoin further functions to the set of initial functions in order that the functions $K(X)$, $L(X)$ & $J(U,V)$ can be defined.

Now $K(X)$, $L(X)$ & $J(U,V)$ are explicitly comprised of the functions

$$Y + X, \qquad Y \dot{-} X, \qquad X.X, \qquad RX.$$

If we are able to define these functions in each of the schemas R_1, R_1^*, R_1^{**} and R_1^{***}, then there would be no need to adjoin further functions to the initial functions. However we shall find that in the case of R_1^{***} we require to adjoin a function to the initial functions, $Q(X)$, which will be defined later.

R_1 This is the defining schema given by

$$F(0,Y) = A(Y)$$

$$F(S_u X,Y) = B_u(X,Y,F(X,Y)) \qquad u=1,\ldots,n$$

$Y + X$ is defined by

$$F(0,Y) = Y$$

$$F(S_u X,Y) = S_u F(X,Y)$$

For $Y \dot{-} X$ we first define $P_u X$ by

$$F(0) = 0$$

$$F(S_v X) = \begin{cases} X & u=v \\ S_v F(X) & u \neq v \end{cases}$$

Then $Y \dot{-} X$ is defined by

$$F(0,Y) = Y$$

$$F(S_u X,Y) = P_u F(X,Y)$$

$C_u X$ is defined by

$$F(0) = 0$$

$$F(S_v X) = \begin{cases} S_v F(X) & u=v \\ F(X) & u \neq v \end{cases}$$

and therefore $X.X$ is defined by

$$F(0) = 0$$

$$F(S_u X) = S_u F(X) + C_u X + C_u X$$

RX is defined by

$$F(0) = 0$$

$$F(S_u X) = F(X) + (S_u 0 \dot{-} ((S_u F(X))^2 \dot{-} S_u X))$$

Thus the initial functions and the defining schema R_1 alone are sufficient to define all primitive recursive functions.

R_1* This is the defining schema given by

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(X,F(X,Y)) \qquad u=1,\ldots,n$$

Defining the functions $Y + X$, $Y \dot{-} X$, $X.X$ and RX as above in R_1 in fact defines all these functions under the reduced schema R_1*. Hence we can say that 'The initial functions and the defining schema R_1* alone are sufficient to define all primitive recursive functions'.

R_1** This is the defining schema given by

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(Y,F(X,Y)) \qquad u=1,\ldots,n$$

$Y + X$ is defined by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = S_uF(X,Y)$$

Now for $Y \dot{-} X$ we have to define P_uX, and to achieve this we proceed as follows.

C_uX is defined by

$$F(0) = 0$$

$$F(S_vX) = \begin{cases} S_vF(X) & u=v \\ F(X) & u\neq v \end{cases}$$

$X.Y$ is defined by

$$F(0,Y) = 0$$

$$F(S_uX,Y) = F(X,Y) + C_uY$$

and $$X.X = F(X,X).$$

$S_u0 \dot{-} X$ is defined by

$$F(0) = S_u0$$

$$F(S_vX) = \begin{cases} 0 & u=v \\ F(X) & u\neq v \end{cases}$$

Define D_kX by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_k0 \dot{-} F(X) & u=k \\ F(X) & u\neq k \end{cases}$$

This takes the value S_k0 if 'S_k' appears an odd number of times in X.

We can therefore define T_kX by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_kF(X).(S_k0 \dot{-} F(X)) + F(X).S_kF(X).(S_k0 \dot{-} (S_k0 \dot{-} F(X))) & u=k \\ F(X) & u\neq k \end{cases}$$

This function is such that when X contains only one of 'S_k' successors then T_kX contains an odd number of 'S_k' successors, otherwise T_kX contains an even number of 'S_k' successors. Hence the function defined by

$$S_k0 \dot{-} D_kT_kX$$

takes the value 0 if X contains only one 'S_k' successors, and S_k0 otherwise.

Define $RM_k(X,2)$ by

$$RM_k(X,2) = D_kX$$

and $RM_k(X,3)$ by

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} (S_k 0 \dot{-} F(X)) + S_k S_k 0.(S_k 0 \dot{-} (S_k 0 \dot{-} D_k T_k(F(X)))) & u=k \\ F(X) & u \neq k \end{cases}$$

This function is the remainder of the number of 'S_k' successors in X when that number is divided by 3, e.g.

$$RM_k(S_k S_k 0, 3) = S_k S_k 0$$

$$RM_k(S_1 S_1 S_2 S_k S_k S_k S_k 0, 3) = S_k 0$$

Consider now the function $N_k X$ defined by

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} (F(X) + S_k 0).(S_k 0 \dot{-} RM_k(F(X),3)) + \\ (F(X) + S_k S_k 0).(S_k 0 \dot{-} (S_k 0 \dot{-} RM_k(F(X),3))) & u=k \\ F(X) & u \neq k \end{cases}$$

In other words

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} \left.\begin{array}{l} F(X) + S_k 0 \quad \text{if} \quad RM_k(F(X),3) = 0 \\ F(X) + S_k S_k 0 \quad \text{if} \quad RM_k(F(X),3) \neq 0 \end{array}\right\} & u=k \\ F(X) & u \neq k \end{cases}$$

Clearly $N_k X = N_k(C_k X)$; that is we are only considering the 'S_k' successors in X to define $N_k X$, if we consider that $1 = S_k 0$, $2 = S_k S_k 0, \ldots,$ etc. and consider X as $C_k X$ which would then be a number: then in normal arithmetical terms

$$N_k X = \begin{cases} 3.X/2 & \text{if X is even} \\ (3.X - 1)/2 & \text{if X is odd} \end{cases}$$

as there are no successor symbols other than 'S_k' in $N_k X$.

This result we shall prove by induction on X; suppose the above is true for X, then

$$N_k(X+1) = \begin{cases} 3.(X+1)/2 & \text{if } X+1 \text{ is even} \\ (3.(X+1)-1)/2 & \text{of } X+1 \text{ is odd} \end{cases}$$

$$N_k(X+1) = \begin{cases} (3.X+3)/2 & \text{if } X \text{ is odd} \\ (3.X+2)/2 & \text{if } X \text{ is even} \end{cases}$$

$$= \begin{cases} (3.X-1)/2+2 & \text{if } X \text{ is odd} \\ 3.X/2+1 & \text{if } X \text{ is even} \end{cases}$$

$$= \begin{cases} 3.X/2+1 & \text{if } X \text{ is even} \\ (3.X-1)/2+2 & \text{if } X \text{ is odd} \end{cases}$$

This is equivalent to

$$N_k(X+1) = \begin{cases} N_kX+1 & \text{if } X \text{ is even} \\ N_kX+2 & \text{if } X \text{ is odd} \end{cases}$$

by hypothesis.

Now if X is even $N_kX = 3.X/2$ and therefore divisible by 3, if X is odd then $N_kX = (3.X-1)/2$ and therefore not divisible by 3. Hence

$$N_k(X+1) = \begin{cases} N_kX+1 & \text{if } RM(N_kX,3) = 0 \\ N_kX+2 & \text{otherwise} \end{cases}$$

The hypothesis is true for $X = 0$; hence N_kX is the function stated. Define now $M_k(X,Y)$ by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = \begin{cases} N_k(F(X,Y)).(S_k0 \dot{-} RM_k(F(X,Y),2)) & u=k \\ F(X,Y) & u\neq k \end{cases}$$

In order to prove the properties of this function we need first to define the function $(Y^X)_k$ by

$$F(0,Y) = S_k 0$$

$$F(S_u X,Y) = \begin{cases} Y.F(X,Y) & u=k \\ F(X,Y) & u\neq k \end{cases}$$

In the notation defined $(Y^X)_k$ is the function Y^X in normal arithmetical notation for the 'S_k' component of both X and Y, e.g.

$$(S_1 S_k S_k 0^{S_2 S_k S_k 0})_k = S_k S_k S_k S_k 0$$

which with the previous notation $S_k 0 = 1$, $S_k S_k 0 = 2$, etc. ... ignoring all other successors, then

$$\begin{aligned}(S_1 S_k S_k 0^{S_2 S_k S_k 0})_k &= 2^2 \quad \text{(normal exponent)} \\ &= 4 \\ &= S_k S_k S_k S_k 0\end{aligned}$$

We analyse the function $M_k(X,Y)$ as follows.

Let $C_k Y$ be non-zero, and let 2^t be the highest power of 2 that divides it exactly; then $M_k(X,Y)$ is such that

if $t < X$ then $M_k(X,Y) = 0$

if $t \geqslant X$ then $M_k(X,Y)$ is non-zero, and 2^{t-X} is the highest power of 2 that divides it exactly.

<u>Proof</u> The proof is by induction on X. The case $X = 0$ gives

$$\begin{aligned}M_k(X,Y) &= M_k(0,Y) \\ &= Y \\ &= 2^t.A \quad (A \neq 0) \\ &= 2^{t-0}.A \quad (\text{as } t \geqslant X)\end{aligned}$$

The result is therefore true for $X = 0$. Let us assume the result for $0 \leqslant X \leqslant m$, and consider the case $m+1$.

Suppose $t < m$. Then by the hypothesis $M_k(m,Y) = 0$, and from the definition we have

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k0 \dot{-} RM_k(M_k(m,Y),2)) \quad \text{as } 1 = S_k0$$

$$= N_k(M_k(m,Y)).S_k0$$

$$= N_k(0).S_k0$$

$$= 0.S_k0$$

$$= 0$$

It is worth noting here that as we are only using the 'k'th components then $A.S_k0 = A$, and $C_kA = A$, as we shall only consider the answer in terms of 'S_k' successors.

Now suppose $t = m$. Then by the hypothesis $M_k(m,Y)$ is non-zero, and $2^{t-m} = 2^0$ is the highest power of 2 that divides it exactly, that is $M_k(m,Y)$ is odd. Hence

$$M_k(m+1),Y) = N_k(M_k(m,Y)).(S_k0 \dot{-} RM_k(M_k(m,Y),2))$$

As $M_k(m,Y)$ is odd then $RM_k(M_k(m,Y),2) = S_k0$ and so

$$M_k(m+1),Y) = N_k(M_k(m,Y)).(S_k0 \dot{-} S_k0)$$

$$= N_k(M_k(m,Y)).0$$

$$= 0 \quad \text{as required as } t < m+1 .$$

Finally, suppose $t > m$. Then by the hypothesis $M_k(m,Y)$ is non-zero and 2^{t-m} is the highest power of 2 that divides it exactly. Hence $M_k(m,Y)$ is even. Therefore

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k0 \dot{-} RM_k(M_k(m,Y),2))$$

$$= N_k(M_k(m,Y)).(S_k0 \dot{-} 0)$$

$$= N_k(M_k(m,Y)).S_k0$$

$$= N_k(M_k(m,Y))$$

$$= 3.M_k(m,Y)/2 \quad \text{as } M_k(m,Y) \text{ is even.}$$

We therefore have

$$M_k(m+1,Y) = 3.M_k(m,Y)/2$$

Hence if 2^{t-m} is the highest power of 2 that divides $M_k(m,Y)$ exactly, then $2^{(t-m)-1}$ is the highest power of 2 that divides $M_k(m+1,Y)$ exactly, that is $2^{t-(m+1)}$ is the highest power of 2 that divides $M_k(m+1,Y)$ exactly. Hence the result.

From this result we can say that the function

$$M_k(X,(S_kS_kC^Y)_k$$

is non-zero in 'S_k' components, if and only if 2^X divides 2^Y exactly (in the previous notation), that is if and only if $X \leqslant Y$, in 'S_k' components. Hence

$$S_k0 \dot{-} M_k(X,(S_kS_k0^Y)_k) = \begin{cases} 0 & \text{if } C_k(X \dot{-} Y) = 0 \\ S_k0 & \text{otherwise} \end{cases}$$

Define now the function $L_k(X,Y)$ by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = \begin{cases} (F(X,Y) + S_k0).(S_k0 \dot{-} M_k(Y,(S_kS_k0^{F(X,Y)})_k)) & u=k \\ S_uF(X,Y) & u\neq k \end{cases}$$

which is such that

$$F(S_kX,Y) = \begin{cases} 0 & \text{if } F(X,Y) \geqslant Y \\ S_kF(X,Y) & \text{otherwise} \end{cases} \quad \text{(in '}S_k\text{' components)}$$

The function $L_k(X,X)$ is therefore defined by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} \left.\begin{matrix} 0 & \text{if } F(X) \geqslant X \\ S_uF(X) & \text{otherwise} \end{matrix}\right\} & \text{(in '}S_k\text{ components)} \quad u=k \\ S_uF(X) & u\neq k \end{cases}$$

P_kX is defined by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} X & u=k \\ S_uF(X) & u\neq k \end{cases}$$

which is the same as

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} \left.\begin{matrix} 0 \text{ if } F(X) \geqslant X \\ S_kF(X) \text{ otherwise} \end{matrix}\right\} & \begin{matrix}\text{(in } S_k \text{ components)} \\ u=k\end{matrix} \\ S_uF(X) & u\neq k \end{cases}$$

Hence by uniqueness $P_kX = L_k(X,X)$, and $L_k(X,X)$ is defined in R_1** by substituting X for Y in the function $L_k(X,Y)$ which is defined under the schema R_1**. For the proof of this in single successor arithmetic see [8].

However we can obtain the result in the following manner.

DX or $RM(X,2)$ is defined by

$$F(0) = 0$$

$$F(S_uX) = S_u0 \dot{-} F(X)$$

TX is defined by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X).(S_u0 \dot{-} F(X)) + F(X).S_uF(X).(S_u0 \dot{-} (S_u0 \dot{-} F(X)))$$

$RM(X,3)$ is defined by

$$F(0) = 0$$

$$F(S_uX) = (S_u0 \dot{-} F(X)) + S_uS_u0.(S_u0 \dot{-} D(T(F(X)))))$$

These functions are such that

$$DX = D_1X + D_2X + \ldots\ldots + D_nX$$

$$RM(X,2) = RM_1(X,2) + RM_2(X,2) + \ldots\ldots + RM_n(X,2)$$

$$TX = T_1X + T_2X + \ldots\ldots + T_nX$$

$$RM(X,3) = RM_1(X,3) + RM_2(X,3) + \ldots\ldots + RM_n(X,3)$$

NX is defined by

$$F(0) = 0$$

$$F(S_u X) = S_u F(X).(S_u 0 \dot{-} RM(F(X),3)) + S_u S_u F(X).(S_u 0 \dot{-} (S_u 0 \dot{-} RM(F(X),3)))$$

and $NX = N_1 X + N_2 X + \ldots\ldots + N_n X$

M(X,Y) is defined by

$$F(0,Y) = Y$$

$$F(S_u X,Y) = N(F(X,Y)).(S_u 0 \dot{-} RM(F(X,Y),2))$$

and $$M(X,Y) = C_1 M_1(X,Y) + C_2 M_2(X,Y) + \ldots\ldots + C_n M_n(X,Y)$$

$(Y^X)_k$ is defined as before.

L(X,Y) is defined by

$$F(0,Y) = Y$$

$$F(S_u X,Y) = S_u F(X,Y).(S_u 0 \dot{-} M(Y,(S_u S_u 0^{F(X,Y)})_u))$$

and $$L(X,Y) = C_1 L_1(X,Y) + C_2 L_2(X,Y) + \ldots\ldots + C_n L_n(X,Y)$$

Define $\bar{C}_k X$ by

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} F(X) & u=k \\ S_u F(X) & u \neq k \end{cases}$$

$\bar{C}_k X$ is therefore all the components of X except the 'k'th component; hence

$$C_k X + \bar{C}_k X = X$$

From this $$P_k X = C_k L(X,X) + \bar{C}_k X .$$

It is interesting to note that in multiple successor arithmetic, if it were necessary to adjoin the functions P_uX $u=1,\ldots,n$ to the initial functions, it would have been sufficient to adjoin the function PX defined by

$$F(0) = 0$$

$$F(S_uX) = X$$

as clearly $P_kX = C_kPX + \bar{C}_kX$, and $PX = L(X,X)$.

We now proceed to define Q_kX, which is the function

$$Q_kX = S_u0 \dot{-} (X \dot{-} (RX)^2)$$

RX standing for $Rt(X)$ as defined in Chapter III, Lemma 65.

Define $Y \dot{-} X$ by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = P_uF(X,Y)$$

Consider now the function $T(X,Y)$ by

$$F(0,Y) = 0$$

$$F(S_uX,Y) = S_uF(X,Y) + (S_u0 \dot{-} ((F(X,Y)^2 \dot{-} Y) + (Y \dot{-} F(X,Y)^2))).$$

This function is such that $T(S_uX,Y)$ is obtained from $T(X,Y)$ by adding S_u0 unless $(T(X,Y))^2 = Y$, in which case we add S_uS_u0. This can happen only if $X^2 = Y$. It is clear therefore that

$$T(X,Y) = \begin{cases} X & \text{if } X^2 \leqslant Y \\ X + QY & \text{if } X^2 > Y \end{cases}$$

where $QX = Q_1X + Q_2X + \ldots\ldots + Q_nX$.

Hence

$$Q_uX = T(S_uX,X) \dot{-} S_uX$$

From Q_uX we obtain RX as follows:

$RM(X,3)$ was defined by

$$F(0) = 0$$

$$F(S_uX) = (S_u0 \dot{-} F(X)) + S_uS_u0.(S_u0 \dot{-} (S_u0 \dot{-} DT(F(X)))))$$

where DX and TX are defined as previous.

We now define UX by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X) + RM_u(F(X),3)$$

Clearly

$$UX = X + Div(X,2)$$

where $Div(X,2)$ is the function defined by

$$F(0) = 0$$

$$F(S_uX) = F(X) + RM_u(X,2)$$

We therefore obtain $Div(X,2)$ in R_1** by the explicit definition

$$Div(X,2) = UX \dot{-} X.$$

Consider the function

$$V_kX = X + S_kS_k0.RX .$$

This can be defined by the schema

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_kF(X) + S_kS_k0.Q_k(S_kX) & u=k \\ S_uF(X) & u\neq k \end{cases}$$

Now the following are equivalent, S_kX is square (that is $(RS_kX)^2 = S_kX$), X is of the form $N^2 + C_kN + C_kN$. Hence from the definition of V_kX we can say that V_kX is of the form $N^2 + C_kN + C_kN + C_kN + C_kN$, and therefore $V_kX + S_kS_kS_kS_k0$ is square.

Our definition of V_kX now becomes

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_kF(X) + S_kS_k0.Q_k(F(X) + S_kS_kS_kS_k0) & u=k \\ S_uF(X) & u\neq k \end{cases}$$

and is therefore defined by schema R_1^{**}.

From this

$$RX = Div(V_1X \dot{-} X,2) + Div(V_2X \dot{-} X,2) + \ldots + Div(V_nX \dot{-} X,2)$$

as $\qquad S_kS_k0.RX = S_kS_k0.C_kRX$

However we can define VX by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X) + S_uS_u0.Q_u(F(X) + S_uS_uS_uS_u0)$$

and

$$RX = Div(VX \dot{-} X,2).$$

For the proof of this result in single successor arithmetic see [7]. However, having defined $Y \dot{-} X$ in R_1^{**} we can define the function $W(X,Y)$ by

$$F(0,Y) = 0$$

$$F(S_uX,Y) = F(X,Y) + (S_u0 \dot{-} ((S_uF(X,Y))^2 \dot{-} Y))$$

and

$$RX = W(X,X)$$

We have now defined the functions

$$Y + X, \quad Y \dot{-} X, \quad X.X, \quad RX,$$

in R_1^{**} without introducing any further functions to the set of initial functions. We can therefore say 'The initial functions and the defining schema R_1^{**} alone are sufficient to define all primitive recursive functions'.

R_1*** This is the defining schema given by

$$F(0,Y) = A(Y)$$

$$F(S_uX,Y) = B_u(F(X,Y)) \qquad u=1,\ldots,n$$

$Y + X$ is defined by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = S_uF(X,Y)$$

In this schema we now suppose that the function QX is adjoined to the initial set of functions.

C_kX is defined by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_kF(X) & u=k \\ F(X) & u\neq k \end{cases}$$

and therefore Q_kX is obtained by

$$Q_kX = C_kQX$$

Next define $S_k0 \dot{-} X$ by

$$F(0) = S_k0$$

$$F(S_uX) = \begin{cases} 0 & u=k \\ F(X) & u\neq k \end{cases}$$

and therefore $Y.(S_k0 - X)$ is defined by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = \begin{cases} 0 & u=k \\ F(X,Y) & u\neq k \end{cases}$$

Before we can proceed it is necessary to prove the following Lemma.

Lemma $(S_v 0 \dot{-} X) = (S_v 0 \dot{-} (S_v 0 \dot{-} (S_v 0 \dot{-} X)))$

<u>Proof</u>

$$L(0) = S_v 0 \dot{-} 0$$

$$= S_v 0 \qquad \text{III.18.}$$

$$R(0) = (S_v 0 \dot{-} (S_v 0 \dot{-} (S_v 0 \dot{-} 0)))$$

$$= (S_v 0 \dot{-} (S_v 0 \dot{-} S_v 0)) \qquad \text{III.18.}$$

$$= S_v 0 \dot{-} 0 \qquad \text{III.22.}$$

$$= S_v 0 \qquad \text{III.18.}$$

$$L(S_u X) = S_v 0 \dot{-} S_u X$$

$$= \begin{cases} 0 & u=v \quad \text{III.61.} \\ S_v 0 \dot{-} X & u \neq v \quad \text{III.61.} \end{cases}$$

$$= \begin{cases} 0 & u=v \\ L(X) & u \neq v \end{cases}$$

$$R(S_u X) = (S_v 0 \dot{-} (S_v 0 \dot{-} (S_v 0 \dot{-} S_u X)))$$

$$= \begin{cases} (S_v 0 \dot{-} (S_v 0 \dot{-} 0)) & u=v \quad \text{III.61.} \\ (S_v 0 \dot{-} (S_v 0 \dot{-} (S_v 0 \dot{-} X))) & u \neq v \quad \text{III.61.} \end{cases}$$

$$= \begin{cases} S_v 0 \dot{-} S_v 0 & u=v \quad \text{III.18.} \\ R(X) & u \neq v \end{cases}$$

$$= \begin{cases} 0 & u=v \quad \text{III.22.} \\ R(X) & u \neq v \end{cases}$$

$$L(X) = R(X) \qquad \text{III.26.}$$

Now in the function $Y.(S_k 0 \dot{-} X)$ substitute $S_k 0 \dot{-} X$ for Y and define $D_x X$ by

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} S_k 0 \dot{-} F(X) & u=k \\ F(X) & u \neq k \end{cases}$$

Now in the function $Y.(S_k0 \dot{-} X)$ substitute D_kX for Y and $S_k0 \dot{-} Q_kX$ for X; therefore as $Q_kX = S_k0 \dot{-} (X \dot{-} (RX)^2)$ we have the function

$$D_kX.(S_k0 \dot{-} (X \dot{-} (RX)^2)) = D_kX.Q_kX$$

and clearly from III.96, which proves that $(RX)^2 \dot{-} X = 0$, we have that

$$Q_kX = (S_k0 \dot{-} |X,(RX)^2|)$$

and therefore

$$Q_kX = \begin{cases} S_k0 & \text{if } X = (RX)^2 \text{ in '}S_k\text{' successors} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_k0 & \text{if } X \text{ is square in '}S_k\text{' successors} \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$D_kX.Q_kX = \begin{cases} S_k0 & \text{if } X \text{ is an odd square in '}S_k\text{' successors} \\ 0 & \text{otherwise} \end{cases}$$

Define therefore the function E_kX by

$$E_kX = S_kS_kX.(S_k0 \dot{-} D_kX.Q_kX)$$

which is obtained in R_1*** by substitution in $Y.(S_k0 \dot{-} X)$.

We can then say that

$$E_kX = \begin{cases} 0 & \text{if } X \text{ is an odd square in '}S_k\text{' successors} \\ C_kX + S_kS_k0 & \text{otherwise} \end{cases}$$

Before we can proceed now we will have to define the function $X.X$ in R_1***. We are able to define V_kX as before by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} S_kF(X) + S_kS_k0.Q_k(F(X) + S_kS_kS_kS_k0) & u=k \\ S_uF(X) & u \neq k \end{cases}$$

and

$$V_kX = X + S_kS_k0.RX$$

From this clearly $X.X$ is defined by

$$F(0) = 0$$

$$F(S_uX) = S_uV_u(F(X))$$

Having now defined $X.X$ in R_1*** consider the function $H_k(X,Y)$, defined by

$$F(0,Y) = Y.Y + C_kY + S_k0$$

$$F(S_uX,Y) = \begin{cases} E_k(F(X,Y)) & u=k \\ F(X,Y) & u\neq k \end{cases}$$

Clearly

$$H_k(X,X) = E_k^X(X.X + C_kX + S_k0)$$

where E_k^XY is defined by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = \begin{cases} E_k(F(X,Y)) & u=k \\ F(X,Y) & u\neq k \end{cases}$$

E_k^XY is therefore C_kX operations of 'E_k' on Y.

Let X contain an even number of 'S_k' successors, then clearly, for $H_k(X,X)$, after $\frac{1}{2}X$ (normal arithmetical expression) operations of 'E_k', as the argument is never an odd square in 'S_k' successors, and every operation of 'E_k' adds S_kS_k0 to the argument

$$H_k(X,X) = E_k^{\frac{1}{2}X}(X.X + C_kX + C_kX + S_k0)$$

At this time the argument is now an odd square in 'S_k' successors and thus,

$$H_k(X,X) = E_k^{\frac{1}{2}X \dot{-} S_k0}(0)$$

$$= C_kX \dot{-} S_kS_k0$$

Define $\bar{C}_kX$ by

$$F(0) = 0$$

$$F(S_uX) = \begin{cases} F(X) & u=k \\ S_uF(X) & u\neq k \end{cases}$$

This function is all components of X except the 'k'th component, that is the inverse of the function C_kX

$$C_kX + \bar{C}_kX = X$$

From $H_k(X,X)$ we obtain C_kP_kX by

$$C_kP_kX = (S_k0 \dot{-} (S_k0 \dot{-} X)).(S_k0 \dot{-} DX).S_kH_k(X,X) + DX.H_k(S_kX,S_kX)$$

which is obtained by repeated substitution into the function

$Y.(S_k0 \dot{-} X)$ (from the definition this is obviously the same as $(S_k0 \dot{-} X).Y$)

and $Y + X$.

From this

$$P_kX = C_kP_kX + \bar{C}_kX$$

and now we can define $Y \dot{-} X$ by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = P_uF(X,Y)$$

TX is defined by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X).(S_u0 \dot{-} F(X)) + F(X).S_uF(X).(S_u0 \dot{-} (S_u0 \dot{-} F(X)))$$

RM(X,3) is defined by

$$F(0) = 0$$

$$F(S_uX) = (S_u0 \dot{-} F(X)) + S_uS_u0.(S_u0 \dot{-} (S_u0 \dot{-} D(T(F(X)))))$$

UX is defined by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X) + RM_u(F(X),3)$$

and $UX = X + Div(X,2)$

We therefore have $Div(X,2)$ in R_1*** by

$$Div(X,2) = UX \dot{-} X$$

VX is defined by

$$F(0) = 0$$

$$F(S_uX) = S_uF(X) + S_uS_u0.Q_u(F(X) + S_uS_uS_uS_u0)$$

$$VX = X + S_1S_1S_2S_2 \ldots\ldots S_nS_n0.RX \quad \text{explicitly,}$$

and $$RX = Div(VX \dot{-} X,2)$$

Hence by adjoining the function QX to the initial functions we have defined the functions

$$Y + X, \quad Y \dot{-} X, \quad X.X, \quad RX,$$

with the schema R_1***.

We can therefore say that 'The initial functions with the function QX and the defining schema R_1*** are sufficient to define all primitive recursive functions'. We have not proved, though, that it is necessary to adjoin the function QX to the initial functions.

For the proof of the above result in single successor arithmetic see [7].

In the normal definition of the initial functions we have the 'n' successor functions $S_u(X)$ u=1,...,n. This set of n functions can in fact be reduced to 2 functions, that is S_10 and a function to generate further successor functions; for this we could use $0\,\sigma_1\,X$. Then

$$S_20 = 0\,\sigma_1\,S_10 \quad \text{etc. .....}$$

In the next section in this chapter we analyse the successor symbols in X, as mentioned in Chapter I.

ANALYSIS OF 'X'

In single successor arithmetic, the successors in X are all the same and thus require no distinction. Further, the number of successor symbols in X can be compared with those in Y simply by use of the functions $X \dot{-} Y$ and $Y \dot{-} X$, which indicate out of X and Y which has the most successor symbols and by how many. In fact, in single successor arithmetic X by itself informs us how many successor symbols it contains.

The problem of how many and which successor symbols X contains is not so easily answered. A question that immediately arises is how does one count the successor symbols. There are 'n' different counting systems; that is one for each successor, but no common counting system.

To achieve a homogeneous counting system, we consider the property

$$S_1S_2\ldots.S_n0.X = X.\,S_1S_2\ldots.S_n0 = X$$

and further

$$S_1S_1S_2S_2\ldots.S_nS_n0.X = X.S_1S_1S_2S_2\ldots..S_nS_n0 = X + X$$

etc.

This property seems to suggest that we consider

$S_1\,S_2\ldots..S_n0$ as 1

and $S_1S_1S_2S_2\ldots..S_nS_n0$ as 2

etc.

We shall therefore use a dummy successor 'S' to count as follows:

S0 to represent $S_1S_2\ldots..S_n0$

SS0 to represent $S_1S_1S_2S_2\ldots..S_nS_n0$

etc.

In order to analyse X in multiple successor arithmetic we wish to answer the following questions:-

(i) How many successor symbols are there in X?

(ii) How many different successor symbols?

(iii) What are these different successor symbols?

(iv) How many are there of each different successor symbol?

(i) How many successor symbols in X?

The answer to this question would obviously partition the set M_n of all numerals in multiple successor arithmetic into the sets $U_n^o, U_n^1, U_n^2, \ldots, U_n^r, \ldots\ldots$ where

$$U_n^o = \{0\}$$

$$U_n^1 = \{S_v 0 : v=1,\ldots,n\}$$

$$U_n^2 = \{S_u S_v 0 : u,v=1,\ldots,n, \ \& \ v \geqslant u\}$$

etc.

The number of successor symbols in X is determined quite simply by the function $\phi(X)$ defined by

$$\phi(0) = 0$$

$$\phi(S_u X) = S_1 S_2, \ldots, S_n \phi(X) \qquad u=1,\ldots,n$$

or in the counting notation

$$\phi(0) = 0$$

$$\phi(S_u X) = S\phi\ (X)$$

We can therefore say

$$X \in U_n^r \text{ if and only if } |r, \phi(X)| = 0.$$

(ii) How many different successor symbols in X?

The answer to this question obviously gives the sets

$D_n^o, D_n^1, D_n^2, \ldots\ldots D_n^r, \ldots\ldots$ where

$$D_n^o = \{0\}$$

$$D_n^1 = \{\overline{S_u S_u \ldots\ldots S_u}^{\,p} 0 : \quad u=1,\ldots,n, \quad \& \quad p=1,2,\ldots\ldots\}$$

$$D_n^2 = \{\overline{S_u S_u \ldots\ldots S_u}^{\,p}\,\overline{S_v S_v \ldots\ldots S_v}^{\,q} 0 : \quad u,v=1,\ldots,n, \quad \& \quad u \neq v \ \& \ p,q=1,2,\ldots\ldots\}$$

etc.

We first define the function introduced in Chapter I:

$$Y^X 0 = 0$$

$$Y^X S_u X = Y^X X + Y \qquad\qquad u=1,\ldots,n.$$

This function is obviously not commutative, as notice is taken of the successor symbols in Y but not those in X, and this particular property makes this function extremely useful.

Now define the function $\psi(X)$ by

$$\psi(0) = 0$$

$$\psi(S_u X) = S_u \psi(X)^X (S_u 0 \dot{-} \psi(X)) + \psi(X)^X (S_u 0 \dot{-} (S_u 0 \dot{-} \psi(X))) \quad u=1,\ldots,n,$$

this function is such that

$$\psi(S_u X) = \begin{cases} S_u \psi(X) & \text{if } \psi(X) \text{ does not contain } 'S_u' \\ \psi(X) & \text{if } \psi(X) \text{ contains } 'S_u' \end{cases}$$

The important property of the function $Y^X X$ which is used in this definition is

$$Y = Y^X S_u 0 \qquad\qquad \text{for all } u$$

and so $\phi(\psi(X))$ gives the number of different successors in X. We can therefore say

$$X \in D_n^r \text{ if and only if } |r, \phi(\psi(X))| = 0$$

and $\psi(X) \in U_n^r$ if and only if $X \in D_n^r$.

(iii) What are these different successor symbols in X?

For this we need to distinguish between one successor symbol and another, so we define $\Sigma(X)$ by

$$\Sigma(0) = 0$$

$$\Sigma(S_u X) = u + \Sigma(X) \qquad u=1,\ldots,n.$$

This definition is possible as n is finite.

$\Sigma(X)$ is the sum of the suffixes of the successor symbols in X. Hence $S_1 0$ can be distinguished from $S_2 0$ by

$$\Sigma(S_2 0) \dot{-} \Sigma(S_1 0) = S0$$

$$\Sigma(S_1 0) \dot{-} \Sigma(S_2 0) = 0$$

and from $S_3 0$ by

$$\Sigma(S_3 0) \dot{-} \Sigma(S_1 0) = SS0$$

$$\Sigma(S_1 0) \dot{-} \Sigma(S_3 0) = 0$$

We can now define $L^1(X)$ and $H^1(X)$ the lowest (suffix) and highest (suffix) of successor symbols of X, by

$$L^1(0) = 0$$

$$L^1(S_u X) = L^1(X).(1 \dot{-} (\Sigma(L^1(X)) \dot{-} \Sigma(S_u 0))) +$$

$$S_u 0.(1 \dot{-} (1 \dot{-} (\Sigma(L^1(X)) \dot{-} \Sigma(S_u 0)))) \qquad u=1,\ldots,n,$$

and

$$H^1(0) = 0$$

$$H^1(S_u X) = H^1(X).(1 \dot{-} (\Sigma(S_u 0) \dot{-} \Sigma(H^1(X)))) +$$

$$S_u 0.(1 \dot{-} (1 \dot{-} (\Sigma(S_u 0) \dot{-} \Sigma(H^1(X))))) \qquad u=1,\ldots,n.$$

That is $L^1(X)$ is defined by

$$L^1(0) = 0$$

$$L^1(S_uX) = \begin{cases} L^1(X) & \text{if the suffix of } S_u0 \geqslant \text{ the suffix of } L^1(X) \\ S_u0 & \text{if the suffix of } S_u0 < \text{ the suffix of } L^1(X) \end{cases}$$

and $H^1(X)$ is defined by

$$H^1(0) = 0$$

$$H^1(S_uX) = \begin{cases} H^1(X) & \text{if the suffix of } S_u0 \leqslant \text{ the suffix of } H^1(X) \\ S_u0 & \text{if the suffix of } S_u0 > \text{ the suffix of } H^1(X) \end{cases}$$

Thus $\Sigma(L^1(X))$ gives the suffix of the lowest successor symbol and $\Sigma(H^1(X))$ gives the suffix of the highest successor symbol.

Now $L^2(X)$, the second lowest successor symbol, is obviously given by

$$L^2(X) = L^1(X \dot{-} L^1(X).\phi(X))$$

that is

$$X \dot{-} L^1(X).\phi(X)$$

gives X without any of its lowest successor symbols, and therefore the lowest successor symbol of this is the second lowest successor symbol of X. Hence $L^3(X)$ is given by

$$L^3(X) = L^2(X \dot{-} L^1(X).\phi(X))$$

etc. ...

The highest successor symbol is therefore given by

$$H^1(X) = \begin{cases} L^1(X) & \text{if } \phi(X) = 1 \\ L^{\phi(\psi(X))}(X) & \text{otherwise} \end{cases}$$

Similarly we have

$$H^2(X) = H^1(X \dot{-} H^1(X).\phi(X))$$

and

$$L^1(X) = \begin{cases} H^1(X) & \text{if } \phi(X) = 1 \\ H^{\phi(\psi(X))}(X) & \text{otherwise} \end{cases}$$

Thus

$$\Sigma(L^1(X)),\ \Sigma(L^2(X)),\ldots\ldots,\Sigma(L^{\phi(\psi(X))}(X))$$

and

$$\Sigma(H^1(X)),\Sigma(H^2(X)),\ldots\ldots\ldots,\Sigma(H^{\phi(\psi(X))}(X))$$

enumerate the suffixes of all the different successor symbols in X, the first series from lowest to highest and the second series from highest to lowest.

(iv) How many are there of each different successor symbol?

We could answer this question by

$$\phi(C_k X) = \text{the number of } 'S_k' \text{ successor symbols in } X$$

though we count the particular successor symbols that are in X by the use of the function defined by

$$\phi\,(X \dot{-} (1 \dot{-} L^k(X))^x \phi(X))$$

which gives the number of the 'k'th lowest successor symbols in X. This is obtained by subtracting $\phi(X)$ of all the other successor symbols in X from X and counting the remainder, as there obviously cannot be more than $\phi(X)$ of any successor symbol in X. Similarly we could have

$$\phi(X \dot{-} (1 \dot{-} H^k(X))^x \phi(X))$$

which gives the number of the 'k'th highest successor symbols in X.

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