A Quantum Langevin Approach to Hawking Radiation

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Abstract

An investigation of Hawking radiation and a method for calculating particle creation in Schwarzschild spacetime using a quantum Langevin approach is presented in this thesis. In particular we shall show that an oscillator confined to a free-fall trajectory in Schwarzschild spacetime radiates as a result of such motions, and this radiation can be interpreted as Hawking radiation. In chapter 1 we present a literature review of the underlying concept: the Unruh effect. We also present some introductory material pertinent to the calculations. Chapter 2 is concerned with the case of a thin collapsing shell to form a black hole in Schwarzschild anti-de Sitter spacetime. We determine the temperature of the black hole to be $T_H = h(r_h)/4\pi = \kappa/2\pi$ where $h(r_h)$ is the factorization of the conformal factor, r is the radial coordinate with the location of the horizon situated at$r = r_h$, and κ the surface gravity. We also calculate the stress tensor at early and late spacetimes which allows us to calculate the renormalized stress-tensor $\langle T_{\mu\nu} \rangle$ which satisfies the semi-classical Einstien field equations. In chapter 3 we examine the case of a harmonic oscillator in 2D Schwarzschild spacetime and we show that the choice of trajectory is responsible for making the oscillator radiate. In chapter 4 we derive a quantum Langevin equation for the oscillator in the Heisenberg picture. By solving this equation using the Wigner-Weiskopff approximation we show that, in the case of an oscillator confined to a free fall trajectory in Schwarzschild spacetime, the oscillator radiates with respect to the Boulware vacuum. In agreement with Hawking[1] we obtain a temperature of the black hole as $T = 1/8\pi M_B$. In chapter 5 we present our conclusions and recommendations for further work.

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Chapter 1

Introduction and Preliminary Discussions

1.1 Introduction

In 1976, W. G. Unruh published his celebrated paper '*Notes on Black Hole Evaporation*'[2]. In the paper, Unruh considers the case of a particle detector which was undergoing a uniform constant acceleration. Using the Rindler coordinate system, Unruh allowed the oscillator to accelerate with respect to the Minkowski vacuum which was of zero-temperature, and, as a result of such motions, the oscillator comes into equilibrium with the Minkowski vacuum at a temperature which is entirely dependent on the acceleration of the oscillator.

Shortly after Hawking published his procedure for demonstrating the existence of the emission of a thermal spectrum of particles from a black hole[1], P C Davies applied the Hawking procedure to Minkowski spacetime using the Rindler coordinate system which represents hyperbolic motion in Minkowski [3]. Using the Hawking procedure, Davies demonstrated that particle creation occurs in that region of Minkowski spacetime known as the *Rindler wedge*, in effect the horizon for observers undergoing hyperbolic motion in

Minkowski spacetime. The result is pertinent to Hawking's result since there is a close analogy between Rindler coordinate and the Schwarzschild system in the exterior part of a spherically symmetric collapsing body.

The results of Hawking's calculations demonstrating particle creation in Schwarzschild and Minkowski spacetime were not generally contentious and are now generally accepted. There is however still controversy as to whether the constantly accelerating oscillators as investigated by Davies and Unruh produce radiation that is actually detectable.

In both his 1976 paper and a later one he co-authored with R. Wald [4], Unruh argued the Rindler particles could be detected. He suggested that 'detection is seen by an inertial observer as the emission of an ordinary Minkowski quantum, this excitation being due to recoil.' Unruh argued that as a result of such excitations, the oscillator would emit a detectable energy signature as it sought to reach equilibrium with the Minkowski vacuum.

These results were widely accepted at the time. However not everyone agreed with this interpretation. It was P. Grove [5] who first put the argument against the prevailing agreement. Grove argued that Unruh's conclusions were an incorrect quantum mechanical interpretation of the results. Grove suggested that the correct interpretation of a constantly accelerating observer interacting with the Minkowski vacuum is that the energy is absorbed from the space of the detector. In short, the main physical effect is the emission of 'negative' energy by the detector.

Grove's alternative interpretation was soon supported by D. Raine, D. Sciama who co authored a paper (along with P Grove) on inertial detectors [6]. In their paper, a uniformly accelerating oscillator is confined to a 2-dimensional wordline in Minkowski spacetime. The oscillator is coupled to a real scalar field and an electrodynamic interaction is used to describe the interaction between the oscillator and the field. The oscillator is driven by the action of a constant force. Raine *et al.* argued that when one takes into account all of the quantum correlations (i.e. the expected change in physical characteristics as a quantum system passes through a point where the oscillator interacts with the field), no radiation is observed.

Unruh responded to this by considering the case of an accelerating oscillator coupled to a thermal bath when both the bath and oscillator are accelerated together [7]. In this paper, Unruh claimed that Raine *et al.* discarded some terms in their autocorrelation function of the field. These terms he argued, would contribute to the excitation of the detector. Unruh does concede however that in the case of 2 dimensions, the difference between free-field radiation and a distortion of the field tied to a particle is a difficult one to make. He further suggested that a more proper consideration of the effect should be to ask if the oscillator makes any measurable changes to the property of the field; something he claimed his paper had demonstrated.

Much more recently, the solution to this problem seems to have been given by G. Ford and R. F. O'Connell. In their paper '*Is There Unruh Radiation*' [8] Ford and O'Connell consider the case of a harmonic oscillator coupled to a real scalar field. They then allow the oscillator to undergo a variety of motions: stationary, moving in the field direction and hyperbolic motion. Ford and O'Connell form an exact analysis of the problem, and the approach they take is as follows. First they form a Lagrangian which describes the oscillator, the free scalar field and the interaction between the two (taken to be electrodynamical in nature). They then use this Lagrangian to find the equation of motion for the oscillator and the scalar field. The solution of the scalar field equation of motion is one of the form $\phi = \phi_h - x(\tau_{\text{ret}})$, where if ϕ is the scalar field, then ϕ_h is taken to be the homogeneous part. The particular integrals turns out to be the solutions to the oscillator equation of motion $x(t)$, evaluated at the retarded time.

Once this is done, the energy flux of the system is calculated by determining the expectation value of the energy-stress tensor component T_1^0. By substituting in the full solution to the free scalar field, Ford and O'Connell are able to split the expectation value of the energy-stress tensor (and hence the overall energy flux of the system) into three components:

$$\langle \mathcal{J} \rangle = \langle J_0 \rangle + \langle J_{\text{int}} \rangle + \langle J_{\text{dir}} \rangle,$$

where $\langle J_0 \rangle$ is the flux arising in the absence of the oscillator (and is shown to be always zero). The term $\langle J_{\text{dir}} \rangle$ is the direct flux term, and represents the flux arising directly from the oscillator alone, while the $\langle J_{\text{int}} \rangle$ is the interference term and represents the flux arising from the result of the interaction between the field and the oscillator.

In all cases of motion considered, Ford and O'Connell demonstrate that there is no overall energy emission. The calculation is clear in why this is the case: the direct flux term always precisely balances the interference term so that when the two terms are added together the total energy flux is zero. Most interestingly, the ability to balance the two terms seems to come from a property of the oscillator. Ford and O'Connell define the oscillator susceptibility function $\alpha(\omega)$ (where ω is the frequency of the field). This function has the property that

$$\Im\{\alpha(\omega_k)\} = \omega\zeta|\alpha(\omega)|^2,$$

for some coupling constant ζ (the fluctuation-dissipation theorem). It is this property that allows the interference term to take on the exact form (but of opposite sign) to the direct flux term and thus giving $\langle \mathcal{J} \rangle = 0$. At the end of their paper, Ford and O'Connell conclude that a system undergoing hyperbolic motion through a zero-temperature vacuum does indeed experience a finite temperature. However they agree with the conclusions of Grove, Raine and Sciama that no overall energy is detected. Furthermore they point out that the argument for no radiation is identical in the case of a stationary oscillator and one undergoing hyperbolic motion. This strongly implies no energy is radiated since the balancing of the direct and interference term always occurs.

Ford and O'Connell then examine Unruh's results. They dismiss Unruh's claims that when a heat bath is introduced and is moving along with the oscillator that radiation is emitted, claiming it does not represent the situation envisaged by experimentalists interested in detecting Unruh radiation. In such cases, only a single particle or a single atomic system is in accelerated motion.

Vanezalla and Matsas [9] seem to argue that the absence of energy in Ford and O'Connell's calculations is due to the model they used. In their paper concerning the interaction between a charged oscillator and the electro-magnetic field, they conclude radiation is observed. However Ford and O'Connell are skeptical of this, and quite rightly point out that their calculation explicitly shows a detailed balance holds in place for both the stationary case and that of a oscillator performing hyperbolic motion. Their conclusion that such a principle is model independent seems very reasonable.

More recently, a good review of the Unruh effect and its applications has been produced by L. Crispino *et al* [10]. In this paper the authors review the Unruh effect and clear up many of the misconceptions regarding the effect. The authors also state that the Unruh effect does not require an experimental confirmation anymore than free quantum field theory does. They also imply that the Unruh effect has applications in information theory, quantum gravity and cosmology and, there are now a number of papers exploring such things [11, 12]. Indeed [11] discusses Unruh-DeWitt detectors and examines the implications in the areas of entanglement dynamics and quantum teleportation.

The Unruh effect has also been of interest to theorests in the field of quantum gravity. D. Bruschi and J. Louko consider the case of a charged Unruh effect on a topological geon black hole [13] (in this case, the charged Reissner-Nordström geon) . In the paper, they conclude that the geon's exterior region contains non-thermal correlations for particle pairs of the same charge. J. Louko and A. Satz consider the transition rate of the Unruh-DeWitt detector in curved spacetime [14]. The Unruh-DeWitt detector is coupled to a real scalar field in 4D spacetime. The authors derive an integral which gives the total excitation probability and show that an instanteous transition can be recovered if a suitable limit is taken. At the end of the paper, Louko and Satz pose a number of interesting questions, including asking if there might be a relationship between the detector's response and a dynamical or evolving horizon.

As we have seen the problem of investigating inertial detectors can be persued from many different avenues and continues to be of interest. We shall take the approach of Raine and Ford, and O'Connell. We shall show that one can use inertial oscillators to formulate Hawking radiation just by using a quantum Langevin description. We shall see that the energy flux associated oscillator confined to a free-fall trajectory in Schwarzschild spacetime delivers a thermal result, this demonstrates that the process of calculating the energy flux of an oscillator on a free-fall trajectory in 1+1 Schwarzschild spacetime is essentially the same as Hawking's calculation. After performing the Ford and O'Connell calculation in 2D Schwarzschild spacetime and observing the presence of radiation, we then use a Hamiltonian which describes the quantum oscillator, the free field and their interaction, and find the equations of motion for the annihilation and creation operators of the field and the oscillator. Using a Wigner-Weisskopff approximation we can determine the expressions for the annihilation and creation operators of the oscillator. We then adopt the same method as Ford and O'Connell in calculating the energy flux of the system. We shall show that when the oscillator is stationary or confined to an inertial trajectory, the indirect flux and direct flux terms exactly balance when we use the fluctuation and dissipation theorem. When the oscillator is put into a free-fall trajectory in Schwarzschild spacetime however, the two terms do not balance and radiation is indeed observed.

The thesis is organized as follows:

- *Introduction.* In the rest of this chapter, we examine the fundamental principles and calculations we shall use: in the next section, we have a brief review of the fundamental principles of Quantum Field Theory. In §3 we examine the Unruh effect and see how the Unruh Temperature is derived. In §4 we examine the Schwarzschild black hole and look at both its spacetime structure and the choice of vacua which are present. We examine the Hawking method result which uses a ray tracing method to find relationships between ingoing and outgoing modes. The Bogoliubov transformations between these modes then give rise to particle creation. We compare this result with the Unruh effect of §3. In §5 we look at the process of obtaining a quantum Langevin equation for a harmonic oscillator coupled to a real scalar field using the approach discussed by Louisell [15]. We discuss the Wigner-Weisskopf approximation needed to solve this equation. In §6 we give the process for deriving the energy calculations used by Ford and O'Connell from the energy-stress tensor, while in the final section we look at the method of solving the Klein-Gordon wave equation using Green's functions. Sections 1.2, 1.3 and 1.4 are all largely based on the excllent set of lecture notes given by L. H. Ford. [16]

- In Chapter 2 we consider the case of a thin collapsing shell in two–dimensional Schwarzschild–anti de Sitter spacetime. Schwarzschild– anti de Sitter is not a globally hyperbolic spacetime, so there is no past and future timelike infinity. As a result we consider early-time modes and late-time modes rather than in and out ones as in the Schwarzschild case. We apply the Hawking-ray tracing process discussed in Chapter 1 and obtain relations between early-time and late-time modes, and by calculating the Bogoliubov coefficients we demonstrate that particle creation occurs. We then use these modes to calculate the forms of the renormalised stress tensor (which satisfies the semi-classical Einstein field equations) at early and late times.

We discuss the problem of renormalization and the various approaches involved. We shall discuss the renormalization method as suggested by Davies, Fulling and Unruh [17, 18] and apply it to obtain an expression for the renormalised stress tensor in 2D Schwarzschild anti-de Sitter spacetime.

- In Chapter 3 we take an oscillator coupled to a real scalar field and put it on a constant trajectory in Schwarzschild spacetime. When we perform the energy flux calculations we see that, as in the case of Ford and O'Connell, the interference term and direct flux term balance to give zero total energy. However, if we put the oscillator on a inertial trajectory in Schwarzschild spacetime, we find that the energy terms no, longer balance, and this is due to the presence of the conformal factor which was previously constant, and now becomes a function of proper time on the inertial trajectory.

- In Chapter 4 we form a Hamiltonian describing the oscillator, free field and their electrodynamic interaction. We solve the equations of motion using a Wigner–Weisskopff approximation. By forming the position operator for the oscillator and solving the field equation for ϕ, we are able to perform the same energy calculations as we did in Chapter 3. We find that if the oscillator is stationary or undergoing hyperbolic motion, no radiation is observed, however if the oscillator is on a free-fall trajectory in Schwarzschild radiation is detected.

- In Chapter 5, we have conclusions and recommendations for further work.

At the end of the thesis there are two appendix sections containing miscellaneous calculations which are referenced in Chapter 4.

1.2 Quantum Field Theory in Curved Spacetimes

In this section we shall review some of the standard results of quantum field theory applied to curved spacetimes. While it is beyond the scope of this work to review all of the many diverse themes and problems associated with the theory, it is relevant to review the framework and results of the theory most pertinent to the calculations. The topics of interest are the principle formalism behind the theory, the Unruh effect and of course Hawking's celebrated result of the thermal emission of black holes. In this section we shall establish the formalism and conventions of quantum field theory, and in the proceeding two sections after this one we shall examine first the Unruh effect and then the Hawking effect.

Although in general there is no problem in generalizing the model from flat space to curved spacetime, it should be noted that there are a number subtle differences which arise when we make the transfer from QFT in Minkowski (flat) spacetime to that of a quantum field in curved space time. In flat spacetime for example, we have a natural choice of vacuum, the Minkowski one. In the case of a curved spacetime, the choice of vacuum will depend upon the exact physical object we are studying. In flat spacetime we can work with particles, however in curved spacetime the notion of a particle becomes ambiguous. There are often more pressing problems; we need to find a suitable coordinate system. Often spacetime geometries are such that a coordinate system covers just part of the manifold, or there may be coordinate singularities within a particular system. In the case of some spacetimes, there is not always a timelike killing vector, so there are problems to be considered when defining positive frequency. We also need to define a suitable scalar product. These things are all essential if we are to define a vacuum state.

In chapter 2 we shall be concerned with renormalization and in particular the calculation the renormalised stress tensor $\langle T_{\mu\nu} \rangle_{\mathrm{ren}}$. The renormalization procedure we adopt must be consistent and take account of absolute energy values. As we shall see in Chapter 2, even where we work with an asymptotically flat spacetime, such renormalization schemes must be applied with great care if we are to obtain a meaningful, renormalised stress tensor satisfying the semi-classical Einstein field equations.

Construction of QFT in curved spacetime. Let $\mathcal{S}$ be a spacetime which can be modelled as a pseudo-Riemannian manifold which is globally hyperbolic, and has metric a tensor $g_{\mu\nu}$. We define the following Lagrangian density:

$$\mathscr{L} = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(m^2 + \xi R\right)\phi^2\right], \tag{1.1}$$

where we have a scalar field ϕ of mass m, R is the Ricci scalar of the spacetime $\mathcal{S}$ and the quantity ξ is the coupling constant. We find two popular choices for ξ:

1. Minimal coupling, in which case $\xi = 0$ (used for the calculations in this thesis)

2. Conformal coupling for which $\xi = 1/6$.

We may use the Euler-Lagrange equations to generate the equation of motion for the scalar field- the Klein Gordon equation:

$$\Box\phi + m^2\phi + R\xi\phi = 0, \tag{1.2}$$

where $\Box$ is the d'Alembertian operator:

$$\Box = \nabla^\mu\nabla_\mu. \tag{1.3}$$

Let $\phi_1(x)$ and $\phi_2(x)$ be two solutions to the Klein-Gordon equation of motion (1.2). We can define an inner product between the two vectors as

$$(\phi_1, \phi_2) = i\int_\Sigma \phi_2^*(x)\overleftrightarrow{\partial}_\mu\phi_1(x)\ d\Sigma^\mu \tag{1.4}$$

where we take the notation

$$\phi_2^*(x)\overleftrightarrow{\partial}_\mu\phi_1(x) = \phi_2^*(x)\partial_\mu(\phi_1(x)) - (\partial_\mu\phi_2^*(x))\phi_1(x)$$

and we have that $d\Sigma^\mu = n^\mu d\Sigma$ and n^μ is a suitably chosen timelike unit vector which is orthogonal to the Cauchy hypersurface Σ. We note the important property that this inner-product is independent of the choice of hypersurface, so

$$(\phi_1(x), \phi_2(x))_{\Sigma_1} = (\phi_1(x), \phi_2(x))_{\Sigma_2}. \tag{1.5}$$

We may also define a quantity called the *canonical momentum*, Π by

$$\Pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}}, \tag{1.6}$$

and this satisfies the canonical commutation relation

$$[\phi(x), \Pi(x')] = i\delta(x, x'). \tag{1.7}$$

We choose a complete orthonormal set of basis mode solutions of equation (1.2), u_i which satisfy the orthogonality conditions:

$$(u_i, u_j) = \delta_{ij},\ (u_i^*, u_j^*) = -\delta_{ij},\ (u_i, u_j^*) = 0, \tag{1.8}$$

and we can expand the ϕ field in terms of these modes:

$$\phi(t, x) = \sum_k [a_k u_k(t, x) + a_k^\dagger u_k^*(t, x)]. \tag{1.9}$$

We can quantize the scalar field by using the commutation relations:

$$[a_k, a_j] = 0,\ [a_k^\dagger, a_j^\dagger] = 0,\ [a_k, a_j^\dagger] = \delta_{kj}. \tag{1.10}$$

In the Heisenberg picture, quantum states span a Hilbert Space. We use the *Fock representation* as a convenient basis for the Hilbert space of quantum states. The normalized ket vectors which we denote $|\ \rangle$, can be constructed from the vacuum state which we denote by $|0\rangle$. The vacuum state has the property that it is annihilated by all the a_k operators:

$$a_k|0\rangle = 0, \quad \forall k. \tag{1.11}$$

It is often the case that we have more than once choice of orthonormal basis modes, and since both sets span a Hilbert space of states then it must be the case that we can write one set in terms of another. Suppose that we have another complete set of orthonormal basis modes, $\overline{u}(x)$ which are a solution of (1.2).We can expand the ϕ field in terms of these new basis modes:

$$\phi = \sum_j [\overline{b}_j \overline{u}_j + \overline{b^\dagger}_j \overline{u}_j{}^*]. \tag{1.12}$$

This new decomposition of the scalar field will also define a new vacuum state $|\overline{0}\rangle$:

$$\bar{b}_j|\overline{0}\rangle, \ \forall j. \tag{1.13}$$

As both sets of modes are complete, we can write the new modes v_j in terms of the the old ones:

$$\overline{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*), \tag{1.14}$$

and conversely, the old modes in terms of the new ones:

$$u_i = \sum_j (\alpha_{ji}^* \overline{u}_j - \beta_{ji} \overline{u}_j{}^*). \tag{1.15}$$

The transformations of (1.14) and (1.15) are called the *Bogoliubov transformations*, and the matrices α_{ij} and β_{ij} are the *Bogoliubov coefficients*. The coefficients can be evaluated using the Klein Gordon inner product since

$$\alpha_{ij} = (\overline{u}_i, u_j), \quad \beta_{ij} = -(\overline{u}_j, u_j^*). \tag{1.16}$$

If we now equate the two field expansions (1.9) and (1.12) then we obtain (using the orthogonality conditions above) that

$$a_i = \sum_j (\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger) \ \text{ and } \ b_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger). \tag{1.17}$$

The Bogoliubov coefficients posses the property:

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{ik}^*) = \delta_{ij} \ \text{ and } \ \sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0. \tag{1.18}$$

We see from(1.17) that the two different Fock spaces which have arisen from two separate choices of modes will be different so long as the Bogoliubov coefficient $\beta_{ji} \neq 0$.

The expectation value of the operator $N_i = b_i^\dagger b_i$ for the number of u_i-mode particles in the alternative vacuum $|\overline{0}\rangle$ is

$$\langle\overline{0}|N_i|\overline{0}\rangle = \sum_j |\beta_{ji}|^2, \tag{1.19}$$

i.e this is a way of calculating the number of expected u_i particles in the second vacuum. The calculation of the Bogoliubov coefficients is an integral part of the Hawking's calculation in showing that black holes have a thermal property and hence, a surface temperature. They also play an important role in the Unruh effect as we shall now see.

1.3 The Unruh Effect

In this section we examine the phenomena known as the *Unruh effect*: a uniformly accelerating observer in Minkowski spacetime sees a thermal spectrum of particle in the vacuum state.

We have the 2D Minkowski metric:

$$ds^2 = dt^2 - dx^2, \tag{1.20}$$

where we use Minkowski spacetime coordinates (t, x); t is the coordinate time and x is the spatial coordinate. We now make the transformation to Rindler coordinates:

$$t = \frac{e^{a\xi}}{a}\sinh(a\tau), \quad \text{and} \quad x = \frac{e^{a\xi}}{a}\cosh(a\tau), \ a > 0. \tag{1.21}$$

Here τ is the proper time experienced on the worldline of the observer ξ =constant. The metric now becomes:

$$ds^2 = e^{2a\xi}[d\tau^2 - d\xi^2]. \tag{1.22}$$

which gives a metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} e^{2a\xi} & 0 \\ 0 & -e^{2a\xi} \end{bmatrix}. \tag{1.23}$$

If we let ξ is constant, then this represents an accelerating observer with proper acceleration $ae^{-a\xi}$ with a proper time $e^{a\xi}\tau$. The trajectory of this observer is shown in figure 1.1. Here we have the trajectory ξ is constant along with the horizons represented by the line $x = -t$ (so $\tau = -\infty$ and $\xi = -\infty$) and line $x = t$ (whereby $\tau = \infty$ and $\xi = -\infty$). The region R^+ will be of interest to us later, and this region on the space time diagram

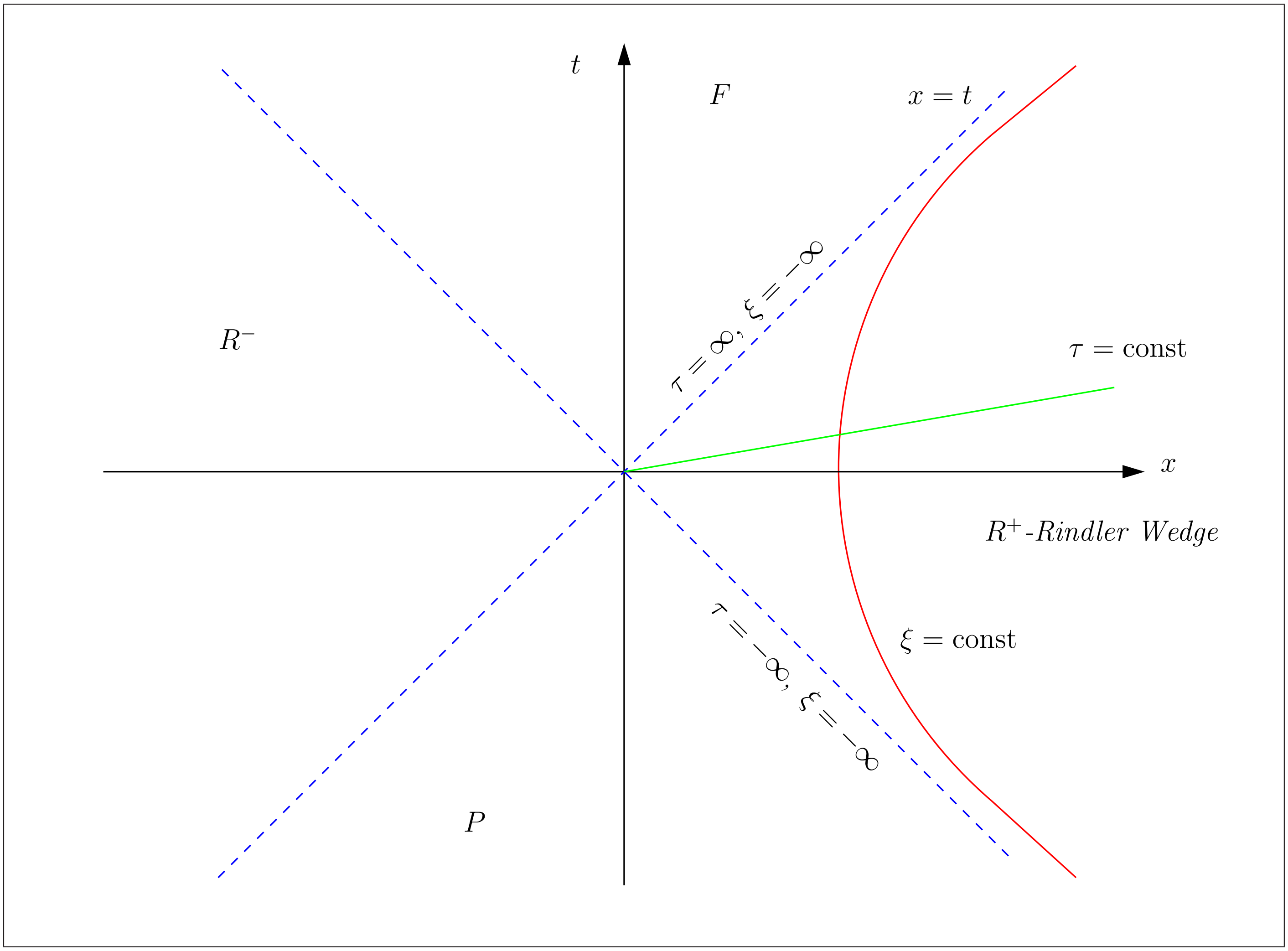

Figure 1.1: A uniformly accelerating oscillator in 2D Minkowski spacetime on the trajectory ξ =const.

is commonly referred to as the *Rindler wedge.*

We now consider the massless scalar field. We shall need to determine the forms of the Rindler modes which we shall denote by u_K^R. We have the covariant form of the wave equation:

$$\partial_\mu \left\{ \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right\} = 0. \tag{1.24}$$

Substituting the inverse of the metric tensor given in (1.23) into (1.24) yields the wave equation:

$$\frac{\partial^2 \Phi}{\partial \tau^2} - \frac{\partial^2 \Phi}{\partial \xi^2} = 0. \tag{1.25}$$

Solving this gives us the Rindler modes of the form

$$u_K^R = \frac{e^{iK\xi - i\varpi\tau}}{\sqrt{4\pi\varpi}}, \quad \text{for} \quad \varpi > 0, \; \varpi^2 = K^2. \tag{1.26}$$

We can analytically extend the modes to cover both the regions F and P in figure 1.1, thus providing us with a complete set of basis modes:

$$u_K^+ = \begin{cases} \frac{1}{\sqrt{4\pi\varpi}} e^{iK\xi - i\varpi\tau} & \text{in } R^+ \\ 0 & \text{in } R^- \end{cases}, \quad \text{and} \quad u_K^- = \begin{cases} 0 & \text{in } R^+ \\ \frac{1}{\sqrt{4\pi\varpi}} e^{iK\xi + i\varpi\tau} & \text{in } R^- \end{cases}. \tag{1.27}$$

We now expand the Φ field in terms of these modes:

$$\Phi = \sum_K b_K^+ u_K^+ + b_K^{\dagger +} u_K^{+*} + b_K^- u_K^- + b_K^{-\dagger} u_K^{-*}, \tag{1.28}$$

where b_k represents the annihilation operator, $b_k^\dagger$ the creation operator of the scalar field. The operators allow us to define the Rindler-Fulling vacuum

$$b_K^+ |0_R\rangle = b_K^- |0_R\rangle = 0. \tag{1.29}$$

Since we are interested in particle creation we shall need to calculate the Bogoliubov coefficients:

$$\beta_{ij} = -(u_k, v_K^*). \tag{1.30}$$

We want particles to be seen by an accelerating observer in the Minkowski vacuum, so the u_i modes correspond to the Minkowski vacuum, and the v_j modes correspond to the

Rindler one. In order to evaluate the product (1.30), we must choose a suitable surface in order to perform the integration. We choose the Cauchy surface $t = 0$ which gives $\tau = 0$. We have the Minkowski modes:

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikx - i\omega t}, \;\; \omega = |k|, \tag{1.31}$$

and the Rindler modes

$$v_K = \frac{1}{\sqrt{4\pi\varpi}} e^{iK\xi - i\varpi\tau}, \;\; \varpi = |K|, \tag{1.32}$$

for $\xi \in (-\infty, \infty)$, i.e. we have that $x > 0$. So, the inner product we wish to evaluate is now

$$\beta_{kK} = i \int_{x=0}^{\infty} u_k \overleftrightarrow{\partial_t} v_K^* \; dx = i \int_0^{\infty} u_k \frac{\partial v_K^*}{\partial t} - \frac{\partial u_k}{\partial t} v_K^* \; dx.$$

We now evaluate each of the derivatives in turn:

$$\frac{\partial u_k}{\partial t} = -\frac{i\omega}{\sqrt{4\pi\omega}} e^{ikx - i\omega t} = -i\omega u_i, \quad \text{on } t = 0, \;\; u_k = \frac{e^{ikx}}{\sqrt{4\pi\omega}}.$$

Using the chain rule we have

$$\begin{aligned}
\frac{\partial v_K^*}{\partial t} &= \frac{\partial \xi}{\partial t} \frac{\partial v_K^*}{\partial \xi} + \frac{\partial \tau}{\partial t} \frac{\partial v_K^*}{\partial \tau}. \\
&= -e^{-a\xi} \sinh(a\tau) \frac{\partial v_j^*}{\partial \xi} + e^{-a\xi} \cosh(a\tau) \frac{\partial v_j^*}{\partial \tau} \\
&= iKe^{-a\xi} \sinh(a\tau) v_j^* + \varpi e^{-a\xi} \cosh(a\tau) v_j^*
\end{aligned}$$

and on the surface $\tau = 0$ this reduces to

$$\frac{\partial v_K^*}{\partial t} = \frac{i\varpi}{ax} v_j^*,$$

and on $t = 0$ we have

$$v_K^* = \frac{e^{-iK\xi}}{\sqrt{4\pi\varpi}} e^{-iK\xi}.$$

Now, our inner product is

$$\beta_{kK} = -\frac{1}{4\pi\sqrt{\omega\varpi}} \int_{x=0}^{\infty} \frac{\varpi}{ax} e^{ikx} e^{-iK\xi} + \omega e^{ikx} e^{-iK\xi} \; dx.$$

We can write the exponential $e^{-iK\xi}$:

$$e^{-iK\xi} = e^{\ln(ax)^{-iK/a}} = (ax)^{-iK/a}.$$

Now make the substitution $y = ax$, and the integrand reduces to

$$\beta_{kK} = -\frac{1}{4\pi\sqrt{\omega\varpi}} \int_0^\infty \left(\frac{\varpi}{y} + \omega\right) e^{iky/a} y^{-iK/a}\, dy.$$

We can write this as two integrals

$$\beta_{kK} = -\frac{1}{4\pi\sqrt{\omega\varpi}}(I_1 + I_2).$$

Dealing with I_1 first:

$$I_1 = \int_0^\infty \varpi e^{iky/a} y^{-1-iK/a}\, dy,$$

which is clearly a Gamma function, and so we have that

$$I_1 = \varpi \left(\frac{a}{ik}\right)^{-iK/a} \Gamma\left(\frac{iK}{a}\right). \tag{1.33}$$

We can do a similar thing for the integral I_2;

$$I_2 = \int_0^\infty \omega e^{iky/a} y^{-iK/a}\, dy = \omega \left(\frac{a}{ik}\right)^{1-iK/a} \Gamma\left(1 + \frac{iK}{a}\right)$$

i.e.

$$I_2 = \omega \left(\frac{a}{ik}\right)^{1-iK/a} \frac{iK}{a} \Gamma\left(iK/a\right).$$

Now, we have that

$$\left(\frac{a}{iK}\right)^{-iK/a} = \exp\left(-\frac{iK}{a}\log\left(\frac{a}{ik}\right)\right) = \exp\left(-\frac{iK}{a}\left(\log\left(\frac{a}{k}\right) - \frac{i\pi}{2}\right)\right) = e^{-\pi K/2a} e^{i\theta}.$$

Hence we have that

$$\beta_{kK} = -\frac{1}{4\pi\sqrt{\omega\varpi}}\left(\varpi + \frac{\omega K}{k}\right) e^{i\theta} e^{-\pi K/2a} \Gamma\left(\frac{iK}{a}\right).$$

We have that β_{kK} is zero if $\varpi = -\omega K/k$, otherwise

$$\beta_{kK} = -\frac{2\pi}{a}\sqrt{\frac{\varpi}{\omega}} e^{i\theta} e^{-\pi K/2a} \Gamma(iK/a).$$

Thus we have that

$$|\beta_{kK}|^2 = \frac{1}{4\pi a\omega} \frac{e^{-\pi\varpi/a}}{\sinh(\pi\varpi/a)} = \frac{1}{2\pi a\omega}(e^{2\pi\varpi/a} - 1)^{-1}. \tag{1.34}$$

This is a thermal spectrum with temperature $T = a/2\pi$. The result is dependent on the acceleration of the observer and not the velocity, and the thermal spectrum of particles are seen in the Rindler wedge. As we have discussed at the start of the chapter, the claim that accelerated observers detect a thermal heat bath is not controversial. However, the claim that the accelerating oscillator is emitting photons and that such radiations are therefore detectable has been questioned. In particular both Raine, Sciama and Grove and Ford and O'Connell have demonstrated that in fact the direct flux arising from the oscillator exactly balances the radiation caused by the interaction of the oscillator with the scalar field. Hence there is no overall energy flux.

1.4 Black Holes and the Hawking Effect.

In this section we shall examine the phenomena of Hawking radiation. The Hawking effect describes the emission of a thermal spectrum of particles by a black hole after its formation from stellar collapse. The emission of such particles (now called Hawking photons) does not depend upon the details of the collapse, or the collapse process itself. In the first part of this section we shall look at the Schwarzschild black hole and its properties. In the second part of this section we shall look at the case of a thin collapsing shell in Schwarzschild spacetime. We shall demonstrate the process used by Hawking whereby early time modes are related to the late time ones by a simple linear function. This function can be found using the geometric optics approximation as the modes pass through the collapsing shell. Calculation of the Bogoliubov coefficients then yields the result of thermal particle creation at late times.

We shall employ this method in Chapter 2 when we come to determine the particle creation involved in the case of a thin collapsing shell, collapsing in 2D Schwarzschild anti de-Sitter spacetime.

1.4.1 2D Schwarzschild Spacetime

In this part, we shall briefly look at the structure of 2D Schwarzschild spacetime. The Schwarzschild metric describes a static black hole which has formed from the symmetrical collapse of a massive star. The manifold is described by the metric

$$ds^2 = \left(1 - \frac{2M_B}{r}\right) dt^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} dr^2. \tag{1.35}$$

Let $r_h = 2M_B$ denote the position of the even horizon, then the above metric describes the exterior of the black hole the region $r \geq r_h$. Here we have that M_B is the mass of the black hole, and we note the existence of a coordinate singularity at the position $r = 2M_B$. All two dimensional spacetimes are conformally flat and we can write the metric (1.35) in the conformally flat form

$$ds^2 = \Omega(dt^2 - dr_*^2), \tag{1.36}$$

where

$$\Omega = 1 - \frac{2M_B}{r} \tag{1.37}$$

is a quantity called the conformal factor, and r_* is the 'tortoise coordinate' satisfying

$$\frac{dr_*}{dr} = \left(1 - \frac{2M_B}{r}\right)^{-1}. \tag{1.38}$$

Integrating this directly gives

$$r_* = r + 2M_B \ln \left| \frac{r}{2M_B} - 1 \right|. \tag{1.39}$$

We may use the 'tortoise coordinate' to define the advanced and retarded null coordinates (respectively):

$$u = t - r_*, \quad \text{and} \quad v = t + r_*. \tag{1.40}$$

Light rays travel along the lines u = constant or v = constant. In terms of the null coordinates, the metric in (1.35) becomes

$$ds^2 = \left(1 - \frac{2M_B}{r}\right) dudv. \tag{1.41}$$

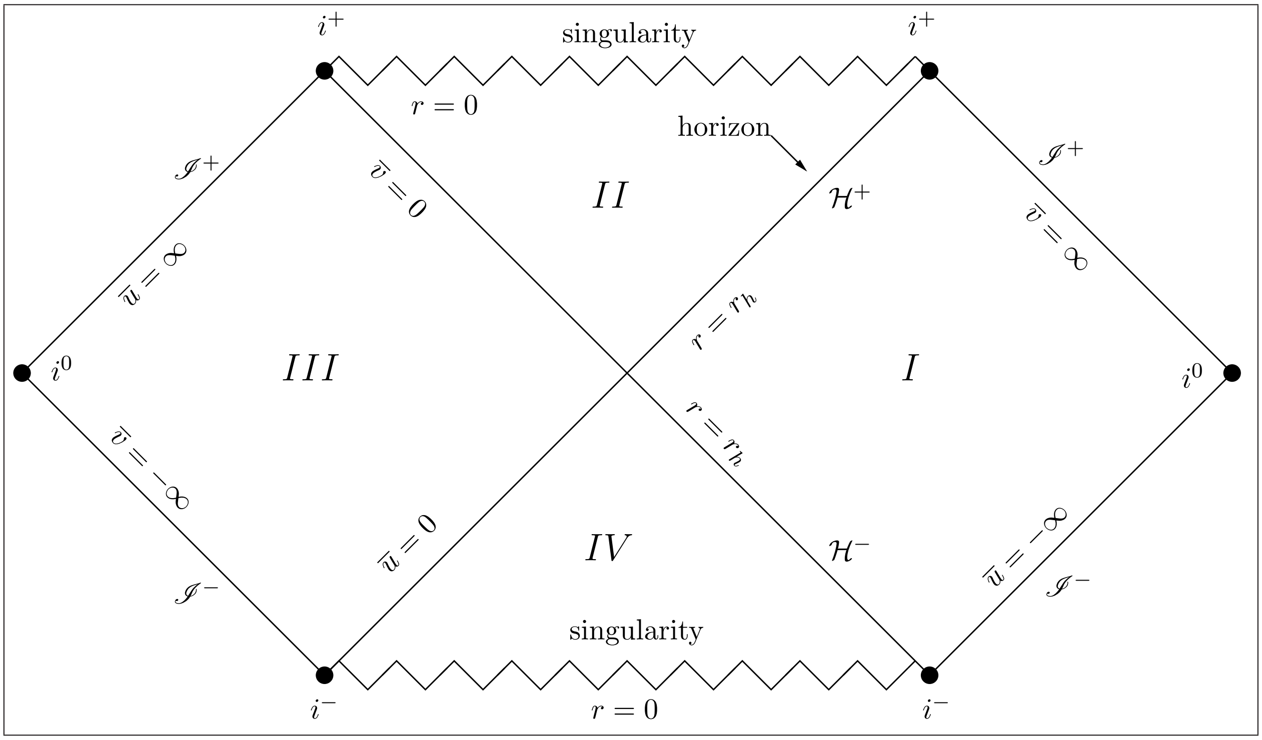

Figure 1.2: A Penrose diagram showing the Maximally extended Kruskal manifold. Timelike geodesics start at i^- and finish at i^+. Future and past timelike infinities $\mathscr{I}^+$ and $\mathscr{I}^-$ are shown, and the Singularity is located at $r = 0$. Past and future event horizons are $\mathcal{H}^-$ and $\mathcal{H}^+$.

We now introduce another coordinate system known as the *Kruskal Coordinates*. These coordinates cover all of the Schwarzschild manifold and make use of the null coordinates previously defined. We let

$$\overline{u} = -4M_B e^{-u/4M_B}, \quad \text{and} \quad \overline{v} = 4M_B e^{v/4M_B}, \ \overline{u} \in (-\infty, 0), \ \overline{v} \in (0, \infty). \tag{1.42}$$

Figure 1.2 gives a Penrose diagram of the Schwarzschild black hole as covered by the Kruskal coordinate system. The intrinsic singularity exists at $r = 0$. Particles are confined to timelike geodesics which start at i^- and terminate at i^+. Light rays travel at $\pi/2$ radians to the origin. We have event horizons situated along the future and past horizons. It will also be noted that we have four regions: I, II, II and IV.

- Region I: This is the exterior region of the black hole, $r > r_h$.
- Region II: The interior of the black hole containing the singularity at $r = 0$.
- Region III: A parallel exterior region
- Region IV: A White hole interior.

We now come to the problem of choosing the candidate vacua. For quantum field theory in flat space, we have a natural candidate: the *Minkowski vacuum*. However, it is the case that, in general, for quantum field theory in curved spacetimes, very often there is no one preferred vacuum, rather the choice of vacuum depends upon in the physical situation we are working with. However, some spacetimes do admit to a natural choice of spacetime. In their paper Kay and Wald [19] show that globally hyperbolic spacetimes possessing a one-parameter group of isometries with a bifurcate Killing horizon do posses quantum states which are both pure and unique. Minkowski, Schwarzshild and de Sitter spacetimes have this property.

Consider now the metric:

$$ds^2 = C(r)\, dudv.$$

We have the massless scalar wave equation:

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0,$$

which gives

$$\partial_\mu\partial_\nu\Phi = 0. \tag{1.43}$$

This equation has a general solution

$$\Phi = f(u) + g(v). \tag{1.44}$$

We can use the null coordinates defined in (1.40) to define positive frequency modes: $e^{-i\omega u}$, $e^{-i\omega v}$ for $\omega > 0$. This leads us to the following basis modes:

- *'Up' modes*: These are basis modes of the form

$$\overleftarrow{u}_s \propto e^{-i\omega u} \tag{1.45}$$

- *'In' modes*: These are basis modes of the form

$$\overrightarrow{u}_s \propto e^{-i\omega v} \tag{1.46}$$

Using the Kruskal coordinate system we can define another set of positive frequency modes:

$$\overrightarrow{u}_k \propto e^{-i\omega\overline{u}},\ \overleftarrow{u}_k \propto e^{-i\omega\overline{v}} \tag{1.47}$$

We now have three possible choices of vacua:

- *Schwarzschild/Boulware Vacuum:* This vacuum, $|0_S\rangle$ represents a vacuum which is empty at infinity. The vacuum is defined using the basis modes of (1.45) and (1.46).

- *Kruskal/Hawking-Hartle Vacuum:* This vacuum, $|0_K\rangle$, represents a vacuum which is in thermal equilibrium at infinity. This vacuum is defined by using the Kruskal modes of (1.47).

- *Unruh Vacuum:* This choice of vacuum, $|0_U\rangle$ represents an evaporating black hole and is defined by using the Schwarzschild mode $\overleftarrow{u}_S$ and the Kruskal mode $\overrightarrow{u}_K$.

It is useful to make a comparison with the Unruh effect of the previous section. The Kruskal coordinates $\overline{u}$ and $\overline{v}$ are null coordinates, so consider a time coordinate defined by

$$\tau = \frac{1}{2}(\overline{u} + \overline{v}).$$

Substituting in the forms for $\overline{u}$ and $\overline{v}$ gives in (1.42):

$$\tau = 4M_B e^{r/4M_B} \left(\frac{r}{2M_B} - 1 \right)^{1/2} \sinh \left(\frac{t}{4M_B} \right).$$

Similarly, we can define a spatial coordinate:

$$\xi = \frac{1}{2}(\overline{v} - \overline{u}) = 4M_B e^{r/4M_B} \left(\frac{r}{2M_B} - 1 \right)^{1/2} \cosh \left(\frac{t}{4M_B} \right).$$

We see that these τ and ξ expressions are the Rindler transformations of the previous section (up to a factor $\sqrt{(r/2M_B - 1)}$ which has appeared due to the fact that Schwarzschild spacetime is curved). The term a has now been replaced with

$$k = \frac{1}{4M_B}, \tag{1.48}$$

and this quantity represents the surface gravity of the black hole. The similarity between the Kruskal coordinates and the Rindler coordinates means that we can compare Region I of the Schwarzschild black hole (see figure 1.2) with the Rindler wedge (region R^+ in figure 1.1) of Minkowski spacetime. So in terms of vacuum, the Rindler vacuum is equivalent to the Boulware vacuum, while the Hawking-Hartle vacuum is equivalent to the Minkowski vacuum. The inevitable conclusion must be that a stationary observer in the Schwarzschild spacetime observes a thermal distribution of particles within the Hawking-Hartle vacuum. (Note that a stationary observer in Schwarzschild spacetime is a uniformly accelerating one).

1.4.2 The Hawking Effect

We now come to the calculation which demonstrates that at late times, far from the black hole event horizon, a thermal spectrum of particles is observed. It should be noted that there are a number of different derivations of the Hawking effect, we present the methodology as used by L. Ford. We will again adopt this procedure in Chapter 2 when we examine the case of a thin collapsing shell in Schwarzschild anti de-Sitter spacetime.

In figure 1.3 we have a Penrose diagram showing the collapse of the matter to form a Schwarzschild black hole. Rays start at timelike infinity and pass through the centre of the collapse. Any ray entering after the ray v_0 (which travels along the event horizon) will hit the singularity. Initially we have the 'in'-modes which define the in-vacuum state which we shall denote $|0_{\text{in}}\rangle$. We have 'in'-modes of the form

$$f_\omega \sim e^{-i\omega v} \text{ as } v \longrightarrow \infty, \tag{1.49}$$

and 'out'-modes of the form

$$F_\omega \sim e^{-i\omega u} \text{ as } u \longrightarrow \infty. \tag{1.50}$$

In order to determine the existence of a thermal spectrum of particles at late times we shall need to calculate the Bogoliubov coefficients. However it is the case that far from the event horizon at late times, very high frequency modes dominate. These modes have arrived from $\mathscr{I}^-$ shortly before the event horizon forms.

The very high frequency nature of the modes as they pass through the collapsing matter means that we can use a *geometric optics* approximation to model the modes. We can also write an asymptotic form for the modes:

$$f_\omega \sim \begin{cases} e^{-i\omega v} & \text{on } \mathscr{I}^- \\ e^{-i\omega G(u)} & \text{on} \mathscr{I}^+ \end{cases} \quad \text{and} \quad F_\omega \sim \begin{cases} e^{-i\omega u} & \text{on} \mathscr{I}^+ \\ e^{-i\omega g(v)} & \text{on } \mathscr{I}^- \end{cases} \tag{1.51}$$

We shall now need to find the forms of the functions g and G. To do this, we shall use a ray-tracing method. First, consider the case of a thin collapsing shell. We let the position of the shell at time t be $r = R(t)$, and so for $r > R(t)$ the spacetime is that of the exterior Schwarzschild spacetime with line element:

$$ds^2 = \left(1 - \frac{2M_B}{r}\right) dt^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} dr^2.$$

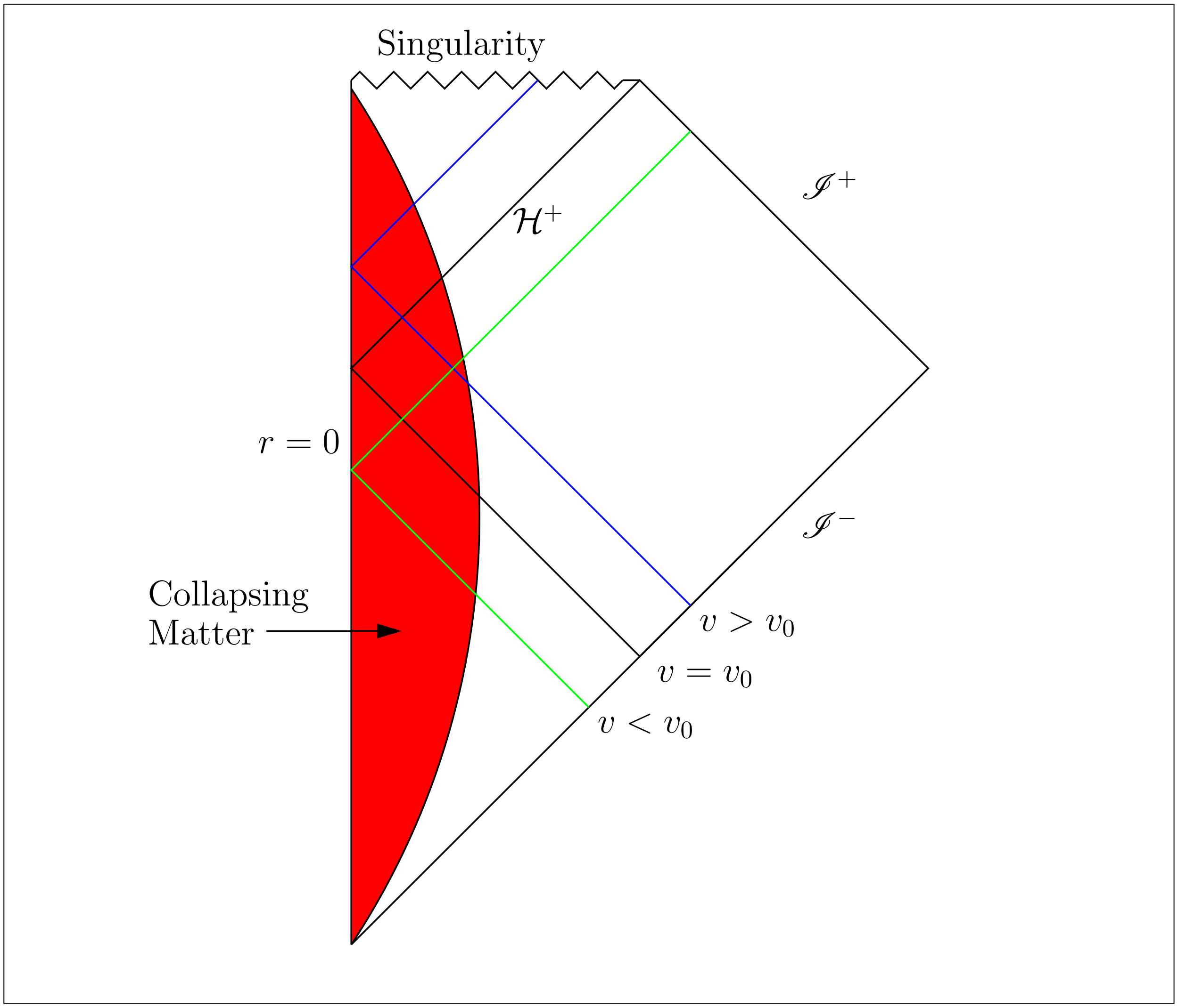

Figure 1.3: A Penrose diagram showing the collapse of a star to form a Schwarzschild black hole. The centre of the collapse is located at $r = 0$. Light rays come in from $\mathscr{I}^-$ and pass through the centre of the collapse. If they pass through the collapse before the ray v_0 then the ray will emerge and travel on to the future timelike infinity, $\mathscr{I}^+$.

Inside the shell $M_B = 0$ and so for $r < R(t)$ the spacetime is flat:

$$ds^2 = dT^2 - dr^2. \tag{1.52}$$

We can also define a set of null coordinates for the interior of the black hole:

$$U = T - r, \quad \text{and} \quad V = T + r. \tag{1.53}$$

The metrics on both sides of the shell must match and so

$$dT^2 - dR^2 = \left(\frac{R - 2M_B}{R}\right) dt^2 - \left(\frac{R - 2M_B}{R}\right)^{-1} dR^2,$$

which we may write as

$$1 - \left(\frac{dR}{dT}\right)^2 = \left(\frac{R - 2M_B}{R}\right)\left(\frac{dt}{dT}\right)^2 - \left(\frac{R - 2M_B}{R}\right)^{-1}\left(\frac{dR}{dT}\right)^2. \tag{1.54}$$

Consider now figure 1.4. Here we have a diagram showing the collapse. We have two neighborhoods of interest. There is the δ-neighbourhood which is a region far from the formation of the event horizon, centered on the point R_1, the position where the ray v_1 enters the collapsing shell. We also have a ϵ-neighbourhood which is a region close to the event horizon centered on the point R_2, the position of the shell when ray v_2 emerges after passing through the collapsing matter. Consider now the ray v_1 shown in the figure. This ray enters the shell when at a position R_1 such that $R_1 - 2M_B$ is finite. We may say, approximately that the quantities

$$\frac{R}{R - 2M_B}, \text{ and } \frac{dR}{dT}$$

are both finite and constant near the point R_1. This means that

$$\frac{dt}{dT} \approx \text{const}, \quad \text{so} \quad t \propto T$$

approximately. We can also say that in the δ-neighbourhood, by definition of the 'tortoise' coordinate in (1.39), that $r_* \propto r$, and so by use of the null coordinates (1.40) and (1.53) we may say that approximately, in the δ-neighbourhood around R_1 that

$$V(v) = av + b, \tag{1.55}$$

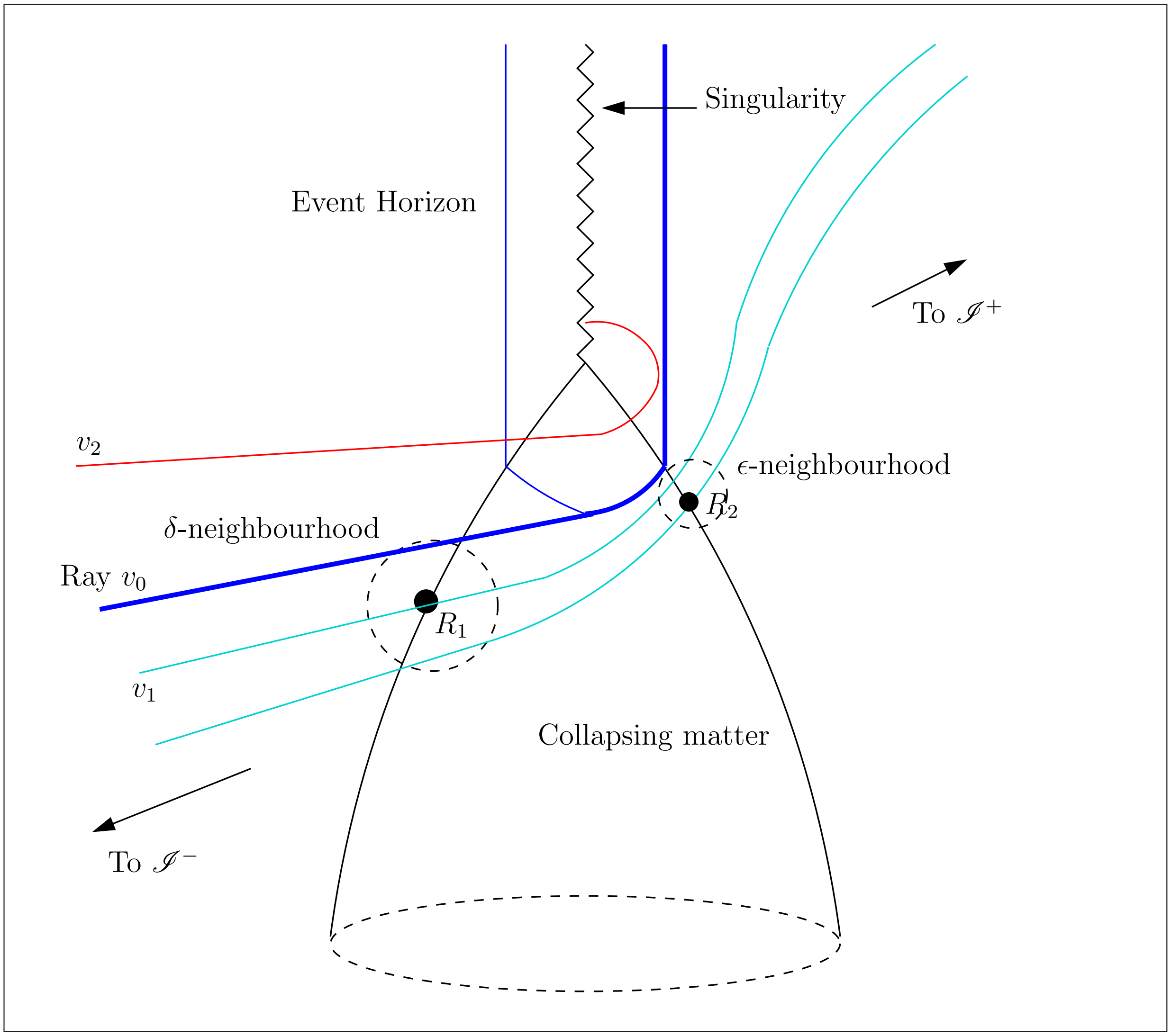

Figure 1.4: A diagram showing the passage of modes from past timelike infinity, through the collapsing matter and on to future timelike infinity. We see that rays which enter the collapse after the ray v_0 are doomed to hit the singularity which has formed after this point.

for some constants a and b.

After entering the shell, the ray v_1 passes through the centre of the collapse. The matching of the null coordinates at the centre is simple, since at the centre of the collapse $r = 0$. Hence we have that at

$$r = 0, \quad U = V. \tag{1.56}$$

Finally, the ray exists the shell at the position R_2. The ϵ-neighbourhood which surrounds the point R_2 is close to the event horizon. If we let T_0 be the time when $R = 2M_B$ (as observed from inside the shell) then near $T = T_0$ we have that

$$R(t) \approx 2M_B + A(T_0 - T), \tag{1.57}$$

where A is a constant. We rearrange (1.54):

$$\left(\frac{dt}{dT}\right)^2 = \left(\frac{R-2M_B}{R}\right)^{-1} - \left(\frac{R-2M_B}{R}\right)^{-1}\left(\frac{dR}{dT}\right)^2 - \left(\frac{R-2M_B}{R}\right)^{-2}\left(\frac{dR}{dT}\right)^2.$$

We shall ignore all but the leading quadratic term since this term is dominant, and so we have approximately that

$$\left(\frac{dt}{dT}\right)^2 \approx \left(\frac{R-2M_B}{R}\right)^{-2}\left(\frac{dR}{dT}\right)^2.$$

We now substitute (1.57) into this expression we obtain that

$$\left(\frac{dt}{dT}\right)^2 \approx \frac{4M_B^2}{(T-T_0)^2}. \tag{1.58}$$

Taking the square root and integrating this equation gives

$$t \approx -2M_B \ln\left(\frac{T_0 - T}{B}\right), \tag{1.59}$$

where B is constant as $T \to T_0$. Similarly, we have from (1.39) that, for $T \to T_0$,

$$r_* \approx 2M_B \ln\left(\frac{A(T_0 - T)}{2M_B}\right). \tag{1.60}$$

From the definition of the null coordinate: $u = t - r_*$ we can say that

$$u \approx -4M_B \ln\left(\frac{T_0 - T}{B'}\right), \tag{1.61}$$

for some constant B'. Now, in this limit we have that

$$U = T - r = T - R(T) \approx (1 + A)T - AT_0 - 2M_B. \tag{1.62}$$

So, from (1.61) we may write that

$$u \approx -4M_B \ln \left(\frac{T - 2M_B - U}{B'} \right).$$

Tracing back through the collapse we have that $U = V$ and so

$$u \approx -4M_B \ln \left(\frac{T - 2M_B - V}{B'} \right),$$

and we have from (1.55) that

$$u \approx -4M_B \ln \left(\frac{T - 2M_B - (av + b)}{B'} \right). \tag{1.63}$$

Now, the argument of the logarithm must vanish on the horizon (i.e. when $v = v_0$ and so

$$T - 2M_B = av_0 + b,$$

and hence we arrive at

$$g(v) = u = -4M_B \ln \left(\frac{v_0 - v}{c} \right), \tag{1.64}$$

where c is a constant for this ray. From (1.50), we see that the out modes when traced back to $\mathscr{I}^-$ have the form

$$F_{\omega lm} \sim \begin{cases} e^{4iM_B \ln((v_0 - v)/c)} & v < v_0 \\ 0 & v > v_0 \end{cases} \tag{1.65}$$

Now let $\{f_i\}$ be positive frequency solutions to the scalar field equation in the past (i.e. the 'in'-region) and we let $\{F_i\}$ be the positive frequency solutions in the future (the 'out'-region). We can choose these sets of modes to be orthonormal, so they satisfy the property:

$$(f_i, f_{i'}) = (F_i, F_{i'}) = \delta_{ii'}, \ (f_i^*, f_{i'}^*) = (F_i^*, F_{i'}^*) = -\delta_{ii'}, \ (f_i, f_{i'}^*) = (F_i, F_{i'}^*) = 0. \tag{1.66}$$

Although we have defined these functions in terms of their asymptotic properties in different regions, they are still solutions of the wave equation everywhere in the spacetime. Thus we may expand the in-modes in terms of the out ones:

$$f_i = \sum_k (\alpha_{ik} F_k + \beta_{ik} F_k^*), \tag{1.67}$$

and vice-versa

$$F_i = \sum_j (\alpha_{ji}^* f_j - \beta_{ji} f_j^*), \tag{1.68}$$

where α and β are the Bogoliubov coefficients. Thus taking the Fourier transform of (1.65) and by using the identities in (1.66) we find the Bogoliubov coefficient:

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{\infty}^{v_0} e^{i\omega' v} e^{4iM_B\omega \ln((v_0 - v)/c)}\, dv$$

If we make the substitution $v' = v_0 - v$ then we have that

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_{0}^{\infty} e^{-i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\, dv \tag{1.69}$$

and

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_{0}^{\infty} e^{i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\, dv. \tag{1.70}$$

We shall now consider the integrand in (1.69). The integrand is analytic everywhere except for a branch cut along the negative real axis as shown in figure 1.5. If we let C be the path

$$C = \gamma_R + \gamma_B + \gamma_\epsilon + \gamma_A,$$

to be traversed in the anticlockwise sense, then we have by Cauchy's theorem that

$$\oint_C e^{-i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\, dv' = 0. \tag{1.71}$$

Consider the integrand now along the path γ_R. We make the substitution

$$v' = R\cos(\theta) + iR\sin(\theta).$$

Thus we have that

$$\int_{\gamma_R} e^{-i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\, dv' =$$
$$\int_0^{\pi} (-R\sin(\theta) + iR\cos(\theta)) \left(\frac{R\cos(\theta) + iR\sin(\theta)}{c}\right) e^{-i\omega'(R\cos(\theta) + iR\sin(\theta))}\, d\theta.$$

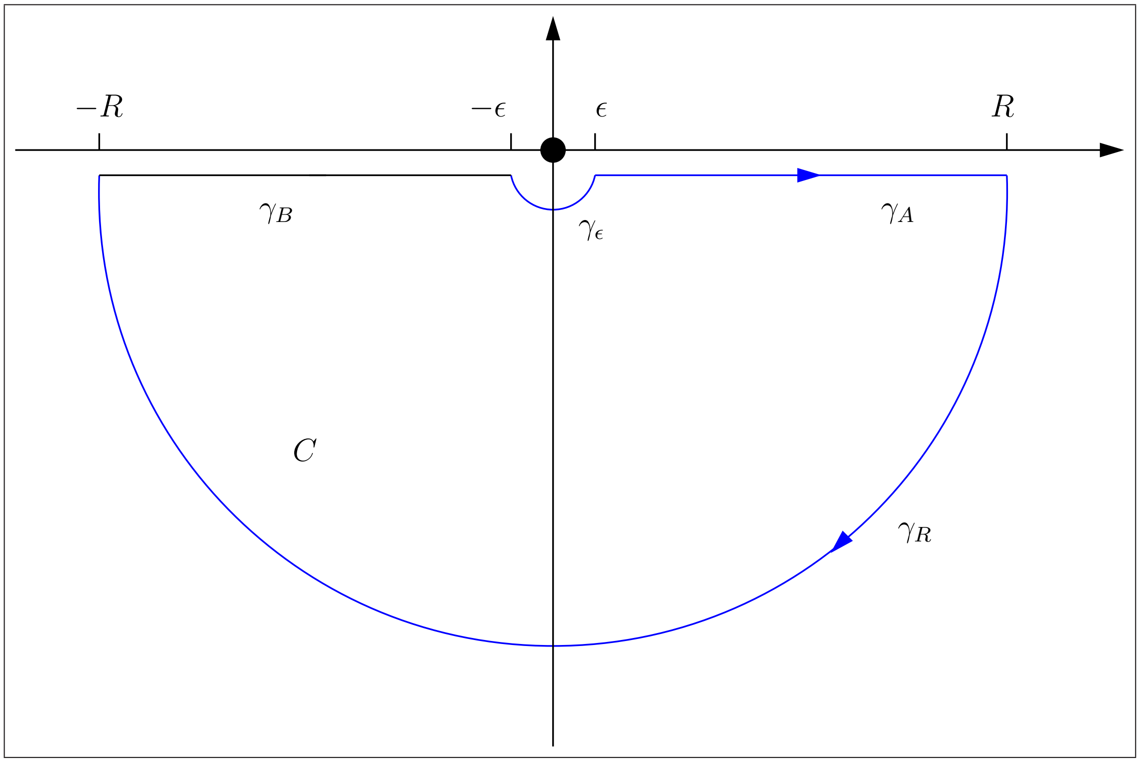

Figure 1.5: The contour of the integrand for the Bogoliubov coefficient $\alpha_{\omega\omega'}$,

Thus

$$\int_{\gamma_R} \sim \mathcal{O}(e^{\omega' R \sin(\theta)}) \to 0 \text{ as } R \to \infty,$$

and so the contribution from $\int_{\gamma_R} = 0$.

Let us now consider the integrand along the small semi-circle γ_ϵ. We make the new substitution

$$v' = \epsilon\cos(\theta) + i\epsilon\sin(\theta),$$

and thus we have that

$$\int_{\gamma_\epsilon} e^{-i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\, dv' =$$
$$\int_0^\pi (-\epsilon\sin(\theta) + i\epsilon\cos(\theta)) \left(\frac{\epsilon\cos(\theta) + i\epsilon\sin(\theta)}{c}\right)^{4iM_B\omega} e^{-i\omega'(\epsilon\sin(\theta)+\epsilon\cos(\theta))}\, d\theta.$$

We have that

$$\int_{\gamma_\epsilon} \sim \mathcal{O}(\epsilon) \to 0, \text{ as } \epsilon \to 0.$$

Hence we must have that

$$\int_{\gamma_A} + \int_{\gamma_B} = 0. \tag{1.72}$$

So, finally we have that

$$\int_0^\infty e^{-i'\omega' v'} e^{4iM_B\omega \ln(v/c)}\ dv' = -\int_0^\infty e^{i\omega' v'} e^{4iM_B\omega \ln(-v'/c - i\epsilon)}\ dv'.$$

Using the fact that

$$\ln\left(-\frac{v'}{c} - i\epsilon\right) = -i\pi + \ln(v'/c),$$

then we have that

$$\int_0^\infty e^{-i'\omega' v'} e^{4iM_B\omega \ln(v/c)}\ dv' = -e^{4\pi M_B\omega} \int_0^\infty e^{i\omega' v'} e^{4iM_B\omega \ln(v'/c)}\ dv'. \tag{1.73}$$

Comparison of (1.73) with (1.70) reveals that

$$|\alpha_{\omega\omega'}| = e^{4\pi M_B\omega} |\beta_{\omega\omega'}|. \tag{1.74}$$

We know that

$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1,$$

and hence from (1.74) we have that

$$\sum_{\omega'} |\beta_{\omega\omega'}|^2 = \frac{1}{e^{8\pi M_B\omega} - 1}. \tag{1.75}$$

This is Hawking's result, and the expression tells that thermal particle creation occurs with a temperature (with $\hbar = c = 1$)

$$T_H = \frac{1}{8\pi M_B}. \tag{1.76}$$

This is known as the *Hawking temperature* and we see that it depends only upon the mass of the black hole. The result we have arrived at does not depend upon the nature of the collapse as we are only concerned with the energy flux at late times, far from the black hole.

1.5 Obtaining a Langevin Equation

In this section, we shall examine the method as presented by Louisell[15] for obtaining a quantum Langevin equation from a Hamiltonian of a damped harmonic oscillator coupled to a real scalar field. This approach makes use of a Wigner-Weisskopff approximation, which we shall also examine. We do this as the process yields forms for the annihilation and creation operators of the oscillator, and hence allows us to define the position and momentum functions for the operator. Once we have these, we can perform the energy calculations of the previous section.

First, let us remind ourselves of some preliminary observations of the harmonic oscillator in the Heisenberg picture. We shall consider a classical harmonic oscillator of unit mass in one dimension whose position is described by the coordinate q, and momentum is described by the coordinate p. This discussion is presented in Louisell's book [15]. The Hamiltonian is

$$\mathscr{H} = \frac{1}{2}(p^2 + \omega_c^2 q^2), \tag{1.77}$$

where ω_c is a constant related to the restoring force of the particle. Using the above Hamiltonian, we can find the equations of motion for position and momenta

$$\frac{dq}{dt} = \frac{\partial \mathscr{H}}{\partial p} = p, \tag{1.78}$$

and

$$\frac{dp}{dt} = -\frac{\partial \mathscr{H}}{\partial q} = -\omega_c^2 q. \tag{1.79}$$

We now have two coupled differential equations, and if we differentiate both sides of (1.78) with respect to t, then we can eliminate dp/dt from (1.79), thus we obtain

$$\frac{d^2 q}{dt^2} = -\omega_c^2 q. \tag{1.80}$$

The solution to this equation is

$$q(t) = A\sin(\omega_c t) + B\cos(\omega_c t), \tag{1.81}$$

where A and B are constants. It is clear that if we substitute this into equation (1.79) and integrate both sides we have that

$$p(t) = -A\omega_c \cos(\omega_c t) + \omega_c B\omega_c \sin(\omega_c t). \tag{1.82}$$

Using the arbitrary boundary condition that at $t = 0$

$$p = p(0) \quad \text{and} \quad q = q(0) \tag{1.83}$$

means that we have

$$q(t) = q(0)\cos(\omega_c t) + \frac{p(0)}{\omega_c}\sin(\omega_c t), \quad \text{and} \quad p(t) = -\omega_c q(0)\sin(\omega_c t) + p(0)\cos(\omega_c t). \tag{1.84}$$

There is an alternative procedure for dealing with the coupled differential equations of (1.78) and (1.79), and this method has a direct relevance to the quantum mechanical treatment of the harmonic oscillator which we shall look at shortly. If we multiply (1.78) through by $\sqrt{\omega_c/2}$ and equation (1.79) by the quantity $\pm i/\sqrt{2\omega_c}$ then add both equations together, we find we obtain the two coupled differential equations:

$$\frac{da}{dt} = -i\omega_c a^*, \tag{1.85}$$

and

$$\frac{da^*}{dt} = i\omega_c a. \tag{1.86}$$

We define

$$a = \frac{1}{\sqrt{2\omega_c}}(\omega_c q + ip), \quad \text{and} \quad a^* = \frac{1}{\sqrt{2\omega_c}}(\omega_c q - ip). \tag{1.87}$$

with a^* being the complex conjugate of a. We may solve these two equations for position and momenta coordinates:

$$q = \frac{1}{\sqrt{2\omega_c}}(a^* + a), \quad \text{and} \quad p = i\sqrt{\frac{\omega_c}{2}}(a^* - a). \tag{1.88}$$

Solving (1.85) and (1.86) is trivial and we have the solutions

$$a(t) = a(0)e^{-i\omega_c t} = \frac{1}{\sqrt{2\omega_c}}\left[\omega_c q(0) + ip(0)\right] e^{-i\omega_c t}, \tag{1.89}$$

and

$$a^*(t) = a^*(0)e^{i\omega_c t} = \frac{2}{\omega_c}\left[\omega_c q(0) - ip(0)\right] e^{i\omega_c t}. \tag{1.90}$$

Thus, the introduction of the quantities a and a^* have made the solutions to (1.78) and (1.79) much simpler. We find that the introduction of a and a^* also simplifies the Hamiltonian. After some algebra we find that the Hamiltonian can now be written in the succinct form

$$\mathscr{H} = \omega_c a^* a. \tag{1.91}$$

Indeed, from this form of the Hamiltonian, we can obtain directly the equations (1.85) and (1.86):

$$i\frac{da}{dt} = \frac{\partial \mathscr{H}}{\partial a^*} = \omega_c a, \quad \text{and} \quad i\frac{da^*}{dt} = -\frac{\partial \mathscr{H}}{da} = -\omega_c a^*.$$

We now turn to the quantum treatment of the oscillator in the Heisenberg picture. From standard quantum theory, we associate hermitian operators with the observables q, p, and $\mathscr{H}$, and furthermore we have that the operators q and p satisfy the commutation relation

$$[q, p] = i\hbar. \tag{1.92}$$

due to (1.87). The Hamiltonian for the system is

$$\mathscr{H} = \frac{1}{2}(p^2 + \omega_c^2 q^2) = \mathscr{H}^\dagger. \tag{1.93}$$

All of the operators above are in the Schrodinger picture, and as such are independent of time. The Schrodinger equation of motion is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathscr{H}|\psi(t)\rangle \tag{1.94}$$

and the solution to this equation is

$$|\psi_S(t)\rangle = U(t,0)|\psi_H(0)\rangle = e^{-i\mathscr{H}t/\hbar}|\psi_H(0)\rangle, \tag{1.95}$$

where U is unitary. The above equation is the transformation law between state vectors in the Schrodinger and Heisenberg pictures. Operators transform between the two pictures by the similarity transform:

$$q_H(t) = U^\dagger(t,0) q_S U(t,0), \quad \text{and} \quad p_H(t) = U^\dagger(t,0) p_S U(t,0). \tag{1.96}$$

Now, for a conservative system, the Hamiltonian in the Heisenberg picture is the same as the one in the Schrodinger one, thus we may write

$$\mathscr{H}_H = \frac{1}{2}[p_H^2(t) + \omega_c^2 q_H^2(t)]. \tag{1.97}$$

Where the subscript 'H' reminds us that the operators are in the Heisenberg picture. The Heisenberg equations of motion for $q_H(t)$ and $p_H(t)$ are

$$\frac{dq_H}{dt} = \frac{\partial \mathscr{H}_H}{\partial p_H} = p_H, \quad \text{and} \quad \frac{dp_H}{dt} = -\frac{\partial \mathscr{H}_H}{\partial q_H} = -\omega_c^2 q_H. \tag{1.98}$$

The only difference between these and the classical equations we derived in (1.78) and (1.79) is that the operators p_H and q_H now satisfy the commutation relation

$$[q_H(t), p_H(t)] = i\hbar. \tag{1.99}$$

We now introduce two convenient operators- the *annihilation operator* a, and it's Hermitian conjugate counterpart, the *creation operator* $a^\dagger$:

$$a = \frac{1}{\sqrt{2\hbar\omega_c}}(\omega_c q + ip), \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2\hbar\omega_c}}(\omega_c q - ip). \tag{1.100}$$

The annihilation and creation operators are related to the position and momentum operators by the relations

$$q = \sqrt{\frac{\hbar}{2\omega_c}}(a^\dagger + a), \quad \text{and} \quad p = i\sqrt{\frac{\hbar\omega_c}{2}}(a^\dagger - a). \tag{1.101}$$

As we would expect, the operators a and $a^\dagger$ do not commute, however they do satisfy the commutation relation

$$[a, a^\dagger] = 1. \tag{1.102}$$

If we use this and substitute (1.101) into (1.97) we find that the Hamiltonian is now

$$\mathscr{H} = \frac{\hbar\omega_c}{2}(aa^\dagger + a^\dagger a) = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right), \tag{1.103}$$

where the term $\hbar\omega_c/2$ is the zero-point energy of the oscillator.

We now turn out attention to the problem of constructing a Langevin equation of motion for a damped harmonic oscillator. As we shall see, such an approach will involve using a Wigner-Weisskopff approximation. Our aim is to find expressions for the annihilation and creation operators of the oscillator, we can then use these to determine the position function q from equation (1.101). In the next section we shall examine the method introduced by Ford and O'Connell for determining the overall energy flux of the system.

We start by introducing the Hamiltonian given in Louisell's text [15]

$$\mathscr{H} = \hbar\omega_c a^\dagger a + \hbar\sum_j \omega_j b_j^\dagger b_j + \hbar\sum_j(\kappa_j b_j a^\dagger + \kappa_j^* b_j^\dagger a). \tag{1.104}$$

where ω_c is the natural frequency of the oscillator which has annihilation and creation operators, a and $a^\dagger$ respectively. Similarly the b_j and $b_j^\dagger$ are the annihilation and creation operators of the scalar field. The annihilation operator a satisfies the Heisenberg equation of motion

$$\frac{da}{dt} = \frac{1}{i\hbar}[a, \mathscr{H}]. \tag{1.105}$$

So we have

$$\frac{da}{dt} = -i\omega_c[a, a^\dagger a] - i\sum_j \kappa_j b_j^\dagger[a, a] - i[a, a^\dagger]\sum_k \kappa_j b_j. \tag{1.106}$$

We may use the identity, for operators $\mathscr{M}$, a and $a^\dagger$ that

$$[\mathscr{M}, a^\dagger, a] = [\mathscr{M}, a^\dagger]a + a^\dagger[\mathscr{M}, a]$$

along with the commutation relations

$$[\mathscr{M}, a] = -\frac{\partial\mathscr{M}}{\partial a^\dagger}, \quad [\mathscr{M}, a^\dagger] = \frac{\partial\mathscr{M}}{\partial a}$$

and so we find that (1.106) now becomes:

$$\frac{da}{dt} = -i\omega_c a - i\sum_j \kappa_j b_j. \tag{1.107}$$

We do the same thing for the scalar field operators:

$$\frac{db_j}{dt} = -i\omega_j b_j - i\kappa_j^* a = -i\omega_j b_j - i\kappa_j^* a. \tag{1.108}$$

Integrating the above equation gives us an expression for the annihilation operator of the scalar field

$$b_j(t) = e^{-i\omega_j t} b_j(0) - i\kappa_j^* \int_0^t a(t') e^{i\omega_j(t'-t)}\ dt', \tag{1.109}$$

where $b_j(0)$ is the value of the operator at time $t = 0$. We now substitute (1.109) and it's complex conjugate back into (1.106) (being mindful of order as the two separate parts of $b_j(t)$ and $b_j^\dagger(t)$ do not commute with all functions of $a(t)$ and $a^\dagger(t)$ in the Heisenberg picture):

$$\frac{da}{dt} = -i\omega_c a + G_a - \sum_j |\kappa_j|^2 \int_0^t a(t') e^{i\omega_j(t'-t)}\ dt', \tag{1.110}$$

where we have that

$$G_a = i \sum_j \kappa_j b_j(0) e^W - i\omega_j t. \tag{1.111}$$

We now remove high-frequency behavior from (1.106). Let

$$a(t) = e^{-i\omega_c t} A(t);\ \text{and still } [a(t), a^\dagger(t)] = [A(t), A^\dagger(t)], \tag{1.112}$$

and hence we have that

$$\frac{dA}{dt} = G_A - \sum_j |\kappa_j|^2 \int_0^t A(t') e^{i(\omega_j - \omega_c)(t'-t)}\ dt, \tag{1.113}$$

with

$$G_A = -i \sum_j \kappa_j b_j(0) e^{-i(\omega_j - \omega_c)t}. \tag{1.114}$$

The equation we have obtained in (1.113) is an integrodifferential equation, and it does not have an exact solution. We must therefore use an approximation to solve it, and this will be the Wigner–Weisskopff approximation. First we multiply both sides of (1.113) by 'e^{-st}' and integrate with respect to t from 0 to infinity (i.e. we take the Laplace transform of both sides):

$$s\overline{A}(s) = \overline{G}_A(s) + \sum_j |\kappa_j|^2 \int_0^\infty e^{i(\omega_c - \omega_j + is)t}\ dt \int_0^t A(t') e^{i(\omega_j - \omega_c)t'}\ dt'.$$

We change the order of integration so that the integrals can be evaluated:

$$s\overline{A}(s) = \overline{G}_A(s) + \sum_j |\kappa_j|^2 \int_{t'}^{\infty} e^{i(\omega_c-\omega_j+is)t}\ dt \int_0^{\infty} A(t')e^{i(\omega_j-\omega_c)t'}\ dt'.$$

Performing the t integral first we now have that:

$$s\overline{A}(s) = \overline{G}_A(s) + \sum_j |\kappa_j|^2 \int_0^{\infty} e^{i(\omega_c-\omega_j+is)t'} A(t')e^{i(\omega_j-\omega_c)t'}\ dt',$$

which simplifies to

$$s\overline{A}(s) = \overline{G}_A(s) + \sum_j |\kappa_j|^2 \overline{A}(s).$$

Re-arranging the above equation and we arrive at the exact form

$$\overline{A}(s) = \frac{\overline{G}_A(s)}{s + \sum_j \frac{|\kappa_j|^2}{s+i(\omega_j-\omega_c)}}, \tag{1.115}$$

where we have that

$$\overline{A}(s) = \int_0^{\infty} e^{-st}A(t)\ dt, \quad \text{and} \quad \overline{G}_A(s) = -i\sum_j \frac{\kappa_j b_j(0)}{s+i(\omega_j-\omega_c)}. \tag{1.116}$$

We now have to deal with the poles in (1.115). To do this we employ the Wigner-Weisskopff approximation. For the case where the atom interaction with the field is small, a zeroth approximation of $s = 0$ can be used. The next approximation consists in calculating the first order shift in the simple pole of (1.115) as is discussed in Louisell's text [15].

Now

$$\begin{aligned}\lim_{s\to 0^+}\left\{\frac{1}{x+is}\right\} &= \lim_{s\to 0^+}\left[\frac{x}{x^2+s^2} - \frac{is}{x^2+s^2}\right] \\ &= \frac{1}{x} - i\pi\delta(x).\end{aligned} \tag{1.117}$$

Since we have that

$$\lim_{s\to 0^+}\left\{\frac{s}{\pi(x^2+s^2)}\right\} = \begin{cases} 0 & \text{for } x \neq 0, \\ \infty & \text{for } x = 0 \end{cases} \quad \text{and} \quad \lim_{s\to 0^+}\left\{\int_{-\infty}^{\infty} \frac{s}{\pi(x^2+s^2)}\ dx\right\} = 1 \tag{1.118}$$

are required properties of the delta function. So, we now write

$$-i\sum_j \frac{|\kappa_j|^2}{(\omega_j-\omega_c)-is} \to \lim_{s\to 0^+}\left\{\int \frac{g(\omega_j)|\kappa(\omega_j)|^2}{(\omega_j-\omega_c)-is}\ d\omega_j\right\}.$$

Using the Wigner-Weisskopff approximation of (1.117) we have that

$$\lim_{s\to 0}\left\{\frac{g(\omega_j)|\kappa(\omega_j)|^2}{(\omega_j-\omega_c)-is}\right\} = -i\int g|\kappa|^2\left(\frac{1}{\omega_j-\omega_c}+i\pi\delta(\omega_j-\omega_c)\right)\,d\omega_j,$$

and thus we can write that

$$-i\sum_j \frac{|\kappa_j|^2}{(\omega_j-\omega_c)-is} = \frac{\gamma}{2}+i\Delta\varpi \tag{1.119}$$

where

$$\gamma = 2\pi g(\omega_c)|\kappa(\omega_c)|^2, \quad \text{and} \quad \Delta\varpi = \int \frac{g(\omega_j)|\kappa(\omega_j)|^2}{\omega_j-\omega_c}\,d\omega_j. \tag{1.120}$$

The effect of applying the Wigner-Weisskopff approximation means that we can replace (1.113) with an exact Langevin equation:

$$\frac{dA}{dt} = -\frac{1}{2}A(t)+G_A(t), \tag{1.121}$$

(ignoring small frequency shifts), and this has the general solution,

$$A(t) = e^{-\gamma t/2}\int_0^t G_A(t')e^{\gamma t'/2}\,dt'. \tag{1.122}$$

So, in order to determine a form for the annihilation operator $A(t)$, we simple substitute in the correct $G_A(t)$ (which in this context represents the random operator Langevin noise source) into (1.122) and perform the integration. Once we have the annihilation and creation operators we are free to determine $q(t)$ from the expressions in (1.101), and we can use it to determine the overall energy flux of the system. We do this using the method as established by Ford and O'Connell [8] and we examine their method in the next section.

1.6 Method of Determining Energy Flux

In this section we consider the problem of determining the energy flux of a harmonic oscillator coupled to a real scalar field undergoing some kind of prescribed motion. We introduce two quantities, the energy density $\mathscr{E}$ and the energy flux $\mathcal{J}$. The energy density represents the amount of energy stored in the system, while the energy flux is the rate of transfer of energy of the system. We have the conservation law:

$$\frac{\partial \mathscr{E}}{\partial t} + \frac{\partial \mathcal{J}}{\partial x} = 0, \tag{1.123}$$

where t is a time coordinate and spatial the coordinate is x. Let ϕ be a real scalar field which satisfies the the Klein- Gordon equation

$$(\Box + m^2)\phi = 0. \tag{1.124}$$

where $\Box$ is the usual D'Alembertian operator

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{1.125}$$

The Klein-Gordon equation may be derived by applying the variational principle to the action

$$S = \int_\Sigma \mathscr{L}(\phi, \partial_\mu\phi)\; d^4x. \tag{1.126}$$

where $\mathscr{L}$ is the Lagrangian density for the scalar field. In the process of taking the variation of the above action, we find an expression for the energy-stress tensor T_μ^ν appears in the expression of the variation of the action for S and has the form

$$T_\mu^\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta_\nu^\mu \mathscr{L}. \tag{1.127}$$

The energy density $\mathscr{E}$ is represented by the component T_0^0 in the stress tensor. So we have from (1.127) that

$$T_0^0 = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}\frac{\partial \phi}{\partial t} - \mathscr{L}, \tag{1.128}$$

where we have used dots to denote differentiation with respect to time. Similarly we have the energy flux $\mathcal{J}$ represented by the component stress tensor component T_1^0:

$$T_1^0 = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}\frac{\partial \phi}{\partial r}. \tag{1.129}$$

We shall be interested determining the energy flux of the system and so we shall need to calculate the expectation value:

$$\langle \mathcal{J} \rangle = \left\langle \frac{\partial \mathscr{L}}{\partial \dot{\phi}}\frac{\partial \phi}{\partial r} \right\rangle. \tag{1.130}$$

As we saw at the start of the chapter, the general Lagrangian density for the free-scalar field is given by

$$\mathscr{L} = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(m^2 + \xi R\right)\phi^2\right].$$

Thus we see that in order to determine the energy flux of the system, we shall need to calculate the expression (1.130). This expression requires the Lagrangian density of the scalar field, which in turn is dependent on the spacetime metric tensor. We shall also require the solution to the scalar field equation, which will have a general form

$$\phi(t,x) = \phi_h(t,x) + \phi_p(t,x), \tag{1.131}$$

where $\phi_h(t,x)$ is the homogeneous solution and $\phi_p(t,x)$ is the particular integral of the field equation of motion. In fact, as we shall see in Chapters 3 and 4 when we come to calculate the energy flux of oscillators coupled to a real (massless) scalar field on various trajectories that the particular integral is in fact the position function of the quantum oscillator, determined by the method given in the previous section. Thus we have a method for calculating the energy flux of a quantum oscillator confined to some prescribed trajectory in a given spacetime:

1. First we form the Hamiltonian which describes the quantum oscillator, the free scalar field and the interaction between both the field and the oscillator.

2. The annihilation and creation operators of both the oscillator and the free field must satisfy the Heisenberg equations of motion, so we use this to determine the equations of motion for the field and oscillator.

3. Solve the equation of motion for the scalar field and obtain expressions for the annihilation and creation operators $b_j(t)$ and $b_j^\dagger(t)$.

4. Substitute these into the equation of motion for the oscillator. Further, let $a(t) = e^{-i\omega_c t}A(t)$ to remove the high frequency behavior of the system and thus obtain a first order integro differential equation for the annihilation operator $A(t)$.

5. Apply the Wigner-Weisskopff approximation to this differential equation so that the differential equation can be replaced with a quantum Langevin equation, the solution of which yields $a(t)$ and $a^{\dagger}(t)$. It is then a straightforward matter to obtain the position function, $q(t)$ from (1.101).

6. Solve the scalar field equation to obtain $\phi(t,x) = \phi_h(t,x) + \phi_p(t,x)$.

7. Using expression for $q(t)$ and $\phi(t,x)$, determine the overall energy flux of the system by calculating the expectation value in (1.130).

We now have almost everything we need to calculate the expectation value (1.130) for a number of different cases. The final thing we need to examine is the process of computing a solution to the inhomogeneous wave equation. As we shall see in the next section, the standard way of doing this is through the use of Green's functions.

1.7 Green's Function for the Wave Equation

In the last section we shall be give the standard method of solving the inhomogeneous wave equation:

$$\Box\phi = f(\mathbf{x},t). \tag{1.132}$$

The quantity $\eta^{\mu\nu}$ is the inverse of the Minkowski metric tensor. The spacetimes we shall examine in this thesis are all two dimensional, and any two dimensional spacetime is conformally flat; one simply writes the metric tensor in terms of a conformal factor and the usual Minkowski tensor $g_{\mu\nu} = \Omega\,\eta_{\mu\nu}$. Thus the method presented here is suitable for conformally flat spacetimes, and we follow the methodology of P. Szekeres text [20]. Here (in this section) we shall let $\mathbf{x} = (x^1, x^2, x^3, x^4)$ with $x^4 = ct$.

The general solution of (1.132) has the form

$$\phi(x) = \phi_{\mathscr{G}} + \phi^h(x), \tag{1.133}$$

where $\phi^h(x)$ is the homogeneous solution, $\Box\phi^h(x) = 0$, and $\phi_{\mathscr{G}}$ is the particular integral to be found using a Green's functions. We shall seek a solution a solution to the equation

$$\Box\mathscr{G}(x - x') = \delta^4(x - x'), \tag{1.134}$$

where we have taken

$$\delta^4(x - x') = \delta(x^1 - x'^1)\delta(x^2 - x'^2)\delta(x^3 - x'^3)\delta(x^4 - x'^4).$$

Every Green's function $\mathscr{G}$ generates a solution $\phi_{\mathscr{G}}(x)$ to the inhomogeneous wave equation of (1.132),

$$\phi_{\mathscr{G}}(x) = \iiiint \mathscr{G}(x - x')f(x')\ d^4x',$$

for

$$\begin{aligned}\Box\phi_{\mathscr{G}} &= \iiiint \Box\mathscr{G}(x - x')f(x')\ d^4x' \\ &= \iiiint \delta^4(x - x')\ d^4x' = f(x).\end{aligned}$$

We shall set

$$\mathscr{G}(x - x') = \frac{1}{(2\pi)^2} \iiiint g(k)e^{ik(x-x')}\ d^4x$$

where we have let $k = (k_1, k_2, k_3, k_4)$ and

$$k(x - x') = k_\mu(x^\mu - x'^\mu).$$

If we write the four-dimensional δ-function as a Fourier Transform we must have that

$$\begin{aligned}\Box\mathscr{G}(x - x') &= \frac{1}{(2\pi)^2} \iiiint -k^2 g(k)e^{ik(x-x')}\ d^4k = \delta^4(x - x') \\ &= \frac{1}{(2\pi)^4} \iiiint e^{ik(x-x')}\ d^4k,\end{aligned}$$

hence we have

$$g(k) = -\frac{1}{4\pi^2 k^2} \tag{1.135}$$

where $k^2 = k.k = k_\mu k^\mu$. So, the Fourier transform of the Green's function is

$$\mathscr{G}(x - x') = -\frac{1}{(2\pi)^4} \iiiint \frac{e^{ik(x-x')}}{k^2}\ d^4k. \tag{1.136}$$

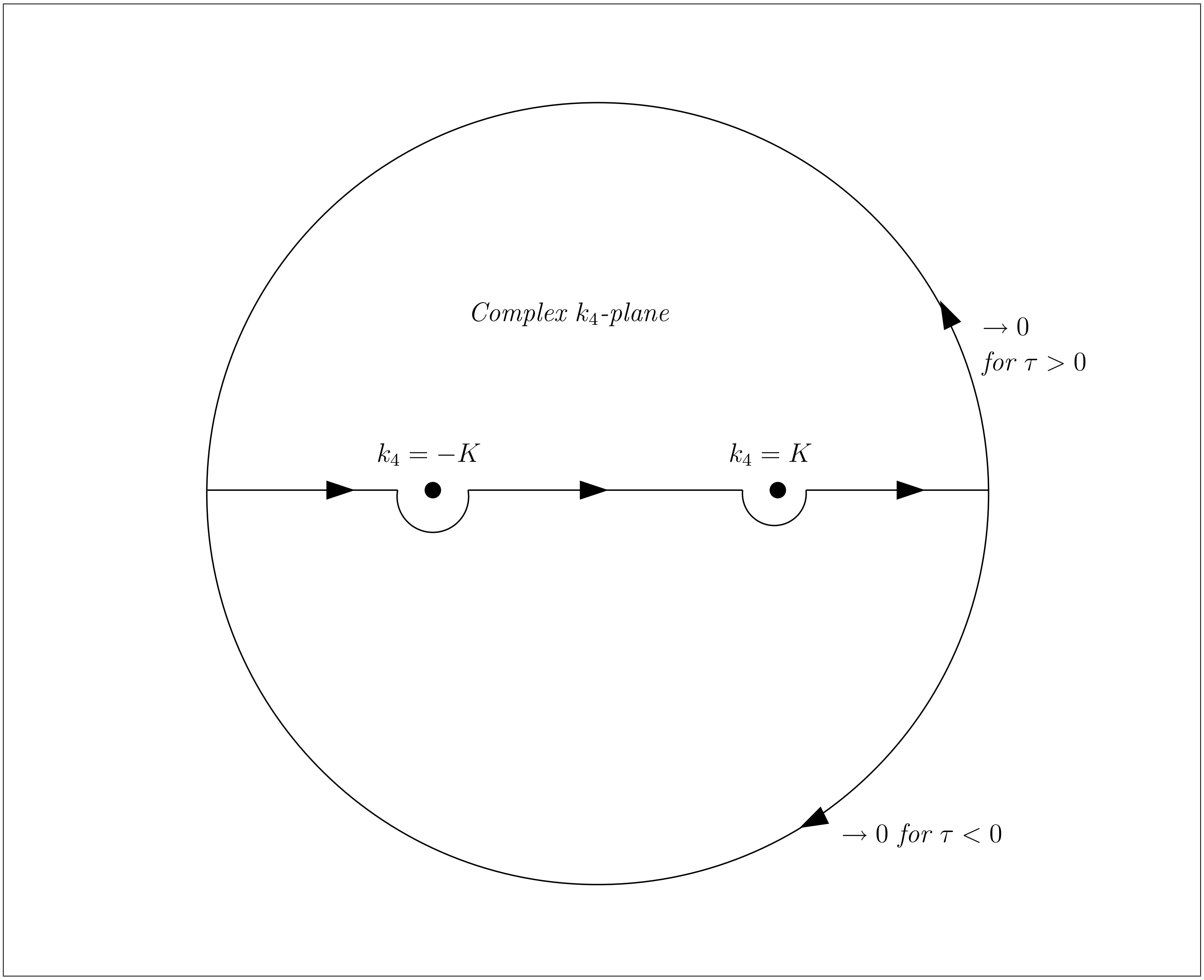

Figure 1.6: The contour in the k_4-plane of the Green's function for the 3-dimensional wave equation.

We shall now need to evaluate this integral. First we shall let

$$\tau = x^4 - x'^4, \quad \mathbf{R} = \mathbf{x} - \mathbf{x}', \quad K = |\mathbf{k}| = \sqrt{\mathbf{k}\cdot\mathbf{k}}$$

and so

$$\mathscr{G}(x - x') = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{ik_4\tau}\ dk_4}{k_4^2 - K^2} \iiint e^{\mathbf{k}\cdot\mathbf{R}}\ d^3k.$$

This is a contour integral and its path in the complex k_4-lane is shown in figure 1.6. There are two possible directions around the path. For $\tau > 0$, the contour is traversed anti-clockwise and we follow the upper semi circle. Now use Cauchy's integral theorem:

Cauchy's Integral Theorem Let $\mathcal{D}$ be a bounded domain in the complex plane with piecewise smooth boundary $\partial\mathcal{D}$. Suppose that $f(z)$ is analytic on $\mathcal{D} \cup \partial\mathcal{D}$ except for a finite number of isolated singularities $z_1, ..., z_m \in \mathcal{D}$. Then

$$\int_{\partial\mathcal{D}} f(z)\ dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}[f(z), z_j],$$

where for a pole of order n at the point z_0 we have that

$$\mathrm{Res}[f(z), z_0] = \frac{1}{(n-1)!} \lim_{z\to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \{(z - z_0)^n f(z)\} \right\}$$

and we find

$$\int_{-\infty}^{\infty} \frac{e^{ik_4\tau}}{k_4^2 - K^2}\ dk_4 = 2\pi i \left[\frac{e^{iK\tau}}{2K} - \frac{e^{-iK\tau}}{2K} \right]. \tag{1.137}$$

For $\tau < 0$, we complete the contour in a clockwise fashion and follow the lower semicircle in figure 1.6. This particular contour does not enclose any poles and so, as a result, the integral must vanish, thus

$$\int_{-\infty}^{\infty} \frac{e^{ik_4\tau}}{k_4^2 - K^2}\ dk_4 = -\frac{2\pi}{K}\Theta(\tau)\sin(K\tau), \tag{1.138}$$

where $\Theta(\tau)$ is the Heaviside function. The bottom contour gives rise to a Green's function which vanishes for $\tau < 0$ i.e for $x^4 < x'^4$. It is called the *retarded Green's function* for a source switched on at the spacetime point $(\mathbf{x}', x'^4)$ and will only affect field points at later times. The corresponding contour above the poles is the *advanced Green's function.*

We can finish the calculation of $\mathscr{G}$ by using polar coordinates in $\mathbf{k}$-space with the k_3 axis being parallel to $\mathbf{R}$ Thus,

$$\begin{aligned}
\mathscr{G}(x-x') &= -\frac{1}{(2\pi)^3}\Theta(\tau)\int_0^{2\pi} d\phi \int_0^{\infty} dK \int_0^{\pi} \frac{K^2 \sin(\theta) e^{iKR\cos(\theta)} \sin(K\tau)}{k}\, d\theta \\
&= -\frac{\Theta(\tau)}{2\pi^2 R}\int_0^{\infty} \sin(K\tau)\sin(KR)\, dK \\
&= -\frac{\Theta(\tau)}{2\pi^2 R}\int_0^{\infty} \frac{\left(e^{iK\tau}-e^{iK\tau}\right)}{2i}\frac{\left(e^{iKR}-e^{-iKR}\right)}{2i}\, dK \\
&= \frac{\Theta(\tau)}{4\pi R}\left(\delta(\tau+R)-\delta(\tau-R)\right).
\end{aligned}$$

Now, the whole expression must vanish for $\tau < 0$ due to the Heaviside function, while for $\tau > 0$ we have that $\delta(\tau + R) = 0$. Hence we find that the Green's function may be written as

$$\mathscr{G}(x-x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|), \tag{1.139}$$

and this expression is only non zero on the future light cone of x'. The solution to the inhomogeneous wave equation (1.132) generated by those Green's function is

$$\phi(\mathbf{x},t) = \iiiint \mathscr{G}(x-x')f(x')\, d^4x' = -\frac{1}{4\pi}\iiint \frac{[f(\mathbf{x}',t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \tag{1.140}$$

where the expression $[f(\mathbf{x}',t')]_{\text{ret}}$ means that f is to be evaluated at the *retarded time* given by

$$\tau'_{\text{ret}} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \tag{1.141}$$

where c is the speed of light in a vacuum. Although we shall be dealing with the 1D case (and the Green's function is different for different dimensions), the method given here is essentially the same.

Chapter 2

Hawking Radiation in $2D$ Schwarzschild anti de-Sitter Spacetime

In this chapter we shall be interested in investigation the Hawking radiation associated with a thin collapsing shell in Schwarzschild anti de-Sitter spacetime, and determining the forms of the renormalised energy-stress tensor at both early and late times. We shall calculate the Hawking radiation arising from the collapse of a thin shell in two dimensional Schwarzschild anti de-Sitter spacetime using the 'geometric optics' approach established by Hawking. This method will allow us to establish a relationship between early modes and late time modes, and hence by calculation of the Bogoliubov coefficients we show that particle creation occurs. The investigation of this problem will provide a familiarity with Hawking radiation in a spacetimetime other than Schwarzschild.

Once we have established a relationship between past and future modes we go on to find the renormalised energy stress tensor. We discuss the process of renormalisation, then use the result established by Davies, Fulling and Unruh which allows us to determine $\langle T_{\mu\nu}\rangle_{\text{ren}}$ both at early and at late times.

We shall adopt the following conventions

- Adopt natural units; $G = c = \hbar = 1$,
- Use a metric signature $(-+++)$
- We shall use the abbreviation SADS_2 to mean *Two Dimensional Schwarzschild–anti de Sitter* spacetime.

2.1 Introduction

It was once the widely held belief that the black hole marked the final end point of gravitational collapse. The supermassive stars which underwent this process formed a collapsed region of spacetime which was forever sealed off from the visible Universe by the event horizon. After this, no further processes took place and the black hole was, for all intents and purposes, a dead and lifeless object.

The idea that black holes were the final evolutioanry dead end for supermassive stars was first challenged by J. D Bekenstein [21]. Bekenstein highlighted the number of similarities between black-hole physics and thermodynamics. in particular, the similarity in the behaviors of black-hole area and of entropy. Bekenstein's argued that objects in the Universe cannot just obey the laws of general relativity, they must also obey all the other rules of physics too, including the second law of thermodynamics. The second law of thermodynamics tells us that the entropy of a closed system always increases. If we identify the event horizon of a black hole as a measure of its entropy, this implied that black holes must have a temperature, and hence they must radiate. As we observed in Chapter 1, Hawking- in his paper concerned with particle creation by black holes [10] - went on to show that the temperature of the black is given by the expression

$$T = \frac{\kappa\hbar}{2\pi k c},$$

where κ is the surface gravity of the black hole, $\hbar = h/2\pi$ where h is Plank's Constant, k is the Boltzmann constant and c is the speed of light in a vacuum. This radiation emitted by black holes is now called *Hawking radiation* and does not depend upon the nature of the collapse.

Soon after this result was announced, the Hawking effect was studied from many different aspects. Boulware's approach was to investigate quantized scalar and Dirac fields around a thin collapsing shell [22] in Schwarzschild spacetime using the Kruskal coordinates. Boulware suggested that the Hawking radiation was result of the collapse process and involved the emissions of pairs of particles which had come from either side of the event horizon. Boulware further demonstrated that any back reaction on the shell would react entirely through the metric and was able to calculate the contributions the radiation made to the energy-stress tensor.

It was not long before the Hawking process was applied to collapse scenarios in other spacetimes. The first alternative spacetime to undergo such investigations was the de Sitter spacetime. The de Sitter spacetime is the maximally entended symmetric solution of the vacuum Einstein field equations with a positive cosmological constant, Λ. Gibbons and Hawking showed that the Killing horizon within de Sitter spacetime has the same quantum properties as a black hole: namely entropy and temperature. They found the temperature was determined by the relationship

$$T_{\text{ds}} = \frac{1}{2\pi l}, \tag{2.1}$$

where l is the curvature radius. In his paper '*Adventures in de Sitter space*' [23], Busso gives a concise summary of the work undertaken by Hawking and Gibbons on the de Sitter manifold.

By this point in the subject's history, all of the spacetimes investigated had been globally hyperbolic manifolds. There is good reason for this: a spacetime which is globally hyperbolic posses a family of Cauchy surfaces and prohibits the existence of closed timelike curves (causality conditions). Thus, everything that happens in such a spacetime can be determined by the equations of motion and some initial boundary data on some specific Cauchy surface. Avis *et al* [24] broke with tradition by considering quantum field theory in anti-de Sitter spacetime a manifold which is not globally hyperbolic. By investigating conformally-coupled massless scalar fields, and the covering spaces of anti-de Sitter (which is conformal to part of the Einstien static Universe), Avis *et al* were able to give three quantization schemes which came in two varieties: one scheme was concerned with transparent boundary conditions (i.e. particles in a transparent box), the other two were concerned with reflective boundary conditions.

It should be noted that undertaking quantum field theory calculations in curved spacetime is far from trivial and many subtleties and difficulties soon occur; a particular problem which shall concern us here is the calculation of the expectation value $\langle 0|T_{\mu\nu}|0\rangle$. The methods for removing divergences from quantum field theory in flat spacetime- for example renormalizing the zero-point energy by an infinite amount, or the use of a ultraviolet regulator function $e^{\alpha|k|}$ cannot be employed since (as we shall see later on when we discuss renormalisation schemes) energy plays an important role in spacetime curvature; energy gravitates and so it cannot be easily discarded. There are a number of elaborate renormalisation schemes in existence.

One approach in dealing with the problem of divergences is to consider the semi-classical field equations:

$$G_{\mu\nu} = -8\pi\langle T_{\mu\nu}\rangle_{\mathrm{ren}},$$

where the stress-energy tensor has been replaced with its renormalised expectation value. This semi-classical approach gives a convincing model of Hawking radiation and provides a method for calculating the back reaction to Hawking radiation. Unfortunately, the calculaton of $\langle T_{\mu\nu}\rangle_{\rm ren}$ is also problematic. For example Brown[25] calculated the stress tensor expectation value of a massive scalar field coupled to an arbitrary classical gravitational field, $\langle T_{\mu\nu}\rangle_{\rm ren}$ was not conformally invariant: its trace contained additional terms which gave rise to the so-called *trace anomaly*.

Davies *et al* [17] and their book *Quantum fields in Curved Space* [26], Birrell and Davies give some reliable methods for calculating the expectation value of the renormalised stress tensor and examine the conformal anomalies for the massless case. In particular, they give a method for calculating $\langle T_{\mu\nu}\rangle_{\rm ren}$ using the Davies Fulling Unruh (DFU) derivative. In this procedure, a 2D spacetime metric is written in the conformal form

$$ds^2 = C(r)\; dudv,$$

where $C(r)$ is the conformal factor and u and v are null coordimates. The components of the renoramlised stress tensor are then computed from the DFU derivative.

Adopting this approach, Tadaki and Takagi [27, 28] demonstrated that by considering a quantized conformal scalar field in 2D asymptotically flat de Sitter spacetime with a collapsing star, the stress tensor in a given quantum state is regular if this quantum state is to be the vacuum state associated with the basis modes defined using the global null coordinate system described above.

More recent investigations of black hole evaporation by Saida *et al* [29] use the DFU method to calculate the radiation power of black holes which are asymptotic to the Einstein Static Universe at spatial and null infinities.

We shall adopt the approach discussed in Birrell and Davies. We start by examining the structure of 2D Schwarzschild anti-de Sitter Spacetime.

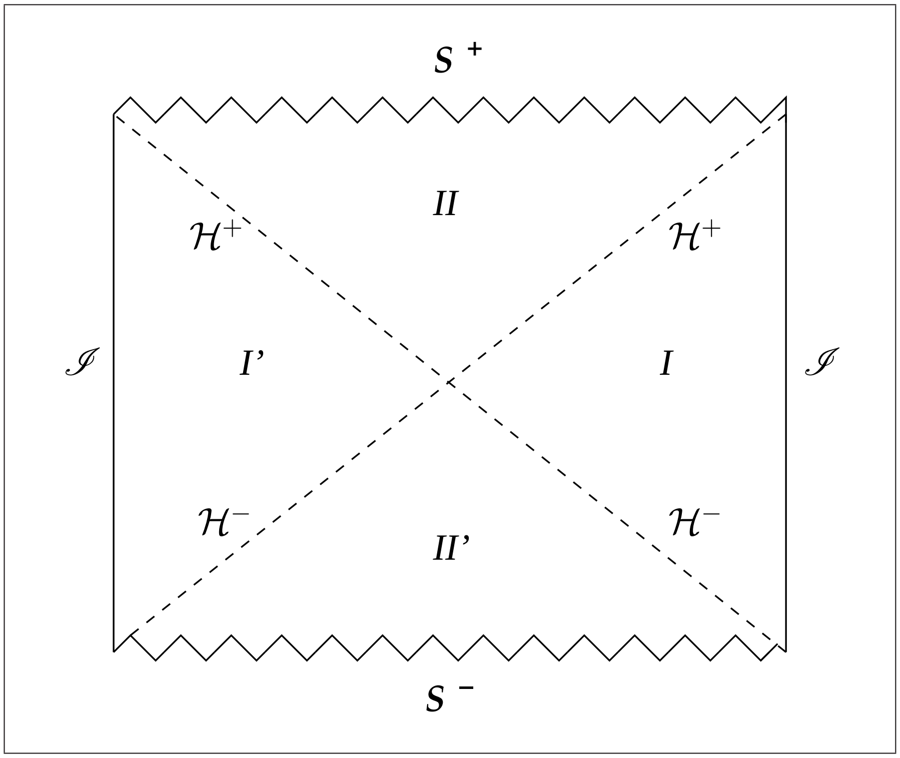

Figure 2.1: Penrose diagram showing the structure of Schwarzschild $2D$ anti de-Sitter.

2.2 The Structure of Two Dimensional Schwarzschild anti-de Sitter Spacetime.

We shall use this section to examine the structure of SADS_2. The metric for this spacetime has the form

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2, \tag{2.2}$$

where M is the mass of the black hole and Λ is the cosmological constant, strictly negative. In figure 3.1 we have a Penrose diagram which shows the structure of SADS_2which is non-globally hyperbolic. [30, 31] On the diagram we have indicated the following:

- $\mathcal{H}^{\pm}$ *(dashed lines)*: These indicate future/past black hole event horizons
- $\mathcal{S}^{\pm}$ *(zig-zag lines)*: Future/past singularities.

We note also that *Region I* covers the region outside the event horizon, and *Region I'* is a copy of *Region I. Regions II and II'* contain the spacelike singularities $\mathcal{S}^+$ and $\mathcal{S}^-$.

Another interesting feature of $SADS_2$ is that there is no null surface $\mathscr{I}^-$ (past time like infinities) or $\mathscr{I}^+$ (future timelike infinities) where null geodesics would normally start and finish, there is only one timelike infinity $\mathscr{I}$.

We can of course write the metric as a conformally flat one:

$$ds^2 = -C(r)(dt^2 - dr_*^2), \tag{2.3}$$

where the conformal factor

$$C(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}. \tag{2.4}$$

There are no cosmological horizons: if we let $C(r) = 0$, then

$$r - 2M - \frac{\Lambda r^3}{3} = 0. \tag{2.5}$$

The equation (2.5) has just one real root solution when $\Lambda < 0$, and hence we have only the event horizon. Furthermore we have that

$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1}, \tag{2.6}$$

then clearly

$$r_* = \int \frac{dr}{C(r)}. \tag{2.7}$$

We shall now write:

$$1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} = (r - r_h)h(r), \tag{2.8}$$

where the event horizon is located at the position $r = r_h$, and we have that the function

$$h(r) = \frac{x}{r} + y + zr, \tag{2.9}$$

where the quantities x, y and z have the forms:

$$x = \frac{2M}{r_h}, \ y = zr_h, \ \text{ and } \ z = -\frac{\Lambda}{3}. \tag{2.10}$$

We can define a set of null coordinates (u, v) with a boundary condition on $\mathscr{I}$:

$$u = t - r_*, \ v = t + r_*; \ r_* = 0 \text{ on } \mathscr{I}, \tag{2.11}$$

and in terms of these advanced and null coordinates, the metric becomes

$$ds^2 = -C(r)\ du\, dv \tag{2.12}$$

where the function $C(r)$ is given in equation (2.4) and is infinitely differentiable and continuous everywhere except at $r = 0$.

Throughout this chapter we shall be working with a massless scalar field which satisfies the wave equation:

$$\Box\phi = 0 \tag{2.13}$$

where, the $\Box$ represents the usual D'Alambertian operator

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu. \tag{2.14}$$

The solution to equation (2.13) are plane wave modes, for which we can define *early time* modes of pure frequency ω':

$$u^{\text{early}}_{\omega'} = \frac{1}{\sqrt{4\pi\omega'}}\left(e^{-i\omega' u} - e^{-i\omega' v}\right), \tag{2.15}$$

where u and v are the advanced and retarded null coordinates defined in equation (2.11). Similarly, we can define *late-time* modes of pure frequency ω;

$$u^{\text{late}}_{\omega} = \frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega U} - e^{-i\omega V}\right), \tag{2.16}$$

where U and V are the advanced and retarded null coordinates at late times

$$U = \overline{T} + R_*, \quad V = \overline{T} - R_*. \tag{2.17}$$

We shall need to impose suitable boundary conditions with the wave equation of (2.13) as SADS_2 is not a globally hyperbolic spacetime. We shall impose *reflective boundary conditions*:

$$\phi = 0 \quad \text{at} \quad r_* = 0, \tag{2.18}$$

which is simply a reflection at null infinity. We can now define a meaningful inner product which is not dependent on any particular choice of hypersurface. It is of course the non-hyperbolicity of $SADS_2$ that has dictated the nomenclature of the two types of modes. As we saw in Chapter 1, in the case of Schwarzschild collapse (where we have $\mathscr{I}^+$ and $\mathscr{I}^-$) we can define in-going and out-going modes. However in $SADS_2$ we just have $\mathscr{I}$ and so the modes are a mixture of ingoing and outgoing modes, so a better distinction will be modes at early times and modes at late times. At early times we shall have that

$$f \sim e^{-i\omega v},$$

and at late times:

$$F \sim e^{-i\omega' g(V)}.$$

Later on, we shall require an expression for the Ricci-scalar, R. We find that, for $SADS_2$

$$R = -\frac{2}{3}\,\frac{\Lambda r^3 + 6M}{r^3}. \tag{2.19}$$

It can be seen that if $\Lambda = 0$, we recover the two-dimensional Schawzrschild Ricci scalar:

$$R = -\frac{4M}{r^3}.$$

One more quantity which will be useful to us later on is the *surface gravity.* This is easily determined from the expression:

$$\kappa = \frac{1}{2}\frac{dC}{dr}\bigg|_{r=r_h} = \frac{M}{r_h^2} - \frac{\Lambda r_h}{3}. \tag{2.20}$$

2.3 Collapse Scenario

We shall now consider the case of a thin uniform collapsing shell in $SADS_2$. Inside the shell the spacetime is simply anti de-Sitter (ADS_2). We shall essentially follow the same argument as given by Ford which we discussed in Chapter 1.

Consider now figure 3.2. Here we have a Penrose diagram which illustrates the collapse scenario (this type of illustration made popular thanks to J. Maladacena). We notice that unlike in the asymptotically flat space, we have no $\mathscr{I}^-$ and no $\mathscr{I}^+$, an incoming ray passes through the collapse and is reflected back as a u ray. Furthermore, we have to arbitrarily choose the start of the collapse which we have done, this is the Cauchy surface $\Sigma_t = 0$ in figure 3.2.

We start by considering now the three rays: v_0, v_1 and v_2. From figure 3.2 it can be seen that:

- The ray v_0 is the ray which passes through the centre of the collapse and then emerges and travels along the event horizon $\mathcal{H}$.
- All rays which enter the collapsing matter before the ray v_0 will pass through the centre of the collapse and back out through the shell.
- The ray v_1 passes through the collapsing shell and emerges as the future ray u_1.
- Any ray entering the shell after v_0 (for example, the ray v_2) will pass through the shell and is doomed eventually to hit the singularity which has formed at the centre of the collapse. Such rays cannot emerge in the future.
- The position of the shell at any time t is given by $R = R(t)$; the ray v_0 enters the shell at point R_1, passes through the collapse and emerges at R_2 along the future event horizon.

Since any ray entering the shell after ray v_0 will hit the singularity and never emerge in the future, we shall only be interested in all rays $v > v_0$.

We now have three regions to consider: outside the shell, on the shell and inside the shell, and we shall consider these three regions separately.

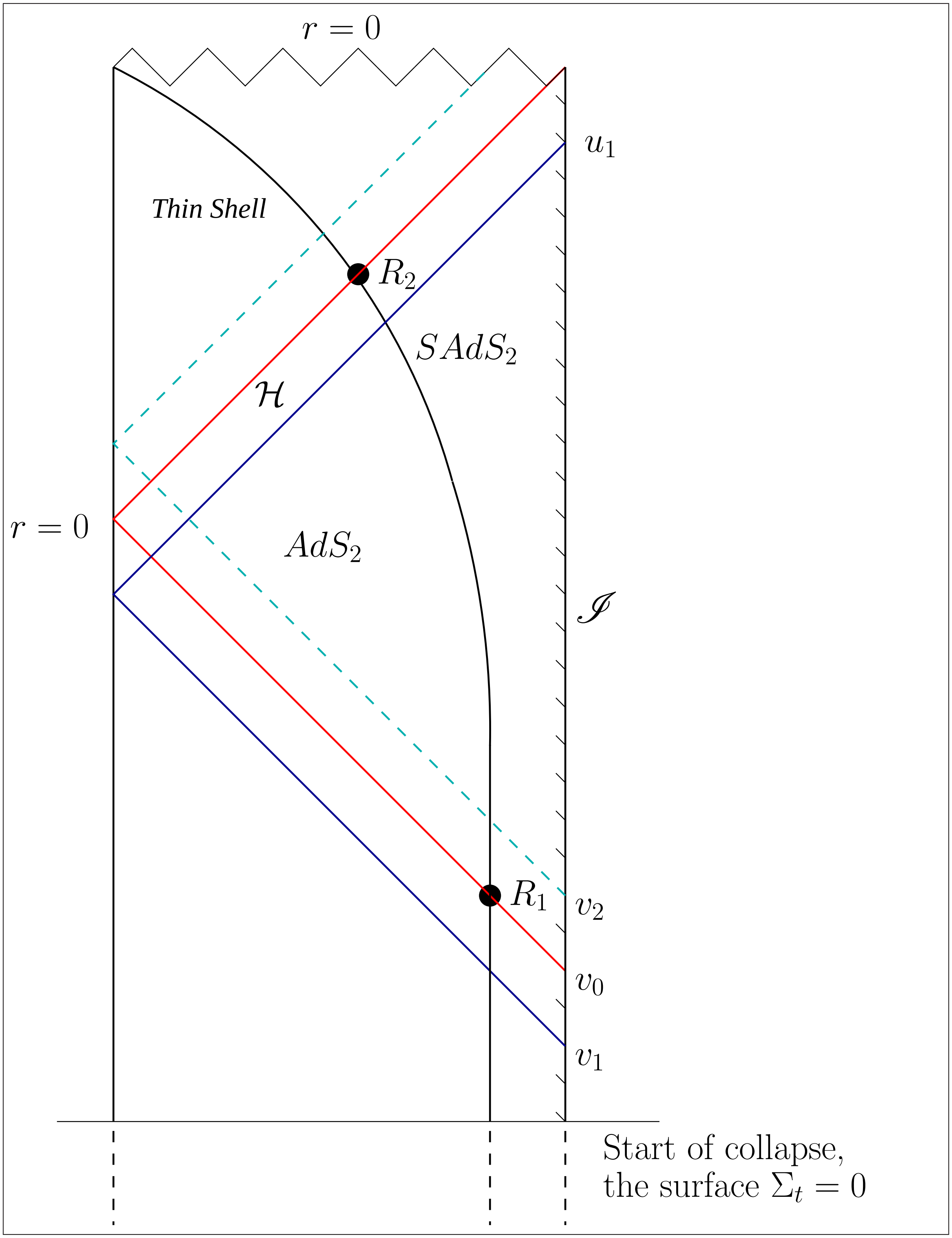

Figure 2.2: Penrose diagram showing the collapse Scenario.

Outside the Shell This is the $SADS_2$ region outside the thin collapsing shell. It is covered by the advanced and retarded null coordinates:

$$u = t - r_*, \ v = t + r_* \tag{2.21}$$

where t is the coordinate time, and the r_* coordinate is given by the expression:

$$r_* = \int \frac{dr}{(r - r_h)h(r)}. \tag{2.22}$$

Inside the Shell The interior of the shell is two dimensional anti de-Sitter, and this has the metric:

$$ds_I^2 = -\left(1 - \frac{\Lambda \overline{r}^2}{3}\right) dT^2 + \left(1 - \frac{\Lambda \overline{r}^2}{3}\right)^{-1} d\overline{r}^2 \tag{2.23}$$

where T and $\overline{r}$ are the coordinate time and radial coordinate respectively for ADS_2. We note that both T and $\overline{r}$ are quite distinct from the t and r. As we did for $SADS_2$, we can define an advanced and retarded null coordinate for the interior. Let

$$\overline{U} = T - \overline{r_*}, \quad \overline{V} = T + \overline{r_*} \tag{2.24}$$

and we also have that

$$\frac{d\overline{r_*}}{dr_*} = -\left(1 - \frac{\Lambda \overline{r}^2}{3}\right)^{-1}. \tag{2.25}$$

On the Shell On the surface of the collapsing shell we have that $r = \overline{r} = R(t)$, where as we defined earlier, $R(t)$ is the position of the shell at any particular coordinate time t.

Now that we have the spacetime structure of the whole scenario established, let us consider the collapse in detail.

2.4 The Ray-Tracing Process

Following the method as outlined by Ford, we shall now use the ray-tracing process to establish a relationship between the early-time and late-time modes. We do this by using the fact that we require the interior metric of the shell to match with the exterior metric across the shell, thus equating the interior metric with the exterior one we have that

$$-\left(1-\frac{\Lambda R^2}{3}\right)+\left(1-\frac{\Lambda R^2}{3}\right)^{-1}\left(\frac{dR}{dT}\right)^2=-(R-r_h)h(R)\left(\frac{dt}{dT}\right)^2+\frac{1}{(R-r_h)h(R)}\left(\frac{dR}{dT}\right)^2. \tag{2.26}$$

From figure 2.3 we observe that a ray which enters the shell at R_1 passes through the ADS_2 region and exits in the future at position R_2. Since the shell is collapsing it must be the case that $R_1 > R_2$. The figure also shows a small ϵ-neighbourhood situated on the collapsing shell a long way from the event horizon, and a small δ-neighbourhood situated near to the future event horizon.

Consider the small ϵ-neighbourhood on the past event horizon. It is located around the point R_1 and we have that $R_1 >> r_h$. We note two things of importance. Firstly we have that $R_1 - r_h$ is constant, and secondly

$$\frac{dR}{dT} \approx \text{constant}.$$

in this ϵ-neighbourhood. So by (2.26),

$$\frac{dR}{dT} \approx \text{const} \;\Rightarrow\; \frac{dt}{dT} \approx \text{const} \;\Rightarrow\; t \approx qT,$$

for some constant q. Since $\frac{dr_*}{d\overline{r_*}}$ is also approximately constant in our ϵ-neighbourhood, we can write

$$\frac{dr_*}{d\overline{r_*}} = \alpha, \;\Rightarrow\; r_* = \alpha\overline{r_*} + \beta.$$

So in our ϵ-neighbourhood we have that

$$u = t - r_*, \text{and } v = t + r_*$$

and therefore we can say that

$$U = kt - \overline{r_*}, \text{ and } V = Kt + \overline{r_*},$$

for arbitrary constants k and K. Since $r_* = \alpha\overline{r_*} + \beta$, we can say that

$$u = t - \alpha\overline{r_*} - \beta, \text{ and } v = t + \alpha\overline{r_*} + \beta$$

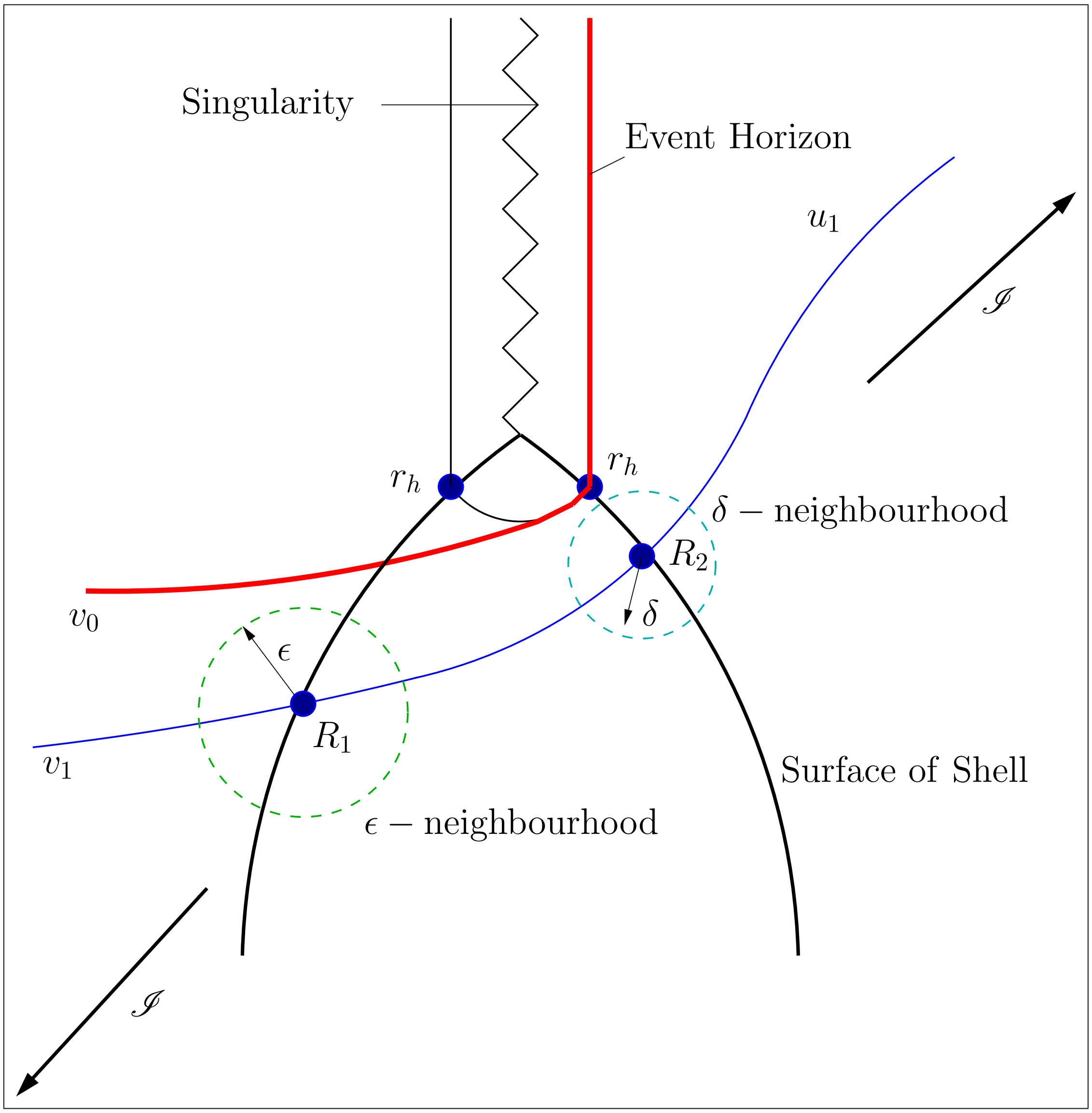

Figure 2.3: Diagram showing the collapse of the shell. The ϵ and δ neighbourhoods are shown.

and hence by some elementary algebra we can write that approximately, in the ϵ-neighbourhood, we have that

$$V = av + b \tag{2.27}$$

for some constants a and b, and similarly

$$U = cu + d, \tag{2.28}$$

again for some choice of constants c and d. Thus we have that $V = g(v)$ and $U = f(u)$ where both the mappings f and g have a simple linear dependence. The ray which started at R_1 passes through the collapse, and since $r = 0$ at the centre of the collapse we have here that $U = V$.

Any ray entering at R_1 exists at R_2. Consider now the δ-neighbourhood in figure 2.3. We shall only be concerned with rays in this neighbourhood which is situated close to the future event horizon located at $R = r_h$. Here rays exit near to point R_2 which is close to the horizon. Since in the δ-neighbourhood

$$\frac{dR}{dt} \approx \text{const}$$

then we may say that

$$R(T) = AT + B, \tag{2.29}$$

where A and B are constants. Now, let us say that at time $T = T_0$, $R = r_h$, so

$$r_h = AT_0 + B, \tag{2.30}$$

and clearly

$$R(T) = r_h + A(T - T_0). \tag{2.31}$$

Re-arranging (2.26) gives

$$(R - r_h)h(R)\left(\frac{dt}{dT}\right)^2 = \left(1 - \frac{\Lambda R^2}{3}\right) - \left(1 - \frac{\Lambda R^2}{3}\right)^{-1}\left(\frac{dR}{dT}\right)^2 + \frac{1}{(R - r_h)h(R)}\left(\frac{dR}{dT}\right)^2$$

So, using (2.29), and some further rearrangement, we have that:

$$\left(\frac{dt}{dT}\right)^2 = \frac{1}{(R-r_h)h(R)}\left(1-\frac{\Lambda R^2}{3}\right) - \left(1-\frac{\Lambda R^2}{3}\right)^{-1}\frac{A^2}{(R-r_h)h(R)} + \underbrace{\frac{A^2}{(R-r_h)^2h^2(R)}}_{*}. \tag{2.32}$$

Now, only the quadratic term (*) makes any significant contribution so we neglect the linear terms and hence

$$\left(\frac{dt}{dT}\right)^2 \approx \frac{A^2}{(R-r_h)^2h^2(r_h)},$$

By expression (2.31),

$$\left(\frac{dt}{dT}\right)^2 \approx \frac{A^2}{A^2(T-T_0)^2h(r_h)^2}.$$

Taking the negative square root because for $T < T_0$ we want R to be positive, we obtain that

$$\frac{dt}{dT} \approx -\frac{1}{(T-T_0)h(r_h)}.$$

Integrating both sides, we have approximately in the δ-neighbourhood that:

$$t = -\frac{1}{h(r_h)}\ln\left|\frac{T-T_0}{\xi}\right|, \tag{2.33}$$

for some constant ξ. We can find the corresponding expression for r_* from (2.22). We have that

$$r_* = \frac{1}{h(r_h)}\int\frac{dR}{R-r_h} = \frac{1}{h(r_h)}\ln\left|\frac{R-r_h}{\xi'}\right|,$$

for some arbitrary constant $\xi'$and hence

$$r_* = \frac{1}{h(r_h)}\ln\left|\frac{A(T-T_0)}{\xi'}\right|. \tag{2.34}$$

(again this is approximately true in the δ-neighbourhood). Since we have defined $u = t - r_*$, we have by (2.33) and (2.34) that

$$u = -\frac{2}{h(r_h)}\ln\left|\frac{T_0-T}{\eta}\right|. \tag{2.35}$$

for some constant η. Now since in the limit $T_0 \approx T$, $U = T - R(t)$, then we must have that

$$A(T_0 - T) = U - T + r_h,$$

which means that we can write

$$u \approx -\frac{2}{h(r_h)} \ln \left| \frac{U - T + r_h}{\eta'} \right| ,$$

(η' is another constant) and so we arrive at:

$$u \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - U}{\mu'} \right| \tag{2.36}$$

where μ' is an arbitrary constant. As we pass through the origin, we have imposed reflective boundary conditions which means $U = V$ and so

$$u \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - V}{\mu'} \right| .$$

and so when we trace along the v-ray we have

$$u \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - (av + b)}{\mu'} \right| . \tag{2.37}$$

Now, on the horizon the argument of the logarithm must vanish, and so

$$T - r_h = av_0 + b.$$

Finally, we arrive at an expression for U in terms of v, namely that

$$U = -\frac{2}{h(r_h)} \ln \left| \frac{v - v_0}{\mu} \right| , \tag{2.38}$$

for some constant μ.

Similarly, following the same argument that we have just used. Figure 2.4 shows an incoming u-ray emerging as an outgoing V-ray. The ray u_0 travels along the horizon. Since $v = t + r_*$, then by (2.33) and (2.34) we can write

$$v = -\frac{2}{h(r_h)} \ln \left| \frac{T_0 - T}{\eta} \right| .$$

Again $A(T - T_0) = R - r_h$, and at $T_0 \approx T$, we have that $V = T + R$ and so we can say that

$$v \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - V}{\mu'} \right| .$$

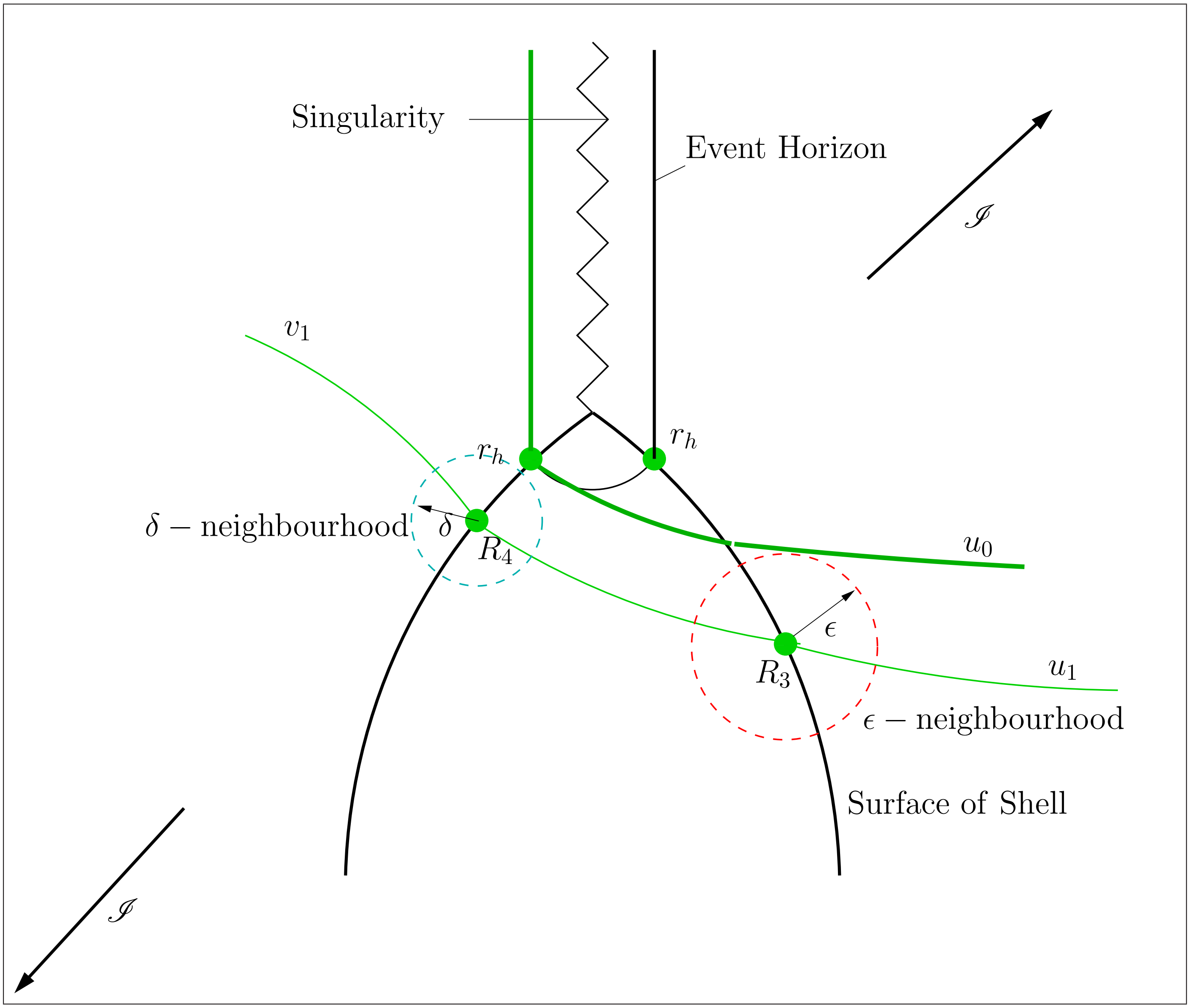

Figure 2.4: Diagram showing the collapse of the shell, this time showing an incoming u_1 (entering the shell at position R_3) which passes through the collapse and emerges as a v_1-ray (at point R_4). The ray u_0 travels along the horizon. Also shown are the corresponding ϵ and δ neighbourhoods.

When we pass through the collapse, $U = V$ and so,

$$v \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - U}{\mu'} \right|,$$

and so tracing along the u-ray:

$$v \approx -\frac{2}{h(r_h)} \ln \left| \frac{T - r_h - (cu + d)}{\mu'} \right|. \tag{2.39}$$

Again on the horizon, the argument of the logarithm vanishes: so

$$T - r_h = cu_0 + d$$

and hence we have that

$$V = -\frac{2}{h(r_h)} \ln \left| \frac{u - u_0}{\mu} \right|. \tag{2.40}$$

We have now established a relationship between early-time modes u and v and late time modes U and V. We see that the ray tracing process has allowed us to write $U = f(v)$ and $V = g(u)$.

2.5 Bogoliubov Coefficients

We shall now calculate the Bogoliubov coefficients for the system. As we observed in Chapter 1, if we have in-going modes of pure frequency ω' and outgoing modes of pure frequency ω we can write one set of modes in terms of the other set via a Fourier transform. Once derived, the Bogoliubov coefficients can be used to determine the Hawking temperature of the black hole.

As we stated earlier, we have early-time modes of pure frequency ω' of the form

$$f_{\omega'}^{\text{early}} = \frac{1}{\sqrt{4\pi\omega'}} \left[e^{-i\omega' u} - e^{-i\omega' v} \right], \tag{2.41}$$

and late-time modes which are of pure frequency ω, and have the form

$$F_{\omega}^{\text{late}} = \frac{1}{\sqrt{4\pi\omega}} \left[e^{-i\omega U} - e^{-i\omega V} \right]. \tag{2.42}$$

From the Hawking ray-tracing method in the previous chapter, we have derived that, for $v < v_0$

$$V = -\frac{2}{h(r_h)} \ln\left|\frac{v_0 - v}{\mu}\right| \quad \text{and} \quad U = -\frac{2}{h(r_h)} \ln\left|\frac{v - v_0}{\mu}\right|$$

for some arbitrary constant μ. We shall now use the Fourier transform result of Chapter 1, where by the F_ω^{late} modes are written in terms of the Fourier transform of the $f_{\omega'}^{\text{early}}$ modes, i.e.

$$F_\omega^{\text{late}} = \int_0^\infty \left(\alpha^*_{\omega'\omega} f_{\omega'}^{\text{early}} - \beta_{\omega'\omega} f_{\omega'}^{\text{early}^*}\right) \, d\omega', \tag{2.43}$$

to calculate the Bogoliubov coefficients $\alpha_{\omega\omega}$ and $\beta_{\omega'\omega}$. We can calculate the coefficent $\alpha_{\omega'\omega}$ by taking the Klein–Gordon inner product of $f_{\omega'}^{\text{early}}$ with (2.43), i.e:

$$\alpha_{\omega',\omega} = (f_{\omega'}^{\text{early}}, F_\omega^{\text{late}}), \tag{2.44}$$

and similarly for $\beta_{\omega'\omega}$ we have that

$$\beta_{\omega'\omega} = -(f_{\omega'}^{\text{early}}, F_\omega^{\text{late}^*}), \tag{2.45}$$

where the Klein-gordon inner product between two vectors ϕ_1 and ϕ_2 is defined to be

$$(\phi_1, \phi_2) = -i \int_\Sigma \phi_1(x) \overleftrightarrow{\partial}_\mu \phi_2(x)^* \sqrt{-g_\Sigma(x)} \; \eta^\mu \; d\Sigma \tag{2.46}$$

and η^μ is the timelike unit vector normal to the Cauchy hypersurface Σ and $d\Sigma$ is the volume element of the Cauchy hypersurface. We note that since our two sets of solutions f^{early} and F^{late} are orthonormal, we must have that

$$\begin{aligned}
(f_\omega^{\text{early}}, f_{\omega'}^{\text{early}}) &= (F_\omega^{\text{late}}, F_{\omega'}^{\text{late}}) = \delta_{\omega\omega'}, \\
(f_\omega^{\text{early}^*}, f_{\omega'}^{\text{early}^*}) &= (F_\omega^{\text{late}^*}, F_{\omega'}^{\text{late}^*}) = -\delta_{\omega\omega'} \text{ and} \\
(f_\omega^{\text{early}}, f_{\omega'}^{\text{early}^*}) &= (F_\omega^{\text{late}}, F_{\omega'}^{\text{late}^*}) = 0
\end{aligned} \tag{2.47}$$

We shall need to choose our hypersurface Σ. Consider again figure 3.2. For early time modes, we are interested in all the modes for which $v < v_0$ (as anything entering the hole after the ray v_0 will hit the singularity), so we form the inner-product given in equation (2.44) and we obtain that

$$\alpha^*_{\omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left|\frac{v_0 - v}{\mu}\right|\right\} \exp\{i\omega' v \; dv\} \; dv, \tag{2.48}$$

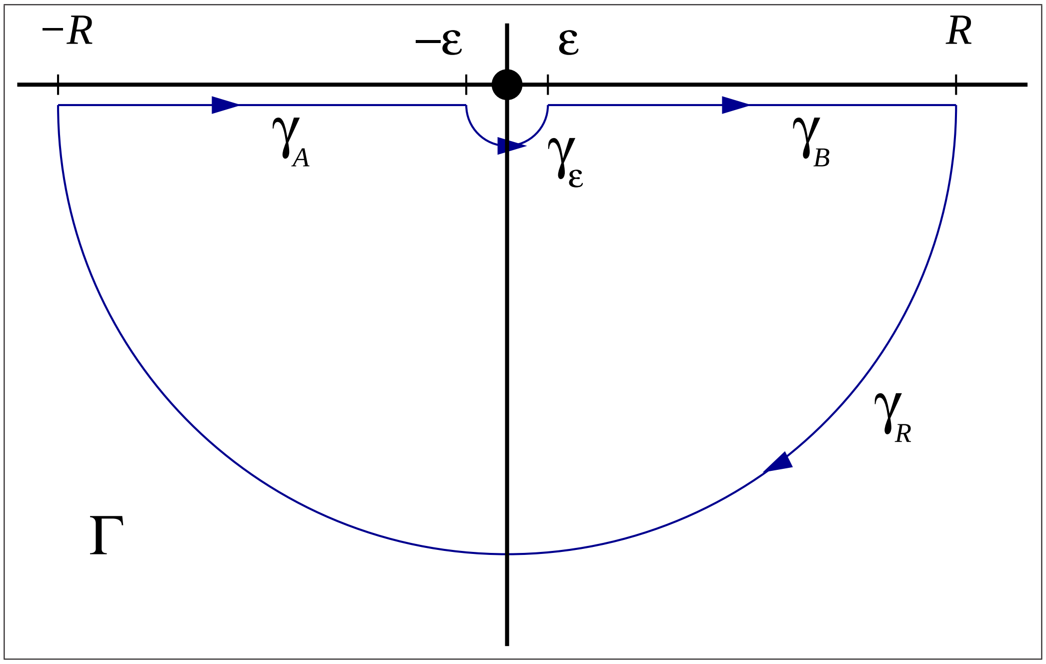

Figure 2.5: Integration over the closed contour Γ. The path Γ itself is the sum of paths γ_A, γ_ϵ, γ_B and γ_R

and correspondingly

$$\beta_{\omega'\omega} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{v_0} \exp\left\{\frac{2i\omega}{h(r_h)}\ln\left|\frac{v_0 - v}{\mu}\right|\right\}\exp\left\{-i\omega' v\right\}\ dv. \tag{2.49}$$

If we make the substitution

$$v' = v_0 - v,$$

then we obtain the Bogoliubov coefficients in the more useful form of

$$\alpha^*_{\omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}e^{i\omega' v_0}\int_0^{\infty} \exp\left\{\frac{2i\omega}{h(r_h)}\ln\left|\frac{v'}{\mu}\right|\right\}\exp\{-i\omega' v\}\ dv', \tag{2.50}$$

and

$$\beta_{\omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}e^{-i\omega' v_0}\int_0^{\infty} \exp\left\{\frac{2i\omega}{h(r_h)}\ln\left|\frac{v'}{\mu}\right|\right\}\exp\{i\omega' v\}\ dv'. \tag{2.51}$$

Let us consider the integrand in (2.50). This is a contour integral and the integrand is analytic everywhere except for a branch cut along the negative real axis. Thus we are integrating over a closed path Γ and by Cauchy's integral theorem we must have that

$$\oint_\Gamma \exp\left\{\frac{2i\omega}{h(r_h)}\ln\left(\frac{v'}{\mu}\right)\right\}\exp\{i\omega' v'\}\ dv' = 0. \tag{2.52}$$

The contour Γ is illustrated in figure 2.5. As can be seen we integrate around the path in a clockwise manner traversing the paths γ_A, γ_ϵ, γ_B and γ_R. We shall now consider the integral over each of the separate paths. We start with the arc γ_R. Let

$$v' = Re^{i\theta}, \quad \text{for } \theta \in (-\pi, 0).$$

So, now

$$\oint_{\gamma_R} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\ dv' = \int_{-\pi}^{0} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{Re^{i\theta}}{\mu}\right)\right\} e^{-i\omega' Re^{i\theta}} iRe^{i\theta}\ d\theta.$$

Now, we know that

$$\ln\left(Re^{i\theta}\right) = \ln(R) + i\theta, \tag{2.53}$$

thus

$$\begin{aligned}\oint_{\gamma_R} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\ dv' &= i\int_{-\pi}^{0} R\exp\left\{\frac{2i\omega}{h(r_h)}\left[\ln\left(\frac{R}{\mu}\right) + i\theta\right]\right\} e^{-i\omega' Re^{i\theta}} e^{i\theta}\ d\theta \\ &= i\int_{-\pi}^{0} Re^{-2\theta\omega/h(r_h)} e^{\frac{2i\omega}{h(r_h)}\ln\left(\frac{R}{\mu}\right)} e^{-i\omega' R\cos(\theta)} e^{\omega R\sin(\theta)} e^{i\theta}\ d\theta.\end{aligned}$$

If we take the limit $R \to \infty$, then

$$\int \mathcal{O}\left(\exp\{R\omega' \sin(\theta)\}\right)\ dv' \to 0,$$

thus the above integrand is zero, and hence we have that

$$\oint_{\gamma_R} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\ dv' = 0. \tag{2.54}$$

Now we shall consider the path γ_ϵ, which is the path around the semicircle contour of radius ϵ. We shall let

$$v' = \epsilon e^{i\theta}, \text{ for } \theta \in (0, \pi)$$

and upon subsituting this into the integrand given in (2.52) we have that

$$\oint_{\gamma_\epsilon} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\ dv' = \int_{0}^{\pi} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{\epsilon e^{i\theta}}{\mu}\right)\right\} e^{-i\omega' \epsilon e^{i\theta}} \epsilon e^{i\theta}\ d\theta.$$

Using (2.53) we can write

$$\oint_{\gamma_\epsilon} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\ dv' = i\int_{0}^{\pi} \epsilon e^{-\theta\omega/h(r_h)} e^{\frac{2i\omega}{h(r_h)}\ln\left(\frac{\epsilon}{\mu}\right)} e^{-i\epsilon\omega' \cos(\theta)} e^{\epsilon\sin(\theta)\omega'} e^{i\theta}\ d\theta.$$

We now let $\epsilon \to 0$, and we find that once more, the integrand is zero, thus

$$\oint_{\gamma_\epsilon} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = 0. \tag{2.55}$$

We now consider the integrand along the path γ_A. Here we must have that

$$\oint_{\gamma_A} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = \int_{-L}^{-\epsilon} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(-\frac{v'}{\epsilon} - i\epsilon\right)\right\} e^{-i\omega'(-v'-i\epsilon)}\, dv'. \tag{2.56}$$

We know that

$$\ln\left(-\frac{v'}{\mu} - i\epsilon\right) = -i\pi + \ln\left(\frac{v'}{\mu}\right), \tag{2.57}$$

so now

$$\oint_{\gamma_A} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = \int_{-L}^{-\epsilon} \exp\left\{\frac{2i\omega}{h(r_h)} \left(-i\pi + \ln\left(\frac{v'}{\mu}\right)\right)\right\} e^{-i\omega'(-v-i\epsilon)}\, dv'.$$

Now, we take the limit $\epsilon \to 0$ and $L \to \infty$, and we have that

$$\oint_{\gamma_A} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = -e^{\frac{2\pi\omega}{h(r_h)}} \int_0^{\infty} \exp\left\{\frac{i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{i\omega' v'}\, dv'. \tag{2.58}$$

Finally, we have the integrand along the path γ_B:

$$\oint_{\gamma_B} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = \int_{\epsilon}^{L} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv'. \tag{2.59}$$

Using a similar argument as we did for integration around the path γ_A, and on letting $\epsilon \to 0$ and $L \to \infty$, we find that

$$\oint_{\gamma_B} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv' = \int_{0}^{\infty} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left(\frac{v'}{\mu}\right)\right\} e^{-i\omega' v'}\, dv'. \tag{2.60}$$

Now, comparing (2.60) and (2.58), it is clear that we have:

$$\int_0^{\infty} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left|\frac{v'}{\mu}\right|\right\} e^{-i\omega' v}\, dv' = -e^{\frac{2\omega\pi}{h(r_h)}} \int_0^{\infty} \exp\left\{\frac{2i\omega}{h(r_h)} \ln\left|\frac{v'}{\mu}\right|\right\} e^{i\omega' v}\, dv', \tag{2.61}$$

and so

$$|\alpha_{\omega'\omega}| = e^{2\pi\omega/h(r_h)}|\beta_{\omega'\omega}|. \tag{2.62}$$

If we square both sides of the above equation we have

$$|\alpha_{\omega'\omega}|^2 = e^{4\pi\omega/h(r_h)}|\beta_{\omega'\omega}|^2.$$

Now since the Bogoliubov coefficients satisfy the condition:

$$\sum_{\omega'} \left(|\alpha_{\omega'\omega}|^2 - |\beta_{\omega'\omega}|^2\right) = 1$$

then

$$\sum_{\omega'} \left(e^{4\pi\omega/h(r_h)} - 1\right) |\beta_{\omega'\omega}|^2 = 1$$

and so we have that:

$$\sum_{\omega'} |\beta_{\omega'\omega}|^2 = \frac{1}{e^{4\pi\omega/h(r_h)} - 1}. \tag{2.63}$$

The mean number of particles created into mode ω is

$$\mathcal{N}_\omega = \sum_{\omega'} |\beta_{\omega'\omega}|^2 = \frac{1}{e^{4\pi\omega/h(r_h)} - 1}, \tag{2.64}$$

which gives us a Hawking Temperature of

$$T_H = \frac{h(r_h)}{4\pi}. \tag{2.65}$$

We have an expression for $h(r_h)$. Using (2.9) and (2.10) we may write

$$h(r_h) = \frac{2M}{r_h^2} - \frac{2\Lambda r_h}{3}.$$

Now comparing this expression with the one we have for the surface gravity in equation (2.20) we see that

$$\kappa = \frac{h(r_h)}{2},$$

and hence, in terms of surface gravity, the Hawking temperature is

$$T_H = \frac{\kappa}{2\pi}. \tag{2.66}$$

We have here a net flux of particles which has arisen because of the choise of the vacuum made (which is dictated by the form opf the modes) at the end of section 2.2

2.6 Stress Tensor Renormalization and the DFU Derivative.

In this section we shall be concerned with calculating the renormalised stress tensor, $\langle T_{\mu\nu}\rangle_{\mathtt{ren}}$. In particular we shall need to find this quantity at early and late times as it is not the same in both regions. In early times, $\langle T_{\mu\nu}\rangle_{\texttt{ren}}$ can be found using the so-called DFU derivative, however, this result must be modified for use in later time regions. In this section we shall derive the DFU derivative, and then use it to find $\langle T_{\mu\nu}\rangle_{\texttt{ren}}$ at early and late times in SAdS. First however, it is appropriate to discuss very briefly the problem of renormalization, which will lead us in to the derivation of the DFU result.

2.6.1 Renormalisation of the Stress Tensor.

It is now widely accepted that changing a gravitational field can produce various quanta, and that only in rare circumstances, does the notion of a 'particle' in curved spacetimes bear any resemblance to the physical concept of the subatomic particle. In short, there is no natural definition of a particle in curved spacetime and so particle detectors respond in a variety of ways.

Part of the problem with the particle concept is that it is concerned with *global* properties: it is defined globally in terms of field modes and is therefore connected with the overall structure of spacetime. From this point of view it becomes clear why we should take an interest in objects defined locally: the stress tensor $T_{\mu\nu}(x)$ at some point x.

Of course, the main problem with Quantum Field Theory (QFT) in curved spacetimes are the many divergences which exist within the theory. The expectation value of the Hamiltonian $\mathscr{H}$ in the Minkowski vacuum state is infinite for example. Moreover, the expectation value $\langle 0|T_{\mu\nu}|0\rangle$ becomes ultraviolet- divergent.

In flat space QFT, the usual procedure is simply to "discard" these divergences. We may use normal ordering, or if the topology is non-trivial (but the geometry is still flat) we use the ultraviolet regulator function $e^{-\alpha|k|}$ to cut off the ultraviolet divergences then take the difference between $\langle T_{\mu\nu}\rangle$ in the topology of interest and its cut off value in Minkowski spacetime, letting $\alpha \to 0$ at the end.

Alternatively, we may use the Green function approach. The divergent Minkowski expression for $G^{(1)}(x,x')$ is subtracted from the $G^{(1)}$ function evaluated on the topology of interest, afterwards the limit $x \to x'$ is taken. Essentially in Minkowski space, we are interested in the differences in the expectation values in 2 states.

Unfortunately, these approaches cannot be used reliably when the spacetime is curved:

1. In non-gravity physics only energy differences are observable. Clearly an infinite vacuum energy is unacceptable, so we renoramlize the zero point by an infinite amount. However, if we take gravity into account, energy is a source for gravity and will bring about local curvature in spacetime, we are not therefore, free to rescale the zero point energy.

2. In the case of discarding the Minkowski type terms, we find that we are still left with divergences. For example, if we calculate the $\langle 0|T^0_0|0\rangle$ for the Robertson-Walker spacetime, we should find that the difference between the Robertson-Walker Universe and Minkowski space is still infinite: the divergence remains even when the Minkowski type terms are discarded.

It is clear then, in order to obtain physically meaningful results, we must remove the divergences. However, this can be done in infinitely many ways, so we must impose some criteria to make the solution unique.

In QED we find that infinite subtractions may be carried out to give finite results (which are in good agreement with experiment), provided the subtractions are performed covariantly. We therefore aim to keep general covariance when handling the divergences of $\langle T_{\mu\nu}\rangle_{\mathtt{ren}}$. We could also insist that $\langle T_{\mu\nu}\rangle_{\mathtt{ren}}$ has a number of physical properties. If we impose enough restrictions on $\langle T_{\mu\nu}\rangle_{\mathtt{ren}}$, then the subtraction procedure can be defined uniquely. Indeed, Wald proposes that any physically meaningful $\langle T_{\mu\nu}\rangle$ should satisfy 4 reasonable axioms:

1. Covariant conservation, i.e: $\langle T_\mu{}^\nu\rangle_{\text{ren},\ \nu} = 0$

2. Causalilty condition: for a fixed 'in' state, $\langle T_{\mu\nu}\rangle$ at some point ξ in spacetime depends only on the spacetime geometry to the causal past of ξ.

3. The standard results are obtained for 'off-diagonal' elements: $\langle \mathbf{\Phi}|T_{\mu\nu}|\mathbf{\Psi}\rangle$ is finite for orthogonal states $\langle \mathbf{\Phi}|\mathbf{\Psi}\rangle$ and the value of this quantity should be the formal one.

4. Standard results in Mikowski space: The normal ordering procedure in Minkowski spacetime should be valid.

Wald has since proved that if any $\langle T_{\mu\nu}\rangle$ satisfies the first three of the above conditions then it is unique to within a local conserved tensor.

An Alternative Approach. Another way is to treat the calculation of $\langle T_{\mu\nu}\rangle_{\mathtt{ren}}$ as being part of a wider dynamical theory which involves gravity. We present the derivation from [18] We shall seek a theory based on the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{2.67}$$

but, we replace the stress tensor with the quantum expectation value $\langle T_{\mu\nu}\rangle$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu}\rangle, \tag{2.68}$$

where Λ_B is the *Bare Cosmological Constant*, never observed, and similarly, G_B is the bare gravitational constant. We may derive (2.67) by taking the action

$$S = S_g + S_m, \tag{2.69}$$

with condition

$$\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0. \tag{2.70}$$

The first term, S_g, in (2.69) is the gravitational action

$$\begin{aligned} S_g &= \int_\Sigma \mathscr{L}_g \sqrt{-g}\; d^n x \\ &= \int_\Sigma \frac{\sqrt{-g}}{16\pi G_B}[R - 2\Lambda_B]\; d^n x, \end{aligned}$$

for which $2(-g)^{-1/2}\delta S_g/\delta g^{\mu\nu}$ yields the left hand side of (2.67) while S_m, the classical matter action, for which

$$\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu} \tag{2.71}$$

yields the right hand side of (2.67).

To make this procedure work in the semi-classical case, we seek the *effective action* for the quantum matter fields (W) which when functionally differentiated gives the desired $\langle T_{\mu\nu}\rangle$:

$$\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu}\rangle. \tag{2.72}$$

2.6.2 The DFU Derivative.

We shall now derive the full DFU result which allows the calculation of the renormalized stress tensor. This derivation is based upon the one given by Birell and Davies in their book [26], and is applied in their paper [17]. First, we shall derive a form for the classical action. We shall want this classical action S to be invariant under conformal transformations, i.e.

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) = \overline{g}_{\mu\nu}(x). \tag{2.73}$$

By the definition of the functional derivative we have

$$S[\overline{g}_{\mu\nu}] = S[g_{\mu\nu}] + \int \frac{\delta S[\overline{g}_{\mu\nu}]}{\delta \overline{g}^{\rho\sigma}(x)} \delta \overline{g}^{\rho\sigma}(x)\; d^n x. \tag{2.74}$$

If we use the fact that

$$\delta \overline{g}_{\mu\nu}(x) = -2\overline{g}^{\mu\nu}(x)\Omega^{-1}(x)\delta(x), \tag{2.75}$$

and (2.71), then we have that

$$S[\overline{g}_{\mu\nu}] = S[g_{\mu\nu}] + \int \frac{\sqrt{-\overline{g}}}{2} T_{\rho\sigma} \left[-2\overline{g}^{\rho\sigma}(x)\Omega^{-1}(x)\delta\Omega(x) \right] \, d^n x,$$

and thus

$$S[\overline{g}_{\mu\nu}] = S[g_{\mu\nu}] - \int \sqrt{-\overline{g}} T^{\rho}_{\rho}[\overline{g}_{\mu\nu}(x)]\Omega^{-1}(x)\delta\Omega(x) \, d^n x, \tag{2.76}$$

where we have that

$$T^{\rho}_{\rho}[\overline{g}_{\mu\nu}(x)] = \frac{-\Omega(x)}{\sqrt{-g(x)}} \frac{\delta S[\overline{g}_{\mu\nu}]}{\delta\Omega(x)} \bigg|_{\Omega=1}. \tag{2.77}$$

We can now use (2.76) directly to determine a form for the renormalised stress tensor. First we replace the classical action S in (2.76) with the renormalised one-loop effective action W_{ren}:

$$W_{\text{ren}}[\overline{g}_{\mu\nu}] = W_{\text{ren}}[g_{\mu\nu}] - \int \sqrt{-\overline{g}(x)} \, \langle T^{\rho}_{\rho}[\overline{g}_{\mu\nu}(x)] \rangle_{\text{ren}} \, \Omega^{-1}(x)\delta\Omega(x) \, d^n x. \tag{2.78}$$

Now, we have that

$$\overline{g}^{\nu\sigma} \frac{\delta}{\delta \overline{g}^{\mu\sigma}} = g^{\nu\sigma} \frac{\delta}{\delta g^{\mu\sigma}}, \tag{2.79}$$

and by using (2.72), we obtain

$$\frac{\delta W_{\text{ren}}}{\delta \overline{g}^{\mu\nu}} = \frac{\delta W_{\text{ren}}}{\delta \overline{g}^{\mu\nu}} - \frac{\delta}{\delta \overline{g}^{\mu\nu}} \left\{ \int \sqrt{-\overline{g}(x)} \, \langle T^{\rho}_{\rho} \left[\overline{g}_{\mu\nu}(x) \right] \rangle_{\text{ren}} \, \Omega^{-1}(x)\delta\Omega(x) \, d^n x \right\},$$

and hence

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{2}{\sqrt{-\overline{g}}} \frac{\delta W_{\text{ren}}[g_{\mu\nu}]}{\delta \overline{g}^{\mu\nu}} - \frac{2}{\sqrt{-\overline{g}}} \frac{\delta}{\delta \overline{g}^{\mu\nu}} \left\{ \int \sqrt{-\overline{g}(x)} \, \langle T^{\rho}_{\rho} \left[\overline{g}_{\mu\nu}(x) \right] \rangle_{\text{ren}} \, \Omega^{-1}(x)\delta\Omega(x) \, d^n x \right\}.$$

Contracting with $\overline{g}^{\mu\sigma}$ and using (2.79) we obtain the expression

$$\begin{aligned} \langle T^{\nu}_{\mu}[\overline{g}_{\kappa\lambda}(x)] \rangle_{\text{ren}} = {} & \sqrt{\frac{g}{\overline{g}}} \langle T^{\nu}_{\mu}[\overline{g}_{\kappa\lambda}(x) \rangle_{\text{ren}} - \frac{2}{\sqrt{-\overline{g}(x)}} \overline{g}^{\mu\sigma}(x) \times \\ & \frac{\delta}{\delta \overline{g}^{\mu\sigma}} \left\{ \int \sqrt{-\overline{g}(x')} \, \langle T^{\rho}_{\rho}[\overline{g}_{\kappa\lambda}(x)] \rangle_{\text{ren}} \, \Omega^{-1}(x')\delta\Omega(x') \, d^n x' \right\}. \end{aligned} \tag{2.80}$$

Now, in the case of conformally invariant field theories, we find that a trace appears in the integrand on the right hand side of the expression given in (2.80). This is known as *the trace anomaly* and it is both local and state dependent. Birrell and Davies show that, if W is conformally invariant in the massless conformally coupled limit, then the expectation value of the trace of the stress tensor is also zero, thus

$$\langle T^{\mu}_{\mu}\rangle\Big|_{m=0,\ \xi=1/6} = -\frac{\Omega(x)}{\sqrt{-g(x)}}\frac{\delta W[\overline{g}_{\mu\nu}]}{\delta\Omega(x)}\Bigg|_{m=0,\ \xi=1/6,\ \Omega=1} \tag{2.81}$$

where m is the mass of the scalar field, and $\xi = \xi(n)$ is the scalar field coupling constant. Birell and Davies demonstrate that if we proceed within the framework of dimensional regularization, then if the divergent portion $\langle T_{\mu\nu}\rangle_{\text{div}}$ acquires a trace, then so must the renormalized residue $\langle T_{\mu\nu}\rangle_{\text{ren}}$, and furthoremore

$$\langle T^{\rho}_{\rho}[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = -\langle T^{\rho}_{\rho}[\overline{g}_{\kappa}\lambda(x)]\rangle_{\text{div}},$$

and hence by equation (2.81) we have

$$\langle T^{\rho}_{\rho}[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = -\frac{\Omega(x)}{\sqrt{-\overline{g}(x)}}\frac{\delta W_{\text{div}}}{\delta\Omega(x)} \tag{2.82}$$

If we substitute this into the integrand in equation (2.80), the integrand becomes

$$-\frac{2}{\sqrt{-g(x)}}\overline{g}^{\nu\sigma}(x)\frac{\delta}{\delta\overline{g}^{\mu\sigma}}\left\{\int\sqrt{-g(x)}\,\langle T^{\rho}_{\rho}[\overline{g}_{\kappa\lambda}(x')\rangle_{\text{ren}}\,\Omega^{-1}(x')\delta\Omega(x')\ d^n x'\right\} =$$
$$-\frac{2}{\sqrt{-g(x)}}\overline{g}^{\nu\sigma}(x)\frac{\delta}{\delta\overline{g}^{\mu\sigma}(x)}W_{\text{div}}[\overline{g}_{\kappa\lambda}] + \frac{2}{\sqrt{-g(x)}}g^{\nu\sigma}\frac{\delta W_{\text{div}}[g_{\kappa\lambda}]}{\delta g^{\mu\sigma}g^{\mu\sigma}(x)},$$

and hence

$$\langle T^{\nu}_{\mu}[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = \sqrt{\frac{g}{\overline{g}}} - \frac{2}{\sqrt{-g(x)}}\overline{g}^{\nu\sigma}(x)\frac{\delta}{\delta\overline{g}^{\mu\sigma}(x)}W_{\text{div}}[\overline{g}_{\kappa\lambda}] + \frac{2}{\sqrt{-g(x)}}g^{\nu\sigma}\frac{\delta W_{\text{div}}[g_{\kappa\lambda}]}{\delta g^{\mu\sigma}g^{\mu\sigma}(x)}. \tag{2.83}$$

Now, in order to proceed further, we shall need to find an expression for W_{div}- the divergent part of the effective action for quantum matter fields. Birrell and Davies give the action involving the effective Lagrangian density $\mathscr{L}_{\text{eff}}$:

$$W = \int\mathscr{L}_{\text{eff}}(x)\ d^n x = \int\sqrt{-g(x)}\mathcal{L}_{\text{eff}}(x)\ d^n x. \tag{2.84}$$

They also show that in n-dimensions, the asymptotic expansion of $\mathcal{L}_{\text{eff}}$ between two separate spacetime points x and x' can be obtained if n is treated as a variable which can be analitically expanded throughout the complex plane, and so by taking the limit $x \to x'$ and after some manipulation, they obtain the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x)(m^2)^{\frac{n}{2}-j}\Gamma(j - n/2). \tag{2.85}$$

Using the above equation with

$$a_1(x) = \left(\frac{1}{6} - \xi\right) R,$$

with the coupling constant $\xi = \xi(2)$ (R is the Ricci scalar), and also the identity:

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{2}{2-n} + \mathcal{O}(1),$$

means that we have:

$$W_{\text{div}}[g_{\kappa\lambda}] = -\frac{1}{4\pi(n-2)} \int \sqrt{-g(x')} a_1[g_{\kappa\lambda}(x')]\ d^n x, \tag{2.86}$$

which we can write in the form:

$$W_{\text{div}}[g_{\kappa\lambda}] = -\frac{1}{24\pi(n-2)} \int \sqrt{-g(x')} R(x')\ d^n x'. \tag{2.87}$$

Now that we have an expression for W_{div} (2.87), we can substitute it into equation (2.83) and we get

$$\begin{aligned}\langle T^{\nu}_{\mu}[\overline{g}_{\kappa\lambda(x)}]\langle_{\text{ren}} &= \sqrt{g\overline{g}}\langle T^{\nu}_{\mu}[g_{\kappa\lambda}(x)]\rangle_{\text{ren}} \\ &+ \frac{2}{\sqrt{-g(x)}}\overline{g}^{\nu\sigma}(x)\frac{\delta}{\delta\overline{g}^{\mu\sigma}(x)}\left\{\frac{1}{24\pi(n-2)}\int\sqrt{-\overline{g}(x')}\overline{R}(x')\ d^n x'\right\} \\ &- \frac{2}{\sqrt{-g(x)}}\overline{g}^{\nu\sigma}(x)\frac{\delta}{\delta\overline{g}^{\mu\sigma}(x)}\left\{\frac{1}{24\pi(n-2)}\int\sqrt{-\overline{g}(x')}R(x')\ d^n x'\right\}\end{aligned}$$

which after some simplification yields,

$$\langle T^{\nu}_{\mu}[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = \sqrt{\frac{g}{\overline{g}}}\langle T^{\nu}_{\mu}[g_{\kappa\lambda}(x)\rangle_{\text{ren}} + \frac{1}{24\pi(n-2)}\left[\int \frac{\delta\overline{R}(x')}{\delta\overline{g}^{\mu\sigma}}\ d^n x' - \int\frac{\delta R(x')}{\delta g^{\mu\sigma}} d^n x'\right].$$

Now, since

$$\int \frac{\delta R(x')}{\delta g^{\mu\sigma}}\ d^n x = R_{\mu\sigma} - \frac{1}{2}Rg_{\mu\sigma},$$

then we have that

$$\langle T_\mu^\nu[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = \sqrt{\frac{g}{\overline{g}}}\langle T_\mu^\nu[g_{\kappa\lambda}(x)\rangle_{\text{ren}} + \frac{1}{12\pi(n-2)}\left[\left(\overline{R}_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^\nu\overline{R}\right) - \left(R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R\right)\right]. \tag{2.88}$$

If we make the conformal transformation,

$$g_{\mu\nu}(x) \to \overline{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)$$

then the Ricci tensor transforms as,

$$R_\mu^\nu \to \overline{R}_\mu^{\ \nu} = \Omega^{-2}R_\mu^\nu - (n-2)\Omega^{-1}(\Omega^{-1})_{;\mu\rho}\, g^{\rho\nu} + \frac{1}{n-2}\omega^{-n}(\Omega^{n-2})_{;\rho\sigma}\, g^{\rho\sigma}\delta_\mu^\nu,$$

and the Ricci scalar as

$$R \to \overline{R} = \Omega^{-2}R + s(n-1)\Omega^{-3}\Omega_{;\mu\nu}\, g^{\mu\nu} + (n-1)(n-4)\Omega^{-4}\Omega_{;\mu}\Omega_{;\nu}\, g^{\mu\nu}.$$

Using the above transformations of the Ricci tensor and scalar, we can now rewrite (2.88) as

$$\begin{aligned}\langle T)\mu^\nu[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} &= \sqrt{\frac{g}{\overline{g}}}\langle T_\mu^\nu[g_{\kappa\lambda}(x)\rangle_{\text{ren}} \\ &+ \frac{1}{12\pi}\left[\left(\Omega^{-3}\Omega_{;\rho\mu} - 2\Omega^{-4}\Omega_{;\rho}\Omega_{;\mu}\right)g^{\rho\nu} + \delta_\mu^\nu g^{\rho\sigma}\left(\frac{3}{2}\Omega^{-4}\Omega_{;\rho}\Omega_{;\sigma} - \Omega^{-3}\Omega_{;\rho\sigma}\right)\right].\end{aligned} \tag{2.89}$$

Now, as we stated at the start of the section, all two dimensional spacetimes are conformally flat and so

$$g_{\mu\nu} = C(x)\eta_{\mu\nu}, \tag{2.90}$$

where $\eta_{\kappa\lambda}$ is the Minkowski spacetime metric tensor. We shall let $\Omega = C^{1/2}$ and we can now write the expectation value of the stress-tensor in any 2D curved spacetime in terms of its flat spacetime expectation value. The result can be written in a rather simple way if we move over to the null coordinate system, and so

$$ds^2 = C(u,v)\; dudv,$$

and now,equation (2.89) becomes the Davies, Fulling and Unruh result:

$$\langle T_\mu^\nu[\overline{g}_{\kappa\lambda}(x)]\rangle_{\text{ren}} = \frac{1}{\sqrt{-g}}\langle T_\mu^\nu[\eta_{\kappa\lambda}(x)]\rangle_{\text{ren}} + \theta_\mu^\nu - \left(\frac{1}{48\pi}\right)R\delta_\mu^\nu, \tag{2.91}$$

and

$$\theta_{uu} = -\frac{\sqrt{C}}{12\pi}\partial_u^2\left\{\frac{1}{\sqrt{C}}\right\}, \quad \text{and} \quad \theta_{vv} = -\frac{\sqrt{C}}{12\pi}\partial_v^2\left\{\frac{1}{\sqrt{C}}\right\}, \text{ with } \theta_{uv} = \theta_{vu} = 0. \qquad (2.92)$$

Remark. The quantity $\langle T_\mu^\nu[\eta_{\kappa\lambda}(x)]\rangle_{\text{ren}}$ is the flat spacetime contribution. Now, if the state used in evaluating the expectation value in flate spacetime is a vacuum state, then the state appearing in the curved spacetime expectation value is a conformal vacuum.

If, however, whether or not the flat spacetime vacuum is the usual Minkowski spacetime vacuum depends upon whether the expectation value of the stress tensor of the curved spacetime is conformal to all of Minkowski spacetime, or just some part of it. If the curved spacetime is indeed conformal to all of Minkowski spacetime, then we find that

$$\langle T_\mu^\nu[\eta_{\kappa\lambda}(x)]\rangle_{\text{ren}} = 0. \qquad (2.93)$$

Otherwise, the quantity is not zero, and its contribution to the curved spacetime expectation value must be determined.

2.7 Calculation of the Renormalised Stress Tensor

The spacetime SADS_2 is a dynamical one, and in order to determine if particle creation occurs, we must calulcate the renormalized stress tensor $\langle T_{\mu\nu}\rangle_{\text{ren}}$ both at early and late times, and subtract the two. In this section we shall determine these two forms of the stress tensor using the expressions derived in (2.91) and (2.92). First though, we shall need to calculate $\langle T_\mu^{\ \nu}[\eta]\rangle_{\text{ren}}$, since we find that SADS_2 is not completely conformal to Minkowski spacetime, but rather it is conformal to half of it.

2.7.1 Calculation of $\langle T_\mu^{\ \nu}[\eta]\rangle_{\text{ren}}$

We shall be interested in finding a form for $\langle T_\mu^{\ \nu}[\eta]\rangle_{\text{ren}}$- that portion of the renormalized stress tensor which comes from the flat space contribution. We shall use the point-

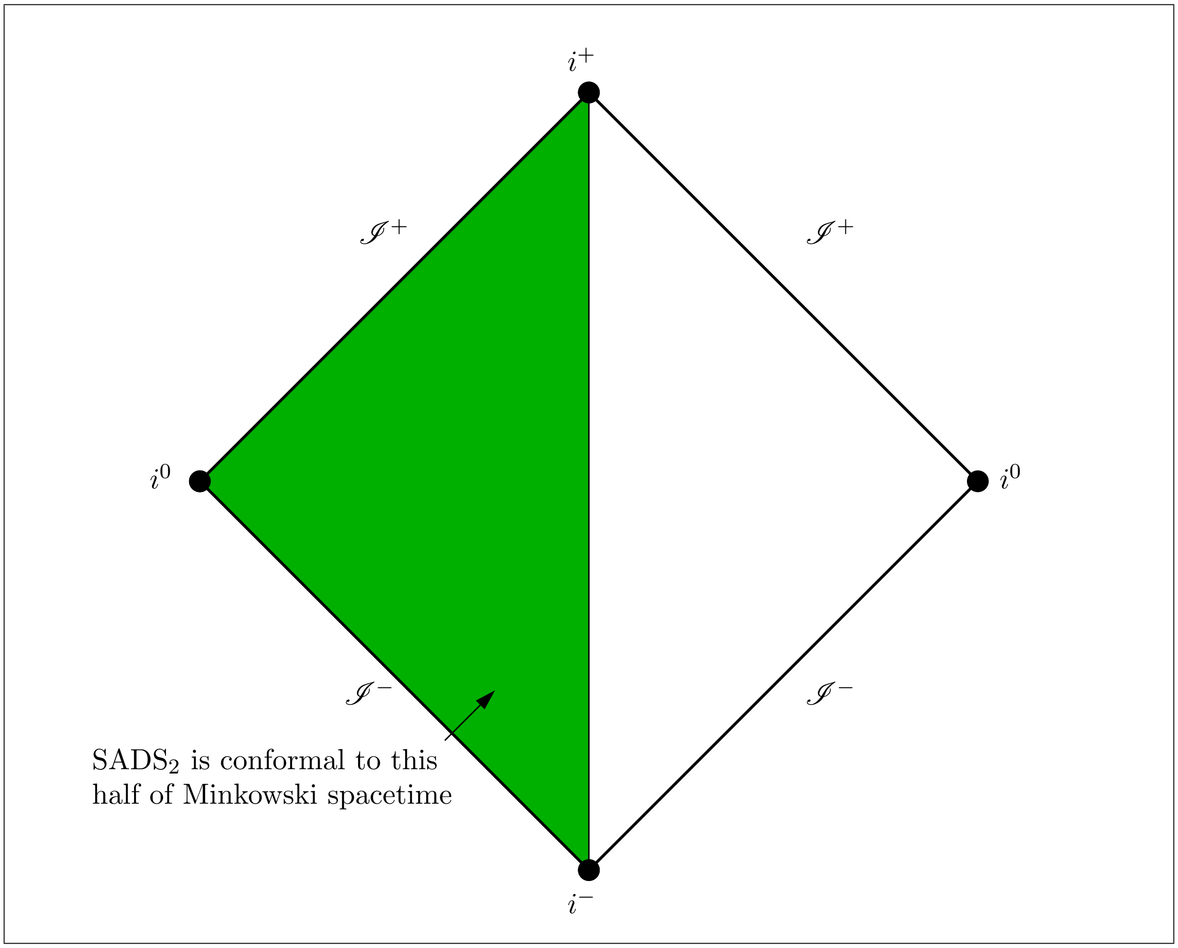

Figure 2.6: Penrose diagram Minkowski spacetime. $SADS_2$ is conformal to half of Minkowski spacetime as indicated

splitting method to calculate the renormalized stress tensor and the methodology to be adopted (discussed in Birrell and Davies) is as follows:

1. Solve the scalar field equation to obtain a complete set of orthonomral modes from which particle states may be defined.

2. Construct $\mathcal{G}^{(1)}(x,x')$ is a mode sum. The function $\mathcal{G}^{(1)}(x,x')$ is a Green's function and moreover, it is a biscalar quantity of two spacetime points x and x'.

3. Form the function

$$\mathcal{G}^{(1)}_{\text{ren}}(x,x') = \mathcal{G}^{(1)}(x,x') - \mathcal{G}^{(1)}_{\text{DS}}(x,x'), \tag{2.94}$$

   where $\mathcal{G}^{(1)}_{\text{DS}}(x,x')$ is the truncated DeWitt–Schwinger expansion.

4. Operate on $\mathcal{G}^{(1)}_{\text{ren}}(x,x')$ to form $\langle 0|T_{\mu\nu}|0\rangle$_ren, discarding any terms which are of adiabatic order greater than n.

5. Take the limit $x \to x'$ and display the finite result $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$.

In our case, we find we can determine $\langle T_{\mu\nu}[\eta]\rangle_{\text{ren}}$ by operating on the Green's function $\mathcal{G}^{(1)}_{\text{ren}}(x,x')$ with a differential operator $\mathcal{D}$ and then take the limit $x \to x'$, i.e:

$$\langle T_{\mu\nu}[\eta]\rangle_{\text{ren}} = \lim_{x\to x'} \mathcal{D}_{\mu\nu}(x,x')\mathcal{G}^{(1)}_{\text{ren}}(x,x'), \tag{2.95}$$

where we define the differential operator

$$\mathcal{D}_{\mu\nu}(x,x') = \partial_\mu \partial_{\nu'} - \frac{1}{2}\eta_{\mu\nu}\left(\eta^{tt}\partial_t\partial_{t'} + \eta^{rr}\partial_r\partial_{r'}\right). \tag{2.96}$$

We shall now consider the unbounded Minkowski case first. We have the Hadamard Green's funcftion (for 2D):

$$\mathcal{G}^{(1)}_U(x,x') = -\frac{1}{4\pi}\ln\left|(t-t')^2 - (r-r')^2\right|. \tag{2.97}$$

Now, we require only half of Minkowski spacetime, and so $\mathcal{G}^{(1)}(t,r;t',0) = 0$, and so the bounded Green's function is

$$\mathcal{G}^{(1)}_B(x,x') = \frac{1}{4\pi}\left[\ln\left|(t-t')^2 - (r+r')^2\right| - \ln\left|(t-t')^2 - (r-r')^2\right|\right] \tag{2.98}$$

and we see that when $r'=0$, $\mathcal{G}_B^{(1)}(x,x')=0$. So now we determine

$$\mathcal{G}_{\text{ren}}^{(1)}(x,x')=\mathcal{G}_B^{(1)}-\mathcal{G}_U^{(1)},$$

and so we have that

$$\mathcal{G}_{\text{ren}}^{(1)}(x,x')=\frac{1}{4\pi}\ln\left|(t-t')^2-(r+r')^2\right|. \tag{2.99}$$

We can now calculate the various components of $\mathcal{D}_{\mu\nu}\mathcal{G}_{\text{ren}}^{(1)}(x,x')$. We start with $\mathcal{D}_{tt}$:

$$\begin{aligned}\mathcal{D}_{tt}\mathcal{G}_{\text{ren}}^{(1)}(x,x') &= \frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t'}+\frac{1}{2}\left[-\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t'}+\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r'}\right], \\ &= \frac{1}{2}\left[\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t'}+\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r'}\right].\end{aligned} \tag{2.100}$$

We shall require the following derivatives:

$$\begin{aligned}\frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t} &= \frac{(t-t')}{2\pi\left[(t-t')^2-(r+r')^2\right]}, & \frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial t'} &= -\frac{(t'-t)}{2\pi\left[(t-t')^2-(r+r')^2\right]} \\ \frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r} &= -\frac{(r+r')}{2\pi\left[(t-t')^2-(r+r')^2\right]}, & \frac{\partial\mathcal{G}_{\text{ren}}^{(1)}}{\partial r'} &= -\frac{(r+r')}{2\pi\left[(t-t')^2-(r+r')^2\right]}\end{aligned} \tag{2.101}$$

and hence we have that

$$\mathcal{D}_{tt}\mathcal{G}_{\text{ren}}^{(1)}(x,x')=\frac{1}{8\pi^2}\left[\frac{(t-t')^2}{\left((t-t')^2-(r+r')^2\right)^2}+\frac{(r+r')^2}{\left((t-t')^2-(r+r')^2\right)^2}\right]. \tag{2.102}$$

Now, to find the $\langle T_{tt}[\eta]\rangle_{\text{ren}}$ component we take the following limit of (2.102):

$$\begin{aligned}\langle T_{tt}[\eta]\rangle_{\text{ren}} &= \lim_{\substack{t\to t'\\ r\to r'}}\left\{\frac{1}{8\pi^2}\left[\frac{(t-t')^2}{\left((t-t')^2-(r+r')^2\right)^2}+\frac{(r+r')^2}{\left((t-t')^2-(r+r')^2\right)^2}\right]\right\} \\ &= \frac{1}{32\pi^2r^2}.\end{aligned} \tag{2.103}$$

Next, we find the component $\langle T_{tr}[\eta]\rangle_{\text{ren}}$. We have that

$$\mathcal{D}_{tr}\mathcal{G}_{\text{ren}}^{(1)}(x,x')=\frac{\partial G_{\text{ren}}^{(1)}(x,x')}{\partial t}\frac{\partial G_{\text{ren}}^{(1)}(x,x')}{\partial r'}=-\frac{(t-t')(r+r')}{4\pi^2\left[(t-t')^2-(r+r')^2\right]^2} \tag{2.104}$$

and so, taking the limit $t\to t'$ and $r\to r'$ of (2.104) we find that

$$\langle T_{tr}[\eta]\rangle_{\text{ren}}=0. \tag{2.105}$$

By symmetry we must also have that

$$\langle T_{rt}[\eta]\rangle_{\text{ren}} = 0. \tag{2.106}$$

We need to calculate one more component: $\langle T_{rr}[\eta]\rangle_{\text{ren}}$. We have then that

$$\begin{aligned}
\mathcal{D}_{tt}\mathcal{G}^{(1)}_{\text{ren}}(x,x') &= \left(\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r}\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r'} + \frac{1}{2}\left[\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial t}\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial t'} - \frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r}\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r'}\right]\right) \\
&= \frac{1}{2}\left(\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial t}\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial t'} + \frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r}\frac{\partial \mathcal{G}^{(1)}_{\text{ren}}}{\partial r'}\right),
\end{aligned}$$

and hence we have that

$$\langle T_{rr}[\eta]\rangle_{\text{ren}} = \frac{1}{32\pi^2 r^2}. \tag{2.107}$$

We can now display all of the components of $\langle T_{\mu\nu}\rangle_{\text{ren}}$ in matrix form, and we find that for SADS_2 we have the flat spacetime contribution:

$$\langle T_{\mu\nu}[\eta]\rangle_{\text{ren}} = \frac{1}{32\pi^2 r^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.108}$$

2.7.2 $\langle T_{\mu\nu}\rangle_{\text{ren}}$ at Early Times

In this section we shall be interested in determining a value for $\langle T_{\mu\nu}\rangle_{\text{ren}}$ at early times. To do this, we use the previously derived expression:

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \theta_{\mu\nu} + \frac{R}{48\pi}g_{\mu\nu}. \tag{2.109}$$

Where R is the Ricci scalar and the quantity $\theta_{\mu\nu}$ is known as the 'DFU (*Davies-Fulling and Unruh*) derivative' and in the case of our two dimensionally conformally flat spacetime, contains two components:

$$\theta_{uu} = -\frac{1}{12\pi}C^{1/2}\frac{\partial^2}{\partial u^2}\left\{\frac{1}{\sqrt{C}}\right\} = -\mathscr{D}_u C \tag{2.110}$$

and

$$\theta_{vv} = -\frac{1}{12\pi}c^{1/2}\frac{\partial^2}{\partial v^2}\left\{\frac{1}{\sqrt{C}}\right\} = -\mathscr{D}_v C. \tag{2.111}$$

Recall from Section 2 that we can write the conformally flat metric of SADS_2 as:

$$ds^2 = C(r)\ dudv$$

where $C(r)$ is the conformal factor

$$C(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}.$$

The quantity $Rg_{\mu\nu}/48\pi$ is known as the *trace anomaly* and we also have that $\theta_{uv} = \theta_{vu} = 0$. We now need to find the two components θ_{uu} and θ_{vv} so that we can determine the form of $\langle T_{\mu\nu}\rangle_{\text{ren}}$ given in (2.109).

Calculation of θ_{uu}. For this component, we have that:

$$\frac{\partial}{\partial u}\left\{\frac{1}{\sqrt{C}}\right\} = -\frac{\frac{\partial C(u,v)}{\partial u}}{2C^{3/2}(u,v)} \tag{2.112}$$

and,

$$\frac{\partial^2}{\partial u^2}\left\{\frac{1}{\sqrt{C}}\right\} = \frac{3\left(\frac{\partial}{\partial u}C(u,v)\right)^2}{4C(u,v)^{5/2}} - \frac{\frac{\partial^2}{\partial u^2}C(u,v)}{2C(u,v)^{3/2}}. \tag{2.113}$$

We must now find the two derivatives $\partial C/\partial u$ and $\partial^2 C/\partial u^2$. We shall determine the $\partial C/\partial u$ derivative first. We have that

$$\frac{\partial C}{\partial u} = \frac{dC}{dr}\frac{dr}{du} = \frac{dC}{dr}\frac{dr}{dr_*}\frac{\partial r_*}{\partial u}.$$

It is clear that,

$$\frac{dC}{dr} = \frac{2M}{r^2} - \frac{2\Lambda r}{3}, \tag{2.114}$$

we have by equation (2.6) that

$$\frac{dr}{dr_*} = C(r),$$

and since,

$$r_* = \frac{u-v}{2}$$

then we see that equation (2.114) is,

$$\frac{\partial C}{\partial u} = -\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)C(r). \tag{2.115}$$

Next we find the second derivative $\partial^2 C/\partial u^2$. Using the chain rule again we have:

$$\begin{aligned}\frac{\partial^2 C}{\partial u^2} &= \frac{d}{dr}\left\{-\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)C(r)\right\}\frac{(-C(r))}{2}\\ &= \frac{C(r)}{2}\left[C(r)\left(-\frac{2M}{r^3}-\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)\right].\end{aligned}$$

Thus we find that if,

$$\frac{\partial^2}{\partial u^2}\left\{\frac{1}{\sqrt{C}}\right\} = \frac{3\left(\frac{\partial}{\partial u}C(u,v)\right)^2}{4C(u,v)^{5/2}} - \frac{\frac{\partial^2}{\partial u^2}C(u,v)}{2C(u,v)^{3/2}}.$$

then

$$\frac{\partial^2}{\partial u^2}\left\{\frac{1}{\sqrt{C}}\right\} = \frac{3}{4\sqrt{C}}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)^2 - \frac{1}{4\sqrt{C}}\left[C\left(-\frac{2M}{r^2}-\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)\right],$$

and hence we have

$$\theta_{uu} = \frac{1}{48\pi}\left[-3\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)^2+\left(C\left(\frac{-2M}{r^3}-\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)\right)\right]. \tag{2.116}$$

Calculation of θ_{vv}. We calculate the other non-zero tensor component, θ_{vv}. As one would expect, this is a similar process to that of finding θ_{uu}. We have that

$$\frac{\partial}{\partial v}\left\{\frac{1}{\sqrt{C}}\right\} = -\frac{\frac{\partial C}{\partial v}}{2C(u,v)^{3/2}} \tag{2.117}$$

and that

$$\frac{\partial^2}{\partial v^2}\left\{\frac{1}{\sqrt{C}}\right\} = \frac{3\left(\frac{\partial C}{\partial v}\right)^2}{4C(u,v)^{5/2}} - \frac{\frac{\partial^2 C}{\partial v^2}}{2C(u,v)^{3/2}}. \tag{2.118}$$

So we must now find expressions for the first order derivative $\frac{\partial C}{\partial v}$ and the second order derivative $\frac{\partial^2 C}{\partial v^2}$. We start with the first order derivative, noting that

$$\frac{\partial C}{\partial v} = \frac{dC}{dr}\frac{dr}{dv} = \frac{dC}{dr}\frac{dr}{dr_*}\frac{\partial r_*}{\partial v}.$$

Substituting in the relevant derivatives we obtain that

$$\frac{\partial C}{\partial v} = \left[\frac{M}{r^2}-\frac{\Lambda r}{3}\right]C(r). \tag{2.119}$$

Now we find $\frac{\partial^2 C}{\partial v^2}$. Again, using the chain rule in the same way as we did for the θ_{vv} component, we have that:

$$\frac{\partial^2 C}{\partial v^2} = \frac{C(r)}{2}\left[C(r)\left(-\frac{2M}{r^3} - \frac{\Lambda}{3}\right) + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2} - \frac{2\Lambda}{3}\right)\right]$$

Hence we see that

$$\frac{\partial^2}{\partial v^2}\left\{\frac{1}{\sqrt{C}}\right\} = \frac{3}{4\sqrt{C}}\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)^2 - \frac{1}{4\sqrt{C}}\left[C\left(\frac{-2M}{r^3} - \frac{\Lambda}{3}\right) + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2} - \frac{\Lambda}{3}\right)\right],$$

and so we must have that

$$\theta_{vv} = \frac{1}{48\pi}\left[-3\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)^2 + \left(C\left(\frac{-2M}{r^3} - \frac{\Lambda}{3}\right) + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2} - \frac{2\Lambda r}{3}\right)\right).\right] \tag{2.120}$$

We should take this oportunity to compare our result with that obtained by P. Davies [32] (and also [18]). Here, Davies calculates $\langle T_{\mu\nu}\rangle_{\text{ren}}$ for the Reissner-Nordström black hole and finds that $\theta_{uu} = \theta_{vv}$

$$\theta_{uu} = \theta_{vv} = \frac{1}{24\pi}\left[-\frac{M}{r^3} + \frac{3}{2}\frac{M^2}{r^4} + \frac{3e^2}{2r^4} - \frac{3Me^2}{r^5} + \frac{e^4}{r^6}\right] \tag{2.121}$$

If we let the charge $e = 0$ we recover the Schwarzschild case:

$$\theta_{uu} = \theta_{vv} = -\frac{M}{24\pi r^3} + \frac{M^2}{16\pi r^4}. \tag{2.122}$$

Here we also have that $\theta_{uu} = \theta_{vv}$, moreover, setting $\Lambda = 0$ in either (2.116) or (2.120) we obtain

$$\begin{aligned}\theta_{vv} &= \frac{1}{48\pi}\left[-\frac{3M^2}{r^4} + \left(1 - \frac{2M}{r}\right)\left(-\frac{2M}{r^3}\right) + \left(\frac{M}{r^2}\right)\left(\frac{2M}{r^2}\right)\right] \\ &= \frac{1}{48\pi}\left[-\frac{2M}{r^3} + \frac{3M^2}{r^4}\right] = -\frac{M}{24\pi r^3} + \frac{M^2}{16\pi r^4},\end{aligned} \tag{2.123}$$

This is exactly the same as the result obtained by Davies for the Schwarzschild case given in (2.122).

We now go on to give the full form for $\langle T_{\mu\nu}\rangle_{\text{ren}}$. Since we have from (2.109) that

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \theta_{\mu\nu} + \frac{R}{48\pi}g_{\mu\nu},$$

where R is the Ricci scalar given in (2.19). We have written the line element in the conformal form

$$ds^2 = -C(r)dudv$$

and so the metric tensor has the form

$$g_{\mu\nu} = \begin{bmatrix} 0 & C(r)/2 \\ C(r)/2 & 0 \end{bmatrix}.$$

Thus we may now write out the components of the $\langle T_{\mu\nu}\rangle_{\text{ren}}$:

$$\begin{aligned} \langle T_{00}\rangle_{\text{ren}} = \theta_{uu} + \frac{Rg_{00}}{48\pi} = \theta_{uu}, \ \text{ and } \ \langle T_{01}\rangle_{\text{ren}} = \theta_{uv} + \frac{Rg_{01}}{48\pi} = -\frac{RC(r)}{48\pi} \\ \langle T_{10}\rangle_{\text{ren}} = \theta_{vu} + \frac{Rg_{10}}{48\pi} = -\frac{RC(r)}{48\pi} \ \text{ and } \ \langle T_{11}\rangle_{\text{ren}} = \theta_{vv} + \frac{Rg_{11}}{48\pi} = \theta_{vv} \end{aligned} \tag{2.124}$$

and hence for early times, the renormalised stress tensor has the form

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \frac{1}{48\pi}\begin{bmatrix} \frac{3M^2}{r^4} + \frac{2M\Lambda}{r} - \frac{2M}{r^3} - \frac{\Lambda}{3} & -\frac{-(6M+\Lambda r^3)(6m-3r+\Lambda r3}{9r^4} \\ -\frac{-(6M+\Lambda r^3)(6m-3r+\Lambda r3}{9r^4} & \frac{3M^2}{r^4} + \frac{2M\Lambda}{r} - \frac{2M}{r^3} - \frac{\Lambda}{3}. \end{bmatrix} \tag{2.125}$$

2.7.3 Calculation of the Renormalised energy Stress Tensor at Late Times

In this section we shall calculate the renormalised stress tensor at late times. To do this we shall need to use the expression stated in the first section of this chapter, namely:

$$\langle T_\mu^\nu[g]\rangle_{\text{ren}} = \sqrt{-g}\langle T_\mu^\nu[\eta]\rangle_{\text{ren}} + \theta_\mu^\nu - \frac{R}{48\pi}\delta_\mu^\nu \tag{2.126}$$

where the quantity $\langle T_\mu^\nu[\eta]\rangle_{\text{ren}}$ is conformal to half of Minkowski spacetime (see figure 2.6), and θ_μ^ν is the DFU derivative and now, as we at late times, will depend on the functions $v = g(U)$ and $u = f(V)$ obtained from the ray tracing process earlier in the chapter. We shall now begin the process of calculating the components of (2.126). We shall start with $\langle T_\mu^\nu\ [\eta]\rangle_{\text{ren}}$.

Calculation of the DFU Derivative at Late Times We now require the two components for the DFU derivative. Recall that the spacetime metric was:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2$$

At late times

$$v = g(U), \text{ and } u = f(V)$$

and so for late times, the metric in terms of null coordinates

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \frac{dg}{dU}\frac{df}{dV} dU dV \tag{2.127}$$

where we define the advanced and retarded null coordinate to be

$$U = T - R_*, \quad V = T + R_*.$$

and from the ray-tracing method earlier in the chapter we had that

$$f(V) = -\frac{1}{h(r_h)} \ln\left|\frac{V - v_0}{\mu}\right|, \quad \text{and} \quad g(U) = -\frac{1}{h(r_h)} \ln\left|\frac{U - u_0}{\mu}\right| \tag{2.128}$$

and hence we have the derivatives

$$\frac{df}{dV} = -\frac{1}{h(r_h)(V - v_0)}, \quad \text{and} \quad \frac{dg}{dU} = -\frac{1}{h(r_h)(U - u_0)}$$

and so now we can write the metric as

$$ds^2 = \frac{C(r)}{h^2(r_h)(V - v_0)(U - u_0)} dU dV$$

and so we now define

$$\overline{C} = \frac{C}{h^2(r_h)(V - v_0)(U - u_0)}, \tag{2.129}$$

then

$$ds^2 = \overline{C}\ dU dV. \tag{2.130}$$

Calculation of θ_{UU}. We have that

$$\theta_{UU} = -\frac{1}{12\pi}\sqrt{\overline{C}}\frac{\partial^2}{\partial U^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} \tag{2.131}$$

where

$$\frac{\partial^2}{\partial U^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} = \frac{3\left(\frac{\partial \overline{C}}{\partial U}\right)^2}{4\overline{C}^{5/2}} - \frac{\frac{\partial^2 \overline{C}}{\partial U^2}}{2\overline{C}^{3/2}}. \tag{2.132}$$

We find the first derivative:

$$\frac{\partial \overline{C}}{\partial U} = \frac{1}{h^2(r_h)(V-v_0)(U-u_0)}\frac{\partial C}{\partial U} - \frac{C(r)}{h^2(r_h)(V-v_0)(U-v_0)^2}. \tag{2.133}$$

Using the same chain rule arguments as before we have

$$\frac{\partial C}{\partial U} = -C(r)\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right),$$

and thus

$$\frac{\partial \overline{C}}{\partial U} = -\overline{C}\left[\frac{M}{r^2} - \frac{\Lambda r}{3} - \frac{1}{U-u_0}\right]. \tag{2.134}$$

Now we find the 2nd derivative. We have

$$\begin{aligned}\frac{\partial^2 C}{\partial U^2} &= -\frac{C}{2}\frac{d}{dr}\left\{-\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)C(r)\right\}\\ &= \frac{C(r)}{2}\left[-C(r)\left(\frac{2M}{r^3} - \frac{\Lambda}{3}\right) + \left(\frac{2M}{r^2} - \frac{2\Lambda r}{3}\right)\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\right].\end{aligned}$$

Next we find:

$$\frac{\partial^2 \overline{C}}{\partial U^2} = \frac{4}{h^2(r_h)(V-v_0)(U-u_0)}\left[\frac{\partial^2 C}{\partial U^2} - \frac{2\frac{\partial C}{\partial U}}{U-u_0} + \frac{2C}{(U-u_0)^2}\right].$$

Substituting these derivatives now into the expression for $\partial^2\overline{C}/\partial U^2$:

$$\begin{aligned}&\frac{\partial^2 \overline{C}}{\partial U^2} =\\ &\quad\frac{4}{h^2(r_h)(V-v_0)(U-u_0)}\left[\frac{C}{2}\left(-C\left(\frac{2M}{r^3} - \frac{\Lambda}{3}\right) + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2} - \frac{2\Lambda r}{3}\right)\right)\right.\\ &\quad\left. - \frac{2}{(U-u_0)}\left(-C\left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\right) + \frac{2C}{(U-u_0)^2}\right],\end{aligned}$$

which we can write as

$$\frac{\partial^2 \overline{C}}{\partial U^2} = \frac{\overline{C}}{2}\left[-C\left(\frac{2M}{r^3} + \frac{\Lambda}{3}\right) + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2} - \frac{2\Lambda r}{3}\right) + \frac{4}{U-u_0}\left(\frac{M}{r^2} - \frac{\Lambda r}{3} + \frac{1}{U-u_0}\right)\right]. \tag{2.135}$$

We now have everything we need to calculate (2.132):

$$\frac{\partial^2}{\partial U^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} = \frac{3}{4\sqrt{\overline{C}}}\left(\frac{M}{r^2} - \frac{\Lambda r}{3} - \frac{1}{U-u_0}\right)^2$$
$$-\frac{1}{4\sqrt{\overline{C}}}\left[-C\left(\frac{2M}{r^3}+\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right) - \frac{4}{U-u_0}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}+\frac{1}{U-u_0}\right)\right],$$

and hence by (2.131) we finally arrive at

$$\theta_{UU} = \frac{1}{48\pi}\left[\frac{2M}{r^3}\left(\frac{3M}{2r}+\Lambda r^2 - 1\right) + \frac{10}{U-u_0}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}+\frac{1}{10(U-u_0)}\right) - \frac{\Lambda}{3}\right]. \tag{2.136}$$

Calculation of θ_{VV} We now calculate the component:

$$\theta_{VV} = -\frac{1}{12\pi}\sqrt{\overline{C}}\frac{\partial^2}{\partial V^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} \tag{2.137}$$

where we have that

$$\frac{\partial^2}{\partial V^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} = \frac{3\left(\frac{\partial \overline{C}}{\partial V}\right)^2}{4\overline{C}^{5/2}} - \frac{\frac{\partial^2 \overline{C}}{\partial V^2}}{2\overline{C}^{3/2}}.$$

Once more we need to comnpute the first and second order derivatives. We have

$$\begin{aligned}\frac{\partial \overline{C}}{\partial V} &= \frac{4}{h^2(r_h)(V-v_0)(U-u_0)}\left[\frac{\partial C}{\partial V} - \frac{C}{V-v_0}\right]\\ &= \overline{C}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}-\frac{1}{V-v_0}\right).\end{aligned}$$

Using exactly the same argument as we did for θ_{UU}, we have that

$$\frac{\partial^2 C}{\partial U^2} = \frac{\overline{C}}{2}\left[-C\left(\frac{2M}{r^2}+\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)\right].$$

We now compute

$$\frac{\partial^2 \overline{C}}{\partial V^2} = \frac{4}{h^2(r_h)(V-v_0)(U-u_0)}\left[\frac{\partial^2 C}{\partial V^2} - \frac{2\frac{\partial C}{\partial V}}{V-v_0} + \frac{2C}{(V-v_0)^2}\right].$$

Subsituting in the derivatives and the function C gives

$$\frac{\partial^2 \overline{C}}{\partial V^2} = \frac{\overline{C}}{2}\left[-C\left(\frac{2M}{r^3}+\frac{\Lambda}{3}\right)+\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)-\frac{4}{V-v_0}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}-\frac{1}{V-v_0}\right)\right].$$

Finally, we have that

$$\frac{\partial^2}{\partial V^2}\left\{\frac{1}{\sqrt{\overline{C}}}\right\} = \frac{3}{4\sqrt{\overline{C}}}\left[\frac{M}{r^2}-\frac{\Lambda r}{3}-\frac{1}{V-v_0}\right]^2$$
$$-\frac{1}{4\sqrt{\overline{C}}}\left[\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right)\left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)-C\left(\frac{2M}{r^3}-\frac{\Lambda}{3}\right)-\frac{4}{V-v_0}\left(\frac{M}{r^2}-\frac{\Lambda r}{3}-\frac{1}{V-v_0}\right)\right],$$

and hence, after some simplification, we have that

$$\theta_{VV} = \frac{1}{48\pi}\left[\frac{2M}{r^3}\left(\frac{3M}{2r} + \Lambda r^2 - 1\right) + \frac{10}{V - v_0}\left(\frac{M}{r^2} - \frac{\Lambda r}{3} - \frac{7}{10(V - v_0)^2}\right) - \frac{\Lambda}{3}\right] \tag{2.138}$$

We now calculate the stress tensor at late times. Since

$$\langle T_\mu^\nu[g]\rangle_{\rm ren} = \sqrt{-g}\langle T_\mu^\nu[\eta]\rangle_{\rm ren} + \theta_\mu^\nu - \frac{R}{48\pi}\delta_\mu^\nu$$

and $\sqrt{-g}\langle T_\mu^\nu[\eta]\rangle_{\rm ren}$ and $\frac{R}{48\pi}\delta_\mu^\nu$ are the same as before, we must have that

$$\langle T_\mu^\nu[g]\rangle_{\rm ren} = \frac{1}{48\pi}\begin{bmatrix} X(r) + \frac{10}{U-u_0}\left[\frac{M}{r^2} - \frac{\Lambda r}{3} + \frac{1}{10(U-u_0)}\right] - \frac{\Lambda}{3} & \frac{(6M+\Lambda r^3)(6M-3r+\Lambda r^3)}{9r^4} \\ \frac{(6M+\Lambda r^3)(6M-3r+\Lambda r^3)}{9r^4} & X(r) + \frac{10}{V-v_0}\left[\frac{M}{r^2} - \frac{\Lambda r}{3} + \frac{7}{10(V-u_0)}\right] - \frac{\Lambda}{3} \end{bmatrix}, \tag{2.139}$$

where we have let

$$X(r) = \frac{3}{\pi r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} + \frac{2M}{r^3}\left(\frac{3M}{2r} + \Lambda r^2 - 1\right). \tag{2.140}$$

This is the value of the stress-tensor at lates times.

Chapter 3

Chapter 3: Does a Quantum Oscillator Radiate in 2D Schwarzschild Spacetime?

3.1 Introduction

In this chapter we shall be interested in calculating the total energy flux associated with a quantum oscillator on various trajectories. Essentially, we shall follow the method as set out by Ford and O'Connell (this was method was reviewed in Chapter 1). This method involves starting with an action for the oscillator (this will be a scalar electro-dynamic action) and a corresponding one for the free scalar field to which the oscillator is coupled. By use of the Euler-Lagrange equations we can determine the equations of motion for both the oscillator and the scalar field. We then solve the scalar field equation by means of an appropriate Green's function, and substitute into the equation of motion of the oscillator to obtain a quantum Langevin equation. The solution of this equation, along with the fluctuation dissipation theorem, allows us to calculate the various expectation values needed (namely an interference term and a direct flux term), and hence, the overall energy flux of the system.

We shall consider two cases: the first case is that of of a quantum oscillator in 2D Schwarzschild spacetime confined to a trajectory $r_*(\tau) = 0$. The second case will involve the oscillator again in 2D Schwarzschild, but this time confined to an inertial trajectory. In the first case we shall obtain the standard result of no overall net radiation arising in the system. In the second case we shall see that particle creation arises due to the net imbalance between the direct flux arising from the oscillator term, and the interference term.

3.2 2D Schwarzschild Spacetime with Constant Trajectory.

In this section we shall show that an accelerating quantum oscillator, which is confined to a constant trajectory in $D = 2$ Schwarzschild spacetime (and in the Boulware vacuum), does not radiate. In particular we derive a quantum Langevin equation, whose solution will be used to compute the energy flux of the system. We shall show that the direct energy flux term, balances the interference term.

3.2.1 Coordinate Systems.

The line element for two dimensional Schwarzschild spacetime is

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{3.1}$$

where t is the coordinate time, r is the radial coordinate, and M is the mass of the black hole. The above line element gives a metric tensor of the form

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{2M}{r} & 0 \\ 0 & -\left(1 - \frac{2M}{r}\right)^{-1} \end{bmatrix}, \tag{3.2}$$

with inverse

$$g^{\mu\nu} = \begin{bmatrix} -\frac{r}{-r+2M} & 0 \\ 0 & -\frac{-r+2M}{r} \end{bmatrix}. \tag{3.3}$$

It is clear that $\sqrt{-g} = 1$. All two dimensional spacetimes are conformally flat, and so we can write (3.1) in the form

$$ds^2 = \Omega(dt^2 - dr_*^2), \tag{3.4}$$

where Ω is the conformal factor given by

$$\Omega = \left(1 - \frac{2M}{r}\right), \tag{3.5}$$

and r_* is the so-called 'tortoise-coordinate' which satisfies the relation

$$\frac{dr}{dr_*} = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{3.6}$$

Integrating this equation directly yields the form

$$r_* = r + 2M\ln|r - 2M| + \xi, \text{ for } \xi \in \mathbb{R}. \tag{3.7}$$

A trajectory of the quantum oscillator in 2D Schwarzschild spacetime is illustrated in fig 3.1. Our oscillator will be coupled to a free scalar field ϕ which satisfies the massless Klein-Gordon equation

$$\Box\phi = 0, \tag{3.8}$$

where we have that $\Box$ represents the usual D'Alembertian operator:

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu. \tag{3.9}$$

The oscillator will be confined to the constant trajectory

$$r_*(\tau) = 0, \ \xi = \text{const}, \quad t = t(\tau), \tag{3.10}$$

where τ is the proper time as measured on the worldline of the oscillator.

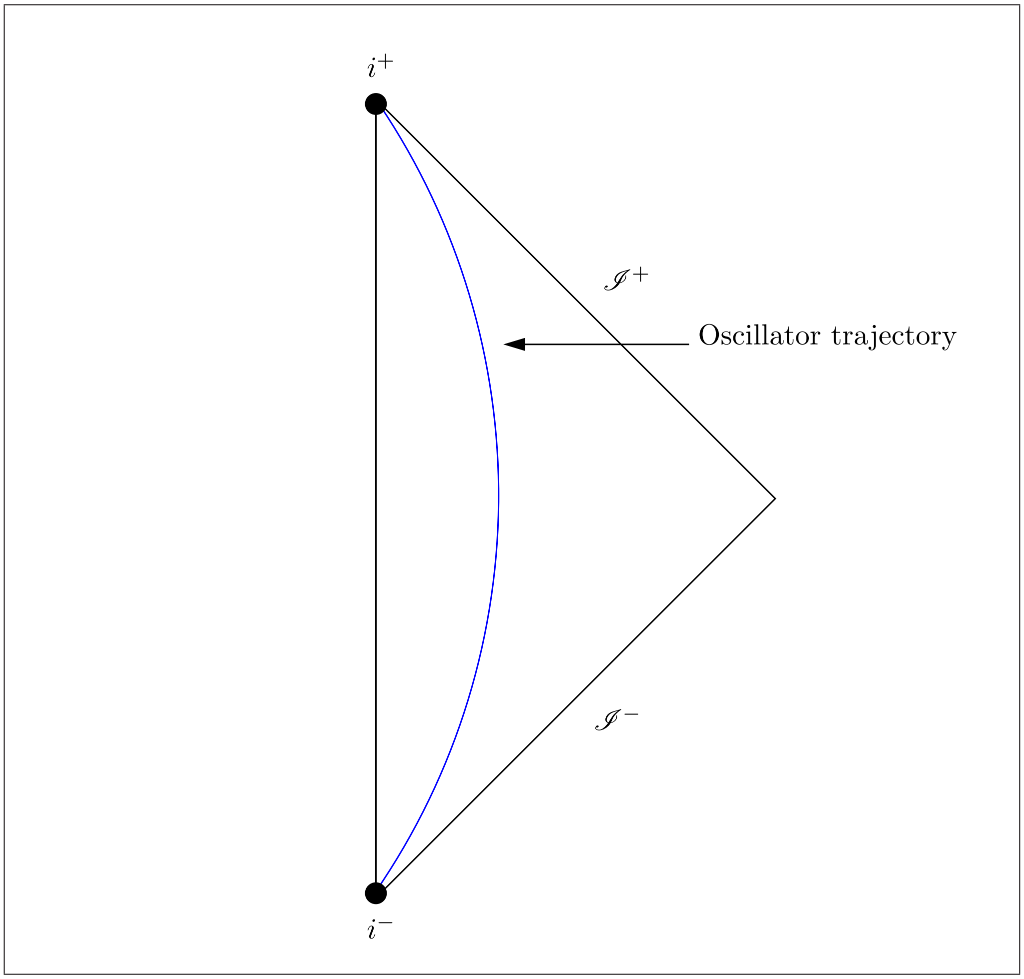

Figure 3.1: Penrose diagram showing the the trajectory of the oscillator. Timelike geodesics start at i^- and terminate at i^+

It is appropriate at this point to make a choice of vacuum. Our choice of vacuum is the vacuum is the Boulware vacuum which is empty at infinity. This means that outgoing modes have the form

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikr_*(\tau) - i\omega t(\tau)} \tag{3.11}$$

where $\omega = |k|$. The modes given in (3.11) are a complete set of solutions to the Klein-Gordon equation given in (3.8), and hence we may expand the ϕ field in a box of volume V in terms of these basis modes:

$$\phi(r_*, t) = \sum_k e_k \left[b_k e^{-i\omega_k t + ikr_*} + b_k^\dagger e^{i\omega_k t - ikr_*} \right] \tag{3.12}$$

where we have let

$$e_k = \sqrt{\frac{2\pi}{\omega_k V}} \tag{3.13}$$

In the expansion of the ϕ field above, the quantities $b_k^\dagger$ and b_k are the annihilation and creation operators (respectively) of the scalar field. The system is quantized in the usual canonical quantization scheme whereby ϕ is treated as a field operator. We define

$$\Pi = \frac{\partial \mathscr{W}}{\partial(\partial_t \phi)} = \partial^t \phi \tag{3.14}$$

where $\mathscr{W}$ is the Lagrangian of the scalar field, and we impose the following equal time commutation relations:

$$[\phi(x,t), \phi(x',t)] = 0, \;\; [\Pi(x,t), \Pi(x',t)] = 0, \;\; \text{and } [\phi(x,t), \phi(x',t)] = i\delta(x - x'). \tag{3.15}$$

For the given choice of vacuum, the field operators satisfy the following expectation values

$$\langle b_k \, b_{k'}^\dagger \rangle = \delta_{kk'}, \;\; \langle b_k^\dagger \, b_{k'} \rangle = 0, \tag{3.16}$$

3.2.2 Equations of Motion.

We shall generate the equations of motion for the oscillator and coupled field by means of an action with the scalar electro-dynamic form

$$S = \int_\Sigma \left(\frac{1}{2} m q(\dot{\tau})^2 - e\phi(r_*, t)\dot{q}(\tau) - V(q) \right) \, d\tau - \mathscr{W}, \tag{3.17}$$

where m is the mass of the oscillator, q is the internal particle coordinate, and is a function of proper time τ along its world line. The quantity $V(q)$ is the potential $-\mu^2 q/2$, while $\mathscr{W}$ is the action for the free scalar field, which has the general form

$$\mathscr{W} = \frac{1}{2}\int_{\Sigma} \sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \; dx^2. \tag{3.18}$$

It is clear that by using (3.3) in (3.18) that:

$$\mathscr{W} = \frac{1}{2}\int_{\Sigma} \left(\frac{r}{r-2M}\right)\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{r-2M}{r}\right)\left(\frac{\partial\phi}{\partial r}\right)^2 \; d^2x$$

Thus,

$$\begin{aligned} S = &\int_{\Sigma} \left(\frac{1}{2}mq(\dot{\tau})^2 - e\phi(r_*,t)\dot{q}(\tau) - \frac{\mu^2 q(\tau)}{2}\right) \; d\tau \\ &- \frac{1}{2}\int_{\Sigma} \left(\frac{r}{r-2M}\right)\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{r-2M}{r}\right)\left(\frac{\partial\phi}{\partial r}\right)^2 \; d^2x. \end{aligned} \tag{3.19}$$

Now we have a Lagrangian density

$$\mathcal{L} = \frac{1}{2}mq(\dot{\tau})^2 - e\phi(r_*,t)\dot{q}(\tau) - \frac{\mu^2 q(\tau)}{2} - \frac{1}{2}\left[\left(\frac{r}{r-2M}\right)\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{r-2M}{r}\right)\left(\frac{\partial\phi}{\partial r}\right)^2\right], \tag{3.20}$$

and so we can now find the equation of motion for the oscillator. Substituting the derivatives

$$\frac{\partial\mathcal{L}}{\partial\dot{q}} = m\dot{q}(\tau) - e\phi(r_*,t), \quad \frac{\partial\mathcal{L}}{\partial q} = -\mu^2 q,$$

into the Euler-Lagrange equations yields the equation of motion for the quantum oscillator:

$$m\frac{d^2q}{d\tau^2} + \mu^2 q = e\frac{d}{d\tau}\left\{\phi(r_*,t)\right\}. \tag{3.21}$$

We now come to find the equation of motion for the free scalar field. From the line element in (3.4), we have the conformally flat metric:

$$ds^2 = \Omega^2(dt^2 - dr_*^2)$$

where

$$\Omega = \left(1 - \frac{2M}{r}\right). \tag{3.22}$$

Note, that since along the oscillator world line we have chosen $r_* = 0$ as our constant trajectory, the conformal factor Ω is also constant.

The conformal metric in (3.4) has a metric tensor (and inverse)

$$\theta_{\mu\nu} = \begin{bmatrix} \Omega^2 & 0 \\ 0 & -\Omega^2 \end{bmatrix}, \quad \text{and} \quad \theta^{\mu\nu} = \begin{bmatrix} 1/\Omega^2 & 0 \\ 0 & -1/\Omega^2 \end{bmatrix}. \tag{3.23}$$

If we use the metric tensor of (3.23) in the covariant form of the wave equation

$$\frac{1}{\sqrt{-\theta}}\left[\frac{\partial}{\partial x^\mu}\left\{\frac{1}{\sqrt{-\theta}}\theta^{\mu\nu}\right\}\frac{\partial \phi}{\partial x^\nu}\right] = 0$$

we obtain

$$\frac{1}{\Omega^2}\left(\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial r_*^2}\right) = 0 \tag{3.24}$$

We will now return to the equation of motion of the scalar field. We have the Euler-Lagrange derivatives

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -e\dot{q}(\tau)\delta(r_*(\tau) - r_*^0(\tau), t(\tau) - t^0(\tau)) \\
\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)} &= \frac{r}{r-2M}\frac{\partial \phi}{\partial t} \\
\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial r}\right)} &= \frac{-r+2M}{r}\frac{\partial \phi}{\partial t}
\end{aligned}$$

Thus, the Euler-Lagrange equations give

$$\frac{r}{r-2M}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial r}\left\{\left(1 - \frac{2M}{r}\right)\frac{\partial \phi}{\partial r}\right\} = -e\dot{q}(\tau)\delta(r - r^0, t - t^0). \tag{3.25}$$

In the form of r_* we can write

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial r_*^2} = -\Omega^2 e\dot{q}(r_* - r_*^0, t - t^0). \tag{3.26}$$

This, then is the equation of motion for the coupled scalar field.

3.2.3 Quantum Langevin Equation.

We shall need to solve the equation of motion for the scalar field, (3.26). The solution to (3.26) will be a solution of the form

$$\phi(r_*, t) = \phi^h(r_*, t) + \phi^{\mathcal{G}}(r_*, t), \tag{3.27}$$

where $\phi^h(r_*, t)$ is the homogeneous solution to (3.26) and $\phi^{\mathcal{G}}(r_*, t)$ is the particular integral to be found from a suitable choice of *retarded Green's function.* Once we have found a form for ϕ we can substitute it into the equation of motion for the oscillator (3.21). The resulting equation will be the quantum Langevin equation.

Solving the equation of motion for the field is a relatively standard procedure which can be found outlined in many texts (for example, P. Szekeres [20]), and the method used here is discussed in Chapter 1. We shall write the particular integral in terms of the retarded Green's function:

$$\phi^{\mathcal{G}}(r_*, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{G}(r_*, r'_*; t, t')\mathcal{F}(r'_*, t(\tau)')dr'_* dt'. \tag{3.28}$$

By inspection of (3.26), and using the method given in section 1.7 on Green's function discussed in Chapter 1, we find that the functions $\mathcal{F}$ and $\mathcal{G}$ have the form

$$\begin{aligned}
\mathcal{G}(r_*, r'_*; t, t') &= \Theta(t(\tau) - t(\tau)' - |r_* - r'_*|) \\
\mathcal{F}(r'_*, t') &= -e\Omega^2\dot{q}(\tau')\delta(r'_* - r^0_*, t' - t^0),
\end{aligned}$$

where Θ is the Heaviside function, and a dot denotes differentiation with respect to proper time τ. Thus (3.28) now has the form

$$\begin{aligned}
\phi^{\mathcal{G}}(r_*, t) &= -e^2\Omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{t-|r_*-r^0_*|} \Theta(t(\tau) - t(\tau)' - |r_*(\tau) - r_*(\tau)')|)\frac{dq(\tau')}{d\tau'} \\
&\quad \delta(r_*(\tau') - r^0_*(\tau), t(\tau') - t^0(\tau'))\ dt'dr'_*.
\end{aligned}$$

Evaluating the integral with respect to r'_* first, we see that

$$\phi^{\mathcal{G}} = -\int_{-\infty}^{\tau_{\rm ret}} \left.\frac{dq(\tau)}{d\tau}\right|_{\substack{r_*(\tau) = r^0_*(\tau), \\ t(\tau) = t^0(\tau)}} \Theta(t(\tau)) - t(\tau') - |r_*(\tau) - r^0_*(\tau)|)e^2\Omega^2(\tau)\ d\tau$$

and so

$$\phi^{\mathcal{G}}(r_*, t) = -e\Omega^2 \int_{-\infty}^{\infty} \frac{dq(\tau')}{d\tau'}\delta(t(\tau) - t(\tau)' - |r_*(\tau) - r^0_*(\tau)|)\ dt'$$

Now, the only contribution comes when $\delta(t(\tau) - t(\tau)' - |r_*(\tau) - r^0_*(\tau)|) \neq 0$, i.e. when

$$t(\tau)' = t(\tau) - |r_*(\tau) - r^0_*(\tau)|,$$

which is the *retarded time* $t(\tau_{\text{ret}})$ and so (using the Chain rule) we write

$$\phi^{\mathcal{G}}(r_*,t) = -e\Omega^2 \int_{-\infty}^{t-|r_*-r_*^0|} \dot{q}(\tau')d\tau'$$

and so

$$\phi^{\mathcal{G}}(r_*,t) = -e\Omega^2 q(\tau_{\text{ret}}) \tag{3.29}$$

where τ_{ret} is the retarded time. So, the solution to the scalar field equation of motion, by (3.27) is

$$\phi^{\mathcal{G}}(r_*,t) = \phi^h(r_*,t) - e\Omega^2 q(\tau_{\text{ret}}). \tag{3.30}$$

It is this function which we will now substitute into the equation of motion for the oscillator. First we note that

$$e\frac{d\phi^{\mathcal{G}}(r_*,t)}{d\tau} = e\frac{d}{d\tau}\phi^h(r_*,t) - e^2\Omega^2\frac{dq}{d\tau}$$

so by (3.21) we have

$$m\frac{d^2q}{d\tau^2} + e^2\Omega^2\frac{dq}{d\tau} + \mu^2 q = F(\tau) \tag{3.31}$$

where

$$F(\tau) = e\frac{d}{d\tau}\phi^h(r_*(\tau),t(\tau)) \tag{3.32}$$

The quantity $F(\tau)$ is called the *fluctuating force operator*. When it is evaluated on the worldline of the oscillator it has the form

$$F(\tau) = e\frac{d}{d\tau}\phi^h(0,t(\tau))$$

3.2.4 Solution of Quantum Langevin Equation.

We now wish to solve the differential equation (3.31). Clearly the solution will be of the form

$$q(\tau) = q^h(\tau) + q^p(\tau),$$

where $q^h(\tau)$ is the homogeneous solution, and $q^p(\tau)$ is the particular integral. In fact, we find that at later times, $q^h(\tau) \to 0$, so for all intents and purposes, the homogeneous solution is zero and we are left with

$$q(\tau) = q^p(\tau).$$

As we stated at the start of the chapter, the scalar field ϕ can be written as an expansion of the outgoing modes, whose form defines the vacuum ($b_k|0\rangle = 0$), (in this case, the Boulware vacuum). Thus, we can write the scalar field as:

$$\phi(r_*(\tau), t(\tau)) = \sum_k \frac{1}{\sqrt{4\pi\omega}} \left[b_k e^{ikr_*(\tau) - i\omega t(\tau)} + b_k^\dagger e^{-ikr_*(\tau) + i\omega t(\tau)} \right], \tag{3.33}$$

with $\omega = |k|$. Using the expansion in (3.33) we can expand this operator to give it the form

$$F(\tau) = e \sum_k \frac{\omega}{\sqrt{4\pi}} \left[-ib_k e^{-i\omega t(\tau)} + ib_k^\dagger e^{i\omega t(\tau)} \right]$$

Using this expansion of the fluctuating force operator in the right hand of (3.31) we now have

$$m\frac{d^2q}{d\tau^2} + e^2\Omega^2\frac{dq}{d\tau} + \mu^2 q = F(\tau) = e \sum_k \frac{\omega}{\sqrt{4\pi\omega}} \left[-ib_k e^{-i\omega t(\tau)} + ib_k^\dagger e^{i\omega t(\tau)} \right]$$

and it can be seen that the solution to the quantum Langevin equation has the expanded form

$$q(\tau) = e \sum_k \frac{\omega}{\sqrt{4\pi}} \left[-i\chi(\omega) b_k e^{-i\omega t(\tau)} + i\chi(\omega)^* b_k^\dagger e^{i\omega t(\tau)} \right] \tag{3.34}$$

where $\chi(\omega)$ is a quantity called the *oscillator susceptibility function* and has the form

$$\chi(\omega) = \frac{1}{-m\omega^2 - ie^2\Omega^2\omega + \mu^2}. \tag{3.35}$$

We can obtain a very useful relationship from (3.35), namely the *Fluctation Dissipation* theorem which states:

$$\mathrm{Im}|\chi(\omega)| = -e^2\Omega^2\omega|\chi(\omega)|^2. \tag{3.36}$$

Now that we have a solution to the quantum Langevin equation, we are in a position to calculate the overall energy flux of the system.

3.2.5 Calculation of Energy Flux.

We shall now calculate the overall energy flux of the system using the energy equations derived by Ford and O'Connell. The energy flux expressions are found from the energy-momentum tensor:

$$T^{\nu}_{\mu} = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\nu_\mu \mathcal{L}\right], \tag{3.37}$$

where $\mathcal{L}$ is the Lagrangian density of the free scalar field:

$$\mathcal{L} = \frac{1}{2}\sqrt{-\theta}\theta^{\mu\nu}\frac{\partial \phi}{\partial x^\mu}\frac{\partial \phi}{\partial x^\nu}.$$

Here $\theta^{\mu\nu}$ is the metric tensor of (3.23). The flux term we require is the expectation value of the T^0_1 component:

$$\langle \mathcal{J} \rangle = \langle T^0_1 \rangle = \mathrm{Re}\left\langle \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\frac{\partial \phi}{\partial r_*}\right\rangle, \tag{3.38}$$

and so

$$\langle \mathcal{J} \rangle = \left\langle \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial r_*}\right\rangle. \tag{3.39}$$

On substituting (3.30) into (3.38) we find that the total energy of the system is given by the relation

$$\langle \mathcal{J}(r_*,t)\rangle = \langle \mathcal{J}_0(r_*,t)\rangle + \langle \mathcal{J}_{dir}(r_*,t)\rangle + \langle \mathcal{J}_{int}(r_*,t),\rangle \tag{3.40}$$

where $\langle \mathcal{J}_0(r_*,t)\rangle$ is the *energy flux in the absence of the oscillator* and here it has the form

$$\langle \mathcal{J}_0(r_*,t)\rangle = \frac{1}{2}e^2\Omega \mathrm{Re}\left\{\left\langle \frac{\partial \phi^h(r_*,t)}{\partial t}\frac{\partial \phi^h(r_*,t)}{\partial r_*}\right\rangle\right\}; \tag{3.41}$$

the quantity $\langle \mathcal{J}_{dir}(r_*,t)\rangle$ is the direct flux arising from the oscillator alone and is determined by

$$\langle \mathcal{J}_{dir}(r_*,t)\rangle = -\frac{1}{2}\frac{r_*}{|r_*|}e^2\Omega\left\langle \dot{q}(\tau_{\mathrm{ret}})^2\right\rangle, \tag{3.42}$$

while the final term, $\langle \mathcal{J}_{int}(r_*,t)\rangle$ is the interference term, and is given by

$$\langle \mathcal{J}_{int}(r_*,t)\rangle = -\frac{1}{2}e^2\Omega \mathrm{Re}\left\{\left\langle \dot{q}(\tau_{\mathrm{ret}})\left[\frac{\partial \phi^h(r_*,t)}{\partial r_*} - \frac{r_*}{|r_*|}\frac{dt}{d\tau}\frac{\partial \phi^h(r_*,t)}{\partial t}\right]\right\rangle\right\}. \tag{3.43}$$

We shall now go on to show that the overall energy flux, $\langle \mathcal{J}(r_*,t)\rangle = 0$.

Proposition 3.2.1 *The energy flux in the absence of the oscillator,* $\langle \mathcal{J}_0(r_*,t)\rangle = 0$.

Proof: We shall prove this by direct calculation. We have that

$$\langle \mathcal{J}_0(r_*,t)\rangle = \frac{1}{2}e^2\Omega\mathrm{Re}\left\{\left\langle \frac{\partial\phi^h(r_*,t)}{\partial t}\frac{\partial\phi^h(r_*,t)}{\partial r_*}\right\rangle\right\}.$$

The expansion of the ϕ field is given by

$$\phi^h(r_*,t) = \sum_k \frac{1}{\sqrt{4\pi\omega}}\left[b_k e^{ikr_*-i\omega t} + b_k^\dagger e^{-ikr_*+i\omega t}\right],$$

and so we have the two derivatives,

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= \sum_k \frac{i\omega}{\sqrt{4\pi\omega}}\left[-b_k e^{ikr_*-i\omega t} + b_k^\dagger e^{-ikr_*+i\omega t}\right],\\ \frac{\partial\phi}{\partial r_*} &= \sum_{k'} \frac{ik'}{\sqrt{4\pi\omega'}}\left[-b_{k'} e^{ik'r_*-i\omega' t} + b_{k'}^\dagger e^{-ik'r_*+i\omega' t}\right].\end{aligned}$$

Thus we find that

$$\left\langle \frac{\partial\phi^h(r_*,t)}{\partial t}\frac{\partial\phi^h(r_*,t)}{\partial r_*}\right\rangle = -\sum_k\sum_{k'}\frac{\omega k'}{4\pi\sqrt{\omega\omega'}}\langle b_k b_{k'}^\dagger\rangle e^{i(k-k')r_*-i(\omega-\omega')t},$$

where we have used the expectation values (3.16), which means that the only non-zero term is $\langle b_k b_{k'}^\dagger\rangle = \delta_{kk'}$. We replace the expectation value with the Kronecker delta, and sum over repeated indices which gives

$$\left\langle \frac{\partial\phi^h(r_*,t)}{\partial t}\frac{\partial\phi^h(r_*,t)}{\partial r_*}\right\rangle = -\frac{1}{4\pi}\sum_k k.$$

Since k is the wave number to be summed over positive and negative k we find that the sum is zero and hence

$$\langle \mathcal{J}_0(r_*,t)\rangle = 0,$$

as desired.

Theorem 3.2.1 *The overall energy flux of the system,* $\langle \mathcal{J}(r_*,t)\rangle = 0$.

Proof: First we calculate the direct flux arising from the oscillator alone:

$$\langle \mathcal{J}_{dir}(r_*,t)\rangle = -\frac{1}{2}\frac{r_*}{|r_*|}e^2\Omega^2\left\langle \dot{q}(\tau_{\mathrm{ret}})^2\right\rangle,$$

We have by (3.34) that

$$q(\tau) = e\sum_k \frac{\omega}{\sqrt{4\pi\omega}}\left[-i\chi(\omega)b_k e^{-i\omega t(\tau)} + i\chi(\omega)^* b_k^\dagger e^{i\omega t(\tau)}\right],$$

and so,

$$\dot{q}(\tau) = -\sum_{k'} \frac{\omega'^2 \dot{t}(\tau)}{\sqrt{4\pi\omega'}}\left[\chi(\omega')b_{k'} e^{-i\omega' t(\tau)} + \chi(\omega')^* b_k^\dagger e^{i\omega' t(\tau)}\right].$$

This means that

$$\langle \dot{q}(\tau_{\mathrm{ret}})^2\rangle = \sum_k\sum_{k'} \frac{\omega^2\omega'^2 \dot{t}(\tau_{\mathrm{ret}})^2}{4\pi\sqrt{\omega\omega'}}\left[\chi(\omega)\chi(\omega')^*\langle b_k b_{k'}^\dagger\rangle e^{-i(\omega-\omega')t(\tau_{\mathrm{ret}})}\right].$$

Again, we use the fact that $\langle b_k b_{k'}^\dagger\rangle = \delta_{kk'}$, and $\omega_k = |k|$ to reduce the above expression to

$$\langle \dot{q}(\tau_{\mathrm{ret}})^2\rangle = \sum_k \frac{\omega^3 \dot{t}(\tau_{\mathrm{ret}})^2}{4\pi}|\chi(\omega)|^2.$$

Hence we have

$$\langle \mathcal{J}_{dir}(r_*,t)\rangle = \frac{e^2\Omega^2\dot{t}(\tau_{\mathrm{ret}})^2}{8\pi}\frac{r_*}{|r_*|}\sum_k \omega^3|\chi(\omega)|^2. \tag{3.44}$$

Finally we calculate the interference term which on the trajectory of the oscillator has the general form

$$\langle \mathcal{J}_{int}(r_*,t)\rangle = -\frac{e^2\Omega^2}{2}\mathrm{Re}\left\{\dot{q}(\tau_{\mathrm{ret}})\frac{r_*}{|r_*|}\dot{t}(\tau_{\mathrm{ret}})\frac{\partial\phi^h(0,t(\tau_{\mathrm{ret}}))}{\partial t}\right\}.$$

Putting in the derivatives and using (3.16) gives,

$$\langle \mathcal{J}_{int}(r_*,t)\rangle = \frac{e^2\Omega^2\dot{t}(\tau_{\mathrm{ret}})^2}{8\pi}\frac{r_*}{|r_*|}\mathrm{Re}\left\{\sum_k \omega^2 i\chi(\omega)^*\right\}.$$

After using the Flux Dissipation theorem expression of (3.36) and some rearrangement, we find that this is simply,

$$\langle \mathcal{J}_{int}(r_*,t)\rangle = -\frac{e^2\Omega^2\dot{t}(\tau_{\mathrm{ret}})^2}{8\pi}\frac{r_*}{|r_*|}\sum_k \omega^3|\chi(\omega)|^2,$$

i.e. we have obtained that

$$\langle \mathcal{J}_{dir}(r_*,t)\rangle = -\langle \mathcal{J}_{int}(r_*,t)\rangle.$$

Hence we see that

$$\begin{aligned} \langle \mathcal{J}(r_*, t) \rangle &= 0 + \langle \mathcal{J}_{dir}(r_*, t) \rangle - \langle \mathcal{J}_{dir}(r_*, t) \rangle \\ &= 0, \end{aligned}$$

and the total energy flux of the system is zero.

The fact that the overall energy flux of the system is zero is of no great surprise. Essentially our oscillator confined to the constant trajectory $r_* = 0$ in conformally flat Schwarzschild spacetime is entirely analogous to the flat spacetime example of Raine et al [6] and Ford and O'Connell [8] as discussed in Chapter 1.

We shall now make one change to our model. We shall change the trajectory of the oscillator from a constant trajectory to one where it is some arbitrary function of proper time; $r_* = r_*(\tau)$. As a result of this, the conformal factor previously constant- now becomes a function of proper time and this will have a substantial effect upon the overall energy flux of the system.

3.3 $2D$ Schwarzschild Spacetime with a non-Constant Trajectory.

3.3.1 Coordinate System.

We shall now examine the second case: that of a quantum oscillator confined to an inertial trajectory. The problem is the same as before except that now the oscillator is confined to the trajectory:

$$r_* = r_*(\tau), \quad t = t(\tau).$$

The trajectory of the oscillator is shown in the Penrose diagram of figure 3.2. As before we have the same line element, and we continue to use the Boulware vacuum. The expansion of the ϕ field is the same as before and the annhilation and creation operators continue to satisfy the commutation relations and expectation values of (3.15) and (3.16). The conformal factor is now a function of proper time, and we have that,

$$\Omega(\tau) = \left(1 - \frac{2M}{r(\tau)}\right). \tag{3.45}$$

3.3.2 Equations of Motion

Once again, we use the scalar electrodynamic action as before and so we have the same equation of motion for the oscillator as before:

$$m\frac{d^2q}{d\tau^2} + \mu^2 q = e\frac{d}{d\tau}\phi(r_*, t),$$

however, the equation of motion for the scalar field is subtly different. As we observed before, if we let $r = r_*$ which we have that

$$\frac{1}{\Omega^2(r)}\left[\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial r_*^2}\right] = -e\dot{q}(\tau)\delta(r_* - r_*^0, t - t^0).$$

Thus, the equation of motion for the scalar field, due to the delta function on the right hand side of the above equation, means that:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial r_*^2} = -e\Omega^2(\tau)\dot{q}(\tau)\delta(r_* - r_*^0, t - t^0). \tag{3.46}$$

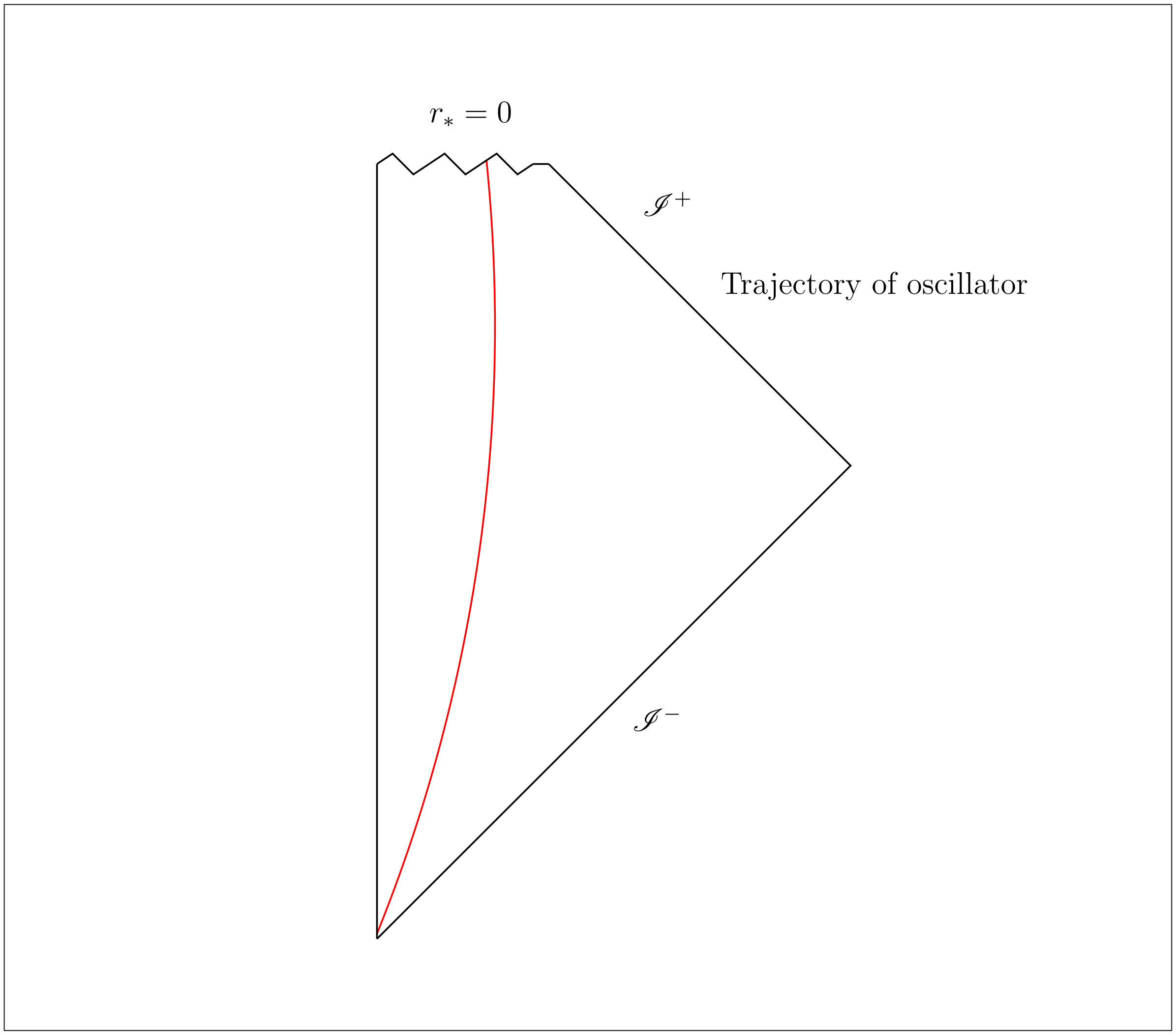

Figure 3.2: Penrose diagram showing the the trajectory of the accelerating oscillator. Now the oscillator eventually falls into the singularity located at $r_* = 0$

3.3.3 Quantum Langevin Equation.

We shall now solve equation (3.46) to determine $\phi(r_*, t)$ Once we have obtained an expression for ϕ, we can substitute it into the equation of motion for the oscillator and determine the Quantum Langevin equation. The general solution to (3.46) is

$$\phi(r_*, t) = \phi^h(r_*, t) + \phi^{\mathcal{G}}(r_*, t), \tag{3.47}$$

where $\phi^h(r_*, t)$ is the homogeneous solution, and $\phi^{\mathcal{G}}(r_*, t)$ is the solution found using the appropriate Green's function. Once again, we shall say that at late times the homogeneous solution give zero contribution, so we are left only with contribution from the Green's function.

So, we have that

$$\begin{aligned} \phi^{\mathcal{G}}(r_* t) &= -e^2 \int_{-\infty}^{\infty} \int_{-\infty}^{t-|r_*-r_*^0|} \Omega^2(\tau)\delta(t(\tau) - t(\tau') - |r_*(\tau) - r_*(\tau')|)\frac{dq(\tau')}{d\tau'} \\ & \quad \delta(r_*(\tau') - r_*^0(\tau), t(\tau') - t^0(\tau'))\ dr_*' dt' \end{aligned}$$

Proceeding as before, this gives

$$\phi(r_*, t) = -e\Omega^2(\tau_{\mathrm{ret}})q(\tau_{\mathrm{ret}}). \tag{3.48}$$

We now now substitute this into the equation of motion for the oscillator, and after some rearrangement, we obtain the new quantum Langevin equation:

$$m\frac{d^2q}{d\tau^2} + e^2\Omega^2(\tau)\frac{dq}{d\tau} + [2e^2\Omega(\tau)\dot{\Omega}(\tau) + \mu^2]q(\tau) = F(\tau), \tag{3.49}$$

where $F(\tau)$ is the fluctuating force operator as before.

3.3.4 Calculation of Energy Flux.

Since we are using the same metric of the previous section, the total energy flux of the system is still:

$$\langle \mathcal{J}(r_*, t)\rangle = \langle \mathcal{J}_0(r_*, t)\rangle + \langle \mathcal{J}_{dir}(r_*, t)\rangle + \langle \mathcal{J}_{int}(r_*, t)\rangle, \tag{3.50}$$

where

$$\langle \mathcal{J}_0(r_*, t)\rangle = \frac{1}{2}\zeta \mathrm{Re}\left\{\left\langle \frac{\partial \phi^h(r_*, t)}{\partial t}\frac{\partial \phi^h(r_*, t)}{\partial r_*}\right\rangle\right\}.$$

The direct flux term, $\langle J_{dir}(r_*, t)\rangle$ is given by

$$\langle \mathcal{J}_{dir}(r_*, t)\rangle = -\frac{\zeta}{2}\mathrm{Re}\left\{\left\langle \frac{\partial q(\tau_{\mathrm{ret}})}{\partial t}\frac{\partial q(\tau_{\mathrm{ret}})}{\partial r_*}\right\rangle\right\}, \tag{3.51}$$

and the interference term is found from the relation

$$\langle \mathcal{J}_{int}(r_*, t)\rangle = \frac{\zeta}{2}\mathrm{Re}\left\{\left\langle \frac{\partial \phi}{\partial t}\frac{\partial q(\tau_{\mathrm{ret}})}{\partial r_*} + \frac{\partial q(\tau_{\mathrm{ret}})}{\partial t}\frac{\partial \phi}{\partial r_*}\right\rangle\right\} \tag{3.52}$$

where $\zeta = e^2\Omega^2(\tau)$ is the damping coefficient. As before, the energy flux in the absence of the oscillator, $\langle \mathcal{J}(r_*, t)\rangle = 0$.

Theorem 3.3.1 *The quantum oscillator radiates as a result of its motion through the spacetime. i.e. the overall energy flux of the system* $\langle \mathcal{J}(r_*, t)\rangle \neq 0$

Proof. We have seen that

$$\langle \mathcal{J}_{dir}(r_*, t)\rangle = -\frac{\zeta}{2}\mathrm{Re}\left\{\left\langle \frac{\partial q(\tau_{\mathrm{ret}})}{\partial t}\frac{\partial q(\tau_{\mathrm{ret}})}{\partial r_*}\right\rangle\right\},$$

Now, using the chain rule we can write

$$\frac{\partial q(\tau_{\mathrm{ret}})}{\partial t} = \frac{dq(\tau_{\mathrm{ret}})}{d\tau_{\mathrm{ret}}}\frac{\partial \tau_{\mathrm{ret}}}{\partial t},$$

and

$$\frac{\partial q(\tau_{\mathrm{ret}})}{\partial r_*} = \frac{dq(\tau_{\mathrm{ret}})}{d\tau_{\mathrm{ret}}}\frac{\partial \tau_{\mathrm{ret}}}{\partial r_*}.$$

Now, using the chain rule on our retarded time relation

$$t - T(\tau_{\mathrm{ret}}) = r_* - R_*(\tau_{\mathrm{ret}}),$$

we obtain the very useful result that

$$\frac{\partial \tau_{\mathrm{ret}}}{\partial r_*} = -\frac{\partial \tau_{\mathrm{ret}}}{\partial t}. \tag{3.53}$$

Now this means that

$$\frac{\partial q(\tau_{\rm ret})}{\partial t} = -\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\frac{\partial \tau_{\rm ret}}{\partial r_*}, \quad \text{and} \quad \frac{\partial q(\tau_{\rm ret})}{\partial r_*} = \frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\frac{\partial \tau_{\rm ret}}{\partial r_*}. \tag{3.54}$$

Now substituting this into (3.51) we see that the direct flux from the source alone is

$$\langle \mathcal{J}_{dir}(r_*, t)\rangle = \frac{\zeta}{2}\mathrm{Re}\left\{\frac{\partial \tau_{\rm ret}}{\partial r_*}\left\langle\left(\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\right)^2\right\rangle\right\}. \tag{3.55}$$

Now we calculate the interference term:

$$\langle \mathcal{J}_{int}(r_*, t)\rangle = \frac{\zeta}{2}\mathrm{Re}\left\{\left\langle\frac{\partial \phi}{\partial t}\frac{\partial q(\tau_{\rm ret})}{\partial r_*} + \frac{\partial q(\tau_{\rm ret})}{\partial t}\frac{\partial \phi}{\partial r_*}\right\rangle\right\}.$$

Using the Chain rule we can write this as

$$\begin{aligned}\langle \mathcal{J}_{int}(r_*, t)\rangle &= \frac{\zeta}{2}\mathrm{Re}\left\{\left\langle\frac{\partial \phi}{\partial t}\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\frac{\partial \tau_{\rm ret}}{\partial r_*} + \frac{dq(\tau_{\rm ret})}{dt}\frac{\partial \tau_{\rm ret}}{\partial t}\frac{\partial \phi}{\partial r_*}\right\rangle\right\} \\ &= \frac{\zeta}{2}\mathrm{Re}\left\{\frac{\partial \tau_{\rm ret}}{\partial r_*}\left\langle\frac{\partial \phi}{\partial t}\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}} - \frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\frac{\partial \phi}{\partial r_*}\right\rangle\right\}\end{aligned}$$

Thus we have obtained

$$\langle \mathcal{J}_{int}(r_*, t)\rangle = \frac{\zeta}{2}\mathrm{Re}\left\{\frac{\partial \tau_{\rm ret}}{\partial r_*}\left\langle\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}}\left(\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial r_*}\right)\right\rangle\right\} \tag{3.56}$$

Comparison of (3.55) with (3.56) shows that unless

$$\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial r_*} = -\frac{dq(\tau_{\rm ret})}{d\tau_{\rm ret}} \tag{3.57}$$

then $\langle \mathcal{J}_{int}(r_*, t)\rangle$ and $\langle \mathcal{J}_{dir}(r_*, t)\rangle$ cannot add to give an overall zero net energy flux. The resulting radiation is due to the conformal factor being a function of time $\Omega = \Omega(\tau)$. We have the solution for the equation of motion for the scalar field as $\phi = \phi^h - e\Omega^2(\tau)q(\tau)$. So,

$$\frac{\partial \phi}{\partial t} = -e\frac{\partial}{\partial t}(\Omega^2(\tau)q(\tau)) = -e\frac{\partial}{\partial \tau}(\Omega^2 q)\frac{d\tau}{dt} = -2e\frac{\partial}{\partial \tau}(\Omega^2 q)\frac{d\tau}{dr_*}.$$

Similarly,

$$\frac{\partial \phi}{\partial r_*} = -e\frac{\partial}{\partial \tau}(\Omega^2 q)\frac{d\tau}{dr_*}.$$

Thus the interference term is now

$$\langle \mathcal{J}_{int}\rangle = -2e\zeta\mathrm{Re}\left\{\frac{d\tau}{dr_*}\left\langle\frac{dq}{d\tau}\frac{\partial}{\partial \tau}(\Omega^2 q)\right\rangle\right\}.$$

It is clear that if Ω^2 is a constant and not a function of proper time τ, then

$$\langle \mathcal{J}_{int} \rangle = -\frac{\zeta}{2}\mathrm{Re}\left\{ \frac{d\tau}{dr_*} \left\langle \left(\frac{dq}{d\tau} \right)^2 \right\rangle \right\},$$

which is the direct energy flux term, $\langle J_{dir} \rangle$. Thus, the radiation depends on the conformal factor being a function of time. $\Box$

We have shown the first important result, a quantum oscillator confined to a general trajectory $r_* = r_*(\tau)$ and $t = t(\tau)$ in two dimensional Schwarzschild spacetime radiates as a result of such motions. The radiation seems to been brought about due to the presence of the conformal factor. In the previous case, Ω was constant, however in the case we have just examined, it was a function of proper time. The conformal factor is dependent on both the trajectory of the oscillator and the spacetime in question. So in fact the radiation has been brought about by a choice of trajectory in this spacetime.

Chapter 4

Quantum Langevin Approach to Hawking Radiation

In this chapter we shall be interested in deriving an expression for Hawking radiation using the method as outlined by Louisell (and discussed in Chapter 1). In particular, we shall use a Hamiltonian to derive expressions for the annhilation and creation operators of the free scalar field and the oscillator. We shall then solve these so that we can form an expression for the position function of the oscillator, and hence calculate the net energy flux of the system as we did in chapter 3. This quantum Langevin approach will be our method for calculating Hawking radiation in the last section of the chapter. However first we shall demonstrate that it produces results in agreement with the standard results. Hence we consider two cases before the Schwarzschild case: on oscillator on an $x =$ constant trajectory, and an oscillator undergoing constant acceleration both in Minkowski spacetime.

4.1 Quantum Oscillator confined to a constant Trajectory in Flat Spacetime using the Heisenberg Picture

4.1.1 Hamiltonian

The Hamiltonian for the system is comprised of three parts: the Hamiltonian for the quantum harmonic oscillator, H_{osc}, the Hamiltonian for the free scalar field H_{sf} and finally the Hamiltonian which describes the interaction between the oscillator and the free scalar field, H_{int}. We shall confine the oscillator to a constant trajectory, so in Minkowski coordinates:

$$x = x(t) = \text{constant}. \tag{4.1}$$

The Hamiltonian for a harmonic oscillator is:

$$H_{\text{osc}} = \hbar\omega_c a^\dagger a \tag{4.2}$$

where ω_c is the angular frequency of the oscillator, and the operators a and $a^\dagger$ are the annihilation and creation operators (respectively) for the quantum oscillator. We can derive the Hamiltonian for the free scalar field from the usual relation

$$H_{\text{sf}} = \Pi\dot{\phi} - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} \tag{4.3}$$

Here we have taken dots to denote derivatives with respect to time, and $\mathcal{L}$ is the Lagrangian density for the scalar field which is given by

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} \tag{4.4}$$

Since we are working in two dimensional Minkowski spacetime, the metric tensor is just

$$\eta_{\mu\nu} = \eta^{\mu\nu}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad \sqrt{-g} = 1,$$

thus we obtain the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2\right]$$

and so by (4.3), the Hamiltonian for the scalar field must be

$$H_{\text{sf}} = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2\right].$$

Now, if we expand the ϕ field in a box of length V and impose periodic boundary conditions we have the expansion:

$$\phi(t,x) = \sum_k \sqrt{\frac{2\pi}{\omega_k V}}[b_k e^{-i\omega_k t + ikx} + b_k^\dagger e^{i\omega_k t - ikx}], \tag{4.5}$$

where b_k and $b_k^\dagger$ are the annihilation and creation operators respectively for the scalar field, x and t are the Minkowski position and time coordinates respectively and ω_k is the angular frequency of the scalar field. It is clear from (4.4) that the Hamiltonian density for the scalar field in (4.3) will be

$$H_{\text{sf}} = \Pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2$$

and so, upon using the expansion for ϕ given above we obtain

$$H_{\text{sf}} = \sum_k \omega_k b_k^\dagger b_k. \tag{4.6}$$

Finally, we come to the interaction of the scalar field and the oscillator. We shall use a scalar electrodynamic form of interaction [6] given by:

$$H_{\text{int}} = \frac{1}{2m}(p + e\phi)^2 + \omega_c^2 q$$

where p is the momentum operator (defined in Chapter 1), and q is the position operator of the oscillator. Substituting in our definitions for p, q and ϕ gives:

$$H_{\text{int}} = \frac{ie}{\sqrt{2\omega_c}m}\sum_k e_k[a^\dagger - a]\left(b_k e^{ikx} + b_k^\dagger e^{-ikx}\right) \tag{4.7}$$

where we have used the abbreviation

$$e_k = \sqrt{\frac{2\pi}{\omega_k V}}, \tag{4.8}$$

and as in Chapter 3, we have the dispersion relation

$$\omega_k = |k|. \tag{4.9}$$

So, we have the Hamiltonian for the model:

$$\mathscr{H} = \omega_c a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + \frac{ie}{\sqrt{2\omega_c m}} \sum_k e_k [a^\dagger - a] \left(b_k e^{ikx} + b_k^\dagger e^{-ikx} \right). \tag{4.10}$$

4.1.2 Equations of Motion.

We now use the Hamiltonian established in the previous subsection to derive the equations of motion for both the oscillator and the scalar field. We shall start with the equation of motion for the oscillator. Using the usual commutation relation, we have that

$$\frac{da}{dt} = \frac{1}{i}[\mathscr{H}, a]$$

which gives

$$\frac{da}{dt} = -i\omega_c[a, a^\dagger a] + \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k([a^\dagger, a] - [a, a]) \left(b_k e^{ikx} + b_k^\dagger e^{-kx} \right).$$

Now, we know that in general, operators $\mathscr{M}$, a and $a^\dagger$ in the Heisenberg picture satisfy the identity

$$[\mathscr{M}, a^\dagger a] = [\mathscr{M}.a^\dagger]a + a^\dagger[\mathscr{M}, a] \tag{4.11}$$

and further, the commutation relations satisfy

$$[\mathscr{M}, a] = -\frac{\partial \mathscr{M}}{\partial a}, \text{ and } [\mathscr{M}, a^\dagger] = \frac{\partial \mathscr{M}}{\partial a} \tag{4.12}$$

Hence our equation of motion for the quantum oscillator is

$$\frac{da}{dt} = -i\omega_c a + \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k \left[b_k e^{ikx} + b_k^\dagger e^{-ikx} \right]. \tag{4.13}$$

Now we find the equation of motion for the free scalar field. Again we have the relation:

$$\frac{db_k}{dt} = \frac{1}{i\hbar}[\mathscr{H}, b_k]. \tag{4.14}$$

So, this gives

$$\frac{db_k}{dt} = -i\sum_k \omega_k [b_k, b_k^\dagger b_k] + \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k (a^\dagger - a)\left[[b_k, b_k] e^{ikx} + [b_k, b_k^\dagger] e^{-ikx} \right],$$

which means the equation of motion for the free scalar field is:

$$\frac{db_k}{dt} = -i\omega_k b_k + \frac{ee_k}{\sqrt{2\omega_c m}} (a^\dagger - a) e^{-ikx}. \tag{4.15}$$

We can solve the differential equation in (4.15) for the field operator b_k by use of an integrating factor in the usual way:

$$\frac{d}{dt}\left\{ e^{i\omega_k t} b_k \right\} = b_k(0) + \frac{ee_k}{\sqrt{2\omega_c m}} e^{-ikx} \int_0^t [a^\dagger(t') - a(t')] e^{i\omega_k t'}\, dt',$$

We now have an expression for the field creation operator:

$$b_k(t) = e^{i\omega_k t} b_k(0) + \frac{ee_k}{\sqrt{2\omega_c m}} e^{ikx} \int_0^t [a(t') - a^\dagger(t')] e^{-i\omega_k (t'-t)}\, dt'. \tag{4.16}$$

Clearly, if we take the hermitian conjugate of (4.16) we obtain the corresponding expression for the field annihilation operator:

$$b_k^\dagger(t) = e^{-i\omega_k t} b_k^\dagger(0) + \frac{ee_k}{\sqrt{2\omega_c m}} e^{-ikx} \int_0^t [a^\dagger(t') - a(t')] e^{i\omega_k (t'-t)}\, dt'. \tag{4.17}$$

We can determine the solution to the scalar field equation of motion using these two operators. We take (4.16) and multiply it by e^{ikx}, and similarly we take (4.17) and multiply it by e^{-ikx} and adding the resulting expressions gives:

$$\begin{aligned} b_k(t)e^{ikx} + b_k^\dagger(t) e^{-ikx} = {} & e^{i\omega_k t} b_k(0) + \frac{ee_k}{\sqrt{2\omega_c m}} e^{ikx} \int_0^t [a(t') - a^\dagger(t')] e^{-i\omega_k (t'-t)}\, dt' \\ & + e^{-i\omega_k t} b_k^\dagger(0) + \frac{ee_k}{\sqrt{2\omega_c m}} e^{-ikx} \int_0^t [a^\dagger(t') - a(t')] e^{i\omega_k (t'-t)}\, dt' \end{aligned}$$

Multiplying through by e_k and summing over k gives:

$$\begin{aligned} \sum_k e_k [b_k(t) e^{ikx} + b_k^\dagger(t) e^{-ikx}] = {} & \sum_k e_k [e^{i\omega_k t} b_k(0) + e^{-i\omega_k t} b_k^\dagger(0)] + \\ & \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k^2 \int_0^t [a^\dagger(t') - a(t')] \left(e^{i\omega_k (t'-t)} - e^{-i\omega_k (t'-t)} \right) dt', \end{aligned}$$

i.e.

$$\phi(t,x) = \phi^h(t,x) + \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k^2 \int_0^t [a^\dagger(t') - a(t')] \left(e^{i\omega_k (t'-t)} - e^{-i\omega_k (t'-t)} \right) dt', \tag{4.18}$$

where ϕ^h is the homogeneous part, of the solution, and the integral after it is the particular integral which we shall now find. Using the relations of chapter 1 which relate position operator q, and momentum operator p , of the oscillator to the annhilation and creation operators:

$$a^\dagger = \frac{1}{\sqrt{2\omega_c}}[\omega_c q - ip], \quad \text{and} \quad a = \frac{1}{\sqrt{2\omega_c}}[\omega_c q + ip],$$

we can now write (4.18) as

$$\phi(t,x) = \phi^h(t,x) - \frac{e}{\omega_c m}\sum_k e_k^2 \int_0^t ip(t')\left(e^{i\omega_k(t'-t)} - e^{-i\omega_k(t'-t)}\right)\,dt'. \tag{4.19}$$

If we convert the sum over k to an integral over $d\omega_k$ using the prescription as $V \to \infty$

$$\sum_k \{...\} = \frac{V}{2\pi}\int ...\,d\omega_k,$$

then (and using the relations between p and q)

$$\phi(t,x) = \phi^h(t,x) - \frac{e}{\omega_c m}\frac{V}{2\pi}\int_{-\infty}^{\infty}\frac{2\pi}{\omega_k V}\,d\omega_k \int_0^t i\dot{q}(t')\left(e^{i\omega_k(t'-t)} - e^{-i\omega_k(t'-t)}\right)\,dt'.$$

We can now write the above exponentials in the integral over $d\omega_k$ as a Heaviside function:

$$\phi(t,x) = \phi^h(t,x) - \frac{e}{\omega_c m}\int_0^t i\dot{q}(t')\Theta(t'-t)dt',$$

and so we have

$$\phi(t,x) = \phi^h(t,x) - \frac{e\pi}{\omega_c m}q(t), \tag{4.20}$$

which we shall write as

$$\phi(t,x) = \phi^h(t,x) - \alpha q(t), \text{ where } \alpha = \frac{e\pi}{\omega_c m}, \tag{4.21}$$

for convenience.

We shall now continue to find a differential equation for the annihilation operator $a(t)$. We substitute (4.16) and (4.17) into (4.13):

$$\frac{da}{dt} = -\omega_c a + \frac{e}{\sqrt{2\omega_c m}}\sum_k e_k\Bigg\{e^{ikx} \times \left[e^{-i\omega_k t}b_k(0) + \frac{ee_k}{\sqrt{2\omega_c m}}e^{-ikx}\int_0^t [a^\dagger - a]e^{i\omega_k(t'-t)}\,dt'\right]$$
$$e^{-ikx} \times \left[e^{i\omega_k t}b_k^\dagger(0) + \frac{ee_k}{\sqrt{2\omega_c m}}e^{ikx}\int_0^t [a - a^\dagger]e^{-i*\omega_k(t'-t)}\,dt'\right]\Bigg\}$$

which, after some simplification we will write as,

$$\frac{da}{dt} = \mathcal{G}_a(t) - i\omega_c a + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \left\{\int_0^t [a^\dagger - a]\left(e^{i\omega_k(t'-t)} - e^{-i\omega_k(t'-t)}\right)\right\} \tag{4.22}$$

where

$$\mathcal{G}_a = \frac{e}{\sqrt{2\omega_c}m}\sum_k e_k \left\{b_k(0)e^{-i\omega_k t + ikx} + b_k^\dagger(0)e^{i\omega_k t - ikx}\right\}. \tag{4.23}$$

We can simplify this equation further by removing the high frequency behavior. This is easily done; let

$$a(t) = e^{-i\omega_c t}A(t), \tag{4.24}$$

and now (4.22) becomes

$$\frac{dA}{dt} = \mathcal{G}_A(t) + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \left\{\int_0^t \left[e^{i\omega_c(t'+t)}A^\dagger - e^{-i\omega_c(t'-t)}A\right]\left(e^{i\omega_k(t'-t)} - e^{-i\omega_k(t'-t)}\right)\,dt'\right\}. \tag{4.25}$$

We have now arrived at a first order differential equation for the operator $A(t)$. We shall now go on to solve this equation and obtain expressions for the annihilation and creation operators of the quantum oscillator.

4.1.3 Calculation of $A(t)$ and $A^\dagger(t)$

In order to solve (4.25), we first multiply out the brackets contained within the sum over k. When we do this, we obtain four exponential terms. In fact we can simplify the expression further by noting that three of the exponential terms we obtain are of the form $e^{i(\omega_c+\omega_k)t}$. The only contribution these exponentials make is to induce rapid oscillations into the system, oscillations which the system cannot adapt and respond to (and so average to zero). So, it is reasonable to neglect these three exponential terms. We also use the rotating-wave approximation which means we can neglect the $a^\dagger$ term, and as a result we arrive at a simpler form of:

$$\frac{dA}{dt} = \mathcal{G}_A(t) - \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \left\{e^{i(\omega_c-\omega_k)t}\int_0^t A(t')e^{-i(\omega_c-\omega_k)t'}\,dt'\right\}. \tag{4.26}$$

Now, we multiply both sides of this equation by the term e^{-st}, then integrate both sides (i.e take the Laplace transform of the equation):

$$s\overline{A}(s) = \overline{\mathcal{G}}_A(s) - \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \left\{ \int_0^\infty e^{i(\omega_c-\omega_k)t-st}\ dt \int_0^t A(t')e^{-i(\omega_c-\omega_k)t'}.\ dt' \right\}.$$

We now change the order of integration for t and t':

$$s\overline{A}(s) = \overline{\mathcal{G}}_A(s) - \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \left\{ \int_{t'}^\infty e^{i(\omega_c-\omega_k+is)t}\ dt \int_0^t A(t')e^{-i(\omega_c-\omega_k)t'}\ dt' \right\}.$$

Performing the t integration first leaves us with

$$s\overline{A}(s) = \overline{\mathcal{G}}_A(s) - \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \int_0^\infty \frac{e^{-st'}A(t')}{i(\omega_c-\omega_k+is)}\ dt',$$

which we can write as

$$\overline{A}(s) = \frac{\overline{\mathcal{G}}(s)}{s + \frac{e^2}{2\omega_c m^2}\sum_k \frac{e_k^2}{i(\omega_c-\omega_k+is)}}. \tag{4.27}$$

Now we apply the technique used by Louisell as discussed in Chapter 1: we convert the sum over k into an integral over $d\omega_k$, and take the limit $s \to 0$, i.e.:

$$-i\sum_k \frac{e_k^2}{\omega_k-\omega_c-is} \to -i\lim_{s\to 0}\left\{ \frac{V}{2\pi}\int \frac{e_k^2}{\omega_k-\omega_c-is}\ d\omega_k \right\},$$

where we have again used the prescription

$$\sum_k \{...\} \to \frac{V}{2\pi}\int\ ...d\omega_k.$$

We perform the Wigner-Weisskopf approximation by taking the above limit and integrating:

$$-i\lim_{s\to 0}\left\{ \frac{V}{2\pi}\int_0^\infty \frac{e_k^2}{\omega_k-\omega_c-is}\ d\omega_k \right\} = -i\frac{V}{2\pi}\int_0^\infty e_k^2 \left[\frac{1}{\omega_k-\omega_c} + i\pi\delta(\omega_k-\omega_c) \right]\ d\omega_k = \frac{\gamma}{2}+i\Delta\omega \tag{4.28}$$

and since $e_k^2 = 2\pi/\omega_k V$, where we have that

$$\gamma = \frac{\pi e^2}{\omega_c^2 m^2},\ \text{and}\ \Delta\omega = -\int \frac{e_k^2}{\omega_k-\omega_c}\ d\omega_k, \tag{4.29}$$

and hence we may now write the expression we obtained in (4.27) in the simpler form:

$$\overline{A}(s) = \frac{\overline{\mathcal{G}}(s)}{s+\frac{\gamma}{2}+i\Delta\omega}. \tag{4.30}$$

We shall now ignore small frequency shifts in the system, and so as a result of applying the Wigner-Weiskpoff approximation, we can in fact now replace our earlier expression for the annihilation operator of the quantum oscillator given in (4.25) with the simple linear first order differential equation:

$$\frac{dA}{dt} = -\frac{\gamma}{2}A(t) + \mathcal{G}_A(t) \tag{4.31}$$

Integrating this equation directly we have

$$A(t) = e^{-\gamma t/2} \int_0^t \mathcal{G}(t')e^{\gamma t'/2}\, dt'.$$

Recalling that

$$\mathcal{G}_A(t') = \frac{e}{\sqrt{2\omega_c} m} \sum_k e_k \left\{ b_k(0) e^{i(\omega_c - \omega_k)t' + ikx} + b_k^\dagger(0) e^{i(\omega_c + \omega_k)t' - ikx} \right\},$$

we have that

$$A(t) = \frac{e\, e^{-\gamma t/2}}{\sqrt{2\omega_c} m} \sum_k e_k \left\{ b_k(0)e^{ikx} \int_0^t e^{i(\omega_c - \omega_k - i\gamma/2)t'}\, dt' + b_k^\dagger(0) e^{-ikx} \int_0^t e^{i(\omega_c + \omega_k - i\gamma/2)t'} dt' \right\}.$$

Note that we can take the e^{ikx} exponentials outside the integrands since in this case, the oscillator is on a constant trajectory and so x has no t dependence. It is a straightforward task to perform the t' integrations:

$$\begin{aligned} A(t) = \frac{e\, e^{-\gamma t/2}}{\sqrt{2\omega_c} m} \sum_k e_k \Bigg\{ & b_k(0)e^{ikx} \left[\frac{e^{i(\omega_c - \omega_k - i\gamma/2)t}}{i(\omega_c - \omega_k - \gamma/2)} - \frac{1}{i(\omega_c - \omega_k - i\gamma/2)} \right] \\ & b_k^\dagger(0)e^{-ikx} \left[\frac{e^{i(\omega_c + \omega_k - i\gamma/2)t}}{i(\omega_c + \omega_k - i\gamma/2)} - \frac{1}{i(\omega_c + \omega_k - i\gamma/2)} \right] \Bigg\}. \end{aligned}$$

Now, we can ignore the two terms in the above expression which have a $e^{-\gamma t/2}$ term in the numerator since any such term will decay to zero as t increases. Further, we shall define the function,

$$\chi(\omega_k) = \frac{1}{\omega_c + \omega_k - i\gamma/2}, \tag{4.32}$$

and hence we finally arrive at expressions for the annihilation and creation operators of the quantum oscillator:

$$A(t) = -\frac{ie}{\sqrt{2\omega_c} m} \sum_k e_k \left\{ b_k(0)\chi(-\omega_k) e^{i(\omega_c - \omega_k)t + ikx} + b_k^\dagger(0)\chi(\omega_k) e^{i(\omega_c + \omega_k)t - ikx} \right\}, \tag{4.33}$$

and

$$A^\dagger(t) = \frac{ie}{\sqrt{2\omega_c}m} \sum_k e_k \left\{ b_k^\dagger(0)\chi^*(-\omega_k)e^{-i(\omega_c-\omega_k)t-ikx} + b_k(0)\chi^*(\omega_k)e^{-i(\omega_c+\omega_k)t-ikx} \right\}. \tag{4.34}$$

Now that we have obtained $A(t)$ and $A^\dagger(t)$, we can go on to calculate the overall energy flux of the system, using exactly the same method as we did in the preceding chapter. We first need to find the expression for the energy flux $\langle \mathcal{J} \rangle$, and then form for the quantum oscillator function $q(t)$. Determining an expression for $q(t)$ is a straightforward process. We know from the definition of the quantum harmonic oscillator that its position equation q is defined as

$$q = \frac{1}{\sqrt{2\omega_c}}[a^\dagger + a]. \tag{4.35}$$

Taking into account (4.24), we must have that,

$$q(t) = \frac{1}{\sqrt{2\omega_c}} \left[e^{i\omega_c t} A^\dagger(t) + e^{-i\omega_c t} A(t) \right],$$

and hence, after substituting in (4.33) and (4.34), and some trivial rearrangement, we have that,

$$q(t) = \frac{ie}{2\omega_c m} \sum_k e_k \left\{ e^{-i\omega_k t + ikx} b_k(0)(\chi^*(\omega_k) - \chi(-\omega_k)) + e^{i\omega_k t - ikx} b_k^\dagger(0)(\chi^*(-\omega_k) - \chi(\omega_k)) \right\}. \tag{4.36}$$

In the proceeding section, we shall require $\dot{q}(t)$, so differentiating (4.132) with respect to t gives,

$$\dot{q}(t) = \frac{-e}{2\omega_c m} \sum_k \omega_k e_k \left\{ -e^{-i\omega_k t + ikx} b_k(0)(\chi^*(\omega_k) - \chi(-\omega_k)) + e^{i\omega_k t - ikx} b_k^\dagger(0)(\chi^*(-\omega_k) - \chi(\omega_k)) \right\}. \tag{4.37}$$

We also see that if we differentiate with respect to the spatial coordinate x, then we have that

$$\frac{dq}{dx} = \pm \dot{q}(t). \tag{4.38}$$

4.1.4 Total Energy Flux of the System.

From Chapter 1, we saw that the energy density $\mathscr{E}$ is the component T_0^0 of the stress tensor and is given by,

$$T_0^0 = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}\dot{\phi} - \mathscr{L},$$

and hence we have that,

$$\mathscr{E} = \frac{1}{2}\left(\dot{\phi}^2 + \phi'^2\right) \tag{4.39}$$

We now find the energy flux which is the $\langle \mathcal{J} \rangle_0 = T_1^0$ component of the energy stress tensor. It is clear that from our earlier definition of T_ν^μ that,

$$\langle \mathcal{J} \rangle = \frac{1}{2}\left\langle \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t} \right\rangle. \tag{4.40}$$

Now, recall the solution to the scalar field wave equation is,

$$\phi(t,x) = \phi_h(t,x) - \alpha q(t), \quad \text{for} \quad \alpha = \frac{e\pi}{\omega_c m}, \tag{4.41}$$

where $q(t)$ is position equation of the oscillator, and $\phi_h(t,x)$ is the homogeneous solution of the wave equation, which when expanded in a box of length V:

$$\phi(t,x)^h = \sum_k e_k[b_k(0)e^{-i\omega_k t + ikx} + b_k^\dagger(0)e^{i\omega_k t - ikx}], \tag{4.42}$$

and,

$$\frac{\partial \phi_h}{\partial t} = i\sum_{k>0} ke_k[-b_k(0)e^{-i\omega_k t+ikx} + b_k^\dagger(0)e^{i\omega_k t-ikx}], \quad \frac{\partial \phi_h}{\partial x} = i\sum_{k>0} ke_k[b_k(0)e^{-i\omega_k t+ikx} - b_k^\dagger(0)e^{i\omega_k t-ikx}].$$

Hence if:

$$\langle \mathcal{J} \rangle = \left\langle \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t} \right\rangle,$$

then,

$$\langle \mathcal{J} \rangle = \left\langle \left(\frac{\partial \phi_h}{\partial x} - \alpha\frac{dq}{dx}\right)\left(\frac{\partial \phi_h}{\partial t} - \alpha\dot{q}(t)\right) \right\rangle.$$

which by (4.38) means that we can write,

$$\langle \mathcal{J} \rangle = \left\langle \left(\frac{\partial \phi_h}{\partial x} + \alpha\dot{q}(t)\right)\left(\frac{\partial \phi_h}{\partial t} - \alpha\dot{q}(t)\right) \right\rangle$$

and so expanding this out we obtain three terms:

$$\langle \mathcal{J} \rangle = \mathrm{Re}\left\{ \left\langle \frac{\partial \phi_h}{\partial x}\frac{\partial \phi_h}{\partial t} \right\rangle \right\} + \mathrm{Re}\left\{ \left\langle \alpha \dot{q}\frac{\partial \phi_h}{\partial t} - \frac{\partial \phi_h}{\partial x}\alpha \dot{q} \right\rangle \right\} - \mathrm{Re}\left\{ \left\langle \alpha^2 \dot{q}^2(t) \right\rangle \right\} \tag{4.43}$$

where as in the preceding chapter $\langle \mathcal{J} \rangle_0$ is the energy flux produced by the scalar field alone,

$$\langle \mathcal{J} \rangle_0 = \mathrm{Re}\left\{ \left\langle \frac{\partial \phi_h}{\partial x}\frac{\partial \phi_h}{\partial t} \right\rangle \right\} \tag{4.44}$$

and this in general is always zero. $\langle \mathcal{J} \rangle_{\mathrm{dir}}$ is the direct energy flux which originates from just the oscillator:

$$\langle \mathcal{J} \rangle_{\mathrm{dir}} = -\alpha^2 \mathrm{Re}\left\{ \left\langle \dot{q}^2(t) \right\rangle \right\}, \tag{4.45}$$

and finally we have the interference term $\langle \mathcal{J} \rangle_{\mathrm{int}}$ which is given by

$$\langle \mathcal{J} \rangle_{\mathrm{int}} = \alpha \mathrm{Re}\left\{ \left\langle \dot{q}\frac{\partial \phi_h}{\partial t} - \frac{\partial \phi_h}{\partial x}\dot{q} \right\rangle \right\}. \tag{4.46}$$

We calculate the direct flux first. From (4.37) that the product

$$\langle \dot{q}\dot{q} \rangle = -\frac{e^2}{4m^2\omega_c^2}\sum_k \omega_k^2 e_k^2 \,\mathrm{Re}\left\{ \chi^*(\omega_k)\chi^*(-\omega_k) - |\chi(-\omega_k)|^2 - |\chi(\omega_k)|^2 + \chi(\omega_k)\chi(-\omega_k) \right\},$$

and so,

$$\langle \mathcal{J} \rangle_{\mathrm{dir}} = \frac{e^2\alpha^2}{4m^2\omega_c^2}\sum_k \omega_k^2 e_k^2 \left\{ -|\chi(-\omega_k)|^2 - |\chi(\omega_k)|^2 + \mathrm{Re}\{\chi^*(\omega_k)\chi^*(-\omega_k)\} + \mathrm{Re}\{\chi(\omega_k)\chi(-\omega_k)\} \right\}.$$

Using the relations for the real part of $\chi(\omega_k)$ in Appendix A, section A.1, we see that we have for outgoing modes (i.e. for $k > 0$)

$$\langle \mathcal{J} \rangle_{\mathrm{dir}} = \frac{e^4\pi^2}{4m^4\omega_c^4}\sum_{k>0} \omega_k^2 e_k^2 \left\{ -|\chi(\omega_k)|^2 - |\chi(-\omega_k)|^2 + \frac{2}{\omega_c^2 - \omega_k^2} \right\}. \tag{4.47}$$

We shall now find the interference term $\langle \mathcal{J} \rangle_{\mathrm{int}}$. After some algebra, we find that

$$\left\langle \dot{q}\frac{\partial \phi^h}{\partial t} - \frac{\partial \phi^h}{\partial x}\dot{q} \right\rangle = \frac{e}{2\omega_c m}\sum_{k>0} \omega_k^2 e_k^2 \,\mathrm{Re}\left\{ i\chi^*(-\omega_k) - i\chi(\omega_k) - i\chi^*(\omega_k) + i\chi(-\omega_k) \right\}$$

and using the relations in Appendix A, we find that

$$\langle \mathcal{J} \rangle_{\mathrm{int}} = \frac{\alpha e\gamma}{4\omega_c m}\sum_{k>0} \omega_k^2 e_k^2 \left[|\chi(\omega_k)|^2 + |\chi(-\omega_k)|^2 - \frac{2}{(\omega_c^2 - \omega_k^2)} \right].$$

Since $\alpha = e\pi/\omega_c m$ and $\gamma = e^2\pi/\omega_c^2 m^2$ we must have that,

$$\langle \mathcal{J} \rangle_{\text{int}} = \frac{e^4\pi^2}{4\omega_c^4 m^4} \sum_{k>0} \omega_k^2 e_k^2 \left[|\chi(\omega_k)|^2 + |\chi(-\omega_k)|^2 - \frac{2}{(\omega_c^2 - \omega_k^2)} \right], \tag{4.48}$$

and hence we have that the total energy flux,

$$\langle \mathcal{J} \rangle = \langle \mathcal{J} \rangle_{\text{dir}} + \langle \mathcal{J} \rangle_{\text{int}} = 0. \tag{4.49}$$

So, the approach of deriving a quantum Langevin equation, and determining the energy flux in the same manner as Ford and O'Connell delivers the expected result of no radiation emitted by the oscillator when coupled to a real scalar field and is confined to a constant trajectory in $\mathbb{M}^2$.

4.2 Uniformly Accelerating Oscillator in $2D$ Minkowski.

We shall now modify the previous model: we shall now place the oscillator on an accelerating trajectory. We shall then show that by deriving, and solving a quantum Langevin equation, that when we come to evaluate the total energy flux of the system, we find it to be zero in accordance with the standard results.

Although this is essentially the same calcualtion performed by Raine *et al.* [6], the methodology employed here is rather different. Instead of considering quantum correlations, we shall adopt the procedure of the preceeding section for calculating the overall energy flux of the system. Thus, we shall need to find expressions for the annihilation and creation operators using the Wigner-Weisskopff approximation, then use the energy-flux method of Ford and O'Connell to show that the interference term again balances with the direct flux term to give a reult of no overall energy flux.

It should be noted that this same calculation is also performed by Ford and O'Connell, although those authors make their calculation in the Shrödinger picture, using a second order differential equation to describe the particle motion. We wish to generalise the methodology of the preceeding section and use as a means of deriving Hawking radiation. Before we do this, we show the method delivers a consistent reuslt with an oscillator undergoing hyperbolic motion.

4.2.1 Hamiltonian

The oscillator is accelerating uniformly, so the 2D trajectory of the oscillator in Minkowski coordinates is,

$$t = \xi \sinh(\tau), \; x = \xi \cosh(\tau), \tag{4.50}$$

where (τ, ξ) are the usual Rindler coordinates with τ being the proper time and we choose $\xi = 1$. We have for $x > x_{\mathrm{ret}}$, the retarded time:

$$\tau_{\mathrm{ret}} = \ln|x - t| = \ln(\nu), \text{ where } \nu = x - t, \tag{4.51}$$

while,

$$\left.\frac{\partial \tau_{\mathrm{ret}}}{\partial x}\right|_t = -\frac{\partial \tau_{\mathrm{ret}}}{\partial t}, \; \frac{\partial \tau_{\mathrm{ret}}}{\partial x} = \frac{1}{\nu}, \text{ and } \left.\frac{\partial \tau_{\mathrm{ret}}}{\partial t}\right|_x = -\frac{1}{\nu}.$$

The Hamiltonian will be,

$$\mathscr{H} d\tau = \hbar\omega_c a^\dagger \; d\tau a + \hbar \sum_k \omega_k b_k^\dagger b_k \; d\tau + \frac{ie\hbar}{\sqrt{2\omega_c m}} \sum_k e_k (a^\dagger - a) \left[b_k e^{ikx\tau} + b_k^\dagger e^{-ikx(\tau)} \right] \; d\tau. \tag{4.52}$$

4.2.2 Equations of Motion.

We find the equations of motion of the oscillator and the scalar field in the same way as in the previous section. For the oscillator we have that,

$$\frac{da}{d\tau} = \frac{1}{i\hbar}[\mathscr{H}, a]$$

and so this yields the expression:

$$\frac{da}{d\tau} = -i\omega_c a + \frac{e}{\sqrt{2\omega_c m}} \sum_k e_k \left[b_k e^{ikx(\tau)} + b_k^\dagger e^{-ikx(\tau)} \right]. \tag{4.53}$$

Similarly we have for the scalar field that,

$$\frac{db_k}{d\tau} = \frac{1}{i\hbar}[\mathscr{H}, b_k],$$

and so we find that,

$$\frac{db_k}{d\tau} = -i\omega_k \frac{dt}{d\tau} + \frac{ee_k}{\sqrt{2\omega_c}m}(a^\dagger - a)e^{-ikx(\tau)}. \tag{4.54}$$

We can solve (4.54) to find expressions for b_k and $b_k^\dagger$. First we write (4.54) as

$$\frac{db_k}{dt}\frac{dt}{d\tau} + i\omega_k b_k \frac{dt}{d\tau} = \frac{ee_k}{\sqrt{2\omega_c}m}(a^\dagger - a)e^{-ikx(\tau)}\frac{d\tau}{dt},$$

and so,

$$e^{i\omega_k t(\tau)}b_k = b_k(0) + \frac{ee_k}{\sqrt{2\omega_c}m}\left(\frac{d\tau}{dt}\right)^2 \int_{-\infty}^{\tau} [a^\dagger(\tau') - a(\tau')]e^{i\omega_k t(\tau') - ikx(\tau')}\, d\tau'.$$

This gives us:

$$b_k(\tau) = e^{-i\omega_k t(\tau)}b_k(0) + \frac{ee_k e^{-i\omega_k t(\tau)}}{\sqrt{2\omega_c}m}\int_{-\infty}^{\tau} [a^\dagger(\tau') - a(\tau')]e^{i\omega_k t(\tau') - ikx(\tau')}\, d\tau', \tag{4.55}$$

and,

$$b_k^\dagger(\tau) = e^{i\omega_k t(\tau)}b_k^\dagger(0) + \frac{ee_k e^{i\omega_k t(\tau)}}{\sqrt{2\omega_c}m}\int_{-\infty}^{\tau} [a(\tau') - a^\dagger(\tau')]e^{-i\omega_k t(\tau') + ikx(\tau')}\, d\tau'. \tag{4.56}$$

We shall now find the full solution to the scalar field equation. We have that,

$$\phi(t,x) = \sum_k e_k[b_k e^{ikx} + b_k^\dagger e^{-ikx}].$$

Subsituting in (4.55) and (4.56) we have that,

$$\begin{aligned}\phi(t,x) = \phi^h(t,x) &+ \frac{e}{\sqrt{2\omega_c}m}\sum_k e_k^2\, e^{-i\omega_k t(\tau) + ikx(\tau)}\int_{-\infty}^{\tau}[a^\dagger - a]e^{i\omega_k t(\tau') - ikx(\tau')}\, d\tau' \\ &+ \frac{e}{\sqrt{2\omega_c}m}\sum_k e_k^2\, e^{i\omega_k t(\tau) - ikx(\tau)}\int_{-\infty}^{\tau}[a - a^\dagger]e^{-i\omega_k t(\tau') + ikx(\tau')}\, d\tau'.\end{aligned}$$

We now subsitute for the functions $e^{-i\omega_k t(\tau) + ikx(\tau)}$ and their complex conjugates given in (4.64) into the above expression, and so we have that,

$$\begin{aligned}\phi(t,x) = \phi^h(t,x) &+ \frac{e}{\sqrt{2\omega_c}m}\int_{\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\tau}\sum_k e_k^2 \alpha_k(k')\alpha_k^*(k'')e^{ik''\tau'}e^{-ik'\tau}(a^\dagger - a)\; dk'dk''d\tau \\ &- \frac{e}{\sqrt{2\omega_c}m}\int_{\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\tau}\sum_k e_k^2 \alpha_k^*(k'')\alpha_k(k')e^{ik''\tau}e^{-ik'\tau}(a^\dagger - a)\; d\tau'.\end{aligned}$$

Using the result in Raine *et al* [6]:

$$\sum_k e_k^2 \alpha_k(k')\alpha_k^*(k'') = \frac{1}{\pi}\delta(k'-k'')|\Gamma(ik')|^2 e^{\pi k'}, \tag{4.57}$$

so we have, after performing the k'' integral:

$$\phi(t,x) = \phi^h(t,x) + \frac{e}{\sqrt{2\omega_c} m\pi}\int_{-\infty}^{\infty} \frac{e^{\pi k'} e^{ik'(\tau'-\tau)}}{k' \sinh(\pi k')}\, dk' \int_{-\infty}^{\tau} (a^\dagger - a)\, d\tau'$$
$$- \frac{e}{\sqrt{2\omega_c} m\pi}\int_{-\infty}^{\infty} \frac{e^{\pi k'} e^{ik'(\tau-\tau')}}{k' \sinh(\pi k')}\, dk' \int_{-\infty}^{\tau} (a^\dagger - a)\, d\tau',$$

where we have used the identity,

$$|\Gamma(iz)|^2 = \frac{\pi}{z\sinh(\pi z)}.$$

Let us now consider the integrand:

$$\int_{-\infty}^{\infty} \frac{e^{\pi k'} e^{ik'(\tau'-\tau)}}{k' \sinh(\pi k')}\, dk'.$$

This is a contour integral with a pole at $k'=0$. We can write

$$\frac{e^{\pi k'}}{\sinh(\pi k')} = \frac{2e^{\pi k'}}{e^{\pi k'} - e^{\pi k'}} = \frac{1}{1-e^{2\pi k}}.$$

We note that

$$\lim_{k\to\infty}\left\{\frac{1}{1-e^{2\pi k}}\right\} = 1 \text{ and } \lim_{k\to-\infty}\left\{\frac{1}{1-e^{2\pi k}}\right\} = 0.$$

This integral is discussed in [6]. Essentially, we evaluate the integral around the contour and the only contribution comes from $k=0$ so we make the subsitution $k = ni + \epsilon$ and now we find,

$$\int_{-\infty}^{\infty} \frac{e^{\pi k'} e^{ik'(\tau'-\tau)}}{k' \sinh(\pi k')}\, dk' = \frac{1}{2\pi}\sum_n \frac{e^{in\pi(\tau'-\tau)}}{n} = \Theta(\tau'-\tau)$$

where $\Theta(\tau-\tau')$ is the Heaviside function. Thus we now have

$$\phi(t,x) = \phi^h(t,x) - \frac{2\pi e}{\sqrt{2\omega_c} m}\int_{-\infty}^{\tau} [a^\dagger - a]\Theta(\tau'-\tau)\, d\tau' - \frac{2\pi e}{\sqrt{2\omega_c} m}\int_{-\infty}^{\tau} [a^\dagger - a]\Theta(\tau-\tau')\, d\tau'$$

and after simplifying and using the relations between p and q we have,

$$\phi(t,x) = \phi^h(t,x) - \frac{\pi e}{\omega_c m}\int_{-\infty}^{\tau} \frac{dq}{d\tau}\Theta(\tau'-\tau)\, d\tau'.$$

and hence,

$$\phi(t,x) = \phi^h(t,x) + \frac{\pi e}{\omega_c m} q(\tau_{\rm ret}) = \phi^h(t,x) - \alpha\phi(t,x), \tag{4.58}$$

where,

$$\alpha = \frac{\pi e}{\omega_c m} \tag{4.59}$$

We now turn our attention back to the oscillator. We now substitute (4.55) and (4.56) into (4.53) and we obtain now the following equation of motion for the oscillator:

$$\frac{da}{d\tau} = -i\omega_c a(\tau) + \mathcal{G}_a(\tau) + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{ e^{-i\omega_k t(\tau)+ikx(\tau)} \int_{-\infty}^{\tau} [a^\dagger(\tau') + a(\tau')] e^{i\omega_k t(\tau') - ikx(\tau')}\, d\tau'$$
$$- e^{i\omega_k t(\tau)-ikx(\tau)} \int_{-\infty}^{\tau} [a^\dagger(\tau') - a(\tau')] e^{-i\omega_k t(\tau')+ikx(\tau')}\, d\tau' \Big\},$$

where,

$$\mathcal{G}_a(\tau) = \frac{e}{\sqrt{2\omega_c} m}\sum_k e_k^2 \left[b_k(0) e^{-i\omega_k t(\tau)+ikx(\tau)} + b_k^\dagger(0) e^{i\omega_k t(\tau)-ikx(\tau)} \right]. \tag{4.60}$$

We now remove the high frequency behavior from the above equation in the same manner as before, only now a is of course a function of proper time τ, so,

$$a(\tau) = e^{-i\omega_c \tau} A(\tau), \tag{4.61}$$

and hence, our equation of motion for the oscillator is now

$$\frac{dA}{d\tau} = \mathcal{G}_A(\tau) + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{ e^{i\omega_c\tau} e^{-i\omega_c t(\tau)+ikx(\tau)} \int_{-\infty}^{\tau} \left[e^{i\omega_c\tau'} A^\dagger(\tau) - e^{-i\omega_c\tau'} A(\tau') \right] e^{i\omega_k t(\tau') - okx(\tau')}\, d\tau'$$
$$- e^{-\omega_c\tau} e^{i\omega_k t(\tau)-ikx(\tau)} \int_{-\infty}^{\tau} \left[e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau') \right] e^{-i\omega_k t(\tau')+ikx(\tau')}\, d\tau' \Big\}. \tag{4.62}$$

In order to proceed further, we shall need to use the Fourier transforms of the functions $e^{i\omega_k t - ikx}$. Let,

$$e^{i\omega_k t(\tau)-ikx(\tau)} = \int_{-\infty}^{\infty} \alpha_k(k') e^{-ik'\tau}\, dk', \quad \text{and,} \tag{4.63}$$
$$e^{-i\omega_k t(\tau)+ikx(\tau)} = \int_{-\infty}^{\infty} \alpha_k^*(k'') e^{ik''\tau} dk''. \tag{4.64}$$

We find that, for $k > 0$ [6]:

$$\alpha_k(k') = \frac{1}{2\pi} k^{ik'} e^{\pi k'/2} \Gamma(-ik'), \tag{4.65}$$

and so,

$$|\alpha(k')|^2 = \frac{e^{\pi k'}}{4\pi^2}|\Gamma(ik')|^2 = \frac{e^{pik'}}{4\pi k' \sinh(\pi k')} = \frac{1}{4\pi k'}[\coth(4\pi k') + 1]. \tag{4.66}$$

So, substituting (4.64) into the equation of motion for the oscillator we now replace the eigenfunctions with an integral over k' and k'', thus:

$$\frac{dA}{d\tau} = \mathcal{G}_A(\tau) + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2$$
$$\Big\{ e^{i\omega_c\tau} e^{-i\omega_k t(\tau)+ikx(\tau)} \int_{-\infty}^{\tau} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau')]d\tau' \int_{-\infty}^{\infty} \alpha_k(k') e^{-ik'\tau'}\ dk'$$
$$- e^{i\omega_c\tau} e^{i\omega_k t(\tau)-ikx(\tau)} \int_{-\infty}^{\tau} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau')]\ d\tau' \int_{-\infty}^{\infty} \alpha_k^*(k'') e^{ik''\tau'}\ dk'' \Big\}.$$

The next stage is to Laplace transform the above differential equation so that we can obtain a form for $\overline{A}(s)$, then $A(\tau)$. So, multiplying both sides by $e^{-s\tau}$ and integrating with respect to τ gives:

$$s\overline{A(s)} = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2$$
$$\Big\{ \int_0^{\infty} e^{i(\omega_c+is)\tau} e^{i\omega_c\tau} e^{-i\omega_k t(\tau)+ikx(\tau)}\ d\tau \int_{-\infty}^{\tau} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau')]d\tau' \int_{-\infty}^{\infty} \alpha_k(k') e^{-ik'\tau'}\ dk'$$
$$- \int_0^{\infty} e^{i(\omega_c+is)\tau} e^{i\omega_c\tau} e^{i\omega_k t(\tau)-ikx(\tau)}\ d\tau \int_{-\infty}^{\tau} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau')]\ d\tau' \int_{-\infty}^{\infty} \alpha_k^*(k'') e^{ik''\tau'}\ dk'' \Big\}.$$

Substituting the Fourier transform expressions for the functions $e^{-i\omega_k t(\tau)+ikx(\tau)}$ and $e^{i\omega_k t(\tau)-ikx(\tau)}$, we have

$$s\overline{A(s)} = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2$$
$$\Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_k(k')\alpha_k^*(k'')\ dk'dk'' \int_0^{\infty} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'}] e^{-ik'\tau'}\ d\tau' \int_0^{\tau'} e^{i(\omega_c+is+k'')\tau}\ d\tau$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_k^*(k'')\alpha_k(k')\ dk'dk'' \int_0^{\infty} [e^{i\omega_c\tau'} A^\dagger(\tau') - e^{-i\omega_c\tau'} A(\tau)] e^{ik''\tau'}\ d\tau' \int_0^{\tau'} e^{i(\omega_c+is-k')\tau}\ d\tau \Big\}.$$

Now we perform the τ integrations:

$$
\begin{aligned}
s\overline{A(s)} = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2} \sum_k e_k^2 \Bigg\{ & \\
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_k(k')\alpha_k^*(k'') \left[\frac{e^{2i\omega_c\tau' - s\tau'}A^\dagger(\tau') - e^{-s\tau'}A(\tau')}{i(\omega_c + is + k'')}\right] e^{i(k''-k')\tau'} \; d\tau' dk' dk'' \\
& - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_k(k')\alpha_k^*(k'') \left[\frac{e^{2i\omega_c\tau' - s\tau'}A^\dagger(\tau') - e^{-s\tau'}A(\tau')}{i(\omega_c + is - k')}\right] e^{i(k''-k')\tau'} \; d\tau' dk' dk'' \Bigg\}.
\end{aligned}
\tag{4.67}
$$

In the above integrands we have that,

$$
\left(\frac{V}{2\pi}\right)^{-1} \sum_{k>0} \frac{e^{i(k'-k'')\ln(k)}}{k} \longrightarrow 0, \text{ unless } k' = k''.
$$

Hence we shall let $k' = k''$ in the abover expression. Furthermore we shall ignore the $A\dagger$ term since we are using a rotating-wave approximation and so we can write (4.67) in a much simpler form:

$$
s\overline{A(s)} = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2} \left\{ \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right) |\alpha_k(k')|^2 \, \overline{A}(s) \left(\frac{1}{i(\omega_c - k' + is)} - \frac{1}{i(\omega_c + k' + is)} \right) \, dk' \right\}.
$$

Factorizing this leaves us with:

$$
\overline{A}(s) = \frac{\overline{\mathcal{G}_A(s)}}{s + \frac{e^2}{2\omega_c m^2} \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right) |\alpha_k(k')|^2 \left(\frac{1}{i(\omega_c - k' + is)} - \frac{1}{i(\omega_c + k' + is)}\right) \, dk'}. \tag{4.68}
$$

Let us now consider the denominator of (4.68); it is the following:

$$
s - \frac{ie^2}{2\omega_c m^2} \left\{ \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right)\left(\frac{2\pi}{V}\right) |\alpha_k(k')|^2 \left(\frac{1}{\omega_c - k' + is}\right) dk' - \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right) |\alpha_k(k')|^2 \left(\frac{1}{\omega_c + k' + is}\right) dk' \right\}.
$$

We can write this as,

$$
s - \frac{ie^2}{2\omega_c m^2} \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right) \frac{|\alpha_k(k')|^2 - |\alpha_k(k')|^2}{\omega_c - k' + is} \, dk'.
$$

Using the expression for (4.66) we find that we have,

$$
s - \frac{ie^2}{2\omega_c m^2} \int_{-\infty}^{\infty} \left(\frac{2\pi}{V}\right) \frac{1}{k'} \left(\frac{1}{\omega_c - k' + is}\right) \, dk' = s - \frac{ie^2}{2\omega_c m^2} \int_{-\infty}^{\infty} \left(\frac{e_{k'}^2}{\omega_c - k' + is}\right) \, dk'.
$$

This is the same denominator as we had in the previous section and so we can write (4.68) in the approximate form of:

$$\overline{A}(s) = \frac{\overline{\mathcal{G}_A(s)}}{s + \frac{\gamma}{2} + \Delta\varpi}. \tag{4.69}$$

where,

$$\frac{\gamma}{2} = \frac{\pi e^2}{2\omega_c^2 m^2}, \quad \text{where} \quad \Delta\varpi = -\int \frac{e_k^2}{\omega_k - \omega_c}\, d\omega_k. \tag{4.70}$$

4.2.3 Calculation of $A(\tau)$ and $A^\dagger(\tau)$

We now have a relatively simple expression for $A(s)$ in the Fourier transform equation of (4.69). If, as before, we ignore small frequency shifts, then we can re-write this expression as a simple linear first order differential equation:

$$\frac{dA}{d\tau} = -\frac{\gamma}{2} A(\tau) + \mathcal{G}_A(\tau), \tag{4.71}$$

where we have let,

$$\mathcal{G}_A(\tau) = \frac{e e^{i\omega_c \tau}}{\sqrt{2\omega_c} m} \sum_k e_k \left[b_k(0) e^{-i\omega_k t(\tau) + ikx(\tau)} + b_k^\dagger(0) e^{i\omega_k t(\tau) - ikx(\tau)} \right].$$

Integrating the above differential equation we have a general solution:

$$A(\tau) = e^{-\gamma\tau/2} \int_0^\tau \mathcal{G}_A(\tau') e^{\gamma\tau'/2}\, d\tau',$$

and hence using the definition $\mathcal{G}_A(\tau)$, and using the Fourier transforms of before we have that,

$$\begin{aligned} A(\tau) = \frac{e e^{-\gamma\tau/2}}{\sqrt{2\omega_c} m} \sum_k e_k \Bigg\{ & b_k(0) \int_{-\infty}^{\infty} \alpha_k^*(k'') e^{ik''\tau}\, dk'' \int_0^\tau e^{i(\omega_c - i\gamma/2)\tau'}\, d\tau' \\ & b_k^\dagger(0) \int_{-\infty}^{\infty} \alpha_k(k') e^{-ik'\tau'}\, dk' \int_0^\tau e^{i(\omega_c - i\gamma/2)\tau'}\, d\tau' \Bigg\}. \end{aligned}$$

Now we change the order of integration:

$$\begin{aligned} A(\tau) = \frac{e^{-\gamma\tau/2}}{\sqrt{2\omega_c m}} \sum_k e_k \Bigg\{ & b_k(0) \int_{-\infty}^{\infty} \alpha_k^*(k'')\, dk'' \int_{-\infty}^\tau e^{i(\omega_c + k'' - i\gamma/2)\tau'}\, d\tau' \\ & + b_k^\dagger(0) \int_{-\infty}^{\infty} \alpha_k(k')\, dk' \int_{\infty}^\tau e^{i(\omega_c - k' - i\gamma/2)\tau'}\, d\tau' \Bigg\}. \end{aligned}$$

Performing the τ' integrations we have,

$$A(\tau) = \frac{ee^{-\gamma\tau/2}}{\sqrt{2\omega_c}m}\sum_k e_k\Bigg\{b_k(0)\int_{-\infty}^{\infty}\alpha_k^*(k'')\left[\frac{e^{i(\omega_c+k''-i\gamma/2)\tau}-1}{i(\omega_c+k''-i\gamma/2)}\right]\,dk''$$
$$+\,b_k^\dagger(0)\int_{-\infty}^{\infty}\alpha_k(k')\left[\frac{e^{i(\omega_c-k'-i\gamma/2)\tau}-1}{i(\omega_c-k'-i\gamma/2)}\right]\,dk'\Bigg\}.$$

We now multiply through by the $e^{-\gamma\tau/2}$ term which is sitting outside the sum in the above expression. When we do this, we find that we have a decaying exponential term in each integrand, i.e. that,

$$\frac{e^{-\gamma\tau/2}}{i(\omega_c+k''-i\gamma/2)}\rightarrow 0,\ \text{and}\ \frac{e^{-\gamma\tau/2}}{i(\omega_c-k'-i\gamma/2)}\rightarrow 0.$$

We shall now define the quantity

$$\chi(k') = \frac{1}{\omega_c+k'-i\gamma/2}, \tag{4.72}$$

and hence, substituting the appropriate relations in for $\alpha_k(k')$ we arrive at expressions for the annihilation and creation operator:

$$\begin{aligned}A(\tau) =& -\frac{ie}{2\pi\sqrt{2\omega_c}m}\sum_k e_k\Bigg\{b_k(0)\int_{-\infty}^{\infty}k^{-ik''}e^{\pi k''/2}\Gamma(ik'')\chi(k'')e^{i(\omega_c+k'')\tau}\,dk''\\ &+b_k^\dagger(0)\int_{\infty}^{\infty}k^{ik'}e^{\pi k'/2}\Gamma(-ik')\chi(-k')e^{i(\omega_c-k)\tau}\,dk'\Bigg\},\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}A^\dagger(\tau) =& \frac{ie}{2\pi\sqrt{2\omega_c}m}\sum_k e_k\Bigg\{b_k^\dagger(0)\int_{-\infty}^{\infty}k^{ik''}e^{\pi k''/2}\Gamma(-ik'')\chi^*(k'')e^{-i(\omega_c+k'')\tau}\,dk''\\ &+b_k(0)\int_{\infty}^{\infty}k^{-ik'}e^{\pi k'/2}\Gamma(ik')\chi^*(-k')e^{-i(\omega_c-k)\tau}\,dk'\Bigg\}.\end{aligned} \tag{4.74}$$

4.2.4 Calculation of Energy Flux

Now that we have the annihilation and creation operators for the quantum oscillator, we can easily determine an expression for $q(\tau)$ and hence calculate the energy flux of the system. As in the preceding section we have that the total energy flux of the system in $\mathbb{M}^2$ is

$$\langle\mathcal{J}\rangle = \left\langle\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right\rangle.$$

Once more the solution to the wave equation is,

$$\phi = \phi_h - \alpha q(\tau_{\text{ret}}), \quad \text{where} \quad \alpha = \frac{\pi e}{\omega_c m} \tag{4.75}$$

So,

$$\langle \mathcal{J} \rangle = \left\langle \left(\frac{\partial \phi_h}{\partial x} - \alpha \frac{\partial q(\tau_{\text{ret}})}{\partial x} \right) \left(\frac{\partial \phi_h}{\partial t} - \alpha \frac{\partial q(\tau_{\text{ret}})}{\partial t} \right) \right\rangle. \tag{4.76}$$

We now go on to calculate the energy flux of the system. Using the chain rule of differentiation we must have that,

$$\frac{\partial q(\tau_{\text{ret}})}{\partial x} = \frac{dq(\tau_{\text{ret}})}{d\tau_{\text{ret}}} \frac{\partial \tau_{\text{ret}}}{\partial x} = \frac{\dot{q}(\tau_{\text{ret}})}{\nu(\tau)}.$$

where $\nu = x - t$, and,

$$\langle \mathcal{J} \rangle = \left\langle \left(\frac{\partial \phi_h}{\partial x} - \alpha \frac{\dot{q}(\tau_{\text{ret}})}{\nu(\tau)} \right) \left(\frac{\partial \phi_h}{\partial t} + \alpha \frac{\dot{q}(\tau_{\text{ret}})}{\nu(\tau)} \right) \right\rangle,$$

giving

$$\langle \mathcal{J} \rangle = \underbrace{\Re \left\{ \left\langle \frac{\partial \phi_h}{\partial x} \frac{\partial \phi_h}{\partial t} \right\rangle \right\}}_{\langle \mathcal{J} \rangle_0 = 0} + \underbrace{\frac{\alpha}{\nu(\tau)} \Re \left\{ \left\langle \frac{\partial \phi_h}{\partial x} \dot{q} - \dot{q} \frac{\partial \phi_h}{\partial t} \right\rangle \right\}}_{\langle \mathcal{J} \rangle_{\text{int}}} - \underbrace{\frac{\alpha^2}{\nu^2(\tau)} \Re \left\{ \langle \dot{q}(\tau_{\text{ret}})^2 \rangle \right\}}_{\langle \mathcal{J} \rangle_{\text{dir}}}. \tag{4.77}$$

As before

$$q = \frac{1}{\sqrt{2\omega_c}} [a^\dagger + a],$$

and so we have,

$$\begin{aligned} q(\tau) = \frac{ie}{4\pi \omega_c m} \sum_k e_k \Bigg\{ \\ & b_k(0) \left[\int_{-\infty}^{\infty} k^{-ik'} e^{\pi k'/2} \Gamma(ik') \chi^*(-k') e^{ik'\tau} \, dk' - \int_{-\infty}^{\infty} k^{-ik''} e^{\pi k''/2} \Gamma(ik'') \chi(k'') e^{ik''\tau} \, dk'' \right] \\ & + b_k^\dagger(0) \left[\int_{\infty}^{\infty} k^{ik''} e^{\pi k''/2} \Gamma(-ik'') \chi^*(k'') e^{-ik''\tau} \, dk'' - \int_{-\infty}^{\infty} k^{ik'} e^{\pi k'/2} \Gamma(-ik') \chi(-k') e^{-ik'\tau} \, dk' \right] \Bigg\}. \end{aligned} \tag{4.78}$$

We shall need $\dot{q}(\tau)$, so differentiating the above expression,

$$\begin{aligned} \dot{q}(\tau) = \frac{-e}{4\pi \omega_c m} \sum_k e_k \Bigg\{ \\ & b_k(0) \left[\int_{-\infty}^{\infty} k' k^{-ik'} e^{\pi k'/2} \Gamma(ik') \chi^*(-k') e^{ik'\tau} \, dk' - \int_{-\infty}^{\infty} k'' k^{-ik''} e^{\pi k''/2} \Gamma(ik'') \chi(k'') e^{ik''\tau} \, dk'' \right] + \\ & b_k^\dagger(0) \left[- \int_{\infty}^{\infty} k'' k^{ik''} e^{\pi k''/2} \Gamma(-ik'') \chi^*(k'') e^{-ik''\tau} \, dk'' + \int_{-\infty}^{\infty} k' k^{ik'} e^{\pi k'/2} \Gamma(-ik') \chi(-k') e^{-ik'\tau} \, dk' \right] \Bigg\}. \end{aligned} \tag{4.79}$$

We shall calculate the direct flux term first. The first step in computing this is to find an expression for the expectation value $\langle \dot{q}\dot{q}\rangle$:

$$\langle \dot{q}\dot{q}\rangle = \frac{e^2}{16\pi\omega_c^2 m^2}\sum_k e_k^2 \,\mathrm{Re}\Bigg\{$$
$$\left[-\int_{\infty}^{\infty} k'' k^{ik''} e^{\pi k''/2}\Gamma(-ik'')\chi^*(k'')e^{-ik''\tau}\ dk'' + \int_{-\infty}^{\infty} k' k^{ik'} e^{\pi k'/2}\Gamma(-ik')\chi(-k')e^{-ik'\tau}\ dk'\right]\times$$
$$\left[\int_{-\infty}^{\infty} k' k^{-ik'} e^{\pi k'/2}\Gamma(ik')\chi^*(-k')e^{ik'\tau}\ dk' - \int_{-\infty}^{\infty} k'' k^{-ik''} e^{\pi k''/2}\Gamma(ik'')\chi(k'')e^{ik''\tau}\ dk''\right]\Bigg\}.$$

Multiplying out,

$$\langle \dot{q}\dot{q}\rangle = \frac{e^2}{16\pi\omega_c^2 m^2}\sum_k e_k^2 \,\mathrm{Re}\Bigg\{$$
$$-\int_{\infty}^{\infty}\int_{\infty}^{\infty} k'k''k^{i(k''-k')}e^{\pi(k'+k'')/2}\Gamma(ik')\Gamma(-ik'')\chi^*(-k')\chi^*(k'')e^{i(k'-k'')\tau_{\mathrm{ret}}}\ dk\ dk''$$
$$-\int_{\infty}^{\infty}\int_{\infty}^{\infty} k'k''k^{i(k''-k')}e^{\pi(k'+k'')/2}\Gamma(-ik')\chi(k'')\chi(-k')e^{i(k''-k')\tau_{\mathrm{ret}}}\ dk'\ dk''$$
$$+\int_{\infty}^{\infty}\int_{\infty}^{\infty} k'k''k^{i(k'-k'')}e^{\pi(k'+k'')/2}\Gamma(ik')\Gamma(-ik'')\chi^*(-k')\chi(-k')e^{i(k'-k'')\tau_{\mathrm{ret}}}\ dk'\ dk''$$
$$+\int_{\infty}^{\infty}\int_{\infty}^{\infty} k'k''k^{i(k'-k'')}e^{\pi(k'+k'')/2}\Gamma(ik'')\Gamma(-ik')\chi(k'')\chi^*(k')e^{i(k''-k')\tau_{\mathrm{ret}}}\ dk'\ dk''\Bigg\}.$$

Next, we note that,

$$\sum_{k>0}\frac{e^{i(k'-k'')\ln(k)}}{k}\to 0,$$

unless $k' = k''$, and taking advantage of this gives a much simpler expression for $\langle \dot{q}\dot{q}\rangle$:

$$\langle \dot{q}\dot{q}\rangle = \frac{e^2}{16\pi\omega_c^2 m^2}\int_{\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik')|^2\Re\left(|\chi(k')|^2 + |\chi(-k')|^2 - \chi^*(-k')\chi^*(k') - \chi(k')\chi(-k')\right)\ dk'.$$

Again, using the relations for $\chi(\omega_k)$ in Appendix A, section A.1, we find that,

$$\langle \mathcal{J}\rangle_{\mathrm{dir}} = \frac{e^2\alpha^2}{16\pi^2\omega_c^2 m^2\nu^2(\tau)}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik)|^2\left(-|\chi(k')|^2 - |\chi(-k')|^2 + \frac{2}{(\omega_c - k'^2)}\right)\ dk'. \tag{4.80}$$

We have now found an expression for the direct flux which arises from the oscillator alone. We will now go on to show that the interference term is precisely $-\langle \mathcal{J}\rangle_{\mathrm{dir}}$. The interference term is given by,

$$\langle \mathcal{J}\rangle_{\mathrm{int}} = \frac{\alpha}{\nu(\tau)}\Re\left\{\left\langle \frac{\partial\phi_h}{\partial x}\dot{q} - \dot{q}\frac{\partial\phi_h}{\partial t}\right\rangle\right\}.$$

We know that the homogeneous form of the ϕ-field is,

$$\phi_h = \sum_k e_k \left[b_k(0) e^{-i\omega_k t + ikx} + b_k^\dagger(0) e^{i\omega_k t - ikx} \right],$$

However, for the interference terms, the field is evaluated on the worldline of the oscillator, and so,

$$\frac{\partial \phi_h}{\partial x} = i \frac{\partial \tau_{\text{ret}}}{\partial x} \sum_{k>0} k e_k \left[b_k e^{-i\omega_k t + ikx} - b_k^\dagger e^{i\omega_k t - ikx} \right],$$

and,

$$\frac{\partial \phi_h}{\partial t} = i \frac{\partial \tau_{\text{ret}}}{\partial t} \sum_{k>0} k e_k \left[-b_k e^{-i\omega_k t + ikx} + b_k^\dagger e^{i\omega_k t - ikx} \right].$$

We now calculate the first part of the expectation value:

$$\left\langle \frac{\partial \phi_h}{\partial x} \dot{q} \right\rangle = \frac{ie}{4\pi \omega_c m \nu(\tau)} \sum_{k>0} k e_k^2 \; \text{Re}\Big\{ e^{i\omega_k t - ikx} \int_{-\infty}^{\infty} k' k^{-ik'} e^{\pi k'/2} \Gamma(ik') \chi^*(-k') e^{ik' \tau_{\text{ret}}} \; dk'$$
$$- e^{i\omega_k t - ikx} \int_{-\infty}^{\infty} k'' k^{-ik''} e^{\pi k''/2} \Gamma(ik'') \chi(k'') e^{ik'' \tau_{\text{ret}}} \; dk'' \Big\}.$$

We now use the Fourier transforms of (4.64) and we have that,

$$\left\langle \frac{\partial \phi_h}{\partial x} \dot{q} \right\rangle = \frac{ie}{8\pi^2 \omega_c m \nu(\tau)} \sum_{k>0} k e_k^2 \; \text{Re}\Big\{$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k' k^{i(k''-k')} e^{\pi(k''+k')/2} \chi^*(-k') \Gamma(-ik') \Gamma(ik') e^{i(k'-k'')\tau_{\text{ret}}} \; dk' dk''$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k'' k^{i(k'-k'')} e^{\pi(k'+k'')/2} \chi(k'') \Gamma(-ik') \Gamma(ik'') e^{i(k''-k')\tau_{\text{ret}}} \; dk' dk'' \Big\}.$$

We have that,

$$\sum_k e^{i(k'-k'')\ln(k)}/k \to 0 \sim \int_{-\infty}^{\infty} e^{i(k'-k'')u} \; du$$

for $u = \ln k$, so unless $k' = k''$ the above expression tends to zero, and so:

$$\left\langle \frac{\partial \phi_h}{\partial x} \dot{q} \right\rangle = \frac{e}{8\pi^2 \omega_c \nu(\tau) m} \int_{-\infty}^{\infty} k'^2 e^{\pi k'} |\Gamma(ik')|^2 \text{Re}\left(i\chi^*(-k') - i\chi(k') \right) \; dk'.$$

Similarly we have that,

$$\left\langle \dot{q} \frac{\partial \phi_h}{\partial t} \right\rangle = \frac{ie}{4\pi \omega_c \nu(\tau) m} \sum_{k>0} k e_k^2 \Big\{$$
$$- e^{-i\omega_k t + ikx} \int_{-\infty}^{\infty} k'' k^{ik''} e^{\pi k''/2} \Gamma(-ik'') \chi^*(k'') e^{-ik'' \tau_{\text{ret}}} \; dk''$$
$$+ e^{-i\omega_k t + ikx} \int_{-\infty}^{\infty} k' k^{ik'} e^{\pi k'/2} \Gamma(-ik') \chi(-k') e^{-ik' \tau_{\text{ret}}} \; dk' \Big\},$$

and hence, using the Fourier transform results and the same technique as above, we arrive at,

$$\left\langle \dot{q}\frac{\partial \phi_h}{\partial t} \right\rangle = \frac{e}{8\pi^2\omega_c\nu(\tau)m}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik')|^2 \ \mathrm{Re}\left(i\chi(-k') - i\chi^*(k')\right) \ dk'.$$

Thus,

$$\left\langle \frac{\partial \phi_h}{\partial x}\dot{q} - \dot{q}\frac{\partial \phi_h}{\partial t} \right\rangle = \frac{e}{8\pi^2\omega_c\nu^2(\tau)m}\int_{\infty}^{\infty} k'^2 e^{\pi k'}\Gamma(ik')|^2 \ \mathrm{Re}\left[i\chi^*(-k') - i\chi(k') - i\chi(-k') + i\chi^*(k')\right] \ dk',$$

and upon taking the real parts of the $\chi(k')$ and their complex conjugates, and using the relations in Appendix A, we see that,

$$\langle \mathcal{J} \rangle_{\text{int}} = \frac{e\alpha\gamma}{16\pi^2\nu^2(\tau)\omega_c m}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik')|^2 \left(|\chi(k')|^2 + |\chi(-k')|^2 - \frac{2}{(\omega_c - k')^2} \right) \ dk'.$$

Now since,

$$\gamma = \frac{\pi e^2}{\omega_c^2 m^2}, \quad \text{and} \quad \alpha = \frac{\pi e}{\omega_c m},$$

then we must have that,

$$\langle \mathcal{J} \rangle_{\text{int}} = \frac{e^4}{16\omega_c^4 m^4 \nu^2(\tau)}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik)|^2 \left(|\chi(k')|^2 + |\chi(-k')|^2 - \frac{2}{(\omega_c - k'^2)} \right) \ dk'. \tag{4.81}$$

Recall that,

$$\begin{aligned}
\langle \mathcal{J} \rangle_{\text{dir}} &= -\frac{e^2\alpha^2}{16\pi^2\omega_c^2 m^2 \nu^2(\tau)}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik)|^2 \left(|\chi(k')|^2 + |\chi(-k')|^2 - \frac{2}{(\omega_c - k'^2)} \right) \ dk' \\
&= \frac{e^4}{16\omega_c^4 m^4 \nu^2(\tau)}\int_{-\infty}^{\infty} k'^2 e^{\pi k'}|\Gamma(ik)|^2 \left(|\chi(k')|^2 + |\chi(-k')|^2 - \frac{2}{(\omega_c - k'^2)} \right),
\end{aligned}$$

and so we see that,

$$\langle \mathcal{J} \rangle_0 = \langle \mathcal{J} \rangle_{\text{int}} + \langle \mathcal{J} \rangle_{\text{dir}} = 0, \tag{4.82}$$

and so the total energy flux of the system is zero. Thus we have shown that by taking the Hamiltonian for a quantum oscillator coupled to a real scalar field and confined to a constant accelerating trajectory in $\mathbb{M}^2$, and deriving the equations of motion, we can use the energy flux calculations of Ford and O'Connell to show that there is no net radiation observed.

4.3 Quantum Oscillator Confined to a Free-Fall Trajectory in $D = 2$ Schwarzschild Spacetime

4.3.1 Two-Dimensional Schwarzschild Spacetime.

As we have discussed earlier, two dimensional Schwarzschild spacetime is a static black hole solution to the Einstein field equations which in two dimensions is conformally flat. We adopt the coordinates (t, r) where t is the coordinate time and r is the radial coordinate, then the line element is:

$$ds^2 = \left(1 - \frac{2M_B}{r}\right) dt^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} dr^2, \tag{4.83}$$

where M_B is the mass of the black hole. We can write this as the conformally flat metric,

$$ds^2 = \Omega(dt^2 - dr_*^2), \tag{4.84}$$

where Ω is the conformal factor:

$$\Omega = 1 - \frac{2M_B}{r}. \tag{4.85}$$

In order to write (4.83) as (4.84) we have made the usual tortoise coordinate substitution:

$$dr_*^2 = \Omega^{-2} dr^2,$$

and integrating this directly gives the standard expression for the tortoise coordinate:

$$r_* = r + 2M_B \ln |r - 2M_B|. \tag{4.86}$$

We shall confine the particle to the free-fall trajectory (see Appendix B):

$$r(\tau) = 2M_B \left(1 + \frac{\tau}{2M_B}\right)^{-1}, \tag{4.87}$$

and thus,

$$r_\star = 2M_B \left(1 + \frac{\tau}{2M_B}\right)^{-1} + 2M_B \ln \left| \frac{2M_B}{1 + \tau/2M_B} - 2M_B \right|,$$

and hence,

$$r_* = 2M_B + \tau + 2M_B \ln |\tau|. \tag{4.88}$$

We also have that (near the horizon),

$$\frac{dt}{d\tau} = \frac{1}{1 - 2M_B/r} \approx -\left(\frac{\tau}{2M_B}\right)^{-1}, \tag{4.89}$$

and so we can write,

$$t(\tau) = 2M_B \ln |\tau|. \tag{4.90}$$

As we have done in the preceding two sections, we shall now go on to formulate a Hamiltonian for the system and from this derive the equations of motion for both the oscillator and the free scalar field. From these we shall be able to obtain expressions for the annihilation and creation operators and hence form an expression for q and then calculate the overall energy flux of the system which as we shall see, is not zero.

4.3.2 Hamiltonian

The Hamiltonian of the system is:

$$H = H_{\text{osc}} + H_{\text{int}} + H_{\text{sf}}. \tag{4.91}$$

The Hamiltonian for the oscillator and the free scalar field are exactly the same as previously: The interaction term has been modified now to take account of the fact that we are no longer in Minkowski space, but conformally flat spacetime. We have the interaction term:

$$H_{\text{int}} = \frac{ie\hbar\Omega(\tau)}{\sqrt{2\omega_c}m} \sum_k e_k (a^\dagger - a) \left[b_k e^{ikr_*(\tau)} + b_k^\dagger e^{-ikr_*(\tau)}. \right]$$

So, the Hamiltonian density now will be

$$\mathscr{H}\ d\tau = \hbar a^\dagger a\ d\tau + \hbar \sum_k \omega_k b_k^\dagger b_k\ d\tau + \frac{ie\hbar\Omega(\tau)}{\sqrt{2\omega_c}m} d\tau. \tag{4.92}$$

4.3.3 Equations of Motion

We now find the equations of motion for the oscillator and the scalar field. We start with the oscillator, and since,

$$\frac{da}{d\tau} = \frac{1}{i\hbar}[\mathscr{H}, a],$$

we can form the differential equation

$$\frac{da}{d\tau} = -i\omega_c[a^\dagger a, a] + \frac{e\Omega}{\sqrt{2\omega_c}m}\sum_k e_k\left([a^\dagger, a] - [a, a]\right)\left[b_k e^{ikr_*(\tau)} + b_k^\dagger e^{-ikr_*(\tau)}\right],$$

and using the commutation relations we have the equation of motion of the oscillator:

$$\frac{da}{d\tau} = -i\omega_c a + \frac{e\Omega(\tau)}{\sqrt{2\omega_c}m}\sum_k e_k\left[b_k e^{ikr_*(\tau)} + b_k^\dagger e^{-ikr_*(\tau)}\right]. \tag{4.93}$$

We do the same thing for the field operators:

$$\frac{db_k}{d\tau} = \frac{1}{i\hbar}[\mathscr{H}, b_k],$$

and so,

$$\frac{db_k}{d\tau} = -i\omega_k b_k\frac{dt}{d\tau} + \frac{ee_k\Omega(\tau)}{\sqrt{2\omega_c}m}(a^\dagger - a)e^{-ikr_*(\tau)}. \tag{4.94}$$

Simply multiplying through and integrating with respect to τ' and we obtain expressions for the creation and annihilation operators for the free field:

$$b_k(\tau) = e^{-i\omega_k t(\tau)}b_k(0) + \frac{ee_k e^{-i\omega_k t(\tau)}}{\sqrt{2\omega_c}m}\int_{-\infty}^{\tau}\Omega(\tau')\left[a^\dagger(\tau') - a(\tau')\right]e^{i\omega_k t(\tau') - ikr_*(\tau')}\,d\tau', \tag{4.95}$$

and,

$$b_k^\dagger(\tau) = e^{i\omega_k t(\tau)}b_k(0) + \frac{ee_k e^{i\omega_k t(\tau)}}{\sqrt{2\omega_c}m}\int_{-\infty}^{\tau}\Omega^*(\tau')[a(\tau') - a^\dagger(\tau')\ e^{-i\omega_k t(\tau') + ikr_*(\tau')}\,d\tau'. \tag{4.96}$$

Substituting (4.95) and (4.96) into (4.93) and we have that,

$$\begin{aligned}\frac{da}{d\tau} = -i\omega_c a(\tau) + G_a(\tau) + \frac{e^2\Omega(\tau)}{2\omega_c m^2}\sum_k e_k^2\Big\{ \\ e^{-i\omega_k t(\tau) + ikr_*(\tau)}\int_{-\infty}^{\tau}\Omega(\tau')\left[a^\dagger(\tau) - a(\tau')\right]e^{i\omega_k t(\tau') - ikr_*(\tau')}\,d\tau' \\ + e^{i\omega_k t(\tau) - ikr_*(\tau)}\int_{-\infty}^{\tau}\Omega^*(\tau')\left[a(\tau') - a^\dagger(\tau')\right]e^{-i\omega_k t(\tau') + ikr_*(\tau')}\,d\tau\Big\}\end{aligned}$$

where we have let,

$$\mathcal{G}_A(\tau) = \frac{e\Omega(\tau)}{\sqrt{2\omega_c}m}\sum_k e_k\left\{b_k(0)e^{-i\omega_k t(\tau)-+ikr_*(\tau)} + b_k^\dagger(0)e^{i\omega_k t(\tau)-ikr_*(\tau)}\right\}. \tag{4.97}$$

As in the previous cases, we once again remove high frequency behavior from the above equation so we shall let,

$$A(\tau) = e^{-i\omega_c\tau)}a(\tau), \tag{4.98}$$

and hence we have that,

$$\begin{aligned}\frac{dA}{d\tau} = \mathcal{G}_A(\tau) + \frac{e^2\Omega(\tau)}{2\omega_c m^2}\sum_k e_k^2\Big\{ \\ e^{i\omega_c\tau}e^{-i\omega_k t(\tau)+ikr_*(\tau)}\int_{-\infty}^{\tau}\Omega(\tau')\left[e^{i\omega_c\tau'}A^\dagger - e^{-i\omega_c\tau'}A\right]e^{i\omega_k t(\tau')-ikr_*(\tau')}\ d\tau' \\ e^{i\omega_c\tau}e^{i\omega_k t(\tau)-ikr_*(\tau)}\int_{-\infty}^{\tau}\Omega^*(\tau')\left[e^{-i\omega_c\tau'}A - e^{i\omega_c\tau'}A^\dagger\right]e^{-i\omega_k t(\tau')+ikr_*(\tau')}\ d\tau'\Big\},\end{aligned} \tag{4.99}$$

with,

$$\mathcal{G}_A(\tau) = \frac{e\Omega(\tau)e^{i\omega_c\tau}}{\sqrt{2\omega_c}m}\sum_k e_k\left[b_k(0)e^{-i\omega_k t+ikr_*} + b_k^\dagger(0)e^{i\omega_k t-ikr_*}\right]. \tag{4.100}$$

4.3.4 Calculation of $A(s)$

We now wish to go on to find a simple form for $A(s)$ which will enable us to determine a modified Langevin equation and from this we will be able to find forms for $A(\tau)$. Before we do this however we shall need to define the Fourier transforms:

$$e^{i\omega_k t(\tau)-ikr_*(\tau)} = \int_{-\infty}^{\infty}\alpha_k(k')e^{-ik'\tau}\ dk', \quad \text{and,} \tag{4.101}$$

$$e^{-i\omega_k t(\tau)+ikr_*(\tau)} = \int_{-\infty}^{\infty}\alpha_k^*(k'')e^{-ik''\tau}\ dk'', \tag{4.102}$$

and as we have the conformal factor as a function of proper time,

$$\Omega(\tau) = \int_{-\infty}^{\infty}\mu(\nu)e^{-i\nu\tau}\ d\nu. \tag{4.103}$$

and from Appendix B we have that,

$$\alpha_k(k') = -\frac{2iM_Bke^{-2iM_Bk}}{\pi}\left(\frac{1}{i(k'-k)}\right)^{1-4iM_Bk}\Gamma(-4iM_Bk); \tag{4.104}$$

$$\alpha_k^*(k') = \frac{2iM_Bke^{2iM_Bk}}{\pi}\left(\frac{1}{-i(k'-k)}\right)^{1+4iM_Bk}\Gamma(4iM_Bk). \tag{4.105}$$

We now take the Laplace transform of both sides of (4.99) which gives us,

$$s\overline{A}(s) = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{$$

$$\int_0^\infty \Omega(\tau)e^{i\omega_c\tau - s\tau}e^{-i\omega_k t + ikr_*}\, d\tau \int_{-\infty}^{\tau} \Omega(\tau')\left[e^{i\omega_c\tau'}A^\dagger - e^{-i\omega_c\tau'}A\right]e^{-ik'\tau'}\, d\tau' \int_{-\infty}^{\infty}\alpha_k(k')\, dk'$$

$$+\int_0^\infty \Omega(\tau)e^{i\omega_c\tau - s\tau}e^{i\omega_k t - ikr_*}\, d\tau \int_{-\infty}^{\tau} \Omega^*(\tau')\left[e^{-i\omega_c\tau'}A - e^{i\omega_c\tau'}A^\dagger\right]e^{-ik'\tau'}\, d\tau' \int_{-\infty}^{\infty}\alpha_k(k'')\, dk''.$$

We now substitute the eigenfunctions and the conformal factor $\Omega(\tau)$ for their respective Fourier transforms and change the order of integration:

$$s\overline{A}(s) = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{$$

$$\Bigg[\int_{\tau'}^\infty e^{i(\omega_c + k'' - \nu + is)\tau}\, d\tau \int_{-\infty}^{\infty}\left[e^{i\omega_c\tau'}A^\dagger - e^{-i\omega_c\tau'}A\right]e^{-i\nu'\tau'}e^{-ik'\tau'}\, d\tau'$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_k(k')\alpha_k^*(k'')'dk'dk'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(\nu)\mu(\nu')\, d\nu d\nu'\Bigg] +$$

$$\Bigg[\int_{-\infty}^{\tau'} e^{i(\omega_c - k' - \nu + is)\tau}\, d\tau \int_{-\infty}^{\infty}\left[e^{-i\omega_c\tau'}A - e^{i\omega_c\tau'}A^\dagger\right]e^{i\nu''\tau'}e^{ik''\tau'}\, d\tau'$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_k^*(k'')\alpha_k^(k')'dk'dk'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(\nu)\mu^*(\nu'')\, d\nu d\nu''\Bigg]\Big\},$$

Performing the τ integrations we have that,

$$s\overline{A}(s) = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{$$

$$\Bigg[\int_{-\infty}^{\infty}\frac{(e^{2i\omega_c\tau'}A^\dagger - A)e^{-s\tau'}}{i(\omega_c + k'' - \nu + is)}\, d\tau' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_k(k')\alpha_k^*(k'')e^{i(k''-k')\tau'}\, dk'dk'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(\nu)\mu(-\nu')e^{-i(\nu-\nu')\tau}\, d\nu d\nu'\Bigg]$$

$$+\Bigg[\int_{-\infty}^{\infty}\frac{(A - e^{2i\omega_c\tau}A^\dagger)e^{-s\tau'}}{i(\omega_c - k' - \nu + is)}\, d\tau' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_k(k')\alpha_k^*(k'')e^{i(k''-k')\tau'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(\nu)\mu^*(\nu'')e^{i(\nu''-\nu)\tau'}\, d\nu d\nu''\Bigg]\Big\},$$

where, to make things simpler we have let $\nu' \to -\nu'$ in the ν' integral first set of square brackets above. As in the previous sections, we assume that $e^{(ik''-k')\tau}$ and $e^{i\nu\tau}$ are rapid oscillating functions, and the rest of the functions in the integrand are slowly varying. Thus we can, for an approximation let $k' = k''$, $\nu = \nu'$ in the first large square bracket, and $\nu = \nu''$ in the second. Once more we use the rotating wave approximation and so ignore the $A^\dagger$ term, and we have,

$$s\overline{A}(s) = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2 \Big\{$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)\overline{A}(s)}{i(\omega_c + k' - \nu + is)}\, dk'd\nu + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)\overline{A}(s)}{i(\omega_c - k' - \nu + is)}\, dk'd\nu\Big\},$$

and hence,

$$s\overline{A}(s) = \overline{\mathcal{G}_A(s)} + \frac{e^2}{2\omega_c m^2}\sum_k e_k^2\Bigg\{$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(k')|^2\mu^2(\nu)\overline{A}(s)\left(-\frac{1}{i(\omega_c-k'-\nu+is)}+\frac{1}{i(\omega_c+k'-\nu+is)}\right)\,d\nu dk'\Bigg\}.$$

After factorizing we have that,

$$\overline{A}(s) = \frac{\overline{\mathcal{G}_A(s)}}{s+\frac{e^2}{2\omega_c m^2}\sum_k e_k^2\Big\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(k')|^2\mu^2(\nu)\overline{A}(s)\left(-\frac{1}{i(\omega_c-k'-\nu+is)}+\frac{1}{i(\omega_c+k'-\nu+is)}\right)\,d\nu dk'\Big\}}. \tag{4.106}$$

We shall want to apply the same Wigner-Weiskopf approximation to the denominator of the above equation as we have done in the preceding sections. To do this, we first write the denominator as,

$$s-\frac{ie^2}{2\omega_c m}\sum_k e_k^2\left\{\underbrace{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c+k'-\nu+is}\,dk'd\nu}_{(\mathcal{I}_1)}+\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c-k'-\nu+is}\,dk'd\nu}_{(\mathcal{I}_2)}\right\}.$$

We now shall need to perform the Wigner-Weiskopff approximation to both $\mathcal{I}_1$ and $\mathcal{I}_2$. Recall we let:

$$\lim_{s\to 0}\left\{\frac{1}{x+is}\right\} = \lim_{s\to 0}\left\{\frac{x}{x^2+s^2}-\frac{is}{x^2+x^2}\right\} = \frac{1}{x}-i\pi\delta(x).$$

We start with $\mathcal{I}_1$:

$$\mathcal{I}_1 = \frac{ie^2}{2\omega_c m^2}\sum_k e_k^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c+k'-\nu+is}\,dk'd\nu, \tag{4.107}$$

which we may write as

$$\mathcal{I}_1 = \frac{ie^2}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e_k^2\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c+k'-\nu+is}\,dk'd\nu dk. \tag{4.108}$$

We now take the limit $s\to 0$ and we now have,

$$\mathcal{I}_1 = \frac{ie^2}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e_k^2\left[\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c+k'-\nu}-i\pi\delta(k'-\nu+\omega_c)\right]\,dk'd\nu dk,$$

and hence:

$$\mathcal{I}_1 = -\frac{e^2\pi}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(\omega_c-\nu)|^2\mu^2(\nu)e_k^2\ dkd\nu$$
$$-\frac{ie}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)e_k^2}{\omega_c+k'-\nu}dk'd\nu dk.$$

If we define:

$$\frac{\gamma'}{2} = \frac{e^2\pi}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(\omega_c-\nu)|^2\mu^2(\nu)e_k^2\ dkd\nu, \tag{4.109}$$

and,

$$\Delta\varpi' = \frac{ie}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha_k(k')|^2\mu^2(\nu)e_k^2}{\omega_c+k'-nu}dk'd\nu dk, \tag{4.110}$$

then we have now that,

$$\mathcal{I}_1 = \frac{\gamma'}{2}+i\Delta\varpi'. \tag{4.111}$$

We note that $\gamma' > 0$. We do the same thing for $\mathcal{I}_2$. We have that,

$$\mathcal{I}_2 = \frac{ie^2}{2\omega_c m^2}\sum_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)}{\omega_c-k'-\nu+is}\ dk'd\nu,$$

We now take the limit $s\to 0$ and we have that,

$$\mathcal{I}_2 = -\frac{e^2\pi}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(\nu-\omega_c)|^2\mu^2(\nu)e_k^2\ d\nu dk$$
$$-\frac{ie^2}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha_k(k')|^2\mu^2(\nu)e_k^2}{k'+\nu-\omega_c}\ dk'd\nu dk.$$

Hence if we define,

$$\frac{\gamma''}{2} = \frac{e^2\pi}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha_k(\nu-\omega_c)|^2\mu^2(\nu)e_k^2\ d\nu dk, \tag{4.112}$$

and,

$$\Delta\varpi'' = \frac{ie^2}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\alpha_k(k')|^2\mu^2(\nu)e_k^2}{k'+\nu-\omega_c}\ dk'd\nu dk, \tag{4.113}$$

then we have that,

$$\mathcal{I}_2 = \frac{\gamma''}{2}+i\Delta\varpi''. \tag{4.114}$$

Furthermore, if we now let,

$$\frac{\gamma}{2} = \frac{\gamma'}{2}+\frac{\gamma''}{2},\quad \text{and}\quad \Delta\varpi = \Delta\varpi'+\Delta\varpi'', \tag{4.115}$$

then we can write (4.106) as,

$$\overline{A}(s) = \frac{\overline{\mathcal{G}_A(s)}}{s + \frac{\gamma}{2} + \Delta\varpi}. \tag{4.116}$$

It is useful to note that γ has the form,

$$\gamma = -\frac{e^2\pi}{2\omega_c m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^2(\nu)e_k^2\left(|\alpha_k(\omega_c-\nu)|^2 + |\alpha_k(\nu-\omega_c)|^2\right)\, d\nu dk, \tag{4.117}$$

and so $\gamma < 0$.

4.3.5 Calculation of $A(\tau)$ and $A^\dagger(\tau)$

As we are able to write (4.106) in the simpler form of (4.116), this means that we can once more write the equation of motion for the quantum oscillator as a modified Langevin equation:

$$\frac{dA}{d\tau} = -\frac{\gamma}{2}A(\tau) + \mathcal{G}_A(\tau), \tag{4.118}$$

where,

$$\mathcal{G}_A(\tau) = \frac{e\Omega(\tau)e^{i\omega_c\tau}}{\sqrt{2\omega_c}m}\sum_k e_k\left[b_k(0)e^{-i\omega_k t + ikr_*} + b_k^\dagger(0)e^{i\omega_k t - ikr_*}\right]. \tag{4.119}$$

Integrating (4.118) we have that,

$$A(\tau) = e^{-\gamma\tau/2}\int_0^\tau \mathcal{G}_A(\tau')e^{\gamma\tau'/2}\, d\tau',$$

and hence

$$A(\tau) = \frac{ee^{-\gamma\tau/2}}{\sqrt{2\omega_c}m}\sum_k e_k\Bigg\{$$
$$b_k(0)\int_0^\tau e^{-i\omega_k t + ikr_*}\Omega(\tau')e^{i(\omega_c - i\gamma/2)\tau'}\, d\tau' + b_k^\dagger(0)\int_0^p te^{i\omega_k t - ikr_*}\Omega(\tau')e^{i(\omega_c - i\gamma/2)\tau'}\, d\tau'\Bigg\}.$$

Now, we substitute the appropriate Fourier transforms for the eigenfunctions and the conformal factor, and this gives:

$$A(\tau) = \frac{ee^{-\gamma\tau/2}}{\sqrt{2\omega_c}m}\sum_k e_k\Bigg\{$$
$$b_k(0)\int_{-\infty}^{\infty}\mu(\nu)\, d\nu\int_{-\infty}^{\infty}\alpha_k^*(k'')\, dk''\int_{-\infty}^{\tau} e^{i(\omega_c + k'' - \nu - i\gamma/2)\tau'}\ \tau'$$
$$+ b_k^\dagger(0)\int_{-\infty}^{\infty}\mu(\nu)\, d\nu\int_{-\infty}^{\infty}\alpha_k(k')\, dk'\int_{-\infty}^{\tau} e^{i(\omega_c - k' - \nu - i\gamma/2)\tau'}\, d\tau'\Bigg\}.$$

We now perform the τ' integrations and we obtain,

$$A(\tau) = \frac{ee^{-\gamma\tau/}}{\sqrt{2\omega_c}m} \sum_k e_k \Bigg\{ \\ b_k(0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\nu)\alpha_k^*(k'') \left[\frac{e^{i(\omega_c+k''-\nu-i\gamma/2)\tau'}}{i(\omega_c+k''-\nu-i\gamma/2} \right] \, dk''d\nu \\ + b_k^\dagger(0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(\nu)\alpha_k(k') \left[\frac{e^{i(\omega_c+-k''-\nu-i\gamma/2)\tau'}}{i(\omega_c-k''-\nu-i\gamma/2} \right] \, dk'd\nu \Bigg\}.$$

We shall now define,

$$\chi(k,\nu) = \frac{1}{\omega_c + k' - \nu - i\gamma/2}. \tag{4.120}$$

We now multiply through by $e^{-\gamma\tau/2}$ and discard the terms which decay to zero and we obtain the expression for the annihilation operator of the oscillator:

$$A(\tau) = -\frac{ie}{\sqrt{2\omega_c}m} \sum_k e_k \Bigg\{ b_k(0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(\nu)\alpha_k^*(k'')\chi(k'',\nu)e^{i(\omega_c+k''-\nu)\tau} \, dk''d\nu \\ + b_k^\dagger(0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(\nu)\alpha_k(k')\chi(-k',\nu)e^{i(\omega_c-k'-\nu)\tau} \, dk'd\nu \Bigg\}, \tag{4.121}$$

and the creation operator:

$$A^\dagger(\tau) = \frac{ie}{\sqrt{2\omega_c}m} \sum_k e_k \Bigg\{ b_k^\dagger(0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\nu)\alpha_k(k'')\chi^*(k'',\nu)e^{-i(\omega_c+k''-\nu)\tau} \, dk''d\nu \\ + b_k(0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\nu)\alpha_k^*(k')\chi^*(-k',\nu)e^{-i(\omega_c-k'-\nu)\tau} \, dk'd\nu \Bigg\}. \tag{4.122}$$

We shall now need to find an expression for the position operator of the oscillator, and its first derivative with respect to proper time. As we have seen previously,

$$q(\tau) = \frac{1}{\sqrt{2\omega_c}} \left[e^{i\omega_c\tau} A^\dagger(\tau) + e^{-i\omega_c\tau} A(\tau) \right],$$

and so we must have that,

$$q(\tau) = \frac{ie}{2\omega_c m} \sum_k e_k \Bigg\{ \\ b_k(0) \Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\nu)\alpha_k^*(k')\chi^*(-k',\nu)e^{i(k'+\nu)\tau} \, dk'd\nu - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\nu)\alpha_k^*(k'')\chi(k'',\nu)e^{i(k''-\nu)\tau} \, dk''d\nu \Bigg] + \\ b_k^\dagger(0) \Bigg[- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(\nu)\alpha_k(k'')\chi^*(k'',\nu)e^{-i(k''-\nu)\tau} \, dk''d\nu + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(\nu)\alpha_k(k')\chi(-k',\nu)e^{-i(k'+\nu)\tau} \, dk'd\nu \Bigg] \Bigg\}. \tag{4.123}$$

Differentiating with respect to proper time τ and letting $-\nu \to \nu$ we obtain:

$$
\begin{aligned}
\dot{q}(\tau) = &-\frac{e}{2\omega_c m}\sum_k e_k\Bigg\{ \\
& b_k(0)\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(k'+\nu)\mu^*(\nu)\alpha_k^*(k')\chi^*(-k',\nu)e^{i(k'+\nu)\tau}\ dk'd\nu \\
& -\int_{-\infty}^{\infty}(k''+\nu)\mu(-\nu)\alpha_k^*(k'')\chi(k'',-\nu)e^{i(k''+\nu)\tau}\ dk''d\nu\Bigg] \\
& b_k^\dagger(0)\Bigg[-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(k''+\nu)\mu^*(-\nu)\alpha_k(k'')\chi^*(k'',-nu)e^{-i(k''+\nu)\tau}\ dk''d\nu \\
& +\int_{\infty}^{\infty}\int_{-\infty}^{\infty}(k'+\nu)\mu(\nu)\alpha_k(k')\chi(-k',\nu)e^{-i(k'+\nu)\tau}\ dk'd\nu\Bigg]\Bigg\}.
\end{aligned}
\tag{4.124}
$$

4.3.6 Energy Flux of the System

Now that we have an expression for $q(\tau)$ and $\dot{q}(\tau)$, all we need to do next is formulate expressions for the energy-flux expectation values. Before we do this however, it will be profitable to consider the trajectory our oscillator is confined to, and the relationship between proper time and retarded time. So, we have:

$$
r_* = r + 2M_B\ln|r-2M_B|, \quad \text{and} \quad t(\tau) = -2M_B\ln|\tau|,
$$

so the relationship between proper time and retarded time is,

$$
t - t(\tau_{\text{ret}}) = r_* - r_*(\tau_{\text{ret}}).
$$

Differentiating;

$$
0 - 2M_B\frac{d\tau_{\text{ret}}}{d\tau_{\text{ret}}} = dr_* - \frac{dr_*}{d\tau_{\text{ret}}}d\tau_{\text{ret}},
$$

and so,

$$
\begin{aligned}
dr_* &= -d\tau_{\text{ret}} + \frac{2M_B}{r-2M_B}(-d\tau_{\text{ret}}) \\
&= -d\tau_{\text{ret}}\left(1+\frac{2M_B}{r-2M_B}\right) = -d\tau_{\text{ret}}\left(1-\frac{2M_B}{\tau_{\text{ret}}}\right).
\end{aligned}
$$

We shall make the approximation,

$$
\frac{dr_*}{d\tau_{\text{ret}}} \approx \frac{2M_B}{\tau_{\text{ret}}}, \tag{4.125}
$$

and hence we have that,

$$\frac{\partial q(\tau_{\text{ret}})}{\partial t} = \frac{\partial q(\tau_{\text{ret}})}{\partial \tau_{\text{ret}}} \left(\frac{\partial t}{\partial \tau_{\text{ret}}}\right)^{-1} = \dot{q} \left[\frac{2M_B}{\tau_{\text{ret}}}\right]^{-1}, \text{ and } \frac{\partial q(\tau_{\text{ret}})}{\partial r_*} = \frac{\partial q(\tau_{\text{ret}})}{\partial \tau_{\text{ret}}} \left(\frac{\partial \tau_{\text{ret}}}{\partial r_*}\right),$$

so in fact,

$$\frac{\partial q(\tau_{\text{ret}})}{\partial t} = \frac{\tau_{\text{ret}}}{2M_B}\dot{q}(\tau_{\text{ret}}), \text{ and } \frac{\partial q(\tau_{\text{ret}})}{\partial r_*} = \frac{\tau_{\text{ret}}}{2M_B}\dot{q}(\tau_{\text{ret}}). \tag{4.126}$$

The solution to the scalar field equation is:

$$\phi = \phi_h - \alpha q(\tau_{\text{ret}}).$$

The total energy flux of the system is given by,

$$\langle \mathcal{J} \rangle = \left\langle \left[\frac{\partial \phi_h}{\partial t} - \alpha \frac{\partial q(\tau_{\text{ret}})}{\partial t}\right] \left[\frac{\partial \phi_h}{\partial r_*} - \alpha \frac{\partial q(\tau_{\text{ret}})}{\partial r_*}\right] \right\rangle,$$

which we may now write as

$$\langle \mathcal{J} \rangle = \left\langle \left[\frac{\partial \phi_h}{\partial t} - \alpha \frac{\tau_{\text{ret}}}{2M_B}\dot{q}(\tau_{\text{ret}})\right] \left[\frac{\partial \phi_h}{\partial r_*} - \alpha \frac{\tau_{\text{ret}}}{2M_B}\dot{q}(\tau_{\text{ret}})\right] \right\rangle. \tag{4.127}$$

Hence we have the direct flux term:

$$\langle \mathcal{J} \rangle_{\text{dir}} = \frac{\alpha^2 \tau_{\text{ret}}^2}{4M_B^2} \left\langle \dot{q}^2(\tau_{\text{ret}}) \right\rangle, \tag{4.128}$$

and the interference term will be

$$\langle \mathcal{J} \rangle_{\text{int}} = \frac{-\alpha \tau_{\text{ret}}}{2M_B} \left\langle \dot{q}(\tau_{\text{ret}}) \frac{\partial \phi_h}{\partial r_*} + \frac{\partial \phi_h}{\partial t}\dot{q}(\tau_{\text{ret}}) \right\rangle. \tag{4.129}$$

We shall rewrite the expression for $q(\tau)$ by making the substitution $\nu = -\nu$, and so we have that,

$$\begin{aligned}
\dot{q}(\tau) = -\frac{e}{2\omega_c m} \sum_k e_k \Bigg\{ \\
& b_k(0)\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k'+\nu)\mu^*(\nu)\alpha_k^*(k')\chi^*(-k',\nu)e^{i(k'+\nu)\tau)}\ dk'd\nu \\
& - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k''+\nu)\mu(-\nu)\alpha_k^*(k'')\chi(k'',-\nu)e^{i(k''+\nu)\tau}\ dk''d\nu\Bigg] \\
& + b_k^\dagger(0)\Bigg[-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k''+\nu)\mu^*(-\nu)\alpha_k(k'')\chi^*(k'',-\nu)e^{-i(k''+\nu)\tau}\ dk''d\nu \\
& - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k'+\nu)\mu(\nu)\alpha_k(k')\chi(-k',\nu)e^{-i(k'+\nu)\tau}\ dk'd\nu\Bigg]\Bigg\},
\end{aligned} \tag{4.130}$$

and we can now go on to calculate $\langle \mathcal{J} \rangle_{\text{dir}}$ and $\langle \mathcal{J} \rangle_{\text{int}}$. We start with the direct flux term.

Calculation of $\langle \mathcal{J} \rangle_{\text{dir}}$ The first step in this calculation is to form the product,

$$\langle \dot{q}\dot{q} \rangle = \frac{e^2}{4\omega_c^2 m^2} \sum_k e_k^2 \operatorname{Re} \Bigg\{ \\ \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k'' + \nu)\mu^*(-\nu)\alpha_k(k'')\chi^*(k'', -\nu)e^{-i(k''+\nu)\tau_{\text{ret}}}\, dk''d\nu \right. \\ \left. + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k' + \nu)\mu(\nu)\alpha)k(k')\chi(-k', \nu)e^{-i(k'+\nu)\tau_{\text{ret}}}\, dk'd\nu \right] \times \\ \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k' + \nu)\mu^*(\nu)\alpha_k^*(k')\chi^*(-k', \nu)e^{i(k'+\nu)\tau_{\text{ret}}}\, dk'd\nu \right. \\ \left. - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (k'' + \nu)\mu(-\nu)\alpha_k^*(k'')\chi(k'', -\nu)e^{i(k''+\nu)\tau_{\text{ret}}} \right] \Bigg\}. \tag{4.131}$$

Multiplying the brackets out, and performing the k'' integral we have that,

$$\langle \dot{q}\dot{q} \rangle = \frac{e^2}{4\omega_c^2 m^2} \sum_k e_k^2 \operatorname{Re} \Bigg\{ \\ \iiint_{-\infty}^{\infty} (k' + \nu)(k' + \nu')|\alpha_k(k')|^2 \mu(-\nu)\mu^*(\nu')\chi^*(k', -\nu)\chi^*(-k, \nu')e^{i(\nu-\nu)\tau_{\text{ret}}}\, dk'd\nu d\nu' \\ - \iiint_{-\infty}^{\infty} (k' + \nu)(k' + \nu')|\alpha_k(k')|^2 \mu^*(-\nu)\mu(-\nu')\chi^*(k', -\nu)\chi(k', -\nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}\, dk'd\nu d\nu' \\ + \iiint_{-\infty}^{\infty} (k' + \nu)(k' + \nu')|\alpha_k(k')|^2 \mu^*(\nu')\mu(\nu)\chi(-k', \nu)\chi^*(-k', \nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}\, dk'd\nu d\nu' \\ - \iiint_{-\infty}^{\infty} (k' + \nu)(k' + \nu')|\alpha_k(k')|^2 \mu(\nu)\mu(-\nu')\chi(-k', \nu)\chi(k', -\nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}\, dk'd\nu d\nu' \Bigg\}. \tag{4.132}$$

We now put in the function $\mu(\nu)$. In fact this function has the form,

$$\mu(\nu) = I(\nu) + \delta(\nu), \tag{4.133}$$

where $I(\nu)$ is the Fourier transform of the Conformal factor (See Appendix B, equation (B.17)), and $\delta(\nu)$ is the usual delta function. If we substitute (4.133) into (4.132), we end up with a very long expression which we can write in the compact form as:

$$\langle \dot{q}\dot{q} \rangle = \frac{e^2}{8\omega_c^2 m^2 \pi} \sum_k \sum_{j=1}^{4} e_k^2 \operatorname{Re}\ \{\mathfrak{I}_j\}\,. \tag{4.134}$$

After performing the $\delta(\nu)$ and $\delta(\nu')$ integrals, we find that we have,

$$\begin{aligned}
\mathfrak{I}_1 = &-\iiint_{-\infty}^{\infty}(k'+\nu)(k'+\nu')I(-\nu)I(\nu)\chi^*(k',-\nu)\chi^*(-k',\nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}|\alpha_k(k')|^2\ dk'd\nu d\nu' \\
&+\iint_{-\infty}^{\infty} ik(k'+\nu)I(-\nu)\chi^*(k',-\nu)\chi^*(-k')e^{-i\nu\tau_{\text{ret}}}|\alpha_k(k')|^2\ dk'd\nu \\
&+\iint_{-\infty}^{\infty} ik'(k'+\nu)I(\nu')\chi^*(k')\chi^*(-k',-\nu')e^{i\nu'\tau_{\text{ret}}}|\alpha_k(k')|^2\ dk'd\nu \\
&+\int_{-\infty}^{\infty} k'^2\chi^*(-k')\chi^*(k')|\alpha_k(k')|^2\ dk',
\end{aligned} \tag{4.135}$$

$$\begin{aligned}
\mathfrak{I}_2 = &\iiint_{-\infty}^{\infty}(k'+\nu)(k'+\nu')I(-\nu)I(-\nu')|\alpha_k(k')|^2\chi^*(k',-\nu)\chi(k',-\nu)e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk'd\nu d\nu' \\
&-\iint_{-\infty}^{\infty} ik'(k'+\nu)|\alpha_k(k')|^2 I(-\nu)\chi^*(k',-\nu)\chi(k')e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
&-\iint_{-\infty}^{\infty} ik'(k'+\nu)|\alpha_k(k')|^2 I(-\nu')\chi^*(k')\chi(k',-\nu')e^{i\nu'\tau_{\text{ret}}}\ dk'd\nu' \\
&-\int_{-\infty}^{\infty} k'^2|\alpha_k(k')|^2\chi^*(k')\chi(k')\ dk',
\end{aligned} \tag{4.136}$$

$$\begin{aligned}
\mathfrak{I}_3 = &-\iiint_{-\infty}^{\infty}(k'+\nu)(k'+\nu')|\alpha_k(k')|^2 I(\nu)I(\nu')\chi(-k',\nu)\chi^*(-k',\nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk'd\nu d\nu' \\
&\iint_{-\infty}^{\infty} ik'(k'+\nu')|\alpha_k(k')|^2 I(\nu')\chi(-k')\chi^*(-k',\nu')e^{i\nu'\tau_{\text{ret}}}\ dk'd\nu' \\
&\iint_{-\infty}^{\infty} ik'(k'+\nu)|\alpha_k(k')|^2 I(\nu)\chi(-k',\nu)\chi^*(-k')e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
&\int_{-\infty}^{\infty} k'^2|\alpha_k(k')|^2\chi(-k')\chi^*(-k)\ dk',
\end{aligned} \tag{4.137}$$

and finally:

$$\begin{aligned}
\mathfrak{I}_4 = &\iiint_{-\infty}^{\infty}(k'+\nu)(k'+\nu')|\alpha_k(k')|^2 I(\nu)I(-\nu')\chi(-k',\nu)\chi(k',-\nu')e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk'd\nu d\nu' \\
&-\iint_{-\infty}^{\infty} ik'(k'+\nu)|\alpha_k(k')|^2 I(\nu)\chi(-k',\nu)\chi(k')e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
&-\iint_{-\infty}^{\infty} ik'(k'+\nu')|\alpha_k(k')|^2 I(-\nu')\chi(-k')\chi(k',-\nu')e^{i\nu'\tau_{\text{ret}}}\ dk'd\nu \\
&-\int_{-\infty}^{\infty} k'^2|\alpha_k(k')|^2\chi(-k')\chi(k')\ dk'.
\end{aligned} \tag{4.138}$$

We now have a large number of integrals to contend with, and at first sight the situation looks very complicated. However, if we adopt a strategy of dealing with the separate classes of integrands (meaning the single, double and triple integrals). Starting with the single integrals over k', we see that these are just the flat spacetime integrals for the direct flux that we obtained in the previous section. These must cancel with their counterparts obtained in the calculation of the interference terms (which we will calculate later). So we have only the double and triple integrals to deal with.

Let us now deal with the double integrals. We have eight integrals in total:

$$
\begin{aligned}
\mathbb{I}_1 &= \iint_{-\infty}^{\infty} ik'(k'+\nu)I(-\nu)\chi^*(k',-\nu)\chi^*(-k')|\alpha_k(k')|^2 e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_2 &= \iint_{-\infty}^{\infty} ik'(k'+\nu)I(\nu)\chi^*(k')\chi^*(-k',\nu)|\alpha_k(k')|^2 e^{i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_3 &= -\iint_{-\infty}^{\infty} ik'(k'+\nu)I(-\nu)\chi^*(k',-\nu)\chi(k')|\alpha_k(k')|^2 e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_4 &= -\iint_{-\infty}^{\infty} ik'(k'+\nu)I(-\nu)\chi^*(k')\chi(k',-\nu')|\alpha_k(k')|^2 e^{i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_5 &= \iint_{-\infty}^{\infty} ik'(k'+\nu)I(\nu)\chi(-k')\chi^*(-k',\nu)|\alpha_k(k')|^2 e^{i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_6 &= \iint_{-\infty}^{\infty} ik'(k'+\nu)I(\nu)\chi(-k',\nu)\chi^*(-k')|\alpha_k(k')|^2 e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_7 &= -\iint_{-\infty}^{\infty} ik'(k'+\nu)I(\nu)\chi(-k',\nu)\chi(k')|\alpha_k(k')|^2 e^{-i\nu\tau_{\text{ret}}}\ dk'd\nu \\
\mathbb{I}_8 &= -\iint_{-\infty}^{\infty} ik'(k'+\nu)I(-\nu)\chi(-k')\chi(k',-\nu')|\alpha_k(k')|^2 e^{i\nu\tau_{\text{ret}}}\ dk'd\nu,
\end{aligned}
\tag{4.139}
$$

where we defined earlier that

$$\chi(k',\nu) = \frac{1}{\omega_c + k' - \nu - i\gamma/2}.$$

We must now consider each of the $\mathbb{I}_j$'s in turn. In fact, we need only perform the ν integrations as we shall see, this will be sufficient to allow us to simplify things greatly. All of the integrals in (4.139) are contour integrals and we may evaluate them using Cauchy's integral theorem which we stated earlier as:

Cauchy's Integral Theorem Let $\mathcal{D}$ be a bounded domain in the complex plane with piecewise smooth boundary $\partial\mathcal{D}$. Suppose that $f(z)$ is analytic on $\mathcal{D} \cup \partial\mathcal{D}$ except for a finite number of isolated singularities $z_1, ..., z_m \in \mathcal{D}$. Then

$$\int_{\partial\mathcal{D}} f(z)\ dz = 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j],$$

where for a pole of order n at the point z_0 we have that

$$\text{Res}[f(z), z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \{(z - z_0)^n f(z)\} \right\}.$$

In the case of $\mathbb{I}_1$ we see that the integral has a pole at $\nu = 0$ and at $\nu = -\omega_c - k' - i\gamma/2$. Thus applying the above theorem we find that,

$$\mathbb{I}_1 = \int_{-\infty}^{\infty} -2\pi i k'^2 \chi^*(-k')\chi^*(k') + 2\pi i k' \left(\omega_c + \frac{i\gamma}{2}\right) \chi^*(-k')\chi^*(k')k'\ dk'. \tag{4.140}$$

In the case of $\mathbb{I}_2$ we have a pole at $\nu = 0$ and $\nu = \omega_c - k' + i\gamma/2$ and hence we find,

$$\mathbb{I}_2 = \int_{-\infty}^{\infty} 2\pi i k'^2 \chi^*(k')\chi^*(-k') - 2\pi i k' \left(\omega_c + \frac{i\gamma}{2}\right) \chi^*(k')\chi^*(-k')\ dk'. \tag{4.141}$$

For the integral $\mathbb{I}_3$ we have a pole at $\nu = 0$ and at $\nu = -\omega_c - k' - i\gamma/2$ and we obtain,

$$\mathbb{I}_3 = \int_{-\infty}^{\infty} 2\pi i k'^2 |\chi(k')|^2 + 2\pi i k' \left(\omega_c + \frac{i\gamma}{2}\right) |\chi(k')|^2\ dk'. \tag{4.142}$$

The integral $\mathbb{I}_4$ has poles at $\nu = 0$ and at $\nu = -\omega_c - k' + i\gamma/2$ and so,

$$\mathbb{I}_4 = \int_{-\infty}^{\infty} 2\pi i k'^2 |\chi(k')|^2 + 2\pi i k' \left(\omega_c - \frac{i\gamma}{2}\right) |\chi(k')|^2\ dk'. \tag{4.143}$$

In the case of $\mathbb{I}_5$ we have a pole at $\nu = 0$ and at $\nu = \omega_c - k' + i\gamma/2$, which gives,

$$\mathbb{I}_5 = \int_{-\infty}^{\infty} 2\pi i k'^2 |\chi(-k')|^2 + 2\pi i k' \left(\omega_c + \frac{i\gamma}{2}\right) |\chi(-k')|^2\ dk'. \tag{4.144}$$

The integral $\mathbb{I}_6$ has a pole at $\nu = 0$ and $\nu = \omega_c - k' - i\gamma/2$, thus,

$$\mathbb{I}_6 = \int_{-\infty}^{\infty} 2\pi i k'^2 |\chi(-k')|^2 + 2\pi i k' \left(\omega_c - \frac{i\gamma}{2}\right) |\chi(-k')|^2\ dk'. \tag{4.145}$$

Coming to the last two integrals, $\mathbb{I}_7$ has a pole at $\nu = 0$ and at $\nu = \omega_c - k' - \gamma/2$, so,

$$\mathbb{I}_7 = \int_{-\infty}^{\infty} -2\pi i k'^2 \chi(k')\chi(-k') + 2\pi i k' \left(\omega_c - \frac{i\gamma}{2}\right) \chi(k')\chi(-k')\ dk', \tag{4.146}$$

and finally we have the integrand $\mathbb{I}_8$ with poles at $\nu = 0$ and $\nu = -\omega_c - k' + i\gamma/2$ and hence,

$$\mathbb{I}_8 = \int_{-\infty}^{\infty} 2\pi i k'^2 \chi(-k')\chi(k') - 2\pi i k' \left(\omega_c - \frac{i\gamma}{2}\right) \chi(-k')\chi(k')\ dk'. \tag{4.147}$$

Now, if we add each of the integrals $\mathbb{I}_1$ to $\mathbb{I}_8$ together and take the real part of each expression, we find after using the fluctuation-dissipation theorem that the terms cancel in pairs (this is done by using the relations for the real parts of the χ functions given in Appendic A.1 as we have done in the preceeding two sections) thus,

$$\sum_{j=1}^{8} \Re\left\{\mathbb{I}_j\right\} = 0. \tag{4.148}$$

We now come to the volume integrals, of which there are four. Indeed, the direct flux term is now just four volume integrals,

$$\langle \mathcal{J} \rangle_{\text{dir}} = \frac{\alpha^2 \tau_{\text{ret}}^2 e^2}{32\pi^2 \omega_c^2 M_B^2 m^2} \sum_{j=1}^{4} \sum_{k} e_k^2\ \Re\{\mathbb{J}_j\}, \tag{4.149}$$

where we have that,

$$\begin{aligned}
\mathbb{J}_1 &= \iiint_{-\infty}^{\infty} \frac{(k'+\nu)(k'+\nu')}{\nu\nu'} |\alpha_k(k')|^2 \chi^*(k', -\nu)\chi^*(-k', \nu') e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk' d\nu d\nu', \\
\mathbb{J}_2 &= \iiint_{-\infty}^{\infty} \frac{(k'+\nu)(k'+\nu')}{\nu\nu'} |\alpha_k(k')|^2 \chi^*(k', \nu)\chi(k', \nu) e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk' d\nu' d\nu, \\
\mathbb{J}_3 &= \iiint_{-\infty}^{\infty} \frac{(k'+\nu)(k'+\nu')}{\nu\nu'} |\alpha_k(k')|^2 \chi(-k', \nu)\chi(k'.\nu') e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk' d\nu' d\nu, \\
\mathbb{J}_4 &= \iiint_{-\infty}^{\infty} \frac{(k'+\nu)(k'+\nu')}{\nu\nu'} |\alpha_k(k')|^2 \chi(-k', \nu)\chi(k', -\nu') e^{i(\nu'-\nu)\tau_{\text{ret}}}\ dk'\ d\nu d\nu'.
\end{aligned} \tag{4.150}$$

In each of the integrals we shall perform the ν and ν' integrations (again using Cauchy's integral theorem) and then let $e^{i(\nu'-\nu)\tau_{\text{ret}}} \to 1$ as $\tau_{\text{ret}} \to 0$. We start with $\mathbb{J}_1$ and we find, for the ν' integration that we have simple poles at $\nu' = 0$ and $\nu = \omega_c - k' + i\gamma/2$, and so after we perform the ν' integration we obtain:

$$\mathbb{J}_1 = 2\pi i \iint_{-\infty}^{\infty} \frac{(k'+\nu)\chi^*(k', -\nu)|\alpha_k(k')|^2}{\nu} \left[k'\chi^*(-k') + (\omega_c + i\gamma/2)\chi^*(-k')\right]\ dk' d\nu. \tag{4.151}$$

Finally, performing the ν integration where we note simple poles at $\nu = 0$ and $\nu = -\omega_c - k' - i\gamma/2$, we find that:

$$\mathcal{I}_1 = -4\pi \int_{-\infty}^{\infty} |\alpha_k(k')|^2 \left[k'\chi^*(-k') + (\omega_c + i\gamma/2)\chi^*(-k') + k'\chi^*(k') + (\omega_c + i\gamma/2)\chi^*(k')\right] dk'. \tag{4.152}$$

Similarly, isolating the simple poles for ν' and ν in $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ and performing the necessary integrations we obtain that

$$\begin{aligned}\mathcal{I}_2 = -4\pi \int_{-\infty}^{\infty} |\alpha_k(k')|^2 (k'\chi(k') - (\omega_c + 2k' - i\gamma/2)\chi(k') + k'\chi^*(k') - \\ (\omega_c + 2k' + i\gamma/2)\chi^*(k'))\, dk',\end{aligned} \tag{4.153}$$

$$\mathcal{I}_3 = -4\pi \int_{-\infty}^{\infty} |\alpha_k(k')|^2 (k'\chi^*(k') + (\omega_c + i\gamma/2)\chi^*(k') + k'\chi(-k') - (\omega_c - i\gamma/2)\chi(-k'))\, dk', \tag{4.154}$$

and finally,

$$\mathcal{I}_4 = -4\pi \int_{-\infty}^{\infty} |\alpha_k(k')|^2 (k'\chi(k') + (\omega_c - i\gamma/2)\chi(k') + k'\chi(-k') - (\omega_c - i\gamma/2)\chi(-k'))\, dk'. \tag{4.155}$$

Finally, adding all of the four $\mathcal{I}_j$'s together and taking the real part using the expressions in Appendix A, section A.2, we arrive at an expression for the direct flux term:

$$\langle \mathcal{J} \rangle_{\text{dir}} = \frac{-\alpha^2 \tau_{\text{ret}}^2 \gamma}{8\pi\omega_c M_B^2 m^2} \sum_k e_k^2 \int_{-\infty}^{\infty} |\alpha_k(k')|^2 \left(\frac{|\chi(k')|^2}{2} + |\chi(-k')|^2 \right) dk'.$$

Substituting in $|\alpha_k(k')|^2$ from Appendix B, section B.3 and using the identities in Appendix A.1 to simplify the bracket inside the integral, we find that we have that,

$$\langle \mathcal{J} \rangle_{\text{dir}} = \frac{\alpha^2 \tau_{\text{ret}}^2 \gamma e^2}{4\pi\omega_c M_B m^2} \sum_k k k' e_k^2 \left[\coth(4\pi M_B k) + 1\right] \int_{-\infty}^{\infty} \left(\frac{1}{(k'-k)} \right)^2 \left[\frac{1}{(\omega_c + k')^2(-\omega_c + k')^2} \right] dk'. \tag{4.156}$$

Calculation of the Interference Term We can now calculate the interference term which is given by the relation,

$$\langle \mathcal{J} \rangle_{\text{dir}} = -\frac{\alpha\tau_{\text{ret}}}{2M_B} \text{Re} \left\{ \left\langle \dot{q} \frac{\partial \phi^h}{\partial r_*} + \frac{\partial \phi^h}{\partial t} \dot{q} \right\rangle \right\}. \tag{4.157}$$

We shall calculate

$$\mathrm{Re}\left\{\dot{q}\frac{\partial\phi^h}{\partial r_*}\right\} \text{ and } \mathrm{Re}\left\{\frac{\partial\phi^h}{\partial t}\dot{q}\right\},$$

separately. As before, the expansion of the homogeneous part of ϕ in a box of length V is given by,

$$\phi^h = \sum_k e_k\left[b_k(0)e^{-i\omega_k t+ikr_*} + b_k^\dagger(0)e^{i\omega_k t-ikr_*}\right], \tag{4.158}$$

and so,

$$\frac{\partial\phi^h}{\partial r_*} = i\sum_k ke_k\left[b_k(0)e^{-i\omega_k t+ikr_*} - b_k^\dagger(0)e^{i\omega_k t-ikr_*}\right] \quad \text{and}$$
$$\frac{\partial\phi^h}{\partial t} = i\sum_{k>0} ke_k\left[-b_k(0)e^{-i\omega_k t+ikr_*} + b_k^\dagger(0e^{i\omega_k t-ikr_*}\right].$$

So, we have

$$\begin{aligned}\mathrm{Re}\left\{\left\langle\dot{q}\frac{\partial\phi^h}{\partial r_*}\right\rangle\right\} = -\frac{ie}{2\omega_c m}\sum_k ke_k^2\mathrm{Re}\Bigg\{ \\ - e^{-i\omega_k t+ikr_*}\iint_{-\infty}^{\infty}(k''+\nu)\mu^*(-\nu)\alpha_k(k')\chi^*(k'',-\nu)e^{-i(k''+\nu)\tau_{\mathrm{ret}}}\,dk''d\nu \\ + e^{-i\omega_k t+ikr_*}\iint_{-\infty}^{\infty}(k'+\nu)\mu(\nu)\alpha_k(k')\chi(-k',\nu)e^{-i(k'+\nu)\tau_{\mathrm{ret}}}\,dk'd\nu\Bigg\}.\end{aligned} \tag{4.159}$$

We now use the Fourier transforms we defined earlier in (4.102). Substituting them into (4.159),

$$\begin{aligned}\mathrm{Re}\left\{\left\langle\dot{q}\frac{\partial\phi^h}{\partial r_*}\right\rangle\right\} = -\frac{ie}{2\omega_c m}\sum_k ke_k^2\mathrm{Re}\Bigg\{ \\ - \iiint_{-\infty}^{\infty}(k''+\nu)\mu^*(-\nu)\alpha_k^*(k')\chi^*(k'',-\nu)e^{i(k'-k'')\tau_{\mathrm{ret}}}e^{-i\nu\tau_{\mathrm{ret}}}\,dk''dk'd\nu \\ + \iiint_{-\infty}^{\infty}(k'+\nu)\mu(\nu)\alpha_k^*(k'')\alpha_k(k')\chi(-k',\nu)e^{i(k''-k')\tau_{\mathrm{ret}}}e^{-i\nu\tau_{\mathrm{ret}}}\,dk'dk''d\nu\Bigg\}.\end{aligned}$$

We perform the k'' as before

$$\begin{aligned}\mathrm{Re}\left\{\left\langle\dot{q}\frac{\partial\phi^h}{\partial r_*}\right\rangle\right\} = -\frac{ie}{2\omega_c m}\sum_k ke_k^2\mathrm{Re}\Bigg\{ \\ - \iint_{-\infty}^{\infty}(k'+\nu)|\alpha_k(K")|^2\mu^*(-\nu)\chi^*(k',-\nu)e^{-i\nu\tau_{\mathrm{ret}}}\,dk'd\nu \\ + \iint_{-\infty}^{\infty}(k'+\nu)|\alpha_k(k')|^2\mu(\nu)\chi(-k',\nu)e^{-i\nu\tau_{\mathrm{ret}}}\,dk'd\nu\Bigg\}.\end{aligned}$$

Next, we substitute in the fact that $\mu(\nu) = \delta(\nu) + I(\nu)$. The resulting $\delta(\nu)$ terms simply become the flat spacetime integrals, and these cancel with their counterparts in the $\langle\mathcal{J}\rangle_{\text{dir}}$ terms. The ones that are left are the terms with $\mu(\nu) = I(\nu)$. Thus we have that

$$\begin{aligned}\text{Re}\left\{\left\langle \dot{q}\frac{\partial\phi^h}{\partial r_*}\right\rangle\right\} = -\frac{ie}{2\omega_c m}\sum_k ke_k^2\text{Re}\Bigg\{ & \\ & \int_{-\infty}^{\infty} |\alpha_k(k')|^2 \, dk' \int_{-\infty}^{\infty} \frac{(k'+\nu)e^{-i\nu\tau_{\text{ret}}}}{\nu(\omega_c + k' + \nu + i\gamma/2)} \, d\nu \\ & + \int_{-\infty}^{\infty} |\alpha_k(k')|^2 \, dk' \int_{-\infty}^{\infty} \frac{(k'+\nu)e^{-i\nu\tau_{\text{ret}}}}{\nu(\omega_c - k' - \nu - i\gamma/2)} \, d\nu \Bigg\}.\end{aligned} \tag{4.160}$$

We have two ν integrals to perform:

$$I_1 = \int_{-\infty}^{\infty} \frac{(k'+\nu)e^{-i\nu\tau_{\text{ret}}}}{\nu(\omega_c + k' + \nu + i\gamma/2)} \, d\nu, \quad \text{and} \quad I_2 = \int_{-\infty}^{\infty} \frac{(k'+\nu)e^{-i\nu\tau_{\text{ret}}}}{\nu(\omega_c - k' - \nu - i\gamma/2)} \, d\nu.$$

We start with I_1. We shall perform the ν integral which is a contour integral with simple poles at $\nu = 0$ and $\nu = -\omega_c - k' - i\gamma/2$. Using Cauchy's integral theorem as before then letting $e^{-i\nu\tau_{\text{ret}}} \to 1$ as $\tau_{\text{ret}} \to 0$ we find that,

$$I_1 = \int_{-\infty}^{\infty} \frac{(k'+\nu)}{\nu(\omega_c + k' + \nu + i\gamma/2)} \, d\nu = 2\pi i k'\chi^*(k') + (\omega_c + i\gamma/2)\chi^*(k')2\pi i. \tag{4.161}$$

We now determine I_2. Like I_1 this is also a contour integral with simple poles located at $\nu = 0$ and $\nu = \omega_c - k' - i\gamma/2$. So, calculating the residues and applying Cauchy's integral theorem once more we have that,

$$I_2 = \int_{-\infty}^{\infty} \frac{(k'+\nu)}{\nu(\omega_c - k' - \nu - i\gamma/2)} \, d\nu = 2\pi i k'\chi(-k') - 2\pi i k'(\omega_c - i\gamma/2)\chi(-k'). \tag{4.162}$$

We now add together the expressions we obtained for I_1 and I_2 and we arrive at,

$$\begin{aligned}\text{Re}\left\{\left\langle \dot{q}\frac{\partial\phi^h}{\partial r_*}\right\rangle\right\} = & \\ & -\frac{ie}{2\omega_c m}\sum_k ke_k^2\text{Re}\Big\{ -2\pi k'\chi^*(k') - 2\pi k'\chi(-k') - 2\pi\omega_c\chi^*(k') + 2\pi k'\omega_c\chi(-k')\Big\}.\end{aligned} \tag{4.163}$$

We now need to find,

$$\begin{aligned}\text{Re}\left\{\left\langle \frac{\partial\phi^h}{\partial t}\dot{q}\right\rangle\right\} = -\frac{e}{2\omega_c m}ke_k^2\text{Re}\Bigg\{ & \\ & e^{i\omega_k t - ikr_*}\int_{-\infty}^{\infty} (k'+\nu)\mu^*(\nu)\alpha_k^*(k')\chi^*(-k',\nu)e^{i(k'+\nu)\tau_{\text{ret}}} \, dk'd\nu \\ & - e^{i\omega_k t - ikr_*}\int_{-\infty}^{\infty} (k'+\nu)\mu(-\nu)\alpha_k^*(k')\chi(k',-\nu)e^{i(k'+\nu)\tau_{\text{ret}}} \, dk'd\nu\Bigg\}.\end{aligned}$$

Using the Fourier transforms of (4.102) then we obtain,

$$\begin{aligned}\mathrm{Re}\left\{\left\langle \frac{\partial \phi^h}{\partial t}\dot{q}\right\rangle\right\} = -\frac{e}{2\omega_c m}ke_k^2\mathrm{Re}\Big\{ \\ \iiint_{-\infty}^{\infty}(k''+\nu)\mu^*(\nu)\alpha_k^*(k')\alpha_k(k'')\chi^*(-k',\nu)e^{i(k'-k'')\tau_{\mathrm{ret}}}e^{i\nu\tau_{\mathrm{ret}}}\ dk'dk''d\nu \\ -\iiint_{-\infty}^{\infty}(k'+\nu)\mu(-\nu)\alpha_k^*(k')\alpha_k(k'')\chi(k',-\nu)e^{i(k'-k'')\tau_{\mathrm{ret}}}e^{i\nu\tau_{\mathrm{ret}}}\ dk'dk''d\nu\Big\}.\end{aligned}$$

Integrating with respect to k'' first we obtain,

$$\begin{aligned}\mathrm{Re}\left\{\left\langle \frac{\partial \phi^h}{\partial t}\dot{q}\right\rangle\right\} = -\frac{e}{2\omega_c m}ke_k^2\mathrm{Re}\Big\{ \\ \iint_{-\infty}^{\infty}(k'+\nu)\mu(\nu)|\alpha_k(k')|^2\chi^*(-k',\nu)e^{i\nu\tau_{\mathrm{ret}}}\ dk'd\nu \\ \iint_{-\infty}^{\infty}(k'+\nu)\mu(-\nu)|\alpha_k(k')|^2\chi(k',-\nu)e^{i\nu\tau_{\mathrm{ret}}}\ dk'd\nu\Big\}.\end{aligned} \tag{4.164}$$

We now have two more ν integrals to find:

$$I_3 = \int_{-\infty}^{\infty}\frac{k'+\nu}{\nu(\omega_c-k'-\nu+i\gamma/2)}\ d\nu \quad \text{and} \quad -I_4 = \int_{-\infty}^{\infty}\frac{k'+\nu}{\nu(\omega_c+k'+\nu-i\gamma/2)}\ d\nu. \tag{4.165}$$

Starting with I_3 we have yet another contour integral with simple poles at $\nu = 0$ and $\nu = \omega_c - k' + i\gamma/2$, and so calculating the residues and applying Cauchy's integral theorem we obtain,

$$I_3 = \int_{-\infty}^{\infty}\frac{k'+\nu}{\nu(\omega_c-k'-\nu+i\gamma/2)}\ d\nu = 2\pi i k'\chi^*(-k') - 2\pi i(\omega_c+i\gamma/2)\chi^*(-k'). \tag{4.166}$$

Similarly for I_4, we have simple poles at $\nu = 0$ and $\nu = -\omega_c - k' + i\gamma/2$ and applying Cauchy's integral theorem once more gives,

$$-I_4 = \int_{-\infty}^{\infty}\frac{k'+\nu}{\nu(\omega_c+k'+\nu-i\gamma/2)}\ d\nu = -2\pi i k'\chi(k') + 2\pi i(\omega_c-i\gamma/2)\chi(k'). \tag{4.167}$$

Hence we have that,

$$\begin{aligned}\mathrm{Re}\left\{\left\langle \frac{\partial \phi^h}{\partial t}\dot{q}\right\rangle\right\} = -\frac{ei}{2\omega_c m}ke_k^2\mathrm{Re}\Big\{ \\ -2\pi k'\chi(-k') + 2\pi\omega_c\chi(-k') + 2\pi k'\chi(k') - 2\pi\omega_c\chi(k')\Big\}.\end{aligned} \tag{4.168}$$

We now have that

$$\mathrm{Re}\left\{\left\langle \dot{q}\frac{\partial\phi^h}{\partial r_*}+\frac{\partial\phi^h}{\partial t}\dot{q}\right\rangle\right\} = -\frac{e}{2\omega_c m}\sum_k ke_k^2\Big[-2\pi k'\mathrm{Re}\{\chi^*(k')+\chi(k')\}+2\pi k'\mathrm{Re}\{-\chi^*(-k')-\chi(-k')\}$$
$$+2\pi\omega_c\mathrm{Re}\{-\chi^*(k')-\chi(k')\}+2\pi\omega_c\mathrm{Re}\{\chi(-k')+\chi(-k')\}\Big]. \tag{4.169}$$

Using the relations in Appendix A section A.2, we find that the above terms cancel in pairs so in fact

$$\langle\mathcal{J}\rangle_{\mathrm{int}} = 0. \tag{4.170}$$

Thus we have that the overall energy flux of the system is not zero, but the only contribution comes directly from the oscillator and has the form:

$$\mathcal{J} = \frac{\alpha^2\tau_{\mathrm{ret}}^2 e^2\gamma}{4\pi\omega_c M_B^2 m^2}\sum_k kk'e_k^2\left[\coth(4\pi M_B k)+1\right]\int_{-\infty}^{\infty}\left(\frac{1}{(k'-k)}\right)^2\left[\frac{1}{(\omega_c+k')^2(-\omega_c+k')^2}\right]dk'. \tag{4.171}$$

Which we shall write as,

$$\mathcal{J} = \frac{\alpha^2\tau_{\mathrm{ret}}^2\gamma}{\omega_c M_B^2 m^2}\sum_k kk'e_k^2\left[\coth(4\pi M_B k)+1\right]Q(k,k'), \tag{4.172}$$

We now write,

$$\coth(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}+1 = \frac{2e^x}{e^x-e^{-2x}} = \frac{2}{1-e^{-x}},$$

Thus,

$$\coth(4\pi M_B k) = \frac{2}{1-e^{-8\pi M_B\omega_k}}$$

Compare this now with Planck's law:

$$\frac{1}{1-\exp\{\frac{-h\nu}{k_BT}\}},$$

where k_B is the Boltzmann constant, and T is the temperature of the black body. The comparison (with $\hbar=1$) shows that we have a temperature of:

$$T = \frac{1}{8\pi M_B k_B}, \tag{4.173}$$

which is the same temperature as the Hawking temperature.

It can be seen that the expression given in (4.172) is infinite. The singularity at $k = k'$ in (4.172) is an artefact of our approximation of the free-fall trajectory which is only valid near the horizon. As it is an artifical singularity introduced by this approximation it does not matter that it makes the expression infinite. If we were to evaluate the expression numerically, the contribution from when $k = k'$ would be small.

Chapter 5

Conclusions and Recommendations for Further Work.

In this chapter we discuss the findings of this thessis and its implications. We also make recommendations on how this work can be continued and pose some questions raised by this work.

5.1 Conclusions.

We now summarize the results of the previous 3 chapters. We shall start with chapter 2 which was concerned with Hawking radiation in Schwarzschild anti-de Sitter spacetime (which will abbreviate to SADS_2). At the start of chapter 2 we considered the spacetime structure of SADS_2. As we observed, unlike the Schwarzschild spacetime, there is no $\mathscr{I}^-$ or $\mathscr{I}^+$, there is only $\mathscr{I}$ in SADS_2. The spacetime is a dynamic one, and we defined two types of modes, early time modes and late time modes. We then considered the case of a thin shell collapsing in 2D to form a black hole in SADS_2. By applying Hawking's method, (as discussed in Chapter 1) and reflective boundary conditions, we were able to use the geometrical optics approximation, and find an expression for the early time modes in terms of the late time modes.

We have a relationship between the early and late time modes enabled us to calculate the Bogoliubov coefficients. Evaluating these coefficients allowed us to demonstrate that the black hole is indeed thermal and has a Hawking temperature of

$$T_H = \frac{h(r_h)}{4\pi\omega} = \frac{\kappa}{2\pi\omega}$$

where $h(r_h)$ is the factorized form of the conformal factor, and r_h is the position of the event horizon.

After we had done this we went on to consider the problem of calculating the renormalised stress tensor. After some consideration of the problems involved in doing this, and choosing a suitable renormalization scheme we use the Davies Fulling and Unruh derivative. This method given by Birrell and Davies gives the form for the renormalised stress tensor as,

$$\langle T_\mu^\nu[\overline{g}_{\kappa\lambda}(x)]\rangle_{\mathrm{ren}} = \frac{1}{\sqrt{-g}}\langle T_\mu^\nu[\eta_{\kappa\lambda}(x)]\rangle_{\mathrm{ren}} + \theta_\mu^\nu - \left(\frac{1}{48\pi}\right) R\delta_\mu^\nu. \tag{5.1}$$

Since $SADS_2$ is conformal to half of Minkowski spacetime, we had to compute the contribution $\langle T_\mu^\nu[\eta_{\kappa\lambda}(x)]\rangle_{\mathrm{ren}}$, which after applying our point-splitting method delivers the result,

$$\langle T_{\mu\nu}[\eta]\rangle_{\mathrm{ren}} = \frac{1}{8\pi^2 r^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

After we found this quantity, we went on to find the components $\theta_{\mu\nu}$. In the case of calculating the stress tensor at early times, we had to calculate the components θ_{uu} and θ_{vv} ($\theta_{uv} = \theta_{vu} = 0$ for all conformally flat spacetimes). This was performed using the DFU derivative and allowed us to form the renormalised stress tensor at early times. We then wished to find the renormalised stress tensor at late times, however the spacetime is a dynamical one, and so we used the relations $u = f(V)$ and $v = g(u)$ found by the ray tracing method. Performing the DFU derivative and using the previously calculated expression for $\langle T_{\mu\nu}[\eta]\rangle_{\mathrm{ren}}$ allowed us to find the renormalised stress tensor at late times.

In chapter 3 we used the methodology of Ford and O'Connell to investigate a quantum oscillator coupled to a real scalar field moving in Schwarzschild spacetime. We examined two specific cases: a $r_* = \text{const}$ trajectory, and an inertial trajectory whereby $r_* = r_*(\tau)$ (τ is the proper time). The result of this experiment showed that, in the case where $r_* = \text{const}$ no radiation was observed; the direct flux arising from the oscillator balances that of the interference term. When we put the oscillator on a general trajectory $r_* = r_*(\tau)$ we find that the direct flux term no longer balances the interference term. This is due to the presence of the conformal factor, Ω, which on a non-constant trajectory, becomes a function of proper time, and as a result it plays a part in the direct flux and interference terms.

One question that seems obvious is, if a harmonic oscillator is radiating in Schwarzschild spacetime, then does this have implications for the formation of black holes? The answer we believe is no. We could model the surface of a collapsing star as a thin shell of such oscillators, and as they fall in under collapse, radiation is generated. Now, it is the case that the collapse takes a finite amount of time in the frame of reference of the oscillators, external observers would see the oscillators falling in, but due to time dilation effects, the external observer never sees the formation of the event horizon[33]. The radiation from the oscillators would become infinitely red shifted and be quite undetectable by the time the hole has formed. Thus one would expect never to be able to detect the radiation of the in-falling oscillators.

The other point to be considered when discussing the likelihood of black hole formation is the reasonable amount of circumstantial evidence which now exists. Obviously it is not possible to observe a black hole directly, but their existence has been revealed in the motions of other stars and other circumstantial evidence. Astronomers now know of many binary systems whereby the one star seems to be under the gravitational influence of an unseen companion. By observing these systems, an orbital period of the unseen companion can be established and hence an lower mass limit imposed on the companion. Such mass limits in observed systems (like Cygnus X1) have placed the mass of the unseen companion above the mass limit of a neutron star. It is true that these limits depend heavily on the assumptions made about the properties of dense matter, and it has recently been proposed that black hole candidates with masses in the range 3.8 to 6 solar masses could in fact be quark stars[34]. Even if this is the case, there are black hole candidates much heavier than this both in binary systems, and at the centres of many galaxies. The indirect evidence for the existence of black holes seems to be reasonable, and growing in body.

In chapter 4 we took a different approach to that of chapter 3. Here we formed a Hamiltonian to describe the oscillator, the scalar field and the electrodynamic interaction. From this we established equations of motion for the annihilation and creation operators of the scalar field and the oscillator. We used the scalar field operators to write the full expressions of $\phi = \phi^h + \phi^p$, and then we used a Wigner-Weisskopff approximation to solve the equation of motion for the operator to obtain expressions for the annihilation and creation operators of the oscillator. We then used these to form an expression for the position function $q(t)$, and then calculate the energy flux of the system as we did in chapter 3. In the case of a $x =$ constant trajectory we found no net radiation was produced, and similarly in the case of constant acceleration there was no radiation detected (in agreement with the results of Grove, Raine and Ford and O'Connell). When we confined the oscillator to a free fall trajectory, we found that radiation was detected. Morever, the temperature of the radiation is the same as the Hawking temperature $T = \frac{1}{8\pi M_B k_B}$. We obtain a result which looks infinite, and this is due to the approximations made in the free-fall trajectory that the oscillator is confined too.

Although this method is perfectly valid, it is in fact much more labor intensive than that of Ford and O'Connell (the method used in Chapter 3). Moreover the result depends upon the correct application of the Wigner-Weisskopff approximation where as the Ford and O'Connell approach does not. What the method does show is that we can take a Hamiltonian and form a quantum Langevin equation and obtain a description of Hawking radiation using a classical physical approach, rather than the usual approach of having to consider the passage of modes from the past passing through the collapsing matter and piling up in the far future. In this setting we are suggesting that black holes radiate because they satisfy the same laws of physics as other radiating objects in the Universe (namely a Langevin equation).

5.2 Recommendation for Further Work

We now make recommendations for further work. In chapter 2, the calculation was carried out in two dimensions. In fact, the Schwarzschild a anti-de Sitter space time is a four dimensional manifold, and it would be interesting to see what contributions the higher dimensions make to the calculation.

In the case of the work carried in chapters 3 and 4, this work could also be extended greatly. Firstly, the calculation concerning an oscillator on a free fall trajectory in Schwarzschild spacetime was again, for simplicity performed in two dimensions. Since the full Schwarzschild spacetime is static, it would be interesting to see what contributions the higher dimensions make to this calculation. It would also be interesting to perform the calculation in a spacetime which is dynamical rather than static. If such a calculation was to be performed, the methodology in Chapter 3 would probably more effective than that used in chapter 4, since it does not involve the Wigner-Weisskopff approximation.

Another interesting extension of this work would be to explore the model with a proper description for the mass of the oscillator, namely the Higgs field for the mass of the oscillator. If we were to incorporate the Higgs field in the 4D calculation discussed above, we should be able to see the mechanism whereby the mass of the oscillator is converted into radiation. Such an investigation would be of interest for researchers concerned with the Transplanckian problem.

Appendix A

Useful Indentities

A.1 Real Parts of Products

We have the function $\chi(\omega_k)$ defined as

$$\chi(\omega_k) = \frac{1}{\omega_c + \omega_k - i\gamma/2}, \tag{A.1}$$

and with this we can now find the real parts of the following products (note we neglect γ^2 terms):

1. $\Re\{\chi^*(-\omega_k)\chi^*(\omega_k)\}$. We have that

$$\begin{aligned}\chi^*(-\omega_k)\chi^*(\omega_k) &= \frac{1}{(\omega_c - \omega_k + i\gamma/2)}\frac{1}{(\omega_c + \omega_k + i\gamma/2)} \\ &= \frac{1}{\omega_c^2 - \omega_k^2 + i\gamma\omega_c},\end{aligned}$$

   and thus

$$\Re\{\chi^*(-\omega_k)\chi^*(\omega_k)\} = \Re\left\{\frac{\omega_c^2 - \omega_k^2 - i\gamma/2}{(\omega_c^2 - \omega_k^2)^2}\right\}.$$

   Hence we have

$$\Re\{\chi^*(-\omega_k)\chi^*(\omega_k)\} = \frac{1}{\omega_c^2 - \omega_k^2}. \tag{A.2}$$

2. $\Re\{\chi(-\omega_k)\chi(\omega_k)\}$. We simply take the complex conjugate of (A.2) and we obtain

$$\Re\{\chi(-\omega_k)\chi(\omega_k)\} = \frac{1}{\omega_c^2 - \omega_k^2}. \tag{A.3}$$

3. $\Re\{\chi(\omega_k)\chi^*(\omega_k)\}$. We have that:

$$\chi(\omega_k)\chi^*(\omega_k) = \frac{1}{(\omega_c + \omega_k - i\gamma/2)} \frac{1}{(\omega_c + \omega_k + i\gamma/2)} = \frac{1}{(\omega_c + \omega_k)^2} = |\chi(\omega_k)|^2 \quad \text{(A.4)}$$

which is of course strictly real.

4. $\Re\{\chi(-\omega_k)\chi^*(-\omega_k)\}$. We have that:

$$\chi(-\omega_k)\chi^*(\omega_k) = \frac{1}{\omega_c - \omega_k - i\gamma/2} \frac{1}{\omega_c - \omega_k + i\gamma/2} = \frac{1}{(\omega_c - \omega_k)^2} = |\chi(-\omega_k)|^2 \quad \text{(A.5)}$$

which again, is strictly real.

A.2 Real Parts of $\chi(k)$, $\chi(-k)$ and their Complex Conjugates

We now determine the real parts of $i\chi(\omega_k)$, $i\chi(-\omega_k)$ and their complex conjugates.

1. $\Re\{i\chi(\omega_k)\}$. We have that

$$\Re\{i\chi(\omega_k)\} = \Re\left\{\frac{i(\omega_c + \omega_k) - \gamma/2}{(\omega_c + \omega_k - i\gamma/3)(\omega_c + \omega_k + i\gamma/2)}\right\}$$

and hence

$$\Re\{i\chi(\omega_k)\} = -\frac{\gamma}{2(\omega_c + \omega_k)^2} \quad \text{(A.6)}$$

2. $\Re\{i\chi^*(\omega_k)\}$. Here

$$\Re\{i\chi^*(\omega_k)\} = \Re\left\{\frac{i(\omega_c + \omega_k) + \gamma/2}{(\omega_c + \omega_k + i\gamma/2)(\omega_c + \omega_k - i\gamma/2)}\right\}$$

thus,

$$\Re\{i\chi^*(\omega_k)\} = \frac{\gamma}{2(\omega_c + \omega_k)^2}. \quad \text{(A.7)}$$

3. $\Re\{\chi(-\omega_k)\}$ Here we have

$$\Re\{\chi(-\omega_k)\} = \Re\left\{\frac{i(\omega_c - \omega_k) - \gamma/2}{(\omega_c - \omega_k - i\gamma/2)(\omega_c - \omega_k + i\gamma/2)}\right\}$$

giving

$$\Re\{\chi(-\omega_k)\} = \frac{-\gamma}{2(\omega_c - \omega_k)^2} \quad \text{(A.8)}$$

4. $\Re\{i\chi^*(-\omega_k)\}$. The final idenity we need:

$$\Re\{i\chi^*(-\omega_k)\} = \Re\left\{\frac{i(\omega_c - \omega_k) + \gamma/2}{(\omega_c - \omega_k + i\gamma/2)(\omega_c - \omega_k - i\gamma/2)}\right\} \tag{A.9}$$

and so

$$\Re\{i\chi^*(-\omega_k)\} = \frac{\gamma}{2(\omega_c - \omega_k)^2} \tag{A.10}$$

Appendix B

Miscellaneous Calculations

B.1 Spacetime Trajectory

In this section we shall calculate a suitable free fall trajectory, and then make an approximation for it to be used in the curved spacetime calculations of Chapter 4. We start with the spacetime metric

$$d\tau^2 = \left(1 - \frac{2M_B}{r}\right) dt^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} dr^2 \tag{B.1}$$

and so we have

$$1 = \left(1 - \frac{2M_B}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 \tag{B.2}$$

We know that

$$\frac{dt}{d\tau} = \frac{\mathcal{E}}{\left(1 - \frac{2M_B}{r}\right)}, \tag{B.3}$$

where $\mathcal{E}$ is the energy of the particle, and so after some rearrangement we have that

$$\left(1 - \frac{2M_B}{r}\right) - \mathcal{E}^2 = -\left(\frac{dr}{d\tau}\right)^2. \tag{B.4}$$

We now let $\mathcal{E}^2 = 1$ which represents a particle falling in from infinity and thus we have that

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M_B}{r}}. \tag{B.5}$$

We now need to integrate this expression. First we make the substitution

$$y = \frac{2M_B}{r} \quad \Rightarrow \quad \frac{dr}{dy} = -\frac{2M_B}{y^2}.$$

Now,

$$\frac{dr}{d\tau} = \frac{dr}{dy} dy d\tau = -\frac{2M_B}{y^2}\frac{dy}{d\tau},$$

and thus

$$-\frac{2M_B}{y^2}\frac{dy}{d\tau} = -\sqrt{y}. \tag{B.6}$$

Integrating both sides gives

$$-\frac{2}{3}y^{-3/2} = \frac{\tau}{2M_B} - \frac{2}{3}, \tag{B.7}$$

and hence we arrive at

$$y = \left(1 - \frac{3\tau}{4M_B}\right)^{-2/3}. \tag{B.8}$$

We now apply the Binomial expansion, using the first two terms we have that

$$\left(1 - \frac{3\tau}{4M_B}\right)^{-2/3} \approx 1 - \frac{2}{3}\left(-\frac{3\tau}{4M_B}\right)$$

and so we have that

$$y \approx 1 + \frac{\tau}{2M_B}. \tag{B.9}$$

This means we can say, approximately that

$$1 - \frac{2M_B}{r} \approx -\frac{\tau}{2M_B}, \tag{B.10}$$

or

$$r = 2M_B\left(1 + \frac{\tau}{2M_B}\right)^{-1} \tag{B.11}$$

B.2 Fourier Transform of the Conformal Factor

From the Schwarzschild metric we have that

$$ds^2 = \left(1 - \frac{2M_B}{r}\right)^2 dt^2 - \left(1 - \frac{2M_B}{r}\right)^{-1} dr^2. \tag{B.12}$$

We may write this as the familiar conformally flat spacetime metric

$$ds^2 = \Omega(dt^2 - dr_*^2), \tag{B.13}$$

where Ω is the conformal factor given by

$$\Omega = 1 - \frac{2M_B}{r}. \tag{B.14}$$

Now, we define the Fourier transform of the conformal factor to be

$$\Omega(\tau) = \int_{-\infty}^{\infty} \mu(\nu) e^{-i\nu\tau}\, d\nu. \tag{B.15}$$

We now wish to find a form for the function $\mu(\nu)$. From (B.10) we may write that, approximately

$$\Omega(\tau) = \left(-\frac{\tau}{2M_B} - 1\right) + 1,$$

and so now

$$\mu(\nu) = \frac{1}{2\pi}\left[\int_{-2M_B}^{0} e^{i\nu\tau}\left(-\frac{\tau}{2M_B} - 1\right) d\tau + \int_{-\infty}^{0} e^{i\nu\tau}\, d\tau\right],$$

or,

$$\mu(\nu) = \frac{1}{2\pi}\left[\int_{-2M_B}^{0} e^{i\nu\tau}\left(-\frac{\tau}{2M_B} - 1\right) d\tau + \delta(\nu)\right].$$

Using integration by parts we have that

$$\mu(\nu) = \frac{1}{2\pi}\left[\left[-\frac{\tau}{2M_B} e^{i\nu\tau} i\nu\right]_{-2M_B}^{0} + \int_{-2M_B}^{0} \frac{e^{i\nu\tau}}{2iM_B\nu}\, d\tau - \int_{-2M_B}^{0} e^{i\nu\tau}\, d\tau + \delta(\nu)\right],$$

giving,

$$\mu(\nu) = \frac{1}{2\pi}\left[\frac{e^{-2iM_B\nu}}{i\nu} + \left[\frac{e^{i\nu\tau}}{-2M_B\nu^2}\right]_{-2M_B}^{0} - \left[\frac{e^{i\nu\tau}}{i\nu}\right]_{-2M_B}^{0} + \delta(\nu)\right],$$

and so,

$$\mu(\nu) = \frac{1}{2\pi}\left[-\frac{1}{i\nu} + \frac{1}{\nu^2}\left(1 - e^{-2iM_B\nu}\right) + \delta(\nu)\right]. \tag{B.16}$$

Consider now the second term in (B.16). We can write

$$\frac{1}{\nu^2}\left(1 - e^{-2iM_B\nu}\right) = \frac{2ie^{-iM_B\nu}\sin(2M_B\nu)}{\nu^2},$$

and thus

$$\mu(\nu) \approx \delta(\nu) - \frac{i}{\nu} + \frac{2ie^{-iM_B\nu}\sin(2M_B\nu)}{\nu^2}.$$

We now consider what happens near $\nu = 0$. We see that

$$\frac{\sin(2M_B\nu)}{\nu} \to 2M_B \quad \text{as} \quad \nu \to 0,$$

while

$$e^{-iM_B\nu} \to 1 \quad \text{as} \quad \nu \to 1,$$

and so we may say that approximately,

$$\mu(\nu) \approx \frac{\delta(\nu)}{2\pi} + \frac{i}{2\pi\nu} = I(\nu) + \frac{\delta(\nu)}{2\pi} \tag{B.17}$$

B.3 Fourier Transform of the Free-Fall Trajectory in Schwarzschild Spacetime

In this section we shall derive an expressions for both $\alpha_k(k')$ and $|\alpha_k(k')|^2$. We start with:

$$r_*(\tau) = r + 2M_B \ln|r - 2M_B|. \tag{B.18}$$

The free-fall trajectory is

$$r(\tau) = 2M_B\left(1 + \frac{\tau}{2M_B}\right)^{-1} \approx 2M_B - \tau, \tag{B.19}$$

and hence we have that

$$r_* = 2M_B - \tau + 2M_B \ln|\tau|, \tag{B.20}$$

with

$$t(\tau) = -2M_B \ln|\tau|. \tag{B.21}$$

By definition we had that

$$\alpha_k(k') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt(\tau) - ikr_*(\tau)}e^{ik'\tau}\, d\tau, \quad \text{for } k > 0. \tag{B.22}$$

However we have that $\tau \in [-\infty, 0]$, and so in fact we can write (B.22) as

$$\alpha_k(k') = \frac{e^{-2iM_B k}}{2\pi} \int_{-\infty}^{0} e^{-4iM_B k \ln(\tau)} e^{i(k'-k)\tau} \, d\tau. \tag{B.23}$$

We now make the subtitution: $u = i(k'-k)\tau$, this allows us to write

$$\alpha_k(k') = -\frac{ie^{2iM_B k}}{2\pi(k'-k)} \int_{-i\infty}^{i0} \left(\frac{u}{i(k'-k)}\right)^{-4iM_B k} e^u \, du,$$

i.e.

$$\alpha_k(k') = \frac{e^{-2iM_B k}}{2\pi} \left(\frac{1}{i(k'-k)}\right)^{1-4iM_B k} \int_{-i\infty}^{i0} u^{-4iM_B k} e^u \, du.$$

We now make one more subtitution; we let $U = -iu$ and this gives

$$\alpha_k(k') = \frac{e^{-2iM_B k}}{2\pi} \left(\frac{1}{i(k'-k)}\right)^{1-4iM_B k} \int_{0}^{\infty} (-iU)^{-4iM_B k} e^{-iU} \, dU. \tag{B.24}$$

Comparision of the above integrand with

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

allows us to write that

$$\alpha_k(k') = \frac{e^{-2iM_B k}}{2\pi} \left(\frac{1}{i(k'-k)}\right)^{1-4iM_B k} \Gamma(1 - 4iM_B k).$$

Using the identity

$$\Gamma(z+1) = z\Gamma(z), \tag{B.25}$$

means that we have

$$\alpha_k(k') = -\frac{2iM_B k e^{-2iM_B k}}{\pi} \left(\frac{1}{i(k'-k))}\right)^{1-4iM_B k} \Gamma(-4iM_B k), \tag{B.26}$$

and

$$\alpha_k^*(k') = \frac{2iM_B k e^{-2iM_B k}}{\pi} \left(\frac{1}{-i(k'-k))}\right)^{1+4iM_B k} \Gamma(4iM_B k). \tag{B.27}$$

If we mutliply togher (B.26) with (B.27) we obtain that

$$|\alpha_k(k')|^2 = \frac{4M_B^2 k^2}{\pi^2} \left(\frac{1}{k'-k}\right)^2 e^{4\pi M_B k} |\Gamma(4iM_B k)|^2.$$

Now, since

$$|\Gamma(4iM_Bk)|^2 = \frac{\pi}{4M_Bk\sinh(4\pi M_Bk)}, \tag{B.28}$$

then we have that

$$|\alpha_k(k')|^2 = \frac{M_Bk}{\pi}\left(\frac{1}{k'-k}\right)^2 \frac{e^{4iM_Bk}}{\sinh(4M_Bk)}$$

and hence we have:

$$|\alpha_k(k')|^2 = \frac{M_Bk}{\pi}\left(\frac{1}{k'-k}\right)^2 [\coth(4\pi M_Bk)+1] \tag{B.29}$$

The singularity at $k = k'$ is a factor that is an artefact of our approximation of the free-fall trajectory which is only valid near the horizon.

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