# Robust discrete time output feedback sliding mode control with application to aircraft systems

Thesis submitted for the degree of Doctor of Philosophy at the University of Leicester

by

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# Abstract

## Robust discrete time output feedback sliding mode control with application to aircraft systems

### Nai One Lai

This thesis describes the development of robust discrete time sliding mode controllers where only output information is available. A connection between discrete time sliding mode controllers and so-called min-max controllers is described. New conditions for the existence of stabilizing output feedback discrete time sliding mode controllers are given for non-square systems with bounded matched uncertainties. A novel sliding surface is described; this in itself is not realizable through outputs alone, but it gives rise to a control law which depends only on outputs. An explicit LMI optimization procedure is described to synthesize a Lyapunov matrix, which satisfies both a discrete Riccati inequality and a structural constraint. This Lyapunov matrix is used to calculate the robustness bounds associated with the closed-loop system.

For systems which are not static output feedback stabilizable, a compensation scheme is proposed and a dynamic output feedback discrete time sliding mode controller is described with a simple parameterisation of the available design freedom.

Initially, a regulation problem is considered. Then a new scheme which incorporates tracking control using integral action is proposed for both the static and dynamic output feedback discrete time sliding mode controller. The scheme requires only that the plant has no poles or zeros at the origin and therefore the controller can be applied to non-minimum phase systems.

The theory described is demonstrated for various engineering systems including implementation on a DC-motor rig in real-time and simulations on a nonlinear, non-minimum phase model of a Planar Vertical Take-Off and Landing aircraft. The effectiveness of the controller is further proven by its application for control of the longitudinal dynamics of a detailed combat aircraft model called the High Incidence Research Model. Simulations with real-time pilot input commands have been carried out on a Real Time All Vehicle Simulator and good results obtained.

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## Chapter 1

# **Introduction and Outline**

## 1.1 Background and Motivation

Whilst much of the control systems literature focuses on the analysis of continuous time systems, increasingly, practising control engineers implement control systems using microprocessors. The controllers can either be implemented from continuous time representations using 'fast sampling' ideas, or the continuous time controllers can be mapped to their discrete time equivalents – so-called emulation (Franklin et al. 1990). Alternatively, discrete time controllers can be designed directly from a discrete time representation of the plant. In certain situations, sensor bandwidth or hardware limitation may make fast sampling impossible. Hence, discrete time controllers designed from discrete time representations of the plant are needed. One thread of the literature has focused on developing discrete time controllers based on Lyapunov ideas to stabilise discrete time uncertain linear systems with bounded uncertainties: for example see (Corless & Manela 1986, Kienitz 1990, Yang & Tomizuka 1990, Sharav-Schapiro, Palmor & Steinberg 1996, Sharav-Schapiro, Palmor & Steinberg 1998, Garcia, Pradin, Tarbouriech & Feng 2003) and the references therein. Most of this early work considered designing (static) state feedback control laws. A technique to design stabilizing state feedback controllers for linear discrete time systems based on the solution of a Riccati-like algebraic equation is given in (Kienitz 1990). The proposed discrete state feedback controllers in (Corless 1985, Yang & Tomizuka 1990) are based on the second method of Lyapunov for single input (Yang & Tomizuka 1990) and multi input (Corless 1985) systems, whereas (Spurgeon 1992) addresses the problem of designing a state feedback controller from the perspective of (discrete time) sliding mode control. In particular the work of Sharav-Schapiro, Palmor & Steinberg (1999) considers a so-called min-max approach.

Another thread of the discrete-time systems literature pursued solutions to the discrete  $\mathcal{H}_2$ and  $\mathcal{H}_{\infty}$  problems: see for example (Green & Limebeer 1995, Zhou, Doyle & Glover 1996) and the references therein. In the output feedback case the approaches described in (Green & Limebeer 1995, Zhou et al. 1996) result in dynamical controllers, driven by the measured outputs, which are the same order as the plant (plus any weighting functions in the case of  $\mathcal{H}_{\infty}$ ). The gains which make up the dynamical controllers are typically obtained from solutions to algebraic Riccati equations formed from the plant state-space description. The synthesis of lower order/fixed structure controllers is largely an open problem. Recently a new numerical algorithm for static output feedback  $\mathcal{H}_{\infty}$  control has appeared (Bara & Boutayeb 2005). This provides sufficient conditions in terms of Linear Matrix Inequalities for the existence of a sub-optimal static output feedback control law which meets a given  $\mathcal{H}_{\infty}$  norm.

This thesis concentrates on the analysis and synthesis of discrete time sliding mode control-

lers (DSMC) to provide robustness to so-called matched uncertainty and in particular develops the state-of-the-art in terms of controller design using only output information. Classically a (continuous time) sliding mode is generated by means of discontinuities in the control signals about a surface in the state space (Utkin 1992). The discontinuity surface (usually known as the sliding surface) is attained from any initial condition in a finite time interval. Provided the controller is designed appropriately, the motion when constrained to the surface (the sliding mode) is completely insensitive to so-called matched uncertainty (Utkin 1992, Edwards & Spurgeon 1998), i.e. uncertainties that lie within the range space of the input distribution matrix. The effective continuous control action necessary to maintain an ideal sliding motion is known as the equivalent control (Utkin 1992). This is not the applied control, which is discontinuous, but is a theoretical quantity representing the continuous, average behaviour of the applied discontinuous control. In digital control implementation, the control signal is held constant during the sample interval and hence it is not possible in general to attain ideal sliding as the required control must switch at infinite frequency. As a result, the invariance properties of continuous time sliding mode control (CSMC) are lost. The obvious solution of sampling at high frequency, which will closely approximate continuous time, is not always possible. For this reason the idea of discrete time sliding mode control (DSMC) has been proposed in (Milosavljević 1985, Sapturk, Istefanopulous & Kaynak 1987, Furuta 1990, Chan 1994, Gao, Wang & Homaifa 1995, Chan 1998).

Much of the early DSMC literature (Sapturk et al. 1987, Gao et al. 1995, Bartoszewicz 1996) focused on establishing a discrete time counterpart to the (continuous time) reachability condition, i.e. the design of the controller to induce sliding (in a discrete time sense). In uncertain discrete time systems it is not possible to ensure the states evolve precisely along a surface within the state space and so the DSMC problem is fundamentally different to its continuous time counterpart (Koshkouei & Zinober 2000). A recent comprehensive overview of this early development is given in (Milosavljević 2004). One key feature is that DSMC does not necessarily require the use of a variable structure discontinuous control strategy (Spurgeon 1992, Hui & Źak 1999, Koshkouei & Zinober 2000). The results presented in (Spurgeon 1992, Hui & Źak 1999) show that an appropriate choice of sliding surface, used with the 'equivalent control', can guarantee a bounded motion about the surface in the presence of bounded matched uncertainty and that the use of a relay/switch in the control law is detrimental to performance. From this point of view, the DSMC problem can be looked at as a robust optimal control problem and is related to discrete time Lyapunov min-max problems (Corless 1985, Manela 1985). Indeed both (Spurgeon 1992) and (Hui & Źak 1999) pose the DSMC problem as an appropriately formulated Lyapunov min-max problem, where the feedback gain is chosen to minimise over all possible controllers, the worse case effect of the uncertainty on the Lyapunov function.

Compared with continuous time sliding mode strategies, the design problem in discrete time is much less mature. Other than early work in (Sira-Ramirez 1991), most of the literature assumes all the states of the plant are directly accessible (Chan 1994, Hui &  $\dot{Z}$ ak 1999, Koshkouei & Zinober 2000, Golo & Milosavljević 2000, Furuta & Pan 2000, Tang & Misawa 2002). This is not very realistic for practical engineering problems. In real systems, it often happens that not all system states are fully available or measurable. The schemes which have restricted themselves to output measurements alone have invariably utilised observers with or without disturbance estimation (Lee & Lee 1999, Tang & Misawa 2000, Mitić & Č. Milosavljević 2002). Recent exceptions have been the work in (Monsees 2002) which considers both static and dynamical output feedback problems, and the discrete time versions of certain higher-order sliding mode control schemes (Bartolini, Pisano & Usai 2001, Bartolini, Pisano & Usai 2000).

In CSMC design using only output information, the system zeros need to be stable- i.e. the system needs to be minimum phase. This is a limitation on the class of system for which the resulting schemes are applicable. In this thesis, novel DSMC strategies which use only measured output information are described. The output feedback discrete time sliding mode controllers (ODSMC) which are proposed apply to uncertain systems (with matched uncertainties) which are not necessarily minimum phase or relative degree one. New sliding surface designs are proposed, which are associated with the equivalent control of the output feedback sliding mode controller. Design freedom is available to select the sliding surface parameters to produce appropriate reduced-order sliding motions.

Initially, static ODSMC will be considered. In order that a stable (ideal) discrete time sliding motion exists, necessary and sufficient conditions will be given in terms of the stabilisability by static output feedback of a fictitious system triple obtained from the real system. This fictitious system can easily be isolated once the real system is transformed into a special canonical form. The stabilisability condition for the fictitious system is the only significant restriction on the class of systems to which the results are applicable. The fact that there is a limitation on the class of systems for which static ODSMC is applicable is not surprising since static output feedback controllers do not exist for all systems (Syrmos, Abdallah, Dorato & Grigoriadis 1997). An explicit procedure is described which shows how a Lyapunov matrix, which satisfies both a discrete Riccati inequality and a structural constraint, can be used to calculate robustness bounds associated with the closed-loop system.

To overcome the static output feedback stabilisability problem, dynamic ODSMC is also presented in this thesis. A compensator design is described which introduces additional degrees of freedom. A simple parameterisation of the available design freedom is proposed. Again a procedure is described to solve both a Riccati inequality and a structural constraint to calculate robustness bounds.

Initially, regulation problems are considered. However, in later chapters tracking is incorporated into the ODSMC framework using integral action. The practicality of the results are demonstrated through the implementation of a static ODSMC in real-time on a small DC-motor test rig. As it is not trivial to incorporate tracking control together with the compensation scheme for the dynamic ODSMC, a separate chapter in the thesis is dedicated to that particular design problem. A nonlinear, non-minimum phase model of a Planar Vertical Take-Off and Landing (PVTOL) aircraft is used as an example.

The effectiveness of the dynamic ODSMC tracking controller is further proven by its application to a detailed combat aircraft model called the *High Incidence Research Model* (HIRM). The HIRM is a good benchmark problem which has been used in the robust flight control design challenge set-up by the Group for Aeronautical Research and Technology in Europe (GARTEUR) (Muir et al. 1997). In the case study, a controller is developed for the longitudinal dynamics of the aircraft model. Simulations with real-time pilot input commands have been carried out on a *Real Time All Vehicle Simulator* (RTAVS) within

the Department of Engineering.

### **1.2 Thesis Structure**

The thesis content is as follows:

**Chapter 2** presents an introduction to (continuous time) sliding mode control and the motivation for DSMC. Basic principles of (continuous time) sliding modes, reaching conditions, invariance properties and a link between DSMC and min-max controllers are given. This link underpins most of the work described later in the thesis.

**Chapter 3** solves the problem of unavailable states by using ODSMC and investigates the conditions necessary for the existence of a static ODSMC design. A controller based on these ideas is designed for an aircraft example. The work in this chapter has been published in (Lai, Edwards & Spurgeon 2003) and (Lai, Edwards & Spurgeon 2004d).

**Chapter 4** considers dynamic ODSMC resulting from the introduction of a compensator developed for systems which are not static output feedback stabilisable. This circumvents some of the restrictions identified in Chapter 3. A numerical example and an aircraft example are given to illustrate the approach. The work in this chapter has been published in (Lai, Edwards & Spurgeon 2004c) and (Edwards, Lai & Spurgeon 2005).

**Chapter 5** treats the problem of tracking with ODSMC. This has been achieved by incorporating integral action into the controller. The control scheme which is developed is based around a static ODSMC from Chapter 3 applied to an augmented system formed from the plant and the integrator states. Two examples are given, one of which is a practical implementation on an experimental DC-motor rig. The work in this chapter has been published in (Lai, Edwards & Spurgeon 2004a) and(Lai, Edwards & Spurgeon 2004b). **Chapter 6** describes a dynamic ODSMC design which incorporates tracking and increases performance by the introduction of a compensator. The robustness of this method is shown by simulations on a Planar Vertical Take-off and Landing aircraft (PVTOL). This emphasises that the results which have been developed in this thesis are applicable to nonminimum phase systems. The work in this chapter has been published in (Lai, Edwards & Spurgeon 2005).

**Chapter 7** is dedicated to a case study of the *High Incidence Research Model* (HIRM)– a good benchmark for the problem of flight control. The methodology from Chapter 6 is applied in the design of a longitudinal controller. Results are also obtained from an implementation of the controller on a *Real Time All Vehicle Simulator* (RTAVS).

**Chapter 8** summarizes the contributions made within the thesis and draws attention to a few areas of interest for future work.

## Chapter 2

# **Sliding Mode Control**

### 2.1 Introduction

This chapter gives an introduction to sliding mode control. It describes the basic concepts of continuous and discrete time sliding mode control and sets up an example to illustrate the motivation for the latter as well as showing the main differences between the two. A brief insight into the key properties of sliding mode control is given. Finally it is argued that discrete sliding mode control problems can be posed in a min-max control setting. This link will underly most of the results which will be described in later chapters.

## 2.2 Continuous Time Sliding Mode Control

Sliding mode control evolved from the ideas of variable structure control which were first brought to light in the early 1960's in Russia. Sliding mode control has been widely utilized because of its robustness properties and the ability to decouple high dimensional systems into a set of independent sub-problems of lower dimension (Utkin 1992).

A classical sliding mode is generated by means of discontinuities in the control signals on a surface in the state space. The discontinuity surface (also called the switching/sliding surface) denoted by S, is attained from initial conditions in a finite-time interval. The structure of the feedback system is changed or switched as the state crosses the discontinuity surface. A sliding motion occurs when the system state repeatedly crosses and immediately re-crosses a switching surface, because all motion in the vicinity of the surface is directed towards the surface. If infinite frequency switching were possible, once the system state reaches the switching surface, it is constrained to lie on the surface and is said to be in an *ideal sliding mode* (Edwards & Spurgeon 1998).

Consider an uncertain linear time invariant system,

$$\dot{x}(t) = Ax(t) + B(u(t) + f(t, x))$$
 (2.2.1)

$$y(t) = Cx(t) \tag{2.2.2}$$

where  $x \in \mathbb{R}^n$  is the system state,  $u \in \mathbb{R}^m$  is the system input, and  $y \in \mathbb{R}^p$  is the system output. The matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$  are the system, input distribution and output distribution matrices respectively. The quantities m, p and n are the number of inputs, outputs and states respectively with  $m \leq p < n$ . The unknown signal  $f(\cdot)$  represents *matched uncertainty*, which is any uncertainty that lies within the range of the control input matrix, B. The pair (A, B) is assumed to be controllable and B is full rank See Appendix B.3.

The design of a state-feedback sliding mode controller consists of two distinct stages:

• Phase 1 (Existence Problem). Define a sliding surface

$$\mathcal{S} = \{ x \in \mathbb{R}^n : s(x) = 0 \}$$
(2.2.3)

which prescribes desired dynamics to the system when the states are confined to the surface S. A common choice of  $s : \mathbb{R}^n \to \mathbb{R}^m$  is a linear function represented as

$$s(x) = Sx$$

where  $S \in \mathbb{R}^{m \times n}$ .

• Phase 2 (Reachability Problem). The determination of a control law to ensure the attainment of the sliding mode on S, i.e. designing a control law that drives the states of the system onto the switching surface in finite time and forces the system states to subsequently remain there.

#### 2.2.1 Existence Problem

In terms of switching function design, one of the most straightforward approaches involves transforming the system in (2.2.1) and (2.2.2) into a suitable canonical form, where the nominal linear system is decomposed into two subsystems. Define an appropriate change of coordinates  $x \mapsto Tx$  where  $T \in \mathbb{R}^{n \times n}$  represents a transformation matrix which is orthogonal. In the new coordinates the system triple  $(\hat{A}, \hat{B}, \hat{C})$  has the form

$$\hat{A} = TAT^{T} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
(2.2.4)

$$\hat{B} = TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$
(2.2.5)

$$\hat{C} = CT^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$
(2.2.6)

where  $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}, B_2 \in \mathbb{R}^{m\times m}$  and  $C_2 \in \mathbb{R}^{p\times m}$ . The square matrix  $B_2$  is nonsingular which follows from the assumption that B is full rank. The matrix T is chosen to provide the partitioned structure in (2.2.5). A simple way to synthesize the required matrix T is by so-called QR reduction (Edwards & Spurgeon 1998).

Partition the states so that

$$Tx = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
(2.2.7)

where  $x_1 \in \mathbb{R}^{n-m}$  and  $x_2 \in \mathbb{R}^m$ . The system (2.2.1) can then be written as

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \tag{2.2.8}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2f(t,x) + B_2u \qquad (2.2.9)$$

This is called *regular form*, where (2.2.8) and (2.2.9) represent what is defined as the *null* space dynamics and the range space dynamics of the system respectively (Utkin 1992).

This is a result of the special structure of the B matrix in (2.2.5).

Let the switching function be defined as

$$s(t) = Sx(t)$$
 (2.2.10)

where  $S \in \mathbb{R}^{m \times n}$  and is partitioned as

$$ST^T = \left[ \begin{array}{cc} S_1 & S_2 \end{array} \right] \tag{2.2.11}$$

where  $S_1 \in \mathbb{R}^{m \times (n-m)}$ ,  $S_2 \in \mathbb{R}^{m \times m}$  and  $S_2$  is nonsingular.

During the sliding motion, s(t) = 0 and (2.2.10) can be re-written as

$$Sx(t) = ST^T T x = S_1 x_1 + S_2 x_2 = 0 (2.2.12)$$

Expressing  $x_2$  in terms of  $x_1$  gives

$$x_2 = -S_2^{-1}S_1x_1 \tag{2.2.13}$$

or

$$x_2 = -Kx_1 \tag{2.2.14}$$

where  $K = S_2^{-1}S_1$ . Substituting (2.2.14) into (2.2.8) yields

$$\dot{x}_1 = (A_{11} - A_{12}K)x_1 \tag{2.2.15}$$

which represents the ideal dynamics in the sliding mode.

Once the sliding mode is attained, the system response is invariant to matched uncertainties. This is because a system with matched uncertainties has all model uncertainties and disturbances entering through the control channel. Thus, the uncertainties can be compensated by suitable control signals through the control input matrix B. During the sliding motion, equation (2.2.15) completely describes the dynamics of the system when sliding, and is apparently independent of the control signal. The system is of order n - m(lower order than the given plant), and the remaining m states can be obtained as a linear combination of the n - m sliding motion states. These are the key properties of sliding mode control.

The existence problem is one of choosing  $K = S_2^{-1}S_1$  so that the eigenvalues of  $A_{11} - A_{12}K$ are stable. This can be viewed as a static state feedback design problem for the pair  $(A_{11}, A_{12})$ . In the literature, there are many ways to go about solving this problem. Robust eigenstructure assignment is one method which effectively minimises the effects of parameter variations lying outside the range space of *B* (unmatched parameter variations) (Edwards & Spurgeon 1998). This approach is sometimes used together with the sensitivity reduction approach (Dorling & Zinober 1988) which minimises a conditioning parameter to reduce the sensitivity of the position of the closed-loop eigenvalues to unmatched parameter variations. A different method of sliding surface design is the quadratic performance approach (Utkin & Yang 1978) where a quadratic cost function is minimised and the strategy is formulated such that it appears as a standard linear quadratic optimal regulator problem. Alternatively, another design approach is to specify a region in the left-hand half-plane within which the eigenvalues must lie. Regions studied include the disk, vertical strip and damping sector in the left-half plane (Woodham & Zinober 1991a, Woodham & Zinober 1991b, Woodham & Zinober 1993).

### 2.2.2 Reachability Problem

In order for a sliding motion to take place, the trajectories of s(t) must be directed towards the sliding surface within a certain domain about the surface. For the single input case, the conditions guaranteeing that an ideal sliding motion will take place can be expressed mathematically as

$$\lim_{s \to 0^+} \dot{s} < 0 \qquad \text{and} \qquad \lim_{s \to 0^-} \dot{s} > 0 \tag{2.2.16}$$

The expression given above is usually replaced by an equivalent criterion

$$\dot{s}s < 0 \tag{2.2.17}$$

These are called the *reachability conditions* (Edwards & Spurgeon 1998). In the multivariable case, a linear reachability condition is

$$\dot{s}(t) = \Phi s(t) \tag{2.2.18}$$

where  $\Phi \in \mathbb{R}^{m \times m}$  is a stable design matrix. This reachability condition however, only ensures that the sliding surface is reached asymptotically: clearly the solution of (2.2.18) is

$$s(t) = e^{\Phi t} s(0) \tag{2.2.19}$$

where s(0) is the initial distance from the sliding surface. Since  $\Phi$  is stable,  $s(t) \to 0$  as  $t \to \infty$  and the sliding motion is not obtained in finite time but asymptotically.

A better condition would be

$$\dot{s}(t) = \Phi s(t) - \rho(t, x) \frac{P_2 s}{\|P_2 s\|}$$
(2.2.20)

where  $\rho(t, x)$  is a scalar design function and  $P_2 \in \mathbb{R}^{m \times m}$  is a symmetric positive definite (s.p.d.) matrix (see Appendix B.2) that satisfies the Lyapunov equation

$$P_2 \Phi + \Phi^T P_2 = -I \tag{2.2.21}$$

Condition (2.2.20) guarantees an ideal sliding motion, i.e s(t) = 0, in finite time. This can be shown by selecting the Lyapunov function  $V(s) = s^T P_2 s > 0$  for  $s \neq 0$ . Then

$$\dot{V}(s) = s^T P_2 \dot{s} + \dot{s}^T P_2 s \tag{2.2.22}$$

Substituting equation (2.2.20) into (2.2.22) gives

$$\dot{V}(s) = s^T (P_2 \Phi + \Phi^T P_2) s - 2\rho(t, x) \parallel P_2 s \parallel$$
(2.2.23)

and from (2.2.21),

$$\dot{V}(s) = - \| s \|^2 - 2\rho(t, x) \| P_2 s \|$$

Since,  $\dot{V}(s) < 0$  when  $s \neq 0$  for all positive  $\rho(t, x)$  and V(s) = 0 in finite time, s = 0 in finite time.

To obtain the control law, substitute (2.2.20) in

$$\dot{s}(t) = S\dot{x}(t) \tag{2.2.24}$$

where the nominal system associated with  $\dot{x}(t)$  from (2.2.1) is used. The control law to ensure that the system attains sliding motion has two distinct parts, the linear part and the non-linear (discontinuous) part, i.e.

$$u(t) = u_l(t) + u_n(t) \tag{2.2.25}$$

Comparing terms yields

$$u_l(t) = -(SB)^{-1}(SA - \Phi S)x(t)$$
(2.2.26)

$$u_n(t) = -\rho(t, x)(SB)^{-1} \frac{P_2 s(t)}{\|P_2 s(t)\|}$$
(2.2.27)

The scalar function  $\rho(t, x)$ , which is chosen depending on the magnitude of the uncertainty (Edwards & Spurgeon 1998), is responsible for the time taken to attain a sliding motion. Therefore, the larger the value of  $\rho(\cdot)$ , the faster the system reaches a sliding motion. However, increasing the value of  $\rho(\cdot)$  would also mean increasing the amplitude of the switching frequency. Practically, this is not ideal as it will mean increasing the wear and tear of actuators. To circumvent this problem, the stable design term  $\Phi$  is employed to effect asymptotic reaching of the sliding mode. Equation (2.2.19) shows that  $\Phi$  affects the rate at which the sliding motion is attained. Therefore  $\rho(\cdot)$  can be chosen to be smaller (to reduce the amplitude of switching) since  $\Phi$  can be chosen to tailor the time taken to achieve the sliding motion. The next section will present a design to illustrate possible solutions to the existence and reachability problem.

### 2.2.3 An Example

The De Havilland-Beaver is a light passenger aircraft with one engine and a maximum speed of approximately 225 km/h. A trimmed and linearised model of the 'DH-Beaver' aircraft was obtained from straight and level flight conditions at a forward speed of  $50ms^{-1}$ . The original aerodynamic coefficients are taken from (Tjee & Mulder 1998). The state-space linear model of the aircraft's lateral dynamics is given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
 (2.2.28)

$$y(t) = Cx(t) \tag{2.2.29}$$

where the states  $x = [\beta \ r \ p \ \phi]^T$  represent the sideslip angle, angular rate of yaw, angular rate of roll and the roll angle respectively. This data is obtained from the *Dhbeaver* aircraft model in (Rauw 1997) linearised at a constant speed of  $50ms^{-1}$ . The system matrices are

$$A = \begin{bmatrix} -0.2110 & -0.9801 & -0.0047 & 0.1950 \\ 0.1046 & -0.3066 & -0.4836 & 0 \\ -2.5016 & 0.9980 & -3.0010 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0076 & 0.0299 \\ -0.1179 & -1.6084 \\ -4.0206 & 0.2457 \\ 0 & 0 \end{bmatrix}$$
(2.2.30)

and  $C = I_4$ .

After transformation into regular form, (2.2.4)-(2.2.5), the matrices

$$A_{11} = \begin{bmatrix} -0.1989 & -0.1947 \\ 0.0024 & 0 \end{bmatrix}$$
(2.2.31)  
$$A_{12} = \begin{bmatrix} -0.0354 & -0.9839 \\ -0.9996 & 0.0293 \end{bmatrix}$$
(2.2.32)  
$$B_{2} = \begin{bmatrix} 4.0223 & -0.1985 \\ 0 & 1.6152 \end{bmatrix}$$
(2.2.33)

The eigenvalues of  $A_{11} - A_{12}K$  are chosen to be -1 and -2. The corresponding value for the matrix  $K = S_2^{-1}S_1$  is

$$K = \begin{bmatrix} -0.0262 & -1.9930\\ & & \\ -0.8133 & 0.2696 \end{bmatrix}$$
(2.2.34)

In the original coordinate system  $S = [K \ I]T$ , where T is the transformation matrix used to attain regular form. This yields

$$S = \begin{bmatrix} 0.0243 & -0.0288 & -0.9996 & -1.9930 \\ 0.8313 & -0.9847 & 0.0273 & 0.2696 \end{bmatrix}$$
(2.2.35)

The design matrix  $\Phi$  was chosen to be

$$\Phi = \begin{bmatrix} -8 & 0\\ 0 & -8 \end{bmatrix}$$
(2.2.36)

The symmetric positive definite matrix  $P_2$  is chosen as the solution to the Lyapunov matrix

equation  $(P_2\Phi + \Phi^T P_2) = -I$  and is found to be

$$P_2 = \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.0625 \end{bmatrix}$$
(2.2.37)

With the choice of  $\rho = 0.1$  and initial conditions as  $x = \begin{bmatrix} 0 & 0 & 0 & 0.1 \end{bmatrix}$ , the aircraft response was simulated and the closed-loop response is shown in Figures 2.2.1,2.2.2 and 2.2.3.



Figure 2.2.1: Plot of states x(t) against time (t)



Figure 2.2.2: Plot of switching function s(t) against time (t)

Practically, the discontinuous control component in (2.2.27) is undesirable. To smooth the discontinuity in the control action, a continuous approximation has been used. A small positive scalar,  $\delta$  has been introduced in the switching function (Edwards & Spurgeon 1998). Equation (2.2.27) becomes

$$u_n(t) = -\rho(t, x)(SB)^{-1} \frac{P_2 s(t)}{\|P_2 s(t)\| + \delta}$$
(2.2.38)

and (2.2.20) is written as

$$\dot{s}(t) = \Phi s(t) - \rho(t, x) \frac{P_2 s(t)}{\|P_2 s(t)\| + \delta}$$
(2.2.39)

The larger the value of  $\delta$ , the smoother the signal. However, too big a  $\delta$  will cause a deviation from ideal performance and this is not desirable either. Therefore, there is a



Figure 2.2.3: Plot of input signal u(t) against time (t)

trade-off in the design between the requirement of maintaining ideal performance and that of ensuring a smooth control action. Figure 2.2.3 shows the control input without the smoothing action. Compare this with Figure 2.2.4 which shows the control input when  $\delta = 0.0001$  has been introduced.

#### 2.2.4 Properties of the Sliding Motion

From the brief introduction to continuous time sliding mode control (CSMC), it can be summarized that the choice of the switching function determines the performance response of the system, whereas the control law is designed to guarantee that a sliding motion will take place in finite time. The key properties of sliding mode control are that whilst sliding, the system experiences order reduction and is completely insensitive to any uncertainty which lies within the range space of the input distribution matrix, i.e. matched uncertain-



Figure 2.2.4: Plot of input signal u(t) against time (t) with smoothing action

ties.

Another important observation is that the reduced-order *ideal sliding mode* is governed by the invariant zeros of the (fictitious) system triple (A, B, S).

**Proposition 2.2.1** The invariant zeros of (A, B, S) are the poles of the sliding motion.

**Proof** (Edwards & Spurgeon 1998) By definition, the invariant zeros of the system (A, B, S) are given by

 $\{z \in \mathbb{C} : P(z) \text{ loses normal rank}\}$ 

where Rosenbrock's system matrix P(z) (Rosenbrock 1970) is given by

$$P(z) = \begin{bmatrix} zI - A & B \\ -S & 0 \end{bmatrix}$$
(2.2.40)

Substituting from equations (2.2.8),(2.2.9) and (2.2.11), Rosenbrock's system matrix loses rank if and only if

$$\det(P(z)) = \det \begin{bmatrix} zI - A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & B_2 \\ -S_1 & -S_2 & 0 \end{bmatrix} = 0$$
(2.2.41)

With the assumption that  $B_2$  is nonsingular

$$\det(P(s)) = 0 \Leftrightarrow \det \begin{bmatrix} zI - A_{11} & -A_{12} \\ -S_1 & -S_2 \end{bmatrix} = 0$$
(2.2.42)

Since

$$\begin{bmatrix} zI - A_{11} & A_{12} \\ S_1 & S_2 \end{bmatrix} \equiv \begin{bmatrix} I & A_{12}S_2^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} zI - (A_{11} - A_{12}K) & 0 \\ 0 & -S_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

and the left and right matrices are both independent of z and have determinant equal to unity,

$$\det(P(z)) = 0 \Leftrightarrow \det \begin{bmatrix} zI - (A_{11} - A_{12}K) & 0\\ 0 & -S_2 \end{bmatrix} = 0$$
(2.2.43)

This means that  $\det(P(z)) = 0 \Leftrightarrow \det(zI - (A_{11} - A_{12}K)) = 0$  since  $\det(S_2) \neq 0$ . Therefore, this proves that the invariant zeros of (A, B, S) are the eigenvalues of  $(A_{11} - A_{12}K)$ , i.e the poles of the reduced-order sliding motion (2.2.15).

This brings about a restriction to the class of systems for which CSMC is applicable when only output information is available. If S depends only on output information then S = FC for some  $F \in \mathbb{R}^{m \times p}$ . Since the invariant zeros of (A, B, C) are a subset of the invariant zeros of (A, B, FC), it follows the system (A, B, C) needs to be minimum phase (with invariant zeros in the open left half plane) in order to use CSMC. Furthermore, in order for det $(SB) \neq 0$  (which is required in (2.2.26) and (2.2.27)) the condition rank(CB) = mmust hold. These restrictions can be described as relative degree one minimum phase conditions. For discrete systems, a new design methodology to overcome both will be presented in later chapters.

### 2.3 Discrete Time Sliding Mode Control

In recent years, a large number of (continuous time) systems are now computer or digitally controlled. This means working with digital signals instead of continuous time ones, i.e. converting continuous time data into sampled or digital data. Sampling is a basic property of computer controlled systems because of the discrete nature of the digital computer. The sampling of a continuous time signal replaces the original signal by a sequence of values at discrete time instances. If the sampling intervals are sufficiently small, not much information is lost and the *signal reconstruction* (see Appendix A.3) is relatively accurate. However, if the sampling points are too far apart, significant information about a signal can be lost. For example, Figure 2.3.1 shows what happens when a sine function is sampled at the rate of two samples per period (Aström & Wittenmark 1984).

A data hold is used to express a sampled signal in a form that closely resembles the continuous time signal. With a zero-order hold (Figure 2.3.2), a value is held constant



Figure 2.3.1: Slow sampling invariably causes loss of information.

until the next sampling instant. Because of its simplicity, it is common in computercontrolled systems. The zero-order hold can be regarded as an extrapolation using a polynomial of degree zero. It is however possible to attain smaller reconstruction errors by using higher order holds (Aström & Wittenmark 1984).

When a sample and hold device is used together with a continuous time plant, it is always desirable to sample at the fastest possible rate to get a good approximation to what is happening in continuous time. This is particularly crucial for sliding mode control systems where, theoretically, sampling at infinite frequency is desirable. However, there is always a limit to how fast one can sample. The question of what happens when a system can only be sampled at an inherently slow sampling rate becomes pertinent. This can be shown by putting a sample and hold on the aircraft simulation in Section 2.2.3, and assuming a relatively slow sampling rate. This means that the control input is carried out only at



Figure 2.3.2: Sampling and zero-order hold reconstruction of a continuous time signal. discrete instants, with a switching frequency equal to or lower than the sampling frequency. Figure 2.3.3 shows the response of the states when the sampling frequency is 0.1s. Figure 2.3.4 is a plot of the switching function against time. Compare these with Figure 2.2.1 and Figure 2.2.2 in the previous section.

With a comparatively slow switching frequency, the system states move in a 'zigzag' manner about the switching surface. This motion, whereby the system trajectories keep crossing and re-crossing the sliding surface, is known as *chattering*. As the sampling rate is reduced, the system behavior changes from sliding on the switching line to excessive 'zigzagging', and further reduction leads to instability. This is illustrated in Figures 2.3.5 and 2.3.6.



Figure 2.3.3: Plot of states against time with sampling rate of 0.1s.



Figure 2.3.4: Plot of switching function against time


Figure 2.3.5: Plot of states against time with sampling rate of 0.25s.



Figure 2.3.6: Plot of states against time with sampling rate of 0.3s.

The effect of sample and hold implementations has prompted the study of sliding mode ideas applied to discrete time systems (Chan 1994, Chan 1998, Furuta 1990, Milosavljević 1985, Sapturk et al. 1987, Spurgeon 1992). Discrete time sliding modes were first named 'quasi-sliding modes' by Milosavljević (1985) after studying the oscillatory characteristics in the neighborhood of the discontinuity surfaces due to the discretization of control signals. This behavior was later called 'pseudo-sliding modes' by Yu (1994) because the similarity between discrete time sliding modes and continuous time sliding modes disappears as the sampling interval increases with the system trajectory 'zigzagging' within a bounded region. The main differences between CSMC and discrete time sliding mode control (DSMC) is in the modelling of the system under control and the implementation of the control law. CSMC invariably uses a non-linear discontinuous control law (with both linear and non-linear component) whereas DSMC may use a purely linear control law and does not necessarily require the use of a variable structure discontinuous control strategy (Hui & Źak 1999, Koshkouei & Zinober 2000, Spurgeon 1992, Su, Drakunov & Özgüner 2000). There is a fundamental difference between CSMC and DSMC: In continuous time it is assumed that the control signal can switch with infinite frequency, i.e. switching is done at any instant, whenever the state trajectories cross the switching surface. This is required in order to obtain robustness to matched uncertainty. In discrete time, this property is lost. The control input is computed at discrete instants and applied at a sampling interval. In other words, the control signal only changes at the sample instances, and the sampling frequency is limited. Therefore, the rate of which sampling is done modulates the extent of the deviation from the ideal sliding mode.

#### 2.3.1 State-feedback Discrete Time Sliding Mode Control

As in CSMC, the design of a DSMC controller can be (though is not necessarily) said to consist of two parts :

- Existence problem/ Hyperplane design : constructing a surface, s(k) = Sx(k), on which the dynamics of the system are stable when s(k) = 0.
- *Reachability phase* : designing a control law which drives the states towards the sliding surface and keeps them as close as possible to the surface.

The first step in the design of a sliding mode controller (in discrete time) is similar to that in CSMC. Here, existing CSMC theories (Edwards & Spurgeon 1998, DeCarlo, Zak & Matthews 1988, Hung, Gao & Hung 1993) can, quite straightforwardly, be extended to discrete time (Monsees 2002).

Consider the nominal system in (2.2.1), discretised at a sampling interval  $\tau$ 

$$x(k+1) = Gx(k) + Hu(k)$$
(2.3.1)

with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ . The pair (G, H) are the discrete time counterparts of (A, B)and are assumed to be fully controllable (see Appendix B.3).

Define

$$s(k) = Sx(k) \tag{2.3.2}$$

where S is a design parameter chosen such that SH is nonsingular. The system shown in (2.3.1) is transformed into a suitable canonical form, where the nominal linear system is decomposed into two subsystems which describe its null space dynamics and range space dynamics, i.e a discrete time version of the regular form from §2.2.1. The nominal system is now written in the form

$$x_1(k+1) = G_{11}x_1(k) + G_{12}x_2(k)$$
(2.3.3)

$$x_2(k+1) = G_{21}x_1(k) + G_{22}x_2(k) + H_2u(k)$$
(2.3.4)

The matrix S is chosen so that the dynamics of the ideal sliding mode (i.e. a situation when Sx(k) = 0 for  $k > k_s$ ) are stable in discrete time, i.e. all poles of the reduced-order system are inside the unit disk.

The construction of a control law which drives the system into the 'sliding mode' is a slightly different problem in discrete time. The definition of the reachability condition is not the same as in continuous time. This will be explored in the next subsection.

# 2.3.2 Reachability Conditions

The main focus of DSMC has always been to find a suitable reachability condition such that when the sample interval,  $\tau$ , tends to zero, the continuous time sliding mode reachability conditions are satisfied. Generally, in continuous time systems with continuous control, the sliding surface can only be reached asymptotically but in discrete time systems with continuous control, the sliding motion may be attained after a finite time interval. Poles of the closed-loop system can be assigned at the origin so that the system is driven to the sliding surface in (at most) n sampling periods, where n is the number of states. This is known as *deadbeat response* (see Appendix A.3). The problem with DSMC is the intersample behaviour. If the sampling interval is large, then there will be a 'zigzagging' effect (Yu 1994) and the motion will not lie close enough to the sliding surface and the states will deviate significantly from the sliding surface.

Discrete time sliding modes were first investigated by Milosavljević (1985). The conditions (2.2.16) and (2.2.17) were translated for the discrete time case as

$$(s(k+1) - s(k))s(k) < 0 \tag{2.3.5}$$

and

$$\lim_{s(k)\to 0^+} \Delta s < 0 \qquad \text{and} \qquad \lim_{s(k)\to 0^-} \Delta s > 0 \tag{2.3.6}$$

where  $\Delta s = s(k+1) - s(k)$ . However, although these conditions are necessary, they are not sufficient for the existence of a discrete time sliding mode and only guarantee that the state trajectories approach (and maybe cross) the sliding surface. They do not ensure convergence of the state trajectories onto the sliding surface itself.

Sapturk et al. (1987) argue that, unlike in the continuous time case where only one bound suffices for the control, in discrete time the control must be upper and lower bounded. They proposed the reaching condition

$$|s(k+1)| < |s(k)| \tag{2.3.7}$$

which can be decomposed into:

$$(s(k+1) - s(k))sign(s(k)) < 0$$
(2.3.8)

$$(s(k+1)+s(k))sign(s(k)) \ge 0$$
 (2.3.9)

Inequalities (2.3.8) and (2.3.9) give an upper and lower bound for the control which, according to Kotta (1989), depends on the distance of the system from the sliding surface. These conditions are sufficient, but according to Spurgeon (1992), not necessary.

The condition in (2.3.7) is not dissimilar to the one proposed by Furuta (1990) in which

$$s^2(k+1) < s^2(k)$$

and Sira-Ramirez (1991) with

$$|s(k+1)s(k)| < s^2(k)$$

Yet another way of approaching this problem was introduced by Gao et al. (1995) who suggested that the closed-loop system should have the following properties:

- Starting from any initial state, the trajectory should move monotonically towards the switching surface and cross it in finite time.
- Once the trajectory has crossed the switching surface for the first time, it should then cross the surface again in every successive sampling interval, resulting in a zigzag motion about the switching surface.
- The size of each successive step should not increase and the trajectory must remain within a specified band.

The above conditions were proposed for a single input system. However, they can easily

be applied to a multi input system by applying the three conditions to the m entries of s(k) independently (Monsees 2002). The reaching law proposed by Gao et al. (1995) for the single input case is given by

$$s(k+1) - s(k) = -q\tau s(k) - \epsilon\tau sign(s(k))$$

$$(2.3.10)$$

where  $\tau$  is the sampling interval,  $\epsilon$ , q are positive constants and  $(1 - q\tau) > 0$ . This can be extended to the multi-input case as

$$s(k+1) = \Phi s(k) - \begin{bmatrix} K_{s,1} sign(s_1(k)) \\ K_{s,2} sign(s_2(k)) \\ \vdots \\ K_{s,m} sign(s_m(k)) \end{bmatrix}$$
(2.3.11)

where  $\Phi \in \mathbb{R}^{m \times m}$  is some diagonal matrix with  $0 \leq \Phi_{i,i} < 1 \quad \forall i = 1 \dots m$  and the gains  $K_{s,i} > 0 \quad \forall i = 1 \dots m$ .

In a multi-input framework, Koshkouei & Zinober (2000) stated that a sufficient condition for the existence of the sliding mode can be given as

$$\|s(k+1)\| < \eta \|s(k)\| \tag{2.3.12}$$

where  $0 < \eta < 1$  is a real number. In this case

$$\|s(k)\| < \|s(0)\|\eta^k \tag{2.3.13}$$

where s(0) is any initial condition. Koshkouei & Zinober (2000) proposed a discrete

time sliding mode controller with a *lattice-wise hyperplane*, a surface on which there is a countable set of points forming a so-called lattice. The velocity with which the state reaches the sliding lattice hyperplane depends on the value of  $\eta$  and the time taken to attain the sliding mode depends on the initial condition x(0).

The (multi-input) reaching law used by Hui &  $\hat{Z}$ ak (1999) for a multi-input discrete time system, is

$$s(k+1) = \Phi s(k) \tag{2.3.14}$$

where  $\Phi \in \mathbb{R}^{m \times m}$  is a diagonal matrix satisfying  $0 \leq \Phi_{i,1} < 1 \forall i = 1...m$ . This is also called the Linear Reaching Law (Monsees 2002). Notice that this reaching law is similar to that in (2.3.11) with the switching term left out. In (Spurgeon 1992) it was shown that a simple linear control law, together with a suitable sliding surface design, can provide better performance, in terms of minimising the bounds around the sliding surface, compared with a more complicated non-linear control structure with an inappropriate choice of sliding surface.

# 2.3.3 Formulation as a Min-Max Control Problem

As argued in the previous section, in discrete time, it is not possible in general to attain ideal sliding as the control signal remains constant between sampling times and is computed at discrete instances. One paradigm is to design the control law to keep the states as close as possible to the sliding surface and the problem becomes one of minimising sensitivity to the system uncertainty. From this point of view, the DSMC problem can be viewed as a robust optimal control problem and is related to discrete time Lyapunov min-max problems (Corless & Manela 1986, Manela 1985). Indeed both (Spurgeon 1992) and (Hui & Zak 1999) pose the DSMC problem as an appropriately formulated Lyapunov min-max problem. These ideas will be expanded here and it will be shown how they relate to DSMC.

Consider an uncertain discrete time system

$$x(k+1) = Gx(k) + H(u(k) + \xi(k))$$
(2.3.15)

with matched uncertainties,  $\xi(k)$ , which are assumed to belong to a 'balanced set' (Corless & Manela 1986)<sup>1</sup>, and where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ . Assume without loss of generality that H is full rank. Define a Lyapunov function candidate as

$$V(k) = x^{T}(k)Px(k)$$
 (2.3.16)

where P > 0 is a symmetric positive definite (s.p.d.) matrix. The Lyapunov difference function is defined as

$$\Delta V(k) = V(k+1) - V(k)$$
(2.3.17)

Consider initially a nominal regulation problem when no uncertainty is present  $(\xi(k) \equiv 0)$ . In the absence of uncertainty, an ideal sliding motion can be attained on the sliding surface

$$S = \{x : Sx = 0\}$$
(2.3.18)

As in (Spurgeon 1992) the class of sliding surfaces will be restricted to those which can be

<sup>&</sup>lt;sup>1</sup>Suppose the uncertainty  $\xi(k) \in \mathcal{F}$ , then  $\mathcal{F}$  is a balanced set if  $\xi(k) \in \mathcal{F} \Rightarrow -\xi(k) \in \mathcal{F}$ 

expressed in the form  $S = H^T P$ . It follows from (2.3.18) that

$$Sx(k+1) = H^T P G x(k) + H^T P H u(k) = 0$$

The equivalent control action necessary to maintain an ideal sliding motion is given by

$$u_{eq}(k) = -(H^T P H)^{-1} H^T P G x(k)$$
(2.3.19)

If P is such that the closed-loop system, obtained from using the control law (2.3.19) in (2.3.15), satisfies  $\Delta V(k) < 0$  for all k, then from standard Lyapunov theory the closedloop system is stable (see Appendix B.1). After some simple algebra, it can be shown that  $\Delta V(k) \equiv -x(k)^T Q x(k)$  where

$$Q := P + G^T P H (H^T P H)^{-1} H^T P G - G^T P G$$
(2.3.20)

Thus if Q > 0, then in the absence of uncertainty,  $x(k) \to 0$  as  $k \to \infty$ . The control law in (2.3.19), where P is such that Q > 0 in (2.3.20), is referred to as a *stabilizing min-max controller*<sup>2</sup>. The stability follows from Lyapunov arguments and is most easily deduced from observing that inequality (2.3.20) is identical to

$$P - G_c^T P G_c > 0 \tag{2.3.21}$$

where the closed-loop system matrix

$$G_c := G - H(H^T P H)^{-1} H^T P G$$
(2.3.22)

<sup>&</sup>lt;sup>2</sup>Sharav-Schapiro et al. refer to this as a Riccati min-max control law (Sharav-Schapiro et al. 1998).

Consequently if in (2.3.20) Q > 0, or equivalently (2.3.21) holds, then the closed-loop system matrix  $G_c$  is stable.

**Proposition 2.3.1** For the uncertain discrete time system in (2.3.15), the control law (2.3.19), with P chosen so that Q from (2.3.20) is s.p.d, has the property that it

- a) induces an ideal sliding motion on S in finite time when  $\xi(k) \equiv 0$ ;
- b) minimises the effect of  $\xi(k)$  on the closed-loop dynamics in a min-max sense i.e. the control law in (2.3.19) uniquely satisfies

$$\min_{u} \left( \max_{\xi \in \mathcal{F}} \Delta V(\xi, u) \right)$$
(2.3.23)

over all possible state feedback controllers.

**Proof** Using the system equation (2.3.15) when  $\xi(k) \equiv 0$  it follows that

$$Sx(k+1) = H^T P G x(k) + H^T P H u(k)$$

and so substituting for u(k) from (2.3.19) ensures Sx(k+1) = 0. This proves that an ideal sliding mode is induced in finite time.

In the uncertain case, by definition

$$V(k+1) = (Gx(k) + H(u(k) + \xi(k)))^T P(Gx(k) + H(u(k) + \xi(k)))$$
(2.3.24)

Suppose the optimal min-max control law i.e. the control law which minimises the worst

case effect of  $\xi$  on  $\Delta V(k)$  over all possible control laws has the form

$$u^{*}(k) = -(H^{T}PH)^{-1}H^{T}PGx(k) + w(k)$$
(2.3.25)

$$= u_{eq}(k) + w(k) (2.3.26)$$

where the component w(k) is yet to be determined. Clearly any (possibly discontinuous) control law can be written in this way for an appropriate choice of w(k). Substituting for (2.3.25) in (2.3.24) and collecting terms yields

$$\Delta V(k) = -x(k)Qx(k) + \xi(k)^{T}(H^{T}PH)\xi(k) + 2\xi(k)^{T}(H^{T}PH)w(k) + w(k)^{T}(H^{T}PH)w(k)$$

where Q is defined in (2.3.20).

For any given w(k) let  $\hat{\xi}$  be the value from the uncertainty set  $\mathcal{F}$  which maximizes  $\Delta V(w,\xi)$ with respect to  $\xi$ . For the optimum  $\hat{\xi}$  it follows that

$$2\hat{\xi}^T (H^T P H) w \ge 0 \tag{2.3.27}$$

To prove this suppose for a contradiction that  $2\hat{\xi}^T(H^T P H)w < 0$ . In this case, since by assumption  $\mathcal{F}$  is a balanced set,  $-\hat{\xi} \in \mathcal{F}$  and so

$$\begin{aligned} \Delta V(w, -\hat{\xi}) &= -x^T Q x + (-\hat{\xi})^T (H^T P H) (-\hat{\xi}) - 2\hat{\xi}^T (H^T P H) w + w^T (H^T P H) w \\ &> -x^T Q x + \hat{\xi}^T (H^T P H) \hat{\xi} + 2\hat{\xi}^T (H^T P H) w + w^T (H^T P H) w \\ &= \Delta V(w, \hat{\xi}) \end{aligned}$$

which contradicts  $\hat{\xi}$  maximizing  $\Delta V(w,\xi)$  over  $\mathcal{F}$  and so (2.3.27) must hold. Also from

the Cauchy-Schwarz inequality

$$\hat{\xi}^T (H^T P H) w = \alpha \| (H^T P H) \hat{\xi} \| \, \| w \|$$

where  $0 \le \alpha \le 1$  is a constant which represents the direction cosine between the vectors  $(H^T P H)\hat{\xi}$  and w. The scalar  $\alpha \ge 0$  because  $\hat{\xi}^T (H^T P H) w \ge 0$ . Consequently

$$\max_{\xi \in \mathcal{F}} \Delta V(w,\xi) = -x^T Q x + \hat{\xi}^T (H^T P H) \hat{\xi} + 2\alpha \| (H^T P H) \hat{\xi} \| \| w \| + w^T (H^T P H) w$$
(2.3.28)

Since  $H^T P H > 0$  and  $\alpha \ge 0$  it follows that

$$2\alpha \| (H^T P H) \hat{\xi} \| \| w \| + w^T (H^T P H) w > 0 \quad \text{for all } w \neq 0$$

and consequently from equation (2.3.28)

$$\min_{w}(\max_{\xi\in\mathcal{F}}\Delta V(w,\xi)) = -x^{T}Qx + \hat{\xi}^{T}(H^{T}PH)\hat{\xi}$$

which is obtained (uniquely) by selecting  $w(k) \equiv 0$ . Thus the unique optimal min-max controller obtained from setting  $w(k) \equiv 0$  in (2.3.25) yields (2.3.19) as claimed.

**Remark 2.3.1** State feedback controllers of the form in (2.3.19) were originally introduced in the context of discrete time optimal control (Corless & Manela 1986, Manela 1985) rather than from a discrete-time sliding mode perspective.

**Remark 2.3.2** Proposition 2.3.1 shows that, in a min-max sense, the control law (2.3.19)

minimises the worst case effect of the uncertainty on  $\Delta V(k)$  over all possible control laws, including controllers with discontinuous switched terms, which are typically used in a continuous time sliding mode context. This justifies the use of a purely linear control law in the discrete time sliding mode scenario (Spurgeon 1992, Hui & Źak 1999, Koshkouei & Zinober 2000).

Hui &  $\hat{Z}$ ak (1999) pose the min-max control problem slightly differently where the distance from the sliding mode is minimised with regards to bounded uncertainty. In Hui &  $\hat{Z}$ ak (1999) the following uncertain system model was considered

$$x(k+1) = Gx(k) + Hu^{*}(k) + \xi(k)$$

with  $\xi(k)$  as the system uncertainties which are assumed to belong to a balanced set. The following is a modification of the results in Hui &  $\hat{Z}$ ak (1999)

**Proposition 2.3.2** The control law (2.3.19) minimises the deviation from the surface

$$\mathcal{S} = \{ Sx \in \mathbb{R}^n : Sx = 0 \}$$

in a min-max sense, where  $S = H^T P$  for some s.p.d. matrix P. Specifically, it solves the problem

$$\min_{u} (\max_{\xi \in \mathcal{F}} \|Sx(k)\|)$$

**Proof** Assume the optimal control law  $u^*(k)$  has the form (2.3.26). Then, using (2.3.2) it

follows

$$s(k+1) = Sx(k+1)$$
  
=  $S(Gx(k) + Hu^*(k) + \xi(k))$   
=  $(SH)((SH)^{-1}SGx(k) + u^*(k) + (SH)^{-1}S\xi(k))$   
=  $(SH)((u^*(k) - u_{eq}(k)) + (SH)^{-1}S\xi(k))$  (2.3.29)

Taking the norm of both sides of the above equation yields

$$\|s(k+1)\|^{2} = \|(SH)(u^{*}(k) - u_{eq}(k))\|^{2} + 2(u^{*}(k) - u_{eq}(k))^{T}(SH)^{T}S\xi(k) + \|S\xi(k)\|^{2}$$
(2.3.30)

Substituting (2.3.26) into (2.3.30) gives

$$||s(k+1)||^{2} = ||(SH)w(k)||^{2} + 2w(k)^{T}(SH)^{T}S\xi(k) + ||S\xi(k)||^{2}$$

As in the proof of Proposition 2.3.1 suppose  $\max_{\xi \in \mathcal{F}} ||s(k+1)||^2$  is given by  $\hat{\xi} \in \mathcal{F}$ . Then, (as argued in Proposition 2.3.1)  $2w^T (SH)^T S\hat{\xi} \ge 0$  and so

$$\max_{\xi \in \mathcal{F}} \|s(k+1)\|^2 = \|(SH)w\|^2 + 2\alpha \|(SH)^T S\hat{\xi}\| \|w\| + \|S\hat{\xi}\|^2$$

where  $0 \leq \alpha < 1$  is the direction cosine between the vector  $(SH)^T S\hat{\xi}$  and w. It follows that

$$\min_{w}(\max_{\xi \in \mathcal{F}} \|s(k+1)\|^2) = \|S\hat{\xi}\|^2$$

obtained from setting w = 0. The min-max controller shown by Hui & Źak (1999) for the distance to the switching surface is therefore  $u^*(k) = u_{eq}(k)$ , which is a linear controller.

# 2.4 Summary

With the advancement of digital technology and computers, implementation of controllers using digital signals has motivated the need for discrete time sliding mode control. In discrete time, ideal sliding cannot be achieved in the presence of uncertainty and so the reaching law must try to maintain the smallest sliding mode boundary layers within which the system states stay. This can be viewed as an optimization problem where the objective is to minimise the effect on the Lyapunov difference function of the worst case uncertainty. Hence, there is a connection between DSMC and so-called min-max controllers. This connection is vital to all the theoretical work which is developed in the subsequent chapters. The implications of this observation will be used in Chapter 3 to develop output DSMC, where a new sliding surface with a direct link to min-max controllers will be introduced.

# Chapter 3

# Discrete Output Feedback Sliding Mode Control

# 3.1 Introduction

In Chapter 2, sliding mode control has been predominantly discussed in a framework in which all the system states are available. This is not very realistic for practical engineering problems. In discrete time, sliding mode schemes which only require output information have been proposed by Sira-Ramirez (1991), Bartolini et al. (2000), Bartolini et al. (2001) and Monsees (2002). As discussed in §2.2.4, in continuous time the use of only output information limits the class of systems for which the sliding mode approach is applicable to relative degree one minimum phase systems (Edwards & Spurgeon 1998). The discrete time results of Sira-Ramirez (1991), Bartolini et al. (2000), Bartolini et al. (2001) and Monsees (2002) also implicitly require minimum phase conditions (in a discrete time

sense). This chapter presents a novel approach to the problem of discrete time output feedback sliding mode controller (ODSMC) design, where the system to be controlled may be non-minimum phase. The conditions which must hold to employ the design strategy are investigated and the importance of a certain subsystem triple is established. The chapter will conclude with an aircraft example to illustrate the application of the approach described.

# 3.2 **Problem Formulation**

Consider the discrete time system with matched uncertainties

$$x(k+1) = Gx(k) + H(u(k) + \xi(k))$$
(3.2.1)

$$y(k) = Cx(k) \tag{3.2.2}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  with  $m \leq p < n$ . Assume that the input and output distribution matrices H and C are full rank. In addition, assume the pair (G, H)is controllable. The matched uncertainties,  $\xi(k)$ , are unknown but are assumed to belong to a balanced set (Corless & Manela 1986).

The objective is to determine an appropriate linear sliding surface of the form

$$S = \{x : Sx = 0\}$$
(3.2.3)

where  $S \in \mathbb{R}^{m \times n}$ , and a control law which depends only on the measured outputs such that:

- for the nominal linear system when  $\xi \equiv 0$  an *ideal* sliding motion is obtained in finite time;
- for uncertain systems the effect of the matched uncertainty  $\xi$  is minimised and an appropriate bounded motion about S is maintained.

Suppose as in (Spurgeon 1992) the sliding surface is chosen as

$$S = H^T P \tag{3.2.4}$$

where P is a s.p.d. matrix and define

$$V(k) = x(k)^T P x(k)$$

as a Lyapunov function candidate. Then as shown in Proposition 2.3.1, the optimal state feedback control law, including all nonlinear state dependent controllers which minimises the Lyapunov function difference  $\Delta V(k) = V(k+1) - V(k)$  with respect to u, for the worst case  $\xi$ , is given by

$$u_{eq}(k) = -(H^T P H)^{-1} H^T P G x(k)$$
(3.2.5)

This means that if P is such that the closed-loop system, obtained from using the control law (3.2.5) in (3.2.1), satisfies  $\Delta V(k) < 0$  for all k, then from standard Lyapunov theory the closed-loop system is stable (see Appendix B.1). By straightforward algebraic manipulation, it can be shown that

$$\Delta V(k) = -x(k)^T (P + G^T P H (H^T P H)^{-1} H^T P G - G^T P G) x(k) + \xi^T(k) H^T P H \xi(k)$$
(3.2.6)

and so if

$$Q := P + G^T P H (H^T P H)^{-1} H^T P G - G^T P G > 0$$
(3.2.7)

then, in the absence of uncertainty,  $x(k) \to 0$  as  $k \to \infty$ .

Generally, all states of the system are required in order to implement the control law in (3.2.5). If the control law is to be realized using only measured outputs, it follows that the right hand side of (3.2.5) must be able to be written in the form

$$(H^T P H)^{-1} H^T P G = Y C$$

for some  $Y \in \mathbb{R}^{m \times p}$ . Since a discontinuous term is not going to be employed here *it is not* a requirement that the switching function matrix (3.2.4) must be expressed in terms of the outputs. This is a key observation which will be used in the rest of this thesis.

Define a sliding surface for the system as

$$S = \{x : FCG^{-1}x = 0\}$$
(3.2.8)

where  $F \in \mathbb{R}^{m \times p}$  is the design freedom in the problem. Comparing the expression for the sliding surface (3.2.8) with the one from (3.2.4), it follows that

$$FCG^{-1} = H^T P \tag{3.2.9}$$

must hold for some s.p.d matrix P.

**Remark 3.2.1** The control structure (3.2.5) is also similar to the one which arises from considering a special case of the discrete quadratic optimal regulation problem under the assumption of 'cheap control'. This is discussed in the recent paper (Garcia et al. 2003) which provides an overview of recent work in the area of robust static output feedback control.

**Remark 3.2.2** Inequality (3.2.7) is the steady-state Riccati inequality associated with a state-feedback discrete LQR problem in the special case in which the penalty on control effort is zero (Ogata 1995) and hence for which the cost function to be minimised is  $J = \sum_{k=0}^{\infty} x(k)^T Q x(k)$ . The static output feedback problem which results from including the constraint  $H^T PG = FC$  from (3.2.9) leads to the static output feedback LQR problem studied in (Garcia et al. 2003).

**Remark 3.2.3** The fact that the uncertainty acts through the input distribution matrix (i.e. it is matched) is exploited when establishing the expression for the control law in (3.2.5). As a result, the focus of this chapter will be to develop a tractable design procedure to solve (3.2.7) subject to the constraint (3.2.9). Because of the explicit solution to the min-max optimization problem embedded in the control law given in (3.2.5), the design procedure in §3.3 primarily requires only knowledge of the nominal plant representation (G, H, C).

In the previous chapter (§2.3.3), a link was made between DSMC and state feedback minmax controllers in which the feedback gain is chosen to minimise over all possible statefeedback controllers the worst case effect of the uncertainty on the Lyapunov difference function. These results were later extended to the static output feedback case in (Sharav-Schapiro et al. 1996, Sharav-Schapiro et al. 1998) where some of the conditions under which static output feedback stabilizing min-max controllers (SOMMC) can be realized are discussed. In particular, this problem requires the simultaneous solution of a structural constraint and a Riccati equation. For square systems, a simple existence test for the existence of a SOMMC is described in (Sharav-Schapiro et al. 1998). In the square case, the SOMMC gain is completely determined by the plant triple. For non-square systems, establishing an equivalent test is an open problem as there is more design freedom which needs to be used appropriately.

Here the open problem of ODSMC/ SOMMC for non-square systems will be considered and so it will be assumed that p > m. The next section considers conditions under which this problem is solvable and proposes a new parameterisation for the design freedom available in F.

# 3.3 Conditions for Realising the Controller

Throughout the remainder of the chapter it will be assumed that

- A1) the state transition matrix G is nonsingular
- A2) the matrix  $CG^{-1}H$  has rank m.

**Remark 3.3.1** Assumption A1 is not a strong assumption and most discrete systems satisfy this requirement.

Remark 3.3.2 Assumption A2 is a system property and is independent of the choice of

coordinate system. It is a necessary condition to find a s.p.d. matrix P and an  $F \in \mathbb{R}^{m \times p}$  to solve (3.2.9). This can be seen as follows: Assuming A1 is satisfied, equation (3.2.9) is satisfied if and only if  $H^T P = FCG^{-1}$  and consequently  $H^T P H = FCG^{-1}H$ . Since P is s.p.d. and H is full rank,  $H^T P H$  is full rank and hence  $rank(FCG^{-1}H) = m$ . This implies both F and  $CG^{-1}H$  must be rank m.

**Remark 3.3.3** Assumption A2 is necessary for the theoretical developments but also has a system theoretic interpretation: if  $\mathcal{G}(z) := C(zI - G)^{-1}H$  is the transfer function representation of the plant then  $\mathcal{G}(0) = -CG^{-1}H$  and so A2 is equivalent to the system (G, H, C) not having any invariant zeros at the origin. Thus assumptions A1 and A2 preclude (G, H, C) from having poles or zeros at the origin. Whilst these are limitations, they are not particularly strong.

**Remark 3.3.4** Assuming there exists a s.p.d. matrix P and an F such that (3.2.7) and (3.2.9) hold, then the discrete output feedback sliding mode control law (3.2.5) can be written as

$$u_{omm}(k) = -(FCG^{-1}H)^{-1}Fy(k)$$
(3.3.1)

The controller design problem may therefore be viewed as the problem of finding a matrix F and a s.p.d. matrix P such that both (3.2.7) and (3.2.9) hold.

Based on assumptions A1 and A2, a change of coordinates will be introduced which facilitates both insight into the class of systems for which this problem is solvable and a constructive design procedure for its solution.

## 3.3.1 Necessary Conditions

From assumption A1,  $det(G) \neq 0$  and so equation (3.2.9) can be re-written as

$$H^T P = F C G^{-1} \tag{3.3.2}$$

Define a new matrix

$$L := CG^{-1} (3.3.3)$$

This matrix will take the role of the output distribution matrix for a new purely fictitious system (G, H, L) which will be necessary for the theoretical developments. In order to facilitate the analysis, a change of coordinates will be introduced for the fictitious system (G, H, L). By definition, and from Assumption 2, rank(LH) = m. As argued in (Edwards & Spurgeon 1995), since rank(LH) = m, there exists a change of coordinates  $x \mapsto \bar{T}x = \bar{x}$ such that

$$\bar{T}G\bar{T}^{-1} = \bar{G} = \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix}, \quad \bar{T}H = \bar{H} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},$$
$$L\bar{T}^{-1} = \bar{L} = \begin{bmatrix} 0_{p \times (n-p)} & T \end{bmatrix}$$
(3.3.4)

where  $\bar{G}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ ,  $H_2 \in \mathbb{R}^{m\times m}$  and is nonsingular and  $T \in \mathbb{R}^{p\times p}$  is orthogonal.

Write the design matrix F from (3.3.1) as

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^T \tag{3.3.5}$$

where  $K \in \mathbb{R}^{m \times (p-m)}$  and  $F_2 \in \mathbb{R}^{m \times m}$  is nonsingular. This effectively re-parameterizes the design freedom in F into two new parameters K and  $F_2$  which are to be determined. It is a particularly convenient parameterisation as K can be interpreted as a state output feedback gain for a particular subsystem. From the definition in (3.3.5) and the structure of  $\overline{L}$  in (3.3.4) it follows that

$$F\bar{L} = \begin{bmatrix} 0 & FT \end{bmatrix} = \begin{bmatrix} 0 & F_2K & F_2 \end{bmatrix} = \begin{bmatrix} F_2K\bar{L}_1 & F_2 \end{bmatrix}$$
(3.3.6)

where

$$\bar{L}_1 := \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{p-m} \end{bmatrix}$$
(3.3.7)

The next step is to pick an appropriate F, by choice of K. The choice of  $F_2$  will be discussed later. It is convenient to introduce a further nonsingular state transformation,  $\bar{x} \mapsto \tilde{T}\bar{x} = \tilde{x}$ , where

$$\tilde{T} := \begin{bmatrix} I_{n-m} & 0\\ K\bar{L}_1 & I_m \end{bmatrix}$$
(3.3.8)

In this new coordinate system, the system triple becomes

$$\tilde{T}\bar{G}\tilde{T}^{-1} = \tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix}, \qquad \tilde{T}\bar{H} = \tilde{H} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix}, \qquad (3.3.9)$$

and

$$F\bar{L}\tilde{T}^{-1} = F\tilde{L} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$$
(3.3.10)

where

$$\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12} K \bar{L}_1 \tag{3.3.11}$$

and  $\tilde{G}_{12} = \bar{G}_{12}$ . The structure associated with  $F\tilde{L}$  follows from the expression in equation (3.3.6) and the definition of  $\tilde{T}$  in equation (3.3.8). The following results show how the choice of matrix K affects the closed-loop system matrix.

**Proposition 3.3.1** The closed-loop matrix  $\tilde{G}_c = \tilde{G} - \tilde{H}(F\tilde{C}\tilde{G}^{-1}\tilde{H})^{-1}F\tilde{C}$  resulting from using the output feedback sliding mode controller (3.3.1) is stable if and only if  $\tilde{G}_{11}$  is stable.

**Proof** Using the definition in (3.3.3), the closed-loop matrix

$$\tilde{G}_c = \tilde{G} - \tilde{H} (F \tilde{C} \tilde{G}^{-1} \tilde{H})^{-1} F \tilde{C} = \tilde{G} - \tilde{H} (F \tilde{L} \tilde{H})^{-1} F \tilde{L} \tilde{G}$$
(3.3.12)

Substituting for  $\tilde{G}$ ,  $\tilde{H}$  and  $F\tilde{L}$  as defined previously in (3.3.10), and after some straightforward algebra, equation (3.3.12) becomes

$$\tilde{G}_{c} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & 0 \end{bmatrix}$$
(3.3.13)

It follows that  $\lambda(\tilde{G}_c) = \lambda(\tilde{G}_{11}) \cup \{0\}^m$  and so  $\tilde{G}_c$  is stable (in a discrete time sense) if and only if the eigenvalues of  $\tilde{G}_{11}$  are within the unit disk.

**Remark 3.3.5** As argued in §3.2, if inequality (3.2.7) holds, then the system matrix  $\tilde{G}_c$  is stable. It follows from Proposition 3.3.1 that  $\tilde{G}_{11}$  is stable, and then from equation

(3.3.11), the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  must be output feedback stabilisable. Thus necessary conditions for the existence of a stabilizing output feedback sliding mode controller are that assumptions A1 and A2 are satisfied and the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  is output feedback stabilisable.

**Remark 3.3.6** As shown in the proof of Proposition 3.3.1 the closed-loop matrix  $\tilde{G}_c$  depends only on K and not  $F_2$ . Indeed the control law (3.3.1) is independent of  $F_2$  since simple algebraic manipulation from the definition of F in (3.3.5) yields

$$u_{omm}(k) = -H_2^{-1} \begin{bmatrix} K & I \end{bmatrix} T^T y(k)$$
(3.3.14)

From the arguments above, the importance of the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  has been established. The following subsection considers the properties of this triple and how they relate to the original system.

#### 3.3.2 System Theoretic Interpretations

The triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  is fictitious and so it is necessary to investigate conditions under which  $(\bar{G}_{11}, \bar{G}_{12})$  is controllable and  $(\bar{G}_{11}, \bar{L}_1)$  is observable.

**Lemma 3.3.1** The pair  $(\bar{G}_{11}, \bar{G}_{12})$  from (3.3.4) is controllable if and only if the pair (G, H) is controllable.

**Proof** From the change of coordinates in (3.3.4) and the fact that  $det(H_2) \neq 0$ , it follows that

$$rank\left[\begin{array}{ccc}zI-G&H\end{array}\right] = rank\left[\begin{array}{ccc}zI-\bar{G}_{11}&\bar{G}_{12}&0\\\\-\bar{G}_{21}&zI-\bar{G}_{22}&H_2\end{array}\right]$$
$$= rank\left[\begin{array}{ccc}zI-\bar{G}_{11}&\bar{G}_{12}\end{array}\right]+m$$

for all  $z \in \mathbb{C}$ , which implies

$$rank\left[\begin{array}{cc} zI-G & H\end{array}
ight]=n \Leftrightarrow rank\left[\begin{array}{cc} zI-\bar{G}_{11} & \bar{G}_{12}\end{array}
ight]=n-m$$

Using the Rosenbrock-Hautus-Popov (RHP) test (see Appendix B.3), (G, H) is controllable if and only if the pair  $(\bar{G}_{11}, \bar{G}_{12})$  is controllable.

Since it has been assumed at the outset that (G, H) is controllable,  $(\bar{G}_{11}, \bar{G}_{12})$  will be taken to be controllable. In the following it will be shown that the observability of  $(\bar{G}_{11}, \bar{L}_1)$  is not guaranteed. Let the submatrix  $\bar{G}_{11}$  from (3.3.4) be partitioned so that

$$\bar{G}_{11} = \begin{bmatrix} \bar{G}_{1111} & \bar{G}_{1112} \\ \bar{G}_{1121} & \bar{G}_{1122} \end{bmatrix}$$
(3.3.15)

where  $\bar{G}_{1111} \in \mathbb{R}^{(n-p) \times (n-p)}$ . Based on this definition, the following results can be proven:

**Lemma 3.3.2** The pair  $(\bar{G}_{11}, \bar{L}_1)$  is observable if and only if  $(\bar{G}_{1111}, \bar{G}_{1121})$  is observable.

**Proof** From the definition of  $\bar{G}_{11}$  and  $\bar{L}_{1}$ :

$$rank \begin{bmatrix} zI - \bar{G}_{11} \\ \bar{L}_1 \end{bmatrix} = rank \begin{bmatrix} zI - \bar{G}_{1111} & -\bar{G}_{1112} \\ -\bar{G}_{1121} & zI - \bar{G}_{1122} \\ 0_{(p-m)\times(n-p)} & I_{p-m} \end{bmatrix}$$
$$= rank \begin{bmatrix} zI - \bar{G}_{1111} \\ -\bar{G}_{1121} \end{bmatrix} + (p-m)$$

Therefore

$$rank \begin{bmatrix} zI - \bar{G}_{11} \\ \bar{L}_1 \end{bmatrix} = n - m \quad \text{if and only if} \quad rank \begin{bmatrix} zI - \bar{G}_{1111} \\ -\bar{G}_{1121} \end{bmatrix} = n - p$$

for all  $z \in \mathbb{C}$ . By the Rosenbrock-Hautus-Popov (RHP) test (see Appendix B.3) it follows that  $(\bar{G}_{1111}, \bar{G}_{1121})$  is observable, if and only if  $(\bar{G}_{11}, \bar{L}_1)$  is observable.

If the pair  $(\bar{G}_{1111}, \bar{G}_{1121})$  is not observable, then there exists a transformation based on a nonsingular matrix  $T_{ob} \in \mathbb{R}^{(n-p) \times (n-p)}$  which puts the pair into the observability canonical form. Specifically:

$$T_{ob}\bar{G}_{1111}T_{ob}^{-1} = \begin{bmatrix} G_{11}^o & G_{12}^o \\ 0 & G_{22}^o \end{bmatrix}, \qquad \bar{G}_{1121}T_{ob}^{-1} = \begin{bmatrix} 0 & G_{21}^o \end{bmatrix}$$
(3.3.16)

where  $G_{11}^o \in \mathbb{R}^{r \times r}$  and  $G_{21}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$ , the pair  $(G_{22}^o, G_{21}^o)$  is observable and r > 0is the number of unobservable states of  $(\bar{G}_{1111}, \bar{G}_{1121})$  (Rosenbrock 1970). From Lemma 3.3.2 it follows that the unobservable modes of  $(\bar{G}_{11}, \bar{L}_1)$  are the eigenvalues of  $G_{11}^o$ .

The following provides a system theoretic interpretation:

**Lemma 3.3.3** The eigenvalues of  $G_{11}^o$  are the invariant zeros of (G, H, L).

**Proof** The invariant zeros of (G, H, L) are  $\{z \in \mathbb{C} : R(z) \text{ loses normal rank}\}$  where Rosenbrock's system matrix R(z) (Rosenbrock 1970) is given by

$$R(z) = \left[ \begin{array}{cc} zI - G & H \\ -L & 0 \end{array} \right]$$

Assuming (G, H, L) takes the structure of (3.3.4) and using the fact that  $det(H_2) \neq 0$  it follows that

$$R(z) \text{ loses rank } \Leftrightarrow \begin{bmatrix} zI - \bar{G}_{11} & -\bar{G}_{12} & 0\\ -\bar{G}_{21} & zI - \bar{G}_{22} & H_2\\ \begin{bmatrix} 0 & -T_1 \end{bmatrix} & -T_2 & 0 \end{bmatrix} \text{ loses rank}$$
$$\Leftrightarrow \begin{bmatrix} zI - \bar{G}_{11} & -\bar{G}_{12}\\ \begin{bmatrix} 0 & -T_1 \end{bmatrix} & -T_2 \end{bmatrix} \text{ loses rank}$$

where the matrix T from (3.3.4) is partitioned into  $T_1 \in \mathbb{R}^{(p-m) \times p}$  and  $T_2 \in \mathbb{R}^{m \times p}$ . Substituting for  $\overline{G}_{11}$  from (3.3.15) and (3.3.16), and repartitioning gives

$$\begin{bmatrix} zI - \bar{G}_{11} & -\bar{G}_{12} \\ 0 & -T_1 \end{bmatrix} - T_2 \end{bmatrix} = \begin{bmatrix} zI - G_{11}^o & -G_{12}^o \\ 0 & zI - G_{22}^o \\ 0 & -G_{21}^o \\ 0 & 0 \end{bmatrix} - T$$

where \* represents a matrix sub-block formed from  $\bar{G}_{1112}$  and  $\bar{G}_{1122}$  which plays no part in the analysis. Since T is full rank,

$$R(z) \text{ loses rank} \Leftrightarrow \begin{bmatrix} zI - G_{11}^o & -G_{12}^o \\ 0 & zI - G_{22}^o \\ 0 & -G_{21}^o \end{bmatrix} \text{ loses rank}$$

By construction, the pair  $(G_{22}^{o}, G_{21}^{o})$  is completely observable and hence from the Rosenbrock-Hautus-Popov (RHP) test for observability,

$$rank \begin{bmatrix} zI - G_{22}^{o} \\ -G_{21}^{o} \end{bmatrix} = n - p - r \quad \text{for all } z \in \mathbb{C}$$

Therefore,

$$R(z)$$
 loses rank  $\Leftrightarrow \det(zI - G_{11}^o) = 0$ 

and so the invariant zeros of  $({\cal G},{\cal H},L)$  are the eigenvalues of  ${\cal G}_{11}^o$  .

A necessary condition for  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  to be output feedback stabilisable is that (G, H, L)is minimum phase. This can be seen from the following argument. It follows from the definition of  $\bar{L}_1$  that from (3.3.15) and (3.3.16)

$$\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12} K \bar{L}_1 = \begin{bmatrix} G_{11}^o & * \\ 0 & * \end{bmatrix}$$
(3.3.17)

where the matrix sub-blocks \* depend on K and the sub-block  $\bar{G}_{11}$  in (3.3.15). From (3.3.17), it follows that  $\lambda(G_{11}^o) \subset \lambda(\tilde{G}_{11})$  for all K and so  $|\lambda(\tilde{G}_{11})| < 1 \Rightarrow |\lambda(G_{11}^o)| < 1$ . From Lemma 3.3.3 the eigenvalues of  $G_{11}^o$  are the invariant zeros of (G, H, L) and hence the claim is proven.

**Remark 3.3.7** As argued above, the stability of  $\tilde{G}_{11}$  depends on the invariant zeros of the fictitious system (G, H, L) where L is defined in (3.3.3). The original system triple (G, H, C), however, need not be minimum phase for  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  to be output feedback stabilisable. This contrasts with the equivalent continuous time situation where (G, H, C)would have to be minimum phase (Steinberg & Corless 1985).

**Remark 3.3.8** For the case of square systems, m = p and no design freedom exists (since the matrix K is an empty matrix) and

$$\lambda(\tilde{G}_c) = \lambda(\bar{G}_{11}) \cup \{0\}^m$$

where  $\bar{G}_{11}$  is the upper left sub-block from (3.3.4). Using arguments similar to those in Lemma 3.3.3, it can be shown that the eigenvalues of  $\bar{G}_{11}$  are the invariant zeros of (G, H, L).

The next subsection proves that assumptions A1, A2 and the output feedback stabilisation requirement on  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$ , are sufficient conditions for the existence of an output stabilizing sliding mode controller.

# 3.3.3 Sufficient Conditions

Suppose that for a given system (3.2.1)-(3.2.2), A1 and A2 are satisfied. Consequently, as a result of a change of coordinates to achieve the canonical form in (3.3.4), the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  can be identified. Assume this triple is output feedback stabilisable and that a  $K \in \mathbb{R}^{m \times (p-m)}$  can be found so that  $(\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1)$  is stable. With this K, define the matrix F according to (3.3.5) where  $F_2$  will be defined shortly. The objective is now to show that there exists a s.p.d. matrix P satisfying (3.2.7) and (3.3.2). An explicit procedure will now be described to obtain P. In the set of coordinates  $\tilde{x}$  associated with the canonical form in (3.3.6), the Lyapunov matrix  $P \mapsto \tilde{T}^{-T}(\bar{T}^{-T}P\bar{T}^{-1})\tilde{T}^{-1} =: \tilde{P}$  and equation (3.3.2) becomes

$$\tilde{H}^T \tilde{P} = F \tilde{L} \tag{3.3.18}$$

Likewise inequality (3.2.7) is equivalent to

$$\tilde{Q} := \tilde{P} - \tilde{G}^T \tilde{P} \tilde{G} + \tilde{G}^T \tilde{P} \tilde{H} (\tilde{H}^T \tilde{P} \tilde{H})^{-1} \tilde{H}^T \tilde{P} \tilde{G} > 0$$
(3.3.19)

As a result of the structures of  $\tilde{H}$  and  $F\tilde{L}$  from (3.3.10), in order to satisfy (3.3.18) the Lyapunov matrix  $\tilde{P}$  must have a block diagonal structure

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$
(3.3.20)

with  $\tilde{P}_1 \in \mathbb{R}^{(n-m)\times(n-m)}$  and  $\tilde{P}_2 \in \mathbb{R}^{m\times m}$ . To establish this suppose

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ \\ \tilde{P}_{12}^T & \tilde{P}_2 \end{bmatrix}$$

It follows by direct algebraic manipulation from the structure of  $\tilde{H}$  from (3.3.9) and  $F\tilde{L}$ 

from (3.3.10) that equation (3.3.18) becomes

$$\left[\begin{array}{cc} H_2^T \tilde{P}_{12}^T & H_2^T \tilde{P}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & F_2 \end{array}\right]$$

However since  $\det(H_2) \neq 0$  this implies  $\tilde{P}_{12} = 0$  and so  $\tilde{P}$  must have the block diagonal structure in (3.3.20) as claimed. Furthermore the matrix  $F_2 = H_2^T \tilde{P}_2$ .

**Proposition 3.3.2** Assume A1 and A2 are satisfied and that there exists a K such that  $\overline{G}_{11} - \overline{G}_{12}K\overline{L}_1$  is stable. If  $F_2 = H_2^T \tilde{P}_2$  then there exists a family of s.p.d. matrices  $\tilde{P}_1$  and  $\tilde{P}_2$  such that (3.3.19) is satisfied.

**Proof** As demonstrated in the proof of Proposition 3.3.1, in the coordinate system of the canonical form (3.3.4)

$$\tilde{G}_{c} = \tilde{G} - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{C} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & 0 \end{bmatrix}$$
(3.3.21)

where the sub-matrix  $\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12}K\bar{L}_1 \in \mathbb{R}^{(n-m)\times(n-m)}$  is stable. Some straightforward algebraic manipulation shows that  $\tilde{Q} \equiv \tilde{P} - \tilde{G}_c^T \tilde{P} \tilde{G}_c$  and so in terms of the partition in (3.3.10) and (3.3.20), inequality (3.3.19) can be written as

$$\tilde{Q} = \begin{bmatrix} \tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} & -\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12} \\ -\tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} & \tilde{P}_2 - \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} \end{bmatrix} > 0$$
(3.3.22)

A family of solutions  $(\tilde{P}_1, \tilde{P}_2)$  to this problem will be shown to exist. Specifically, let

 $\tilde{P}_1 > 0$  be a solution to

$$\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} > 0 \tag{3.3.23}$$

Such a solution is guaranteed to exist since  $\tilde{G}_{11}$  is stable. Then from the Schur complement (Boyd, El-Ghaoui, Feron & Balakrishnan 1994), inequality (3.3.22) is satisfied if and only if

$$\tilde{P}_2 > \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} + \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} (\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11})^{-1} (\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12})$$
(3.3.24)

Consequently, the claim is proven.

The main result may now be summarized as follows:

There exists a solution to the output stabilizing sliding mode control problem if and only if assumptions A1 and A2 hold and there exists a  $K \in \mathbb{R}^{m \times (p-m)}$  such that  $\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1$ is stable where  $\bar{G}_{11}$  and  $\bar{G}_{12}$  are defined in (3.3.4) and  $\bar{L}_1$  is defined in (3.3.7).

**Remark 3.3.9** Testing whether conditions A1 and A2 are satisfied is straightforward. Establishing whether a gain  $K \in \mathbb{R}^{m \times (p-m)}$  exists so that  $\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12}K\bar{L}_1$  is stable is, in general, an open research problem. No algorithm is available to guarantee the synthesis of a stabilizing gain. It is, however, an area for which there is a wealth of literature (Syrmos et al. 1997). Thus, from a practical viewpoint, any algorithm of choice may be employed by the designer to implement these ideas.

#### 3.3.4 Key Steps for the ODSMC Design

1. Check that the system satisfies assumptions A1 and A2, and that the pair (G, H) is controllable.

- 2. Form an output distribution matrix,  $L = CG^{-1}$  to produce the 'fictitious' system (G, H, L) and check that the invariant zeros of the system triple are inside the unit disk.
- 3. Perform a change of coordinates as in (3.3.4) to obtain  $(\bar{G}, \bar{H}, \bar{L})$ .
- 4. Select the design parameter K to stabilise the reduced-order system  $(\bar{G}_{11} \bar{G}_{12}K\bar{L}_1)$ , where  $\bar{L}_1$  is defined in (3.3.7).
- 5. The control can be calculated from (3.3.14).

# 3.3.5 Optimal Choices of Lyapunov Function

As shown in Proposition 3.3.2, for a given gain K there is still considerable freedom in the choice of P (as parameterized by  $\tilde{P}_1$  and  $\tilde{P}_2$ ): Suppose the matched uncertainty  $\xi(k)$ in (3.2.1) satisfies

$$\|\xi(k)\| < \rho_1 \|x(k)\| + \rho_0 \tag{3.3.25}$$

where  $\rho_1$  and  $\rho_0$  are positive constants. For a given gain K which makes  $\tilde{G}_{11}$  stable, a possibility is to choose  $\tilde{P}_1$  and  $\tilde{P}_2$  to maximize

$$\eta = \frac{\lambda_{min}(Q)}{\lambda_{max}(H^T P H)} \tag{3.3.26}$$

where  $Q = \bar{T}^T \tilde{T} \tilde{Q} \tilde{T} \tilde{T}$  and represents the first term on the r.h.s of equation (3.2.6). As argued in (Sharav-Schapiro et al. 1998), this quantity is a measure of the allowable cone bounded parametric uncertainty since if  $\rho_1 < \sqrt{\eta}$ , then in the absence of external disturbances ( $\rho_0 = 0$ ), the controller (3.3.1) guarantees asymptotic stability of the system (3.2.1) and (3.2.2).
From the expression in (3.3.26), the problem of maximizing the allowable uncertainty is equivalent to minimizing

 $\lambda_{max}(H_2^T \tilde{P}_2 H_2)/\lambda_{min}(Q)$  with respect to  $\tilde{P}_1$  and  $\tilde{P}_2$ . For simplicity, let

$$r = \lambda_{max}(H_2^T P_2 H_2) \quad \text{and} \quad q = \lambda_{min}(Q)$$

If P is replaced by  $\alpha P$ , where  $\alpha$  is a positive scalar, then the ratio of r/q is unaffected, since, by definition and from the structure of  $\tilde{Q}$  in (3.3.22), both q and r are scaled by  $\alpha$ . Consequently, the problem of minimising the ratio of r/q with respect to P is equivalent to minimizing r(P) subject to q(P) = 1. This is in turn equivalent to

$$\min \mu: \qquad \bar{T}^T \tilde{T}^T \tilde{Q} \tilde{T} \tilde{T} > I \qquad (3.3.27)$$

$$H_2^T \tilde{P}_2 H_2 < \mu I \tag{3.3.28}$$

$$P > 0$$
 (3.3.29)

where  $\mu$  is a positive scalar and  $\tilde{Q}$  is defined as in (3.3.19). This represents a convex optimization problem with decision variables  $\tilde{P}_1$ ,  $\tilde{P}_2$  and  $\mu$ . Linear Matrix Inequality (LMI) (see Appendix B.2) optimization (Boyd et al. 1994) can be used to obtain the optimal P matrix as a generalized eigenvalue problem (gevp).

#### **3.4** Design for an Aircraft Example

The design and application of the discrete output sliding mode controller will now be illustrated with an example. Consider the 4th order lateral dynamics of the De Havilland-Beaver aircraft from Chapter 2 in §2.2.3, discretised at a sample interval of 0.1s. This is a sensible choice of sampling interval according to (Aström & Wittenmark 1984) who argue that there should be between 5-20 samples in a step response of the closed-loop system. The system matrices associated with the discrete system are

$$G = \begin{bmatrix} 0.9729 & 0.0200 & -0.0945 & 0.0095 \\ -1.1463 & 0.8416 & 0.1211 & -0.0057 \\ 0.1728 & -0.0054 & 0.9727 & 0.0008 \\ -0.0574 & 0.0920 & 0.0262 & 0.9998 \end{bmatrix}, \quad H = \begin{bmatrix} -0.0008 & 0.0174 \\ -0.6082 & 0.3055 \\ -0.0771 & -0.1958 \\ -0.0320 & 0.0139 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The states represent sideslip angle, roll rate, yaw rate and roll angle respectively. The inputs of the system are the differential tailplane and rudder position. The output distribution matrix implies that only the quantities of roll rate, yaw rate and roll angle are available for use in the control law. The system triple (G, H, C) satisfies assumptions A1-A2 and the fictitious system (G, H, L), where  $L := CG^{-1}$ , does not have invariant zeros. Changing coordinates to obtain the canonical form in (3.3.4) yields

$$\bar{G}_{11} = \begin{bmatrix} 0.9611 & 0.0115 \\ -0.0735 & 0.9997 \end{bmatrix}, \quad \bar{G}_{12} = \begin{bmatrix} 0.1453 & -0.0058 \\ -0.0186 & -0.0880 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0.0487 & 0.1190 & -0.9917 \\ 0.0109 & -0.9929 & -0.1186 \\ 0.9988 & 0.0050 & 0.0497 \end{bmatrix}$$

Since  $\bar{G}_{12}$  is square and invertible, some simplification occurs. Equation (3.3.11) can be re-written as

$$\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1 = \bar{G}_{11} - W\bar{L}_1$$

where  $W = \bar{G}_{12}K$ . The choice of gain W represents an observer gain selection problem for the pair  $(\bar{G}_{11}, \bar{L}_1)$ . As argued in §3.3.2 this pair is observable since for this example the system (G, H, L) does not possess invariant zeros. Once W has been determined, an appropriate gain  $K = \bar{G}_{12}^{-1}W$ . Here, pole placement was used to locate the eigenvalues of  $\tilde{G}_{11}$  at (0.9, 0.95). This results in

$$W = \begin{bmatrix} 0.0023\\ 0.1108 \end{bmatrix} \quad \text{and hence} \quad K = \begin{bmatrix} -0.0339\\ -1.2512 \end{bmatrix}$$

Once K is obtained, perform the next change of coordinates as in (3.3.10) to obtain

$$\tilde{G}_{11} = \begin{bmatrix} 0.9611 & 0.00092 \\ -0.0735 & 0.8889 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 0 & 0.2413 \\ 0.7092 & -0.3534 \end{bmatrix}$$

The sub-matrices  $\tilde{P}_1$  and  $\tilde{P}_2$  returned by the LMI solver applied to the minimization problem subject to (3.3.27)-(3.3.29) are

$$\tilde{P}_1 = \begin{bmatrix} 28.7906 & 0.0551 \\ 0.0551 & 22.1791 \end{bmatrix} \text{ and } \tilde{P}_2 = \begin{bmatrix} 20.3046 & 2.7401 \\ 2.7401 & 1.9955 \end{bmatrix}$$

Following this, simple calculations show that the matrices

$$F_2 = H_2^T \tilde{P}_2 = \begin{bmatrix} 1.9434 & 1.4153\\ 3.9305 & -0.0442 \end{bmatrix}$$

and

$$\tilde{Q} = \begin{bmatrix} 2.0863 & 1.2020 & -4.0504 & 0.0218 \\ 1.2020 & 4.6497 & 0.3208 & 1.7373 \\ -4.0504 & 0.3208 & 19.6890 & 2.7288 \\ 0.0218 & 1.7373 & 2.7288 & 1.8226 \end{bmatrix}$$

The value obtained for  $\lambda_{max}(H_2^T \tilde{P}_2 H_2) = 1.0210$ . The maximum computable allowable bound on the uncertainty for the closed-loop system to remain stable is thus  $\sqrt{1/1.0210} =$ 0.9794. This indicates a good level of robustness in this case since this value is large relative to the size of the elements in the system and input distribution matrices G and H for this example. Finally, the output feedback control gain from (3.3.14) is given by

$$H_2^{-1} \begin{bmatrix} K & I \end{bmatrix} T^T = \begin{bmatrix} -1.2419 & -2.2378 & -1.7515 \\ 0.4862 & -4.1168 & -0.1196 \end{bmatrix}$$

### 3.5 Summary

A new design procedure has been presented to synthesize ODSMC's. A novel switching function is described; this in itself is not realizable through outputs alone, but it gives rise to a control law which depends only on outputs. The discrete time reduced-order sliding motion associated with this novel choice of switching function is not governed by

#### 3.5 Summary

the invariant zeros of the system - which therefore are not required to be minimum phase. The class of systems to which this approach is applicable is easily identified. As a result, new conditions for the existence of a stabilizing ODSMC have been given, for non-square systems with bounded matched uncertainties.

Thus far, it has been shown that for a stable ODSMC to exist, a certain subsystem triple has to be output feedback stabilisable. The next chapter will address the situation when this subsystem is not static output feedback stabilisable and proposes a method to solve this case.

## Chapter 4

# Dynamic Discrete Output Feedback Sliding Mode Controllers

## 4.1 Introduction

The chapter considers dynamic discrete time output feedback sliding mode controllers (dynamic ODSMC's) for non-square discrete time uncertain linear systems. It has been demonstrated in Chapter 3 that for many systems, it is not possible to obtain stability by static ODSMC. This of course is not surprising since not all systems are static output feedback stabilisable (Syrmos et al. 1997). To broaden the class, a compensator based framework is proposed to introduce additional degrees of freedom. For square systems, this problem has been addressed in (Sharav-Schapiro et al. 1999), from the point of view of discrete min-max controllers, with the introduction of a compensator. Here, a compensator scheme is proposed and a dynamic ODSMC is described for non-square systems.

This represents a new solution to an open problem (from a min-max perspective (Sharav-Schapiro et al. 1998)). The conditions for the existence of such dynamic ODSMC are given in this chapter. They are shown to be relatively mild and easily tested. Furthermore, a simple parameterisation of the available design freedom is proposed. An explicit procedure is also described which shows how a Lyapunov matrix, which satisfies both a discrete Riccati inequality and a structural constraint, can be obtained using LMI optimization. This Lyapunov matrix is used to calculate the robustness bounds associated with the closed-loop system. The efficacy of the method is demonstrated with an engineering example taken from the literature.

## 4.2 **Problem Formulation**

As in §3.2, consider the discrete time system with matched uncertainties

$$x(k+1) = Gx(k) + H(u(k) + \xi(k))$$
(4.2.1)

$$y(k) = Cx(k) \tag{4.2.2}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  with m . Assume that the input and outputdistribution matrices <math>H and C are full rank. In addition, assume the triple (G, H, C) is minimal. The matched uncertainties,  $\xi(k)$ , are assumed to be unknown but are required to belong to a 'balanced set'. Associate with this system a candidate Lyapunov function

$$V(k) = x^{T}(k)Px(k)$$
 (4.2.3)

where P is a s.p.d. matrix.

As stated in Chapter 3, the optimal state feedback control law, which minimises the Lyapunov difference  $\Delta V(k) = V(k+1) - V(k)$  with respect to u, for the worst case uncertainty  $\xi$ , is

$$u_s(k) = -(H^T P H)^{-1} H^T P G x(k)$$
(4.2.4)

It was shown in §3.2 that the controller can be realised through the outputs, as

$$u_o(k) = -(FCG^{-1}H)^{-1}Fy(k)$$
(4.2.5)

if the constraint

$$H^T P G = F C \tag{4.2.6}$$

is satisfied. The design problem is one of finding a matrix P and an F such that

$$Q := P + G^T P H (H^T P H)^{-1} H^T P G - G^T P G$$
(4.2.7)

and (4.2.6) holds for some  $F \in \mathbb{R}^{m \times p}$  and Q > 0 in (4.2.7). As argued in §3.3, a solution to this problem represents an ODSMC for the system in (4.2.1)-(4.2.2).

Throughout this chapter it will be assumed that:

- A1) the plant state transition matrix G is nonsingular
- A2) the matrix  $CG^{-1}H$  has rank m.

**Remark 4.2.1** These conditions are entirely compatible with those for square systems described in (Sharav-Schapiro et al. 1999) (argued from a min-max point of view) and indeed A2 is in fact just a generalization of the transfer function requirement  $det(G(0)) \neq 0$ 

where  $G(z) = C(zI - G)^{-1}H$  used in (Sharav-Schapiro et al. 1999).

Based on assumptions A1 and A2 as in  $\S3.3$ , a change of coordinates will be introduced which facilitates insight into the class of systems for which this problem is solvable. As in  $\S3.3.1$ , define a matrix

$$L := CG^{-1} (4.2.8)$$

which will be the new output distribution matrix for a purely fictitious system (G, H, L). By definition, and from Assumption 2, rank(LH) = m and there exists a change of coordinates such that

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & T \end{bmatrix}$$
(4.2.9)

where  $G_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ ,  $H_2 \in \mathbb{R}^{m\times m}$  and is nonsingular and  $T \in \mathbb{R}^{p\times p}$  is orthogonal.

The work in §3.3 shows that necessary and sufficient conditions for the solution of the problem of establishing a s.p.d. matrix P and an F satisfying (4.2.7) and (4.2.6), is that assumptions A1 and A2 are satisfied and the fictitious triple  $(G_{11}, G_{12}, L_1)$  where

$$L_{1} := \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{p-m} \end{bmatrix}$$
(4.2.10)

is static output feedback stabilisable i.e. there exists a  $K \in \mathbb{R}^{m \times (p-m)}$  such that the matrix  $(G_{11} - G_{12}KL_1)$  is stable. Clearly this is a limitation on the class of systems to which static ODSMC is applicable.

## 4.3 Dynamic Sliding Mode Controllers

This section explores the scenario in which it is not possible to synthesize a gain K so that  $(G_{11} - G_{12}KL_1)$  is stable. Here a compensator will be used to introduce additional dynamics to provide more degrees of freedom.

#### 4.3.1 Compensator Design

For notational convenience let the plant to be controlled be represented by

$$x_p(k+1) = G_p x_p(k) + H_p(u(k) + \xi(k))$$
(4.3.1)

$$y_p(k) = C_p x_p(k) \tag{4.3.2}$$

where  $G_p \in \mathbb{R}^{n_p \times n_p}$ ,  $H_p \in \mathbb{R}^{n_p \times m}$  and  $C_p \in \mathbb{R}^{p \times n_p}$  are the state, input and output distribution matrices respectively. Consider a compensator of the form

$$x_c(k+1) = \Phi x_c(k) + \Gamma y_p(k)$$
(4.3.3)

where  $\Phi \in \mathbb{R}^{q \times q}$  and  $\Gamma \in \mathbb{R}^{q \times p}$  are to be determined under the restriction that  $\det(\Phi) \neq 0$ . Here it will be assumed that  $q = n_p - m$ . An augmented plant description involving the plant and the compensator can be established:

$$\begin{bmatrix} x_c(k+1) \\ x_p(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma C_p \\ 0 & G_p \end{bmatrix} \begin{bmatrix} x_c(k) \\ x_p(k) \end{bmatrix} + \begin{bmatrix} 0 \\ H_p \end{bmatrix} (u(k) + \xi(k))$$
(4.3.4)

Define  $x(k) := \operatorname{col}(x_c(k), x_p(k))$ , then the new augmented system matrices are

$$G = \begin{bmatrix} \Phi & \Gamma C_p \\ 0 & G_p \end{bmatrix}, \qquad H = \begin{bmatrix} 0 \\ H_p \end{bmatrix}$$
(4.3.5)

Define the output distribution matrix for the augmented system as

$$C := \begin{bmatrix} I_q & 0\\ 0 & C_p \end{bmatrix}$$
(4.3.6)

The objective is now to establish that a (static) ODSMC exists for the triple (G, H, C)which of course constitutes a dynamic output feedback controller for the original plant  $(G_p, H_p, C_p)$ .

Notice from the definition in (4.3.4), that  $det(G) \neq 0$  if  $det(G_p) \neq 0$  since  $det(\Phi) \neq 0$ . Also

$$CG^{-1}H = \begin{bmatrix} \Phi^{-1}\Gamma C_p G_p^{-1} H_p \\ C_p G_p^{-1} H_p \end{bmatrix}$$

and so rank $(CG^{-1}H) = m$  provided rank $(C_pG_p^{-1}H_p) = m$ . Thus conditions A1 and A2 from §4.2 hold for the augmented triple (G, H, C) provided they hold for the original plant  $(G_p, H_p, C_p)$ . Define

$$L := CG^{-1} = \begin{bmatrix} \Phi^{-1} & -\Phi^{-1}\Gamma C_p G_p^{-1} \\ 0 & C_p G_p^{-1} \end{bmatrix} = \begin{bmatrix} \Phi^{-1} & -\Phi^{-1}\Gamma L_p \\ 0 & L_p \end{bmatrix}$$
(4.3.7)

where  $L_p := C_p G_p^{-1}$ . Also define a gain matrix

$$F := \tilde{F} \left[ \begin{array}{cc} F_c \Phi & F_o \end{array} \right] \tag{4.3.8}$$

where  $F_c \in \mathbb{R}^{m \times q}$  and  $F_o \in \mathbb{R}^{m \times p}$  are to be determined and  $\tilde{F} \in \mathbb{R}^{m \times m}$  is a nonsingular matrix which will be specified later. The objective is to establish an equation of the form

$$H^T P G = F C \tag{4.3.9}$$

for some s.p.d matrix  $P \in \mathbb{R}^{(2n_p-m)\times(2n_p-m)}$  for the augmented system such that the control law

$$u(k) = -(FCG^{-1}H)^{-1}Fy(k)$$
(4.3.10)

where  $y(k) := col(x_c(k), y_p(k))$  is a static ODSMC for the augmented uncertain system (4.3.4)-(4.3.6).

For the subsequent analysis, assume the plant system  $(G_p, H_p, L_p)$  has the canonical structure in (4.2.9), so that

$$G_p = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H_p = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \qquad L_p = \begin{bmatrix} 0 & T \end{bmatrix} \qquad (4.3.11)$$

where  $G_{11} \in \mathbb{R}^{(n_p-m)\times(n_p-m)}$ ,  $H_2 \in \mathbb{R}^{m\times m}$  and is nonsingular and  $T \in \mathbb{R}^{p\times p}$  is orthogonal. To facilitate the development which follows write

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = F_o T \quad \text{and} \quad \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} = \Gamma T \tag{4.3.12}$$

where  $T \in \mathbb{R}^{p \times p}$  is the orthogonal matrix from (4.3.11),  $F_1 \in \mathbb{R}^{m \times (p-m)}$  and  $\Gamma_1 \in \mathbb{R}^{q \times (p-m)}$ . Then it follows

$$FL = \tilde{F} \begin{bmatrix} F_c & (F_o L_p - F_c \Gamma L_p) \end{bmatrix}$$
$$= \tilde{F} \begin{bmatrix} F_c & (F_1 - F_c \Gamma_1) L_1 & (F_2 - F_c \Gamma_2) \end{bmatrix}$$
(4.3.13)

where

$$L_{1} = \begin{bmatrix} 0_{(p-m),(n_{p}-p)} & I_{p-m} \end{bmatrix}$$
(4.3.14)

Choose

$$F_2 = F_c \Gamma_2 + I_m \tag{4.3.15}$$

which makes the last sub-block in (4.3.13) equal to the identity matrix. Then choose

$$F_1 = F_c \Gamma_1 \tag{4.3.16}$$

which nullifies the second term in (4.3.13). Equations (4.3.15)-(4.3.16) completely determine  $F_o$  in (4.3.8) once  $F_c$  and  $\Gamma$  are synthesized. Partition the output distribution matrix of the plant in the canonical form in (4.3.11) as

$$C_p = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right] \tag{4.3.17}$$

where  $C_1 \in \mathbb{R}^{p \times (n_p - m)}$ . Perform a change of coordinates  $x \mapsto \tilde{T}x = \tilde{x}$  according to the

nonsingular matrix

$$\tilde{T} = \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_{n_p - m} & 0 \\ F_c & 0 & I_m \end{bmatrix}$$
(4.3.18)

then the system (G,H,FL) becomes  $(\tilde{G},\tilde{H},F\tilde{L})$  where

$$\tilde{G} = \begin{bmatrix} \Phi - \Gamma C_2 F_c & \Gamma C_1 & \Gamma C_2 \\ -G_{12} F_c & G_{11} & G_{12} \\ F_c \Phi - (F_c \Gamma C_2 + G_{22}) F_c & F_c \Gamma C_1 + G_{21} & F_c \Gamma C_2 + G_{22} \end{bmatrix}$$
(4.3.19)

$$\tilde{H} = \begin{bmatrix} 0\\ 0\\ H_2 \end{bmatrix} \quad \text{and} \quad F\tilde{L} = \begin{bmatrix} 0 & 0 & \tilde{F} \end{bmatrix}$$
(4.3.20)

The closed-loop system matrix associated with (4.3.4) and (4.3.10) (in the new coordinates) is

$$\tilde{G}_{c} \equiv \tilde{G} - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{C} = \begin{bmatrix} \Phi - \Gamma C_{2}F_{c} & \Gamma C_{1} & \Gamma C_{2} \\ -G_{12}F_{c} & G_{11} & G_{12} \\ \hline 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & 0 \end{bmatrix}$$
(4.3.21)

This representation is most directly established by noticing that  $\tilde{G}_c \equiv (I - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{L})\tilde{G}$ from the definition of  $\tilde{L} = \tilde{C}\tilde{G}^{-1}$ . The projection operator which pre-multiplies  $\tilde{G}$  to form  $\tilde{G}_c$  has the form diag $(I_{(n_p-m)}, I_{(n_p-m)}, 0_{m,m})$  because of the special form of  $\tilde{H}$  and  $F\tilde{L}$ from (4.3.20).

Before proving the main result, two lemmas are required, Lemma 3.3.1 from §3.3.2 and

another which is proved as follows.

**Lemma 4.3.1** If the plant  $(G_p, H_p, C_p)$  is minimal and  $det(G_p) \neq 0$  then the pair  $(G_{11}, C_1)$  is observable.

**Proof**: By definition  $L_p = C_p G_p^{-1}$  and in the canonical form of (4.3.11),

$$L_p = \begin{bmatrix} 0_{p \times (n_p - p)} & T \end{bmatrix}$$
(4.3.22)

where  $T \in \mathbb{R}^{p \times p}$  is orthogonal. It follows that

$$C_p = T \begin{bmatrix} 0_{p \times (n_p - p)} & I_p \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where, as in (4.3.11), the sub-block  $G_{11} \in \mathbb{R}^{(n_p-m)\times(n_p-m)}$ . From the definition of  $C_1$  in equation (4.3.17) it can be shown that

$$C_1 = T \left[ \begin{array}{c} G_{112} \\ G_{21} \end{array} \right]$$

where  $G_{112} \in \mathbb{R}^{(p-m) \times (n_p-m)}$  represents the last p-m rows of  $G_{11}$  and T is the orthogonal matrix from (4.3.22). The observability of the pair  $(G_{11}, C_1)$  will now be shown using the Rosenbrock-Hautus-Popov (Rosenbrock 1970) test. To begin with, clearly

$$\operatorname{rank} \left[ \begin{array}{c} zI_q - G_{11} \\ C_1 \end{array} \right] = \operatorname{rank} \left[ \begin{array}{c} zI_q - G_{11} \\ G_{112} \\ G_{21} \end{array} \right]$$

Two cases will now be considered: when z = 0 and when  $z \neq 0$ . If z = 0 then

$$\operatorname{rank} \begin{bmatrix} zI_q - G_{11} \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} G_{11} \\ G_{112} \\ G_{21} \end{bmatrix}$$
$$= \operatorname{rank} \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}$$
$$= n_p - m$$

since by definition  $G_{112}$  represents the bottom p-m rows of  $G_{11}$  and the last rank equality follows because, by assumption,  $G_p$  is nonsingular. For  $z \neq 0$  then

$$\operatorname{rank} \begin{bmatrix} zI_{q} - G_{11} \\ G_{112} \\ G_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} zI_{q} - G_{11} \\ -zL_{1} \\ G_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} zI_{q} - G_{11} \\ G_{21} \\ L_{1} \end{bmatrix}$$

where  $L_1$  is defined in (4.3.14). To obtain the second expression the last p - m rows from  $zI - G_{11}$  have been subtracted from  $G_{112}$ ; and to obtain the third, the fact that  $z \neq 0$  has been used. Furthermore

$$\operatorname{rank} \begin{bmatrix} zI_q - G_{11} \\ G_{21} \\ L_1 \end{bmatrix} = n_p - m \iff \operatorname{rank} \begin{bmatrix} zI_q - G_{11} & -G_{12} \\ -G_{21} & zI_m - G_{22} \\ L_1 & 0 \\ 0 & I_m \end{bmatrix} = n_p$$
$$\Leftrightarrow \operatorname{rank} \begin{bmatrix} zI_{n_p} - G_p \\ L_p \end{bmatrix} = n_p$$

From the Rosenbrock-Hautus-Popov test the last expression is equivalent to  $(G_p, L_p)$  being observable. Since det $(G_p) \neq 0$  and  $L_{pg} = C_p G_p^{-1}$ ,  $(G_p, L_p)$  is observable if and only if  $(G_p, C_p)$  is observable and so

$$\operatorname{rank} \left[ \begin{array}{c} zI_q - G_{11} \\ C_1 \end{array} \right] = n_p - m \quad \text{for all } z \in \mathbb{C}$$

and the pair  $(G_{11}, C_1)$  is observable as claimed.

**Proposition 4.3.1** Providing assumptions  $det(G_p) \neq 0$  and  $rank(C_pG_p^{-1}H_p) = m$  hold, and  $F_c$  in (4.3.8) is chosen so that  $G_{11} - G_{12}F_c$  is stable and  $\Gamma$  is chosen so that  $G_{11} - \Gamma C_1$ is stable, then if the compensator state transition matrix is chosen as

$$\Phi = \Gamma C_2 F_c - G_{12} F_c - \Gamma C_1 + G_{11} \tag{4.3.23}$$

the closed-loop system matrix (4.3.21) will be stable.

**Proof**: The system matrix in (4.3.21) is stable if and only if the top left sub-block  $\tilde{G}_{11}$  is stable. Notice this is parameterized solely by three matrices still to be determined, namely  $\Phi$ ,  $\Gamma$  and  $F_c$ . Also notice that (4.3.21) is not dependent on  $\tilde{F}$  (which is also yet to be determined). Let  $x_1$  represent the first  $n_p - m$  components of the state vector x. Then the change of coordinates  $(x_c, x_1) \mapsto (x_c - x_1, x_1)$ , which is possible since  $q = n_p - m$ , facilitates the selection of  $\Phi$ ,  $\Gamma$  and  $F_c$ . Specifically, it is easy to check that the top left sub-block of (4.3.21) becomes

$$\begin{bmatrix} I_q & -I_q \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \Phi - \Gamma C_2 F_c & \Gamma C_1 \\ -G_{12} F_c & G_{11} \end{bmatrix} \begin{bmatrix} I_q & I_q \\ 0 & I_q \end{bmatrix} \equiv \begin{bmatrix} \Phi - \Gamma C_2 F_c + G_{12} F_c & \Phi^* \\ -G_{12} F_c & G_{11} - G_{12} F_c \end{bmatrix}$$

where  $\Phi^* := \Phi - \Gamma C_2 F_c + G_{12} F_c + \Gamma C_1 - G_{11}$ . Then choosing the design parameter  $\Phi$  as in (4.3.23) makes  $\Phi^* = 0$  and thus

$$\begin{bmatrix} I_q & -I_q \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \Phi - \Gamma C_2 F_c & \Gamma C_1 \\ -G_{12} F_c & G_{11} \end{bmatrix} \begin{bmatrix} I_q & I_q \\ 0 & I_q \end{bmatrix} \equiv \begin{bmatrix} G_{11} - \Gamma C_1 & 0 \\ -G_{12} F_c & G_{11} - G_{12} F_c \end{bmatrix}$$

Consequently  $\sigma(\tilde{G}_{11}) = \sigma(G_{11} - G_{12}F_c) \cup \sigma(G_{11} - \Gamma C_1)$ . From Lemmas 3.3.1 and 4.3.1 the triple  $(G_{11}, G_{12}, C_1)$  is minimal and so it is always possible to choose  $F_c$  and  $\Gamma$  to make  $\tilde{G}_{11}$  stable.

**Proposition 4.3.2** The control law (4.3.10) in conjunction with the compensator (4.3.3) is a stabilising dynamic output feedback sliding mode controller for (4.3.1)-(4.3.2).

**Proof**: In the new coordinate system, equation (4.3.9) can be written as

$$\tilde{H}^T \tilde{P} = F \tilde{L} \tag{4.3.24}$$

The objective now is to show that there exists a s.p.d. matrix  $\tilde{P}$  satisfying (4.3.24) and

$$\tilde{P} - \tilde{G}_c^T \tilde{P} \tilde{G}_c > 0 \tag{4.3.25}$$

where  $\tilde{G}_c$  is the closed-loop system matrix from (4.3.21). An explicit procedure will be described to obtain  $\tilde{P}$ .

As a result of the structures of  $\tilde{H}$  and  $F\tilde{L}$  from (4.3.20), in order to satisfy equation (4.3.24), the Lyapunov matrix  $\tilde{P}$  must have a block diagonal structure

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$
(4.3.26)

with  $\tilde{P}_1 \in \mathbb{R}^{2(n_p-m)\times 2(n_p-m)}$  and  $\tilde{P}_2 \in \mathbb{R}^{m\times m}$ . To establish this suppose

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_2 \end{bmatrix}$$

It follows by direct algebraic manipulation from the structure of  $\tilde{H}$  and  $F\tilde{L}$  from (4.3.20) that equation (4.3.24) becomes

$$\left[\begin{array}{cc} H_2^T \tilde{P}_{12}^T & H_2^T \tilde{P}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & \tilde{F} \end{array}\right]$$

However since det $(H_2) \neq 0$  this implies  $\tilde{P}_{12} = 0$  and so  $\tilde{P}$  must have the block diagonal structure in (4.3.26) as claimed. Furthermore the scaling matrix  $\tilde{F}$  from (4.3.8) must satisfy  $\tilde{F} = H_2^T \tilde{P}_2$ . (Note: this scaling term is independent of the coordinate system.)

Some straightforward algebraic manipulation shows that in terms of the partition in (4.3.21) and (4.3.26)

$$\tilde{Q} = \tilde{P} - \tilde{G}_{c}^{T} \tilde{P} \tilde{G}_{c} = \begin{bmatrix} \tilde{P}_{1} - \tilde{G}_{11}^{T} \tilde{P}_{1} \tilde{G}_{11} & -\tilde{G}_{11}^{T} \tilde{P}_{1} \tilde{G}_{12} \\ -\tilde{G}_{12}^{T} \tilde{P}_{1} \tilde{G}_{11} & \tilde{P}_{2} - \tilde{G}_{12}^{T} \tilde{P}_{1} \tilde{G}_{12} \end{bmatrix}$$
(4.3.27)

A family of solutions  $(\tilde{P}_1, \tilde{P}_2)$  to make  $\tilde{Q} > 0$  will be shown to exist. Specifically, let  $\tilde{P}_1 > 0$  be a solution to

$$\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} > 0 \tag{4.3.28}$$

From Lyapunov theory such a solution is guaranteed to exist since  $\tilde{G}_{11}$  is stable. Then from the Schur complement, inequality (4.3.27) is satisfied if and only if

$$\tilde{P}_2 > \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} (\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11})^{-1} (\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12}) + \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12}$$
(4.3.29)

Thus the existence of a s.p.d  $\tilde{P}$  satisfying both (4.3.25) and (4.3.24) has been shown which means the stabilizing controller (4.3.10) is a min-max controller for (G, H, C) from (4.3.4) and (4.3.6). Consequently (4.3.10) in conjunction with (4.3.3) is a dynamic min-max controller for (4.3.1)-(4.3.2).

**Remark 4.3.1** The (dynamic) control law from (4.3.10) can be written as

$$u(k) = -H_2^{-1} \begin{bmatrix} F_c \Phi & F_o \end{bmatrix} y(k)$$
 (4.3.30)

where  $y(k) = \operatorname{col}(x_c(k), y_p(k)).$ 

**Remark 4.3.2** The choice  $q = n_p - m$  is judicious for the developments described in this chapter and is required for the coordinate change undertaken in Proposition 4.3.1. As demonstrated in this section, it allows a very elegant parameterisation of the closedloop system poles in terms of a 'separation principle' result. Such a choice of q does not immediately agree with natural intuition. One interpretation of this choice however is that the developments described here are appropriate for a sliding mode framework since the control structure in (4.3.10) has an 'equivalent control' interpretation (Edwards & Spurgeon 1998). Viewed from this perspective the compensator effectively acts as a full order estimator for the reduced-order motion traditionally associated with sliding mode control (Edwards & Spurgeon 1998). The change of coordinates associated with Proposition 4.3.1 occurs in the  $x_1$  states which are of dimension  $n_p - m$ , which because of the inherent matched uncertainty assumption, are independent of the uncertainty. There is no requirement in the development here to estimate the states  $x_2$  which are associated with the 'dead-beat' behaviour (and which are directly affected by the uncertainty). Also the results in this chapter only require relatively weak assumptions (A1 and A2) and not the relative-degree-one minimum phase conditions typically required by continuous time output feedback sliding mode controllers for systems with matched uncertainty (Edwards & Spurgeon 1998).

#### 4.3.2 Key Steps for the Dynamic ODSMC Design

An explicit design algorithm to realize the compensator can be described as follows:

- 1. For a given discrete time representation  $(G_p, H_p, C_p)$  check that  $\det(G_p) \neq 0$  and rank  $(C_p G_p^{-1} H_p) = m$ . If these conditions are not satisfied, then a dynamic compensator does not exist.
- Form the fictitious triple (G<sub>p</sub>, H<sub>p</sub>, L<sub>p</sub>) where L<sub>p</sub> = C<sub>p</sub>G<sub>p</sub><sup>-1</sup> and change coordinates to obtain the canonical form given in (4.3.11). (See (Edwards & Spurgeon 1995) for details.) Also establish a representation for the (real) output distribution matrix C<sub>p</sub> in these coordinates.

- Determine the matrices G<sub>11</sub>, G<sub>12</sub>, C<sub>1</sub> and T by partitioning G<sub>p</sub>, C<sub>p</sub> and L<sub>p</sub> according to the structures in (4.3.11) and (4.3.17). Choose F<sub>c</sub> and Γ using any algorithm of choice so that G<sub>11</sub> G<sub>12</sub>F<sub>c</sub> and G<sub>11</sub> ΓC<sub>1</sub> are stable. (By construction the triple (G<sub>11</sub>, G<sub>12</sub>, C<sub>1</sub>) is a minimal realization and so F<sub>c</sub> and Γ always exist.)
- 4. From (4.3.12) determine  $\Gamma_1$  and  $\Gamma_2$  as a partition of  $\Gamma T^T$  and calculate  $F_1$  and  $F_2$  from (4.3.15)-(4.3.16). The matrix  $F_o$  can then be calculated from (4.3.12).
- 5. Once  $\Gamma$  and  $F_c$  have been calculated the system matrix of the compensator  $\Phi$  can be computed from (4.3.23) and the dynamical system in (4.3.3) is completely specified.
- 6. The expression for the dynamic output feedback sliding mode control law is then given by (4.3.30).

**Remark 4.3.3** This chapter has been directed towards non-square systems. In the square case i.e when p = m, the same approach is applicable. Some minor modifications can be made to §4.3.1 and in particular  $F_1$  and  $\Gamma_1$  become empty matrices and the second term in (4.3.13) reduces to  $(F_1 - F_c\Gamma_1)L_1 = 0$ . The remainder of the analysis proceeds with no further modification.

#### 4.3.3 Robustness Analysis

This subsection considers the robustness properties associated with the proposed dynamic output feedback sliding mode controller. Suppose the matched uncertainty  $\xi(k)$  in (4.3.4) satisfies

$$\|\xi(k)\| < \rho_1 \|x_p(k)\| + \rho_0 \tag{4.3.31}$$

where  $\rho_1$  and  $\rho_0$  are positive constants. In §4.3.1 the design freedom associated with the Lyapunov matrix was shown to be represented by the pair of s.p.d. matrices  $\tilde{P}_1$  and  $\tilde{P}_2$ . Whilst the pair  $(\tilde{P}_1, \tilde{P}_2)$  must satisfy the matrix inequalities (4.3.28) and (4.3.29), there is some inherent design freedom. The selection of these matrices has no effect on the compensator dynamics (4.3.3) or indeed the control law (4.3.30). Assume the design parameters  $\Phi, \Gamma, F_c$  and  $F_o$  have been selected to ensure  $\tilde{G}_c$  (and in particular  $\tilde{G}_{11}$  from (4.3.21)) is stable. Let

$$V(k) := x(k)^T P x(k)$$
(4.3.32)

where  $P = \tilde{T}^T \tilde{P} \tilde{T}$  with  $\tilde{P}$  defined in (4.3.26) and  $\tilde{T}$  is given in (4.3.18). Then from (4.3.4)

$$\Delta V(k) := V(k+1) - V(k) = -x(k)^T \tilde{T}^T \tilde{Q} \tilde{T} x(k) + \xi(k)^T H^T P H \xi(k)$$
(4.3.33)

where  $\tilde{Q}$  is defined in (4.3.27). A slightly different approach will be adopted here than the one in §3.3.5 because the uncertainty is cone bounded with respect to  $x_p$  rather than the augmented state x. To reflect this let

$$N := \begin{bmatrix} 0_{n_p \times (n_p - m)} & I_{n_p} \end{bmatrix}$$
(4.3.34)

then by construction  $Nx = x_p$ . Consider

$$\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \alpha) := -x^T \tilde{T}^T \tilde{Q} \tilde{T} x + \xi^T H^T P H \xi + \alpha (\frac{1}{\mu} x^T N^T N x - \xi^T \xi)$$
(4.3.35)

where the scalars  $\alpha, \mu > 0$ . It is easy to see if  $\tilde{P}_1, \tilde{P}_2$  and  $\alpha$  are chosen so that  $\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \alpha) < 0$  then  $\|\xi\| < \sqrt{1/\mu} \|x_p\|$  implies  $\Delta V(k) < 0$ . Consequently if  $\rho_1 < \sqrt{1/\mu}$ , in the absence of external disturbances (i.e. when  $\rho_0 = 0$ ), asymptotic stability of the closed-loop system

is guaranteed.

The condition  $\mathcal{L}(\tilde{P}_1, \tilde{P}_2, \alpha) < 0$  is guaranteed if

$$\alpha N^T N < \mu \tilde{T}^T \tilde{Q} \tilde{T} \tag{4.3.36}$$

and

$$H^T P H < \alpha I_m \tag{4.3.37}$$

are satisfied subject to P > I and  $\alpha > 0$ . A logical way to proceed is to choose  $\tilde{P}_1, \tilde{P}_2$ and  $\alpha$  to minimise  $\mu$  (and thus maximize  $1/\mu$ ). This represents a convex optimization problem with decision variables  $\tilde{P}_1, \tilde{P}_2, \alpha$  and  $\mu$ . Linear Matrix Inequality (LMI) methods (Boyd et al. 1994) can be used to obtain the optimal values of the decision matrices as a generalized eigenvalue problem. Then from the arguments above, if  $\rho_1 < \sqrt{1/\mu}$  then the uncertain closed-loop system will retain asymptotic stability.

#### 4.4 Examples

The theoretical results from §4.3 will be demonstrated on two examples. The first of which shows that the approach is applicable to practical problems, which may or may not be output feedback stabilisable, and provides good regulation. Even in the case when  $(G_{11}, G_{12}, L_1)$  is output feedback stabilisable (and so theoretically a dynamic output feedback sliding mode controller is not required) it may be beneficial to design a compensator to improve robustness and performance. However, the robustness measure is still typically less than that which can be achieved from using state-feedback. This will be demonstrated in §4.4.2 (Example 2).

#### 4.4.1 Example 1: Application to a High Incidence Research Model

The theoretical results from §4.3 will be demonstrated on an example which represents the linearised longitudinal dynamics of a High Incidence Research Model (HIRM) aircraft (Magni, Bennani & Terlouw 1997) about an operating condition of 0.8Mach and 5,000ft. A discretised version using a sampling time of 0.025 secs is given by

$$G_p = \begin{bmatrix} 0.9619 & 0.0238 & 0 \\ -0.1374 & 0.9730 & 0 \\ -0.0017 & 0.0247 & 1.0000 \end{bmatrix} \qquad H_p = \begin{bmatrix} -0.0143 \\ -0.5528 \\ -0.0069 \end{bmatrix}$$

with

$$C_p = \left[ \begin{array}{rrr} -1 & 0 & 1 \\ & & \\ 0 & 1 & 0 \end{array} \right]$$

the states represent the angle of attack (rad), pitch rate (rad/s) and pitch angle (rad); the outputs are flight path angle and pitch rate; the control input is the elevator deflection angle (rad). In the canonical form of (4.3.11) the system and input distribution matrices

$$G_p = \begin{bmatrix} 0.9798 & 0.0149 & 0.0336 \\ 0.0248 & 0.9823 & 0.0005 \\ -0.0989 & 0.0704 & 0.9728 \end{bmatrix} \qquad H_p = \begin{bmatrix} 0 \\ 0 \\ -0.5683 \end{bmatrix}$$

and the output distribution matrix

$$C_p = \begin{bmatrix} 0.0261 & 0.9812 & -0.0129 \\ -0.0986 & 0.0839 & 0.9727 \end{bmatrix}$$

The orthogonal matrix associated with  $L_p = C_p G_p^{-1}$  is

$$T = \begin{bmatrix} 0.9999 & -0.0137 \\ 0.0137 & 0.9999 \end{bmatrix}$$

It follows from the above that the partitions of  $G_p$  and  $C_p$  in (4.3.11) and (4.3.17)

$$G_{11} = \begin{bmatrix} 0.9798 & 0.0149 \\ 0.0248 & 0.9823 \end{bmatrix} \qquad G_{12} = \begin{bmatrix} 0.0336 \\ 0.0005 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 0.0261 & 0.9812 \\ -0.0986 & 0.0839 \end{bmatrix} \qquad C_2 = \begin{bmatrix} -0.0129 \\ 0.9727 \end{bmatrix}$$

Using simple pole assignment (for demonstration purposes) it can be verified that

$$\Gamma = \begin{bmatrix} 0.0401 & -0.2913 \\ 0.5016 & -0.1183 \end{bmatrix}$$

and

$$F_c = \begin{bmatrix} 3.0030 & 2.6203 \end{bmatrix}$$

assign  $\sigma(G_{11} - G_{12}F_c) = \{ 0.90, 0.96 \}$  and  $\sigma(G_{11} - \Gamma C_1) = \{ 0.50, 0.95 \}$  respectively.

The compensator system matrix can be computed from (4.3.23) as

$$\Phi = \left[ \begin{array}{rrr} -0.0033 & -0.8318 \\ -0.3664 & 0.1803 \end{array} \right]$$

and in particular det( $\Phi$ )  $\neq 0$ . Furthermore this choice of  $\Gamma$  and  $F_c$  gives a stable  $\Phi$ . With  $\Gamma$  and  $F_c$  calculated, from (4.3.15) and (4.3.16),  $F_1 = 1.4185$  and  $F_2 = -0.2045$  and so from (4.3.12)

$$F_o = \left[ \begin{array}{cc} F_1 & F_2 \end{array} \right] T^T = \left[ \begin{array}{cc} 1.4212 & -0.1850 \end{array} \right]$$

Finally from (4.3.8) the output gain matrix

$$F = \left[ \begin{array}{ccc} -0.9702 & -2.0256 & 1.4212 & -0.1850 \end{array} \right]$$

and the design is complete. The LMI optimization used to obtain the uncertainty bounds

yields

$$P = \begin{bmatrix} 43.2292 & -8.8980 & -40.1084 & 10.7694 & 39.1222 \\ -8.8980 & 233.6303 & 236.0424 & 5.6228 & -222.0218 \\ -40.1084 & 236.0424 & 268.3925 & -3.2092 & -252.1647 \\ 10.7694 & 5.6228 & -3.2092 & 3.5270 & 3.4233 \\ 39.1222 & -222.0218 & -252.1647 & 3.4233 & 241.5858 \end{bmatrix}$$

and an associated value of  $\tilde{F} = -1.8823$ . The optimal values of  $\mu = 18.3554$  which gives a bound on the allowable uncertainty of  $\rho_1 = \sqrt{1/18.3554} = 0.2334$ .

The final control law is given by

$$u(k) = \begin{bmatrix} -1.7073 & -3.5647 & 2.5010 & -0.3255 \end{bmatrix} y(k)$$
(4.4.1)

the design is complete. Figure 4.4.1 shows the response of the nominal system to the initial conditions [0, 0, 0.1].



Figure 4.4.1: closed-loop response of the nominal system

Suppose instead

$$u(k) = \begin{bmatrix} -3.7741 & -5.3136 & 2.5822 & -1.0101 \end{bmatrix} y(k)$$
(4.4.2)

The closed-loop poles are now located at { 0.40, 0.50, 0.90, 0.94, 0.95 }. The intention here has been to deliberately remove the 'min-max' character of the control law by replacing the closed-loop pole at the origin with one at 0.4 whilst leaving the remaining nonzero closed-loop poles associated with (4.4.1) at their original values. To demonstrate the performance of the dynamic ODSMC, the closed-loop system has been subjected to a matched disturbance in the form of a sine wave of frequency 1 rad/s.

In Figure 4.4.2 the quantity  $V(k) = x(k)^T P x(k)$  has been plotted for both the dynamic ODSMC/ 'min-max' controller in (4.4.1) and the controller associated with (4.4.2). The quantity V(x) is a measure of the deviation of the states from zero since P is s.p.d by construction. It can be clearly seen that the disturbance is better attenuated by the ODSMC/'min-max' controller. Figure 4.4.3 is a plot of the absolute value of  $\Delta V(k)$  for both the ODSMC controller in (4.4.1) and the controller associated with (4.4.2). The dynamic ODSMC controller is specifically designed to minimise the deviation of  $\Delta V(k)$  for worst case disturbances. It can be seen from Figure 4.4.3 that, even for a sine wave disturbance, the dynamic ODSMC controller out-performs the controller associated with (4.4.2).



Figure 4.4.2: Plot of the Lyapunov function V(s)

### 4.4.2 Example 2: Numerical Example

For the purpose of demonstrating the improved robustness that is brought upon by the dynamic ODSMC method described in this chapter, the following system, taken from (Sharav-Schapiro et al. 1999), will be used:

$$x(k+1) = G_p x(k) + H_p(u(k) + \xi(k))$$
(4.4.3)

$$y(k) = C_p x(k) \tag{4.4.4}$$



Figure 4.4.3: Plot of  $|\Delta V(s)|$ 

where

$$\|\xi(k)\| < \rho_1 \|x(k)\| \tag{4.4.5}$$

and  $\rho_1$  is a positive scalar. The system matrices are

$$G_p = \begin{bmatrix} 0 & 1 \\ 0.24 & 0.2 \end{bmatrix}, \qquad H_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

As argued in (Sharav-Schapiro et al. 1999), with a state feedback controller, the maximum achievable level of robustness is 0.5 (i.e. if  $\rho_1 < 0.5$ , then closed-loop stability is guaranteed). This can be shown by choosing Q = I and solving the Riccati equation in (4.2.7) which results in

$$P = \left[ \begin{array}{rrr} 1 & 0 \\ 0 & 2 \end{array} \right]$$

Hence, the optimal uncertainty bound is

$$\eta = \frac{\lambda_{min}(Q)}{\lambda_{max}(H^T P H)} = 0.5$$

The static ODSMC does exist for this example and can be obtained using the method from Chapter 3. The calculated uncertainty bound is 0.0528. As predicted, this bound is smaller than the one obtained for the state feedback case. Next, the dynamic ODSMC will be considered where the design parameters  $\Gamma = -0.2040$ ,  $F_c = -4.0858$  and  $\Phi = -1.0000$ . The Lyapunov matrix

$$P = \begin{bmatrix} 154.4254 & 30.1044 & -26.1362 \\ 30.1044 & 16.7941 & -5.3308 \\ -26.1362 & -5.3308 & 26.6538 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 26.6948 & 4.3133 & -1.2590 \\ 18.4712 & 18.0037 & -0.2568 \\ 2.7137 & 1.5258 & 10.9258 \end{bmatrix}$$

The matrix  $F = [4.0858\ 0.8333]$  and the uncertainty bound obtained is 0.4046. Note that this bound is significantly larger than the one obtained for the static ODSMC but is smaller than the maximum calculated from the state feedback case.

## 4.5 Summary

This chapter has considered the design of dynamic ODSMC's for a class of non-square discrete time uncertain linear systems subject to bounded matched uncertainty. The scenario which has been considered here is a realistic one in which only outputs are measured and the states of the system are unknown. The dynamic controller that has been proposed relies purely on measured outputs. New, necessary and sufficient conditions for the solution of this problem has been proposed. These are relatively mild and can easily be verified. A particular compensator, of order less than the original plant, has been suggested which is parameterized in a way that is constructive from the point of view of synthesis. Furthermore an explicit design algorithm has been proposed which synthesizes the parameters of the compensator and provides a Lyapunov matrix which satisfies both a discrete Riccati inequality and a structural constraint. The efficacy of the approach has been demonstrated by considering a system triple relating to a practical aircraft system.

So far in the thesis, only the problem of plant regulation has been considered. The requirement to incorporate tracking control, which is often needed in practical applications, will be addressed in the following chapter.

## Chapter 5

# Discrete Output Feedback Sliding Mode Control with Integral Action

## 5.1 Introduction

The two previous chapters have considered the ODSMC problem, where the objective is to drive the system states to zero. This chapter will examine the problem of designing a static ODSMC which utilises integral action to provide tracking. The simplicity of the resulting scheme will be apparent which is very advantageous from the point of view of practical implementation. The conditions necessary in order to realize such a scheme are established. A numerical example is used to describe the theoretical implications of the approach and real-time implementation of the technique for control of a DC-motor system is used to illustrate the ODSMC tracking framework.

## 5.2 System Description and Problem Formulation

As in Chapter 3, consider the discrete time system with matched uncertainties

$$x_p(k+1) = G_p x_p(k) + H_p(u(k) + \xi(k))$$
(5.2.1)

$$y_p(k) = C_p x_p(k) \tag{5.2.2}$$

which is assumed to be square. The vectors  $x_p \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y_p \in \mathbb{R}^m$  with m < n, represent the states, inputs and outputs respectively. Assume that the input and output distribution matrices  $H_p$  and  $C_p$  are full rank. In addition, assume the pair  $(G_p, H_p)$  is controllable and the pair  $(G_p, C_p)$  is observable. A more specific structure to the uncertainty will be provided later. The assumption that the system is square helps simplify the analysis. For non-square systems a suitable subset of 'controlled outputs' would need to be identified.

In this chapter, an integral action approach will be considered in the controller design. Consider the nominal system in (5.2.1) and (5.2.2). Additional states,  $x_r \in \mathbb{R}^m$ , are introduced which satisfy

$$x_r(k+1) = x_r(k) + \tau(r(k) - y_p(k))$$
(5.2.3)

where r(k) is a reference signal and  $\tau$  is the sampling interval.

The states of the system are augmented with the integral action states to obtain

$$x(k) = \begin{bmatrix} x_r(k) \\ x_p(k) \end{bmatrix}$$
(5.2.4)

Using (5.2.1), (5.2.2) and (5.2.3) the augmented system can now be written as

$$x(k+1) = Gx(k) + H(u(k) + \xi(k)) + H_r r(k)$$
(5.2.5)

$$y(k) = Cx(k) \tag{5.2.6}$$

where  $x \in \mathbb{R}^{(n+m)}$  and the augmented system matrices are

$$G = \begin{bmatrix} I_m & -\tau C_p \\ 0 & G_p \end{bmatrix}, \qquad H = \begin{bmatrix} 0 \\ H_p \end{bmatrix}, \qquad H_r = \begin{bmatrix} \tau I_m \\ 0 \end{bmatrix}$$
(5.2.7)

and the output distribution matrix

$$C = \begin{bmatrix} I_m & 0\\ 0 & C_p \end{bmatrix}$$
(5.2.8)

which implies  $y(k) = col(x_r(k), y_p(k))$ , i.e. the integral action states plus the measured outputs.

As in the two previous chapters, it will be assumed that

- A1) the augmented state transition matrix G from (5.2.7) is nonsingular (which is equivalent to  $G_p$  from (5.2.1) being nonsingular)
- A2)  $CG^{-1}H$  has rank m (which is equivalent to  $C_pG_p^{-1}H_p$  being rank m).

These two assumptions are perfectly in keeping with the assumptions from Chapter 3 and Chapter 4. Assume initially that r(k) = 0. This assumption will be relaxed later in the chapter. Also assume a static ODSMC is sought for the augmented system (5.2.7)-(5.2.8). From here, the set-up is similar to that in §3.2. A sliding surface for the (augmented) system is defined as

$$S = \{x : FCG^{-1}x = 0\}$$
(5.2.9)

where  $F \in \mathbb{R}^{m \times 2m}$  is the design freedom. In the selection of the design parameter F, the condition

$$FCG^{-1} = H^T P$$
 (5.2.10)

must hold for some s.p.d matrix P (which will play the role of a Lyapunov matrix for the system). Since by assumption  $CG^{-1}H$  is rank m and  $F \in \mathbb{R}^{m \times 2m}$ , the design freedom F can be chosen so that  $\det(FCG^{-1}H) \neq 0$ . As in Chapter 3 the proposed discrete time sliding mode static output feedback control law is given by

$$u(k) = -(FCG^{-1}H)^{-1}FCx(k)$$
(5.2.11)

In order for the closed-loop system to be stable, the system matrix

$$G_c = G - H(FCG^{-1}H)^{-1}FC (5.2.12)$$

must be stable and P must be a Lyapunov function matrix for (5.2.12), i.e.

$$G_c^T P G_c - P < 0 (5.2.13)$$
The problem is now to find conditions under which it is possible to select an  $F \in \mathbb{R}^{m \times 2m}$ and a s.p.d.  $P \in \mathbb{R}^{(n+m) \times (n+m)}$  such that (5.2.10) and (5.2.13) hold. This is explored in the next section.

## 5.3 The Hyperplane Synthesis Problem

This section develops necessary and sufficient conditions under which it is possible to select an  $F \in \mathbb{R}^{m \times 2m}$  which parameterizes the design freedom in (5.2.11) such that (5.2.10) and (5.2.13) hold for a s.p.d.  $P \in \mathbb{R}^{(n+m) \times (n+m)}$  which serves as a Lyapunov matrix. As in §3.3.1, define  $L = CG^{-1}$  and consider a new fictitious system (G, H, L). Using assumptions A1 and A2, as in §3.3 a change of coordinates will be introduced to facilitate the subsequent analysis, where  $x \mapsto \overline{T}x = \overline{x}$  such that

$$\bar{G} = \bar{T}G\bar{T}^{-1} = \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix}, \qquad \bar{H} = \bar{T}H = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},$$

$$\bar{L} = L\bar{T}^{-1} = \begin{bmatrix} 0_{2m \times (n-m)} & T \end{bmatrix}$$
 (5.3.1)

where  $\bar{G}_{11} \in \mathbb{R}^{n \times n}$ ,  $H_2 \in \mathbb{R}^{m \times m}$  is nonsingular,  $T \in \mathbb{R}^{2m \times 2m}$  is orthogonal.

The design matrix F from (5.2.9) is reparameterised as

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^T$$
(5.3.2)

where  $K \in \mathbb{R}^{m \times m}$  and  $F_2 \in \mathbb{R}^{m \times m}$  is nonsingular. Again this is similar to the approach

adopted in §3.3.1. From the definition in (5.3.2) and the structure of  $\overline{L}$  in (5.3.1) it follows that

$$F\bar{L} = \begin{bmatrix} 0_{m \times (n-m)} & FT \end{bmatrix} = \begin{bmatrix} 0_{m \times (n-m)} & F_2K & F_2 \end{bmatrix} = \begin{bmatrix} F_2K\bar{L}_1 & F_2 \end{bmatrix}$$
(5.3.3)

where

$$\bar{L}_1 := \begin{bmatrix} 0_{m \times (n-m)} & I_m \end{bmatrix}$$
(5.3.4)

As in §3.3.1, for convenience, a further nonsingular state transformation is introduced,  $\bar{x} \mapsto \tilde{T}\bar{x} = \tilde{x}$ , where

$$\tilde{T} := \begin{bmatrix} I_n & 0\\ \\ K\bar{L}_1 & I_m \end{bmatrix}$$
(5.3.5)

In this new coordinate system, the system triple becomes

$$\tilde{G} = \tilde{T}\bar{G}\tilde{T}^{-1} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix}, \qquad \qquad \tilde{H} = \tilde{T}\bar{H} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},$$

$$F\tilde{L} = F\bar{L}\tilde{T}^{-1} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$$
(5.3.6)

where

$$\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12} K \bar{L}_1 \tag{5.3.7}$$

The structure associated with  $F\tilde{L}$  in (5.3.6) follows from the expression in equation (5.3.3) and the definition of  $\tilde{T}$  in equation (5.3.5). The matrix  $\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12}K\bar{L}_1$  is crucial to the developments that follow. Before proving the main result, some lemmas need to be established relating to the fictitious triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$ . Using arguments similar to Lemma 3.3.1 in Chapter 3, it can be shown that because of the structure of  $\bar{G}$  and  $\bar{H}$ in (5.3.1), the pair  $(\bar{G}_{11}, \bar{G}_{12})$  is controllable if and only if (G, H) is controllable. The next lemma shows that controllability of the augmented pair (G, H) follows from the controllability of  $(G_p, H_p)$  provided  $(G_p, H_p, C_p)$  does not have any invariant zeros at unity.

**Lemma 5.3.1** If  $(G_p, H_p, C_p)$  is completely controllable and has no invariant zeros at unity, then the pair (G, H) is completely controllable.

**Proof** Express Rosenbrock's system matrix as

$$R(z) = \begin{bmatrix} zI - G_p & H_p \\ -C_p & 0 \end{bmatrix}$$
(5.3.8)

The invariant zeros of the triple  $(G_p, H_p, C_p)$  are given by

$$\{z \in \mathbb{C} : \det R(z) = 0\}$$

where R(z) is defined in (5.3.8). Hence, the system  $(G_p, H_p, C_p)$  has zeros at unity if and only if det R(1) = 0, i.e.

$$\det R(1) = 0 \quad \Leftrightarrow \quad \det \begin{bmatrix} -C_p & 0\\ I_n - G_p & H_p \end{bmatrix} = 0$$
$$\Leftrightarrow \quad \det \begin{bmatrix} C_p & 0\\ I_n - G_p & H_p \end{bmatrix} = 0$$

Using the RHP rank test, the pair (G, H) is completely controllable if and only if

$$rank\left[\begin{array}{cc} zI-G & H\end{array}\right] = n+m \quad \text{for all } z \in \mathbb{C}$$
 (5.3.9)

If z = 1, then from (5.2.7)

$$rank \begin{bmatrix} zI - G & H \end{bmatrix} = n + m \iff rank \begin{bmatrix} 0 & \tau C_p & 0 \\ 0 & I_n - G_p & H_p \end{bmatrix} = n + m$$
$$\Leftrightarrow \det \begin{bmatrix} \tau C_p & 0 \\ I_n - G_p & H_p \end{bmatrix} \neq 0$$
$$\Leftrightarrow \det \begin{bmatrix} C_p & 0 \\ I_n - G_p & H_p \end{bmatrix} \neq 0$$
$$\Leftrightarrow (G_p, H_p, C_p) \text{ has no invariant zeros at}$$

Otherwise  $z \neq 1$  and

$$rank \begin{bmatrix} zI - G & H \end{bmatrix} = n + m \iff rank \begin{bmatrix} zI_m - I_m & \tau C_p & 0 \\ 0 & zI_n - G_p & H_p \end{bmatrix} = n + m$$
$$\Leftrightarrow rank \begin{bmatrix} zI_n - G_p & H_p \end{bmatrix} = n$$

Since it has been assumed in §5.2 that  $(G_p, H_p)$  is controllable, by assumption

$$rank \left[ \begin{array}{cc} zI_n - G_p & H_p \end{array} \right] = n \quad \text{for all } z$$

from the RHP rank test (applied to the pair  $(G_p, H_p)$ ). Therefore (5.3.9) is true and (G, H) is controllable.

unity

Generally, the pair  $(\bar{G}_{11}, \bar{L}_1)$  is not guaranteed to be observable. (In fact, as shown in Lemma 3.3.2 and 3.3.3, the unobservable modes of  $(\bar{G}_{11}, \bar{L}_1)$  are the invariant zeros of (G, H, L).) An important observation will now be made which shows that under mild conditions the fictitious system (G, H, L) has no invariant zeros (and therefore  $(\bar{G}_{11}, \bar{L}_1)$ is observable).

**Lemma 5.3.2** Suppose the triple  $(G_p, H_p, C_p)$  is minimal and that conditions A1 and A2 hold. If in addition, none of the invariant zeros of  $(G_p, H_p, C_p)$  are zero, then the augmented system (G, H, L) has no invariant zeros.

**Proof** For a square plant  $(G_p, H_p, C_p)$ , under the assumption that rank $(C_pH_p) = m$ , a coordinate system can be chosen so that

$$G_p = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H_p = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \qquad C_p = \begin{bmatrix} 0 & I_m \end{bmatrix} \qquad (5.3.10)$$

where  $H_2 \in \mathbb{R}^{m \times m}$  is nonsingular and  $G_{11} \in \mathbb{R}^{n \times n}$ . Furthermore the eigenvalues of  $G_{11}$  are the invariant zeros of  $(G_p, H_p, C_p)$ . This is an extension of the ideas in §5.4 of (Edwards & Spurgeon 1998) to discrete systems and is a special case of Lemma 3.3.2 and 3.3.3.

From the definition of L and using the definitions of (G, H, C) from (5.2.7)-(5.2.8), it follows that in the coordinates associated with (5.3.10):

$$L = CG^{-1} = \begin{bmatrix} I_m & \tau C_p G_p^{-1} \\ 0 & C_p G_p^{-1} \end{bmatrix}$$
(5.3.11)

The invariant zeros of (G, H, L) are values of  $z \in \mathbb{C}$  for which Rosenbrock's system matrix

(Rosenbrock 1970) given by

$$R(z) := \left[ egin{array}{cc} zI - G & H \ -L & 0 \end{array} 
ight]$$

loses normal rank.

It follows by substituting from (5.2.7), (5.3.10) and (5.3.11) that

$$\operatorname{rank} R(z) = \operatorname{rank} \begin{bmatrix} z - 1 & \tau C_p & 0 \\ 0 & zI_n - G_p & H_p \\ -I_m & -\tau C_p G_p^{-1} & 0 \\ 0 & -C_p G_p^{-1} & 0 \end{bmatrix}$$
(5.3.12)  
$$= \operatorname{rank} \begin{bmatrix} z - 1 & \tau C_p \\ 0 & [zI - G_{11} & -G_{12}] \\ -I_m & -\tau C_p G_p^{-1} \\ 0 & -C_p G_p^{-1} \end{bmatrix} + m$$
(5.3.13)  
$$= \operatorname{rank} \begin{bmatrix} z - 1 & \tau C_p \\ 0 & [zI - G_{11} & -G_{12}] \\ -I_m & 0 \\ 0 & -C_p G_p^{-1} \end{bmatrix} + m$$
(5.3.14)

and so

$$R(z) \operatorname{loses \ rank} \Leftrightarrow \left[ \begin{array}{c} \tau C_p \\ zI - G_{11} & -G_{12} \\ -C_p G_p^{-1} \end{array} \right] \operatorname{loses \ rank}$$

Exploiting the structure of  $C_p$  from (5.3.10) it follows

$$R(z)$$
 loses rank  $\Leftrightarrow \begin{bmatrix} zI - G_{11} \\ -X_1 \end{bmatrix}$  loses rank

where  $X_1 \in \mathbb{R}^{m \times (n-m)}$  represents the first n - m columns of  $C_p G_p^{-1}$ .

The remainder of the proof will show that in fact

$$\operatorname{rank} \begin{bmatrix} zI - G_{11} \\ -X_1 \end{bmatrix} = n - m \text{ for all } z \in \mathbb{C}$$
(5.3.15)

i.e. R(z) does not lose rank, and consequently (G, H, L) does not possess invariant zeros.

Suppose for a contradiction this is not the case and the expression in (5.3.15) is not rank n-m. Consequently there exists a non-zero vector  $v \in \mathbb{R}^{n-m}$  such that

$$(zI - G_{11})v = 0$$
 and  $X_1v = 0$ 

From the definition of  $X_1$ , it follows that

$$C_p G_p^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix} = 0 \quad \Rightarrow \quad G_p^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{N}(C_p) \quad \Rightarrow \quad G_p^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix}$$

for some  $w \in \mathbb{R}^{n-m}$  where  $\mathcal{N}(C_p)$  represents the null-space of  $C_p$  and further, the structure of  $C_p$  from (5.3.10) has been exploited to obtain the last inequality. Since by assumption  $v \neq 0$  the last equation implies  $w \neq 0$  from the inevitability of  $G_p$ . The last equation can be re-written as

$$G_p \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} \Rightarrow G_{11}w = v \text{ and } G_{21}w = 0$$

from the definition of  $G_p$  in (5.3.10). By construction  $(zI - G_{11})v = 0$  which means

$$0 = (zI - G_{11})G_{11}w = G_{11}(zI - G_{11})w \quad \Rightarrow (zI - G_{11})w = 0$$

using the fact that  $G_{11}$  is nonsingular. Since  $G_{12}w = 0$  it follows that

$$\operatorname{rank}\left[\begin{array}{c} zI - G_{11} \\ \\ G_{21} \end{array}\right] \neq n - m$$

This means that the pair  $(G_{11}, G_{21})$  is not observable which in turn implies the pair  $(G_p, C_p)$  is not observable. This is a contradiction.

-	

It has been shown that under mild conditions the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  is minimal. This is a slightly different result to the generic case in Chapter 3 where invariant zeros of the plant manifest themselves as unobservable modes of  $(\bar{G}_{11}, \bar{L}_1)$ . The main result of this chapter will now be proved.

**Proposition 5.3.1** Necessary and sufficient conditions for the existence of an ideal stable sliding mode on the surface  $S = \{x : FCG^{-1}x = 0\}$ , where F is defined in (5.3.2) is that the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  is output feedback stabilisable. **Proof** In the coordinate system associated with (5.3.6), following the coordinate transformation (5.3.5), it is straightforward to identify the sliding motion: let  $(\tilde{x}_1, \tilde{x}_2)$  represent a partition of the states  $\tilde{x}$  with  $\tilde{x}_2 \in \mathbb{R}^m$ . During an ideal sliding motion, because of the structure of  $F\tilde{L}$  in (5.3.6),  $F\tilde{C}\tilde{G}^{-1}\tilde{x} \equiv F\tilde{L}\tilde{x} = 0$  implies  $\tilde{x}_2 = 0$  and so the sliding motion is governed by  $\tilde{x}_1(k+1) = \tilde{G}_{11}\tilde{x}_1(k)$ . By definition  $\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12}K\bar{L}_1$  and so the sliding mode is stable if and only if the static output feedback problem for the fictitious triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  is feasible.

To prove the converse suppose there exists a K such that  $\tilde{G}_{11} = \bar{G}_{11} - \bar{G}_{12}K\bar{L}_1$  is stable. When implementing the discrete output feedback controller (5.2.11), the closed-loop dynamics are governed by the system matrix

$$\tilde{G}_c = \tilde{G} - \tilde{H} (F \tilde{L} \tilde{H})^{-1} F \tilde{L} \tilde{G}$$
(5.3.16)

In the new set of coordinates induced by (5.3.5), the Lyapunov matrix

$$P \mapsto (\tilde{T}^{-1})^T (\bar{T}^{-1})^T P \bar{T}^{-1} \tilde{T}^{-1} =: \tilde{P}$$
(5.3.17)

Using the definition of  $\tilde{L}$ , equation (5.2.10) becomes

$$\tilde{H}^T \tilde{P}^T = F \tilde{C} \tilde{G}^{-1} = F \tilde{L} \tag{5.3.18}$$

In order to show that (5.2.11) is a min-max controller, a  $\tilde{P}$  must be found to satisfy (5.3.18) which makes

$$\tilde{Q} := \tilde{P} - \tilde{G}_c^T \tilde{P} \tilde{G}_c > 0 \tag{5.3.19}$$

It can be seen from the structures of  $\tilde{H}$  and  $\tilde{L}$  in (5.3.6) that in order to satisfy (5.3.18)  $\tilde{P}$  must have a block diagonal structure:

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$
(5.3.20)

where  $\tilde{P}_1 \in \mathbb{R}^{n \times n}$ ,  $\tilde{P}_2 \in \mathbb{R}^{m \times m}$  and

$$F_2 = H_2^T \tilde{P}_2 \tag{5.3.21}$$

From here, the proof is similar to Proposition 3.3.2. It can be shown that

$$\tilde{Q} = \begin{bmatrix} \tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} & -\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12} \\ -\tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} & \tilde{P}_2 - \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} \end{bmatrix}$$
(5.3.22)

which can be made positive definite if  $\tilde{P}_1$  and  $\tilde{P}_2$  are chosen to satisfy

$$\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} > 0 \tag{5.3.23}$$

and

$$\tilde{P}_2 > \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} + \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} (\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11})^{-1} (\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12})$$
(5.3.24)

This is always possible since  $\tilde{G}_{11}$  is stable.

**Remark 5.3.1** Due to the order reduction which takes place when establishing the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$ , the stabilisability problem for this lower order system sometimes reduces

to root locus arguments, as will be seen in the DC-motor implementation study or even observer-like arguments if  $\operatorname{rank}(\bar{G}_{12}) = m > n - m$  (as seen in the DH-Beaver aircraft example in §3.4).

**Remark 5.3.2** The restrictions on the class of systems for which sliding surfaces of the form (5.2.9) are appropriate (in the sense that they produce stable ideal sliding motions) depends solely on the stabilisability of the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$ . There is no obvious dependence on the invariant zeros of (G, H, C). As such the method is applicable to non-minimum phase systems. This is in direct contrast to the continuous time sliding mode situation where in order for classical output feedback sliding mode controllers to work, the system must be minimum phase and relative degree one (Edwards & Spurgeon 1998). The relative degree assumption is needed to make the first derivative of the switching variable discontinuous whilst the minimum phase requirement follows because the reduced-order dynamics of the system, when sliding, have the invariant zeros of the system amongst the poles of the closed-loop system, and hence unstable transmission zeros will result in unstable sliding mode dynamics (Edwards & Spurgeon 1998).

Suppose the reference signal  $r(k) = r_s = \text{const}$  for all  $k > k_s = \text{const}$ . Modify the control law (5.2.11) to be

$$u(k) = -(FCG^{-1}H)^{-1}FCx(k) + F_r r(k)$$
(5.3.25)

where  $F_r \in \mathbb{R}^{m \times m}$  is a feedforward term to be determined. The steady state value of x(k)as  $k \to \infty$ , obtained by using the control law (5.3.25) is

$$x_s = (I - G_c)^{-1} (H_r + HF_r) r_s (5.3.26)$$

The expression in (5.3.26) is well defined since by design the eigenvalues of the closed-loop system matrix  $G_c = G - H(FCG^{-1}H)FC$  are inside the unit disk and so  $(I - G_c)$  is invertible. Using (5.3.25) and defining  $e(k) = x(k) - x_s$  it follows from simple algebraic manipulation that

$$e(k+1) = G_c e(k) + H\xi(k)$$
(5.3.27)

and all the analysis follows through as for the regulation case in Chapter 3 for (5.3.27). In the absence of uncertainty,  $e(k) \to 0$  as  $k \to \infty$ , and since steady state is achieved, it follows from the first p equations in (5.2.5) that  $y_p(k) = r_s$  and so tracking is achieved. Furthermore it can be shown that

$$FCG^{-1}x_{s} = FCG^{-1}(I - G_{c})^{-1}(H_{r} + HF_{r})r_{s} \equiv FCG^{-1}(H_{r} + HF_{r})r_{s}$$

and consequently if  $F_r$  in (5.3.25) is chosen as

$$F_r = -(FCG^{-1}H)^{-1}FCG^{-1}H_r (5.3.28)$$

then  $FCG^{-1}x_s = 0$  and so  $x_s \in S$  from (5.2.9). Using  $F_r$  defined in (5.3.28) the control law can then be written as

$$u(k) = -(FCG^{-1}H)^{-1}FCx(k) - (FCG^{-1}H)^{-1}FCG^{-1}H_rr(k)$$
(5.3.29)

**Remark 5.3.3** The control law in (5.3.29) can be written as

$$u(k) = -H_2^{-1} \begin{bmatrix} K & I_m \end{bmatrix} T^T y(k) + H_2^{-1} \begin{bmatrix} K & I_m \end{bmatrix} T^T C G^{-1} H_r r(k)$$
(5.3.30)

which is clearly independent of the Lyapunov matrix P and is parameterized solely by the choice of K.

#### 5.3.1 Key Steps for Designing the Tracking Controller

- 1. Check that the system satisfies assumptions A1 and A2. These conditions are easily tested numerically. If either are not met, then the approach is not valid.
- 2. Augment the nominal system with integral action states to obtain the system in the form of (5.2.5). Check the controllability of the pair (G, H) which is guaranteed providing the triple  $(G_p, H_p, C_p)$  has no invariant zeros at unity.
- 3. Form an output distribution matrix,  $L = CG^{-1}$  to produce the 'fictitious' system (G, H, L).
- 4. Perform the change of coordinates in (5.3.1) to obtain the triple (\$\bar{G}\_{11}\$, \$\bar{G}\_{12}\$, \$\bar{L}\_1\$) and select the design parameter \$K\$ to stabilise the reduced-order system (\$\bar{G}\_{11}\$-\$\bar{G}\_{12}\$K\$\bar{L}\_1\$). (In the general case this is not trivial but there is a wealth of literature and algorithms which consider this problem; see (Syrmos et al. 1997).)
- 5. The control can be calculated from (5.3.30).

#### 5.4 Closed-Loop Analysis

The control law synthesis in §5.3 requires only knowledge of the nominal linear system and the matched structure of the uncertainty/disturbances represented by  $\xi(k)$ . In this section the effect of  $\xi(k)$  on the closed-loop dynamics will be explored. When uncertainty or external disturbances are present, asymptotic stability is (usually) lost. However if  $\xi(k)$  is bounded, as argued in (Corless 1985, Spurgeon 1992) for example, suitable ultimate boundedness sets can be calculated for the states x(k) into which the system must enter. If  $\xi(k)$  represents an exogenous disturbance then it can easily be shown that, for the closed-loop system, the states x(k) evolve in such a way that

$$Sx(k+1) = SH\xi(k), \quad k = 1, 2, \dots$$
 (5.4.1)

and thus  $||SH\xi(k)||$  is the deviation from the ideal sliding surface  $S = \{x : Sx = 0\}$ . If  $\xi(k)$  is bounded then  $\max_{\xi(k)\in\mathcal{F}} ||SH\xi(k)||$  represents the boundary layer about S into which the states x(k) ultimately enter. The discrete time output feedback sliding mode control law (5.2.11) is derived from the min-max control (2.3.19) in Chapter 2. As argued in §2.3.3, the choice of the control law in (2.3.19) minimises the worst case deviation from S over all possible controllers. Also, as demonstrated in Proposition 2.3.1 (Chapter 2), the control law (2.3.19) minimises the worst case deviation from the nominal ideal sliding mode dynamics in a min-max sense.

Assuming  $\xi(k)$  represents matched uncertainty and satisfies

$$\|\xi(k)\| < \rho_1 \|x(k)\| + \rho_0 \tag{5.4.2}$$

where  $\rho_1$  and  $\rho_0$  are positive constants, the same arguments as those in §3.3.5 can be applied here to formally analyse the closed-loop stability.

**Remark 5.4.1** From (5.4.1) and (5.4.2) it follows that

$$||Sx(k+1)|| \le \rho_1 ||SH|| ||x(k)|| + \rho_0 ||SH||$$

and so the states evolve in a conic sector around S. This correlates with the work of (Furuta & Pan 2000).

**Remark 5.4.2** In the case when  $r(k) \neq 0$  and a tracking problem is being considered, the same arguments can be applied to the uncertain situation by considering the e(k) states satisfying (5.3.27).

#### 5.5 Examples

Two examples will be considered: one to demonstrate the properties of the discrete time sliding mode controllers from a theoretical perspective; and the second to demonstrate the practicality of the approach by considering a real-time implementation on a DC-motor rig.

#### 5.5.1 Example 1

Consider the second-order discrete system used in (Sharav-Schapiro et al. 1996, Sharav-Schapiro et al. 1998) given by

$$G_p = \begin{bmatrix} 0 & 1.0000 \\ 0.4000 & 0.2000 \end{bmatrix} \qquad H_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_p = \begin{bmatrix} 1.5000 & 1.0000 \end{bmatrix} \qquad (5.5.1)$$

It can be shown that this system is non-minimum phase and that in fact the system has a zero at -1.5. For simulation purposes, it has been assumed that the system has been obtained from a continuous system by a sample and hold operation with sample interval  $\tau = 0.1$ . After augmenting with an integrator to obtain a system of the form described in (5.2.5), and changing coordinates, it can be shown that

$$\bar{G} = \begin{bmatrix} -0.0669 & -0.0247 & 0.2659 \\ -0.1423 & 1.0029 & -0.0264 \\ 1.4231 & 0.0711 & 0.2640 \end{bmatrix} \quad \bar{H} = \begin{bmatrix} 0 \\ 0 \\ -3.7687 \end{bmatrix} \quad (5.5.2)$$
$$\bar{C} = \begin{bmatrix} -0.0000 & -1.0050 & 0.0000 \\ -1.4302 & 0.0291 & -0.2653 \end{bmatrix} \quad (5.5.3)$$

The fictitious output distribution matrix is given by

$$\bar{L} = \bar{C}\bar{G}^{-1} = \begin{bmatrix} 0 & T \end{bmatrix} = \begin{bmatrix} 0 & -0.9950 & -0.0995 \\ 0 & 0.0995 & -0.9950 \end{bmatrix}$$
(5.5.4)

It follows from  $\overline{G}$  in (5.5.2) that

$$\bar{G}_{11} = \begin{bmatrix} -0.0669 & -0.0247 \\ -0.1423 & 1.0029 \end{bmatrix} \quad \bar{G}_{12} = \begin{bmatrix} 0.2659 \\ -0.0264 \end{bmatrix} \quad \bar{L}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The design procedure involves selecting a K to ensure  $(\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1)$  is stable. From the root-locus (Figure 5.5.1), the matrix  $(\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1)$  is stable in a discrete-time sense for the range of values -27.0 < K < -0.1. If K = -5, then using T from (5.5.4) and the definition in (5.3.2), the design matrix associated with the sliding surface is

$$F = F_2 \left[ \begin{array}{cc} -4.2417 & 1.2985 \end{array} \right]$$

To calculate the parameter  $F_2$ , an LMI optimization problem similar to (3.3.27)-(3.3.29)must be solved for the augmented system. Using the LMItool (Gahinet, Nemirovski, Laub



Figure 5.5.1: Root locus plot of reduced-order system  $(\bar{G}_{11}, \bar{G}_{12}, -\bar{L}_1)$ 

& Chilali 1995) in MATLAB yields  $\mu = 3.2793$  and a value of  $F_2 = H_2^T \tilde{P}_2 = -0.8700$  and thus

$$S = FCG^{-1} = \left[ \begin{array}{cc} -4.2417 & 0.2186 & 3.2787 \end{array} \right]$$

The associated Lyapunov matrix

$$P = \begin{bmatrix} 20.8984 & -1.6151 & -4.2417 \\ -1.6151 & 1.4551 & 0.2186 \\ -4.2417 & 0.2186 & 3.2787 \end{bmatrix}$$
(5.5.5)

The control law (5.3.29) is given by

$$u(k) = \begin{bmatrix} 1.2937 & -0.3960 \end{bmatrix} y(k) + 0.1294r(k)$$
 (5.5.6)

where  $y(k) = col(y_p(k), x_r(k))$ . The corresponding stability margin is  $\sqrt{1/\mu} = 0.5522$ .

In the following simulation, the control law in (5.5.6) has been compared with a discontinuous one of the form

$$u(k) = \left[ 1.2937 \quad -0.3960 \right] y(k) + 0.1294r(k) - \eta (\bar{H}^T P \bar{H})^{-1} \text{sign}(Sx(k))$$
(5.5.7)

for different values of the scalar gain  $\eta$ . The controller in (5.5.7) satisfies the three conditions of Gao (Gao et al. 1995) and as argued by (Monsees 2002) is an optimal special case of the discrete time controllers proposed in (Gao et al. 1995). Of course, as argued earlier, generally Sx(k) cannot be calculated based on output measurements alone and so the control law in (5.5.7) technically cannot be implemented. Here the purpose is to show that the inclusion of the sign term does not improve the control performance and so the linear control law (5.5.6) is more appropriate in the case of discrete-time output feedback sliding mode control. In the following simulations the matched disturbance from (5.2.5) is given by  $\xi(k) = 0.05 \sin(5\tau k)$  for k = 1, 2, 3... and the reference signal  $r(k) \equiv 0$ .

Figure 5.5.2 shows plots of the Lyapunov difference function  $\Delta V(k) = V(k+1) - V(k)$ where  $V(k) = x(k)^T P x(k)$  for the value of P given in (5.5.5). The linear min-max controller minimises the effect of the worst case  $\xi$  on  $\Delta V(k)$ . It can be seen that the linear control law (5.5.6) comfortably out-performs the nonlinear one (5.5.7). Another measure of performance is to examine the evolution of V(k) as this may be viewed as a weighted norm of the states x(k). Again it can be seen from Figure 5.5.3 that there is no advantage to adding a switched term. Figure 5.5.4 show plots of the switching function Sx(k). Again, as predicted in §2.3.3, the linear controller provides a smaller boundary layer around Sx = 0.



Figure 5.5.2: Plots of  $|\Delta V(k)|$  for different values of  $\eta$ 



Figure 5.5.3: Plots of V(k) for different values of  $\eta$ 



Figure 5.5.4: Plots of Sx(k) for different values of  $\eta$ 

#### 5.5.2 Example 2 (Experimental Results)

A discrete output feedback sliding mode integral action controller has been designed for a small 30W DC-motor system. The requirement is for the motor to rotate an output shaft to a specified reference. In the experimental setup a potentiometer is connected to the motor shaft and produces an output voltage proportional to the shaft angle. The potentiometer has a track angle of 300 deg and the total voltage across it is 30V, giving 1V/10 deg. An eddy-current disc brake is mounted on the shaft and a disturbance torque in the form of a magnet assembly can be engaged and disengaged at will.

A discrete model of the motor was obtained using system identification at a sample time of 0.03s. This is a sensible choice of sampling interval according to (Aström & Wittenmark 1984) who argue that there should be between 5-20 samples in a step response of the closed-

loop system. The third order system has position (rad), velocity (rad/s) and armature current (Amp) as the states. In the experiments which follow, only the shaft angular position measurement has been used for control purposes. The system matrices are as shown below

$$G_{p} = \begin{bmatrix} 1.0000 & 0.0646 & 0.0065 \\ 0 & 0.6282 & 0.0791 \\ 0 & -0.3591 & -0.0430 \end{bmatrix} \qquad H_{p} = \begin{bmatrix} 0.0405 \\ 1.1060 \\ 1.5651 \end{bmatrix} \qquad C_{p} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

**Remark 5.5.1** The invariant zeros of  $(G_p, H_p, C_p)$  are  $\{-1.3911, -0.0376\}$  and therefore the system has one unstable transmission zero. For the augmented system it is not possible to use a sliding surface of the form

$$S_o = \{x : F_o C x = 0\}$$
(5.5.8)

where  $F_o \in \mathbb{R}^{m \times 2m}$  since for all choices of  $F_o$ 

$$\{-1.3911, -0.0376\} \subset \sigma(G - H(F_o CH)^{-1} F_o CG)$$

and hence the ideal sliding motion (and thus the closed-loop system) will always be unstable. This follows from the fact that, as argued in §3.3.2, output based sliding surfaces of the form (5.5.8) always have the invariant zeros of (G, H, C) as part of the dynamics of the ideal sliding motion. This can be easily seen from Lemma 3.3.2 and 3.3.3 which indicates that the invariant zeros of (G, H, C) will appear as unobservable modes of the triple governing the sliding motion. Remark 5.5.2 A standard 3rd order model in continuous time relating the control signal to angular position is relative degree three and so most classical continuous time sliding mode output based schemes are not applicable. For example the output based scheme described in (Edwards & Spurgeon 1998) which use only output measurements (without computing the derivatives of the measured signals), and which provide complete insensitivity to matched uncertainty is only applicable to minimum phase relative degree one systems. Also only position information is used here compared to the full state-information required in the scheme described for servo-position control in (Golo & Milosavljević 2000). Sliding mode schemes for DC drive control have also been proposed in (Damiano, Gatto, Marongiu & Pisano 2004) but this work is concerned with speed control (and hence is only relative degree two) and uses output derivative estimation schemes to provide additional information to the control system.

The augmented system matrices were formed as in (5.2.7)-(5.2.8) and after the first change of coordinates in (5.3.1)

$$\bar{G} = \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix} = \begin{bmatrix} 0.6244 & 0.4456 & -0.0084 & 0.5965 \\ -0.0904 & 0.9341 & 0.0012 & -0.1244 \\ 0.0000 & 0.0188 & 1.0005 & -0.0008 \\ \hline 0.0000 & -0.6269 & 0.0118 & 0.0261 \end{bmatrix}$$
$$\bar{H} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ \hline -1.5514 \end{bmatrix}$$

The fictitious output is given by

$$\bar{L} = \begin{bmatrix} 0_{2 \times 2} & T \end{bmatrix} = \begin{bmatrix} -0.0000 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix} \begin{bmatrix} -0.9996 & -0.0300 \\ 0.0300 & -0.9996 \end{bmatrix}$$
(5.5.9)

The design freedom in the controller involves selecting the parameter K to ensure that the eigenvalues of  $(\bar{G}_{11} - \bar{G}_{12}K\bar{L}_1)$  lie inside the unit disk. In this instance, K is a scalar and from root locus plots of the single input single output reduced-order system, it is found that the range of values for K which stabilise the system are -14.5 < K < 0.

The parameter K = -1.2 stabilises the sliding motion where the corresponding closed-loop poles are {  $0.8018 \pm 0.1032i$ , 0.9545 }. The final control law from (5.3.30) is

$$u(k) = \left[ \begin{array}{cc} 0.7538 & -0.6675 \end{array} \right] y(k) + 0.0226r(k)$$

Solving the LMI optimization problem yields

$$P = \begin{bmatrix} 331.5509 & -47.6944 & -10.5957 & 0.6462 \\ -47.6944 & 42.8278 & 2.3982 & 4.1054 \\ -10.5957 & 2.3982 & 3.7165 & -0.4869 \\ 0.6462 & 4.1054 & -0.4869 & 5.3318 \end{bmatrix}$$

and an optimal value of  $\mu = 16.7278$ . The associated values of  $F_2 = H_2^T \tilde{P}_2 = -10.8075$ and so

$$F = \left[ \begin{array}{cc} -12.6391 & 11.1915 \end{array} \right]$$

The associated robustness bound is  $\rho_1 < \sqrt{1/16.7278} = 0.2445$ . The design described



Figure 5.5.5: Plots of output (position) and control signal against time in response to an external disturbance

above has been implemented in real-time via dSPACE<sup>1</sup> on a test rig. The system was tested for disturbance rejection as well as tracking. Figure 5.5.5 shows the response of the system to an external torque applied to displace the shaft position from zero. The shaft was manually displaced and held at a non-zero angle whilst under closed-loop control. During this time, there is a buildup of the current level in the armature coils and integral windup occurs in the states  $x_r$ . The point 'A' on the graph indicates the point at which the shaft was released. The controller is seen to regulate the shaft back to zero in approximately two seconds. In Figure 5.5.6, a ramped signal is used as the reference input to the system and the tracking behavior is shown. The maximum current drawn in this case was 0.4346Amps. The 'zig-zag' phenomenon observed in the control signal is not so-called chattering (the control law is linear), but results from the stick-slip behaviour associated with the test rig.

<sup>&</sup>lt;sup>1</sup>This is a registered trademark of dSpace GmbH.



Figure 5.5.6: Plots of output (position) and control signal tracking a ramped signal

#### 5.6 Summary

Theoretical development and a design synthesis procedure for a ODSMC which incorporates integral action have been presented. The methodology introduces additional states to form a new augmented system which utilizes and builds on the new design approaches from Chapter 3 to produce a controller with a tracking capability. The efficacy of the approach has been demonstrated with a real engineering example. Furthermore the scheme has been implemented on a rig in real-time and very good results have been obtained. Because the controller (thought of as controlling the plant augmented with appropriate integrators) is static output feedback in nature, as in Chapter 3, in order for the controller to be stabilising it is necessary to be able to solve a certain reduced-order classical static output feedback pole placement problem for a certain (fictitious) triple. This is the most significant restriction on its applicability. The next chapter will investigate improving system performance and solving the tracking

control problem when the fictitious subsystem is not static output feedback stabilisable.

# Chapter 6

# Output Tracking Using Dynamic Discrete Output Feedback Sliding Mode Controllers

### 6.1 Introduction

The results proposed in chapter 5 using static output feedback sliding mode control required certain criteria on the discrete time system to be met. In particular, a certain triple must satisfy an output feedback criteria (Proposition 5.3.1). It is well known that it is not possible to stabilise all systems using static output feedback. One way of overcoming this is to introduce additional dynamics, or a compensator, so that the conditions are satisfied by the augmented system.

This chapter builds on the work described in Chapter 5 and proposes a specific com-

pensator structure to circumvent the 'output feedback stabilisability' restriction when incorporating tracking control. The resulting controller is applied to a planar vertical takeoff and landing (PVTOL) aircraft model. The PVTOL has previously elicited interest from the control community (Al-Hiddabi & McClamroch 1998, Lu, Spurgeon & Postlethwaite 1997, Fantoni, Zavala & Lozano 2002) because the simplified two degree of freedom dynamics are nonlinear and non-minimum phase. It is an interesting theoretical problem associated with a nonlinear system clearly motivated by an application. Simulation results are presented to verify the robustness of this method.

#### 6.2 **Problem Formulation**

Consider the discrete time square system with matched uncertainties

$$x_{p}(k+1) = Gx_{p}(k) + H(u(k) + \xi(k))$$
(6.2.1)

$$y(k) = Cx_p(k) \tag{6.2.2}$$

where  $x_p \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^m$  with m < n. Assume that the input and output distribution matrices H and C are full rank. In addition, assume the triple (G, H, C) is minimal. The matched uncertainties,  $\xi(k)$ , are assumed to be unknown but bounded.

As before, the objective is to determine an appropriate sliding surface, S, and a control law which depends only on the outputs. The nominal linear system must achieve an *ideal sliding motion* in finite time when  $\xi \equiv 0$ , whereas in the presence of matched uncertainty, the effect of  $\xi$  is minimised and an appropriate bound about the sliding surface is maintained by the system trajectories. As in chapter 5, introduce integral action, where the difference equations

$$x_r(k+1) = x_r(k) + \tau(r(k) - Cx(k))$$
(6.2.3)

are added to plant representation in (6.2.1). In equation (6.2.3),  $\tau$  represents the sample interval. The quantity r(k) represents the signal to be tracked by the output. Furthermore assume  $r(k) = r_s = \text{const}$  for  $k > k_s$ . As in Chapter 3-5 assume

- A1) the plant state transition matrix G is nonsingular.
- A2) the matrix  $CG^{-1}H$  has rank m.

In chapter 5 a static output sliding mode controller was designed for the augmented system formulated from (6.2.1)-(6.2.3). This required that the triple  $(\bar{G}_{11}, \bar{G}_{12}, \bar{L}_1)$  obtained from the augmented system defined in Proposition 5.3.1 to be output feedback stabilisable (in addition to assumptions A1 and A2). The stabilisability requirement on  $(\bar{G}_{11}, \bar{G}_{12}, C_1)$  will be removed in this chapter by the introduction of a dynamical output feedback controller. As in earlier chapters, define a new output distribution matrix  $L := CG^{-1}$  to generate a new purely fictitious system (G, H, L). In order to facilitate the analysis, a change of coordinates will be introduced for the fictitious system (G, H, L). From assumption A2  $rank(CG^{-1}H) = rank(LH) = m$ , there exists a change of coordinates such that

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & T \end{bmatrix}$$
(6.2.4)

where  $G_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ ,  $H_2 \in \mathbb{R}^{m\times m}$  and is nonsingular and  $T \in \mathbb{R}^{m\times m}$  is orthogonal. Partition the state vector  $x_p$  conformably as  $\operatorname{col}(x_1, x_2)$  where  $x_1 \in \mathbb{R}^{(n-m)}$ . It follows from the canonical form (6.2.4) that the true output distribution matrix

$$C = LG = \begin{bmatrix} TG_{21} & TG_{22} \end{bmatrix}$$
(6.2.5)

### 6.3 Controller Design

Also introduce additional states  $x_c \in \mathbb{R}^{(n-m)}$ , which under certain circumstances (which will be explained later) represent an estimate of the states  $x_1$ .

The intention is to induce an ideal sliding motion on the surface

$$S = \{ (x_1, x_c, x_r, x_2) : K_1 x_c + K_r x_r + x_2 + S_r r_s) = 0 \}$$
(6.3.1)

where  $K_1 \in \mathbb{R}^{m \times (n-m)}$  and  $K_r \in \mathbb{R}^{m \times m}$  together with  $S_r \in \mathbb{R}^{m \times m}$  represent design freedom. This is a slightly more complex surface than the one in §5.2 since it is reference dependent. However, since  $S_r$  is a design parameter, choosing  $S_r = 0$  recovers a sliding surface more akin to the one in Chapter 5.

Let the compensator take the form

$$x_c(k+1) = G_{11}x_c + G_{12}x_2 + \Omega(y - \hat{y})$$
(6.3.2)

where

$$\hat{y}(k) = TG_{21}x_c(k) + TG_{22}x_2(k) \tag{6.3.3}$$

and  $\Omega \in \mathbb{R}^{(n-m) \times m}$  is a design variable.

During an ideal sliding motion, from (6.3.1)

$$x_2(k) = -K_1 x_c(k) - K_r x_r(k) - S_r r(k)$$

and so after some algebraic manipulation

$$x_c(k+1) = \Phi x_c(k) + \Gamma_1 y(k) + \Gamma_2 x_r(k) + \Gamma_3 r(k)$$
(6.3.4)

where

$$\Phi = G_{11} - \Omega T G_{21} - G_{21} K_1 + \Omega T G_{22} K_1$$
(6.3.5)

$$\Gamma_1 = \Omega \tag{6.3.6}$$

$$\Gamma_2 = -G_{12}K_r + \Omega T G_{22}K_r \tag{6.3.7}$$

$$\Gamma_3 = -G_{12}S_r + \Omega T G_{22}S_r \tag{6.3.8}$$

It is assumed as part of the design process that  $\Omega$  is chosen to guarantee that det  $\Phi \neq 0$ .

Augment the system in (6.2.4) with the integral and compensator states from (6.2.3) and (6.3.4) to obtain:

$$x_a(k+1) = G_a x_a(k) + H_a(u(k) + \xi(k)) + H_r r(k)$$
(6.3.9)

where  $x_a = col(x_1, x_c, x_r, x_2)$ . At first sight, this represents a non-intuitive arrangement of the states but it leads to a simplification in the presentation.

The measurable outputs associated with this system are  $y_a = col(x_c, x_r, y)$ . It is easily

verified that

$$G_{a} = \begin{bmatrix} G_{11} & 0 & 0 & G_{12} \\ \Gamma_{1}TG_{21} & \Phi & \Gamma_{2} & \Gamma_{1}TG_{22} \\ -\tau TG_{21} & 0 & I_{m} & -\tau TG_{22} \\ G_{21} & 0 & 0 & G_{22} \end{bmatrix}, \quad H_{r} = \begin{bmatrix} 0 \\ \Gamma_{3} \\ \tau I_{m} \\ 0 \end{bmatrix}, \quad H_{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ H_{2} \end{bmatrix}$$
(6.3.10)

and the output distribution matrix

$$C_{a} = \begin{bmatrix} 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_{m} & 0 \\ TG_{21} & 0 & 0 & TG_{22} \end{bmatrix}$$
(6.3.11)

where  $y_a := C_a x_a$ .

A output feedback sliding mode controller of the form

$$u(k) = -(FC_a G_a^{-1} H_a)^{-1} FC_a x_a(k) + F_r r(k)$$
(6.3.12)

will now be developed for the augmented system, where both F and  $F_r \in \mathbb{R}^{m \times m}$  are to be determined (in terms of  $\Omega$ ,  $K_1$ ,  $K_r$  and  $S_r$ ).

The objective is to select F and a parameter  $F_2 \in \mathbb{R}^{m \times m}$  so that the surface

$$S_a = \{x_a : FC_a G_a^{-1} x_a + F_2 S_r r_s = 0\}$$
(6.3.13)

is identical to the surface S in (6.3.1), and then to select  $K_1$ ,  $K_r$  and  $\Omega$  to ensure a stable ideal sliding motion when confined to S.

Providing the design matrix F is chosen to ensure the eigenvalues of

$$G_c = G_a - H_a (FC_a G_a^{-1} H_a) FC_a$$
(6.3.14)

are inside the unit disk,  $(I - G_c)$  is invertible. As in §5.3, define

$$x_s = (I - G_c)^{-1} (H_r + H_a F_r) r_s aga{6.3.15}$$

Then using (6.3.12) and defining  $e(k) = x_a(k) - x_s$  it follows from simple algebraic manipulation that

$$e(k+1) = G_c e(k) + H_a \xi(k)$$
(6.3.16)

In the absence of uncertainty  $e(k) \to 0$  as  $k \to \infty$ , and since steady state is achieved, it follows from (6.2.3) that  $y_p(k) = r_s$  and so tracking is achieved. Furthermore it can be shown that

$$FC_a G_a^{-1} x_s = FC_a G_a^{-1} (I - G_c)^{-1} (H_r + H_a F_r) r_s \equiv FC_a G_a^{-1} (H_r + H_a F_r) r_s$$

and consequently if  $F_r$  is chosen as

$$F_r = -(FC_a G_a^{-1} H_a)^{-1} \left( FC_a G_a^{-1} H_r + F_2 S_r \right)$$
(6.3.17)

then

$$FC_a G_a^{-1} x_s + F_2 S_r r_s = 0$$

and so  $x_s \in S$ . The control law can then be written

$$u(k) = -(FC_a G_a^{-1} H_a)^{-1} \left( FC_a x_a(k) + (FC_a G_a^{-1} H_r + F_2 S_r) r(k) \right)$$
(6.3.18)

In order to make (6.3.12) an output feedback sliding mode controller, the problem is therefore to find an F and a s.p.d. matrix  $P_a \in \mathbb{R}^{2n \times 2n}$  such that

$$FC_a = H_a^T P_a G_a \tag{6.3.19}$$

and

$$G_c^T P_a G_c - P_a < 0 (6.3.20)$$

As in §6.2 define for the augmented system  $L_a := C_a G_a^{-1}$ . From (6.3.10)-(6.3.11) and after some algebra

$$L_{a} = \begin{bmatrix} 0 & \Phi^{-1} & -\Phi^{-1}\Gamma_{2} & -\Phi^{-1}\Gamma_{1}T - \tau\Phi^{-1}\Gamma_{2}T \\ 0 & 0 & I_{m} & \tau T \\ 0 & 0 & 0 & T \end{bmatrix}$$
(6.3.21)

Define

$$T_{a} := \begin{bmatrix} \Phi^{-1} & -\Phi^{-1}\Gamma_{2} & -\Phi^{-1}\Gamma_{1}T - \Phi^{-1}\Gamma_{2}\tau T \\ 0 & I_{m} & \tau T \\ 0 & 0 & T \end{bmatrix}$$
(6.3.22)

and observe that  $\det T_a \neq 0$ . Then

$$L_a = \left[ \begin{array}{cc} 0 & T_a \end{array} \right] \tag{6.3.23}$$

and the triple  $(G_a, H_a, L_a)$  has the form of the canonical form from (5.3.1) except  $T_a$  is nonsingular rather than orthogonal. Define a matrix

$$F = F_{2} \begin{bmatrix} K_{1} & K_{r} & I_{m} \end{bmatrix} T_{a}^{-1}$$

$$= \begin{bmatrix} F_{2}K_{1} & F_{2}K_{r} & F_{2} \end{bmatrix} \begin{bmatrix} \Phi & \Gamma_{2} & \Gamma_{1} \\ 0 & I_{m} & -\tau I_{m} \\ 0 & 0 & T^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2}K_{1}\Phi & F_{2}K_{1}\Gamma_{2} + F_{2}K_{r} & F_{2}K_{1}\Gamma_{1} - F_{2}K_{r}\tau + F_{2}T^{-1} \end{bmatrix} (6.3.24)$$

where  $F_2 \in \mathbb{R}^{m \times m}$  and is nonsingular. This variable has no effect on the dynamics of the reduced-order sliding motion but is required to solve the constraint (6.3.19).

After a little algebra it can be shown that

$$FL_a = FC_a G_a^{-1} = F_2 \left[ \begin{array}{ccc} 0 & K_1 & K_r & I_m \end{array} \right]$$
(6.3.25)

and so the sliding surface  $S_a$  in (6.3.13) is identical to the one in (6.3.1) because by definition  $F_2$  is nonsingular.

To facilitate choosing the parameters  $\Omega$ ,  $K_1$  and  $K_r$  change coordinates according to the

transformation  $x_a \mapsto \tilde{T}x_a =: \tilde{x}$  where

$$\tilde{T} := \begin{bmatrix} I_{n-m} & -I_{n-m} & 0 & 0 \\ 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & K_1 & K_r & I_m \end{bmatrix}$$
(6.3.26)

This effectively forces the last m states of the new coordinates to represent what in continuous time sliding mode control would be called the 'switching function'  $\sigma = K_r x_r + K_1 x_c + x_2$  associated with S in (6.3.1). It follows  $\tilde{G} = \tilde{T} G_a \tilde{T}^{-1}$ ,  $\tilde{H} = \tilde{T} H_a$ ,  $\tilde{H}_r = \tilde{T} H_r$ ,  $\tilde{C} = C_a \tilde{T}^{-1}$  and  $\tilde{L} = L_a \tilde{T}^{-1}$ . Using the definition of  $\phi$ ,  $\Gamma_1$  and  $\Gamma_2$  in (6.3.5), (6.3.6) and (6.3.7) respectively,

$$\tilde{G} = \begin{bmatrix} G_{11} - \Omega T G_{21} & 0 & 0 \\ \Omega T G_{21} & G_{11} - G_{12} K_1 & -G_{12} K_r \\ -\tau T G_{21} & -\tau T G_{21} + \tau T G_{22} K_1 & I + \tau T G_{22} K_r \\ K_1 L T G_{21} - K_r \tau T G_{21} + G_{21} & \tilde{G}_{42} & \tilde{G}_{43} \end{bmatrix}$$

$$G_{11} - \Omega T G_{22} \\ \Omega T G_{22} \\ -\tau T G_{22} \\ K_1 L T G_{22} - K_r \tau T G_{22} + G_{22} \end{bmatrix}$$

$$(6.3.27)$$

where

$$\tilde{G}_{42} = -K_r \tau T G_{21} + G_{21} + K_1 G_{11} - K_1 G_{12} + K_r \tau T G_{22} K_1 - G_{22} K_1$$
$$\tilde{G}_{43} = K_r - K_1 G_{12} K_r + K_r \tau T G_{22} K_r - G_{22} K_r$$
Also

$$\tilde{H} = \begin{bmatrix} 0\\ 0\\ 0\\ H_2 \end{bmatrix}, \qquad \tilde{H}_r = \begin{bmatrix} -\Gamma_3\\ \Gamma_3\\ \tau I_m\\ K_1\Gamma_3 + \tau K_r \end{bmatrix}$$
(6.3.28)

$$\tilde{C} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ TG_{21} & TG_{21} - TG_{22}K_1 & -TG_{22}K_r & TG_{22} \end{bmatrix}$$

From equation (6.3.25)

$$F\tilde{L} = \left[\begin{array}{cccc} 0 & 0 & 0 & F_2 \end{array}\right] \tag{6.3.29}$$

Some algebra reveals the closed-loop system matrix

$$\tilde{G}_{c} = \tilde{G} - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{C}$$

$$= \begin{bmatrix}
G_{11} - \Omega TG_{21} & 0 & 0 & G_{11} - \Omega TG_{22} \\
\Omega TG_{21} & G_{11} - G_{12}K_{1} & -G_{12}K_{r} & \Omega TG_{22} \\
-\tau TG_{21} & -\tau TG_{21} + \tau TG_{22}K_{1} & I + \tau TG_{22}K_{r} & -\tau TG_{22} \\
\hline
0 & 0 & 0 & 0
\end{bmatrix}$$

$$=: \begin{bmatrix}
\tilde{G}_{11} & \tilde{G}_{12} \\
0 & 0
\end{bmatrix}$$
(6.3.30)

This is most easily seen from the definition of  $\tilde{L} = \tilde{C}\tilde{G}^{-1}$  and the fact that

$$\tilde{G}_c = (I - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{L})\tilde{G}$$

From (6.3.28) and (6.3.29) it can be easily shown that

$$(I - \tilde{H}(F\tilde{L}\tilde{H})^{-1}F\tilde{L}) = \operatorname{diag}(I_{n-m}, I_{n-m}, I_m, 0_{m \times m})$$

and hence the structure in (6.3.30) follows immediately. The matrix  $\tilde{G}_{11}$  can be written as

where

$$\tilde{G}_m := \begin{bmatrix} G_{11} - G_{12}K_1 & -G_{12}K_r \\ -\tau TG_{21} + \tau TG_{22}K_1 & I_m + \tau TG_{22}K_r \end{bmatrix}$$
(6.3.32)

It is clear from (6.3.30) and (6.3.31) that

$$\sigma(\tilde{G}_c) = \{0\}^m \cup \sigma(G_{11} - \Omega T G_{21}) \cup \sigma(\tilde{G}_m)$$

where  $\tilde{G}_m$  from (6.3.32) can be decomposed as

$$\tilde{G}_{m} = \underbrace{\begin{bmatrix} G_{11} & 0 \\ -\tau T G_{21} & I_{m} \end{bmatrix}}_{G_{11}^{a}} - \underbrace{\begin{bmatrix} G_{12} \\ -\tau T G_{22} \end{bmatrix}}_{G_{12}^{a}} \begin{bmatrix} K_{1} & K_{r} \end{bmatrix}$$
(6.3.33)

Since the matrix pair  $(G_{11}, G_{21})$  is observable (see §5.3) and T is nonsingular, the pair  $(G_{11}, TG_{21})$  is observable. Consequently an  $\Omega$  can always be found which makes  $(G_{11} - \Omega TG_{21})$  stable. Likewise it can be shown that provided (G, H, C) does not have any invariant zeroes at unity, the pair  $(G_{11}^a, G_{12}^a)$  is controllable and hence the choice of the parameters  $K_1$  and  $K_r$  constitutes a state-feedback problem. Consequently  $K_1$ ,  $K_r$  and  $\Omega$  can be chosen to make  $\tilde{G}_{11}$  from (6.3.31) stable.

In the new set of coordinates  $\tilde{x}$ , let the Lyapunov matrix be represented by  $\tilde{P}$ . Using the definition of  $\tilde{L}$ , equation (6.3.19) becomes

$$\tilde{H}^T \tilde{P} = F \tilde{C} \tilde{G}^{-1} = F \tilde{L} \tag{6.3.34}$$

In order to show that  $\tilde{P}$  is a Lyapunov matrix for  $\tilde{G}_c$  it must be established that

$$\tilde{Q} := \tilde{P} - \tilde{G}_c^T \tilde{P} \tilde{G}_c > 0 \tag{6.3.35}$$

It can be seen from the structures of  $\tilde{H}$  and  $F\tilde{L}$  in (6.3.28) and (6.3.29) and from the fact that det  $H_2 \neq 0$  that in order to satisfy (6.3.34)  $\tilde{P}$  must have a block diagonal structure:

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$
(6.3.36)

where  $\tilde{P}_1 \in \mathbb{R}^{(2n-m) \times (2n-m)}$ ,  $\tilde{P}_2 \in \mathbb{R}^{m \times m}$  and

$$F_2 = H_2^T \tilde{P}_2 \tag{6.3.37}$$

In terms of the partition in (6.3.30), (6.3.35) can be written as

$$\tilde{Q} = \begin{bmatrix} \tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} & -\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12} \\ -\tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} & \tilde{P}_2 - \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} \end{bmatrix}$$
(6.3.38)

As shown in Proposition 5.3.1, a family of solutions  $(\tilde{P}_1, \tilde{P}_2)$  exists to make  $\tilde{Q} > 0$ . Specifically, let  $\tilde{P}_1 > 0$  be a solution to

$$\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} > 0 \tag{6.3.39}$$

Such a solution  $\tilde{P}_1$  is guaranteed to exist since  $\tilde{G}_{11}$  is stable. Then from the Schur complement, inequality (6.3.38) is satisfied if and only if

$$\tilde{P}_2 > \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} (\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11})^{-1} (\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12}) + \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12}$$
(6.3.40)

Any pair  $(\tilde{P}_1, \tilde{P}_2)$  satisfying (6.3.39) and (6.3.40) ensures  $\tilde{P}$  from (6.3.36) satisfies (6.3.34) and (6.3.35).

#### 6.3.1 Key Steps for Designing Dynamic ODSMC with Tracking

The approach to designing a dynamic ODSMC which incorporates tracking is summarized as follows:

1. Check that for a discrete time system, represented by (G, H, C), the state transition matrix G is nonsingular and rank  $(CG^{-1}H) = m$ . If these conditions are not met,

then a dynamic ODSMC does not exist.

- 2. Form the fictitious triple (G, H, L) where  $L = CG^{-1}$  and change coordinates to obtain the canonical form given in (6.2.4). From (6.2.4) identify the matrices  $G_{11}, G_{12}$ and  $G_{21}$ . Also determine the true output distribution matrix, C in these coordinates.
- 3. Select  $\Omega$  using any algorithm of choice so that  $(G_{11} \Omega T G_{21})$  is stable where T is defined in (6.2.4). Determine  $K_1$  and  $K_r$  from the state feedback problem in (6.3.33).
- 4. Once  $\Omega$ ,  $K_1$  and  $K_r$  have been selected, the system matrices of the compensator  $\Phi$ ,  $\Gamma_1$  and  $\Gamma_2$  can be computed from (6.3.5)-(6.3.7). Calculate F and  $F_r$  from (6.3.24) and (6.3.17) respectively.
- 5. The control law can be calculated from (6.3.18).

## 6.4 PVTOL Aircraft Simulations

The planar vertical takeoff and landing (PVTOL) aircraft is an example of a nonlinear, non-minimum phase system (Al-Hiddabi & McClamroch 1998, Lu et al. 1997, Fantoni et al. 2002). The coupling between the rolling moment and the lateral acceleration of the aircraft is taken into account in the model by means of a coefficient  $\epsilon$ . The inputs of the system are the roll acceleration  $(u_1)$  and the thrust acceleration  $(u_2)$ . The two outputs are the horizontal position, x(k), and the vertical position, y(k) (altitude). The inputs, outputs and tracking configuration used here are the standard framework for the PVTOL system. The nonlinear equations are given by

$$\ddot{x} = -\sin\theta u_2 + \epsilon\cos\theta u_1 \tag{6.4.1}$$

$$\ddot{y} = \cos\theta u_2 + \epsilon \sin\theta u_1 - g \tag{6.4.2}$$

$$\ddot{\theta} = u_1 \tag{6.4.3}$$

A linearisation of the aircraft system about the equilibrium point

 $x = y = \theta = \dot{\theta} = \dot{y} = \dot{x} = 0$ 

-

with  $u_1 = 0$  and  $u_2 = g$  is given by

where g = 9.81 is the acceleration of gravity and  $\epsilon \neq 0$  is the coupling coefficient. This linearisation has zeros at  $\pm 3.3015$  and so is (significantly) non-minimum phase. It is also not relative degree one because CB = 0. The system was discretised at a sample interval of t = 0.2s and the controller was designed with a nominal value of  $\epsilon = 0.9$ . The discretetime zeros are at { -1, -1, 1.9359, 0.5166 } so the discretization is also significantly non-minimum phase.

The control task as in (Al-Hiddabi & McClamroch 1998) is to track the reference trajectories, which are step commands:  $x(k) \rightarrow 20m$  while maintaining  $y(k) \rightarrow 30m$  and the roll angle,  $\theta \rightarrow 0$ .

				1	-				1	•	٦
					2.0213	1.0326	0.0863	-0.0675	0	13.1365	
					0.7447	1.9820	-0.0829	-0.3434	1.6918	7.4590	
G =	$G_{11}$	0	- 712	_	0.0448	-0.0849	2.9930	-0.0177	-20.3255	0.4927	
u –	$G_{21}$	0	7 <sub>22</sub>		3.1491	0.0267	0.0131	1.0037	-0.1106	23.5350	,
					0	-0.0163	0.1954	0.0011	-1	0	
					-0.2701	0	0	0	0	-1	

A change of coordinates is performed and the canonical form in (6.2.4) is obtained, where

The closed-loop poles associated with  $G_{11} - \Omega T G_{21}$  were selected to be { 0.8, 0.82, 0.86, 0.88 }

which gives the design parameter

$$\Omega = \begin{bmatrix} 9.1734 & -46.6902 \\ 8.0573 & -58.9717 \\ 0.6065 & 6.1411 \\ 11.7521 & 2.9799 \end{bmatrix}$$

The matrices obtained from (6.3.33) are

$$G_{11}^{a} = \begin{bmatrix} 2.0213 & 1.0326 & 0.0863 & -0.0675 & 0 & 0 \\ 0.7447 & 1.9820 & -0.0829 & -0.3434 & 0 & 0 \\ 0.0448 & -0.0849 & 2.9930 & -0.0177 & 0 & 0 \\ 3.1491 & 0.0267 & 0.0131 & 1.0037 & 0 & 0 \\ -0.0540 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0033 & -0.0391 & -0.0002 & 0 & 1.0000 \end{bmatrix}$$
(6.4.4)  
$$G_{12}^{a} = \begin{bmatrix} 0 & 13.1365 \\ 1.6918 & 7.4590 \\ -20.3255 & 0.4927 \\ -0.1106 & 23.5350 \\ 0 & -0.2000 \\ 0.2000 & 0 \end{bmatrix}$$
(6.4.5)

The closed-loop poles associated with the matrix  $\tilde{G}_m$  are { 0.8, 0.85, 0.875, 0.825, 0.9, 0.95 }

and the design matrices  $K_1, K_r$  from (6.3.33) are

$$K_1 = \begin{bmatrix} 0.6725 & -1.2162 & -0.2143 & 0.2002 \\ 0.1108 & -0.1477 & -0.0128 & 0.0921 \end{bmatrix} \text{ and } K_r = \begin{bmatrix} 0.0393 & 0.0490 \\ 0.0010 & 0 \end{bmatrix}$$

From equations (6.3.5)-(6.3.7), the following matrices (which constitute the compensator)

are obtained

$$\Phi = \begin{bmatrix} 30.5027 & -55.9246 & -0.7423 & 8.9653 \\ 37.1544 & -68.7275 & -0.8386 & 11.2427 \\ 9.4323 & -17.2524 & -1.2479 & 2.8261 \\ -3.2600 & 5.3050 & 0.1963 & -0.6588 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 9.1734 & -46.6902 \\ 8.0573 & -58.9717 \\ 0.6065 & 6.1411 \\ 11.7521 & 2.9799 \end{bmatrix} \text{ and } \Gamma_2 = \begin{bmatrix} 1.8288 & 2.2862 \\ 2.2488 & 2.8049 \\ 0.5568 & 0.6946 \\ -0.1240 & -0.1411 \end{bmatrix}$$

This gives the matrix

$$F = \begin{bmatrix} -27.3473 & 50.7349 & 0.8275 & -8.3815 & -1.6100 & -2.0020 & -1.4150 & 40.5923 \\ -2.5277 & 4.6616 & 0.0756 & -0.7637 & -0.1470 & -0.1827 & -0.0992 & 3.7307 \end{bmatrix}$$

from equation (6.3.24). Finally the control law, denoted by  $u(k) = -G_y y(k) - G_r r(k)$  is

given by

The simulation results are shown in Figures 6.4.1, 6.4.2 and 6.4.3. From the graphs, as  $\epsilon$  varies from its nominal value, the horizontal position is affected very minimally. It is observed that tracking performance is good and  $\theta$  is well regulated.

These results can be compared with those from existing sliding mode methods in continuous time (Lu et al. 1997), where a nonlinear control law is used. Due to the undesirable non-minimum phase characteristics of the PVTOL model, Lu et al. (1997) keep  $\theta$  bounded by using a second control law. The results in this section, however, are obtained from a linear control law which proves to be robust and gives good tracking.

## 6.5 Summary

In this chapter, a new dynamic ODSMC scheme incorporating tracking has been proposed. The scheme requires only that the plant has no poles or zeros at the origin. This shows that with an appropriate choice of surface, discrete time sliding mode control can be applied to non-minimum phase systems. The scheme which has been proposed here includes a



Figure 6.4.1: Tracking control of the PVTOL for  $\epsilon = 0.9$ .  $\theta$  is well regulated.



Figure 6.4.2: Tracking control of the PVTOL for  $\epsilon = 0.5$ .  $\theta$  is well regulated.



Figure 6.4.3: Tracking control of the PVTOL for  $\epsilon = 0.25$ .  $\theta$  is well regulated.

compensator and simple parameterisation of the design freedom has been obtained. As a demonstration it has been applied to a discretised version of a linearisation of the PVTOL aircraft, which has unstable zero dynamics. The resulting discrete time control law has been tested on the nonlinear model to verify the robustness. Very good results have been obtained.

## Chapter 7

# Case study - High Incidence Research Model (HIRM)

## 7.1 Introduction

The thesis thus far has been concerned with the development of ODSMC. This chapter will focus on a case study, a high incidence research model (HIRM), which is a detailed nonlinear fighter aircraft model, and use it as a test bed for the ODSMC. The HIRM model was developed as a benchmark problem for the Group for Aeronautical Research and Technology in Europe (GARTEUR). Although the aircraft is stable both longitudinally and laterally, there are combinations of angle of attack and control surface deflections which may cause the aircraft to be unstable. The HIRM model represents an ideal case study for the theoretical work proposed in this thesis. Longitudinal control of the aircraft model will be considered and in particular pitch control. Various different control methods from the literature have considered the HIRM design problem (Bag 1997, Papageorgiou & Glover 1999, Harkegard & Glad 2000, Amato et al. 2000). However, with the exception of (Harkegard & Glad 2000), the results in the literature look at commanding a different output to that used in this chapter. Hence, it is not strictly possible to directly compare the results obtained in this chapter with the ones found in the literature. However, the aim of this chapter is to demonstrate the robustness of the ODSMC technique by means of simulation on detailed linear and nonlinear models of the aircraft. Simulations of the HIRM have also been carried out on a Real Time All Vehicle Simulator (RTAVS) where manual pilot input commands have been used.

## 7.2 High Incidence Research Model (HIRM)

The HIRM is a mathematical model of a generic fighter aircraft developed by the Group for Aeronautical Research and Technology in Europe (GARTEUR). The HIRM is based on aerodynamic data obtained from wind tunnel tests and flight testing of an unpowered, scaled drop model. The aerodynamics contain nonlinearities. Engine and actuator models have been added to create a representative, nonlinear simulation of a twin engined modern fighter aircraft of F-18 proportions (Muir et al. 1997). The model is one of the three benchmark military aircraft models within GARTEUR. The model was set-up to investigate flight at high angles of attack ( $-50^{\circ}$  to  $+120^{\circ}$ ) and over a wide sideslip range (-50deg to +50deg), but does not include compressibility effects resulting from high subsonic speeds (Muir et al. 1997).

The aircraft model has 16 states (including 4 engine states), 11 inputs (including wind turbulence) and 20 measured outputs (although some are available only for simulation



Figure 7.2.1: Block diagram of nonlinear HIRM aircraft system.

purposes). The Simulink block diagram of the six degrees of freedom (see Appendix C.2) nonlinear HIRM is shown in Figure 7.2.1. The states, inputs and outputs of the aircraft system are given in Tables C.1.1, C.1.2 and C.1.3 of Appendix C. The available control surfaces and sensor information used in the HIRM aircraft model are:

Control Surfaces	Sensors	
Rudder	Body axis angular rates, $p, q$ and $r$	
Differential canard	Body axis attitudes, $\theta$ , $\phi$ and $\psi$	
Differential tailplane	Airspeed	
Symmetric canard	Mach number	
Symmetric tailplane	Altitude	
Engine throttle	Angles of attack and sideslip, $\alpha$ and $\beta$	

Full details of the HIRM's aircraft dynamics, actuator and sensor models can be found in (Muir et al. 1997).

#### 7.2.1 Control Problem Definition

In this section, a controller will be designed for the longitudinal dynamics of the HIRM model, with a (pilot) command input controlling the pitch demand via deflection of symmetrical tailplane. The controller should track a series of manoeuvres with good response times (Amato et al. 2000, Harkegard & Glad 2000) within the flight envelope shown in Figure 7.2.2. These manoeuvres will be discussed in detail in later sections. When designing the controller for a pitch demand system, the following limitations must also be addressed :

- D1)  $-10^{\circ}$  and  $+30^{\circ}$  for the angle of attack.
- D2)  $-40^{\circ}$  and  $10^{\circ}$  for the symmetrical tailplane deflection.

If there are overshoots that exceed any of the limits (D1 and D2), the aircraft should recover and within 2 seconds return to a slight regime which satisfies the limits.

#### 7.2.2 Robustness Considerations

The control system should maintain good performance and robustness across the flight envelope (Figure 7.2.2). The design envelope for the HIRM control law is:



Figure 7.2.2: Flight envelope of the HIRM. (Sets of points marked are used in the testing of the controller on the nonlinear model in subsequent sections.)

## 7.3 Design of an ODSMC for Pitch Control

In this section, the longitudinal dynamics of a linear model of the HIRM will be considered. The nonlinear model of the aircraft system has been linearised at an operating point of Mach number 0.3 and a height of 5000 ft. This flight condition is within the recommended design envelope for the HIRM control laws in Muir et al. (1997) (see Figure 7.2.2) and is one of the linearised design points suggested for the design of control laws for the HIRM (Moormann & Bennett 1999).

The states selected from the complete model of the linearised aircraft model to form a 3rd order system are angle of attack ( $\alpha$ ), pitch rate (q) and pitch angle ( $\theta$ ). The input is the symmetrical tailplane deflection (dts) and the output is pitch angle. In continuous time, the system matrices which have been obtained are

$$A = \begin{bmatrix} -0.5430 & 0.9826 & 0 \\ -1.0674 & -0.4144 & 0 \\ 0 & 1.0000 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1114 \\ -3.2599 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

The 3rd order system has been discretised using a sample interval of  $\tau = 0.025$  and the discrete counterparts of (A, B, C) are

$$G_p = \begin{bmatrix} 0.9862 & 0.0243 & 0 \\ -0.0264 & 0.9894 & 0 \\ -0.0003 & 0.0249 & 1.0000 \end{bmatrix}, \quad H_p = \begin{bmatrix} -0.0038 \\ -0.0810 \\ -0.0010 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

The discrete time system has open loop poles at  $\{1.0000, 0.9878 \pm 0.0252i\}$  and transmission zeros at  $\{-0.9962, 0.9874\}$ . The system is then subjected to a change of coordinates

(6.2.4) and the resulting canonical form is given by

$$T_r G_p T_r^{-1} = \begin{bmatrix} 0.9874 & -0.1166 & 9.1989 \\ -0.0003 & 2.9806 & -157.0127 \\ -0.0003 & 0.0251 & -0.9925 \end{bmatrix}, \quad T_r H_p = \begin{bmatrix} 0 \\ 0 \\ 0.0010 \end{bmatrix}$$
(7.3.1)  
$$L = C_p G_p^{-1} T_r^{-1} = \begin{bmatrix} 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$
(7.3.2)

where  $L = CG^{-1}$  is the new fictitious output distribution matrix and the transformation matrix to achieve the canonical form is

$$T_r = \begin{bmatrix} -0.9988 & 0.0924 & -3.6769 \\ -0.0266 & -0.9916 & 79.2729 \\ -0.0003 & -0.0251 & 1.0000 \end{bmatrix}$$

#### 7.3.1 Controller Design

A dynamic ODSMC law which incorporates tracking will be designed by using the methodology proposed in §6.3. For the design of the controller, the sub-matrices  $G_{11}$ ,  $G_{12}$ ,  $G_{21}$ and T are obtained by partitioning the canonical form in 7.3.1 to give (in the notation of 6.2.4) :

$$G_{11} = \begin{bmatrix} 0.9874 & -0.1166 \\ -0.0003 & 2.9806 \end{bmatrix}, \quad G_{12} = \begin{bmatrix} 9.1989 \\ -157.0127 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} -0.0003 & 0.0251 \end{bmatrix}$$

and T = 1. Once  $G_{11}$  and  $G_{12}$  have been isolated

$$G_{11}^{a} = \begin{bmatrix} 0.9874 & -0.1166 & 0 \\ -0.0003 & 2.9806 & 0 \\ 0 & -0.0006 & 1.0000 \end{bmatrix} \text{ and } G_{12}^{a} = \begin{bmatrix} 9.1989 \\ -157.0127 \\ 0.0248 \end{bmatrix}$$

from (6.3.33). The design matrices  $\Omega$ ,  $K_1$  and  $K_r$  have been selected so that

$$\sigma(G_{11} - \Omega T G_{21}) = \{0.97, 0.98\}$$
 and  $\sigma(\tilde{G}_m) = \{0.8, 0.9, 0.98\}$ 

and hence (6.3.31) is stable. Using pole placement techniques, the matrices were selected to be

$$\Omega = \begin{bmatrix} -4.5025\\ 80.2795 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.4004 & 0.0090 \end{bmatrix}, \quad K_r = 0.6418$$

The system matrices of the compensator  $\Phi$ ,  $\Gamma_1$  and  $\Gamma_2$  can be computed from (6.3.5)-(6.3.7) to be

$$\Phi = \begin{bmatrix} -0.9082 & -0.0460 \\ 30.9949 & 1.6593 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} -4.5025 \\ 80.2795 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} -3.0357 \\ 49.6316 \end{bmatrix}$$

The matrix F, which defines the hyperplane, calculated from (6.3.24), is

$$\hat{F} = \begin{bmatrix} -0.1277 & -0.0851 & -0.0035 & -0.0974 \end{bmatrix}$$

The term  $S_r$ , from the definition of the sliding surface, is a design matrix that affects the

transient response and has been selected to be 0.005. The poles of the closed-loop system are { 0, 0.8, 0.9, 0.97, 0.98, 0.98 }.

**Remark 7.3.1** In this particular example, the bigger the value of  $S_r$ , the faster the response of the system. However, it also brings about bigger overshoots. The final selection of  $S_r = 0.005$  has been made as a trade-off.

This particular design, in terms of the choice of the poles for  $G_{11} - \Omega T G_{21}$  and  $\tilde{G}_m$ , has been selected because it stabilises the compensator and does not result in too large values of the controller gain. The control law, from (6.3.18), is given by

$$u(k) = \begin{bmatrix} 124.8638 & 83.1914 & 3.4452 & 95.2110 \end{bmatrix} y(k) - 14.7186r(k)$$
(7.3.3)

The robustness measure can be obtained from LMI optimisation methods in §4.3.3: here the calculated optimal value of  $\mu = 2.7792$ .

#### 7.3.2 Simulation Results

The controller (7.3.3) was first applied to the 3rd order linear HIRM model in Simulink. Three different sets of results are shown here, each for a different tracking demand. Table 7.3.1 gives a summary of the manoeuvres, L1-L3, and the respective results.

For the first manoeuvre, a step input to 5° (L1), the simulation results in Figure 7.3.1 show that tracking is achieved with relatively small overshoots (less than 0.2°) with a rise time of approximately 2.5 seconds. Maximum deflection of the symmetrical tailplane (dts) is  $\pm 4^{\circ}$ , which is well within the limits [-40°, 10°] given in §7.2.1. Figure 7.3.2 gives results

Manoeuvre	Description	Results	
L1	Step inputs to pitch demand +5° at $t = 2$ seconds -5° at $t = 10$ seconds	Figure 7.3.1	
L2	Step inputs to pitch demand +8° at $t = 1$ seconds -5° at $t = 7$ seconds	Figure 7.3.2	
L3	Ramp input to pitch demand slope of $+1.5^{\circ}/s$ Cut-off at $3^{\circ}$	Figure 7.3.3	

Table 7.3.1: Simulated manoeuvres on linear model



Figure 7.3.1: Manoeuvre L1 Plot of output (pitch angle) and reference signal against time.



Figure 7.3.2: Manoeuvre L2 Plot of output (pitch angle) and reference signal against time.



Figure 7.3.3: Manoeuvre L3 Plot of output (pitch angle) and reference signal against time.

corresponding to L2, a bigger step input command and also a slightly quicker manoeuvre. The signal has less time to settle initially on the step input to 8° but overall tracking is still good. The response time can be improved by increasing the parameter  $S_r$ . However, the trade off will be bigger overshoot and increased control effort. Figure 7.3.3 shows that a relatively small *dts* signal is needed to track the ramped input given in L3.

## 7.4 Nonlinear Model

In this section, the linear controller from §7.3.1 is implemented on the full nonlinear HIRM model in Simulink shown in Figure 7.4.1. The 52-state model includes 12 states from flight dynamics, 4 states from engine dynamics, 13 states from actuator dynamics and 23 states from sensor dynamics. The block diagram shows the closed-loop HIRM set-up together with the ODSMC controller. The ODSMC has been integrated into the closed-loop model of the HIRM together with the Robust Inverse Dynamics Estimation (RIDE) controller, designed by GARTEUR (Moormann & Bennett 1999) to form an augmented controller block diagram.

The ODSMC controls pitch angle via the symmetrical tailplane deflection (dts) and the RIDE controller controls the other 7 control inputs, u(2)-u(8) (see Table C.1.1 in Appendix C), which includes the lateral dynamics of the aircraft model.

The aircraft flight dynamics are represented by the *HIRM dynamics* block in Figure 7.4.1. This block is expanded in Figure 7.4.2. In Figure 7.4.1, the block *trim inputs* includes the actuator input trim settings from Table 7.4.1.

#### 7.4 Nonlinear Model



Figure 7.4.1: Block diagram of nonlinear HIRM with ODSMC controller.



Figure 7.4.2: Block diagram of HIRM dynamics model.

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Description	M = 0.3	M = 0.8
symmetrical tailplane deflection $(dts)$	$-7.7619^{\circ}$	$-5.8036^{\circ}$
differential tailplane deflection $(dtd)$	0°	0°
rudder deflection $(dr)$	0°	0°
pitch angle $(\theta)$	11.9786°	3.0849°
angle of attack $(\alpha)$	11.9786°	3.0849°
flight path angle $(\gamma)$	0°	0°
pitch rate $(q)$	0°	0°
true airspeed $(V)$	100.3181m/s	267.5148m/s
thrust of engines $(F_p)$	10787.0474N	32605.8407N

Table 7.4.1: Straight and level flight : Trimmed values of the HIRM at M = 0.3 and M = 0.8 (h = 5000 ft)

Manoeuvre	Flight Condition	Description	Results
N1 N2	M = 0.3, h = 5000 ft M = 0.8, h = 5000 ft	Step inputs to pitch demand +8° at $t = 1$ seconds -5° at $t = 7$ seconds	Figure 7.4.3 Figure 7.4.4
N3 N4	M = 0.3, h = 5000 ft M = 0.8, h = 5000 ft	Ramp input to pitch demand slope of $+1.5^{\circ}/s$ Cut-off at $3^{\circ}$	Figure 7.4.5 Figure 7.4.6

Table 7.4.2: Simulated manoeuvres on nonlinear HIRM model

#### 7.4.1 Simulation Results

Similar manoeuvres to L2 and L3 as described in Table 7.3.1 are used for easy comparison with the linear model. In addition, different flight conditions, in the form of Mach number (M) and height (h), are also simulated to show the robustness of the ODSMC. This is summarised in Table 7.4.2. The controller, designed based on the HIRM model at trimmed flight conditions M = 0.3 and h = 5000 ft, is applied to the nonlinear HIRM model at both M = 0.3 and M = 0.8 flight conditions. Both these flight cases are within the HIRM flight envelope, with M = 0.8 at h = 5000 ft a point in the flight envelope used for the testing of nonlinear response criterion (Moormann & Bennett 1999). In the simulations, the wind components, ugust, wgust and vgust are set to zero.



Figure 7.4.3: Manoeuvre N1 Plot of output (pitch angle) and reference signal against time.



Figure 7.4.4: Manoeuvre N2 Plot of output (pitch angle) and reference signal against time.



Figure 7.4.5: Manoeuvre N3 Plot of output (pitch angle) and reference signal against time.



Figure 7.4.6: Manoeuvre N4 Plot of output (pitch angle) and reference signal against time.

The discrete nature of the controller can be seen from the control signal in Figures 7.4.3-7.4.6. The response for N1 and N3 (Figure 7.4.3 and Figure 7.4.5) are similar to that of their linear counterparts L2 and L3 (compare with Figure 7.3.2 and Figure 7.3.3). Although there is a difference between the linear and nonlinear simulation results in the control effort needed, the margin is not big. The maximum deflection of the symmetrical tailplane (*dts*) relative to the trim points for manoeuvre N1 (Figure 7.4.3) in both directions are  $-9.2797^{\circ}$  and  $6.6045^{\circ}$ . For N3 (Figure 7.4.5) the maximum *dts* is  $-0.6781^{\circ}$ . These are well below the system limits given in §7.2.1. The results from N1 (Figure 7.4.3) are comparable with those from Harkegard & Glad (2000), who use backstepping control. The response shown in Figure 7.4.3 has a slower rise time but gives less overshoot than that of (Harkegard & Glad 2000).

N2 (Figure 7.4.4) and N4 (Figure 7.4.6) are similar manoeuvres to N1 and N3 but at a different operating point (M = 0.8, h = 5000 ft). The simulation results are obtained by using the same controller design, with the exception of the value of  $S_r$ , which has been increased slightly to 0.05. Figure 7.4.4 and Figure 7.4.6 show that the ODSMC controller is robust enough to control the dynamics at a different trim point, even in this case, M=0.8 and h = 5000 ft which borders the flight envelope (see Figure 7.2.2).

## 7.5 Implementation on a Real Time All Vehicle Simulator

Real Time All Vehicle Simulators (RTAVSs) are used for varying applications, from flight testing new aircraft to full mission rehearsal for existing or future vehicles. They make it possible to evaluate and assess systems in a representative, safe and controlled environment. They can be used for any vehicle and any problem, as long as a computer generated

#### 7.5 Implementation on a Real Time All Vehicle Simulator



Figure 7.5.1: Photograph of the RTAVS.

model can be created. Systems can be added and modifications can be made. The RTAVS includes PC driven head-down displays and a monitor that reproduces an 'outside world' representation. In an aircraft representation a seat is configured to represent a basic cockpit. Longitudinal and lateral motion are controlled through a joystick and foot pedals. Figure 7.5.1 is a photograph taken of the RTAVS.

The nonlinear HIRM model has been run on the RTAVS to simulate flight with manual pilot commands (via the joystick) and not a fixed tracking signal as previously. This was achieved via the Real-Time Workshop (RTW). RTW generates and executes standalone C code for developing and testing algorithms modelled in Simulink. The resulting code can be for an entire model or for an individual subsystem and can be run on any microprocessor or real-time operating system.

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A series of manoeuvres will be shown with real-time pitch demand and control.





Figure 7.5.2: Block diagram of modified HIRM dynamics model used for the RTAVS.

Manoeuvre	Description	Results/Plots
R1	Pitch up $(8^{\circ})$ followed by pitch down $(5^{\circ})$	Figure 7.5.3 and
		Figure 7.5.4
R2	Pitch up to $+30^{\circ}$ and down to $-10^{\circ}$	Figure 7.5.5 and
7.1.3- Mana	Same a second second second second	Figure 7.5.6
R3	Series of steps up and down	Figure 7.5.7 and
		Figure 7.5.8

Table 7.5.1: Simulated manoeuvres on the RTAVS

#### 7.5.1 Flight Conditions and RTAVS Settings

In the results which follow, the operating point is given by Mach number M = 0.3, height h = 5000ft (1524m) with initial conditions as given in Table 7.4.1. The nonlinear model of the HIRM (Figure 7.4.1) has been implemented on the RTAVS with a few modifications to the flight dynamics (see Figure 7.5.2).

#### 7.5.2 Manoeuvres and Results

In this section, 3 different manoeuvres will be shown as described in Table 7.5.1, where the input command is given via the joystick.



Figure 7.5.3: Manoeuvre R1 From top: Plots of pilot demands, output pitch angle and tracking error against time.



Figure 7.5.4: Manoeuvre R1 Top: Plot of control signal, dts against time. Bottom: Plot of pitch rate, q against time.



Figure 7.5.5: Manoeuvre R2 From top: Plots of pilot demands, output pitch angle and tracking error against time.



Figure 7.5.6: Manoeuvre R2 Top: Plot of control signal, dts against time. Bottom: Plot of pitch rate, q against time.



Figure 7.5.7: Manoeuvre R3 From top: Plots of pilot demands, output pitch angle and tracking error against time.



Figure 7.5.8: Manoeuvre R3 Top: Plot of control signal, dts against time. Bottom: Plot of pitch rate, q against time.

**Remark 7.5.1** The commanded input for pitch is given by the user/pilot via a joystick which is connected to sensors. In the plots there is high frequency noise from the sensors which does not reflect the input command from the user.

The manoeuvre R1 (Figure 7.5.3) is an attempt to emulate the one carried out in N1 (Figure 7.4.3) in real-time so that comparisons can be made. The control signals (Figure 7.5.3) appear to be slightly smaller than that of the nonlinear simulation (N1). This may possibly be because of the 'gradual' increase in input command (imagine the user pushing the joystick forwards/ pulling it back) compared to a step-input command block in Simulink.

The high frequency component in the control signals arises from noise in the input channel and is not 'chattering' of the controller. This is clear when comparing R1 (Figures 7.5.3-7.5.4) and R2 (Figures 7.5.5- 7.5.6), where there is less noise in the input channel of R2 (Figure 7.5.6) and hence the control signal for R2 is 'smoother' than that for R1 (Figure 7.5.4).

Figures 7.5.5 and 7.5.6 show the results for R2. In this manoeuvre, the limits of the control surface is tested where the symmetrical tailplane deflection dts exceeds the 10° maximum limit set in §7.2.1. However, the guidelines are still met since the aircraft recovers and returns to stay inside the limiting values within 2 seconds. Tracking is achieved in good time even with a 30° step-input command and the tracking error decays to zero after 3 seconds.

R3 is a series of step-input commands and the results of the real-time simulation are shown in Figure 7.5.7. The controller copes well with the 5 consecutive input commands. The tracking error goes to zero within 4 seconds of each step-input. The control signal always remains within acceptable bounds.

Although pitch rate (q) is not a controlled output, Figures 7.5.4, 7.5.6 and 7.5.8 have included a plot of q against time. This is so the results can be put into perspective with those found in the literature: linear quadratic optimal control (Amato et al. 2000) and robust gain scheduled control (Papageorgiou & Glover 1999). Amato et al. (2000) and Papageorgiou & Glover (1999) have been designed for controlling pitch rate instead of pitch angle.

Even with the noise and amateur handling of the pilot joystick, the system proves to be robust and good tracking is maintained throughout. From the plots, it can be observed that the tracking error for each manoeuvre is usually close to zero or moves to zero quickly after a step-input command.

## 7.6 Summary

This chapter has considered the application of ODSMC techniques to a flight control problem. The ODSMC has been implemented on a linear and nonlinear model of the HIRM. In both cases, strong results show good tracking was obtained. Real time simulations on the RTAVS have emphasized further the robustness properties of the controller: the results show that the controller can produce good tracking at different operating points or flight conditions within the flight envelope. The ODSMC design is also relatively insensitive to noise in the pilot input/demand channel. It has been demonstrated that the theoretical developments previously described are suitable for practical application.
### Chapter 8

# **Conclusions and Future Work**

#### 8.1 Conclusions and Contributions

This thesis has considered the problem of designing robust output feedback sliding mode controllers in discrete time. The development of a static output feedback discrete time sliding mode controller (static ODSMC) was presented, where no additional dynamics have been used in the controller. Conditions under which such controllers exist have been established. For systems which do not satisfy the conditions for the static output feedback methodology, a dynamic output feedback discrete time sliding mode controller (dynamic ODSMC) has been introduced. The latter approach uses a compensator to give additional freedom in the design and performance of the controller. Tracking has been incorporated into both the static and dynamic case by using an integral action approach. Whilst examples have been given in each chapter to illustrate the theoretical developments and design methodology described, a detailed case study has also been undertaken which focuses on the longitudinal control of an aircraft system model. This controller has been tested in a 'piloted' situation on an aircraft simulator.

The major contributions of this thesis are outlined below:

- It has been argued that discrete time sliding mode control problems can be posed in a min-max control setting. This is because in discrete time, ideal sliding cannot be achieved in the presence of uncertainty and so the reaching law must try to attain the smallest sliding mode boundary layers within which the system states stay. This can be viewed as an optimization problem where the objective is to minimise the effect on the Lyapunov difference function of the worst case uncertainty. This idea has been extended to the case of ODSMC where a new sliding surface with a direct link to min-max controllers was introduced.
- A novel sliding surface has been described, which in itself is not realisable through the outputs alone but gives rise to a control law which depends only on the outputs. Using this approach, the discrete time reduced-order sliding motion is not governed by the invariant zeros of the system. Therefore, requirements of relative degree and minimum phaseness are overcome and the method presented is applicable to nonminimum phase systems. This is significant because typically sliding mode schemes require minimum phase conditions which limits the class of systems to which the controllers can be applied. The work in this thesis therefore broadens significantly the class of systems for which DSMC can be employed.
- New conditions for the existence of a stabilizing static ODSMC have been given in Chapter 3, for non-square systems with bounded matched uncertainties. It has been shown that for a stabilising static ODSMC to exists, a certain subsystem triple has to be output feedback stabilisable. A new design procedure has been presented

to synthesize ODSMC's. The design problem involves finding a matrix F and a Lyapunov matrix P which simultaneously solve a structural constraint as well as a discrete Riccati inequality. A novel parameterisation for the design matrix F which allows existing static output feedback pole-placement algorithms to be used to obtain the stabilizing gain has been shown. For a given computed stabilizing gain, there is still design freedom in the choice of Lyapunov matrix P. An LMI optimization procedure has been proposed to optimally select P. This Lyapunov matrix is used to calculate the level of robustness associated with the closed-loop system.

- Because of the link which has been established between ODSMC and output minmax controllers, the results in Chapter 3 may be viewed as providing a solution to the open problem of designing output min-max controllers for non-square systems as posed in (Sharav-Schapiro et al. 1998).
- Since static output feedback controllers do not exist for all systems, a compensation scheme has been proposed in Chapter 4 and a dynamic ODSMC has been described for non-square systems. A particular compensator, of order less than the original plant, has been suggested which is parameterized in a way that is constructive from the point of view of synthesis. An explicit design algorithm has been given which synthesizes the parameters of the compensator. The examples given also show that the introduction of the compensator to the ODSMC improved robustness.
- Again because of the link between ODSMC and output min-max controllers, the results of Chapter 4 may be viewed as representing a solution to the open problem of designing dynamic output min-max controllers as posed in (Sharav-Schapiro et al. 1999) and the conditions for the existence of such dynamic ODSMC are given.

- The requirement to incorporate tracking control, which is often needed in practical applications, has been addressed. The problem of designing an ODSMC which utilises integral action to provide tracking is examined individually for the static and dynamic case. In the static case (in the sense that a static controller is designed for the augmented plant formed from incorporating integrators with the original plant dynamics) the dependency on the invariant zeros is apparently circumvented. This potentially broadens the applicability of the results even more although a static output feedback pole placement problem must still be solved. The methodology introduced in Chapter 5 involves the addition of integrators to form a new augmented system which utilises and builds on the new design approaches from Chapters 3 to produce a controller with a tracking capability. The simplicity of the resulting scheme will be apparent and is very advantageous from the point of view of practical implementation. Again the conditions necessary in order to realize the tracking control law have been established. The examples in Chapter 5 demonstrate the robustness of the approach: tracking an output shaft angular position of a DC-motor rig in real-time.
- Chapter 6 has introduced a new dynamic ODSMC incorporating integral action. The developments in Chapter 6 are quire different from Chapter 3-5. A quite different control structure has been obtained which requires only very mild assumptions on the plant (basically no poles or zeros at the origin). The ideas from this chapter have been used as a basis for a controller for the nonlinear planar vertical take-off and landing (PVTOL) aircraft which has unstable zero dynamics. The PVTOL aircraft is a nonminimum phase system which has been widely studied in the literature. Traditional sliding mode controllers cannot be applied to this system because it is

not relative degree one and minimum phase. Good results have been obtained.

• To verify the practicality of the controller design, the methodologies proposed have been applied to a High Incidence Research Model (HIRM) aircraft. The synthesis of a dynamic controller for longitudinal control of the aircraft model was shown and the resulting linear and nonlinear simulations were compared with other design methods in the literature. The nonlinear aircraft model was implemented on a Real Time All Vehicle Simulator (RTAVS) to allow the results to be explored on a reasonably realistic platform. Good results have been obtained.

Significant theoretical developments in the area of robust discrete time output feedback sliding mode control have been shown and their practicality as well as robustness substantiated by the examples. All the results, numerical, simulated and 'real-time', demonstrate the applicability and effectiveness of method for real systems in the industry, and specifically for a flight control problem case study.

#### 8.2 Recommendations for Future Work

This thesis has presented a theoretical basis for robust output feedback sliding mode control design in discrete time which is novel and practical. It can be viewed as a platform for further research and development.

• The design, in terms of selecting the controller parameters, has been achieved using existing techniques, for example pole placement and root locus methods. This is reasonable for low order systems but is not ideal for more complex systems. Further work would need to develop more systematic methods, perhaps building on Linear Matrix Inequality (LMI) aspects and using these ideas as a synthesis tool, to take advantage of the design freedom and show how it can be fully and best utilised.

- So far, only linear and linearised systems have been considered in the development of the control law. A challenging route for future research would be to extend the approach in this thesis and explore optimal ideas for nonlinear systems, perhaps initially in terms of nonlinear systems affine in the control.
- In the implementation of the robust discrete time output feedback sliding mode controller on the HIRM and RTAVS (Chapter 7), only longitudinal control was considered. Future work can include the design of a controller for the whole aircraft model, i.e. a complete design for both longitudinal and lateral control without using the existing Robust Inverse Dynamics Estimation (RIDE) controller.

# References

- Al-Hiddabi, S. A. & McClamroch, N. H. (1998). Output tracking for nonlinear non-minimum phase vtol aircraft, Proc 37th IEEE Conference on Decision and Control, Florida pp. 4573-4578.
- Amato, F. et al. (2000). Robust flight control design for the HIRM based on linear quadratic control, Aerospace Sci. Technol. 4: 423–438.
- Aström, K. & Wittenmark, B. (1984). Computer Controlled Systems: Theory and Design, Prentice Hall, Englewood Cliffs, N.J.
- Bag, S. K. (1997). Robust sliding mode control using output information, PhD thesis, University of Leicester, U.K.
- Bara, G. I. & Boutayeb, M. (2005). Static output feddback stabilization with  $\mathcal{H}_{\infty}$  performance for linear discrete-time systems., *IEEE Transactions on Automatic Control* **50**: 250– 254.
- Bartolini, G., Pisano, A. & Usai, E. (2000). Digital sliding mode control with O(t<sup>3</sup>) accuracy, in X. Yu & J. X. Xu (eds), Advances in Variable Structure Systems - Analysis, Integration and Application, Proceedings of the 6th IEEE International Workshop on VSS, pp. 103-112.

- Bartolini, G., Pisano, A. & Usai, E. (2001). Digital second-order sliding mode control for uncertain nonlinear systems, *Automatica* **37**: 1371–1377.
- Bartoszewicz, A. (1996). Comments on 'discrete-time variable structure control systems', *IEEE Transactions on Industrial Electronics* **43**: 235–238.
- Boyd, S., El-Ghaoui, L., Feron, E. & Balakrishnan, V. (1994). Linear Matrix Inequalities in Systems and Control Theory, Society for Industrial and Applied Mathematics, Philadelphia.
- Chan, C. (1994). Robust discrete-time sliding mode controller, Systems and Control Letters 23: 371–374.
- Chan, C. (1998). Discrete adaptive sliding mode control of a state-space system with a bounded disturbance, *Automatica* **34**: 1631–1635.
- Corless, M. (1985). Stabilization of uncertain discrete-time systems, Proc IFAC Workshop on Model Error Concepts and Compensation, Boston.
- Corless, M. & Manela, J. (1986). Control of uncertain discrete-time systems, Proc American Control Conference, Seattle pp. 515–520.
- Damiano, A., Gatto, G. L., Marongiu, I. & Pisano, A. (2004). Second order sliding mode control of DC drives, *IEEE Transactions on Industrial Electronics* 51: 364–373.
- DeCarlo, R. A., Źak, S. H. & Matthews, G. P. (1988). Variable structure control of nonlinear multivariable systems: A tutorial, *Proceedings of the IEEE* **76** (3): 212–232.
- Dorling, C. M. & Zinober, A. S. I. (1988). Robust hyperplane design in multivariable variable structure control systems, *International Journal of Control* **48**: 2043–2054.

- Edwards, C., Lai, N. O. & Spurgeon, S. K. (2005). On discrete dynamic output feedback min-max controllers, *Automatica* **40**: 1783–1790.
- Edwards, C. & Spurgeon, S. (1998). Sliding Mode Control: Theory and Applications, Taylor & Francis.
- Edwards, C. & Spurgeon, S. K. (1995). Sliding mode stabilization of uncertain systems using only output information, *International Journal of Control* **62**: 1129–1144.
- Fantoni, I., Zavala, A. & Lozano, R. (2002). Global stabilization of a pvtol aircraft with bounded thrust, Proc 41st IEEE Conference on Decision and Control, Las Vagas pp. 4462-4467.
- Franklin, G., Powell, J. & Emami-Naeini, A. (2002). Feedback Control of Dynamic Systems, Prentice Hall International Edition, 4th Edition.
- Franklin, G. et al. (1990). Digital Control of Dynamic Systems, Addison-Wesley, 2nd Edition, Reading, MA.
- Furuta, K. (1990). Sliding-mode control of a discrete system, Systems and Control Letters 14: 145–152.
- Furuta, K. & Pan, Y. (2000). Variable structure control with sliding sector, Automatica 36: 211–228.
- Gahinet, P., Nemirovski, A., Laub, A. & Chilali, M. (1995). LMI Control Toolbox, Users Guide, The MathWorks, Inc.
- Gao, W., Wang, Y. & Homaifa, A. (1995). Discrete-time variable structure control systems, IEEE Transactions on Industrial Electronics 42: 117–122.

- Garcia, G., Pradin, B., Tarbouriech, S. & Feng, F. Y. (2003). Robust stabilization and guaranteed cost control for discrete-time linear systems by static output feedback, Automatica 39: 1635–1641.
- Golo, G. & Milosavljević, C. (2000). Robust discrete-time chattering free sliding mode control, Systems and Control Letters 41: 19–28.
- Green, M. & Limebeer, D. (1995). *Linear Robust Control*, Prentice Hall, Englewood Cliffs, N.J.
- Harkegard, O. & Glad, T. (2000). Aircraft pitch attitude control using backstepping, Technical Report LiTH-ISY-R-2243, Department of Electrical Engineering, Linkoping University, Sweden.
- Horn, R. A. & Johnson, C. (1985). Matrix Analysis, Cambridge University Press.
- Hui, S. & Źak, S. (1999). On discrete-time variable structure sliding mode control, Systems and Control Letters 38: 283–288.
- Hung, J. Y., Gao, W. & Hung, J. C. (1993). Variable structure control: A survey, IEEE Transactions on Industrial Electronics 40: 2-21.
- Kienitz, K. H. (1990). Guaranteed cost control approach to the stabilization of uncertain linear discrete-time systems, International Journal of Systems Sc. 21: 643–652.
- Koshkouei, A. J. & Zinober, A. S. I. (2000). Sliding mode control of discrete-time systems, Journal of Dynamic Systems, Measurement, and Control 122: 793–802.
- Kotta, U. (1989). Comments on the stability of discrete-time sliding mode control systems, IEEE Transactions on Automatic Control 34: 1021–1022.

- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2003). On discrete time output feedback sliding mode-like control for nonminimum phase systems, Proc 42nd IEEE Conference on Decision and Control, Hawaii pp. 1374–1379.
- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2004a). Discrete output feedback sliding mode based tracking control, Proc 43rd IEEE Conference on Decision and Control, Bahamas pp. 237-242.
- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2004b). Discrete output feedback sliding mode control with integral action, International Journal of Robust and Nonlinear Control. (Accepted for publication).
- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2004c). On discrete dynamic output feedback min-max controllers, Proc 43rd IEEE Conference on Decision and Control, Bahamas pp. 1836–1841.
- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2004d). On discrete time output feedback min-max controllers, International Journal of Control 77: 554–561.
- Lai, N. O., Edwards, C. & Spurgeon, S. K. (2005). On output tracking using dynamic output feedback discrete-time sliding mode controllers, 44th IEEE Conference on Decision and Control and European Control Conference ECC, Spain. (Accepted for publication).
- Lee, S. & Lee, B. (1999). A discrete time sliding mode controller and observer with computation delay, Control Engineering Practice 7: 2943–2955.
- Lu, X. Y., Spurgeon, S. K. & Postlethwaite, I. (1997). Robust variable structire control of a pvtol aircraft, International Journal of Systems Science 28(6): 547-558.

- Magni, J. F., Bennani, S. & Terlouw, J. (1997). Robust flight control: a design challenge, Springer-Verlag.
- Manela, J. (1985). Deterministic control of uncertain linear discrete and sampled-data systems, PhD thesis, University of California, Berkeley.
- Milosavljević, C. (1985). General conditions for the existence of a quasisliding mode on the switching hyperplane in discrete variable structure systems, Automation and Remote Control 46: 307-314.
- Milosavljević, C. (2004). Discrete-time vss, in A. Sabanovic, L. Fridman & S. Spurgeon (eds), Variable Structure Systems: From Principles to Implementation, Institution of Electrical Engineers, U.K., chapter 5.
- Mitić, D. & Č. Milosavljević (2002). Sliding mode based generalised minimum variance control with O(T<sup>3</sup>) accuracy, In Šabanović, A. (Ed.): Advances in Variable Structure Systems- Theory and application, Proc. of the 7th Int. workshop on VSS, University of Sarajevo (Bosnia and Herzagovina) pp. 69–76.
- Monsees, G. (2002). Discrete-Time Sliding Mode Control, PhD thesis, Delft University of Technology.
- Moormann, D. & Bennett, D. (1999). The HIRMplus aircraft model and control laws development, *Technical Report TP-119-2 v1*, Group for Aeronautical Research and Technology in Europe (GARTEUR).
- Muir, E. et al. (1997). Robust flight control design challenge- problem formulation and manual: the high incidence research model (HIRM), *Technical Report TP-088-4 v3*, Group for Aeronautical Research and Technology in Europe (GARTEUR).

Ogata, K. (1995). Discrete-Time Control Systems, Prentice-Hall International, Inc.

- Papageorgiou, G. & Glover, K. (1999). Design of a robust gain scheduled controller for the High Incidence Research Model, *Technical report*, University of Cambridge, U.K.
- Rauw, M. (1997). FDC 1.2- A Simulink toolbox for flight dynamics and control analysis, *Technical report*, Zeist, The Netherlands.
- Rosenbrock, H. H. (1970). State-space and Multivariable Theory, Nelson, London.
- Sapturk, S., Istefanopulous, Y. & Kaynak, O. (1987). On the stability of discrete-time sliding mode control systems, *IEEE Transactions on Automatic Control* **32**: 930–932.
- Sharav-Schapiro, N., Palmor, Z. J. & Steinberg, A. (1996). Robust output feedback stabilizing control for discrete uncertain SISO systems, *IEEE Transactions on Automatic Control* 41: 1377–1381.
- Sharav-Schapiro, N., Palmor, Z. J. & Steinberg, A. (1998). Output stabilizing robust control for discrete uncertain systems, *Automatica* **34**: 731–739.
- Sharav-Schapiro, N., Palmor, Z. J. & Steinberg, A. (1999). Dynamic robust output min-max control for discrete uncertain systems, Journal of Optimization Theory and Applications 103: 421-439.
- Sira-Ramirez, H. (1991). Non-linear discrete variable structure systems in quasi-sliding mode, International Journal of Control 54: 1171–1187.
- Spurgeon, S. (1992). Hyperplane design techniques for discrete-time variable structure control systems, *International Journal of Control* 55: 445–456.
- Steinberg, A. & Corless, M. (1985). Output feedback stabilization of uncertain dynamical systems, IEEE Transactions on Automatic Control 30: 1025–1027.

Strang, G. (1993). Introduction to Linear Algebra, Wellesley-Cambridge Press.

- Su, W., Drakunov, S. & Özgüner, U. (2000). An  $O(T^2)$  boundary layer in sliding mode for sampled-data systems, *IEEE Transactions on Automatic Control* **45(3)**: 482–485.
- Syrmos, V., Abdallah, P., Dorato, P. & Grigoriadis, K. (1997). Static output feedback: A survey, Automatica 33: 125–137.
- Tang, C. & Misawa, E. (2000). Discrete variable structure control for linear multivariable systems., Journal of Dynamic Systems, Measurement and Control 122: 783-792.
- Tang, C. & Misawa, E. (2002). Sliding surface design for discrete VSS using LQR technique with a preset eigenvalue, Systems and Control Letters 45: 1–7.
- Tjee, R. & Mulder, J. (1998). Stability and control derivatives of the De Havilland DHC-2 'Beaver' aircraft, *Technical Report LR-556*, Delft University of Technology, Faculty of Aerospace Engineering, Delft, The Netherlands.
- Utkin, V. (1992). Sliding Modes in Control Optimization, Springer-Verlag, Berlin.
- Utkin, V. I. & Yang, K. D. (1978). Methods for constructing discontinuity planes in multidimensional variable structure systems, *Autom. Remote Control* **39**: 1466–1470.
- Woodham, C. A. & Zinober, A. S. I. (1991a). Eigenvalue assignment for the sliding hyperplanes, Proc IEE Control Conference, Edinburgh pp. 982–988.
- Woodham, C. A. & Zinober, A. S. I. (1991b). Robust eigenvalue assignment techniques for the sliding mode, *IFAC Symposium on Control System Design*, Zurich pp. 529–533.
- Woodham, C. A. & Zinober, A. S. I. (1993). Eigenvalue placement in a specified sector for variable structure control systems, *International Journal of Control* 57: 1021–1037.

- Yang, W. C. & Tomizuka, M. (1990). Discrete time robust control via state feedback for single input systems, *IEEE Transactions on Automatic Control* 35: 590–598.
- Yu, X. (1994). Digital variable structure control with pseudo-sliding modes, Lecture Notes in Control and Information Sciences 193: 134–159.
- Zhou, K., Doyle, J. C. & Glover, K. (1996). Robust and Optimal Control, Prentice Hall, New Jersey.

### Appendix A

## Notation

#### A.1 Mathematical Notation

$\mathbb{R}$	$\operatorname{the}$	field	of	real	numbers

- $\mathbb{C}$  the field of complex numbers
- $\mathcal{N}(A)$  the null space of the matrix A
- $I_n$  the  $n \times n$  identity matrix
- $(\cdot)^T$  the transpose of a matrix
- $(\cdot)^{-1}$  the inverse of a square matrix
- $det(\cdot)$  the determinant of a square matrix
- $rank(\cdot)$  the rank of a matrix
- $\|\cdot\|$  the Euclidean norm of a vector and the spectral norm of a matrix
- $\sigma(\cdot)$  the spectrum of a matrix
- $\lambda_{max}(\cdot)$  the maximum eigenvalue of a square matrix
- $\lambda_{min}(\cdot)$  the minimum eigenvalue of a square matrix
- $\forall$  for all
- s.p.d. symmetric positive definite

#### A.2 Acronyms

- LMI Linear Matrix Inequality
- HIRM High Incidence Research Model
- RTAVS Real Time All Vehicle Simulator
- CSMC Continuous Time Sliding Mode Control
- DSMC Discrete Time Sliding Mode Control
- ODSMC Output Discrete Time Sliding Mode Control
- PVTOL Planar Vertical Take-Off and Landing
- SOMMC Static Output Min-Max Controller

### A.3 Definitions

$Deadbeat\ response$	The response of a closed-loop control system to a step in-			
	put which exhibits the minimum possible settling time, no			
	steady-state error, and no ripples between the sampling in-			
	tervals (§4-7 in (Ogata 1995)).			
Rank	The rank of a matrix is the number of linearly independant			
	rows or columns (Strang 1993).			
Signal reconstruction	The determination of the analog signal that has been trans-			
	mitted as a train of pulse samples ( $\S1-4$ in (Ogata 1995)).			
Symmetric Matrix	A is a symmetric matrix if it is a square matrix and satisfies			
	$A^T = A$ where $A^T$ denotes the transpose. This also implies			
	$A^{-1}A^T = I$ , where I is the identity matrix (§4.0 in (Horn &			
	Johnson 1985)).			
Positive Definite	A is positive definite if $x^T A x > 0$ , where $x^T A x$ is a quadratic			
	form, for all nonzero $x \in \mathbb{R}^n$ . All eigenvalues have to be			
	real and positive, and all determinants associated with all			
	upper-left submatrices are positive (§7.1 in (Horn & Johnson			
	1985)).			

### Appendix B

# **Mathematical Preliminaries**

### **B.1** Lyapunov Equations for Stability Analysis

Consider the discrete time system described by

$$x(k+1) = Gx(k) \tag{B.1.1}$$

where  $x \in \mathbb{R}^n$  is a state vector and  $G \in \mathbb{R}^{n \times n}$  is a constant nonsingular matrix. The origin x = 0 is the equilibrium state. Define a Lyapunov function

$$V(x(k)) = x^{T}(k)Px(k)$$
(B.1.2)

 $\triangle$ 

where  $P \in \mathbb{R}^{n \times n}$  is a symmetric positive definite matrix. The function (B.1.2) is non-zero except at the origin. Then

$$V(x(k)) = V(x(k+1)) - V(x(k))$$
  
=  $x^{T}(k+1)Px(k+1) - x^{T}(k)Px(k)$   
=  $(Gx(k))^{T}P(Gx(k)) - x^{T}(k)Px(k)$   
=  $x^{T}(k)(G^{T}PG - P)x(k)$ 

Since V(x(k)) is chosen to be positive definite, to obtain asymptotic stability,  $\Delta V(x(k))$  has to be negative definite. Therefore,

$$\Delta V(x(k)) = -x^{T}(k)Qx(k), \quad \text{where} \quad Q = P - G^{T}PG$$

is positive definite. Hence, for asymptotic stability of the discrete time system in (B.1.1), it is sufficient that Q be positive definite (see Theorem 5-6 in (Ogata 1995)).

#### B.2 Linear Matrix Inequalities (LMI's)

A linear matrix inequality (LMI) is any constraint of the form

$$A(x) := A_0 + x_1 A_1 + \dots + x_N A_N > 0$$

where

•  $x = (x_1, ... x_N)$  is a vector of unknown scalars (decision/optimization variables)

- $A_0, ... A_N$  are given symmetric matrices
- '> 0' means positive definite

The main strength of LMI formulations is the ability to combine various design constraints or objectives in a numerically tractable manner

Nonlinear inequalities can be converted to LMI form using *Schur complements*, i.e the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0$$
(B.2.1)

where  $Q(x) = Q(x)^T$ ,  $R(x) = R(x)^T$  and S(x) depend affinely on x, is equivalent to

$$R(x) > 0 \tag{B.2.2}$$

$$Q(x) - S(x)R(x)^{-1}S(x)^{T} > 0$$
(B.2.3)

In other words, the set of nonlinear inequalities (B.2.2) - (B.2.3) can be represented as the LMI (B.2.1). For example,

$$AP^{T} + PA + PBR^{-1}B^{T}P + Q < 0 (B.2.4)$$

can be converted into the LMI

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0$$
(B.2.5)

where  $Q = Q^T$ ,  $R = R^T$  and  $P = P^T$  is the optimization variable. Notice that variable P only appears in linear form in the LMI.

For more details and further information on LMI's in control, see (Gahinet et al. 1995) and (Boyd et al. 1994).

#### **B.3** Controllability and Observability

Consider the linear system

$$x(k+1) = Gx(k) + Hu(k)$$
 (B.3.1)

$$y(k) = Cx(k) \tag{B.3.2}$$

where  $G \in \mathbb{R}^{n \times n}$ ,  $H \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$ . The system (B.3.1) is said to be controllable if it is possible to move the system from any arbitrary initial state to a desired state in finite time by a suitable choice of control signals (Franklin, Powell & Emami-Naeini 2002). The Rosenbrock-Hautus-Popov (RHP) test (Rosenbrock 1970) states that the system (G, H) is controllable if the matrix  $[zI - G \ H]$  has full rank, i.e

$$rank [zI - G \ H] = n$$
 for all  $z \in \mathbb{C}$ 

Observability refers to the ability to determine information on all the modes of the system from only the sensed outputs (Note: Unobservability means one or more mode or subsystem has no effect on the output). The concept of observability is the dual to that of controllability and the RHP test can be applied by substituting the transpose  $G^T$  for G and  $H^T$  for C, where C is the output distribution matrix of the system. The other properties that are dual to controllability can be found in Chapter 8 of Franklin et al. (2002).

# Appendix C

# **Aircraft Flight Dynamics**

### C.1 Nomenclature

Alphanumeric	Symbol	Name	Unit
u(1)	dts	symmetrical tailplane deflection	rad
u(2)	dtd	differential tailplane deflection	rad
u(3)	dcs	symmetrical canard deflection	rad
u(4)	dcd	differential canard deflection	rad
u(5)	dr	rudder deflection	rad
u(6)	suction	nose suction	-
u(7)	throttle1	left engine throttle	-
u(8)	throttle2	right engine throttle	-
u(9)	ugust	longitudinal wind	m/s
u(10)	vgust	lateral wind	m/s
u(11)	wgust	normal wind	m/s

Table C.	1.1: D	efinition	of	aircraft	model	inputs
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Symbol	Name	Unit
u	longitudinal velocity	m/sec
v	lateral velocity	m/sec
w	normal velocity	m/sec
p	roll rate	rad/sec
q	pitch rate	rad/sec
r	yaw rate	rad/sec
$\phi$	roll angle	rad
heta	pitch angle	rad
$\psi$	heading angle	rad
x	x-position of the center of gravity	m
y	y-position of the center of gravity	m
z	z-position of the center of gravity	m
$engine1_F$	first state of engine 1 (thrust)	N
$engine1_{F_1}$	second state of engine 1 (time derivative of thrust)	N/s
$engine 2_F$	first state of engine 2 (thrust)	N
$engine2_{F_1}$	second state of engine 2 (time derivative of thrust)	N/s

Table C.1.2: Definition of aircraft model states

Symbol	Name	Unit
Measured		
p	roll rate	rad/sec
q	pitch rate	rad/sec
r	yaw rate	rad/sec
$\theta$	pitch attitude	rad
$\phi$	roll angle	rad
$\psi$	heading angle	rad
anx	x-accelerometer output in body axes	$m/s^2$
any	y-accelerometer output in body axes	$m/s^2$
anz	z-accelerometer output in body axes	$m/s^2$
V	total velocity (true airspeed)	m/s
M	Mach number	-
h	$\operatorname{height}$	m
α	angle of attack	rad
β	sideslip angle	rad
Simulated		
$\gamma$	flight path angle	rad
Vground	total ground speed (magnitude)	m/s
$Fp1$	Thrust of engine 1	N
Fp2	Thrust of engine 2	N

Table C.1.3: Definition of aircraft model outputs

### C.2 Degrees of Freedom

Figure C.2.1-C.2.3 show the primary flight controls of an aircraft and illustrate its degrees of freedom. For a full description of the differential equations of motion for a rigid body with six degrees of freedom see §2.3.2 of (Muir et al. 1997).



Figure C.2.1: Aircraft diagram showing pitch



Figure C.2.2: Aircraft diagram showing yaw



Figure C.2.3: Aircraft diagram showing roll