INNER IDEALS OF SIMPLE LOCALLY FINITE LIE ALGEBRAS

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ABSTRACT. Inner ideals of simple locally finite dimensional Lie algebras over an algebraically closed field of characteristic 0 are described. In particular, it is shown that a simple locally finite dimensional Lie algebra has a non-zero proper inner ideal if and only if it is of diagonal type. Regular inner ideals of diagonal type Lie algebras are characterized in terms of left and right ideals of the enveloping algebra. Regular inner ideals of finitary simple Lie algebras are described. Inner ideals of some finite dimensional Lie algebras are studied. Maximal inner ideals of simple plain locally finite dimensional Lie algebras are classified.

1. Introduction

An inner ideal of a Lie algebra L is a subspace I of L such that $[I[I,L]] \subseteq I$. Inner ideals were first systematically studied by Benkart [12, 13] and proved to be useful in classifying simple Lie algebras, both of finite and infinite dimension. They play a role similar to one-sided ideals of associative algebras in developing Artinian structure theory for Lie algebras [17]. They are also useful in constructing gradings of Lie algebras [19].

The sections 2 to 5 consist mainly of joint work with Alexander Baranov and are found in [3]. In the paper we study inner ideals of simple locally finite Lie algebras over an algebraically closed field F of characteristic zero. Recall that an algebra is called *locally finite* if every finitely generated subalgebra is finite dimensional. All locally finite algebras will be considered to be infinite dimensional. Although full classification of simple locally finite Lie algebras seem to be impossible to obtain, there are two classes of these algebras which have especially nice properties and can be characterized in many different ways. Those are finitary simple Lie algebras and diagonal simple locally finite Lie algebras. Recall that an infinite dimensional Lie algebra is called *finitary* if it consists of finite-rank linear transformations of a vector space. It is easy to see that finitary Lie algebras are locally finite. Diagonal locally finite Lie algebras were introduced in [4] and are defined as limits of “diagonal” embeddings of finite dimensional Lie algebras (see Definition 2.4 for details). They can be also characterized as Lie subalgebras of locally finite associative algebras [5, Corollary 3.9].

In Section 3 we prove the following theorem, which is one of our main results.

Theorem 1.1. *A simple locally finite Lie algebra over F has a proper non-zero inner ideal if and only if it is diagonal.*

The theorem shows that non-trivial inner ideals appear only in diagonal Lie algebras and gives another characterisation of this class of algebras. The complete classification of diagonal simple locally finite Lie algebras was obtained in [1] and we need some notation to state it here.

Let A be an associative enveloping algebra of a Lie algebra L (i.e. L is a Lie subalgebra of A and A is generated by L as an associative algebra). We say that A is a $\mathfrak{P}$-*enveloping algebra* of L if $[A, A] = L$. Assume now that A has an involution (which will be always denoted by $*$). Then the set $\mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$ of skew symmetric elements of A is a Lie subalgebra of A. Let $\mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$ denote the commutator subalgebra of $\mathfrak{u}^*(A)$. We say that A is a $\mathfrak{P}^*$-*enveloping algebra* of L if $\mathfrak{su}^*(A) = L$. It is shown in [1, 1.3-1.6] that every simple diagonal locally finite Lie algebra L has a unique involution simple $\mathfrak{P}^*$-*enveloping algebra* $A(L)$ (which is necessarily locally finite). Moreover, the mapping $L \mapsto A(L)$ is a bijective correspondence between the set of all (up to isomorphism) infinite dimensional simple diagonal locally finite Lie algebras and the set of all (up to isomorphism) infinite dimensional involution simple locally finite associative algebras (the inverse map is $A \mapsto \mathfrak{su}^*(A)$). Similarly, every simple plain (see Definition 2.4) locally finite Lie algebra L has a unique *(* up to isomorphism and antiisomorphism) simple $\mathfrak{P}$-*enveloping algebra* $A(L)$ (which is necessarily locally finite). Moreover, the mapping $L \mapsto A(L)$ is a bijective correspondence between the set of all (up to isomorphism) infinite dimensional simple plain locally finite Lie algebras and the set of all (up to isomorphism and antiisomorphism) infinite dimensional simple locally finite associative algebras (the inverse map is $A \mapsto [A, A]$).

In Section 4 we introduce and describe basic properties of so-called regular inner ideals of simple diagonal locally finite Lie algebras. Those correspond to left and right ideals of the $\mathfrak{P}$- (and $\mathfrak{P}^*$*)*-enveloping algebras. We believe that the following conjecture is true.

Conjecture 1.2. *Let L be a simple diagonal locally finite Lie algebra. Assume that L is not finitary orthogonal. Then every inner ideal of L is regular.*

We prove some partial results towards the conjecture (see Theorem 4.13) and show that the conjecture holds in the case of locally semisimple diagonal Lie algebras (see Corollary 4.16). We also show that $I^2 = 0$ for every inner ideal I of L, which actually means that I is a Jordan-Lie inner ideal as defined in [21].

In section 5 we apply our results to the finitary simple Lie algebras. Over a field of zero characteristic those were classified in [6]. In particular, there are just three finitary simple Lie algebras over F of infinite countable dimension: $\mathfrak{sl}_\infty(F)$, $\mathfrak{so}_\infty(F)$ and $\mathfrak{sp}_\infty(F)$. Since finitary simple Lie algebras are both diagonal and locally semisimple, by Corollary 4.16, all their inner ideals are regular, except in the finitary orthogonal case. The classification of inner ideals of finitary simple Lie algebras was first obtained by López, García and Lozano [16] (over arbitrary fields of characteristic zero), with Benkart and López [14] settling later the missing case for orthogonal algebras. We provide an alternative proof for the case of special linear and symplectic algebras over an algebraically closed field of characteristic zero (see Theorem 5.2). In the case of orthogonal algebras we describe only regular inner ideals.

It follows from a general result, proved for nondegenerate Lie algebras by Draper, López, García and Lozano, that a simple locally finite Lie algebra contains proper minimal inner ideals if and only if it is finitary (see [15, Theorems 5.1 and 5.3]). We prove a version of this result for regular inner ideals, see Corollary 5.7.

In section 6 we study inner ideals of the Lie algebra $L = [A, A]$ where A is a strongly perfect finite dimensional associative algebra with $(\mathrm{Rad}\, A)^2 = 0$. In particular we show that every inner ideal I of L with $I^2 = 0$, splits, i.e. $I = I_S \oplus I_R$ where S is a Levi subalgebra of A, $R = \mathrm{Rad}\, A$, $I_S = I \cap S$ and $I_R = I \cap R$. We also show that I is regular in certain cases and give an example of a non-regular inner ideal.

In section 7 the maximal inner ideals of simple plain locally finite Lie algebras are classified in terms of special maximal pairs of left and right ideals of the plain enveloping algebra. The case of finitary Lie algebras is considered.

2. Preliminaries

Recall that a Lie algebra L is called *perfect* if $[L, L] = L$. Similarly, an associative algebra A is *perfect* if $AA = A$ (which is always true if A contains the identity). Let L be a perfect finite-dimensional Lie algebra. Then its solvable radical $\operatorname{Rad} L$ annihilates every simple L-module and $L/\operatorname{Rad} L \cong Q_1 \oplus \cdots \oplus Q_n$ is the sum of simple components Q_i. Denote by V_i the first fundamental Q_i-module (so V_i is natural and $Q_i \cong \mathfrak{sl}(V_i)$, $\mathfrak{so}(V_i)$, $\mathfrak{sp}(V_i)$ if Q_i is of classical type). The modules V_i can be considered as L-modules in an obvious way and are called the *natural* L-modules. Assume that all Q_i are of classical type. An L-module V is called *diagonal* if each non-trivial composition factor of V is a natural or co-natural module (i.e. dual to natural) of L. Otherwise V is called *non-diagonal.* A diagonal L-module V is called *plain* if all Q_i are of type A and each non-trivial composition factor of V is a natural L-module. Let L' be another perfect finite dimensional Lie algebra containing L. If W is an L'-module we denote by $W \downarrow L$ the module W restricted to L. Let $V_1', \dots, V_k'$ be the natural L'-modules. The embedding $L \subseteq L'$ is called *diagonal* (respectively *plain*) if $(V_1' \oplus \cdots \oplus V_k') \downarrow L$ is a diagonal (respectively plain) L-module. By the *rank* of a perfect finite dimensional Lie algebra we mean the smallest rank of the simple components of $L/\operatorname{Rad} L$.

We will frequently use the following lemma from [1].

Lemma 2.1. [1, Lemma 2.5] *Let $L_1 \subseteq L_2 \subseteq L_3$ be three perfect finite dimensional Lie algebras. Suppose that the ranks of L_1 and L_3 are greater than 10 and the embedding $L_1 \subseteq L_3$ is diagonal. Then the embedding $L_1 \subseteq L_2$ is diagonal. Moreover, if the restriction of each natural L_2-module to L_1 is non-trivial then both embeddings $L_1 \subseteq L_2$ and $L_2 \subseteq L_3$ are diagonal.*

We will also use the following obvious property of perfect finite dimensional Lie algebras.

Lemma 2.2. *Let L be a perfect finite dimensional Lie algebra and let $Q_1, \dots, Q_n$ be the simple components of $L/\operatorname{Rad} L$. Then L has exactly n maximal ideals $M_1, \dots, M_n$ and $L/M_i \cong Q_i$.*

Proof. Let M_i be the kernel of the natural epimorphism $L \to Q_i$. Then M_i is a maximal ideal of L assume now that M is another maximal ideal of L. We need to show $M \cong M_i$ for some i. Since L is perfect, the quotient L/M is perfect, so L/M is a simple Lie algebra. This implies $\operatorname{Rad} L \subseteq M$ and $M \cong M_i$ for some i. □

Definition 2.3. A system of finite dimensional subalgebras $\mathfrak{L} = (L_\alpha)_{\alpha\in\Gamma}$ of a Lie (or associative) algebra L is called a *local system* for L if the following are satisfied:
(1) $L = \bigcup_{\alpha\in\Gamma} L_\alpha$
(2) for $\alpha, \beta \in \Gamma$ there exists $\gamma \in \Gamma$ such that $L_\alpha, L_\beta \subseteq L_\gamma$.

Put $\alpha \le \beta$ if $L_\alpha \subseteq L_\beta$. Then Γ is a directed set and $L = \varinjlim L_\alpha$. We say that a local system is *perfect* (resp. *semisimple*) if it consists of perfect (resp. semisimple) subalgebras.

Definition 2.4. A perfect local system $(L_\alpha)_{\alpha\in\Gamma}$ is called *diagonal* (resp. *plain*) if for all $\alpha \le \beta$ the embedding $L_\alpha \subseteq L_\beta$ is diagonal (resp. plain). A simple locally finite Lie algebra L is called *diagonal* (resp. *plain*) if it has a diagonal (resp. plain) local system. Otherwise, L is called *non-diagonal*.

Note that plain locally finite Lie algebras are diagonal.

Lemma 2.5. [2, Theorem 3.2 and Lemma 3] *Let L be a simple locally finite Lie (or associative) algebra. Then L has a perfect local system and if $(L_\alpha)_{\alpha\in\Gamma}$ is a perfect local system for L then for every $\alpha \in \Gamma$ there exists $\alpha' \in \Gamma$ such that for all $\beta \ge \alpha'$ one has* $\operatorname{Rad} L_\beta \cap L_\alpha = 0$.

Proof. In the case of Lie algebras this was proved in [2, Theorem 3.2 and Lemma 3]. Proof of the associative case is similar. □

Definition 2.6. A perfect local system $(L_\alpha)_{\alpha\in\Gamma}$ is called *conical* if Γ contains the smallest element 1 such that

(1) $L_1 \subseteq L_\alpha$ for all $\alpha \in \Gamma$;

(2) L_1 is simple;

(3) for each $\alpha \in \Gamma$ the restriction of any natural L_α-module to L_1 has a non-trivial composition factor.

By the *rank* of a conical system we mean the rank of the simple Lie algebra L_1. Note that property (3) of the definition implies that for every $\alpha \in \Gamma$ and every simple component S of a Levi subalgebra of L_α one has $\operatorname{rk} S \geq \operatorname{rk} L_1$. In particular, all these simple components are classical if $\operatorname{rk} L_1 \geq 9$.

Proposition 2.7. [1, Proposition 3.1] *Let L be a simple locally finite Lie algebra and let $\mathfrak{L} = (L_\alpha)_{\alpha\in\Gamma}$ be a perfect local system of L. Let Q be a finite dimensional simple subalgebra of L. Fix any $\beta \in \Gamma$ such that $Q \subseteq L_\beta$. For $\gamma \geq \beta$, denote by L_γ^Q the ideal of L_γ generated by Q. Put $L_1^Q = Q$ and $\Gamma^Q = \{\gamma \in \Gamma \mid \gamma \geq \beta\} \cup \{1\}$. Then $\mathfrak{L}^Q = (L_\alpha^Q)_{\alpha\in\Gamma^Q}$ is a conical local system of L and the following hold.*

(1) Every natural L_α^Q-module is the restriction of a natural L_α-module. In particular, the embedding $L_\alpha^Q \subseteq L_\alpha$ is diagonal.

(2) If the local system $\mathfrak{L}$ is diagonal (resp. plain) then the local system $\mathfrak{L}^Q$ is diagonal (resp. plain).

(3) If the local system $\mathfrak{L}$ is semisimple then the local system $\mathfrak{L}^Q$ is semisimple.

Proof. Parts (1) and (2) were proved in [1]. Part (3) is obvious. $\square$

Proposition 2.8. [1, Corollary 3.3] *Simple locally finite Lie algebras have conical local systems of arbitrary large rank.*

Remark 2.9. Similar results hold for locally finite associative algebras. In particular, every (involution) simple locally finite associative algebra A has a conical ($*$-invariant) local system of subalgebras, see [1, Proposition 2.9]. Moreover, this system will be semisimple if A is locally semisimple.

The following two results were essentially proved in [5, Corollary 3.4].

Theorem 2.10. *Let L be a simple locally finite Lie algebra and let $(L_\alpha)_{\alpha\in\Gamma}$ be a conical local system for L. Then for every $\alpha \in \Gamma$ there is $\alpha' \in \Gamma$ such that for all $\beta \geq \alpha'$ and all maximal ideals M of L_β one has $L_\alpha \cap M = 0$. In particular, for every simple component Q of $L_\beta / \operatorname{Rad} L_\beta$ one has $\dim Q \geq \dim L_\alpha$.*

Proof. For each $\gamma \in \Gamma$ we denote by R_γ the solvable radical of L_γ, by S_γ the semisimple quotient L_γ/R_γ and by $S_\gamma^1, \dots, S_\gamma^{k_\gamma}$ the simple components of S_γ. In particular, $R_1 = 0$ and $L_1 = S_1 = S_1^1$. Fix any $\alpha \in \Gamma$. By Lemma 2.5, there is $\gamma > \alpha$ such that $R_\gamma \cap L_\alpha = 0$ and by [5, Corollary 3.4] there is $\alpha' > \gamma$ such that the sets of $S_1^1-, S_\gamma^1-, S_\gamma^2-, \dots, S_\gamma^{k_\gamma}-$accessible simple components on level β coincide for all $\beta \geq \alpha'$. Recall that for $\beta > \gamma$, a component S_β^i is S_γ^j-accessible if the restriction of the natural L_β-module V_β^i to L_γ has a composition factor which is non-trivial as a S_γ^j-module. Fix any $\beta \geq \alpha'$. Let M be a maximal ideal of L_β. Then by Lemma 2.2, $L_\beta/M \cong S_\beta^i$ for some i. More exactly, M is the annihilator of the natural L_β-module V_β^i. Note that all components of S_β are S_1^1-accessible by the definition of conical systems (property (3)). This means that S_β^i is S_γ^j-accessible for all j, i.e. all simple components of S_γ act non-trivially on V_β^i and cannot be in its annihilator M. Therefore $M \cap L_\gamma \subset R_\gamma$. Since $R_\gamma \cap L_\alpha = 0$, one has that $M \cap L_\alpha = 0$, as required. $\square$

Corollary 2.11. *Let L be a simple locally finite Lie algebra and let $(L_\alpha)_{\alpha\in\Gamma}$ be a conical local system for L. Then for every finite-dimensional simple subalgebra Q of L there exists $\alpha' \in \Gamma$ such that for all $\beta \geq \alpha'$, $Q \subseteq L_\beta$ and the restriction of every* *natural L_β-module V to Q has a non-trivial composition factor, i.e. $\{Q, L_\beta \mid \beta \geq \alpha'\}$ is a conical local system of L.*

Proof. Fix any $\alpha \in \Gamma$ such that $Q \subseteq L_\alpha$. By Theorem 2.10, there is $\alpha' \in \Gamma$ such that for all $\beta \geq \alpha'$ and all maximal ideals M of L_β one has $L_\alpha \cap M = 0$. Let V be a natural L_β-module. Then its annihilator M is a maximal ideal of L_β. Since $Q \cap M = 0$, Q acts non-trivially on V. □

We will need a version of the above theorem for associative algebras.

Theorem 2.12. *Let A be a (involution) simple locally finite associative algebra and let $(A_\alpha)_{\alpha\in\Gamma}$ be a conical perfect ($*$-invariant) local system for A. Then for every $\alpha \in \Gamma$ there is $\alpha' \in \Gamma$ such that for all $\beta \geq \alpha'$ and all ($*$-invariant) maximal ideals M of A_β one has $A_\alpha \cap M = 0$.*

Proof. The proof is similar to that of the previous theorem. □

Proposition 2.13. *Let L be a simple diagonal locally finite Lie algebra and let $(L_\alpha)_{\alpha\in\Gamma}$ be a conical local system of L. Then for every $n \in \mathbb{N}$ there is $\alpha' \in \Gamma$ and a simple subalgebra Q of L with $\operatorname{rk} Q > n$ such that $Q \subseteq L_\beta$ for all $\beta \geq \alpha'$ and $\{Q, L_\beta \mid \beta \geq \alpha'\}$ is a conical diagonal local system of L of rank $> n$.*

Proof. Since L is diagonal, by [5, Theorem 3.8] L has a conical diagonal local system $(M_\delta)_{\delta\in\Delta}$ of rank $> \max\{10, n\}$. Note that M_1 is simple of rank $> \max\{10, n\}$. Put $Q = M_1$. By Corollary 2.11, there is $\alpha' \in \Gamma$ such that $Q \subseteq L_{\alpha'}$ and for all $\beta \geq \alpha'$ the restriction of every natural L_β-module V to Q has a non-trivial composition factor. It remains to prove that the embeddings $Q \subseteq L_{\beta_1}$ and $L_{\beta_1} \subseteq L_{\beta_2}$ are diagonal for all $\beta_2 > \beta_1 \geq \alpha'$. Fix any $\delta \in \Delta$ such that $L_{\beta_2} \subseteq M_\delta$, so we have a chain of embeddings

$$Q = M_1 \subseteq L_{\beta_1} \subseteq L_{\beta_2} \subseteq M_\delta.$$

Since $\operatorname{rk} Q > 10$ and the embedding $Q \subseteq M_\delta$ is diagonal, by Lemma 2.1, the embedding $Q \subseteq L_{\beta_2}$ is diagonal. Applying this lemma again to the triple $Q \subseteq L_{\beta_1} \subseteq L_{\beta_2}$, we get that the embeddings $Q \subseteq L_{\beta_1}$ and $L_{\beta_1} \subseteq L_{\beta_2}$ are diagonal, as required. □

Theorem 2.14. *Let L be a simple diagonal locally finite Lie algebra and let $(L_\alpha)_{\alpha\in\Gamma}$ be a perfect local system for L. Assume that there is $\alpha \in \Gamma$, a non-zero $x \in L_\alpha$ and a natural number k such that for all $\beta \geq \alpha$, the rank of x is $\leq k$ on every natural L_β-module. Then L is finitary.*

Proof. By Proposition 2.13, we can assume that $(L_\alpha)_{\alpha\in\Gamma}$ is a conical diagonal local system for L of rank > 10. Let A be its involution simple associative $\mathfrak{P}^*$-envelope and let A_α be the subalgebra of A generated by L_α. Then it follows from the construction of A (see proof of Theorem 1.3 in [1]), that $(A_\alpha)_{\alpha\in\Gamma}$ is a conical diagonal local system for A, $\mathfrak{su}^*(A_\alpha) = L_\alpha$, every natural L_α-module is lifted to A_α and every irreducible A_α-module is either natural or conatural L_α-module. Let B be the ideal of A generated by x. Since $x^* = -x$, B is $*$-invariant, so $B = A$. Note that $xAx \neq 0$. Indeed, otherwise $A = A^3 = BAB = 0$. Therefore x acts nontrivially on the left A- (and L-) module $V = Ax$. We claim that $\dim xAx \leq 2k^2$. It is enough to show that $\dim xA_\beta x \leq 2k^2$ for all large β. By Theorem 2.12, there is $\gamma > \beta$ and a maximal $*$-invariant ideal M of A_γ such that $M \cap A_\beta = 0$. Note that the quotient $Q = A_\gamma/M$ is either simple or the direct sum of two simple components, so Q is isomorphic to $\operatorname{End} U$ or $\operatorname{End} W_1 \oplus \operatorname{End} W_2$ where U and W_1 are natural L_γ-modules and W_2 is conatural. Since $M \cap A_\beta = 0$, we have an isomorphic image of A_β in Q. Assume first that $Q \cong \operatorname{End} U$. Since x is of rank $\leq k$ on U, it is easy to see that $\dim xQx \leq k^2$ (e.g. by using the Jordan canonical form of x). Similarly, if $Q \cong \operatorname{End} W_1 \oplus \operatorname{End} W_2$, we get that $\dim xQx \leq 2k^2$. Therefore, $\dim xA_\beta x \leq 2k^2$ and $\dim xAx \leq 2k^2$, as required. Thus, x is a finite rank transformation of $V = Ax$.

Note that all finite rank transformations of V in L form an ideal of L. Since L is simple, V is a non-trivial finitary module for L, so L is finitary. $\square$

Definition 2.15. Let L be a Lie algebra. An *inner ideal* of L is a subspace I of L such that $[I,[I,L]] \subseteq I$.

Although inner ideals are not ideals in general (not even subalgebras) it is easy to see that they are well-behaved with respect to subalgebras and factor algebras:

Lemma 2.16. *Let I be an inner ideal of a Lie algebra L.*

(1) Let H be a subalgebra of L. Then $I \cap H$ is an inner ideal of H.

(2) Let J be an ideal of L then $(I+J)/J$ is an inner ideal of L/J.

The following classifies the inner ideals of the classical finite dimensional Lie algebras over F. This is only a very particular case of the results proven in [12, 14].

Theorem 2.17. [12, Theorem 5.1][14, Theorem 6.3(i)] *Let V be a finite dimensional vector space over an algebraically closed field F of characteristic zero. Let $A = \operatorname{End} V$ and Φ (resp. Ψ) be a non-degenerate symmetric (resp. skew-symmetric) form on V. Let $*$ be the involution of A induced by either Φ or Ψ.*

(1) Let $L = \mathfrak{sl}(V)$. A subspace I of L is a proper inner ideal of L if and only if there exist idempotents e and f in A such that $I = eAf$ and $fe = 0$.

(2) Let $L = \mathfrak{sp}(V,\Psi)$ and $\dim V > 4$. A subspace I of L is a proper inner ideal of L if and only if there exists an idempotent e in A such that $I = eLe^*$ and $e^*e = 0$ (equivalently, $I = [U,U] = span\{u^*v + v^*u | u, v \in U\}$ where U is a totally isotropic subspace of V and $u^*v \in \operatorname{End} V$ is defined as $(u^*v)(w) = \Psi(w,u)v$ for all $w \in V$).

(3) Let $L = \mathfrak{o}(V,\Phi)$ and $\dim V > 4$. A subspace I of L is a proper inner ideal of L if and only if one of the following holds.

(i) $I = eLe^*$ where $e \in A$ is an idempotent such that $e^*e = 0$.

(ii) $I = [v, H^\perp]$ where $v \in V$ is a nonzero isotropic vector of H, and H is a 2-dimensional subspace of V such that the restriction of Φ to H is nondegenerate

(equivalently, there is a basis $\{x_1, \dots, x_n\}$ *of* V *such that* I *is the* F*-span of the matrix units* $e_{1j} - e_{j2}$*,* $j \geq 3$*, with respect to this basis* [14, 4.1]*).*

(iii) I *is a Type 1 point space of dimension greater than 1.*

Recall that a subspace P of a Lie algebra L is called a *point space* if $[P, P] = 0$ and $ad_x^2 L = Fx$ for every nonzero element $x \in P$. Moreover, a point subspace P of $\mathfrak{o}(V, \Phi)$ is said to be of *Type 1* if there is a non-zero vector u in the image of every non-zero $a \in P$.

Lemma 2.18. [13, Lemma 1.13] *Let* L *be a finite dimensional simple Lie algebra and let* I *be a proper inner ideal of* L*. Then* $[I, I] = 0$*, i.e.* I *is abelian.*

The following two facts are well-known, see for example [18, Proposition 2.3].

Lemma 2.19. *Let* L *be a finite dimensional simple Lie algebra and let* I *be an inner ideal of* L*. Then* $[I, [I, L]] = I$*.*

Proof. Let I be an inner ideal of L. If $I = L$ then this is obviously true. Assume that I is proper. Then by Lemma 2.18, I is abelian. Let $x \in I$. Then

$$[x, [x, [x, L]]] \subseteq [x, I] = 0,$$

so x is ad-nilpotent. By the Jacobson-Morozov Theorem, there exist $y, h \in L$ such that $\{x, y, h\}$ form an $\mathfrak{sl}_2$-triple. Note that

$$[x, [x, y]] = [x, h] = -2x,$$

so $x \in [I, [I, L]]$. This implies $I = [I, [I, L]]$, as required. □

Lemma 2.20. *Let* L *be a finite dimensional semisimple Lie algebra. Let* $Q_1, \dots, Q_n$ *be the simple components of* L*. Let* I *be an inner ideal of* L *and* $I_i = I \cap Q_i$*. Then* $I = I_1 \oplus \cdots \oplus I_n$*.*

Proof. Let $\psi_k : L \to Q_k$, $\psi_k((q_1, \ldots, q_n)) = q_k$, be the natural projection and let $J_k = \psi_k(I)$. We need to show $J_k = I_k$. Indeed, by Lemma 2.16, J_k is an inner ideal of Q_k. It is clear that $I_k \subseteq J_k$. On the other hand, by Lemma 2.19,

$$J_k = [J_k, [J_k, Q_k]] = [I, [I, Q_k]] \subseteq I_k.$$

Therefore $I_k = J_k$ for all k, so $I = I_1 \oplus \cdots \oplus I_n$. □

Proposition 2.21. *Let L be a perfect finite dimensional Lie algebra and let I be an inner ideal of L. Assume that $(I + M)/M = L/M$ for every maximal ideal M of L. Then $I = L$.*

Proof. Without loss of generality we can assume that I is minimal among all inner ideals of L satisfying this assumption. By [13, Lemma 1.1(4)], for every inner ideal J of L the subspace $J^{[3]} = [J, [J, J]]$ is also an inner ideal of L. Note that $I^{[3]}$ satisfies the assumption of the proposition since L is perfect. Moreover, $I^{[3]}$ is contained in I:

$$I^{[3]} \subseteq [I, [I, L]] \subseteq I.$$

Therefore $I^{[3]} = I$. Now

$$[L, I] = [L, [I, [I, I]]] \subseteq [I, [I, L]] \subseteq I$$

so I is an ideal of L. Since I is not contained in any maximal ideal, $I = L$, as required. □

Lemma 2.22. [12, Lemma 4.23] *Let L be a classical simple finite dimensional Lie algebra and let V be the natural module for L. Let I be a proper inner ideal of L. Then $I^3V = 0$. In particular, $x^3V = 0$ for all $x \in I$.*

Proof. This was proved in [12] but also follows from the classification of inner ideals given in Theorem 2.17. Indeed, referring to the notation of the theorem, suppose

$I = eAf$ or $I = eLe^*$ as in cases 1, 2 and 3 part (i). Then $fe = 0$ or $e^*e = 0$, so $I^2 = 0$. Now suppose $I = [v, H^\perp]$ as in case 3 part (ii). Then I is the F-span of the matrix units $e_{1j} - e_{j2}$, $j \geq 3$. Note that $I^2 = Fe_{12}$ and $I^3 = 0$. Finally consider case 3 part (iii). If I is a point space of type 1 then I is a subspace of eLe^* for some idempotent e with $e^*e = 0$ (see [14, Proposition 4.3]). Thus again $I^2 = 0$. $\square$

3. Non-diagonal locally finite Lie algebras

The aim of this section is to prove Theorem 1.1: a simple locally finite Lie algebra over F has a proper nonzero inner ideal if and only if it is diagonal. First we are going to show that every simple diagonal locally finite Lie algebra has a non-zero proper inner ideal. This will be generalized in the next section where we describe all regular inner ideals of diagonal Lie algebras.

Proposition 3.1. *Every simple diagonal locally finite Lie algebra has a proper non-zero inner ideal.*

Proof. Let L be a simple diagonal locally finite Lie algebra. By [1, Theorems 1.1 and 1.2] there exists an involution simple locally finite associative algebra A such that $L = \mathfrak{su}^*(A)$. By [1, Corollary 2.11], for every integer m, A contains an involution simple finite dimensional subalgebra A_1 of dimension greater than m. It is well known A_1 is isomorphic to a matrix algebra $M_n(F)$ with orthogonal or symplectic involution or the direct sum of two copies of $M_n(F)$ with involution permuting the components and $L_1 = \mathfrak{su}^*(A_1)$ is a finite dimensional classical Lie algebra isomorphic to $\mathfrak{sl}_n$, $\mathfrak{sp}_n$, or $\mathfrak{o}_n$ (see for example [9, Lemmas 2.1 and 2.2]). Fix any idempotent e in A_1 such that $e^*e = 0$ and $eL_1e^* \neq 0$ (see Theorem 2.17). Put $I = eAe^* \cap \mathfrak{su}^*(A)$. Note that $I^2 = 0$, so I is a proper non-zero subspace of L. Since $(eAe^*)A(eAe^*) \subseteq eAe^*$, one has $[I, [I, L]] \subseteq I$, so I is an inner ideal of L. □

Remark 3.2. As it was mentioned to us by Fernández López, the proposition also follows from [22, Corollary 2.3], because every simple diagonal locally finite Lie algebra L has an algebraic adjoint representation [5, Corollary 3.9(6)], and hence a non-zero abelian inner ideal.

Lemma 3.3. *Let L be a non-diagonal simple locally finite Lie algebra. Let $(L_\alpha)_{\alpha\in\Gamma}$ be a conical local system of L of rank > 10. Then for every β there exists $\beta' \geq \beta$ such that for all $\gamma \geq \beta'$ the embedding $L_\beta \subseteq L_\gamma$ is non-diagonal.*

Proof. Let $\beta \in \Gamma$. Suppose to the contrary that for every $\beta' \geq \beta$ there is $\gamma \geq \beta'$ such that the embedding $L_\beta \subseteq L_\gamma$ is diagonal. Since $L_\beta \subseteq L_{\beta'} \subseteq L_\gamma$, by Lemma 2.1 the embedding $L_\beta \subseteq L_{\beta'}$ is diagonal for all $\beta' \geq \beta$. Fix any simple component Q of a Levi subalgebra of L_β. Then $\operatorname{rk} Q > 10$ and by Corollary 2.11, there is $\alpha' > \beta$ such that $\mathfrak{L} = \{Q, L_\gamma \mid \gamma \geq \alpha'\}$ is a conical local system of L. We are going to prove that $\mathfrak{L}$ is a diagonal local system, so L is a diagonal Lie algebra, which is a contradiction. We already know that all embeddings $Q \subseteq L_\gamma$, $\gamma \geq \alpha'$, are diagonal. Consider any $\xi > \zeta \geq \alpha'$. Then we have a chain of embeddings $Q \subseteq L_\zeta \subseteq L_\xi$. By construction both $Q \subseteq L_\zeta$ and $Q \subseteq L_\xi$ are diagonal. Since $\mathfrak{L}$ is conical and $\operatorname{rk} Q > 10$, by Lemma 2.1 the embedding $L_\zeta \subseteq L_\xi$ is diagonal, as required. □

Lemma 3.4. [4, Lemma 4.5] *Let $L_1 \subset L_2$ be finite dimensional Lie algebras; let S_1 and S_2 be Levi subalgebras of L_1 and L_2, respectively. Then there exists an automorphism θ of L_2 such that $\theta(S_1) \subseteq S_2$ and $\theta(l) = l + r(l)$ for all $l \in L_2$, with $r(l)$ being in the nilpotent radical of L_2. Moreover the monomorphism $S_1 \subseteq S_2$ induced by θ does not depend on the choice of such θ.*

In what follows we will use the function δ introduced in [4]. This is a function defined on the weights (and modules) of simple Lie algebras. The function δ takes integral (and half-integral in the case of algebras of type B) values only. The function is linear and is defined by writing down its values on fundamental weights. Let L be a finite dimensional simple Lie algebra of rank m. Denote by $\omega_1, \dots, \omega_m$ the fundamental weights of L and $\alpha_1, \dots \alpha_m$ the simple roots of L. In the following list $\delta = (p_1, \dots, p_m)$ means that $\delta(\omega_i) = p_i$ for $i = 1, 2, \dots, m$.

$\delta = (1, 2, \dots, k, k, \dots, 2, 1)$ (A_{2k});

$\delta = (1, 2, \dots, k+1, \dots, 2, 1)$ (A_{2k+1});

$\delta = (1, 2, \dots, m-2, m-1, m)$ $(C_m,\ m \geq 2)$;

$\delta = (1, 2, \dots, m-2, m-1, [\frac{m}{2}])$ $(B_m,\ m \geq 3)$;

$\delta = (1, 2, \dots, 2k-2, k-1, k)$ $(D_{2k},\ k \geq 2)$;

$\delta = (1, 2, \dots, 2k-1, k, k)$ $(D_{2k+1},\ k \geq 2)$;

$\delta = (2, 2, 3, 4, 3, 2)$ (E_6);

$\delta = (2, 2, 3, 4, 3, 2, 1)$ (E_7);

$\delta = (4, 5, 7, 10, 8, 6, 4, 2)$ (E_8);

$\delta = (2, 3, 2, 1)$ (F_4);

$\delta = (1, 2)$ (G_2).

It is easy to verify that $\delta(\alpha_i) \geq 0$ for all $i = 1, \dots m$. Let V be an L-module and M be its set of weights then set $\delta_L(V) = sup\{\delta(\mu)\}_{\mu \in M}$. We will write $\delta(V)$ instead of $\delta_L(V)$ if L is fixed. Since the value of δ on the simple roots is ≥ 0 this means $\delta(V) = \delta(\mu_h)$ where μ_h is the highest weight of V. Let S be a finite dimensional simple Lie algebra of rank > 10 and let V be an S-module. Then V is trivial if and only if $\delta(V) = 0$; V is non-trivial diagonal if and only if $\delta(V) = 1$; V is non-diagonal if and only if $\delta(V) \geq 2$ (see [4, Section 6] for details).

Lemma 3.5. *Let $L_1 \subseteq L_2 \subseteq L_3$ be three perfect finite dimensional Lie algebras such that L_1 is simple and $\operatorname{rk} L_1 > 10$. Suppose that the embedding $L_2 \subseteq L_3$ is non-diagonal and the restriction of every natural L_2-module to L_1 is non-trivial. Then there is a natural L_3-module V such that $\delta(V \downarrow L_1) > 1$. In particular, the restriction of V to L_1 is non-diagonal.*

Proof. By using Levi-Malcev Theorem and Lemma 3.4 we can reduce this to the case of Levi subalgebras, one embedded into the next, so we can assume that the L_i are semisimple. Since the embedding $L_2 \subseteq L_3$ is non-diagonal, there exists a natural L_3-module, say, V such that $V \downarrow L_2$ has an irreducible component W which is non-trivial, non-natural and non-conatural. We have

$$\delta(V \downarrow L_1) = \delta((V \downarrow L_2) \downarrow L_1) \geq \delta(W \downarrow L_1)$$

It remains to show that $\delta(W \downarrow L_1) > 1$. The module W can be represented in the form $W = W_1 \otimes \cdots \otimes W_k$ where each W_i is a non-trivial irreducible module for a simple component S_i of L_2. Then we have two cases: either at least two W_i are non-trivial or at least one W_i is non-trivial non-natural and non-conatural. For the first case, without loss of generality we may assume that there are just two non-trivial W_i, so that $W = W_1 \otimes W_2$. Using [4, Lemma 7.2] we see that

$$\delta(W \downarrow L_1) \geq \delta((W \downarrow S_1) \downarrow L_1) + \delta((W \downarrow S_2) \downarrow L_1) \geq 2.$$

In the second case we may assume that $W = W_1$ where W_1 is a non-trivial, non-natural and non-conatural S_1-module. Then using [4, Lemma 6.7], we get

$$\delta(W \downarrow L_1) \geq \delta((W \downarrow S_1) \downarrow L_1) \geq \delta(W_1 \downarrow L_1) > \delta(V_1 \downarrow L_1) \geq 1$$

where V_1 is the natural S_1-module. In both cases $\delta(V \downarrow L_1) > 1$, so V is a non-diagonal L_1-module. □

Lemma 3.6. *Let L be a non-diagonal simple locally finite Lie algebra and let $\mathfrak{L}$ be a conical perfect local system for L of rank > 10. Let n be a positive integer and let S be a finite dimensional simple subalgebra of L. Then there exists a chain of subalgebras $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$ of L and subalgebras $S_i \subseteq M_i$, $1 \leq i \leq n$, such that $M_1 = S_1 = S$, for each $i = 2, \ldots, n$, $M_i \in \mathfrak{L}$, S_i is a simple component of a Levi subalgebra of M_i and the restriction $V_i \downarrow S_{i-1}$ is a non-diagonal S_{i-1}-module where V_i is the natural M_i-module corresponding to S_i. Moreover, $\delta(V_n \downarrow S) > n/2$.*

Proof. We construct the algebras M_i and S_i by induction. Recall that $M_1 = S_1 = S$. Assume that M_{i-1} and S_{i-1} have been constructed. By Corollary 2.11, there is an algebra $Q_i \in \mathfrak{L}$ such that $S_{i-1} \subseteq Q_i$ and the restriction of every natural Q_i-module to S_{i-1} is non-trivial. By Lemma 3.3, there is $M_i \in \mathfrak{L}$ such that $Q_i \subseteq M_i$ and the embedding $Q_i \subseteq M_i$ is non-diagonal. Therefore by Lemma 3.5, there is a simple component S_i of a Levi subalgebra of M_i such that the restriction $V_i \downarrow S_{i-1}$ is a non-diagonal S_{i-1}-module and $\delta(V_i \downarrow S_{i-1}) > 1$ where V_i is the natural M_i-module corresponding to S_i. Let W_{i-1} be any non-diagonal composition factor of the restriction $V_i \downarrow S_{i-1}$. Then W_{i-1} can be viewed as both S_{i-1} and M_{i-1}-module. Similarly to the proof of Lemma 3.5, using [4, Lemma 6.7], we get that

$$\delta(V_i \downarrow S_1) = \delta((V_i \downarrow M_{i-1}) \downarrow S_1) \geq \delta(W_{i-1} \downarrow S_1) > \delta(V_{i-1} \downarrow S_1)$$

Therefore

$$\delta(V_n \downarrow S_1) > \delta(V_{n-1} \downarrow S_1) > \cdots > \delta(V_1 \downarrow S_1) = 1.$$

Since δ has half-integer values only, this implies $\delta(V_n \downarrow S_1) > n/2$. □

Proposition 3.7. *Let L be a non-diagonal simple locally finite Lie algebra and let $\mathfrak{L}$ be a conical perfect local system for L of rank > 10. Let n be a positive integer and let S be a finite dimensional simple subalgebra of L. Then there exists a subalgebra $M \in \mathfrak{L}$ containing S such that for every $M' \in \mathfrak{L}$ containing M and every natural M'-module V one has $\delta(V \downarrow S) > n$.*

Proof. By Lemma 3.6, there exists $Q_1 \in \mathfrak{L}$ containing S and a simple component S_1 of a Levi subalgebra of Q_1 such that $\delta(V_1 \downarrow S) > n$ where V_1 is the natural Q_1-module corresponding to S_1. By Theorem 2.10, there exists $M \in \mathfrak{L}$ containing Q_1 such that for every $M' \in \mathfrak{L}$ containing M and every maximal ideal N of M' one has $Q_1 \cap N = 0$, so $V \downarrow S_1$ is a non-trivial S_1-module for every natural M'-module V. It remains to show that $\delta(V \downarrow S) > n$. Let W_1 be any non-trivial composition factor of the restriction $V \downarrow S_1$. Then W_1 can be viewed as both S_1 and Q_1-module. Similarly to the proof of Lemma 3.5, using [4, Lemma 6.7], we get that

$$\delta(V \downarrow S) = \delta((V \downarrow Q_1) \downarrow S) \geq \delta(W_1 \downarrow S) \geq \delta(V_1 \downarrow S) > n.$$

□

Proposition 3.8. *Let L be a simple non-diagonal locally finite Lie algebra. Then L has no non-zero proper inner ideals.*

Proof. Let $(L_\alpha)_{\alpha\in\Gamma}$ be a perfect conical local system for L of rank > 10. Let R_α be the solvable radical of L_α and let S_α be a Levi subalgebra of L_α so that $L_\alpha = S_\alpha \oplus R_\alpha$ for $\alpha \in \Gamma$. Assume I is a proper non-zero inner ideal of L. For $\alpha \in \Gamma$ put $I_\alpha = I \cap L_\alpha$. Then I_α is an inner ideal of L_α by Lemma 2.16. Fix $\alpha_1 \in \Gamma$ such that I_{α_1} is a proper non-zero inner ideal of L_{α_1}. By Lemma 2.5, there is $\alpha_2 \geq \alpha_1$ such that $L_{\alpha_1} \cap R_{\alpha_2} = 0$, so $I_{\alpha_2} \not\subseteq R_{\alpha_2}$ and the image $\overline{I_{\alpha_2}}$ of I_{α_2} in the semisimple quotient $\overline{L_{\alpha_2}} = L_{\alpha_2}/R_{\alpha_2}$ is a non-zero inner ideal of $\overline{L_{\alpha_2}} \cong S_{\alpha_2}$. It follows from Lemmas 2.20 and 2.22 that $\overline{I_{\alpha_2}}$ contains a non-zero ad-nilpotent element. Therefore there exist a non-zero ad-nilpotent $s \in S_{\alpha_2}$ and an $r \in R_{\alpha_2}$ such that $x = s + r \in I_{\alpha_2}$. By the Jacobson-Morozov Theorem, there exists a subalgebra S of S_{α_2} isomorphic to $\mathfrak{sl}_2$ containing s. Consider the subalgebra $\hat{S} = S + R_{\alpha_2}$ of L_{α_2}. Then $\operatorname{Rad}\hat{S} = R_{\alpha_2}$ and $I_0 = I \cap \hat{S}$ is an inner ideal of $\hat{S}$ containing x. By Proposition 3.7, there exists $\alpha_3 \in \Gamma$ such that $\hat{S} \subset L_{\alpha_3}$ and for every natural L_{α_3}-module V one has $\delta(V \downarrow S) > 2$. Fix any such module V. Note that all composition factors of $V \downarrow \hat{S}$ are irreducible modules for S, so $V \downarrow \hat{S}$ has a composition factor W, which is also an irreducible module for $S \cong \mathfrak{sl}_2$ with $\delta(W) > 2$. It follows from the definition of the function δ that $\dim W = \delta(W) + 1 > 3$ (see [4, Section 6] for details), so $s^3 W \neq 0$ as s is a basic ad-nilpotent element of S. Since $r \in \operatorname{Rad}\hat{S} = R_{\alpha_2}$ and R_{α_2} annihilates every composition factor of $V \downarrow L_{\alpha_2}$ one has $rW = 0$, so

$$x^3W = (s + r)^3W = s^3W \neq 0.$$

Therefore, $x^3V \neq 0$. Let M be the annihilator of V in L_{α_3}. Then M is a maximal ideal of L_{α_3} and let $a \mapsto \bar{a}$ be the natural homomorphism $L_{\alpha_3} \to \overline{L_{\alpha_3}} = L_{\alpha_3}/M$. Then $\overline{L_{\alpha_3}}$ is a classical simple Lie algebra of rank > 10, $\overline{I_{\alpha_3}}$ is an inner ideal of $\overline{L_{\alpha_3}}$ and $\overline{x} \in \overline{I_{\alpha_3}}$. Note that $\overline{x}^3V = x^3V \neq 0$. Since V is a natural module for $\overline{L_{\alpha_3}}$, by Lemma 2.22, one has $\overline{I_{\alpha_3}} = \overline{L_{\alpha_3}}$. Since this is true for every natural L_{α_3}-module V (and so for every maximal ideal M of L_{α_3}), by Proposition 2.21, $I_{\alpha_3} = L_{\alpha_3}$. This implies that $I_{\alpha_1} = L_{\alpha_1}$, which contradicts the assumption that I_{α_1} is a proper inner ideal of L_{α_1}. □

Proof of Theorem 1.1. This follows from Propositions 3.1 and 3.8.

4. Regular Inner Ideals and Diagonal Lie Algebras

In this section we discuss inner ideals of simple diagonal locally finite Lie algebras.

Lemma 4.1. *Let A be an associative algebra and let $L = [A, A]$. Let I be a subspace of L such that $I^2 = 0$. Then the following hold.*

(1) I is an inner ideal of L if and only if $ixj + jxi \in I$ for all $i, j \in I$ and all $x \in L$.

(2) $IAI \subseteq L$.

(3) If $IAI \subseteq I$, then I is an inner ideal of L.

Proof. (1) Recall that I is an inner ideal of L if and only if $[i, [j, x]] \in I$ for all $i, j \in I$ and all $x \in L$. It remains to note that

$$[i, [j, x]] = ijx - ixj - jxi + xji = -ixj - jxi.$$

(2) Indeed,

$$iaj = i(aj) - (aj)i = [i, aj] \subseteq [A, A] = L$$

for all $i, j \in I$ and all $a \in A$.

(3) This follows from (1). □

Lemma 4.2. *Let A be an associative algebra with involution and let $K = \mathfrak{su}^*(A)$. Let I be a subspace of K such that $I^2 = 0$. Then the following hold.*

(1) $\mathfrak{u}^*(IAI) \subseteq K$.

(2) $\mathfrak{u}^*(IAI) = IAI \cap K$.

(3) If $\mathfrak{u}^*(IAI) \subseteq I$, then I is an inner ideal of K.

Proof. (1) Note that IAI is $*$-invariant, so $\mathfrak{u}^*(IAI) = \{q - q^* \mid q \in IAI\}$. It remains to note that

$$iaj - (iaj)^* = iaj - ja^*i = i(aj + ja^*) - (aj + ja^*)i = [i, aj - (aj)^*] \in [\mathfrak{u}^*(A), \mathfrak{u}^*(A)] = K$$

for all $i, j \in I$ and all $a \in A$.

(2) This is obvious.

(3) By Lemma 4.1(1), it is enough to check that $ixj + jxi \in I$ for all $i, j \in I$ and all $x \in K$. One has

$$ixj + jxi = ixj - (ixj)^* \in \mathfrak{u}^*(IAI) \subseteq I$$

as required. $\square$

We will show (see Theorem 4.13) that for every inner ideal I of an infinite dimensional simple locally finite Lie algebra L one has $I^2 = 0$, (the only exception being the finitary orthogonal algebras). Thus Lemmas 4.1 and 4.2 justify the following definition.

Definition 4.3. (1) Let A be an associative algebra and let $L = [A, A]$. Let I be a subspace of L such that $I^2 = 0$. We say that I is a *regular* inner ideal of L (with respect to A) if and only if $IAI \subseteq I$.

(2) Let A be an associative algebra with involution and let $K = \mathfrak{su}^*(A)$. Let I be a subspace of K such that $I^2 = 0$. We say that I is a $*$-*regular* (or, simply, *regular*) inner ideal of K (with respect to A) if and only if $\mathfrak{u}^*(IAI) \subseteq I$.

Remark 4.4. Note that regular inner ideals are always abelian (since $[I, I] \subseteq I^2 = 0$), so they are proper inner ideals of L (if L is not abelian).

If B is an associative algebra denote by $B^{(-)}$ the Lie algebra obtained from the vector space B with the new multiplication $[x, y] = xy - yx$.

We will use the following well-known facts.

Lemma 4.5. *Let A be an associative algebra.*

(1) If A is involution simple then A is either simple or $A = B \oplus B^*$ where B is a simple ideal.

(2) Assume $A = B \oplus B^*$. Then $\mathfrak{u}^*(A) = \{(b, -b^*) \mid b \in B\}$. Let φ be the projection of A on B. Then the restriction of φ to $\mathfrak{u}^*(A)$ is an isomorphism of the Lie algebras $\mathfrak{u}^*(A)$ and $B^{(-)}$. Moreover, if C is a $*$-invariant subalgebra of A then $\varphi(\mathfrak{u}^*(C)) = \varphi(C)^{(-)}$.

Proof. (1) Suppose A is not simple. So A has a proper non-zero ideal B. Then $B+B^*$ and $B \cap B^*$ are $*$-invariant ideals of A. Since A is involution simple, $B + B^* = A$ and $B \cap B^* = 0$. So $A = B \oplus B^*$ and B is a simple ideal.

(2) This is obvious. □

Lemma 4.6. *Let $A = B \oplus B^*$ and let $\varphi : \mathfrak{su}^*(A) \to [B, B]$ be the isomorphism as in Lemma 4.5. Then I is a regular inner ideal of $\mathfrak{su}^*(A)$ if and only if $\varphi(I)$ is a regular inner ideal of $[B, B]$.*

Proof. We need to show that $\mathfrak{u}^*(IAI) \subseteq I$ if and only if $\varphi(I)B\varphi(I) \subseteq \varphi(I)$. Since both $\mathfrak{u}^*(IAI)$ and I are subspaces of $\mathfrak{u}^*(A)$, the first inclusion is equivalent to $\varphi(\mathfrak{u}^*(IAI)) \subseteq \varphi(I)$. Note that

$$\varphi(\mathfrak{u}^*(IAI)) = \varphi(IAI) = \varphi(I)B\varphi(I),$$

so this can be rewritten as

$$\varphi(I)B\varphi(I) \subseteq \varphi(I),$$

as required. □

Lemma 4.7. *Let A be a simple associative ring and let $\mathcal{L}$ (resp. $\mathcal{R}$) be a left (resp. right) non-zero ideal of A. Then the following holds.*

(1) $\mathcal{L}A = A$, $A\mathcal{R} = A$, and $\mathcal{L}A\mathcal{R} = A$.

(2) $\mathcal{R}\mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$.

(3) If $\mathcal{L}\mathcal{R} = 0$ then $\mathcal{R}\mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L} \cap [A, A]$.

(4) $\mathcal{R}\mathcal{L}$ and $\mathcal{R} \cap \mathcal{L}$ are non-zero.

Proof. (1) Assume $\mathcal{L}A \neq A$. Since $\mathcal{L}A$ is a two-sided ideal of A and A is simple, $\mathcal{L}A = 0$. This implies that $\mathcal{L}$ is a two-sided ideal of A. Since $\mathcal{L}$ is non-zero, $\mathcal{L} = A$, so $\mathcal{L}A = A^2 = A \neq 0$, which is a contradiction. Proof for $\mathcal{R}$ is similar. Now $\mathcal{L}A\mathcal{R} = (\mathcal{L}A)(A\mathcal{R}) = A^2 = A$.

(2) It is enough to note that $\mathcal{R}\mathcal{L} \subseteq \mathcal{R}$ and $\mathcal{R}\mathcal{L} \subseteq \mathcal{L}$;

(3) This is obvious.

(4) Assume $\mathcal{R}\mathcal{L} = 0$. Then $A = A^2 = (A\mathcal{R})(\mathcal{L}A) = A(\mathcal{R}\mathcal{L})A = 0$, which is a contradiction. □

Let A be an associative ring. An element $x \in A$ is called *von Neumann regular* if there is $y \in A$ such that $xyx = x$. The ring A is called *von Neumann regular* if every element of A is von Neumann regular. We are grateful to Miguel Gómez Lozano for the following observation.

Proposition 4.8. *Let A be an associative ring.*

(1) $\mathcal{R}\mathcal{L} = \mathcal{R} \cap \mathcal{L}$ for all left and right ideals $\mathcal{L}$ and $\mathcal{R}$, respectively, in A if and only if A is von Neumann regular.

(2) $\mathcal{R}\mathcal{L} = \mathcal{R} \cap \mathcal{L}$ for all left and right ideals $\mathcal{L}$ and $\mathcal{R}$, respectively, in A such that $\mathcal{L}\mathcal{R} = 0$ if and only if every x in A with $x^2 = 0$ is von Neumann regular.

Proof. (1) Suppose $\mathcal{R}\mathcal{L} = \mathcal{R} \cap \mathcal{L}$ for all left and right ideals $\mathcal{L}$ and $\mathcal{R}$, respectively. Let $x \in A$. Consider the ideals $\mathcal{R} = xA + \mathbb{Z}x$ and $\mathcal{L} = Ax + \mathbb{Z}x$. Note that

$$x \in \mathcal{R} \cap \mathcal{L} = \mathcal{R}\mathcal{L} = xAx + \mathbb{Z}x^2,$$

Hence $xy'x = x$ for some $y' \in A'$ where the ring $A' = A + \mathbb{Z}\mathbf{1}$ is obtained from A by adding the identity $\mathbf{1}$. Since A is an ideal of A', one has $xyx = x$ for $y = y'xy' \in A$. Therefore A is von Neumann regular.

Assume now A is von Neumann regular. Let $\mathcal{L}$ and $\mathcal{R}$ be a left and right ideal of A respectively. Clearly $\mathcal{RL} \subseteq \mathcal{R} \cap \mathcal{L}$. Let $x \in \mathcal{R} \cap \mathcal{L}$. Then there exists $y \in A$ such that $x = xyx = x(yx) \in \mathcal{RL}$, So $\mathcal{RL} = \mathcal{R} \cap \mathcal{L}$.

(2) Suppose $\mathcal{RL} = \mathcal{R} \cap \mathcal{L}$ for all left and right ideals $\mathcal{L}$ and $\mathcal{R}$, respectively, such that $\mathcal{LR} = 0$. Let $x \in A$ with $x^2 = 0$. Then $\mathcal{R} = xA + \mathbb{Z}x$ and $\mathcal{L} = Ax + \mathbb{Z}x$ are a left and right ideal of A with $\mathcal{LR} = 0$. The rest of the argument follows as in (1).

Assume now every $x \in A$ for which $x^2 = 0$ is von Neumann regular. Let $\mathcal{L}$ and $\mathcal{R}$ be a left and right ideal of A respectively with $\mathcal{LR} = 0$. Clearly $\mathcal{RL} \subseteq \mathcal{R} \cap \mathcal{L}$. Let $x \in \mathcal{R} \cap \mathcal{L}$. Note that $x^2 \in \mathcal{LR} = 0$ so x by assumption is von Neumann regular. So there exists $y \in A$ such that $x = xyx = x(yx) \in \mathcal{RL}$. Therefore $\mathcal{RL} = \mathcal{R} \cap \mathcal{L}$. □

Now we are in position to describe regular inner ideals.

Proposition 4.9. *Let A be an associative algebra and let $L = [A, A]$. Let I be a subspace of L. Then I is a regular inner ideal of L if and only if there exist $\mathcal{L}$ and $\mathcal{R}$ where $\mathcal{L}$ (resp. $\mathcal{R}$) is a left (resp. right) ideal of A such that $\mathcal{LR} = 0$ and*

$$\mathcal{RL} \subseteq I \subseteq \mathcal{R} \cap \mathcal{L} \cap L. \tag{4.1}$$

In particular, if A is von Neumann regular then every regular inner ideal of L is of the shape $I = \mathcal{RL}$ ($= \mathcal{R} \cap \mathcal{L}$).

Proof. Assume first that I is a regular inner ideal of L. Then $I^2 = 0$ and $IAI \subseteq I$. Put $\mathcal{L} = AI + I$ and $\mathcal{R} = IA + I$. Then $\mathcal{L}$ (resp. $\mathcal{R}$) is a left (resp. right) ideal of A with $\mathcal{LR} = 0$ and

$$\mathcal{RL} \subseteq IAI + I = I \subseteq \mathcal{R} \cap \mathcal{L} \cap L,$$

as required.

Now assume that $\mathcal{R}\mathcal{L} \subseteq I \subseteq \mathcal{R} \cap \mathcal{L} \cap L$. Then $I^2 \subseteq \mathcal{L}\mathcal{R} = 0$ and

$$IAI \subseteq \mathcal{R}A\mathcal{L} \subseteq \mathcal{R}\mathcal{L} \subseteq I,$$

so I is a regular inner ideal. $\square$

If A is simple then one can show that the ideals $\mathcal{L}$ and $\mathcal{R}$ are defined by I almost uniquely. More exactly we have the following.

Lemma 4.10. *Let A be a simple associative algebra and let $L = [A, A]$. If I is a regular inner ideal of L and a pair of ideals $(\mathcal{L}, \mathcal{R})$ satisfies (4.1) then $A\mathcal{L} = AI$ and $\mathcal{R}A = IA$.*

Proof. Assume the pair of ideals $(\mathcal{L}, \mathcal{R})$ satisfies (4.1). Then $I \subseteq \mathcal{L}$, so $AI \subseteq A\mathcal{L}$. On the other hand, by Lemma 4.7(1),

$$A\mathcal{L} = (\mathcal{L}A\mathcal{R})\mathcal{L} = \mathcal{L}A(\mathcal{R}\mathcal{L}) \subseteq AI,$$

so $A\mathcal{L} = AI$. Similarly, $\mathcal{R}A = IA$. $\square$

The next proposition describes regular inner ideals in the case of algebras with involution.

Proposition 4.11. *Let A be an associative algebra with involution and let $K = \mathfrak{su}^*(A)$. Let I be a subspace of K. Then I is a regular inner ideal of K if and only if there exists a left ideal $\mathcal{L}$ of A such that $\mathcal{L}\mathcal{L}^* = 0$ and*

$$\mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) \subseteq I \subseteq \mathcal{L}^* \cap \mathcal{L} \cap K. \tag{4.2}$$

In particular, if A is von Neumann regular then every regular inner ideal of L is of the shape $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ $(= \mathfrak{u}^*(\mathcal{L}^* \cap \mathcal{L}))$.

Proof. Assume first that I is a regular inner ideal of K. Then $I^2 = 0$ and $\mathfrak{u}^*(IAI) \subseteq I$. Put $\mathcal{L} = AI + I$. Then $\mathcal{L}$ is a left ideal of A, $\mathcal{L}^* = IA + I$, $\mathcal{L}\mathcal{L}^* = 0$ and

$$\mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) \subseteq \mathfrak{u}^*(IAI + I) \subseteq I \subseteq \mathcal{L}^* \cap \mathcal{L} \cap K,$$

as required.

Now assume that $\mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) \subseteq I \subseteq \mathcal{L}^* \cap \mathcal{L} \cap K$. Then $I^2 \subseteq \mathcal{L}\mathcal{L}^* = 0$ and

$$\mathfrak{u}^*(IAI) \subseteq \mathfrak{u}^*(\mathcal{L}^*A\mathcal{L}) \subseteq \mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) \subseteq I,$$

so I is a regular inner ideal. □

Proposition 4.12. *Let A be a finite dimensional semisimple associative algebra and let $L = [A, A]$. Then every proper inner ideal I of L is regular. More exactly, $I = \mathcal{R}\mathcal{L}$ ($= \mathcal{L} \cap \mathcal{R}$) where $\mathcal{L}$ is a left ideal of A and $\mathcal{R}$ is a right ideal of A with $\mathcal{L}\mathcal{R} = 0$.*

Proof. Suppose I is an inner ideal of L. Note that L is semisimple. Therefore by Propositions 2.20 and 2.17(1), $I = eAf$ for a pair of idempotents e and f of A such that $fe = 0$. Define $\mathcal{L} = Af$ and $\mathcal{R} = eA$. Then

$$\mathcal{R}\mathcal{L} = eAAf = eAf = I,$$

as required. □

Recall that every simple diagonal locally finite Lie algebra can be represented as $\mathfrak{su}^*(A)$ where A is an involution simple locally finite associative algebra (A is actually unique and called the $\mathfrak{P}^*$-enveloping algebra of L, see introduction). Moreover, if L is plain then $L = [A, A]$ where A is simple. Thus, the following theorem.

Theorem 4.13. *(1) Let A be a simple locally finite associative algebra and let $(A_\alpha)_{\alpha\in\Gamma}$ be a perfect conical local system for A of rank > 4. Let $L = [A, A]$ and let I be a proper inner ideal of L. Put $L_\alpha = [A_\alpha, A_\alpha]$ and $I_\alpha = I \cap L_\alpha$. Let $\overline{I_\alpha}$ be the* *image of* I_α *in* $\overline{L_\alpha} = L_\alpha / \operatorname{Rad} L_\alpha$*. Then* $I^2 = 0$ *and for every* $\alpha \in \Gamma$*,* $\overline{I_\alpha}$ *is a regular inner ideal of* $\overline{L_\alpha}$*.*

(2) Let A *be an involution simple locally finite associative algebra and let* $(A_\alpha)_{\alpha\in\Gamma}$ *be a perfect conical* $*$*-invariant local system for* A *of rank* > 36*. Let* $L = \mathfrak{su}^*(A)$ *and let* I *be an inner ideal of* L*. Put* $L_\alpha = \mathfrak{su}^*(A_\alpha)$ *and* $I_\alpha = I \cap L_\alpha$*. Let* $\overline{I_\alpha}$ *be the image of* I_α *in* $\overline{L_\alpha} = L_\alpha / \operatorname{Rad} L_\alpha$*. If* A *is not finitary with orthogonal involution (i.e.* L *is not finitary orthogonal) then* $I^2 = 0$ *and there is* $\alpha_0 \in \Gamma$ *such that for every* $\alpha \geq \alpha_0$*,* $\overline{I_\alpha}$ *is a regular inner ideal of* $\overline{L_\alpha}$*.*

Proof. (1) By [8, Theorem 6.3(1)] (see also [1, Theorem 2.12(1)] and its proof), $(L_\alpha)_{\alpha\in\Gamma}$ is a perfect conical local system for L. By Proposition 2.16 I_α is an inner ideal of L_α. By Lemma 2.5, for every α there exists β such that $\operatorname{Rad} A_\beta \cap A_\alpha = 0$. Let $^- : A_\beta \to A_\beta / \operatorname{Rad} A_\beta$ be the canonical surjection. Note that $\operatorname{Rad} L_\beta \subseteq \operatorname{Rad} A_\beta$, $\overline{L_\beta} = [\overline{A_\beta}, \overline{A_\beta}]$, and by Lemma 2.16, $\overline{I_\beta}$ is an inner ideal of $\overline{L_\beta}$. Moreover $\overline{I_\beta}$ is regular by Proposition 4.12 and $\overline{I_\beta}^2 = 0$. Since $A_\alpha \cap \operatorname{Rad} A_\beta = 0$, $\overline{A}_\alpha \cong A_\alpha$, so A_α can be considered as a subalgebra of $\overline{A_\beta}$ and $I_\alpha \subseteq \overline{I_\beta}$. Therefore $I_\alpha^2 \subseteq \overline{I_\beta}^2 = 0$, so $I_\alpha^2 = 0$. Since $I = \varinjlim I_\alpha$, we conclude that $I^2 = 0$. This implies that $\overline{I_\alpha}$ is a proper inner ideal of $\overline{L_\alpha}$ for every $\alpha \in \Gamma$, so $\overline{I_\alpha}$ is regular by Proposition 4.12.

(2) By [9, Theorem 6.3] (see also [1, Theorem 2.12(2)] and its proof), $(L_\alpha)_{\alpha\in\Gamma}$ is a perfect conical local system for L. By Proposition 2.16, I_α is an inner ideal of L_α. Assume first that A is not simple. Then by Lemma 4.5 $A = B \oplus B^*$ where B is a simple ideal of A. Moreover, if φ is the projection of A on B then φ is an isomorphism of the Lie algebras $\mathfrak{su}^*(A)$ and $[B, B]$ and the result follows from part (1) of the theorem. Thus, we can suppose that A is simple.

Assume now that L is finitary. Since A is simple and the involution is not orthogonal, it must be symplectic. Therefore there is a local system $(S_\delta)_{\delta\in\Delta}$ of naturally embedded finite dimensional symplectic subalgebras of L. Fix any $\delta \in \Delta$ and $\alpha_0 \in \Gamma$ such that S_δ is of rank > 10 and $L_1 \subseteq S_\delta \subseteq L_{\alpha_0}$. We claim that $\overline{L_\alpha} = L_\alpha / \operatorname{Rad} L_\alpha$

is symplectic for all $\alpha \geq \alpha_0$. Indeed, consider any Levi subalgebra Q of L_α which contains S_δ and fix δ' such that $Q \subseteq S_{\delta'}$. We have a chain of embeddings

$$L_1 \subseteq S_\delta \subseteq Q \subseteq S_{\delta'}.$$

Since the embedding $S_\delta \subseteq S_{\delta'}$ is diagonal, by Lemma 2.1, both embeddings $S_\delta \subseteq Q$ and $Q \subseteq S_{\delta'}$ are diagonal. Moreover, since $S_\delta \subseteq S_{\delta'}$ is natural, Q must be simple and both embeddings $S_\delta \subseteq Q$ and $Q \subseteq S_{\delta'}$ must be natural. This implies that Q is symplectic (see for example [10, Proposition 2.3]), so $\overline{L_\alpha} \cong Q$ is symplectic. Therefore $\overline{I_\alpha}$ is a regular inner ideal of $\overline{L_\alpha}$ and $\overline{I_\alpha}^2 = 0$ for all $\alpha \geq \alpha_0$. As in the proof of part (1), fix any β such that $\operatorname{Rad} A_\beta \cap A_\alpha = 0$. Then $I_\alpha^2 \subseteq \overline{I_\beta}^2 = 0$, so $I^2 = 0$.

Suppose now that L is not finitary. First we are going to show that $I^2 = 0$. Assume $I^2 \neq 0$. Fix any $x, y \in I$ such that $xy \neq 0$. Since $I^2 = \varinjlim I_\alpha^2$, there is $\beta \in \Gamma$ such that $x, y \in I_\gamma$ and $xy \notin \operatorname{Rad} A_\gamma$ for all $\gamma \geq \beta$. Let M be a $*$-invariant maximal ideal of A_γ with $xy \notin M$. Note that $Q = A_\gamma/M$ is involution simple and $K = \mathfrak{su}^*(Q)$ is isomorphic to one of the simple components of $L_\gamma/\operatorname{Rad} L_\gamma$. Let V be the corresponding natural module for K and L_γ and let J be the image of I_γ in K. Then J is an inner ideal of K. Since $xy \notin M$, J^2 is nonzero in Q. Therefore J is as in Theorem 2.17(3)(ii), i.e. spanned by the matrix units $e_{1j} - e_{j2}$, $j \geq 3$. In particular, x (and y) is of rank 2 on V. Thus x acts as zero or a rank 2 linear transformation on every natural L_γ-module for all $\gamma \geq \beta$. Therefore by Theorem 2.14, L is finitary, which contradicts the assumption.

Fix any non-zero $x \in I$ and any $\beta \in \Gamma$ such that $x \in I_\gamma$ and $x \notin \operatorname{Rad} L_\gamma$ for all $\gamma \geq \beta$. One can also assume that x is of rank greater than 2 on some natural L_β-module V (otherwise L is finitary by Theorem 2.14). Let Q be the corresponding simple component of a Levi subalgebra of L_β (so V is a natural Q-module). By Corollary 2.11 there exists $\alpha_0 \in \Gamma$ such that for all $\alpha \geq \alpha_0$ the restriction of every natural L_α-module W to Q has a non-trivial composition factor. Since the embedding

$L_\beta \subseteq L_\alpha$ is diagonal, this implies that the restriction of W to L_β contains V or V^* as a composition factor, so rank of x on W is greater than 2. Let M be the annihilator of W in L_α. Then M is a maximal ideal. Note that the image J of I_α in $S = L_\alpha/M$ is a regular inner ideal of S because it contains the non-zero image of x and rank of x is greater than 2 on the natural S-module W. Since the intersection of all maximal ideals of L_α is the radical of L_α this implies that $\overline{I_\alpha}$ is a regular inner ideal of $\overline{L_\alpha}$. □

Remark 4.14. As it was reported to us by Antonio Fernández López, $I^2 = 0$ implies in this case that I is a Jordan-Lie inner ideal, as defined in [21]. Thus Theorem 4.13 shows that every inner ideal of a simple diagonal locally finite Lie algebra is Jordan-Lie.

We say that an associative algebra with involution is $*$-*locally semisimple* if it has a local system of $*$-invariant semisimple finite dimensional subalgebras.

Proposition 4.15. *(1) Let L be a simple diagonal Lie algebra and let A be its involution simple associative $\mathfrak{P}^*$-envelope. Then L is locally semisimple if and only if A is $*$-locally semisimple.*

(2) Let L be a simple plain Lie algebra and let A be its simple associative $\mathfrak{P}$-envelope. Then L is locally semisimple if and only if A is locally semisimple.

Proof. We will only prove the first part. The proof of the second statement is similar. Assume first that A is $*$-locally semisimple. Then A has a local system $(A_\alpha)_{\alpha\in\Gamma}$ such that all A_α are $*$-invariant semisimple finite dimensional algebras. Let $L_\alpha = \mathfrak{su}^*(A_\alpha)$. Then L_α is a semisimple finite dimensional Lie algebra for each α (see [9, Lemma 2.3] for example). Therefore $(L_\alpha)_{\alpha\in\Gamma}$ is a semisimple local system for L and L is locally semisimple.

Assume now that L is locally semisimple. By Proposition 2.13, L has a diagonal semisimple conical local system $(L_\alpha)_{\alpha\in\Gamma}$ of rank >10. It follows from the construction of A as a quotient of the universal enveloping algebra $U(L)$ by the annihilator of a diagonal inductive system for L (see proof of [1, Theorem 1.3]) that A is $*$-locally semisimple. □

Corollary 4.16. *(1) Let L be a simple plain Lie algebra and let A be its simple associative $\mathfrak{P}$-envelope, so $L = [A, A]$. Assume that L is locally semisimple. Then the following hold.*

(i) A is locally semisimple and von Neumann regular.

(ii) Every proper inner ideal I of L is regular, i.e. $I = \mathcal{RL}$ $(= \mathcal{L} \cap \mathcal{R})$ where $\mathcal{L}$ is a left ideal of A and $\mathcal{R}$ is a right ideal of A with $\mathcal{LR} = 0$.

(iii) A subspace I of L is a proper inner ideal of L if and only if $I = \varinjlim e_\alpha A f_\alpha$ where $\{e_\alpha, f_\alpha \mid \alpha \in B\}$ is a directed system of idempotents in A such that $f_\alpha e_\alpha = 0$, $e_\beta e_\alpha = e_\alpha$ and $f_\alpha f_\beta = f_\alpha$ for all α, β with $\alpha \leq \beta$.

(2) Let L be a simple diagonal Lie algebra and let A be its involution simple associative $\mathfrak{P}^*$-envelope, so $L = \mathfrak{su}^*(A)$. Assume that L is locally semisimple. Then the following hold.

(i) A is $*$-locally semisimple and von Neumann regular.

(ii) If L is not finitary orthogonal then every proper inner ideal I of L is regular, i.e. $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ $(= \mathfrak{u}^*(\mathcal{L}^* \cap \mathcal{L}))$ where $\mathcal{L}$ is a left ideal of A with $\mathcal{LL}^* = 0$.

(iii) If L is not finitary orthogonal then a subspace I of L is an inner ideal of L if and only if $I = \varinjlim \mathfrak{u}^*(e_\alpha A e_\alpha^*)$ where $\{e_\alpha \mid \alpha \in B\}$ is a directed system of idempotents in A such that $e_\alpha^* e_\alpha = 0$ and $e_\beta e_\alpha = e_\alpha$ for all α, β with $\alpha \leq \beta$.

Proof. We will prove part (2) only. Proof of part (1) is similar.

(i) By Proposition 4.15, A is $*$-locally semisimple, so von Neumann regular.

(ii) Let $(A_\alpha)_{\alpha\in\Gamma}$ be a $*$-invariant semisimple local system for A. By [1, 2.9-2.11] we can assume that this local system is conical of rank > 36. Then the Lie algebras $L_\alpha = \mathfrak{su}^*(A_\alpha)$ are semisimple for all α and $(L_\alpha)_{\alpha\in\Gamma}$ is a conical local system of L. Let I be any inner ideal of L and let $I_\alpha = I \cap L_\alpha$. By Theorem 4.13(2), there is $\alpha_0 \in \Gamma$

such that $I_\alpha = \overline{I_\alpha}$ is a regular inner ideal of L_α for all $\alpha \geq \alpha_0$. We need to show that I is regular, i.e. $\mathfrak{u}^*(IAI) \subseteq I$. Consider any element $x \in \mathfrak{u}^*(IAI)$. Then there exists $\alpha \geq \alpha_0$ such that $x \in \mathfrak{u}^*(I_\alpha A_\alpha I_\alpha)$. Since I_α is a regular inner ideal, $x \in I_\alpha \subseteq I$. Hence I is a regular inner ideal. By Proposition 4.11 all regular inner ideals of L are of the shape $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ ($= \mathfrak{u}^*(\mathcal{L}^* \cap \mathcal{L})$) where $\mathcal{L}$ is a left ideal of A with $\mathcal{L}\mathcal{L}^* = 0$.

(iii) Assume first that I is an inner ideal of L and let $(A_\alpha)_{\alpha\in\Gamma}$ be a $*$-invariant semisimple local system for A. Then I is regular by part (ii), so $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ where $\mathcal{L}$ is a left ideal of A with $\mathcal{L}\mathcal{L}^* = 0$. Let $\mathcal{L}_\alpha = \mathcal{L} \cap A_\alpha$. Since every one-sided ideal of a finite dimensional semisimple algebra is generated by an idempotent, $\mathcal{L}_\alpha = A_\alpha e_\alpha^*$ and $\mathcal{L}_\alpha^* = e_\alpha A_\alpha$ for some idempotent e_α of A_α. We claim that the system $\{e_\alpha \mid \alpha \in \Gamma\}$ satisfies the required conditions. Let $\beta \geq \alpha$. Recall that A_α is semisimple so it contains identity $\mathbf{1}$, so $e_\alpha = e_\alpha \mathbf{1} \in e_\alpha A_\alpha \subseteq e_\beta A_\beta$. Since $e_\beta x = x$ for all $x \in \mathcal{L}_\beta^* = e_\beta A_\beta$ we have that $e_\beta e_\alpha = e_\alpha$. Also we have

$$e_\alpha^* e_\alpha \in A_\alpha e_\alpha^* e_\alpha A_\alpha = \mathcal{L}_\alpha \mathcal{L}_\alpha^* \subseteq \mathcal{L}\mathcal{L}^* = 0$$

so $e_\alpha^* e_\alpha = 0$. Note that $e_\alpha A_\beta = e_\beta e_\alpha A_\beta \subseteq e_\beta A_\beta$ for all $\beta \geq \alpha$, so $e_\alpha A_\beta e_\alpha^* \subseteq e_\beta A_\beta e_\beta^*$. Therefore

$$I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) = \varinjlim \mathfrak{u}^*(\mathcal{L}_\alpha^*\mathcal{L}_\alpha) = \varinjlim \mathfrak{u}^*(e_\alpha A_\alpha e_\alpha^*) = \varinjlim \mathfrak{u}^*(e_\alpha A e_\alpha^*),$$

as required.

Assume now that $\{e_\alpha \mid \alpha \in B\}$ is a directed system of idempotents in A such that $e_\alpha^* e_\alpha = 0$ and $e_\beta e_\alpha = e_\alpha$ for all α, β with $\alpha \leq \beta$. Then $e_\alpha A$ is a right ideal of A and $e_\alpha A = e_\beta e_\alpha A \subseteq e_\beta A$ for all $\beta \geq \alpha$. Therefore the one-sided ideals $\mathcal{L} = \varinjlim A e_\alpha^*$ and $\mathcal{L}^* = \varinjlim e_\alpha A$ are well-defined. Note that $\mathcal{L}\mathcal{L}^* = 0$, so $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ is a regular inner ideal of L by Proposition 4.11. $\square$

5. Finitary Lie algebras

Recall that an algebra is called *finitary* if it consists of finite-rank linear transformations of a vector space. First we define the classical finitary simple Lie algebras, see [6] and [16] for details.

A pair of dual vector spaces (X, Y, g) consists of vector spaces X and Y over F and a non-degenerate bilinear form $g : X \times Y \to F$. A linear transformation $a : X \to X$ is *continuous* (relative to Y) if there exists $a^{\#} : Y \to Y$, necessarily unique, such that $g(ax, y) = g(x, a^{\#}y)$ for all $x \in X$, $y \in Y$. Note that Y can be identified with a *total subspace* (i.e. $\mathrm{Ann}_X Y = 0$) of the dual vector space X^*. In that case $a^{\#}\varphi = \varphi a$ for all $\varphi \in X^*$ and a is continuous if and only if $a^{\#}Y \subseteq Y$.

Denote by $\mathcal{F}(X, Y)$ the algebra of all continuous (relative to Y) finite rank linear transformations of X. Then $\mathcal{F}(X, Y)$ is a simple associative algebra with minimal left ideals. For $u \in X$, $w \in Y$ we denote by w^*u the linear transformation $w^*u(x) = g(x, w)u$, $x \in X$, and for subspaces $U \subseteq X$ and $W \subseteq Y$ we denote by W^*U the set of all finite sums of $w_i^*u_i$, $u_i \in U$, $w_i \in W$. Note that

$$(y_2^*x_2)(y_1^*x_1) = g(x_1, y_2)y_1^*x_2,$$

for $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\mathcal{F}(X, Y) = Y^*X$.

The *finitary special linear Lie algebra* $\mathfrak{fsl}(X, Y)$ is defined to be $[\mathcal{F}(X, Y), \mathcal{F}(X, Y)]$.

Let Φ be a nondegenerate symmetric or skew-symmetric form on X, $\Phi(y, x) = \epsilon\Phi(x, y)$, $\epsilon = \pm 1$. Then X becomes a self-dual vector space with respect to Φ and the algebra $\mathcal{F}(X, X)$ of continuous linear transformations on X has an involution $a \mapsto a^*$ given by $\Phi(ax, y) = \Phi(x, a^*y)$, for all $x, y \in X$. As before, we denote by $\mathfrak{u}^*(\mathcal{F}(X, X)) = \{a \in \mathcal{F}(X, X) \mid a^* = -a\}$ the set of skew-symmetric elements of $\mathcal{F}(X, X)$ and by $\mathfrak{su}^*(\mathcal{F}(X, X))$ its derived subalgebra. For $x, y \in X$, define $[x, y] = x^*y - \epsilon y^*x \in \mathcal{F}(X, X)$. One can check that $(x^*y)^* = \epsilon y^*x$, so $[x, y] \in \mathfrak{u}^*(\mathcal{F}(X, X))$. If U, W are subspaces of X, then $[U, W]$ will denote the set of all finite sums of $[u_i, w_i]$,

$u_i \in U$, $w_i \in W$. Note that

$$\begin{aligned} \mathfrak{u}^*(\mathcal{F}(X,X)) &= \{b - b^* \mid b \in \mathcal{F}(X,X)\} \\ &= \{x^*y - \epsilon y^*x \mid x, y \in X\} \\ &= [X,X]. \end{aligned}$$

If Φ is a symmetric bilinear form, then $\mathfrak{u}^*(\mathcal{F}(X,X)) = \mathfrak{su}^*(\mathcal{F}(X,X))$ is the *finitary orthogonal algebra* $\mathfrak{fo}(X,\Phi)$.

If Φ is a skew-symmetric bilinear form, then $\mathfrak{u}^*(\mathcal{F}(X,X)) = \mathfrak{su}^*(\mathcal{F}(X,X))$ is the *finitary symplectic algebra* $\mathfrak{fsp}(X,\Phi)$.

Theorem 5.1. [6, Corollary 1.2] *Any infinite dimensional finitary simple Lie algebra over F is isomorphic to one of the following:*

(1) A finitary special linear Lie algebra $\mathfrak{fsl}(X,Y)$.

(2) A finitary symplectic algebra $\mathfrak{fsp}(X,\Phi)$.

(3) A finitary orthogonal algebra $\mathfrak{fo}(X,\Phi)$.

In [7] this result was extended to positive characteristic.

The classification of inner ideals of finitary simple Lie algebras was first obtained by López, García and Lozano [16] (over arbitrary fields of characteristic zero), with Benkart and López [14] settling later the missing case for orthogonal algebras. We provide an alternative proof for the case of special linear and symplectic algebras over an algebraically closed field of characteristic zero. In the case of orthogonal algebras we can only describe regular inner ideals.

Theorem 5.2. [16, results 2.5, 3.6, 3.8][14, Theorem 6.6] *Let (X,Y,g) be a dual pair of infinite dimensional vector spaces over F and let Φ (resp. Ψ) be a nondegenerate symmetric (resp. skew-symmetric) form on X.*

(1) A subspace I is a proper inner ideal of $\mathfrak{fsl}(X,Y)$ if and only if $I = W^*U$ where the subspaces $U \subseteq X$ and $W \subseteq Y$ are mutually orthogonal (i.e. $g(U,W) = 0$) (or equivalently, I is a regular inner ideal).

(2) A subspace I is a proper inner ideal of $\mathfrak{fsp}(X,\Psi)$ if and only if $I = [U,U]$ for some totally isotropic subspace U of X (i.e. $\Psi(U,U) = 0$) (or equivalently, I is a regular inner ideal).

(3) A subspace I is a proper inner ideal of $\mathfrak{fo}(X,\Phi)$ if and only if I satisfies one of the following.

(i) $I = [U,U]$ for some totally isotropic subspace $U \subseteq X$ (or equivalently, I is a regular inner ideal).

(ii) I is a Type 1 point space of dimension greater than 1.

(iii) $I = [x, H^\perp]$ where H is a 2-dimensional subspace of X such that the restriction of Φ to H is nondegenerate and x is a non-zero isotropic vector in H.

Proof. Note that the simple infinite dimensional finitary Lie algebras are locally semisimple Lie algebras, so we can use Theorem 4.16. The associative algebras $\mathcal{F}(X,Y)$ are simple, with minimal one-sided ideals, and, in particular, they are locally finite dimensional (see for example [20, Theorem 4.15.3]).

(1) Recall that $\mathfrak{fsl}(X,Y) = [\mathcal{F}(X,Y), \mathcal{F}(X,Y)]$. In particular, $\mathfrak{fsl}(X,Y)$ is plain and $\mathcal{F}(X,Y)$ is its simple associative $\mathfrak{P}$-envelope. By Corollary 4.16(1) a subspace I of $\mathfrak{fsl}(X,Y)$ is a proper inner ideal if and only if it is a regular inner ideal, i.e. there exists a left ideal and a right ideal of $\mathcal{F}(X,Y)$, say $\mathcal{L}$ and $\mathcal{R}$, such that $I = \mathcal{RL} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{LR} = 0$. By [20, Theorem 4.16.1], every right ideal of $\mathcal{F}(X,Y)$ is of the shape $\mathcal{R} = Y^*U = \{a \in \mathcal{F}(X,Y) \mid aX \subseteq U\}$ for some subspace $U \subseteq X$ and every left ideal is of the shape $\mathcal{L} = W^*X = \{a \in \mathcal{F}(X,Y) \mid a^\# Y \subseteq W\}$ for some subspace $W \subseteq Y$. Then

$$0 = \mathcal{LR} = (W^*X)(Y^*U) = g(U,W)Y^*X$$

if and only if $g(U,W)=0$. And

$$I=\mathcal{R}\mathcal{L}=(Y^*U)(W^*X)=g(X,Y)W^*U=W^*U$$

(2) Recall $\mathfrak{fsp}(X,\Psi)=\mathfrak{u}^*(\mathcal{F}(X,X))=\mathfrak{su}^*(\mathcal{F}(X,X))=[X,X]$. In particular $\mathfrak{fsp}(X,\Psi)$ is diagonal and $\mathcal{F}(X,X)$ is its simple associative $\mathfrak{P}^*$-envelope. By Corollary 4.16(2) a subspace I of $\mathfrak{su}^*(\mathcal{F}(X,X))$ is a proper inner ideal if and only if it is a regular inner ideal, i.e. $I=\mathfrak{u}^*(\mathcal{R}\mathcal{R}^*)$ for some right ideal $\mathcal{R}$ of $\mathcal{F}(X,X)$. As in part (1), every right ideal is of the shape $\mathcal{R}=X^*U=\{a\in\mathcal{F}(X,X)\mid aX\subseteq U\}$ for some subspace U of X. Therefore $\mathcal{R}^*=U^*X=\{a\in\mathcal{F}(X,X)\mid a^*X\subseteq U\}$ and this is a left ideal. One has $\mathcal{R}^*\mathcal{R}=0$ if and only if $\Phi(U,U)=0$, i.e. U is a totally isotropic subspace. Now

$$I=\mathfrak{u}^*(\mathcal{R}\mathcal{R}^*)=\mathfrak{u}^*((X^*U)(U^*X))=\mathfrak{u}^*(\Phi(X,X)U^*U)$$
$$=\mathfrak{u}^*(U^*U)=\{a-a^*\mid a\in U^*U\}=[U,U],$$

as required.

(3) Recall

$$\mathfrak{fo}(X,\Phi)=\mathfrak{u}^*(\mathcal{F}(X,X))=\mathfrak{su}^*(\mathcal{F}(X,X))=[X,X].$$

In particular $\mathfrak{fo}(X,\Phi)$ is diagonal and $\mathcal{F}(X,X)$ is its simple associative $\mathfrak{P}^*$-envelope. By Corollary 4.16(2), $\mathcal{F}(X,X)$ is von Neumann regular. Then by Proposition 4.11, I is a regular inner ideal of $\mathfrak{fo}(X,\Phi)$ if and only if $I=u^*(\mathcal{R}\mathcal{R}^*)$ where $\mathcal{R}$ is a right ideal of $\mathcal{F}(X,X)$ with $\mathcal{R}^*\mathcal{R}=0$. As in the proof of part (2), this is equivalent to say that $I=[U,U]$ for some totally isotropic subspace $U\subseteq X$. The case of non-regular inner ideals in $\mathfrak{fo}(X,\Phi)$ is fully considered in [16, 2.5, 3.6, 3.8] and [14, Theorem 6.6]. $\square$

A Lie algebra is L is called *nondegenerate* if and only if $(ad\,x)^2 \neq 0$ for all nonzero $x \in L$.

Proposition 5.3. *Let L be a simple locally finite Lie algebra over F. Then L is nondegenerate.*

Proof. Assume L is simple. Let $x \in L$ nonzero. Let $L = \varinjlim L_\alpha$. Find L_α such that $x \in L_\alpha$ and $\beta > \alpha$ such that $L_\alpha \cap \operatorname{Rad} L_\beta = 0$. Let $^- := L_\beta \to L_\beta / \operatorname{Rad} L_\beta$. Then $\bar{x} \neq 0$. Since $\overline{L_\beta}$ is a finite dimensional semisimple Lie algebra, $\overline{L_\beta}$ is nondegenerate, so $(ad\,\overline{x})^2 \neq 0$, so $(ad\,x)^2 \neq 0$, hence L is nondegenerate. □

It follows from a general result, proved for nondegenerate Lie algebras by Draper, López, García and Lozano, that a simple locally finite Lie algebra contains proper minimal inner ideals if and only if it is finitary (see [15, Theorems 5.1 and 5.3]). As can be seen by Proposition 5.3 simple locally finite Lie algebras are nondegenerate. We are going to prove a version of this result for regular inner ideals. We will need the following facts.

Proposition 5.4. *Let A be a simple associative ring and let $L = [A, A]$. Then L has a minimal regular inner ideal if and only if A has a proper minimal left ideal.*

Proof. Suppose first that A has a proper minimal left ideal. Since A is simple with non-zero socle, by [20, 4.9], there is a pair (X, Y, g) of dual vector spaces over a division ring Δ such that A is isomorphic to the ring $\mathcal{F}(X, Y)$ of all continuous (relative to Y) finite rank linear transformations of X. Moreover, $\dim_\Delta X > 1$ (otherwise A is a division ring and doesn't have proper non-zero left ideals). Take any one-dimensional subspaces $W \subset Y$ and $V \subset X$ such that $g(V, W) = 0$. Then $I = W^*V$ will be a minimal regular inner ideal of $\mathfrak{fsl}(X, Y) = [\mathcal{F}(X, Y), \mathcal{F}(X, Y)]$ (see [16, Theorem 2.5] or Theorem 5.2(1) above for the case $\Delta = F$).

Suppose now that L has a minimal regular inner ideal I. Then $\mathcal{L} = AI$ (resp. $\mathcal{R} = IA$) is a left (resp. right) ideal of A. We claim that both ideals are non-zero.

Indeed, if, say, $AI = 0$, then IA is a two-sided ideal of A with $(IA)^2 = 0$. Since A is simple, this implies that $IA = 0$ and so I is a non-zero two-sided ideal of A, which is obviously a contradiction because $I^2 = 0$. Therefore $\mathcal{L} \neq 0$ and $\mathcal{R} \neq 0$. Note that $\mathcal{L}$ is a proper ideal of A (otherwise $A = AI = (AI)I = AI^2 = 0$). We claim that $\mathcal{L}$ is a minimal left ideal of A. Indeed, assume there exists a left ideal $\mathcal{L}_1$ of A such that $0 \neq \mathcal{L}_1 \subseteq \mathcal{L}$. By Proposition 4.9, $I_1 = \mathcal{R}\mathcal{L}_1$ is a regular inner ideal of L and it is non-zero by Lemma 4.7(4). Note that

$$I_1 = \mathcal{R}\mathcal{L}_1 \subseteq IAAI \subseteq I$$

Since I is minimal, $I_1 = I$. Therefore $\mathcal{L}_1 \supseteq A\mathcal{R}\mathcal{L}_1 = AI_1 = AI = \mathcal{L}$, which is a contradiction. $\square$

A similar result holds for rings with involutions. We need the following analogue of Lemma 4.7(4).

Lemma 5.5. *Let A be a simple associative ring with involution and let $\mathcal{L}$ be a non-zero left ideal of A such that $\mathcal{L}\mathcal{L}^* = 0$. Assume that the socle of A is zero, i.e. A doesn't have minimal left ideals. Then $\mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ is non-zero.*

Proof. Assume to the contrary that $\mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) = 0$. Take any non-zero $a \in \mathcal{L}$. Then $a^*Aa \subseteq \mathcal{L}^*\mathcal{L}$, so $\mathfrak{u}^*(a^*Aa) = 0$. Note that $a^*(x - x^*)a \in \mathfrak{u}^*(a^*Aa)$ for all $x \in A$. Therefore $a^*(x - x^*)a = 0$ for all $x \in A$, i.e. A satisfies a non-trivial generalized identity with involution. Therefore A has a non-zero socle (see for example [11, 6.2.4 and 6.1.6]), which is a contradiction. $\square$

Proposition 5.6. *Let A be an infinite dimensional simple associative algebra over F with involution and let $L = \mathfrak{su}^*(A)$. Then L has a minimal regular inner ideal if and only if A has a proper minimal left ideal.*

Proof. ($\Leftarrow$) Since A is simple with non-zero socle, by [20, 4.9, 4.12], $A = \mathcal{F}(X, X)$ where X is a self-dual vector space over F with respect to a nondegenerate symmetric or skew-symmetric form Φ and the involution $a \mapsto a^*$ of A is given by $\Phi(ax, y) = \Phi(x, a^*y)$, for all $x, y \in X$. Assume first that Φ is skew-symmetric. Then $L = \mathfrak{su}^*(A) = \mathfrak{fsp}(X, \Phi)$. Take any non-zero isotropic vector v in X. Then

$$I = [Fv, Fv] = F[v, v] = F(v^*v + v^*v) = Fv^*v$$

is a one-dimensional regular inner ideal by Theorem 5.2(2). Assume now that Φ is symmetric. Then $L = \mathfrak{su}^*(A) = \mathfrak{fo}(X, \Phi)$ Take any two-dimensional totally isotropic subspace U of X (this is always possible because the ground field F is algebraically closed) and let $\{x, y\}$ be its basis. Then $I = [U, U] = F[x, y]$ is again a one-dimensional regular inner ideal by Theorem 5.2(3)(i). So in both cases there exists one-dimensional (hence minimal) regular inner ideal.

($\Rightarrow$) Let I be a minimal regular inner ideal of L and assume that A has no proper minimal left ideals. By Proposition 4.11, there exists a left ideal $\mathcal{L}$ of A such that $\mathcal{L}\mathcal{L}^* = 0$ and $\mathfrak{u}^*(\mathcal{L}^*\mathcal{L}) \subseteq I \subseteq \mathcal{L}^* \cap \mathcal{L} \cap L$. Note that $\mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$ is a non-zero regular inner ideal by Lemma 5.5, so $I = \mathfrak{u}^*(\mathcal{L}^*\mathcal{L})$. Let $x \in I$ be a non-zero element. We claim that there exists a left ideal $\mathcal{L}_1$ of A such that $0 \neq \mathcal{L}_1 \subset \mathcal{L}$, $x \notin \mathcal{L}_1$. Indeed, suppose x is an element in every non-zero left ideal contained in $\mathcal{L}$. Let $\mathcal{J} = \bigcap\{$ non-zero left ideals $\mathcal{H} \mid \mathcal{H} \subset \mathcal{L}\}$. Then $x \in \mathcal{J}$, so $\mathcal{J}$ is non-zero. It is clear that $\mathcal{J}$ is a minimal left ideal of A giving a contradiction. By Proposition 4.11 and Lemma 5.5, $I_1 = \mathfrak{u}^*(\mathcal{L}_1^*\mathcal{L}_1)$ is a non-zero regular inner ideal of L. Note that $I_1 \subseteq \mathcal{L}_1$, so $x \notin I_1$. Therefore I_1 is properly contained in I. Hence I is not minimal. □

Corollary 5.7. *Let L be an infinite dimensional locally finite simple Lie algebra over F. Then L is finitary if and only if it has a minimal regular inner ideal.*

Proof. Suppose first that L is finitary. Then by Theorem 5.1, $L = [\mathcal{F}(X,Y), \mathcal{F}(X,Y)]$ or $\mathfrak{su}^*(\mathcal{F}(X,X))$. Both $\mathcal{F}(X,Y)$ and $\mathcal{F}(X,X)$ are infinite dimensional and have proper minimal left ideals. Therefore by Propositions 5.4 and 5.6, L has a minimal regular inner ideal.

Suppose now that L has a proper minimal regular inner ideal I. Since non-diagonal Lie algebras have no proper non-zero inner ideals (see Theorem 3.8), L must be diagonal. Therefore by [1, Section 1], L is either plain, i.e. $L = [A, A]$ for some simple locally finite associative algebra A, or L is non-plain diagonal and $L = \mathfrak{su}^*(A)$ for some simple locally finite associative algebra A with involution. By Propositions 5.4 and 5.6, A has a proper minimal left ideal. By [20, 4.9, 4.12], $A = \mathcal{F}(X,Y)$ or $\mathcal{F}(X,X)$, so L is finitary. $\square$

6. Inner ideals of finite dimensional non-semisimple Lie algebras

In this section the inner ideals of non-semisimple finite dimensional Lie algebras are studied in the case of $L = [A, A]$ for a finite dimensional associative algebra A with $(\operatorname{Rad} A)^2 = 0$. Define a finite dimensional associative algebra A to be *strongly perfect* if A is perfect and for every maximal ideal M of A one has $\dim A/M > 4$.

Lemma 6.1. *Let A be a strongly perfect associative algebra with $(\operatorname{Rad} A)^2 = 0$, and let Q be a Levi subalgebra of A. Put $L = [A, A]$, $S = [Q, Q]$ and $R = [Q, \operatorname{Rad} A]$. Then L is a perfect Lie algebra, $\operatorname{Rad} L = R$ and S is a Levi subalgebra of L.*

Proof. By [8, corollary 6.4], L is perfect. It remains to note that S is a semisimple Lie algebra, R is a solvable ideal of L (in fact, [R,R]=0 since $(\operatorname{Rad} A)^2 = 0$) and $[A, A] = [Q, Q] \oplus [Q, \operatorname{Rad} A] = S \oplus R$. □

Definition 6.2. Let I be an inner ideal of a finite dimensional Lie algebra L, and $R = \operatorname{Rad} L$. We say that I *splits* if there is a Levi subalgebra S of L such that $I = I_S \oplus I_R$ where $I_S = I \cap S$ and $I_R = I \cap R$.

The following theorem will be proven at the end of the section.

Theorem 6.3. *Let A be a strongly perfect finite dimensional associative algebra. Assume $(\operatorname{Rad} A)^2 = 0$. Let $L = [A, A]$ and let I be a proper inner ideal of L with $I^2 = 0$ then there is a Levi subalgebra S of L such that $I = I_S \oplus I_R$ where $I_S = I \cap S$ and $I_R = I \cap \operatorname{Rad} L$ (i.e. I splits).*

Lemma 6.4. *Let V be a vector space over F, and let e be an idempotent in $\operatorname{End} V$. Then there is a basis $\{e_1, \dots, e_n\}$ of V such that $e = diag(1, \dots, 1, 0, \dots, 0)$ in the corresponding matrix realization of $\operatorname{End} V$.*

Proof. Since $e = e^2$ it is easy to see that the Jordan Normal form of e must be a diagonal matrix and the only possible non zero entries are 1. □

The following simple facts are well known.

Lemma 6.5. *Let e, f be idempotents in* $\operatorname{End} V$ *with* $ef = fe = 0$. *Then there is a basis* $\{e_1, \ldots, e_n\}$ *of* V *such that* $e = diag(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$ *and* $f = diag(0, \ldots, 0, \underbrace{1, \ldots, 1}_{l})$ *with* $k + l \leq n$.

Proof. By Lemma 6.4, there is a basis $\{x_i\}$ of $\operatorname{End} V$ such that $e = diag(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$. Since $ef = fe = 0$ one has $f = \begin{pmatrix} 0 & \\ & X \end{pmatrix}$ where X is an $n-k$ by $n-k$ matrix. So write $V = V_1 \oplus V_2$ where e acts as zero on V_2 and the identity on V_1. Changing basis in V_2 using Lemma 6.4 we find the desired matrix realization of f without affecting that of e. □

Lemma 6.6. *Let A be a finite dimensional semisimple associative algebra. Let $L = [A, A]$ and I be an inner ideal of L. Then $I = eAf$ for some e and f idempotents of A with $fe = ef = 0$.*

Proof. By Theorem 2.17 and 2.20 $I = e_1 A f_1$ where $e_1 = e_1^2$, $f_1^2 = f_1$, and $f_1 e_1 = 0$. Let $e_2 = 1 - e_1$ let $f_2 = e_2 f_1 e_2 = e_2 f_1 (1 - e_1) = e_2 f_1$. Then f_2 is an idempotent since

$$\begin{aligned} f_2^2 &= (e_2 f_1 e_2)(e_2 f_1 e_2) \\ &= e_2 f_1 e_2 f_1 e_2 \\ &= e_2 f_1 (1 - e_1) f_1 e_2 \\ &= e_2 f_1 e_2 = f_2. \end{aligned}$$

Note that $e_1 f_2 = f_2 e_1 = 0$ so it remains to show that $I = e_1 A f_2$. Indeed

$$e_1 A f_2 = e_1 A (1 - e_1) f_1 (1 - e_1) \subseteq e_1 A f_1.$$

Let $e_1af_1 \in I$ where $a \in A$, then

$$e_1af_1 = e_1af_1(1-e_1)f_1 = e_1(e_1af_1)f_2 \in e_1Af_2$$

so $I \subseteq e_1Af_2$. □

Lemma 6.7. *Let $A = Q \oplus \operatorname{Rad} A$ be an associative algebra with $Q \cong M_n(F)$ and I be an inner ideal of $L = [A, A]$. Let $R = \operatorname{Rad} L$ and let $\overline{I} = (I+R)/R$, the inner ideal of $\overline{L} = L/R$. Then there are integers $1 \leq k < l \leq n$ such that I is the space spanned by the matrix units $\{e_{ij} | 1 \leq i \leq k < l \leq j \leq n\} \subset Q$.*

Proof. By Lemma 2.16, $\overline{I} = (I+R)/R$ is an inner ideal of $\overline{L}$. Using Lemma 6.6 $\overline{I} = eAf$ where e, f are idempotents with $fe = ef = 0$. By Lemma 6.5 we easily identify $\overline{I}$ with the space spanned by $\{e_{ij} | 1 \leq i \leq k < l \leq j \leq n\} \subset Q$ for some integers k and l. □

Our aim now is to prove Theorem 6.3. Recall that A is a perfect finite dimensional associative algebra and I is a an inner ideal of A with $I^2 = 0$. Let Q be a Levi subalgebra of A and $R = \operatorname{Rad} A$, with $R^2 = 0$. Note that R is a Q-bimodule. We will first consider 3 special cases of the theorem in the following propositions. The completion of the proof will follow, using induction on the length of the bimodule R.

Proposition 6.8. *Theorem 6.3 holds in the case when A/R is simple and $RA = 0$, moreover I is regular.*

Proof. Let Q be a Levi subalgebra of A. Then $Q \cong A/R$ is simple and we can identify Q with $M_n(F)$ for some n. Since A is perfect, $QR = R$. This implies that R, as a left $Q-$module, is the direct sum of copies of the natural $n-$dimensional $Q-$module V. So we identify R with $V \otimes Z$ where Z is a finite dimensional vector space. Let $\{e_1, \ldots, e_n\}$ be the natural basis of V. Then

$$V \otimes Z = \{e_1 \otimes z_1 + \cdots + e_n \otimes z_n | z_i \in Z\}.$$

By Lemma 6.7 we identify the inner ideal $\overline{I} = (I + R)/R$ of $\overline{L}$ with the space spanned by the matrix units $\{e_{ij} | 1 \le i \le k < l \le j \le n\} \subset Q$ for some integers l and k. Assume $y = e_1 \otimes z_1 + \cdots + e_n \otimes z_n \in I \cap R$. Let $Z_y = span\{z_1, \ldots, z_n\} \subseteq Z$ and denote by

$$Z_I = \sum_{y \in I \cap R} Z_y.$$

We claim first $e_i \otimes z \in I$ for all $i \le k$ and $z \in Z_I$. Indeed consider again an element $y = e_1 \otimes z_1 + \cdots + e_n \otimes z_n \in I \cap R$. Fix any $r \in R$ such that $x = e_{in} + r \in I$, and consider $e_{nj} \in Q$. Then $ye_{nj}x = 0$, so $xe_{nj}y \in I$ by Lemma 4.1(1). So

$$xe_{nj}y = (e_{in} + r)e_{nj}(e_1 \otimes z_1 + \cdots + e_n \otimes z_n) = e_i \otimes z_j \in I \tag{6.1}$$

for all $1 \le i \le k$ and $1 \le j \le n$, hence $V_1 \otimes Z_I \subseteq I$ where $V_1 = span\{e_1, \ldots, e_k\}$. We claim

$$I \cap V_1 \otimes Z = V_1 \otimes Z_I. \tag{6.2}$$

Indeed, obviously $I \cap V_1 \otimes Z \supseteq V_1 \otimes Z_I$. Now let $y \in I \cap V_1 \otimes Z$, then $y = e_1 \otimes z_1 + \cdots + e_n \otimes z_n$. Since $y \in V_1 \otimes Z$, $z_{k+1} = \cdots = z_n = 0$. Also $y \in I \cap R$ giving $z_i \in Z_I$. So $y \in V_1 \otimes Z_I$. Let $Z^\perp$ be a subspace of Z such that $Z = Z^\perp \oplus Z_I$, so $R = V \otimes Z^\perp \oplus V \otimes Z_I$. Fix any subset

$$\{x_{ij}^{(1)} = e_{ij} + r_{ij}^{(1)} | 1 \le i \le k < l \le j \le n\} \subseteq I.$$

Then by Lemma 4.1(1)

$$x_{ij}^{(2)} = x_{ij}^{(1)} e_{ji} x_{ij}^{(1)} = e_{ij} e_{ji} (e_{ij} + r_{ij}^{(1)}) = e_{ij} + e_{ii} r_{ij}^{(1)} \in I$$

Let $r_{ij}^{(2)} = e_{ii} r_{ij}^{(1)}$. Then $r_{ij}^{(2)} \in e_i \otimes Z \subseteq V_1 \otimes Z$ for $(1 \le i \le k)$. Hence $x_{ij}^{(2)} = e_{ij} + r_{ij}^{(2)}$ with $r_{ij}^{(2)} \in e_i \otimes Z$. Note $e_i \otimes Z = e_i \otimes Z^\perp \oplus e_i \otimes Z_I$. So $r_{ij}^{(2)} = r_{ij}^{(3)} + r_{ij}^{(3)'}$ where

$r_{ij}^{(3)} \in e_i \otimes Z^{\perp}$ and $r_{ij}^{(3)'} \in e_i \otimes Z_I \subseteq I$. Put $x_{ij}^{(3)} = x_{ij}^{(2)} - r_{ij}^{(3)'} = e_{ij} + r_{ij}^{(3)} \in I$, where $r_{ij}^{(3)} = e_i \otimes z_{ij}$ with $z_{ij} \in Z^{\perp}$. So $x_{ij}^{(3)} = e_{ij} + e_i \otimes z_{ij}$. Note that for $x_{ij}^{(3)}$, $x_{sj}^{(3)}$ we have

$$\begin{aligned} X &= x_{ij}^{(3)} e_{js} x_{sj}^{(3)} + x_{sj}^{(3)} e_{js} x_{ij}^{(3)} \\ &= (e_{ij} + e_i \otimes z_{ij}) e_{js} (e_{sj} + e_s \otimes z_{sj}) + (e_{sj} + e_s \otimes z_{sj}) e_{js} (e_{ij} + e_i \otimes z_{ij}) \\ &= e_{is}(e_{sj} + e_s \otimes z_{sj}) + e_{ss}(e_{ij} + e_i \otimes z_{ij}) \\ &= e_{ij} + e_i \otimes z_{sj} \in I. \end{aligned}$$

Therefore by equation 6.2

$$x_{ij}^{(3)} - X = e_i \otimes (z_{ij} - z_{sj}) \in I \cap V_1 \otimes Z^{\perp} = 0.$$

So $z_{ij} = z_{sj}$. For each $x_{ij}^{(3)}$ and $x_{sj}^{(3)}$ we have $z_{ij} = z_{sj}$, so denote $z_{ij} = z_j$ for all i. Therefore

$$\{x_{ij} = e_{ij} + e_i \otimes z_j | 1 \leq i \leq k < l \leq j \leq n\} \subset I.$$

Let

$$q = \sum_{j=l}^{n} e_j \otimes z_j \in R.$$

Note $q^2 = 0$, so the map $\varphi : A \to A$ given by $\varphi(a) = (1+q)a(1-q)$ is a special automorphism of A and L. Note that $\varphi(x_{ij}) = e_{ij}$ for all i and j. Indeed

$$\begin{aligned} \varphi(x_{ij}) = (1 + \sum_{j=l}^{n} e_j \otimes z_j) x_{ij} (1 - \sum_{j=l}^{n} e_j \otimes z_j) &= (e_{ij} + e_i \otimes z_j)(1 - \sum_{j=l}^{n} e_j \otimes z_j) \\ &= e_{ij} + e_i \otimes z_j - e_i \otimes z_j = e_{ij}. \end{aligned}$$

Also note $\varphi(r) = r$ for all $r \in \operatorname{Rad} A$ and $\varphi(I)$, so $\varphi(I) = \varphi(I)_S \oplus \varphi(I)_R$ where $\varphi(I)_S = \varphi(I) \cap S$ and $\varphi(I)_R = \varphi(I) \cap R = I \cap R = I_R$. By changing Levi subalgebra Q to $\varphi^{-1}(Q)$ we get the required properties for I. It remains to prove that I is regular, i.e. $xay \in I$ for all $x, y \in I$ and for all $a \in A$. Denote by a_s (resp. a_r)

the projection of a on Q (resp. R) . We have shown I splits, so $x = x_s + x_r$ and $y = y_s + y_r$ where $x_s, y_s \in I_S$ and $x_r, y_r \in I_R$. Since $\overline{I}$ is regular $y_s a_s x_s \in I_S$. Also $y_s a_s x_r = y_s a_s x_r + x_r a_s y_s \in I$. Therefore

$$xay = xay + yax - y_s a_s x_s - y_s a_s x_r \in I.$$

□

Proposition 6.9. *Theorem 6.3 holds in the case when A/R is simple and $AR = 0$, moreover I is regular.*

Proof. This is identical to the proof of Proposition 6.8 if we replace the left natural $Q-$module V by the right natural $Q-$module. □

Proposition 6.10. *Theorem 6.3 holds for the case where $A = Q \oplus R$ where $Q \cong M_n$ as an algebra and $R \cong M_n$ as a vector space. The multiplication is defined by R being considered as a natural Q-bimodule.*

Proof. Let I be an inner ideal of $L = [A, A]$. We will show that there is a special automorphism φ of A such that the theorem holds for the inner ideal $\varphi(I)$. This will imply the theorem holds for I as well. As in the proof of Proposition 6.8 fix any standard basis $\{e_{ij} | 1 \le i, j < n\}$ (resp. $\{f_{ij} | 1 \le i, j \le n\}$) of Q (resp. R), consisting of matrix units, such that the action of Q on R corresponds to matrix multiplication and $\overline{I}$ is spanned by $\{e_{ij} | 1 \le i \le k < l \le j \le n\} \subset Q$. Fix any subset

$$J = \{x_{ij}^{(1)} = e_{ij} + \sum \alpha_{st}^{ij} f_{st} | 1 \le i \le k < l \le j \le n\} \subseteq I.$$

Note $x_{ij}^{(1)} f_{ji} x_{ij}^{(1)} = e_{ij} f_{ji} e_{ij} = f_{ij} \in I$ for all $x_{ij}^{(1)} \in J$. Therefore

$$I_0 = span\{f_{ij} | 1 \le i \le k < l \le j \le n\} \subseteq I \cap R.$$

For $x_{ij} = e_{ij} + \sum \alpha^{ij}_{st} f_{st} \in I$ denote

$$\theta(x_{ij}) = e_{ij} + \sum_{s>k} \alpha^{ij}_{sj} f_{sj} + \sum_{t<l} \alpha^{ij}_{it} f_{it}.$$

We claim $\theta(x_{ij}) \in I$. Indeed by Lemma 4.1(1)

$$\begin{aligned} x_{ij}e_{ji}x_{ij} &= (e_{ij} + \sum \alpha^{ij}_{st} f_{st})e_{ji}(e_{ij} + \sum \alpha^{ij}_{st} f_{st}) \\ &= e_{ij} + e_{ii} \sum \alpha^{ij}_{st} f_{st} + (\sum \alpha^{ij}_{st} f_{st})e_{jj} \\ &= e_{ij} + \sum_{t=1}^{n} \alpha^{ij}_{it} f_{it} + \sum_{s=1}^{n} \alpha^{ij}_{sj} f_{sj} \\ &= \theta(x_{ij}) + r \in I \end{aligned}$$

where $r \in I_0 \subseteq I \cap R$. So $\theta(x_{ij}) \in I$. Let $x^{(2)}_{ij} = \theta(x^{(1)}_{ij})$. Define the automorphism φ_1 on A and L by $\varphi_1(a) = (1 + q_1)a(1 - q_1)$ for $a \in A$ where

$$q_1 = -\sum_{s>k} \alpha^{1n}_{sn} f_{s1} + \sum_{t<l} \alpha^{1n}_{1t} f_{nt} \in R.$$

Put $x^{(3)}_{ij} = (\varphi_1(x^{(2)}_{ij})) \in I_1 = \varphi_1(I)$. Then

$$\begin{aligned} x^{(3)}_{1n} &= (1 + q_1)x^{(3)}_{1n}(1 - q_1) \\ &= (1 - \sum_{s>k} \alpha^{1n}_{sn} f_{s1} + \sum_{t<l} \alpha^{1n}_{1t} f_{nt})(e_{1n} + \sum_{s>k} \alpha^{1n}_{sn} f_{sn} + \sum_{t<l} \alpha^{1n}_{1t} f_{1t})(1 - q_1) \\ &= (e_{1n} + \sum_{s>k} \alpha^{1n}_{sn} f_{s1} - \sum_{s>k} \alpha^{1n}_{sn} f_{s1} + \sum_{t<l} \alpha^{1n}_{1t} f_{1t} + \alpha^{1n}_{11} f_{nn})(1 - q_1) \\ &= (e_{1n} + \sum_{t<l} \alpha^{1n}_{1t} f_{1t} + \alpha^{1n}_{11} f_{nn})(1 + \sum_{s>k} \alpha^{1n}_{sn} f_{s1} - \sum_{t<l} \alpha^{1n}_{1t} f_{nt}) \\ &= e_{1n} + \sum_{t<l} \alpha^{1n}_{1t} f_{1t} + \alpha^{1n}_{11} f_{nn} + \alpha^{1n}_{nn} f_{11} - \sum_{t<l} \alpha^{1n}_{1t} f_{1t} \\ &= e_{1n} + \alpha^{1n}_{11} f_{nn} + \alpha^{1n}_{nn} f_{11}. \end{aligned}$$

Since $I^2 = 0$,

$$\begin{aligned} 0 &= (x_{1n}^{(3)})^2 \\ &= (e_{1n} + \alpha_{11}^{1n} f_{nn} + \alpha_{nn}^{1n} f_{11})(e_{1n} + \alpha_{11}^{1n} f_{nn} + \alpha_{nn}^{1n} f_{11}) \\ &= \alpha_{11}^{1n} f_{1n} + \alpha_{nn}^{1n} f_{1n} = (\alpha_{11}^{1n} + \alpha_{nn}^{1n}) f_{1n} \end{aligned}$$

so $x_{1n}^{(3)} = e_{1n} + \alpha f_{11} - \alpha f_{nn}$ for some $\alpha \in F$. Put $q_2 = \alpha f_{n1}$ and consider the special automorphism φ_2 defined $\varphi_2(a) = (1 + q_2)a(1 - q_2)$. Then

$$\begin{aligned} x_{1n}^{(4)} &= \varphi_2(x_{1n}^{(3)}) \\ &= (1 + \alpha f_{n1})(e_{1n} + \alpha f_{11} - \alpha f_{nn})(1 - \alpha f_{n1}) \\ &= e_{1n} + \alpha f_{nn} + \alpha f_{11} - \alpha f_{nn} = e_{1n} \in I_2 = \varphi_2(I_1). \end{aligned}$$

Put $x_{ij}^{(4)} = \theta(\varphi_2(x_{ij}^{(3)})) \in I_2$. Now set $x_{1n}^{(5)} = x_{1n}^{(4)}$ and $x_{ij}^{(5)} = x_{ij}^{(4)}$ for $j \neq n$. For $j = n$ and $i \neq 1$ put

$$\begin{aligned} x_{in}^{(5)} &= x_{in}^{(4)} e_{n1} e_{1n} + e_{1n} e_{n1} x_{in}^{(4)} \\ &= x_{in}^{(4)} e_{nn} + e_{11} x_{in}^{(4)} \\ &= (e_{in} + \sum_{s>k} \beta_{sn}^{in} f_{sn} + \sum_{t<l} \beta_{it}^{in} f_{it}) e_{nn} + e_{11}(e_{in} + \sum_{s>k} \beta_{sn}^{in} f_{sn} + \sum_{t<l} \beta_{it}^{in} f_{it}) \\ &= (e_{in} + \sum_{s>k} \beta_{sn}^{in} f_{sn} + \sum_{t<l} \beta_{it}^{in} f_{it}) e_{nn} \\ &= e_{in} + \sum_{s>k} \beta_{sn}^{in} f_{sn} \in \varphi_2(\varphi_1(I)). \end{aligned}$$

Define the special automorphism φ_3 of A and L by $\varphi_3(a) = (1 + q_3)a(1 - q_3)$ for $a \in A$ where

$$q_3 = -\sum_{s>k} \sum_{t=2}^{k} \beta_{sn}^{tn} f_{st}$$

Then

$$\begin{aligned}
x_{in}^{(6)} = \varphi(x_{in}^{(5)}) &= (1+q_3)x_{in}^{(5)}(1-q_3) \\
&= (1-\sum_{s>k}\sum_{t=2}^{k}\beta_{sn}^{tn}f_{st})(e_{in}+\sum_{s>k}\beta_{sn}^{in}f_{sn})(1+\sum_{s>k}\sum_{t=2}^{k}\beta_{sn}^{tn}f_{st}) \\
&= (e_{in}+\sum_{s>k}\beta_{sn}^{in}f_{sn}-\sum_{s>k}\beta_{sn}^{in}f_{sn})(1+\sum_{s>k}\sum_{t=2}^{k}\beta_{sn}^{tn}f_{st}) \\
&= e_{in}(1+\sum_{s>k}\sum_{t=2}^{k}\beta_{sn}^{tn}f_{st}) \\
&= e_{in}+\sum_{t=2}^{k}\beta_{nn}^{tn}f_{it} \in I_3 = \varphi_3(I_2).
\end{aligned}$$

Then for $1 \leq i, p \leq k$,

$$x_{in}^{(6)}x_{pn}^{(6)} = (e_{in}+\sum_{t=2}^{k}\beta_{nn}^{tn}f_{it})(e_{pn}+\sum_{t=2}^{k}\beta_{nn}^{tn}f_{pt}) = \beta_{nn}^{pn}f_{in}$$

Since $I^2 = 0$ this implies $\beta_{nn}^{pn} = 0$ for all $p = 1, \dots k$. So $x_{in}^{(6)} = e_{in}$ for $i = 1, \dots k$. Put $x_{ij}^{(6)} = \theta(\varphi_3(x_{ij}^{(5)})) \in I_3$. Now for $j \neq n$ set

$$\begin{aligned}
x_{ij}^{(7)} &= x_{in}^{(6)}e_{ni}x_{ij}^{(6)} + x_{ij}^{(6)}e_{ni}x_{in}^{(6)} \\
&= e_{in}e_{ni}x_{ij}^{(6)} + x_{ij}^{(6)}e_{ni}e_{in} \\
&= e_{ii}x_{ij}^{(5)} + x_{ij}^{(5)}e_{nn} \\
&= e_{ii}(e_{ij}+\sum_{s>k}\gamma_{sj}^{ij}f_{sj}+\sum_{t<l}\gamma_{it}^{ij}f_{it}) + (e_{ij}+\sum_{s>k}\gamma_{sj}^{ij}f_{sj}+\sum_{t<l}\gamma_{it}^{ij}f_{it})e_{nn} \\
&= e_{ij}+\sum_{t<l}\gamma_{it}^{ij}f_{it} \in I_3
\end{aligned}$$

and put $x_{in}^{(7)} = x_{in}^{(6)} = e_{in} \in I_2$ for $i = 1, \dots k$. Note for $i \neq 1$ we have that

$$\begin{aligned} x_{in}^{(7)} e_{n1} x_{1j}^{(7)} + x_{1j}^{(7)} e_{n1} x_{in}^{(7)} &= e_{in} e_{n1} x_{1j}^{(7)} + x_{1j}^{(7)} e_{n1} e_{in} \\ &= e_{i1} x_{1j}^{(7)} \\ &= e_{i1}(e_{1j} + \sum_{t<l} \gamma_{it}^{ij} f_{it}) \\ &= e_{ij} + \sum_{t<l} \gamma_{1t}^{1j} f_{it} \in I_2. \end{aligned} \tag{6.3}$$

Let $x_{ij}^{(8)} = e_{ij} + \sum_{t<l} \gamma_{1t}^{1j} f_{it}$ for $j \neq n$, and for $j = n$, $x_{in}^{(8)} = x_{in}^{(7)} = e_{in}$. Then equation 6.3 implies $x_{ij}^{(8)} \in I_2$ for $1 \leq i \leq k < l \leq j \leq n$. We define a final special automorphism φ_4 on A and L by $\varphi_4(a) = (1 + q_4)a(1 - q_4)$ where

$$q_4 = \sum_{s=l}^{n-1} \sum_{t<l} \gamma_{1t}^{1s} f_{st}.$$

Put

$$\begin{aligned} x_{ij}^{(9)} = \varphi_4(x_{ij}^{(8)}) &= (1 + \sum_{s=l}^{n-1} \sum_{t<l} \gamma_{1t}^{1s} f_{st})(e_{ij} + \sum_{t>l} \gamma_{1t}^{1j} f_{it})(1 - \sum_{s=1}^{n-1} \sum_{t<l} \alpha_{1t}^{1s} f_{st}) \\ &= (e_{ij} + \sum_{t>l} \gamma_{1t}^{1j} f_{it} + \sum_{s=l}^{n-1} \gamma_{1i}^{1s} f_{sj})(1 - \sum_{s=1}^{n-1} \sum_{t<l} \alpha_{1t}^{1s} f_{st}) \\ &= e_{ij} + \sum_{t>l} \gamma_{1t}^{1j} f_{it} + \sum_{s=l}^{n-1} \gamma_{1i}^{1s} f_{sj} - \sum_{t<l} \gamma_{1t}^{1j} \\ &= e_{ij} + \sum_{s=l}^{n-1} \gamma_{1i}^{1s} f_{sj}. \end{aligned}$$

Note

$$x_{ij}^{(9)} x_{pq}^{(9)} = (e_{ij} + \sum_{s=l}^{n-1} \gamma_{1i}^{1s} f_{sj})(e_{pq} + \sum_{s=l}^{n-1} \gamma_{1p}^{1s} f_{sq}) = e_{ij} \sum_{s=l}^{n-1} \gamma_{1p}^{1s} f_{sq} = \gamma_{1p}^{1j} f_{jq},$$

and since $I^2 = 0$, $\gamma_{1p}^{1j} = 0$ for $1 \leq p \leq k < l \leq j \leq n$. Therefore $x_{ij}^{(9)} = e_{ij} \in I_4 = \varphi_4(I_3)$ for $1 \leq i \leq k < l \leq j \leq n$. So I_4 splits. Hence I splits. $\square$

The following example shows that I may not be regular in Proposition 6.10.

Example 6.11. Let A be the associative algebra as described in Proposition 6.10. Let

$$I = span\{e_{1n}, f_{1n}, \alpha f_{1n-1} + \beta f_{2n}\}$$

where $\alpha, \beta \in F$ are non-zero. Then $I^2 = 0$ and $xay + yax \in I$ for all $x, y \in I$ and for all $a \in A$ so I is indeed an inner ideal. However $e_{1n}e_{n1}(\alpha f_{1n-1} + \beta f_{2n}) = \alpha f_{1n-1} \notin I$, so $xay \notin I$ for some $x, y \in I$ and $a \in A$. Therefore I is not regular.

Proposition 6.12. *Theorem 6.3 holds for $A = Q \oplus R$ where $Q = Q_1 \oplus Q_2$, with $Q_1 \cong M_n$, $Q_2 \cong M_m$ and $R \cong M_{nm}$ is a left Q_1- right Q_2- bimodule. Moreover any inner ideal I of A is regular.*

Proof. Let B_1 and B_2 be the two maximal ideals of A. Then $B_s = Q_s + R$, $s = 1, 2$ and $B_1 + B_2 = A$. Let $^-\colon A \to A/R \cong Q_1 \oplus Q_2$ be the natural homomorphism and let I be an inner ideal of A with $I^2 = 0$. Since $\overline{A}$ is semisimple $\overline{I} = \overline{I}_1 \oplus \overline{I}_2$ where $\overline{I}_s$ is an inner ideal of $\overline{B}_s \cong Q_s$, $(s = 1, 2)$. Consider the map $I \to \overline{I} = \overline{I}_1 \oplus \overline{I}_2$. Let I_s be the full preimage of $\overline{I}_s$ in I. Then $I_s \subseteq I$ and $I_1 + I_2 = I$. Since $\overline{I}_s \subseteq \overline{B}_s$ and B_s is the full preimage in A of $\overline{B}_s$, $I_s \subseteq B_s$. Therefore $I_s \subseteq J_s = B_s \cap I$. On the other hand, $\overline{J}_s \subseteq \overline{B}_s \cap \overline{I} = \overline{I}_s$, so $J_s \subseteq I_s$. Thus, $I_s = B_s \cap I$. Note that I_s is an inner ideal of B_s and $I_s^2 = 0$. Since B_s satisfies the conditions of Proposition 6.8 or 6.9, one can assume that Q_s is a Levi subalgebra of B_s which splits I_s. Since $I = I_1 + I_2$, the Levi subalgebra $Q = Q_1 + Q_2$ of A splits I. $\square$

We are now ready to provide the proof of Theorem 6.3.

Proof. (of Theorem 6.3) Let Q be a Levi subalgebra of A and let $Q_1, \ldots, Q_t$ be the simple components of Q. Note R is a $Q-Q-$bimodule. Since $A^2 = A$, $R = QR + RQ$ so R is the direct sum of copies of natural left Q_i modules V_i, natural right Q_i-modules W_i, and natural left Q_i- right Q_j- bimodules U_{ij} (with $Q_iU_{ij}Q_j = U_{ij}$, $Q_sU_{ij} = 0$

for $s \neq i$, and $U_{ij}Q_s = 0$ for $s \neq j$). The proof follows by induction on the length m of the bimodule R. Indeed if $m = 1$, then this is proved in Propositions 6.8, 6.9, 6.10 and 6.12. Assume that $m > 1$. Since $\overline{I}$ is an inner ideal of the semisimple algebra $Q_1 \oplus \cdots \oplus Q_t$, $\overline{I} = \overline{I}_1 \oplus \cdots \oplus \overline{I}_t$ with $\overline{I}_s = span\, E^s$ where

$$E^s = \{e^s_{ij} | 1 \leq i \leq k_s,\, l_S \leq j \leq n_s\}$$

for some choice of matrix presentations for Q_s. Put $E = E^1 \cup \cdots \cup E^t$. Note that I splits if and only if there exists a special automorphism φ of A such that $E \subseteq \varphi(I)$. Fix any preimage $x^s_{ij} \in I$ of e^s_{ij}, i.e. $x^s_{ij} = e^s_{ij} + r^s_{ij}$ with $r^s_{ij} \in R$ (for all i, j, s). Consider any submodule T of R of length $m-1$. Note T is an ideal of A. Let $B = A/T$ and let I_B be the image of I in B and E_B the image of E in B. By the case $m = 1$, there is a special automorphism φ_1 of B such that $E_B \subseteq \varphi_1(I_B)$. Note that φ_1 is induced by a special automorphism φ_2 of A, so $X = \{x^s_{ij} = e^s_{ij} + t^s_{ij} | \forall s, i, j\} \subseteq \varphi_2(I)$. Denote $I_2 = \varphi_2(I) \cap (Q \oplus T)$. Then I_2 is an inner ideal of $Q \oplus T$. Since the length of T is less than m, by induction I_2 splits, so there is a special automorphism φ_3 of $Q \oplus T$ (and A), such that $E \subseteq \varphi_3(I_2) \subseteq \varphi_3(\varphi_2(I))$, so I splits. $\square$

7. Classification of Maximal Inner Ideals

In this chapter it will be shown that maximal inner ideals exist and are classified in plain simple locally finite Lie algebras and finitary Lie algebras. First finitary Lie algebras are considered and we need the following two facts. We use the notation of section 5.

Lemma 7.1. [16, Theorem 2.5(iii)]*Let $L = \mathfrak{fsl}(X,Y)$. If $I = W^*U$ is an inner ideal where $U \subseteq X$ and $W \subseteq Y$ are mutually orthogonal subspaces (i.e. $g(V,W) = 0$). Then $U = IX$ and $W = I^{\#}Y$.*

Proposition 7.2. *Let $L = \mathfrak{fsl}(X,Y)$. Let $U_1 \subseteq X, W_1 \subseteq Y$ and $U_2 \subseteq X, W_2 \subseteq Y$ be two mutually orthogonal pairs of subspaces then $U_1^*W_1 = U_2^*W_2$ if and only if $U_1 = U_2$ and $W_1 = W_2$.*

Proof. Follows from the above Lemma. □

Theorem 7.3. *(1) I is a maximal inner ideal of $\mathfrak{fsl}(X,Y)$ if and only if $I = W^*U$ where $0 \neq U \subseteq X$ and $0 \neq W \subseteq Y$ are subspaces such that $W = Ann\, U$ and $U = Ann\, W$.*

(2) I is a maximal inner ideal of $\mathfrak{fsp}(X,\Psi)$ if and only if $I = [U,U]$ for a maximal totally isotropic subspace U of X.

(3) I is a maximal inner ideal of $\mathfrak{fso}(X,\Phi)$ if and only if $I = [U,U]$ for a maximal totally isotropic subspace U of X or $I = [x, H^\perp]$ for a 2-dimensional subspace H of X such that the restriction of Φ to H is nondegenerate and an isotropic vector $x \in H$.

Proof. (1) Suppose I is maximal. By Theorem 5.2, $I = W^*U$ for a pair of mutually orthogonal subspaces $U \subseteq X$ and $W \subseteq Y$. It is clear that if $B = W_1^*U_2$ is another inner ideal with $B \subseteq I$ then $U_1 \subseteq U$ and $W_1 \subseteq W$. Therefore if I is maximal we must have $W = Ann\, U$ and $U = Ann\, W$ so that U and W are maximal mutually orthogonal subspaces. The converse is clear.

(2) Let I be an inner ideal of L. By Theorem 5.2, $I = [U, U]$ for some totally isotropic subspace U. Clearly I is maximal if and only if U is maximal.

(3) Let I be a maximal inner ideal of $\mathfrak{fso}(X, \Phi)$. By Theorem 5.2 $I = [U, U]$ for a totally isotropic subspace, or $I = [x, H^{\perp}]$ for a 2-dimensional subspace H of X with nondegenerate restriction of Φ to H and isotropic vector $x \in H$, or I is a type 1 point space. Note if I is a point space of type 1 then I is a subspace of $[U, U]$ for some totally isotropic subspace U and so cannot be maximal (see [14, Proposition 4.3]). It is clear that if $I = [U, U]$ then U must be maximal. Finally $[x, H^{\perp}]$ as above is always maximal (see proof of 3.8 in [16]). The converse is clear. $\square$

Now the case of the plain locally finite Lie algebras is considered. The following results and definitions will be needed.

Lemma 7.4. *Let $L = [A, A]$ and I be an inner ideal with $I^2 = 0$. Denote $\hat{I} = IAI + I$. Then the following are true.*

(1) $I \subseteq \hat{I}$

(2) $\hat{I}^2 = 0$

(3) $\hat{I}A\hat{I} \subseteq IAI \subseteq \hat{I}$, so $\hat{I}$ is an inner ideal of L (see 4.1(3)).

(4) $\hat{I}$ is a regular inner ideal of L (see def 4.3).

Definition 7.5. Let $\mathcal{L}$ and $\mathcal{R}$ be left and right ideals of an associative algebra A. Then $(\mathcal{L}, \mathcal{R})$ is called a maximal pair of ideals if $\mathcal{L} = LAnn\, \mathcal{R} = \{a \in A | a\mathcal{R} = 0\}$ (the left annihilator of $\mathcal{R}$) and $\mathcal{R} = RAnn\, \mathcal{L} = \{a \in A | \mathcal{L}a = 0\}$ (the right annihilator of $\mathcal{L}$).

Let $(\mathcal{L}, \mathcal{R})$ be a maximal pair. Then $\mathcal{L}\mathcal{R} = 0$, so $I = \mathcal{L} \cap \mathcal{R} \cap L$ is an inner ideal of $[A, A]$ by Proposition 4.9. Our aim is to show that every maximal inner ideal is obtained in this way.

Proposition 7.6. *Let $(\mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{L}_2, \mathcal{R}_2)$ be two maximal pairs of a simple locally finite associative algebra A and let $L = [A, A]$. Let $I_1 = \mathcal{L}_1 \cap \mathcal{R}_1 \cap L$ and $I_2 = \mathcal{L}_2 \cap \mathcal{R}_2 \cap L$. Then $I_1 \subseteq I_2$ if and only if $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{R}_1 = \mathcal{R}_2$.*

Proof. Assume $I_1 \subseteq I_2$. Consider the left ideal $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. By Lemma 4.10,

$$A\mathcal{L} = A\mathcal{L}_1 + A\mathcal{L}_2 = AI_1 + AI_2 = AI_2 = A\mathcal{L}_2 \subseteq \mathcal{L}_2.$$

Assume $\mathcal{L}\mathcal{R}_2 \neq 0$. Since A is simple and $\mathcal{L}\mathcal{R}_2$ is a two sided ideal of A, $\mathcal{L}\mathcal{R}_2 = A$. So $A\mathcal{L}\mathcal{R}_2 = A^2 = A$. But $A\mathcal{L}\mathcal{R}_2 \subseteq \mathcal{L}\mathcal{R}_2 = 0$. Thus $\mathcal{L}\mathcal{R}_2 = 0$, so $\mathcal{L} \subseteq LAnn\,\mathcal{R}_2 = \mathcal{L}_2$, so $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Similarly $\mathcal{R}_1 \subseteq \mathcal{R}_2$. But $\mathcal{L}_1 = LAnn\,\mathcal{R}_1 \supseteq LAnn\,\mathcal{R}_2 = \mathcal{L}_2$ this implies $\mathcal{L}_1 = \mathcal{L}_2$. Similarly, $\mathcal{R}_1 = \mathcal{R}_2$. □

Theorem 7.7. *Let A be a simple locally finite associative algebra and let $L = [A, A]$. Then*

(1) For every maximal pair of ideals $(\mathcal{L}, \mathcal{R})$, $I_{\mathcal{L}\mathcal{R}} = \mathcal{L} \cap \mathcal{R} \cap L$ is an inner ideal of L.

(2) The map $(\mathcal{L}, \mathcal{R}) \to I_{\mathcal{L}\mathcal{R}}$ is a bijection between the set of all maximal pairs of ideals and the maximal inner ideals of L.

Proof. (1) Is obvious.

(2) Let φ be the map $(\mathcal{L}, \mathcal{R}) \to I_{\mathcal{L}\mathcal{R}}$. First by proposition 7.6, φ is injective. Assume now $I_{\mathcal{L}\mathcal{R}}$ is not maximal. Then $I_{\mathcal{L}\mathcal{R}}$ is properly contained in another inner ideal I. Since I is maximal by Lemma 7.4, $\hat{I} = I$, so I is regular. Therefore there exists a pair of ideals $(\mathcal{L}', \mathcal{R}')$ so that

$$I_{\mathcal{L}\mathcal{R}} \subseteq I \subseteq I^{'} = \mathcal{L}' \cap \mathcal{R}' \cap L.$$

Let $\mathcal{R}'' = RAnn\,\mathcal{L}'$, and $\mathcal{L}'' = LAnn\,\mathcal{R}''$ so that $(\mathcal{L}'', \mathcal{R}'')$ is a maximal pair of ideals. Note $\mathcal{R}' \subseteq \mathcal{R}''$, and $\mathcal{L}' \subseteq \mathcal{L}''$. So $I_{\mathcal{L}\mathcal{R}} \subsetneq I \subseteq I_{\mathcal{L}''\mathcal{R}''}$. By proposition 7.6, $\mathcal{L} = \mathcal{L}''$ and $\mathcal{R} = \mathcal{R}''$ so $I = I_{\mathcal{L}\mathcal{R}}$ giving a contradiction. Finally assume I is a maximal inner ideal of L then $I \subseteq \hat{I} \subseteq I_{\mathcal{L}\mathcal{R}}$ where $(\mathcal{L}, \mathcal{R})$ is a maximal pair of ideals. So every maximal inner ideal is of the shape $I_{\mathcal{L}\mathcal{R}}$. $\square$

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