PERFORMANCE BASED COORDINATION CONTROL OF MULTI-AGENT SYSTEMS SUBJECT TO TIME DELAYS

Thesis submitted for the degree of

Doctor of Philosophy

at the University of Leicester

by

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August, 2012

Abstract

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This thesis considers the design of distributed state and output feedback control algorithms for linear multi-agent systems with performance guarantees in the presence of delays. The multi-agent systems considered are assumed to exchange relative information over an information network. As a first contribution, a novel distributed state feedback control design method with a sub-optimal LQR performance is developed for a network of multiple agents. For the control design process, it is assumed that the exchange of relative information is instantaneous. A stability analysis of the proposed control law is performed by incorporating delays in relative information to ascertain the maximum possible delay that can be accommodated by the communication network. Subsequently, the assumption of the exchange of instantaneous relative information in the control design process is relaxed and the relative information is assumed to be delayed. The system is then represented as a time-delay system. Distributed state feedback control synthesis methods are then developed for the system with a certain level of LQR performance. In the above contributions, the time delay analysis and the development of delay based control methods, it is implicitly assumed that delays are detrimental to achieving cooperative tasks for a multi-agent system. Subsequently, positive effects of delays in communication of relative information are explored. For this a network of vehicles described by double integrator dynamics, which cannot be stabilized by static output feedback without delays, is considered. A novel control design method to achieve exponential stabilization of such a multi-agent system by static output feedback using delayed relative information is developed. Conclusions are drawn from the results of the research presented in this thesis and a few directions for future work are identified.

Acknowledgements

I am indebted to my advisors Prof. Chris Edwards and Dr. Prathyush Menon. They have have provided me with constant motivation and inspiration during the course of my research. I am thankful to Prof. Ian Postlethwaite and the UKIERI for this unique opportunity. I thank Prof. Da-Wei Gu for his support during my research. I extend my thanks to Prof. Debasish Ghose for his support during my visit to Indian Institute of Science, Bangalore, India.

I am grateful to my labmates Bharani, Mangal, Indira, Peter, Mirza and Rihan for making my stay in Leicester memorable. I would like to thank my wingies from IIT Bombay, especially Sujay, Shantanu, and Saurabh. I would also like to thank my friends Swarada, Aniruddha, Manisha, Bilwa and Rajlaxmi for their help and support during my stay in Leicester. I also express my gratitude to my aunt Dr. Seema, cousins Nikhil and Deven, Kashmira, Abhijeet and Natasha for the invaluable experiences during my studies in the United Kingdom. I thank my uncle Dr. Chandrashekhar Gadgil for his advice and encouragement over these years. I express my deepest gratitude and unconditional love for my parents, my sister and my family for their relentless support and encouragement. I thank Lord Ganesha for bestowing me with his blessings and the ability to reach where I have today.

Contents

List of Tables

List of Figures

Nomenclature

ARE	Algebraic Riccati Equation
DIRECT	DIviding RECTangle algorithm
LMI	Linear Matrix Inequality
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
ODE	Ordinary Differential Equation
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Set of n-dimensional real vectors
$\mathbb{R}^{n\times n}$	Set of $n\times$ real matrices
$\mathrm{C}[a,b]$	Set of $\mathbb{R}^n$-valued functions on $[a,b]$
A^T	Transpose of matrix A
$S > 0$	A positive definite matrix S
P	Lyapunov matrix
$A \otimes B$	Kronecker product of A and B
$\mathscr{C}ol(.)$	Column vector
$\mathscr{D}iag(.)$	Diagonal matrix
I_n	Identity matrix of dimension $n \times n$
$\mathscr{L}$	Graph Laplacian
$\mathscr{V}$	Set of vertices
$\mathscr{E}$	Set of edges
τ	Time delay

Chapter 1

Introduction

Research in consensus and coordination of multi-agent systems has received a good deal of attention over the past decade. A multi-agent system consists of several dynamical systems and are required to operate in agreement, i.e. in a synchronized manner. Each individual agent in a multi-agent system shares information with a set of agents of the multi-agent system. The topology of such a network of interconnections within a multi-agent system can be represented by a graph. A wide variety of studies dealing with multi-agent systems have employed algebraic graph theory. By combining graph theory with systems and control theory, many researchers have made significant contributions in this area of research.

1.1 Research Motivation

It is fair to say that initially less attention was paid to performance issues associated with stabilization and agreement problems in multi-agent systems. However, recently there has been progress in this direction. Performance in consensus and coordination problems has been studied in two perspectives. The first perspective focuses on achieving faster convergence of the multiple agents and is based on optimizing the communication topology among the agents in a network. The second perspective focuses on using standard performance techniques such as LQR/$\mathscr{H}_2$ and $\mathscr{H}_\infty$. Few researchers have studied the problem of designing distributed controllers for multi-agent systems with LQR performance. In this thesis novel distributed control design procedures for stabilization of a network of identical linear agents, with a certain level of LQR performance, are developed.

Most of the literature on design of distributed control methodologies for multi-agent systems with LQR performance assumes communication of instantaneous information. This assumption is idealistic and exchange of information over an information network is bound to incur delays. Delays can cause the agents in a multi-agent system to deviate away from the cooperative task or the system may fail to achieve consensus of information. In this scenario a study of multi-agent systems from the perspective of time-delay systems becomes necessary. In this thesis, the primary focus is on studying multi-agent systems by representing them as time-delay systems where the delays arise from the communication of relative information. Novel control synthesis techniques are developed to achieve a certain level of LQR performance in coordination of multi-agent systems in the presence of delays.

In the case of multi-agent systems delays are mostly detrimental to the task of achieving cooperative goals. Delays can affect the problem of achieving consensus of information and can cause divergence from achieving cooperative goals such as maintaining a formation. Most of the literature on consensus and coordination of multi-agent systems implicitly assumes that delays have a negative effect on the system. In the literature on time-delay systems some researchers have shown that delays can also have a positive effect on some systems. Particularly, it has been shown that introduction of delays can stabilize some systems which are otherwise not stabilizable without delays. In this thesis a multi-agent system comprising a network of vehicles, with the dynamics of each described by double integrator dynamics, is considered. The system, which cannot be stabilized by static output feedback without delays, is stabilized using delayed relative output information.

1.2 Contributions of this Thesis

1. **A novel distributed state feedback control design methodology for identical dynamically coupled linear multi-agent systems with suboptimal LQR performance**: The objective here is to design distributed control laws for a network of linear dynamical systems while minimizing a network level LQR cost. The dynamical agents are coupled by the relative state information communicated over the network. The communication of relative state information over the network is assumed to be bidirectional and instantaneous. Distributed control laws are obtained by posing the control problem as an LMI based optimization problem. An analysis of the closed loop system obtained from the distributed control laws is performed to ascertain the the maximum level of delay that can be accommodated by the network in communication of relative information.

2. **Development of distributed state feedback control design methodologies for multi-agent systems in the presence of delays with LQR performance**: As a logical sequel to the previous contribution, the assumption of instantaneous relative state information in the control design process is relaxed and the multi-agent system is represented as a time-delay system by incorporating delays in relative state information. Both cases of delays, fixed and time-varying are considered. It is assumed that the delays are identical across all communication links. Then distributed state feedback control design methodologies are developed for the system. For the case of fixed delays, delay independent and delay dependent distributed control design methods are developed for the stabilization of the multi-agent system with LQR performance. For the case of time-varying delays a delay dependent control design method is developed for the stabilization of the multi-agent system with LQR performance. Existing Lyapunov-Krasovskii based stability analysis techniques (see Appendix B for details) for linear time delay systems are modified to obtain the control synthesis techniques for multi-agent systems with LQR performance.

3. **Formation stabilization of multi-agent systems described by double integra-**

tor dynamics using delayed static output feedback: In this study positive effect of delays in communication of relative output information on design of static output feedback control laws is explored. A network of vehicles described by double integrator dynamics, which cannot be stabilized by static output feedback without delays, is considered. Delayed relative output information is then used to stabilize the network of vehicles into a formation with a guaranteed rate of exponential decay.

1.3 Thesis Organization

The thesis is organized as follows:

- **Chapter 2**: This chapter presents a brief introduction to the area of research considered in this thesis. In this chapter various control design methodologies for multi-agent systems are discussed. The motivation of designing distributed control laws for multi-agent systems using graph theory is explained. The preliminaries required to understand the concepts in this thesis are stated. After a brief introduction to the broad research area, the chapter proceeds to describe the literature on the effects of delays on consensus and coordination of multi-agent systems and optimal performance of multi-agent systems.

- **Chapter 3**: This chapter presents a novel distributed state feedback controller design method for multi-agent systems with sub-optimal LQR performance. The novelty of this chapter, along with the control design, is the use of an augmented LQR cost function, incorporating the Laplacian of the network topology, to guarantee sub-optimal LQR performance. The maximum level of time delay that can be accommodated in the communication of relative information is ascertained with the help of existing Lyapunov-Krasovskii based stability analysis techniques for linear time-delay systems.

- **Chapter 4**: This chapter presents novel distributed state feedback controller design methods for multi-agent systems with LQR performance in the presence of delays. The novelty of this chapter is the development of delay independent and delay dependent distributed control design methodologies with LQR performance for multi-agent systems by modification of existing Lyapunov-Krasovskii based stability analysis techniques.

- **Chapter 5**: This chapter explores the positive effects of delays in communication of relative information in multi-agent systems. A network of vehicles, moving in a two dimensional plane, described by double integrator dynamics is not stabilizable by static output feedback (without derivative information). Delayed relative position information is then used to stabilize the network. The controller gains are obtained by employing DIRECT search optimization, a global Lipschitzian optimization technique, together with LMI constraints.

- **Chapter 6**: This chapter presents the conclusions of this study. A few future research directions are also explained in this chapter.

1.4 Publications

The research work presented in this thesis has yielded the following publications:

1. P. Deshpande, P. P. Menon, C. Edwards and I. Postlethwaite, "A sub-optimal distributed control law with H2 performance for identical dynamically coupled linear systems," IET Control Theory and Applications, Vol. 6, Iss. 16, pp. 2509-2517, 2012.

2. P. Deshpande, P. P. Menon, C. Edwards and I. Postlethwaite, "Formation control of multi-agent systems with double integrator dynamics using delayed static output feedback," Proceedings of the IEEE Conference on Decision and Control, Orlando, December 2011.

3. P. Deshpande, P. P. Menon, C. Edwards and I. Postlethwaite, "A distributed control law with guaranteed LQR cost for identical dynamically coupled linear systems," Proceedings of the American Control Conference, San Francisco, June 2011.

Chapter 2

Control of Multi-Agent Systems

2.1 Introduction

Initially researchers attempted to model observed behaviour in nature, such as flocking of birds, herds of animals or schools of fish. In [1] such behaviour was modelled to generate computer animations. Gradually researchers realized that the control of independent systems to perform various collective behaviours, has many potential engineering applications. Early research involved developing simple navigation strategies for multiple agents: for example, navigation strategies for multiple robots were presented in [2]. With significant technological advances in control methodologies over the past two decades, and development of sophisticated communication and signal processing techniques control of engineering systems comprising multiple dynamical systems interacting with one another became feasible [4]. Such systems are usually referred to as 'Multi-agent Systems', where each agent in the multi-agent system represents an independent dynamical system. Multi-agent systems have significant advantages over their single large scale system counterparts. The use of multiple agents can break down a large complex system into multiple simple systems, increases the possibility of incorporating more functionality into the overall system, and more importantly, reduces costs and the risk of failure if a single system is used to achieve a desired goal [3]. The control of such multi-agent systems to achieve a collective or common goal is known as 'Cooperative Control' [5]. Such strategies focus on achieving consensus of information, defined as the agreement or convergence to a common value, and/or the coordination of multiple agents to perform tasks such as attaining a rendezvous (meeting at a point), or maintaining a formation i.e. achieving a fixed configuration relative to each other. Cooperative control strategies have been designed to achieve various objectives such as de-mining of fields using multiple ground robots [9], oceanographic sampling and minesweeping using autonomous underwater vehicles [10, 11], decentralized cooperative air surveillance using fixed wing unmanned air vehicles [12], maintaining spacecraft formations [13] and control of automated highway systems [14]. Figure 2.1 shows a cluster of satellites that can be used to achieve cooperative goals such as earth observation [1].

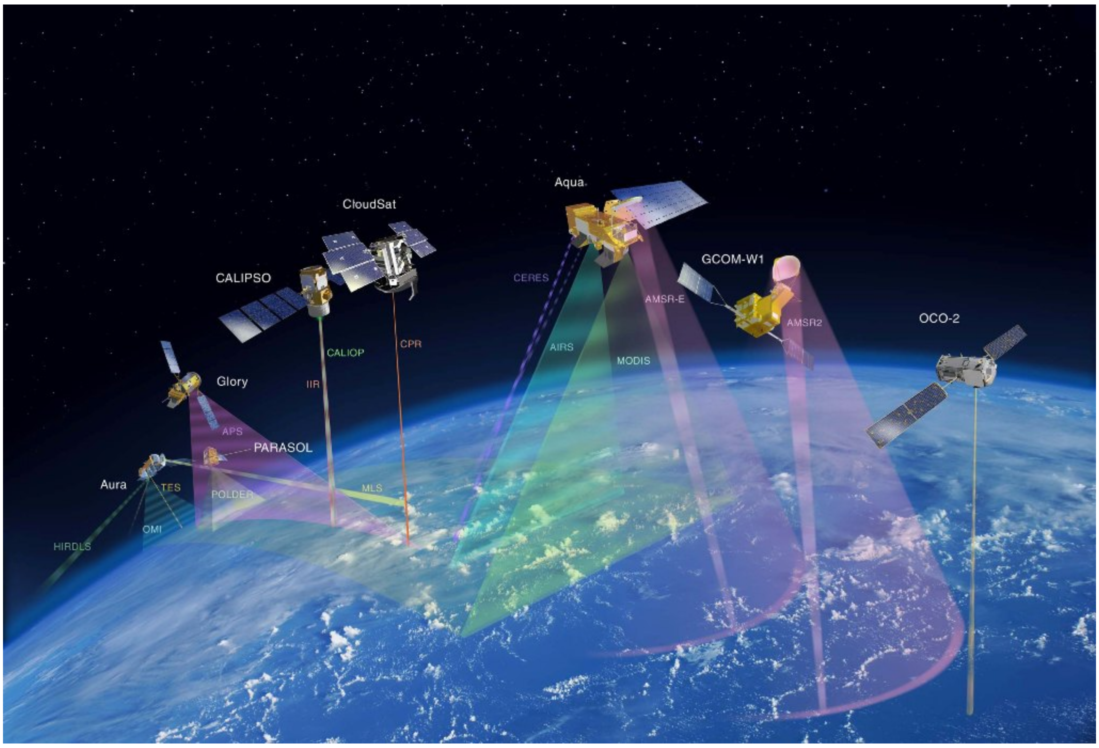

Figure 2.1: A formation of earth observing satellites

Cooperative control strategies generally fall under three broad methods, namely, centralized control, decentralized control and distributed control. A *centralized control* obtains and utilizes all the state information, computes the control signal and relays the relevant control commands to each agent. For a large scale system composed of a network of subsystems, it essentially means a single control law is designed and implemented for the system as a whole. A major disadvantage is that this places a large cost

[1]NASA's proposed A-Train satellite mission *http://www.giss.nasa.gov/research/news/20110210/*

on communication of information for the system. Also it might not be possible to obtain or measure the complete state or output information required by a centralized controller. Another disadvantage of centralized control is that it is not possible to include more subsystems into the existing system once the control has been designed, i.e centralized control methods are not scalable. To avoid the problems of large costs in terms of communication and scalability, *decentralized control* techniques have been proposed where controllers are developed for the sub-systems of a large scale system and they require access to only a part of the complete system's states. Generally there is no communication between the sub-systems or agents in this method. Various decentralized control techniques have been developed over the last four decades and it is also a current area of interest. The reader is referred to [15]-[21] for further information on decentralized control. In a system which is composed of multiple decoupled dynamical sub-systems *distributed control* methods can be employed to achieve consensus and coordination. In this methodology there exists an exchange of information between some of the sub-systems. This information is used to design local feedback control laws which stabilize the sub-systems along with the overall system. Distributed control methods are scalable and more robust as compared to their centralized control counterparts [5]. Figure

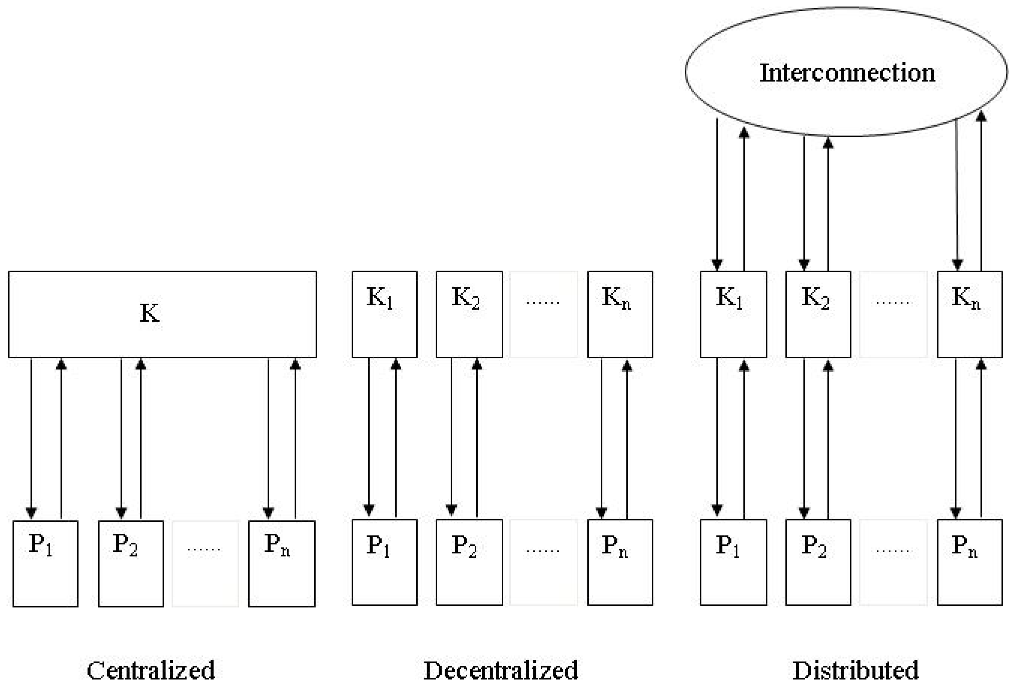

Figure 2.2: Centralized, decentralized and distributed control [22]

2.2 shows a schematic for the three types of cooperative control methods [22]. The centralized controller implements a single controller K for a plant which consists of N subsystems $P_1, P_2, \ldots, P_N$. The decentralized controller has N controllers for the same plant with no interconnections whereas in the distributed control formulation there are N independent subsystems $P_1, \ldots, P_N$ controlled by N controllers $K_1, \ldots, K_N$ which have access to the states or outputs of some of the subsystems $P_1, P_2, \ldots, P_N$. Research in co-operative control of multi-agent systems is now mostly focussed on designing distributed control algorithms to achieve consensus and coordination [3][2].

In the next section different control strategies for consensus and coordination of multi-agent systems are explained in brief. The advantages and disadvantages of various strategies are discussed. The control methods based on graph theory, which are the focus of this research, are explained in detail.

2.2 Control Strategies for Multi-Agent Systems

For cooperative control of multi-agent systems researchers have proposed various strategies such as the leader-follower approach [24]-[31], the behaviour-based methods[36]-[39], the virtual structure approach [40]-[47], artificial potential based control design [49]-[56], and graph theoretic methods [72]-[77]. The leader-follower methods and the virtual structure approach are in most cases centralized, though recently some researchers have proposed a decentralized virtual structure scheme [44] and distributed leader follower approaches [35]. In contrast, potential function approaches can be centralized, decentralized or distributed. Graph theoretic methods generally employ distributed control techniques, exploiting the properties of the underlying communication topology. Recently, graph theoretic methods have also been applied to leader-follower and potential function methods. In this section all these approaches are described in brief, and some of their advantages and disadvantages are explained.

[2]In control applications for multi-agent systems references to ‘decentralized’ and ‘distributed’ methods have been used interchangeably [23, 105]

2.2.1 Leader-Follower Approach

In the leader-follower approach, agents are differentiated and identified as leaders or followers [25]. The leaders follow preassigned trajectories while each follower tracks the trajectory of its leader, maintaining a relative distance. In Figure 2.3 a leader-follower architecture with two mobile robots is shown. The objective is to control the relative distance l between the leader and the follower and the relative orientation ψ, defined as the difference between the heading angles of the leader and the follower The leader-follower

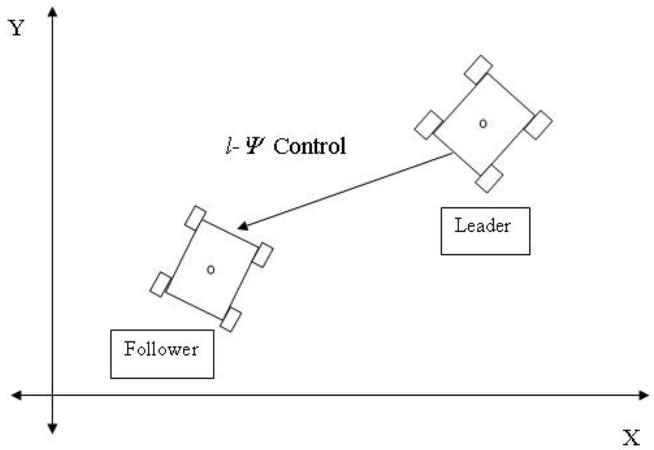

Figure 2.3: Leader follower architecture with two mobile robots

strategy was first used in [24] to develop navigation strategies for multiple autonomous robots moving in a formation. A single leader was identified to specify the desired formation to the following robots. Other followers then use nearest neighbour tracking, i.e. tracking the trajectory of the closest robot, to move in the desired formation. This work was subsequently extended to maintain formations of multiple microspacecraft [25]. In [25] the authors considered a fleet of microspacecraft divided into groups with each group having a leader. Each group then used nearest neighbour tracking to achieve the desired goal. The authors subsequently incorporated adaptive control techniques into the above control methods to achieve formation flying of spacecraft [26]. In [27] adaptive control laws for formation flying of multiple spacecraft were extended to include nonlinear dynamics. The methods in [24]-[27] all have a centralized architecture. As a natural progression, researchers developed decentralized control methods. In [28] decentralized leader-follower control was designed for two robotic manipulators to achieve the cooperative task of moving a box. In [29] a leader-follower formation control strategy for mobile robots maintaining desired relative distance and orientation was proposed. The authors showed that the leader-follower assignment can be represented as a graph and can be used to model changes in formation. Leader-follower assignment using graphs has also been used in [30] for formation flying of spacecraft. The approach incorporates ideas from LMIs and controller switching, i.e. selecting the 'on-line' controller from two or more available controllers. In [31], the effects of leader behaviour on the errors in formation, defined as the deviation of agents from their desired positions, is studied. In this work error propagation in a leader follower network and methods to improve the safety, robustness and performance of a formation is considered. Recently, distributed control methods for leader-follower approaches based on graph theory [3] have been the focus of research of a few researchers [32]-[35].

The main advantage of the leader-follower strategy is its ease of implementation [43]. Another advantage is that stability of an individual agent implies stability of the formation [72] and the multi-agent system can be coordinated by specifying the trajectory of the leader. This is also one disadvantage since the leader becomes the single point of failure. Any failure of the leader to trace its defined trajectory, or communication loss from the leader, will result in failure of the control strategy to achieve its desired goal. Another disadvantage is the lack of feedback from the followers to the leader. This may cause the followers to deviate from the desired path if there are errors in tracking the leader's trajectory [64].

2.2.2 Behaviour-Based Methods

In behaviour-based control methods, different desired behaviours for each agent are considered together. The resulting control action is obtained by weighting these behaviours with respect to a particular task. In [36] the behaviour based approach is used to design distributed control laws to realize geometric patterns in multiple robots. In [37] several objectives such as 'move-to-goal', 'avoid-static-obstacle', 'maintain-formation' etc. are prescribed to each agent. The objectives are called 'motor schemas' as each one gen-

[3] Graph theory based approaches are explained later in this section.

erates appropriate motor commands to execute the task. Each agent then utilizes these schemas to achieve the overall goal for the multi-agent system. In [38] techniques for path planning and control of a group of autonomous vehicles are presented. The approach presented regulates the global behaviour of a platoon of vehicles while giving limited autonomy to each vehicle to satisfy individual trajectory requirements. In [39], a behaviour based approach is used for formation manoeuvres of a group of mobile robots.

The major advantage of behaviour-based strategies is that several objective behaviors can be specified to the system [43]. In most cases explicit dynamical models of the agents are not required. Also behaviour based control techniques require less communication compared to virtual structure or leader-following approaches. However a major disadvantage of behaviour based approaches is that they are difficult to analyze mathematically, and have limited ability in terms of precision [39].

2.2.3 Virtual Structure Approach

The Virtual Structure approach is employed in applications such as formation flying or stabilization of multiple agents into a desired formation. In the virtual structure approach the entire desired formation is considered as a single structure. In this approach the agents align their positions as per a rigid virtual structure, and then follow the virtual structure thereafter.

Figure 2.4 shows a virtual structure in the form of a triangle. The 'Virtual Center' (VC) of the triangle then follows a desired trajectory for the formation. The three agents try to align as a rigid formation and move as a formation thereafter. The concept of a virtual structure was introduced in [40] to design formation control laws for mobile robots with rigid geometric formations. The method was employed for a collection of 3 differential drive mobile robots to ensure that they behave as particles embedded in a rigid structure. In [41], spacecraft formation flying was achieved by the use of virtual structures called constellation templates. In [42] a method is proposed to blend the ideas of leader-following, behavioral and virtual structures, to improve spacecraft formation flying algorithms. The method is then demonstrated in a virtual structure framework to achieve coordination of multiple spacecraft. In [43] a virtual structure with formation

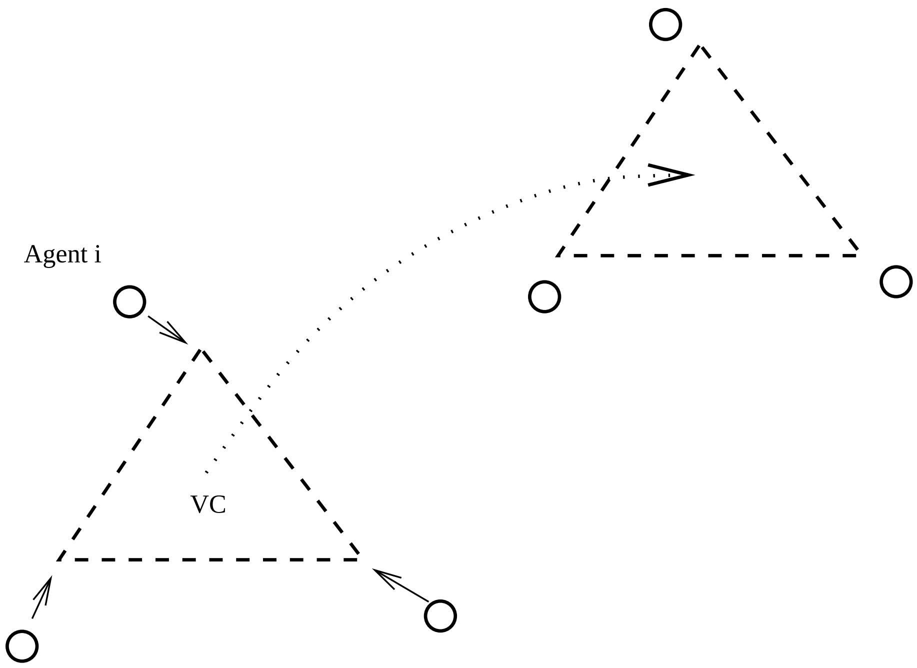

Figure 2.4: Formation of agents tracking a Virtual Structure

feedback from multiple spacecraft is used to overcome the effects of internal or external disturbances. As a natural progression to the centralized architecture in [43], the authors have proposed a decentralized scheme for formation flying of spacecraft using the virtual structure approach [44]. In [44], a method to align multiple spacecraft into a formation, by providing coordination vectors to each spacecraft, has been presented. In [45], a non-linear formation control methodology making use of a combination of virtual structure and path planning approaches has been presented. In [46], consensus and flocking with collision avoidance of multiple vehicles is achieved using feedback linearized virtual vehicles for the network. Recently in [47], a distributed virtual structure control strategy has been proposed to achieve a formation control of unicyclic mobile robots.

In the virtual structure approach it is easy to assign coordinated behaviour for the group. The virtual structure can evolve as a rigid structure in any given direction which is advantageous in maintaining a formation [44]. The disadvantage of this approach is that it can be implemented on only a limited number of applications as the complete network of agents has to act as the virtual structure and it is computationally intensive and complex if the desired formation is time-varying [45].

2.2.4 Potential Function Approach

Here the strategy is to design artificial potentials incorporating the desired cooperative goals. In this approach the desired configuration of agents is generally the unique minimum for the potential function. This approach was first presented in [48] to achieve exact path planning and control of robots. An example of a potential function between two agents denoted by i and j, is given by

$$V_p := \begin{cases} k(\ln(r_{ij}) + \frac{d_0}{r_{ij}}) & 0 < r_{ij} < d_1 \\ k(\ln(d_1) + \frac{d_0}{d_1}) & r_{ij} \geq d_1 \end{cases} \tag{2.1}$$

where r_{ij} is the relative distance between i^{th} and the j^{th} agent, k is a scalar gain, and $d_0 < d_1$ are positive scalars [49]. Here d_0 is the desired separation between the two agents. The force of interaction, defined as the force acting on each agent towards or away from the other, is given by

$$F_p := \begin{cases} \nabla_{r_{ij}} V_p & 0 < r_{ij} < d_1 \\ 0 & r_{ij} \geq d_1 \end{cases} \tag{2.2}$$

where $\nabla_{r_{ij}}$ is the gradient with respect to r_{ij} given by $\nabla_{r_{ij}} = \frac{\partial}{\partial r_{ij}}$. The force at the non-smooth point $r_{ij} = d_1$ is considered to be $F_p = 0$. Let $d_0 = 0.1$, $d_1 = 1$ and $k = 1000$. Figure 2.5 shows that V_p is minimum at d_0. The potential V_p is constant for $r_{ij} > d_1$. The force of interaction between the two agents is shown in Figure 2.6. The force of interaction between the two agents is $F_p < 0$ for $r_{ij} < d_0$ and the two agents are repelled from each other. When $d_0 < r_{ij} < d_1$ the force of interaction $F_p > 0$ and the two agents are attracted towards each other. When $r_{ij} > d_1$, $F_p = 0$ indicating that the two agents are not affected by each other. Also, $F_p = 0$ at $r_{ij} = d_0$ as V_p has a global minimum at d_0. In [49] artificial potentials are constructed to define the interacting forces between neighbouring vehicles. The control forces are designed to drive the agents to a geometry which corresponds to a minimum of the total potential for the agents. The stability and robustness of the group motion has been proved by Lyapunov techniques. In [50] a group of autonomous robots was controlled by using potential functions and their gradients to

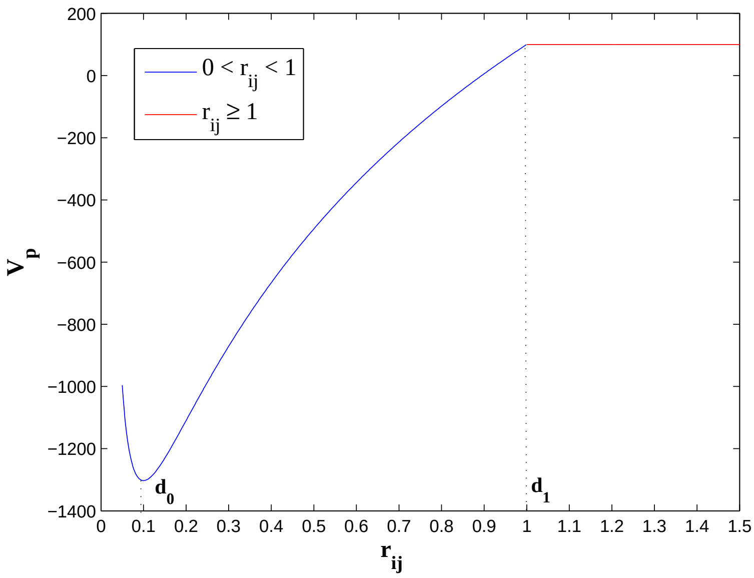

Figure 2.5: The potential as a function of relative distance

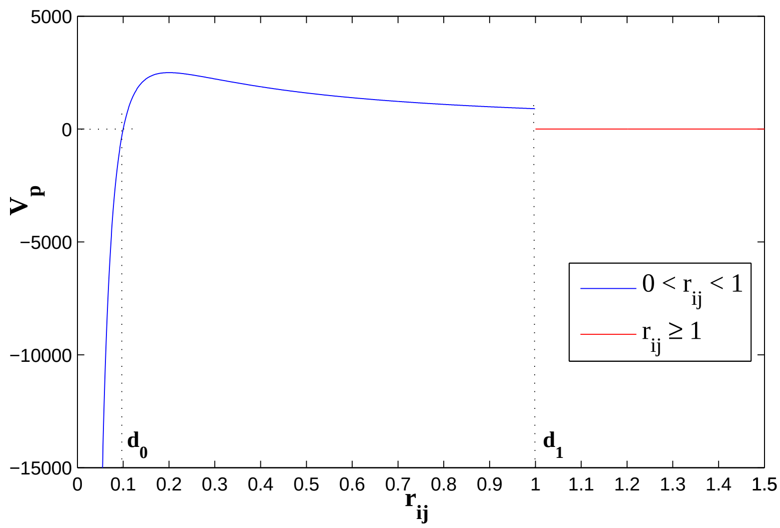

Figure 2.6: The force as a function of relative distance

perform collective activities such as foraging. The agents move in the direction of the negative gradient of the potential functions. In [51] the gradient based technique is used to achieve cooperative goals for a group of mobile wheeled robots. The authors have also derived conditions for robust stability of the group behaviour in the presence of delays in sensing information and noise. In [52], artificial potentials are used to design control laws for cooperative reconfiguration of mobile sensor networks. The control law is obtained as a sum of the gradient of potential functions along with a damping term. In [54] potential functions are constructed to realize flocking behaviour of multi-agent systems. Flocking algorithms incorporating various tasks such as motion in narrow spaces and obstacle avoidance, have been developed for multi-agent systems. A stability analysis of the proposed algorithms has been presented with the help of Hamiltonian formulations depending on the desired structure of the flock. In [55], distributed control laws for maintaining formations of multiple autonomous vehicles are obtained using potential functions incorporating the structure of the desired formation. Concepts from graph theory have been used to construct the potential functions. In [58] a stability analysis of a multi-agent formation control system based on artificial potentials is provided. The authors demonstrate how unwanted equilibria for a multi-agent system can be avoided in an artificial potential based control architecture. In [56] a potential function approach is used to analyze stability properties in the flocking of mobile agents. Stability of the group of agents in the case of switching communication topologies is analyzed with the help of concepts from graph theory. Switching occurs when communication links between a group of mobile agents are established or broken when an agent lies within the range or is beyond the range of communication of another agent. The communication topology in such a scenario is time-varying. In [56] it was established that stabilization of the network is guaranteed as long as the network graph is connected.

The advantage of the potential functional approach is that several objectives like collision avoidance and maintaining formation can be easily incorporated in the potential function. The disadvantage of potential functions is that they may have several local minima (especially when relative distances are used to construct the potential functions) and this may lead to undesired formation stabilizations [57].

2.2.5 Graph Theoretic Methods

A network of multiple agents can be represented by an equivalent graph with nodes (or vertices) and edges. The nodes represent the agents and the edges represent communication links between the agents. Graph theoretic methods facilitate the design of distributed control algorithms by exploiting the properties of this graph. Graph theory perspectives have also been combined with some of the methods discussed earlier. The major advantages of graph based control methods are the ease of analysis and design of the associated control methods, incorporating various communication topologies and scalability. In this section a review of the methods which have utilized the properties of a graph to design *distributed consensus and coordination algorithms* is presented. Before proceeding to the control methods, the next section describes basic concepts of graph theory useful in representing and analyzing multi-agent systems.

2.2.5.1 Graph Laplacian and Its Properties

The representation of a network of multiple agents as the nodes and the communication links as the edges, results in two types of graphs: Directed graphs and Undirected (Bidirected) graphs [7]. Directed graphs result when an agent i shares information with agent j but not vice-versa. In an undirected graph, if an edge exists between two agents then both agents send information to each other. For a graph, the edge or the line segment joining two vertices shows a communication link between the two corresponding agents. Figure 2.7 shows a directed graph with 6 vertices. In the case of the directed graph there

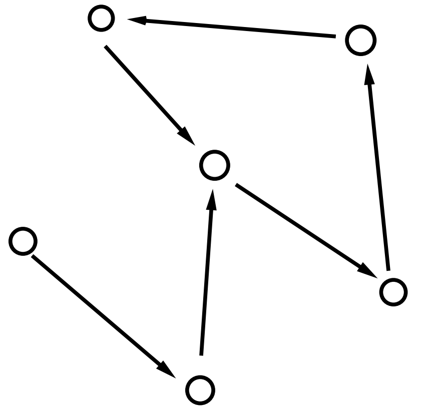

Figure 2.7: A directed graph with 6 vertices

is communication only from agent j to i and hence the edge is shown by an arrowhead in the direction of information flow. The set of vertices for a graph representing the agents is denoted by $\mathscr{V}$, and a set of edges is denoted by $\mathscr{E} \subset \mathscr{V}^2$ where $e = (i, j) \in \mathscr{E}$ denotes an edge. For the edge e, i is called the *tail* and j is called the *head* which implies that there is information flow from agent i to agent j. The in (out) degree of a vertex i is the number of edges with vertex i as the head (tail). For an undirected graph whenever the edge $(i,j) \in \mathscr{E}$, the edge (j,i) belongs to $\mathscr{E}$. An undirected graph can be represented by showing bidirectional arrows or no arrows. Figure 2.8 shows an undirected graph with bidirectional arrows for each node. A network $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, represents a simple, finite

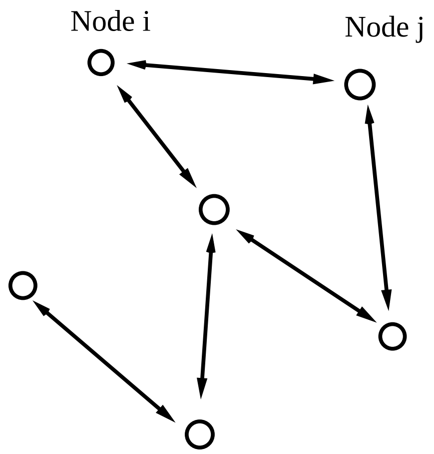

Figure 2.8: An undirected graph with 6 vertices

graph consisting of N vertices and k edges. The following properties of directed and undirected graphs are important with respect to the discussions and contributions of this thesis:

- If there exists a path between any two vertices i and j of a graph $\mathscr{G}$ with vertex i as the tail and j as the head then the graph $\mathscr{G}$ is connected. In the case of directed graphs, a directed graph is called strongly connected if there also exists a path with j as the tail and i as the head.

- A graph $\mathscr{G}_2$ is a subgraph of $\mathscr{G}_1$ if $\mathscr{V}(\mathscr{G}_2) \subseteq \mathscr{V}(\mathscr{G}_1)$ and $\mathscr{E}(\mathscr{G}_2) \subseteq \mathscr{E}(\mathscr{G}_1)$.

- If $\mathscr{V}(\mathscr{G}_2) = \mathscr{V}(\mathscr{G}_1)$ then $\mathscr{G}_2$ is a spanning subgraph of $\mathscr{G}_1$.

- A connected graph $\mathscr{G}$ where each vertex has at least two neighbours is said to contain a cycle, i.e. starting from a vertex i as the tail in the graph $\mathscr{G}$, it is possible to have a path with finite number of edges to arrive at the same vertex i as the head.

- A spanning subgraph with no cycles is called as a *spanning tree*.

In most of the literature on multi-agent systems, and also in this thesis it is assumed that the graph has no multiple identical edges and does not contain any loops. Control applications of graph theory for multi-agent systems make use of the matrices associated with the graph. Assuming N agents in a multi-agent system, for a graph $\mathscr{G}$, an adjacency matrix is defined as $\mathscr{A}(\mathscr{G}) = [a_{ij}]$, where a_{ij} is the weight for the communication link with agent i as the tail. The weight $a_{ij} = 1$ if $(i, j) \in \mathscr{E}$, and $a_{ij} = 0$ otherwise for $i, j \in \{1, \ldots, N\}$. The degree matrix, a diagonal matrix, is defined as $\Delta(\mathscr{G}) = [\delta_{ij}]$, where $\delta_{ii} = \sum_j a_{ij}$ is the out degree of the vertex i and $\delta_{ij} = 0$ if $i \neq j$. The Laplacian of $\mathscr{G}$, $\mathscr{L}$, is defined as the difference $\Delta(\mathscr{G}) - \mathscr{A}(\mathscr{G})$. Laplacian matrices can also be created using non-unitary weights, where $0 \leq a_{ij} \leq 1$. For example in normalized Laplacians, row i of the Laplacian matrix is divided by the corresponding degree of vertex i. For examples of this see [87, 72]. In this thesis Laplacians based on unitary weights have been used. With unitary weights for the directed graph in Figure 2.7, the Adjacency matrix and the Laplacian with respect to the out degree at each node is given by

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathscr{L} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.3}$$

For an undirected or bidirectional graph, the in degree is the same as the out degree for all vertices, and hence the Laplacian for an undirected graph is symmetric. For the undirected graph in Figure 2.8 the Adjacency and the Laplacian matrices are given by

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathscr{L} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.4}$$

The Laplacian for an undirected graph has properties which have been exploited by researchers to devise control laws for multi-agent systems. These properties are stated as follows:

- The smallest eigenvalue of $\mathscr{L}$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1} = \mathscr{C}ol(1,\dots 1)$.
- The Laplacian $\mathscr{L}$ is always rank deficient and positive semi-definite.
- The rank of $\mathscr{L}$ is $N-1$ if for $\mathscr{G}$ there exists a path from every vertex to all other vertices, i.e. the graph is connected.

Please refer to [7] for further reading on graph theory. In this thesis the network topologies are assumed to have bidirectional information exchange. The Laplacian's used to develop control laws are consequently symmetric and its properties are exploited to decompose network level representations into node level subsystems. In the subsections that follow, research involving both directed and undirected graphs is discussed.

2.2.5.2 Kinematic Models

In cooperative control, the information shared over the network is important to achieve coordination of the multi-agent system. In many cooperative control problems it is important for the agents to achieve consensus in terms of information. The problem of convergence to a common value is defined as *a consensus or agreement problem* in the literature on cooperative control. Consensus and coordination problems in multi-agent systems have mostly been studied throughout the literature via kinematic models. Two of the most important kinematic models used in the literature are the single integrator and double integrator dynamics. In this section, a review of these kinematic models and their applications is presented.

The single integrator model was the first and the most widely used model to study consensus and cooperation in multi-agent systems. The single integrator kinematics associated with the i^{th} agent is given by

$$\dot{x}_i = u_i \tag{2.5}$$

where $x_i, u_i \in \mathbb{R}$. The standard consensus algorithm for a network of N such agents is given by

$$u_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t) - x_j(t)) \tag{2.6}$$

for $i = 1, \ldots, N$. Here $\mathscr{J}_i \subset \{1, 2, \ldots N\}/\{i\}$ denotes the agents for which the i^{th} agent has information. For a connected graph $\mathscr{J}_i$ is a non-empty set for all $i = 1, \ldots, N$. The network of single integrators can be represented as

$$\dot{X}(t) = -\mathscr{L}X(t) \tag{2.7}$$

where $X = \mathscr{C}ol(x_1(t), \ldots, x_N(t))$ and $\mathscr{L}$ is the Laplacian associated with the underlying topology [93]. For a bidirectional communication topology the Laplacian for the underlying graph is symmetric positive semi-definite with a unit eigenvector. Hence all the eigenvalues of $-\mathscr{L}$ in (2.7) are negative (except one zero eigenvalue). Due to this property the consensus protocol in (2.6) can be shown to converge asymptotically to the average of the initial values for the individual integrator agents [93]. In [59] single integrator models representing the headings of autonomous agents are used to achieve consensus. In [93] this model has been used in both continuous and discrete time for convergence analysis in the case of switching topologies, i.e. changing communication topologies described by forming and breaking of communication links, and also in the presence of delays. In [60] a comprehensive review of consensus algorithms based on single integrator dynamics has been provided. In [61], consensus of agents with single integrator dynamics tracking a time-varying reference state has been studied. In [62], a method to incorporate asynchronicity, i.e. in a situation where the state evolution of multiple agents is not synchronized to a common clock, is provided. An overview of information consensus in multivehicle cooperative control using single integrator model is provided in [63]. In [64] formation control strategies developed using this model are implemented on a robotic platform.

Although the single integrator model has been used extensively, it has a few limitations. For example it is insufficient to describe motion in a plane with acceleration as an input to the system. However a double integrator model can be used to model the motions of wheeled mobile robots [65], single axis spacecraft rotation [66], etc. The double integrator model is represented as

$$\ddot{x}_i(t) = u_i(t) \tag{2.8}$$

where $x_i, u_i \in \mathbb{R}$. In [67], agents with point mass models are described by double integrator dynamics and the average consensus problem of state agreement and velocity stabilization has been studied. In [68] necessary and sufficient conditions for consensus of double integrators via unidirectional local information exchange is studied. In this work a formal analysis of information exchange topologies for consensus of double integrators has been presented. In [69], conditions for consensus of agents described by double integrator dynamics with constraints on measurements and control inputs are presented. In [70] a stability criterion for the consensus of a network of double integrators in the presence of delays is presented. Consensus algorithm for agents modelled as double integrator have been extended to the problem of stabilization of formations in [42, 72, 86, 71]. In [42] behaviour based approaches are applied to agents with input-output dynamics represented by a double integrator to perform complex formation maneuvers. In [72] an example of a double integrator with time-delay is used to demonstrate the effects of Laplacian eigenvalues on the stability of formations. In [86] a double integrator example is used to demonstrate necessary and sufficient conditions for stabilization of vehicle formations. In [71] methods for stabilization of such agents to maintain a formation with actuator saturation and collision avoidance are presented.

2.2.5.3 Consensus Seeking in Multi-Agent Systems

In this section an overview of the problems associated with consensus algorithms and their application is provided.

The pioneering work in terms of the development of a theoretical framework for problems of consensus among multi-agent systems was done in [72, 73]. In [72] the effects of communication topologies on consensus are analyzed for multi-agent systems. The effect of the eigenvalues of the network graph associated with a communication topology, on the stability of the information flow, is illustrated. The same authors have extended their work to develop linear and nonlinear consensus protocols for multi-agent systems, and then studied the effects of communication delays and distortions or filtering effects in the communication links, using Lyapunov based techniques [73]. In [74], the same authors developed average consensus protocols for directed graphs with switching topologies based on Lyapunov techniques. The work in [72] is further extended to study the effects of increased or decreased communication links in [75]. In [75], necessary and sufficient conditions for state agreement among a network of agents are presented. It is shown that more communication, in the case of directed graphs, does not necessarily lead to better convergence. In [76], it is argued that lack of consensus of inputs in distributed control algorithms can adversely affect coordination of multiple agents. Subsequently conditions on the network topology for consensus of information are presented. It is proved that asymptotic consensus is achievable if and only if the communication topology has a directed spanning tree. In [77] linear iterations for distributed averaging consensus are studied. The authors propose semi-definite programming methods to find the fastest converging iteration. In [78] information consensus in multi-agent systems is studied under limited and unreliable communication. It is proved that for dynamically changing topologies, consensus is achieved asymptotically if the union of the directed interactions graphs have a spanning tree 'frequently' over the system evolution. In [79] the consensus problem was extended to asynchronous multi-agent systems. Specifically, the authors illustrated how the problem of consensus in synchronous multi-agent systems with communication topologies changing as a function of time can be considered as a special case of consensus of asynchronous multiple agents with no delays in communication of information. In [80] consensus algorithms were applied to the problem of estimation. Consensus protocols along with local Kalman filters were used to develop a decentralized estimator in [80]. Recently researchers have looked at incorporating ideas such as convex analysis and linear programming to achieve consensus in multi-agent systems. In [81] the problem of achieving optimal consensus by optimizing local objective cost functions, with nonlinear protocols for networks with directed communication topologies, is investigated. Convergence to the optimal solution set, corresponding to an objective function which is the sum of local objective functions, and consensus analysis are provided with the help of convex and nonsmooth analysis. In [82] a distributed algorithm using a modified simplex algorithm for distribution of various goals among the agents in a multi-agent system, is proposed. The algorithm is applicable to networks with asynchronous agents where the updates schemes for the agents are not synchronized to a common clock, and has very few requirements on the communication structure. In [83] the problem of minimal sensing and control requirements to achieve consensus in a multi-agent system is explored. The proposed control law achieves a rendezvous of multiple vehicles, described as Dubin's Car, without state estimation, using coordinates.

It is clear from all of the past research mentioned above that the communication topology plays a central role in consensus and coordination of multi-agent systems. Also in this thesis the Laplacian and its eigenvalues will be important to the design and development of distributed state and output feedback control laws.

2.2.5.4 Formation of Vehicles

Consensus algorithms generally focus on state agreement or convergence of the states to a common value. However consensus algorithms can be used to maintain formations of multi-agent systems. In this section, literature focussed on formations of multi-agent systems is reviewed.

Even in the case of formations, the communication topology and information flow play a major role in achieving the desired objectives. In [84] the eigenvalues of a graph Laplacian representing the communication topology are used to design a Nyquist like stability criterion for vehicle formations. The authors correlate the location of the Laplacian eigenvalues to the graph structure, to identify desirable and undesirable communication topologies. In [85] Lyapunov stability based arguments are used to prove convergence as compared to the Nyquist methods in [84]. In [86] it is proved that necessary and sufficient conditions on the communication topologies to achieve formations of multiple agents, is the existence of a directed spanning tree. In [87] the problem of multi-agent coordination and formation control while ensuring the connectedness of communication topologies changing over time, is studied. In [88], the feasibility problem of achieving a desired geometric formation is studied. Necessary and sufficient conditions on the information flow in a network for guaranteeing asymptotic convergence are provided. In [89], the problem of convergence of underactuated surface vehicles to a desired formation with the same orientation is studied. In [90] global stabilization of relative sensing networks is achieved by a distributed algorithm. The algorithm guarantees convergence to a desired formation even in cases where some of the agents in a network are not synchronized with the others in terms of a common clock. In [91], formation control of spacecraft with nonlinear dynamics is achieved by designing adaptive time-varying gains. In [92] a nonlinear super-twist sliding mode observer based control strategy is presented for spacecraft formation flying. The controller gains are designed by polytopic system representation dependent on the graph Laplacian. The analysis of state estimation and convergence is provided using Lyapunov stability methods.

2.2.5.5 Effect of Delays

A review of the literature on consensus in the presence of delays is presented in this section. In Section 2.2.5.2, a single integrator based consensus algorithm was described in equation (2.6) as

$$\dot{x}_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t) - x_j(t))$$

for $i = 1, \ldots, N$, where $x \in \mathbb{R}$. In (2.6) it is implicitly assumed that the i^{th} agent has instantaneous information from the j^{th} agent. This assumption is idealistic, as information communicated over a network is bound to incur delays. Two types of delay can occur in consensus algorithms:

- Delays in communication of information from an agent's neighbours
- Delays in sensing of relative information

Delays in communication of information from an agent's neighbours is given by

$$\dot{x}_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t) - x_j(t - \tau(t))) \tag{2.9}$$

for $i = 1, \ldots, N$, where $x_i \in \mathbb{R}$. Here $\tau(t) > 0$ is a delay in communication of information from agent j to agent i. The delay $\tau(t)$ can be fixed or time-varying. The delays are assumed to be equal for each agent. This is done to exploit the properties of the resulting adjacency and Laplacian matrices explained in Section 2.2.5.1. In real engineering systems the delays for different agents in most cases will be unequal. In such a case buffers can be added during the communication of information to equalize the delays. Hence the assumption of equal delays in stability analysis and design of consensus protocols is justified. Stability criteria associated with the consensus dynamics in a network of agents in the presence of communication delays were subsequently developed in [111] using frequency domain methods and Lyapunov Krasovskii based techniques. The strong dependency of the magnitude of delay and the initial conditions on the consensus value was established in [111]. In [111] it is illustrated by an example of a communication topology that introduction of delay may improve convergence rate. In [112], flocking or rendezvous of a network of second order dynamical systems with heterogeneous delays, i.e. unequal delays across communication between any two agents, is obtained using decentralized control. Both frequency and time domain approaches are utilized in [112] to establish delay dependent and delay independent stability for the network. Moreover, it is shown that the maximum bound on the delay depends on the gain of the local controller at each node. Subsequently the theory was extended in [113] to the case of a network formed from a certain class of nonlinear systems. The robustness of linear consensus algorithms is addressed and conditions for convergence subject to communication delays are developed and reported in [114] building on the research described in [112] and [113]. Scalable delay dependent controller design algorithms for consensus in linear multi-agent networks is proposed in [114]. Reference [115] reports an independent attempt to achieve consensus of second order systems using delayed position and velocity information. Stability criteria for guaranteed convergence were obtained using frequency domain methods in [115]. This work was further extended to higher order systems with time delays in [115]. Recently another methodology based on the treatment of roots of the characteristic equation for the closed loop system, has been proposed in [116] to study the effect of large and uniform delays in second order consensus problems with undirected graphs.

Delays in sensing of relative information for an agent in a network are given by

$$\dot{x}_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t-\tau(t)) - x_j(t-\tau(t))) \tag{2.10}$$

for $i = 1,\ldots,N$, where $x_i \in \mathbb{R}$. Here $\tau(t) > 0$ is a delay in sensing of relative information for an agent i. The delay $\tau(t)$ may be fixed or time-varying. Necessary and sufficient conditions for average consensus problems in networks of linear agents in the presence of fixed delays in relative sensing have been derived [93]. In [93] it is shown that the upper bound on the maximum permissible delay in a consensus protocol is inversely proportional to the largest eigenvalue of the Laplacian representing the information flow. Moreover, the authors have argued in [93] that networks with higher communication links will not be able to tolerate higher time-delays. This work was further extended to include bounded time-varying delays in [117]. In [117] necessary and sufficient conditions for the existence of average consensus for a network of single integrators with delayed protocols given by partial difference equations are provided. In [113] delay dependent criteria for the robustness of consensus algorithms for multi-agent systems in the presence of relative sensing delays were developed. The case of asymmetric delays in sensing of information is also considered in [113]. Conceptually, the case of asymmetric delays is given by

$$\dot{x}_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t-\tau_1) - x_j(t-\tau_2)) \tag{2.11}$$

for $i = 1,\ldots,N$, where $x \in \mathbb{R}$. Here $\tau_1, \tau_2 > 0$ and $\tau_1 \neq \tau_2$ are fixed delays in the sensing of relative information from agent j to agent i. Recently, in [118] the performance of consensus algorithms in terms of providing a fast convergence rate and involving communication delays, was studied for second order multi-agent systems. In [119], using Lyapunov-Krasovskii methods and an integral inequality approach, sufficient conditions for robust $\mathscr{H}_\infty$ consensus are developed for the case of directed graphs consisting of linear dynamical systems. This work accounts for node level disturbances, uncertainties and time varying delays.

2.3 Performance Analysis of Multi-Agent Systems

The previous section has discussed the various control methods for consensus and formation control in multi-agent systems. Initially less attention was paid to the performance aspects when designing the consensus and coordination control algorithms. Recently a few researchers have tackled the problem of designing control laws with performance guarantees. Optimization techniques and performance analysis in graph based methods for multi-agent systems are largely based on two perspectives: i) convergence analysis based on graph properties, and ii) optimal control design techniques. The convergence analysis techniques explore the effects of the communication topology and Laplacian eigenvalues whereas optimal control design techniques employ traditional control techniques such as LQR, $\mathscr{H}_2$, and $\mathscr{H}_\infty$ methods. Convergence analysis methods are described briefly in this section. In keeping with the motivation of this thesis, the optimal control methods are described in detail with a focus on LQR control methods.

2.3.1 Topology Based Convergence Analysis

Topology based convergence analysis is mainly based on analyzing the information flow in a network of multiple agents. Researchers have achieved optimum communication links for faster convergence in a network, by weighting the Laplacian associated with the network topology. In [93] it is shown that the convergence of a consensus protocol for a multi-agent system is related to the second smallest eigenvalue, also called the Fiedler eigenvalue [109], of the Laplacian. It argues that the higher the magnitude of the Fiedler eigenvalue the faster the convergence of the consensus algorithm. For undirected graphs, the Fiedler eigenvalue is higher for denser communication topologies and this implies faster convergence. In [72] information exchange strategies among multi-agent systems to improve formation stability and performance are discussed. The formation stability and performance has been analyzed with respect to the Laplacian eigenvalues. In [77] algorithms for improved convergence are designed by constructing a fastest distributed linear averaging consensus (FDLA) problem. The subsequent problem of finding optimal weights in the Laplacian matrix is posed as an optimization problem with the weighted Laplacian as the optimization variable. The authors also tackle the problem of solving this problem for a large scale system with a large number edges by using computational methods such as interior point methods and gradient based methods. Interior point methods for this problem are quicker in terms of obtaining a solution but are computationally intensive. The proposed gradient based methods are slower but are simple to implement. In [94] robustness and improved stability properties are achieved by maximizing the Fiedler eigenvalue of the Laplacian of a network graph. Semi-definite programming algorithms are then proposed by the authors to maximize the Fiedler eigenvalue. In [95] strategies to achieve average consensus by obtaining an optimal topology of communication between the agents are presented.

2.3.2 Optimal Control of Multi-Agent Systems

The methods discussed in the previous section only provide optimization of the rate of convergence by 'improving' the communication topologies. However there are issues such as disturbance rejection, robustness, optimizing control effort, simultaneous convergence etc. which also need to be tackled for consensus and coordination of multi-agent systems. The design of control laws for multi-agent systems with performance criteria based on LQR, $\mathscr{H}_2$, and $\mathscr{H}_\infty$ has recently been undertaken by researchers.

A distributed control methodology ensuring LQR performance in the case of a network of linear identical systems is presented in [96]. The robust stability of the collective dynamics with respect to the robustness of the local node level controllers and the underlying topology of the interconnections, is also established in [96]. A decentralized receding horizon controller with guaranteed LQR performance for coordination problems is proposed in [97] and the efficacy is demonstrated by an application to attain coordination for a group of unmanned air vehicles. In [98], the relationship between the interconnection graph and closed-loop performance in the design of distributed control laws, is studied using an LQR cost function. In [99] procedures to design distributed controllers with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance have been proposed for a network of identical dynamically coupled systems. In [99] a comparison of centralized, decentralized, and distributed control design methods for multi-agent systems with guaranteed $\mathscr{H}_2$ performance was made and the results favoured distributed control design methods. In [100], decentralized static output feedback controllers are used to stabilize a network of dynamical systems with guaranteed overall $\mathscr{H}_2$ performance, depending only on the node level quadratic performance. LQ optimal control laws for a wide class of systems, known as spatially distributed large scale systems, are developed in [101] by making use of an approximation method. In [102], LQR optimal algorithms for continuous as well as discrete time consensus are developed, where the agent dynamics are restricted to be single integrators. However, interesting relationships between the optimality in LQR performance and the Laplacian matrix of the underlying graph are also developed. In [103] $\mathscr{H}_\infty$ control methods are used to design robust distributed formation controllers for multi-agent systems. The proposed control methods guarantee robustness in the face of changes in the communication topologies. In [104] optimal formation control strategies for multi-agent systems with tunable communication topologies and interaction parameters are proposed. LQR control design techniques are then employed to design optimal controllers. In [105] relative sensing networks with: a) each agent in the network having described by the same set of linear state dynamics, i.e a network of homogeneous agents, and b) the agents in the network can be described by different linear state dynamics, i.e. a network of heterogeneous agents, are considered. It is shown that the $\mathscr{H}_2$ performance depends on the number of edges in the graph, and the $\mathscr{H}_\infty$ performance is dependent on the topology and is related to the spectral radius i.e. the largest eigenvalue of the graph Laplacian.

2.4 Concluding Remarks

Research in cooperative control of multi-agent systems has received considerable attention over the last two decades. Various control strategies have been applied for control of multi-agent systems. A review of some of the important strategies such as leader-follower methods, behaviour based control, the potential function approach, the virtual structure approach and graph theoretic methods has been presented in this chapter. Over the last decade researchers have been especially interested in distributed control methods for multi-agent systems. These methods are favoured for their various perceived advantages such as scalability, flexibility, robustness as compared to centralized and decentralized control. The graph theoretic methods facilitate the design and analysis of distributed control methods and hence many researchers have exploited these ideas to obtain novel results in terms of the control of multi-agent systems. Two important control problems associated with graph theoretic methods for multi-agent systems are consensus and maintaining a formation. Consensus problems have mostly been studied with the help of single integrator models. Another model which has been used to study consensus is the double integrator kinematics model which is particularly useful in modeling planar motion with acceleration as the feedback input.

Performance analysis related to graph theoretic methods can broadly be classified into two categories. The first category focuses on obtaining optimal topologies and Laplacian eigenvalues to maximize the rate of convergence. The second category involves incorporating performance measures such as LQR, $\mathscr{H}_\infty$, etc. Consensus and coordination of multi-agent systems involves communication of information over a network. This exchange of information over a network is bound to incur communication delays. Research in the field of multi-agent systems subject to delays is ongoing. In this thesis these two aspects in control of multi-agent systems are addressed: the design of controllers to provide LQR performance at a network level, and delays in communication of information will be considered.

Chapter 3

Optimal LQR Performance of Multi-Agent Systems

3.1 Introduction

In Section 2.3.2, a review of the literature on optimal control of multi-agent systems was presented. One of the optimality measures used for the multi-agent systems is Linear Quadratic Regulator (LQR) performance. The LQR control method obtains controller gains by minimizing a quadratic cost. The quadratic cost generally includes quadratic measures of the error/states of the system along with quadratic measures of the control action which limits the energy spent by the controller. Specifically, in [96] a distributed control methodology was designed for multi-agent systems where the node level controllers were sub-optimal in terms of a network level LQR cost. In this chapter, a novel state feedback distributed control design methodology, which considers a network level LQR cost function , is developed for a network of identical dynamical systems with LQR performance. The identical agents are coupled by the control law using relative information exchanged over a communication network. A two step control design procedure to obtain distributed control laws for the network of agents is proposed. The proposed control laws guarantee sub-optimal LQR performance at a network level. The results are illustrated with the help of a numerical example. A comparison of the proposed control law with the control law obtained from the method proposed in [96] is presented for the same numerical example. A time-delay analysis is then carried out for the network with the proposed control laws. A bound on the permissible delay in the exchange of relative information is obtained for fixed delays and also for the case of time-varying delays.

The organization of this chapter is as follows: In Section 3.2 the problem definition is stated. The control objective and the adopted state feedback control design methodology is presented in this section. In Section 3.3 the two-step control design procedure is explained. In Section 3.4, a time-delay analysis of the proposed design methodologies to ascertain a bound on the maximum amount of delay in relative sensing that can be accommodated, is presented. In Section 3.5 a detailed review of the control design method in [96], which has a few similarities to the control design methodology adopted in this chapter, is provided and comparisons are made. In Section 3.6, the proposed methodologies and the subsequent time-delay analysis is illustrated with the help of a numerical example. Concluding remarks are given in Section 3.7.

3.2 Problem Formulation

In this section the LQR control problem considered in this chapter is formulated. The system (represented by a network of identical linear agents), the structure of the control law and the subsequent LQR cost function to be minimized are presented in this section.

3.2.1 Linear System Model

A collection of N identical n-dimensional linear dynamical systems indexed as $1,2,\ldots,N$ are considered. Each system is assumed to have access to its own state measurements together with relative external measurements with respect to the other dynamical systems which it can sense or interact with. The communication between neighbours is assumed to be bi-directional. As explained in Section 2.2.5.1, this interconnected system can be represented as a graph, with N vertices or nodes, each representing an $n-$dimensional dynamical system. The existence of relative sensing among the dynamical systems is indicated by an edge in this graph.

The dynamics of the i^{th} individual node are given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3.1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the states and the control inputs. The constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and it is assumed that the pair (A,B) is controllable. The exchange of relative information available at each node is assumed to have the form

$$z_i(t) = \sum_{j\in \mathscr{J}_i} (x_i(t) - x_j(t)) \tag{3.2}$$

for $i = 1 \dots N$. The signal $z_i(t)$ represents aggregated external measurements relative to the other dynamical systems which the i^{th} node exchanges information with. The signal $z_i(t)$ is equivalent to the consensus algorithm for single integrator kinematics as explained in Section 2.2.5.2. The nonempty set $\mathscr{J}_i \subset \{1,2,\dots N\}/\{i\}$ denotes the indices of the dynamical systems for which the i^{th} dynamical system has information. Making use of Kronecker products, at a network level, the system given in (3.1) is represented by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{3.3}$$

where

$$X(t) = \mathscr{C}ol(x_1(t), ..., x_N(t)) \tag{3.4}$$

$$U(t) = \mathscr{C}ol(u_1(t), ..., u_N(t)) \tag{3.5}$$

At a network level, the equations in (3.2) can be represented as

$$Z(t) = (\mathscr{L} \otimes I_n)X(t) \tag{3.6}$$

where $Z(t) = \mathscr{C}ol(z_1(t),\dots,z_N(t))$ and $\mathscr{L}$ is the Laplacian for the network. Here, an assumption is made that the bi-directional network topology is connected. As explained in Section 2.2.5.1, this implies that each dynamical system has information about at least one other system which ensures $rank(\mathscr{L}) = N - 1$

3.2.2 Control Objective

The problem considered in this chapter is the design of state feedback control laws of the form

$$u_i(t) = -Kx_i(t) - \Phi K z_i(t) \tag{3.7}$$

for $i = 1, \ldots, N$, where $K \in \mathbb{R}^{m\times n}$ and $\Phi \in \mathbb{R}^{m\times m}$, such that, the cost function

$$J = \int_0^\infty (X(t)^{\mathrm{T}}((I_N \otimes Q_1) + (\mathscr{L} \otimes Q_2))X(t) + U(t)^{\mathrm{T}}(I_N \otimes R)U(t))dt \tag{3.8}$$

where $Q_1 = Q_1^{\mathrm{T}} \geq 0 \in \mathbb{R}^{n\times n}$, $Q_2 = Q_2^{\mathrm{T}} \geq 0 \in \mathbb{R}^{n\times n}$, $R = R^{\mathrm{T}} > 0 \in \mathbb{R}^{m\times m}$, is minimized. The motivation behind the form of control laws in (3.7) and the cost function in (3.8) are explained in Remark 3.5 and Remark 3.6 at the end of the control design procedure in Section 3.3. The solution to this problem is obtained in a sub-optimal way via a two step optimization process. The details of the two step design process are as follows:

- **Step 1**: First the control gain matrix K in (3.7) is synthesized to optimize the LQR performance at a decoupled node level by solving a classical LQR problem [8], i.e the gain matrix K is obtained such that the decoupled node level system

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3.9}$$

  is stabilized by the control law $u_i(t) = -Kx_i(t)$ while simultaneously minimizing the node level cost function given by

$$J_i = \int_0^\infty (x_i(t)^{\mathrm{T}} Q_1 x_i(t) + u_i(t)^{\mathrm{T}} R u_i(t))dt \tag{3.10}$$

  where Q_1 and R are associated with the LQR cost functional given in (3.8). In this step no interactions between the agents are considered and hence the cost considered can be viewed as the case when $Q_2 = 0$ in the cost function in (3.8).

- **Step 2**: Once K has been synthesized, a design matrix $\Phi \in \mathbb{R}^{m\times m}$ is chosen such that collection of systems

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3.11}$$

are stabilized by the distributed control laws given in (3.7) for $i = 1, ..., N$, where $z_i(t)$ is given in (3.2). The objective of this step is to obtain the design matrix Φ such that the network level cost function in equation (3.8) is minimized. Using (3.5) and (3.6), the control law at the network level is represented by

$$U(t) = -(I_N \otimes K)X(t) - (\mathscr{L} \otimes \Phi K)X(t) \tag{3.12}$$

The control design objective here is to achieve an asymptotic rendezvous of the N agents described in (3.1). In this thesis rendezvous is considered to be a consensus problem of driving the agents to the origin. Asymptotic consensus is an interesting problem studied by many researchers over the past decade. This is a starting point to modifying consensus problems into formation stabilization problems and also consensus problems with finite time stabilization which have applications in spacecraft formation flying, spacecraft interferometry, surveillance and reconnaissance and rescue operations as stated in Chapter 2.

Remark 3.1: In (3.8), the term $\mathscr{L} \otimes Q_2$ is introduced to penalize the relative information. This makes an attempt to obtain simultaneous convergence of the multiple agents. Since undirected or bidirectional communication topologies are considered for the inter-agent communication, the Laplacian $\mathscr{L}$ is symmetric positive semi-definite with row sum equalling zero as explained in Section 2.2.5.1. This row sum property is preserved by $\mathscr{L} \otimes Q_2$. At convergence if $x_i = x_s$, where x_s is the desired steady state, for $i = 1, \ldots, N$ then $X^T(\mathscr{L} \otimes Q_2)X = 0$ and the individual decoupled node level LQR performance, as in (3.10), is recovered.

Remark 3.2: An LQR cost function similar to (3.8) is also considered in [96]. The LQR control problem in [96] is discussed in Section 3.5. The approach in [96] to solve the associated LQR problem is different. The suboptimal distributed LQR problem is posed as a single LQR problem exploiting the properties of the graph associated with the communication topology. The order of the LQR problem to be solved depends on the maximum vertex degree (plus one), but not on the total number of nodes in the network. In the method proposed in this chapter, the sub-optimal distributed LQR problem is solved systematically in two steps: the first step involves solving a node level LQR problem; and the second step involves obtaining a scaling matrix, addressing the distributed control part, which creates an optimization problem which depends on the number of nodes of the graph.

3.3 Control Design Procedure

In this section the two step control design procedure is explained in detail.

3.3.1 Details of Step 1

It is assumed without loss of generality that the input distribution matrix from (3.1) has the form

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tag{3.13}$$

where $B_2 \in \mathbb{R}^{m\times m}$. This is so-called regular form [107]. By solving the standard LQR problem posed in Step 1, the optimal feedback gain is given by

$$K = -R^{-1}B^T P \tag{3.14}$$

where the symmetric positive definite Lyapunov matrix P is obtained as a solution to the corresponding Algebraic Riccati Equation (ARE) given by

$$PA + A^T P + Q_1 - PBR^{-1}B^{\mathrm{T}}P = 0 \tag{3.15}$$

The reader is referred to [8] for the derivation of the Algebraic Riccati Equation in (3.15).

3.3.2 Details of Step 2

In order to solve the optimization problem in Step 2, a change of coordinates given by $x \mapsto \hat{T}x = \hat{x}$ is introduced. The coordinate transformation matrix $\hat{T}$ is given by

$$\hat{T} := \begin{pmatrix} I_{(n-m)\times(n-m)} & 0 \\ P_{22}^{-1}P_{12}^T & I_m \end{pmatrix} \tag{3.16}$$

The matrices P_{12} and P_{22} in (3.16) are obtained by decomposing the Lyapunov matrix P from (3.15) so that

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \tag{3.17}$$

where $P_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_{22} \in \mathbb{R}^{m\times m}$. Because P is symmetric positive definite, the sub-matrix P_{22} is symmetric positive definite and therefore nonsingular. Clearly the transformation $\hat{T}$ is nonsingular. Following the change of coordinates, $(A,B,K,P) \mapsto (\hat{A},\hat{B},\hat{K},\hat{P})$ where the matrices $\hat{A} = \hat{T}A\hat{T}^{-1}$, $\hat{B} = \hat{T}B$, $\hat{K} = K\hat{T}^{-1}$ and $\hat{P} = (\hat{T}^{-1})^T P\hat{T}^{-1}$, it can easily be verified that the Lyapunov matrix in the new coordinates has the block diagonal form

$$\hat{P} = \begin{pmatrix} P_{11} - P_{12}P_{22}^{-1}P_{12}^T & 0 \\ 0 & P_{22} \end{pmatrix} \tag{3.18}$$

It is also easy to see that $\hat{B} = B$, i.e the input distribution matrix is invariant under the transformation $\hat{T}$. In these coordinates it can be verified by direct computation that the feedback gain matrix $\hat{K}$ has the structure

$$\hat{K} = \begin{pmatrix} 0 & \hat{K}_2 \end{pmatrix} \tag{3.19}$$

where $\hat{K}_2 \in \mathbb{R}^{m\times m}$ and $\det(\hat{K}_2) \neq 0$. (This follows easily from the structures of $\hat{B}$ and $\hat{P}$ because $\hat{K} = -R^{-1}\hat{B}^T\hat{P}$). In the new coordinate system, the node level LQR cost functions are given by

$$J_l = \int_0^\infty (\hat{x}_l(t)^{\mathrm{T}}\hat{Q}_1\hat{x}_l(t) + u_l(t)^{\mathrm{T}}Ru_l(t))dt \tag{3.20}$$

where $\hat{Q}_1 := (\hat{T}^{-1})^T Q_1 \hat{T}^{-1}$. Further, for the multi-agent system

$$\dot{\hat{x}}_l(t) = \hat{A}\hat{x}_l(t) + \hat{B}u_l(t) \tag{3.21}$$

the distributed state feedback control law given by

$$u_l(t) = -\hat{K}\hat{x}_i(t) - \Phi\hat{K}\hat{z}_l(t) \tag{3.22}$$

for $l = 1, \ldots, N$, is considered. The relative information in the new coordinates is given by

$$\hat{z}_l(t) = \sum_{j \in \mathscr{J}_l} (\hat{x}_l(t) - \hat{x}_k(t)) \tag{3.23}$$

Remark 3.3: The control law in (3.22) for the transformed system in (3.21) is equivalent to the control law (3.7) for the system in (3.1) since $\hat{K} = K\hat{T}^{-1}$ and $\hat{x}_l(t) = \hat{T}x_i(t)$ and $\hat{x}_k(t) = \hat{T}x_j(t)$ from (3.2). The reader should note that each state of the decomposed node contains a combination of the states of the agent represented by that node. For example each state in $\hat{x}_1(t)$ contains a combination of the states of $x_1(t)$.
The states of the network in the transformed coordinates are given by

$$\hat{X}(t) = (I_N \otimes \hat{T})X(t) \tag{3.24}$$

where

$$\hat{X}(t) = \mathscr{C}ol(\hat{x}_1(t), ..., \hat{x}_N(t)) \tag{3.25}$$

In the $\hat{X}$ coordinates, from (3.21), the system at the network level is given by

$$\dot{\hat{X}}(t) = (I_N \otimes \hat{A})\hat{X}(t) + (I_N \otimes \hat{B})U(t) \tag{3.26}$$

The control law in (3.22) at a network level is given by

$$U(t) = -(I_N \otimes \hat{K})\hat{X}(t) - (\mathscr{L} \otimes \Phi\hat{K})\hat{X}(t) \tag{3.27}$$

Substituting (3.27) in (3.26), the closed loop system is given by

$$\dot{\hat{X}}(t) = ((I_N \otimes (\hat{A} - \hat{B}\hat{K})) - (\mathscr{L} \otimes \hat{B}\Phi\hat{K}))\hat{X}(t) \tag{3.28}$$

The cost function in (3.8) in the new coordinates is given by

$$J = \int_0^\infty (\hat{X}(t)^T((I_N \otimes \hat{Q}_1) + (\mathscr{L} \otimes \hat{Q}_2))\hat{X}(t) + U(t)^T(I_N \otimes R)U(t))dt \tag{3.29}$$

where $\hat{Q}_2 := (\hat{T}^{-1})^T Q_2 \hat{T}^{-1}$.

Remark 3.4: The cost function in (3.29) is equivalent to (3.8). This can be verified by a substitution of (3.24), (3.27), and $\hat{K} = K\hat{T}^{-1}$ in (3.29).

In this step $\hat{K} \in \mathbb{R}^{m \times n}$ is considered to be fixed, and $\Phi \in \mathbb{R}^{m \times m}$ represents the available design freedom. In this step the gain matrix Φ is obtained such that the LQR cost function in (3.29) (in the transformed coordinates) is minimized. The objective of this step is to achieve a rendezvous such that all the states of the agents are driven to the origin while simultaneously minimizing the inter-agent distances as the agents approach asymptotic stabilization.

The gain matrix Φ is obtained by the following theorem:

Theorem 3.3.1. *For selected weighting matrices Q_1, $Q_2 \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$, the control laws in (3.22) stabilize the node level systems in (3.21) if there exist symmetric matrices $W_l > 0$ and $Z_l > 0 \in \mathbb{R}^{n \times n}$ with the structures*

$$W_l = \begin{pmatrix} W_{l1} & 0 \\ 0 & W_2 \end{pmatrix} \; Z_l = \begin{pmatrix} Z_{l1} & 0 \\ 0 & Z_2 \end{pmatrix} \tag{3.30}$$

for $l = 1, \ldots, N$, where the matrices $W_{l1} \in \mathbb{R}^{(n-m)\times(n-m)}$, W_2 and $Z_2 \in \mathbb{R}^{m \times m}$ are symmetric positive definite, and a matrix $\hat{Y} \in \mathbb{R}^{m \times n}$ given by the structure

$$\hat{Y} = \begin{pmatrix} 0 & \hat{Y}_2 \end{pmatrix} \tag{3.31}$$

where $\hat{Y}_2 \in \mathbb{R}^{m \times m}$ is nonsingular, such that the following matrix inequalities are satisfied *for $l = 1, \ldots, N$*

$$\begin{pmatrix} -Z_l & I_{n\times n} \\ I_{n\times n} & -W_l \end{pmatrix} < 0 \tag{3.32}$$

$$\begin{pmatrix} \Psi_l & W_l(\hat{Q}_1 + \lambda_l \hat{Q}_2)^{1/2} & (\lambda_l \hat{Y} + \hat{K} W_l)^T \\ * & -I_n & 0 \\ * & * & -R^{-1} \end{pmatrix} < 0 \tag{3.33}$$

where $\Psi_l = \hat{A}_c W_l - \lambda_l \hat{B}\hat{Y} + W_l \hat{A}_c^T - \lambda_l \hat{Y}^T \hat{B}^T$ for $l = 1, \ldots, N$ and the matrix $\hat{A}_c := \hat{A} - \hat{B}\hat{K}$. The scalars λ_l $l = 1, \ldots, N$ are the eigenvalues of the Laplacian $\mathscr{L}$ in (3.28). The matrices $\hat{Q}_1 := (\hat{T}^{-1})^T Q_1 \hat{T}^{-1}$ and $\hat{Q}_2 := (\hat{T}^{-1})^T Q_2 \hat{T}^{-1}$ in (3.33). The scaling matrix $\Phi \in \mathbb{R}^{m\times m}$ is given by $\Phi = \hat{Y}_2 W_2^{-1} \hat{K}_2^{-1}$. Furthermore, a bound on the LQR cost J from (3.29) is obtained if $\sum_{l=1}^{N} trace(Z_l)$ is minimized subject to the conditions (3.32)-(3.33).

Proof. Since $\mathscr{L}$ is symmetric positive semi-definite, by spectral decomposition [7] $\mathscr{L} = V\Lambda V^T$ where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathscr{L}$ and $\Lambda = \mathscr{D}iag(\lambda_1 = 0, \ldots \lambda_N)$ is the matrix of the eigenvalues of $\mathscr{L}$. An orthogonal state transformation

$$\hat{X}(t) \mapsto (V^T \otimes I_n)\hat{X} = \tilde{X}(t) \tag{3.34}$$

is employed on the system in (3.28). In the new coordinates the system is represented as

$$\dot{\tilde{X}}(t) = (I_N \otimes (\hat{A} - \hat{B}\hat{K}))\tilde{X}(t) - (\Lambda \otimes \hat{B}\Phi\hat{K})\tilde{X}(t) \tag{3.35}$$

The weighting matrices $(I_N \otimes \hat{Q}_1)$ and $(\mathscr{L} \otimes \hat{Q}_2)$ from (3.29) map to

$$(V \otimes I_N)^T (I_N \otimes \hat{Q}_1)(V \otimes I_N) = (V^T V \otimes \hat{Q}_1) = (I_N \otimes \hat{Q}_1) \tag{3.36}$$

$$(V \otimes I_N)^T (\mathscr{L} \otimes \hat{Q}_2)(V \otimes I_N) = (V^T \mathscr{L} V \otimes \hat{Q}_2) = (\Lambda \otimes \hat{Q}_2) \tag{3.37}$$

Using (3.36) and (3.37), the quadratic performance at the network level in (3.29) is represented as

$$J = \int_0^\infty (\tilde{X}(t)^T((I_N \otimes \hat{Q}_1) + (\Lambda \otimes \hat{Q}_2))\tilde{X}(t) + U(t)^T(I_N \otimes R)U(t))dt \qquad (3.38)$$

Since Λ is a diagonal matrix, (3.35) is represented at a node level in the transformed coordinates as

$$\dot{\tilde{x}}_l = (\hat{A} - \hat{B}\hat{K} - \lambda_l \hat{B}\Phi\hat{K})\tilde{x}_l \qquad (3.39)$$

for $l = 1, \ldots, N$ and (3.38) as

$$J = \sum_{l=1}^{N} \int_0^\infty (\tilde{x}_l(t)^{\mathrm{T}}(\hat{Q}_1 + \lambda_l \hat{Q}_2)\tilde{x}_l(t) + u_l(t)^{\mathrm{T}} R u_l(t))dt \qquad (3.40)$$

The node level cost function is represented by

$$\tilde{J}_l = \int_0^\infty (\tilde{x}_l(t)^{\mathrm{T}}(\hat{Q}_1 + \lambda_l \hat{Q}_2)\tilde{x}_l(t) + u_l(t)^{\mathrm{T}} R u_l(t))dt \qquad (3.41)$$

For each of the decoupled node level systems in (3.39), quadratic Lyapunov functions given by

$$\tilde{V}_l = \tilde{x}_l^T P_l \tilde{x}_l \qquad (3.42)$$

for $l = 1, \ldots, N$ are considered. The Lyapunov matrices P_l are assumed to have the structure

$$P_l = \begin{pmatrix} P_{i11} & 0 \\ 0 & P_2 \end{pmatrix} \qquad (3.43)$$

for $l = 1, \ldots, N$, where $P_{l11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_2 \in \mathbb{R}^{m\times m}$ are symmetric positive definite. In the $\hat{x}$ coordinates the Lyapunov matrix associated with the optimum LQR cost for the $(\hat{A}, \hat{B})$ pair has a block-diagonal form as shown in (3.18). In (3.43) the block-diagonal structure is retained for the subsequent optimization. Note that P_2 is defined to be the same for all $l = 1, \ldots, N$. Differentiating (3.42), it is required that

$$\dot{\tilde{V}}_l(\tilde{x}_l(t)) + \frac{d}{dt}\int_0^t (\tilde{x}_l(t)^{\mathrm{T}}(\hat{Q}_1 + \lambda_l \hat{Q}_2)\tilde{x}_l(t) + u_l(t)^{\mathrm{T}} R u_l(t))dt < 0 \qquad (3.44)$$

which implies

$$\dot{\tilde{V}}_l(\tilde{x}_l(t)) \leq -\frac{d}{dt}\int_0^t (\tilde{x}_l(t)^{\mathrm{T}}(\hat{Q}_1+\lambda_l\hat{Q}_2)\tilde{x}_l(t)+u_l(t)^{\mathrm{T}}Ru_l(t))dt < 0 \tag{3.45}$$

Substituting (3.39), (3.41), and (3.42) in (3.44) yields

$$\begin{aligned} &P_l(\hat{A}-\hat{B}\hat{K}-\lambda_l\hat{B}\Phi\hat{K})+(\hat{A}-\hat{B}\hat{K}-\lambda_l\hat{B}\Phi\hat{K})^T P_l \\ &+(\hat{Q}_1+\lambda_l\hat{Q}_2)+(\hat{K}+\lambda_l\Phi\hat{K})^T R(\hat{K}+\lambda_l\Phi\hat{K}) < 0 \end{aligned} \tag{3.46}$$

where $P_l > 0$, for $l = 1,\ldots,N$. Integrating (3.44) over $[0,t]$ gives

$$\tilde{V}_l(\tilde{x}_l(t)) - \tilde{V}_l(\tilde{x}_l(0)) \leq -\int_0^t (\tilde{x}_l(t)^{\mathrm{T}}(\hat{Q}_1+\lambda_l\hat{Q}_2)\tilde{x}_l(t)+u_l(t)^{\mathrm{T}}Ru_l(t))dt \tag{3.47}$$

for $l = 1,\ldots,N$. Since $\tilde{V}_l(\tilde{x}(t))$ tends to zero as $t \to \infty$, which implies

$$\tilde{V}_l(\tilde{x}(0)) \geq \tilde{J}_l \tag{3.48}$$

for $l = 1,\ldots,N$. This gives

$$\tilde{J}_l \leq \tilde{x}_l^T(0)P_l\tilde{x}_l(0) \tag{3.49}$$

for $l = 1,\ldots,N$. Thus the network level cost J in (3.40)

$$J \leq \sum_{l=1}^{N} \tilde{x}_l^T(0)P_l\tilde{x}_l(0) \tag{3.50}$$

Thus the following minimization problem constitutes minimizing an upper bound on the LQR cost J at a network level

$\boxed{\text{Minimize } \sum_{l=1}^{N} trace(P_l) \text{ subject to (3.46)}}$

The matrix inequality stated in (3.46) is not a convex representation. To develop a convex representation of the optimization problem stated above, a set of matrices W_l given by (3.30) for $l = 1,\ldots,N$, where $W_{l1} = P_{l11}^{-1}$ and $W_2 = P_2^{-1}$, are defined. Pre and post multiplying (3.46) by W_l yields

$$(\hat{A}-\hat{B}\hat{K}-\lambda_l\hat{B}\Phi\hat{K})W_l+W_l(\hat{A}-\hat{B}\hat{K}-\lambda_l\hat{B}\Phi\hat{K})^T+W_l(\hat{Q}_1+\lambda_l\hat{Q}_2)W_l$$
$$+W_l(\hat{K}+\lambda_l\Phi\hat{K})^T R(\hat{K}+\lambda_l\Phi\hat{K})W_l<0 \qquad (3.51)$$

for $l=1\ldots N$. Because of the structures inherent in the terms $\hat{K}$, and W_l, it follows that

$$\Phi\hat{K}W_l=\begin{pmatrix} 0 & \Phi\hat{K}_2W_2 \end{pmatrix} \qquad (3.52)$$

To generate an LMI representation as explained in Appendix A, a new variable given by

$$\hat{Y}=\begin{pmatrix} 0 & \hat{Y}_2 \end{pmatrix} \qquad (3.53)$$

where $\hat{Y}_2=\Phi\hat{K}_2W_2$ is defined. Then by the use of the Schur decomposition from Appendix A.1, inequality (3.51) is written as (3.33). Also the set of matrices Z_l, for $l=1,\ldots,N$ are defined conformably with the definition of W_l as shown in (3.30). Then minimization of $\sum_{l=1}^{N} trace(Z_l)$ subject to (3.32) and (3.33) is equivalent to minimizing

$$\sum_{l=1}^{N} trace(W_l^{-1})=\sum_{l=1}^{N} trace(P_l)$$

since (3.32) is equivalent to $W_l^{-1}<Z_l$. Hence the bound the LQR cost in (3.29) is minimized by the optimization problem

$\boxed{\text{Minimize } \sum_{l=1}^{N} trace(Z_l) \text{ subject to (3.32)-(3.33)and } W_i>0}$

This is a convex optimization problem in terms of the LMI variables W_l, $\hat{Y}$ and Z_l. The design matrix Φ can then be obtained from $\hat{Y}$ as $\Phi=\hat{Y}_2W_2^{-1}\hat{K}_2^{-1}$. □

Note that:

- The block diagonal structure of P_l in (3.43) is enforced to help with the formulation of a convex representation of the problem. However this does induce conservatism in the solution that will be obtained.

- For $\Phi=0$, choosing $P_l=\hat{P}$ from (3.18) obtained from Step 1, is a feasible solution to the optimization problem. Thus the minimization problem is guaranteed to have a meaningful solution.

- The coordinate transformation $\hat{T}$ in (3.16) was employed to enable the definition of P_l in the form of (3.43). This facilitates the use of a different Lyapunov matrices at the decomposed node level in (3.42). This reduces the excess conservatism that would have been imparted in case a single Lyapunov matrix P was chosen for each node. The orthogonal coordinate transformation in (3.34) was employed to decompose the symmetric Laplacian $\mathscr{L}$ and obtain the node level equations in (3.39). It can now be stated that this decomposition of the symmetric Laplacian is the main reason for the assumption of bidirectional topologies in Theorem 3.3.1. The Laplacian of a directed graph is not symmetric and hence the equations in (3.39) cannot be obtained purely in the node level states.

- This theorem obtains the gain matrix ΦK for the relative information to be used along with the local information such that simultaneous stabilization is achieved while minimizing an augmented LQR cost. The main drawback of this theorem is that it requires that N LMIs in (3.30) - (3.33) are satisfied simultaneously. In (3.33), $(\hat{Q}_1 + \lambda_l \hat{Q}_2)$ is positive definite for $l = 1, \ldots, N$ and hence (3.33) is equivalent to

$$\begin{pmatrix} \Psi_l & W_l(\hat{Q}_1 + \lambda_l \hat{Q}_2) & (\lambda_l \hat{Y} + \hat{K} W_l)^T \\ * & -(\hat{Q}_1 + \lambda_l \hat{Q}_2) & 0 \\ * & * & -R^{-1} \end{pmatrix} < 0 \tag{3.54}$$

with the terms as stated in Theorem 3.3.1. In [93] it is proved that the eigenvalues of the graph Laplacian $\mathscr{L} \in [0, 2\delta_{max}(\mathscr{G})]$, where $\delta_{max}(\mathscr{G})$ is the maximum degree of the graph Laplacian. Hence the LMIs in (3.54) are affine with respect to the eigenvalues λ_l for $l = 1, \ldots, N$. Thus the Theorem 3.3.1, with (3.54) instead of (3.33), can then be solved for $\lambda = 0$ and $2\delta_{max}(\mathscr{G})$. This will reduce the number of LMIs to be simultaneously satisfied but may introduce some conservatism as the second LMI is satisfied for the maximum possible eigenvalue $2\delta_{max}(\mathscr{G})$. To summarize, the Theorem 3.3.1 should be used when the number of agents is small and satisfying the LMIs for each node is not computationally intensive whereas the approach with (3.54) should be used when the multi-agent system has a large number of agents.

Remark 3.5: First the gain matrix K is obtained from Step 1 by solving the standard LQR problem which minimizes the cost function in (3.10). To minimize the cost function in (3.8), the gain matrix matrix associated with the relative information z_i, $i = 1,\dots,N$ in (3.7) is considered to be ΦK, where Φ is a scaling matrix for the node level gain matrix K. This has been introduced to enable the use of different Lyapunov matrices in the control design procedure. If a full gain matrix K_f had been considered instead of ΦK one will have to consider same Lyapunov matrix for the decomposed node levels in (3.42) in Theorem 3.3.1 from Section 3.3. This would have added more conservatism to the design process.

Remark 3.6: The term $I_N \otimes Q_1$ in (3.8) is from the Step 1 of the control design procedure. In the absence of the term $\mathscr{L} \otimes Q_2$, i.e $Q_2 = 0$, the solution obtained from Step 2 will be $\Phi = 0$. The term $\mathscr{L} \otimes Q_2$ is introduced to penalize the relative information and aid simultaneous convergence as explained in Remark 3.1. The weighting for the control effort given by the term $I_N \otimes R$ is kept the same. In Step 1, R is the weighting used to obtained the gain matrix K. In Step 2, the same weighting R is used for the control effort (3.22) employed on the decomposed systems in (3.21). The gain matrix ΦK then reflects the excess control effort required from the use of relative information to compensate for the term $\mathscr{L} \otimes Q_2$ when the weighting for the control effort is the same.

3.4 Analyzing Effects of Time-Delays

In this section the effects of delays in exchange of relative information are analyzed. An analysis with both fixed and time-varying delays, to ascertain the maximum bound on the delay that can be accommodated in the network, is presented.

The aggregated relative measurements in (3.2) were given by

$$z_i(t) = \sum_{j \in \mathscr{J}_i} (x_i(t) - x_j(t)) \tag{3.55}$$

This is very idealized and in reality delays will be present in the information network. In this section, the effect of time delays across all communication links will be analyzed. The time delays are assumed to be equal across all communication links. The relative information available at each node is now given by

$$z_i(t-\tau) = \sum_{j \in \mathscr{J}_i} (x_i(t-\tau) - x_j(t-\tau)) \tag{3.56}$$

for $i = 1, \ldots, N$. The assumption behind delayed relative information of the form (3.56) comes from the scenario where a multi-agent system has only relative state information sensors. In this scenario, the system representation becomes

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3.57}$$

$$u_i(t) = -Kx_i(t) - \Phi K z_i(t-\tau) \tag{3.58}$$

where the scalar τ (representing the delay) can be a fixed or time-varying. This is more realistic than the earlier formulation with no delay. A more realistic case arises when the delays across different communication links are considered to be different. This will not be considered in this thesis as a central theme of this thesis is to exploit the property of the resulting symmetric Laplacian when (3.56) is written at a network level. Also the use of buffers in the case of unequal delays to equalize the delays has been explained in Section 2.2.5.5.

With the use of Kronecker algebra, the control law for the overall system can be written as

$$U(t) = -(I_N \otimes K)X(t) - (\mathscr{L} \otimes B\Phi K)X(t-\tau) \tag{3.59}$$

and the closed loop system is then given by

$$\dot{X}(t) = A_0 X(t) + A_1 X(t-\tau) \tag{3.60}$$

where

$$A_0 = (I_N \otimes A) - (I_N \otimes BK) \tag{3.61}$$

$$A_1 = -(\mathscr{L} \otimes B\Phi K) \tag{3.62}$$

The class of systems represented by (3.60) have been studied extensively in the literature to develop stability criteria for both fixed and time-varying delays. This class of time-delay systems and the related stability criteria are explained in Appendix B. The reader is referred to [120]-[129] for further reading on this class of systems.

Remark 3.7: The time delays are assumed to be equal across all communication links to represent the system given by (3.57)-(3.58) at a network level in the form of (3.60). With unequal delays the system cannot be represented by (3.60) and the stability criteria from Appendix B cannot be applied.

Two types of delay τ in the system (3.60) are considered

- Case A : τ is fixed and unknown in $[0\ \hat{\tau}]$
- Case B : τ is time-varying subject to $0 \leq \tau(t) \leq \hat{\tau}$ for all $t \geq 0$

The system in (3.60) is analyzed to ascertain the maximum possible delay $\hat{\tau}$ such that the system is stable for all $i = 1, \ldots, N$ and $\tau \in [0\ \hat{\tau}]$.

3.4.1 Fixed Delay

For *Case A*, the system in (3.60) is analyzed using the results of *Proposition 4* and *Proposition 5* in the Appendix B.3.2 and Appendix B.4 respectively. *Proposition 4* ascertains stability with the help of a quadratic Lyapunov-Krasovskii functional as explained in Appendix B.3.2. The stability of the system in (3.60) using *Proposition 4* is given by the following proposition:

Proposition 3.4.1. *The system in (3.60) is asymptotically stable if there exist real matri-*

ces $Y_1^T = Y_1$, $S^T = S$, $Y_2, \in \mathbb{R}^{Nn\times Nn}$ and $P^T = P \in \mathbb{R}^{Nn\times Nn}$ such that

$$P > 0 \tag{3.63}$$

$$\begin{pmatrix} \hat{N} & PA_1 - Y_2 & -A_0^T Y_2^T \\ * & -S & -A_1^T Y_2^T \\ * & * & -\frac{1}{\tau} Y_1 \end{pmatrix} < 0 \tag{3.64}$$

where

$$\hat{N} = PA_0 + A_0^T P + S + \tau Y_1 + Y_2 + Y_2^T \tag{3.65}$$

Proof. The Lyapunov-Krasovskii functional considered for the proof of this theorem is given by

$$V(X_t) = X^T(t)PX(t) + \int_{t-\tau}^{t} X^T(\theta)SX(\theta)d\theta + \int_{t-\tau}^{t}\int_{\theta}^{t} g^T(\zeta)\bar{Z}g(\zeta)d\zeta d\theta \tag{3.66}$$

where $t-\tau \le \theta \le t$, $-\tau \le \zeta \le 0$, $X_t = X(t+\theta)$ for $\theta \in [-\tau, 0]$, $\bar{Z} \in \mathbb{R}^{n\times n}$ is symmetric and

$$g(\zeta) = A_0X(t) + A_1X(t+\zeta) \tag{3.67}$$

The LMI in (3.64) is obtained from $\dot{V}(X_t) < 0$. The reader is referred to *Proposition 5.17 in [129]* for the proof. □

The objective of the time delay analysis with *Proposition 3.4.1* is to maximize τ to obtain $\hat{\tau}$ such that there exist matrices $Y_1^T = Y_1$, $S^T = S$, $P^T = P$ and $Y_2, \in \mathbb{R}^{Nn\times Nn}$ satisfying (3.63)-(3.64).

The LMIs associated with this method have 4 matrix variables each of dimension $Nn \times Nn$. The three variables Y_1, S and P are symmetric and have $\frac{Nn(Nn+1)}{2}$ decision variables each. The matrix variable Y_2 has $(Nn)^2$ decision variables. The total number of decision variables is given by

$$\frac{5(Nn)^2 + 3Nn}{2}$$

The maximum possible delay $\hat{\tau}$ obtained by this result is conservative [129]. This is due to the choice of the Lyapunov-Krosovskii functional which considers a single positive matrix S over the full delay interval $t-\tau \leq \theta \leq t$. *Proposition 5* reduces the conservatism by dividing the interval $[0\ \tau]$ into n_p equal parts and establishes stability with the help of a discretized Lyapunov-Krasovskii functional. The stability of the system in (3.60) using *Proposition 5* is given by the following proposition:

Proposition 3.4.2. *The system described in (3.60) is asymptotically stable if there exist $Nn \times Nn$ matrices $P = P^T$; $\bar{Q}_p$, $S_p = S_p^T$, $p = 0, \ldots, n_p$; $R_{pq} = R_{qp}^T$, $p = 0, \ldots, n_p$, $q = 0, \ldots, n_p$ and $h = \tau/n_p$ such that*

$$\begin{pmatrix} P & \tilde{F} \\ * & \tilde{R}+\tilde{S} \end{pmatrix} > 0 \tag{3.68}$$

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ * & R_d + S_d & 0 \\ * & * & 3S_d \end{pmatrix} > 0 \tag{3.69}$$

where

$$\tilde{F} = \begin{pmatrix} \bar{Q}_0 & \bar{Q}_1 & \ldots & \bar{Q}_{n_p} \end{pmatrix} \tag{3.70}$$

$$\tilde{R} = \begin{pmatrix} R_{00} & R_{01} & \ldots & R_{0n_p} \\ R_{10} & R_{11} & \ldots & R_{1n_p} \\ . & . & \ldots & . \\ R_{n_p0} & R_{n_p1} & \ldots & R_{n_pn_p} \end{pmatrix} \tag{3.71}$$

$$\tilde{S} = \begin{pmatrix} \frac{1}{h}S_0 & \frac{1}{h}S_1 & \ldots & \frac{1}{h}S_{n_p} \end{pmatrix} \tag{3.72}$$

and

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} \\ * & \Delta_{11} \end{pmatrix} \tag{3.73}$$

$$\Delta_{00} = -PA_0 - A_0^T P - \bar{Q}_0 - \bar{Q}_0^T - S_0 \tag{3.74}$$

$$\Delta_{01} = \bar{Q}_{n_p} - PA_1 \tag{3.75}$$

$$\Delta_{11} = S_{n_p} \tag{3.76}$$

$$
\begin{aligned}
S_d &= \mathscr{D}iag\begin{pmatrix} S_{d1} & S_{d2} & \dots & S_{dn_p} \end{pmatrix} && (3.77)\\
S_{dp} &= S_{(p-1)} - S_p && (3.78)\\
R_d &= \begin{pmatrix} R_{d11} & R_{d12} & \dots & R_{d1n_p} \\ R_{d21} & R_{d22} & \dots & R_{d2n_p} \\ . & . & \dots & . \\ R_{dn_p1} & R_{dn_p2} & \dots & R_{dn_pn_p} \end{pmatrix} && (3.79)\\
R_{dpq} &= h(R_{(p-1,q-1)} - R_{pq}) && (3.80)\\
D^s &= \begin{pmatrix} D_1^s & D_2^s & \dots & D_{n_p}^s \end{pmatrix} && (3.81)\\
D_p^s &= \begin{pmatrix} D_{0p}^s \\ D_{1p}^s \end{pmatrix} && (3.82)\\
D_{0p}^s &= \frac{h}{2}A_0^T(\bar{Q}_{(p-1)} + \bar{Q}_p) + \frac{h}{2}(R_{(0,p-1)} + R_{0p}) - (\bar{Q}_{(p-1)} - \bar{Q}_p) && (3.83)\\
D_{1p}^s &= \frac{h}{2}A_1^T(\bar{Q}_{(p-1)} + \bar{Q}_p) - \frac{h}{2}(R_{(n_p,p-1)} + R_{n_pp}) && (3.84)\\
D^a &= \begin{pmatrix} D_1^a & D_2^a & \dots & D_{n_p}^a \end{pmatrix} && (3.85)\\
D_p^a &= \begin{pmatrix} D_{0p}^a \\ D_{1p}^a \end{pmatrix} && (3.86)\\
D_{0p}^a &= -\frac{h}{2}A_0^T(\bar{Q}_{(p-1)} - \bar{Q}_p) - \frac{h}{2}(R_{(0,p-1)} - R_{0p}) && (3.87)\\
D_{1p}^a &= -\frac{h}{2}A_1^T(\bar{Q}_{(p-1)} + \bar{Q}_p) + \frac{h}{2}(R_{(n_p,p-1)} - R_{n_pp}) && (3.88)
\end{aligned}
$$

Proof. The delay interval $[-\tau, 0]$ is divided into n_p segments of equal length $h = \frac{\tau}{n_p}$ given by $[\theta_p, \theta_{p-1}]$, $p = 1, \dots, n_p$, where $\theta_p = -ph$ and $\theta_0 = 0$. This divides the square $\mathscr{S} = [-\tau, 0] \times [-\tau, 0]$ into $n_p \times n_p$ smaller squares $\mathscr{S}_{pq} = [\theta_p, \theta_{p-1}] \times [\theta_p, \theta_{p-1}]$. Each small square is further divided into two smaller rectangles. The reader is referred to [129] for further details.

The discretized Lyapunov-Krasovskii functional employed for this proof is given by

$$
V(X_t) = X^T(t)PX(t) + 2X^T(t)\sum_{p=1}^{n_p} V_{\bar{Q}} + \sum_{p=1}^{n_p}\sum_{q=1}^{n_p} V_R + \sum_{p=1}^{n_p} V_S \tag{3.89}
$$

where

$$
\begin{aligned}
V_{\bar{Q}} &= \int_0^1 \bar{Q}^{(p)}(\zeta)X(t+\theta_p+\zeta h)hd\zeta &(3.90)\\
V_R &= \int_0^1 (\int_0^1 X^T(t+\theta_p+\zeta h)R^{(pq)}(\zeta,\eta)X(t+\theta_p+\zeta h)hd\eta)hd\zeta &(3.91)\\
V_S &= \int_0^1 X^T(t+\theta_p+\zeta h)S^{(p)}(\zeta)X(t+\theta_p+\zeta h)hd\zeta &(3.92)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{Q}^{(p)}(\zeta) &= (1-\zeta)\bar{Q}_p+\zeta\bar{Q}_{p-1} &(3.93)\\
R^{(pq)}(\zeta,\eta) &:= \begin{cases} (1-\zeta)R_{pq}+\eta R_{p-1,q-1}+(\zeta-\eta)R_{p-1,q} & \zeta\geq\eta \\ (1-\eta)R_{pq}+\zeta R_{p-1,q-1}+(\eta-\zeta)R_{p,q-1} & \zeta<\eta \end{cases} &(3.94)\\
S^{(p)}(\zeta) &= (1-\zeta)S_p+\zeta S_{p-1} &(3.95)
\end{aligned}
$$

for $p=1,\ldots,n_p$, $q=1,\ldots,n_p$, $0\leq\zeta\leq1$ and $0\leq\eta\leq1$. The matrix inequality in (3.68) is obtained from the condition $V(X_t)>0$ and (3.69) is obtained from the condition $\dot{V}(X_t)<0$. The reader is referred to Proposition 5.22 [129] for the proof of this theorem.

□

The objective of the time delay analysis with *Proposition 3.4.2* is to maximize τ to obtain $\hat{\tau}$ such that there exist $Nn\times Nn$ matrices $P=P^T$; $\bar{Q}_p$, $S_p=S_p^T$, $p=0,\ldots,n_p$; $R_{pq}=R_{qp}^T$, $p=0,\ldots,n_p$, $q=0,\ldots,n_p$ satisfying (3.68)-(3.69).

The matrices R_{pq} above contribute to $\frac{1}{2}(n_p+1)(n_p+2)$ matrices. The LMIs for this method in total have

$$\frac{(n_p^2+7n_p+8)}{2}$$

matrix decision variables each of dimension $Nn\times Nn$. The total number of decision variables given by

$$\frac{(n_p^2+5n_p+5)(Nn)^2+(2n_p+3)(Nn)}{2}$$

It is important to note that the number decision variables can be very large depending of the number of partitions n_p, the number of agents N in the multi-agent system and the number of states n of an agent. Thus this method may be computationally unsolvable for a very large number of agents N.

3.4.2 Time-Varying Delay

For *Case B*, the system (3.60) is analyzed with a time varying delay $\tau(t)$ using the result of Proposition 6 from Appendix B.5. The stability of the system in (3.60) using *Proposition 6* employs a descriptor representation of the system as explained in Appendix B.5. The Lyapunov-Krasovskii functional approach as explained in Appendix B.1 is then employed to obtained the stability criteria.

The objective of the time delay analysis with *Proposition 6* is to maximize τ to obtain $\hat{\tau}$ such that there exist symmetric matrices P_1 and $\bar{R} \in \mathbb{R}^{Nn \times Nn}$ and matrices P_2, P_3, Z_1, Z_2 and $Z_3 \in \mathbb{R}^{n \times n}$ satisfying (B.56)-(B.59) in *Proposition 6* Appendix B.5.

The corollary when used appropriately for the system in (3.60) has 7 matrix variables each of dimension $Nn \times Nn$ of which 2 are symmetric matrix variables. The total number of decision variables is given by $6(Nn)^2 + Nn$.

3.4.3 Bisection Algorithm

In the three methods described in Section 3.4.1 and Section 3.4.2 the matrix inequalities of (B.20), (B.25) and (B.59) should be satisfied for *Proposition 4*, *Proposition 5*, and *Proposition 6* in Appendix B respectively. The time delay τ appears in these LMIs. The value of τ is fixed to retain convexity, and the LMIs are subsequently tested for feasibility. A bisection algorithm is employed along with the three methods stated above to find the largest possible value of the feasible delay $\hat{\tau}$. A Matlab algorithm is given in Table 3.1.

Remark 3.8: Note that the three propositions in Section 3.4.1 and Section 3.4.2 are used to obtain the maximum possible delay. In those propositions the delay τ appears in the matrix inequalities. The bisection algorithm first fixes a value of delay in the given range $[a,b]$ and then checks for the feasibility of the propositions at that level of delay. This retains the convexity of the LMIs. The bisection algorithm will stop when the given tolerance level tol is reached.

Table 3.1: Algorithm 1

```
tol = 1e-6;
a = 0.000001;
b = 2;
fa = timedelaylmi(a); % Feasibility of LMIs
fb = timedelaylmi(b);
if (sign(fa)==sign(fb))
   fprintf('Change the interval considered \n')
   return
end
while abs(a-b)>tol
   c =(a+b)/2;
   fc = timedelaylmi(c);
   if sign(fa)==sign(fc)
      a=c;
      fa=fc;
   else
      b=c;
      fb=fc;
   end
end
```

3.5 Previous Work on LQR Control of Identical Linear Agents

In [96] a state feedback LQR control design method for identical decoupled LTI systems is proposed. The control design method achieves sub-optimal LQR performance with the use of a network level LQR cost function. The structure of the LQR cost function used is similar to one that is used in this chapter.

In [96] the LQR problem for a network of identical linear agents is posed as a single LQR problem exploiting the properties of the graph associated with the corresponding communication topology. Each linear agent is described by the following system dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.96}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. A network of N such identical decoupled agents is considered in [96]. This multi-agent system with N agents is assumed to have a fixed undirected communication topology described by a graph $\mathscr{G}$. For the communication topology $\delta_{max}(\mathscr{G})$ represents the maximum degree for the graph $\mathscr{G}$. In

[96], in the first step of the control design procedure, an LQR problem is solved for a reduced system with N_L identical agents, where $N_L = \delta_{max}(\mathscr{G}) + 1$. The system with N_L identical linear dynamical systems is given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3.97}$$

$$x_i(0) = x_{i0} \tag{3.98}$$

for $i = 1, \ldots, N_L$, where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ are the states and control respectively. Using Kronecker products and column concatenation, the system at a network level can be represented as

$$\dot{X}_{N_L}(t) = (I_{N_L} \otimes A)X_{N_L}(t) + (I_{N_L} \otimes B)U(t) \tag{3.99}$$

$$X_{N_L}(0) = X_0 \triangleq [x_{10}, \ldots, x_{N_L 0}]^T \tag{3.100}$$

where

$$X_{N_L}(t) = \mathscr{C}ol(x_1(t), \ldots, x_{N_L}(t)) \tag{3.101}$$

$$U(t) = \mathscr{C}ol(u_1(t), \ldots, u_{N_L}(t)) \tag{3.102}$$

The LQR control problem for the first step, stated in *Theorem 2* [96], is to minimize

$$J(U(t), X_{N_L}(0)) = \int_0^\infty (X_{N_L}^T(t)\tilde{Q}X(t) + U(t)\tilde{R}U(t))dt \tag{3.103}$$

with $U(t) = -\tilde{K}X_{N_L}(t)$ where the matrices $\tilde{Q} \in \mathbb{R}^{nN_L \times nN_L}$ and $\tilde{R} \in \mathbb{R}^{mN_L \times mN_L}$ are defined as

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & -\tilde{Q}_2 & \cdots & -\tilde{Q}_2 \\ -\tilde{Q}_2 & \tilde{Q}_{11} & \cdots & -\tilde{Q}_2 \\ \vdots & \ddots & \ddots & \vdots \\ -\tilde{Q}_2 & \cdots & \cdots & \tilde{Q}_{11} \end{pmatrix} \quad \tilde{R} = \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & R \end{pmatrix} \tag{3.104}$$

where $\tilde{Q}_{11} = \tilde{Q}_1 + (N_L - 1)\tilde{Q}_2$, and $\tilde{Q}_1, \tilde{Q}_2 \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive definite. The gain matrix $\tilde{K}$ is obtained by solving the corresponding Algebraic Riccati Equation and is given by $\tilde{K} = \tilde{R}^{-1}(I_{N_L} \otimes B)^T \tilde{P}$. The matrix $\tilde{P} \in \mathbb{R}^{nN_L \times nN_L}$ is the solution to the ARE given by

$$(I_{N_L} \otimes A)^T \tilde{P} + \tilde{P}(I_{N_L} \otimes A) - \tilde{P}(I_{N_L} \otimes B)^T R^{-1}(I_{N_L} \otimes B)\tilde{P} + \tilde{Q} = 0 \tag{3.105}$$

The matrix $\tilde{P}$ is shown in [96] to have the structure

$$\tilde{P} = \begin{pmatrix} P_1 & P_2 & \cdots & P_2 \\ P_2 & P_1 & \cdots & P_2 \\ \vdots & \ddots & \ddots & \vdots \\ P_2 & \cdots & \cdots & P_1 \end{pmatrix} \tag{3.106}$$

where $P_1, P_2 \in \mathbb{R}^{n \times n}$. The structure (3.106) arises due to the selection of equal non-diagonal weights for $\tilde{Q}$ and zero diagonal weights in $\tilde{R}$ in (3.104). The optimal LQR controller thus obtained is shown to have the following structure

$$\tilde{K} = \begin{pmatrix} K_1 & K_2 & \cdots & K_2 \\ K_2 & K_1 & \cdots & K_2 \\ \vdots & \ddots & \ddots & \vdots \\ K_2 & \cdots & \cdots & K_1 \end{pmatrix} \tag{3.107}$$

where $K_1, K_2 \in \mathbb{R}^{m \times n}$.

In the second step the gains in (3.107), which are obtained by solving a reduced agent problem in the first step with $N_L = \delta_{max}(\mathscr{G}) + 1$, are used to design a feedback controller for the complete network of N identical agents in Theorem 4 [96]. The system given in (3.97) can be represented for the network of N identical dynamical agents as (3.3) given by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{3.108}$$

where

$$X(t) = \mathscr{C}ol(x_1(t), \dots, x_N(t)) \tag{3.109}$$

$$U(t) = \mathscr{C}ol(u_1(t), \dots, u_N(t)) \tag{3.110}$$

Note that the control law $U(t)$ in (3.110) is now for the complete network of N agents. The objective of the second step is to minimize the cost function given by

$$J = \int_0^\infty (X^T(t)\hat{Q}X(t) + U(t)\hat{R}U(t))dt \tag{3.111}$$

where $\hat{Q} \in \mathbb{R}^{nN \times nN}$ and $\hat{R} \in \mathbb{R}^{mN \times mN}$. The control law is given by $U(t) = \hat{K}_c X(t)$ where $\hat{K}_c \in \mathbb{R}^{mN \times nN}$. The weighting matrix $\hat{Q}$ is defined as

$$\hat{Q}_{ij} = \hat{Q}[(i-1)n : in, (j-1)n, jn] \tag{3.112}$$

for $i, j = 2, \dots, N$, where $\hat{Q}_{ij} = 0$ if agent i is not a neighbour of j and $\hat{Q}_{ij} \neq 0$ for $i = j$ and agent i is a neighbour of j.. The matrix $\hat{R} = \mathscr{D}iag(R, \dots, R)$. It is required that the gain $\hat{K}_c$ has the form

$$\hat{K}_{cij} = \hat{K}_c[(i-1)m : im, (j-1)n, jn] \tag{3.113}$$

for $i, j = 2, \dots, N$, where $\hat{K}_{cij} = 0$ if agent i is not a neighbour of j and $\hat{K}_{cij} \neq 0$ for $i = j$ and agent i is a neighbour of j. The optimization problem of minimizing the cost function in (3.111), with the structures $\hat{Q}$ and $\hat{K}$ given by (3.112) and (3.113), is stated as NP hard in [96]. Hence the authors have proposed a distributed suboptimal controller. In [96], the suboptimal controller $U(t) = \hat{K}_c X(t)$ is designed for the system in (3.108), where $\hat{K}_c$ is given by

$$\hat{K}_c = -I_N \otimes R^{-1} B^T \tilde{P}_1 + \tilde{M} \otimes R^{-1} B^T P_2 \tag{3.114}$$

where $\tilde{P}_1 = P_1 + 2P_2$ and $P_2 \in \mathbb{R}^{n \times n}$ are Lyapunov matrices obtained from $\tilde{P}$. The matrix

$\tilde{P}_1$ is also the symmetric positive definite solution of the ARE associated with

$$A^T\tilde{P}_1 + \tilde{P}_1 A - \tilde{P}_1 B^T R^{-1} B\tilde{P}_1 + \tilde{Q}_1 = 0 \tag{3.115}$$

The matrix $\tilde{M} \in \mathbb{R}^{N\times N}$ is symmetric and has the structure of the Laplacian of the corresponding communication topology where $\tilde{M}(i,j) \neq 0$ if agent i is a neighbour of agent j and $\tilde{M}(i,j) = 0$ otherwise. In [96] it is stated that when the matrix $\tilde{M}$ is chosen such that the eigenvalues of $\tilde{M}$ satisfy

$$\lambda_i(\tilde{M}) > \frac{N_L}{2} \tag{3.116}$$

for $i = 1,\ldots,N$, the system in (3.108) is asymptotically stabilized by the controller in (3.114). The requirement in (3.116) is further relaxed to

$$\lambda_i(\tilde{M}) \geq 0 \tag{3.117}$$

if the following condition is satisfied

Condition 1: For the matrices $\tilde{P}_1$ and P_2 in (3.114), the matrix

$$A - BR^{-1}B^T\tilde{P}_1 + \rho N_L BR^{-1}B^T P_2 \tag{3.118}$$

is stable for $\rho \in [0, \frac{1}{2}]$. The reader is referred to [96] for further reading.

Remark 3.9: In the first step of the procedure in [96] a reduced order problem was solved for $N_L = \delta_{max}(\mathcal{G}) + 1$ in (3.99). The gains obtained from (3.107) were used in the second step for the stabilization of full network with N agents given in (3.108). The cost function considered for the second problem is given by (3.111). The class of weighting matrices considered for the cost function are given in (3.112). It is important to note that the cost matrix considered in Section 3.2, given by $(I_N \otimes Q_1) + (\mathcal{L} \otimes Q_2)$ in (3.8) is of the form of (3.112). But the second step of the control design procedure in [96] is to design a controller of the form (3.113). This is stated to be NP hard in [96]. The second step of the method in [96], described in this section, just guaranties asymptotic stabilization. The major advantage of the method in [96] is that it considers generic cost function in (3.112) whereas the drawback is that network level optimization problem is NP hard.

The major advantage of the method proposed in this Chapter in Section 3.2 is that the second step minimizes the cost function with $(I_N \otimes Q_1) + (\mathscr{L} \otimes Q_2)$ in (3.8) using an LMI formulation. Also though $(I_N \otimes Q_1) + (\mathscr{L} \otimes Q_2)$ is one of the possibilities of the matrix in (3.112), it was identified in this research to aid simultaneous convergence as explained in Remark 3.1.

3.6 Numerical Example

A network of 5 planar vehicles each described by a double integrator in each of the directions x and y is considered. The system is then represented by

$$\dot{\zeta}_i(t) = A\zeta_i(t) + Bu_i(t) \tag{3.119}$$

where ζ_i represents the states of the i^{th} vehicle, and consists of the x and y plane positions and velocities. The plant matrix and input distribution matrix are given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.120}$$

The matrices in the LQR cost function in (3.8) have been chosen as $Q_1 = 10I_4$, $Q_2 = 25I_4$ and $R = I_2$.

3.6.1 Communication Topology

A nearest neighbour interconnection topology as shown in Fig. 3.1 is assumed between the five agents.

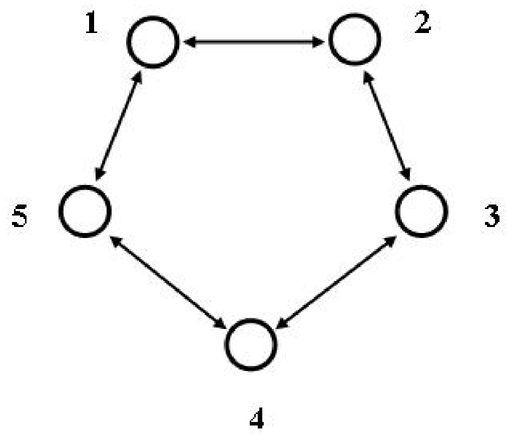

Figure 3.1: 5 agents with cyclic interconnection

The Laplacian for this network is given by

$$\mathscr{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{3.121}$$

The Laplacian in (3.121) is used to design the control law such that the network described in Fig. 3.1 is stabilized to achieve a rendezvous.

3.6.2 Control Gains

The proposed two step design procedure from Section 3.3 is followed to obtain the control gain matrix K and the scaling matrix Φ. The matrix B in (3.120) is already in the regular form described in (3.13). As a first step, a control law of the form in Step 1 is obtained by solving the standard LQR problem, using the Matlab command `'lqr'` giving the control gain matrix in (3.7) as

$$K = \begin{pmatrix} 3.1623 & 0 & 4.0404 & 0 \\ 0 & 3.1623 & 0 & 4.0404 \end{pmatrix} \tag{3.122}$$

The LQR performance cost J_i in (3.10) associated with the controller is 33.63. The Lyapunov matrix P is given by

$$P = \begin{pmatrix} 12.7767 & 0 & 3.1623 & 0 \\ 0 & 12.7767 & 0 & 3.1623 \\ 3.1623 & 0 & 4.0404 & 0 \\ 0 & 3.1623 & 0 & 4.0404 \end{pmatrix} \tag{3.123}$$

Subsequently, the transformation $\hat{T}$ in (3.16) is obtained using the Lyapunov matrix in (3.123) as

$$\hat{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.7827 & 0 & 1 & 0 \\ 0 & 0.7827 & 0 & 1 \end{pmatrix} \tag{3.124}$$

The matrices $\hat{A}$, $\hat{B}$ and $\hat{K}$ are given by

$$\hat{A} = \begin{pmatrix} -0.7827 & 0 & 1 & 0 \\ 0 & -0.7827 & 0 & 1 \\ -0.6126 & 0 & 0.7827 & 0 \\ 0 & -0.6126 & 0 & 0.7827 \end{pmatrix} \quad \hat{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.125}$$

$$\hat{K} = \begin{pmatrix} 0 & 0 & 4.0404 & 0 \\ 0 & 0 & 0 & 4.0404 \end{pmatrix} \tag{3.126}$$

A distributed control law as in (3.7) is designed using the LMIs provided in (3.32) - (3.33), and $W_i > 0 \; \forall i = 1, \ldots, N$. The scaling matrix Φ obtained from Theorem 3.3.1 is given by

$$\Phi = \begin{pmatrix} 0.6736 & 0 \\ 0 & 0.6736 \end{pmatrix} \tag{3.127}$$

3.6.3 Measure of Sub-Optimality

In Section 3.5 an alternative method from [96] was presented. The first step of the control design process involves solving an LQR control synthesis problem with reduced number of nodes given by $N_L = \delta_{max}(\mathscr{G}) + 1$, where δ_{max} is the maximum degree of the underlying topology. For the communication topology in Fig. 3.1 the maximum degree $\delta_{max} = 2$ and $N_L = 3$. The matrix $\tilde{Q}_1$ and $\tilde{Q}_2$ are chosen as $\tilde{Q}_1 = Q_1$ and $\tilde{Q}_2 = Q_2$ respectively. The matrix $\tilde{P}$ is then obtained from the ARE in (3.105). It is evident from (3.115) that the first term of the controller in (3.114) is equivalent to the Step 1 of the control design process in Section 3.3. The term $R^{-1}B^T P_2$ in (3.114) is given by

$$R^{-1}B^T P_2 = \begin{pmatrix} -2.0191 & 0 & -2.0434 & 0 \\ 0 & -2.0191 & 0 & -2.0434 \end{pmatrix} \tag{3.128}$$

For this example, the Condition 1 in Section 3.5 is satisfied. In this scenario the symmetric matrix $\tilde{M}$ in (3.114) can be selected such that $\tilde{\lambda}_i(\tilde{M}) \geq 0$, where $\tilde{\lambda}_i(\tilde{M})$ are the eigenvalues of $\tilde{M}$. Then the controller in (3.114) asymptotically stabilizes the system in (3.108). The matrix ΦK in (3.12) from Section 3.3 is given by

$$\Phi K = \begin{pmatrix} 2.1300 & 0 & 2.7214 & 0 \\ 0 & 2.1300 & 0 & 2.7214 \end{pmatrix} \tag{3.129}$$

If $\tilde{M}$ is chosen as the Laplacian $\mathscr{L}$, then the controllers obtained from the method proposed in Section 3.3 and from Section 3.5 differ only in the term represented by (3.128) and (3.129).

For the system in (3.3) let $K^* \in \mathbb{R}^{Nm \times Nn}$ and $P^* \in \mathbb{R}^{Nn \times Nn}$ be the optimal LQR controller and the corresponding solution to the ARE associated with the optimal LQR problem for the system with (3.8) as the cost function given by

$$(I_N \otimes A)^T P^* + P^*(I_N \otimes A) - P^*(I_N \otimes B)(I_N \otimes R)^{-1}(I_N \otimes B)^T P^* + (I_N \otimes Q_1 + \mathscr{L} \otimes Q_2) = 0 \tag{3.130}$$

For the system given in (3.3) the control law $U(t) = -K^* X(t)$, where $K^* = (I_N \otimes$

$R)^{-1}(I_N \otimes B)^T P^*$ then minimizes the cost function in (3.8). As explained in Appendix A.2, the optimal cost is given by $Trace(P^*) = 751.19$. For the controller designed in Section 3.3 a special structure of the Lyapunov matrix given by (3.43) was used. The cost obtained from (3.50) by solving the optimization problem in Theorem 3.3.1 is 772.21. For the gain matrix K and the scaling matrix Φ obtained from the design procedure in Section 3.3, the closed loop system is given by

$$\dot{X}(t) = (I_N \otimes A - I_N \otimes BK - \mathscr{L} \otimes B\Phi K)X(t) \tag{3.131}$$

As explained in [96], for the closed loop system given by (3.131), a minimum bound on the sub-optimal cost with the control law given in (3.12) is given by $J = Trace(P_s)$ where $P_s \in \mathbb{R}^{Nn \times Nn}$ is the positive definite solution of the Lyapunov equation

$$\begin{aligned}(I_N \otimes A - I_N \otimes BK - \mathscr{L} \otimes B\Phi K)^T P_s + P_s(I_N \otimes A - I_N \otimes BK - \mathscr{L} \otimes B\Phi K) + \\ (I_N \otimes Q_1 + \mathscr{L} \otimes Q_2) + (I_N \otimes K + \mathscr{L} \otimes \Phi K)^T (I_N \otimes R)(I_N \otimes BK + \mathscr{L} \otimes \Phi K) = 0\end{aligned} \tag{3.132}$$

The sub-optimal cost thus obtained is characterized by $Trace(P_s) = 759.1$. Similarly, for the controller $\hat{K}_c$ in (3.114) obtained from the method proposed in [96], the sub-optimal cost is given by $Trace(P_k)$ where P_k is the positive definite solution of the Lyapunov equation given by

$$\begin{aligned}(I_N \otimes A + (I_N \otimes B)\hat{K}_c)^T P_k + P_k(I_N \otimes A + (I_N \otimes B)\hat{K}_c) + \\ (I_N \otimes Q_1 + \mathscr{L} \otimes Q_2) + \hat{K}_c^T (I_N \otimes R)\hat{K}_c = 0\end{aligned} \tag{3.133}$$

The sub-optimal cost obtained is $J = Trace(P_k) = 751.66$.

The bounds for the LQR cost in (3.8) for the sub-optimal controllers in (3.12) and (3.114) are 751.66 and 759.1 respectively which is comparable to 751.19 obtained from the optimal controller $U(t) = -K^* X(t)$. The cost obtained from Theorem 3.3.1 is higher since a block diagonal structure for the Lyapunov matrix given by (3.43) was employed. A comparison of the costs obtained from the control structures in (3.12) and (3.114) is given in Table 3.2.

Table 3.2: Comparison of control structures

Control Structure	Bounds
(3.12)	$Trace(P_s) = 759.1$ (3.132)
(3.114)	$Trace(P_k) = 751.66$ (3.133)

To summarize the cost obtained from the method proposed in Theorem 3.3.1 is comparable to the method proposed in [96] for the example considered and has been obtained through an LMI based optimization procedure.

3.6.4 Time-delay Analysis

An analysis of the distributed control laws in the presence of time delays in communications is carried out in the sequel to understand the maximum permissible delay that can be tolerated. The maximum permissible time delay is found using the bisection algorithm given in Table 1, *Propositions 4*, *Proposition 5* and *Proposition 6* in Appendix B. As suggested in [129], the maximum permissible delay bound obtained by *Proposition 4* is conservative. When the discretized Lyapunov method of *Proposition 5* even when the number of partitions $n_p = 1$ is applied, the conservatism is considerably reduced. The time-varying delay analysis using *Proposition 6* gives the same bound as that of *Proposition 5* in Appendix B. This may be due to the maximum permissible delay being small. A comparison table of the delays is shown in Table 3.3.

Table 3.3: Time-delay analysis

Type of Delay	Method	$\hat{\tau}_{max}$(sec)	No. of Decision Variables
Fixed	Proposition 3.4.1	0.145	1030
Fixed	Proposition 3.4.2	0.20	2250
Time-varying	Proposition 6 (Appendix)	0.20	2420

3.6.5 Simulation Results

The simulations in this section were done using SIMULINK with a fixed-step size of $0.001s$ and *ode4(Runge-Kutta)* solver.

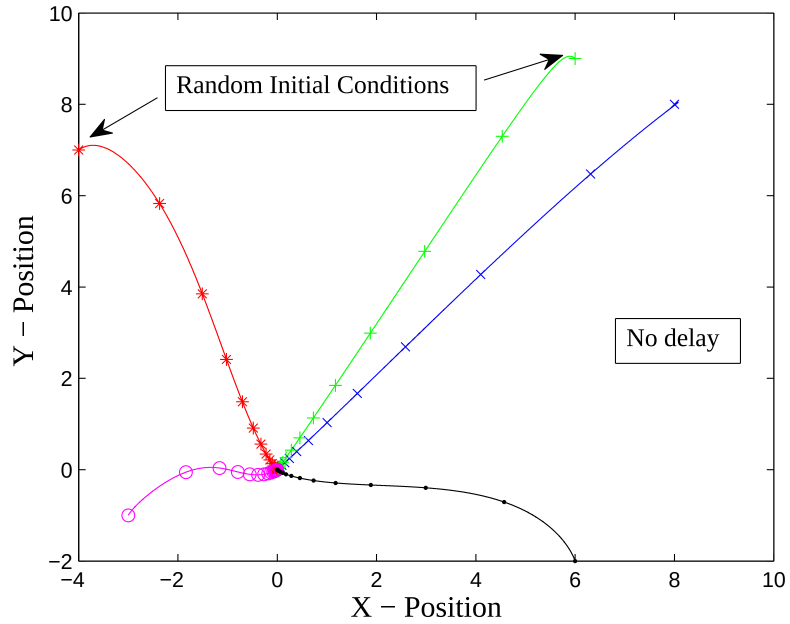

Figure 3.2: Rendezvous with no delay

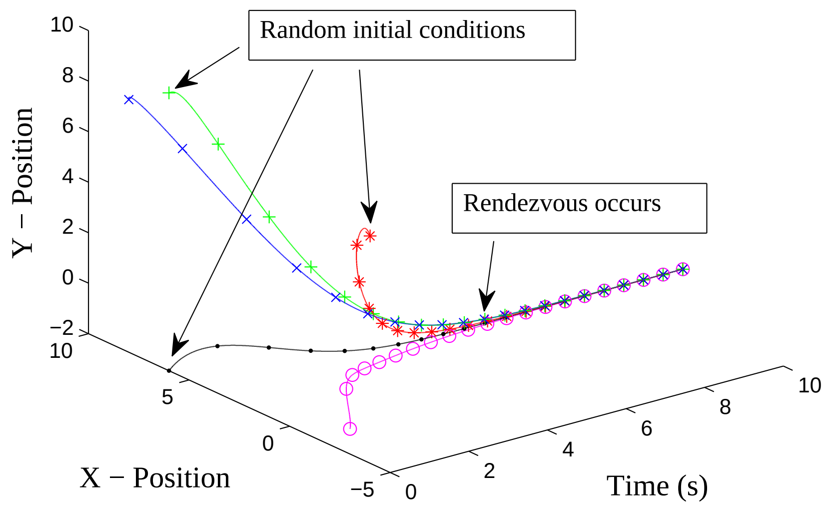

Figure 3.3: Rendezvous with no delay plotted against time

The objective is to create a rendezvous situation for the five agents, each kept at a randomly chosen initial condition. The same initial conditions are chosen for each simulation presented in this section. For the controller (3.7) obtained by employing the procedure in Section 3.3, Figure 3.2 shows 5 agents attaining a rendezvous when no delay is present in the communication in a 2-dimensional plane.

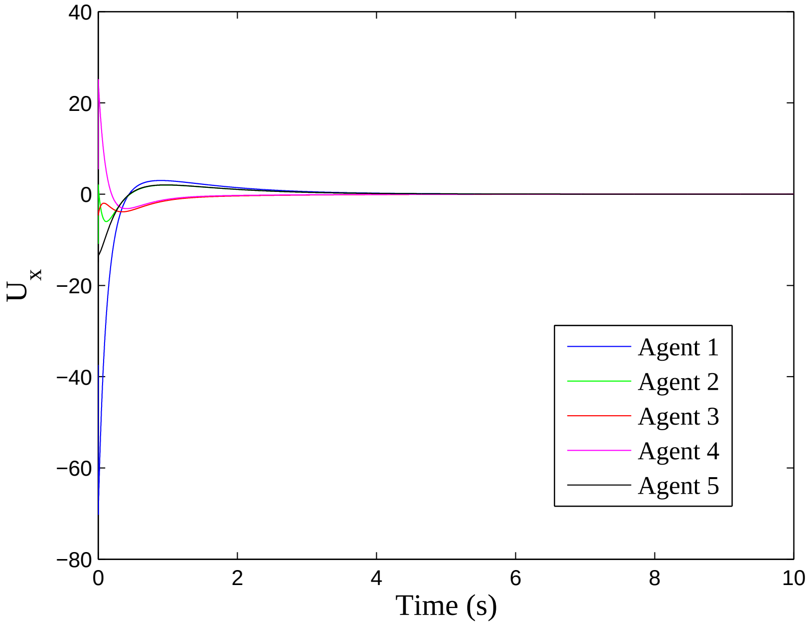

Figure 3.4: Control effort for agents in x direction - No delay

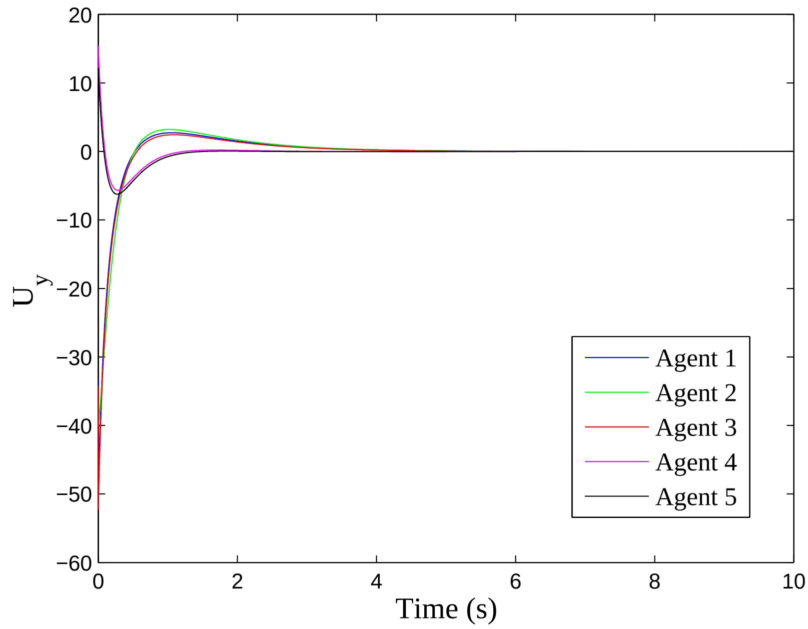

Figure 3.5: Control effort for agents in y direction - No delay

Figure 3.3 shows the plots of the simulation without any delay in the relative information. It can be seen in the figure that after time $t = 4s$ the agents meet at a point and stay at this point for all subsequent time. The control signals U_x and U_y for the agents in the x and y directions without delays are shown in Figure 3.4 and Figure 3.5. If no relative information had been used in (3.7), i.e. $\Phi = 0$ and just the Step 1 of the control design procedure had been solved, it would have resulted in a rendezvous of individual agents given in Figure 3.6 and Figure 3.7.

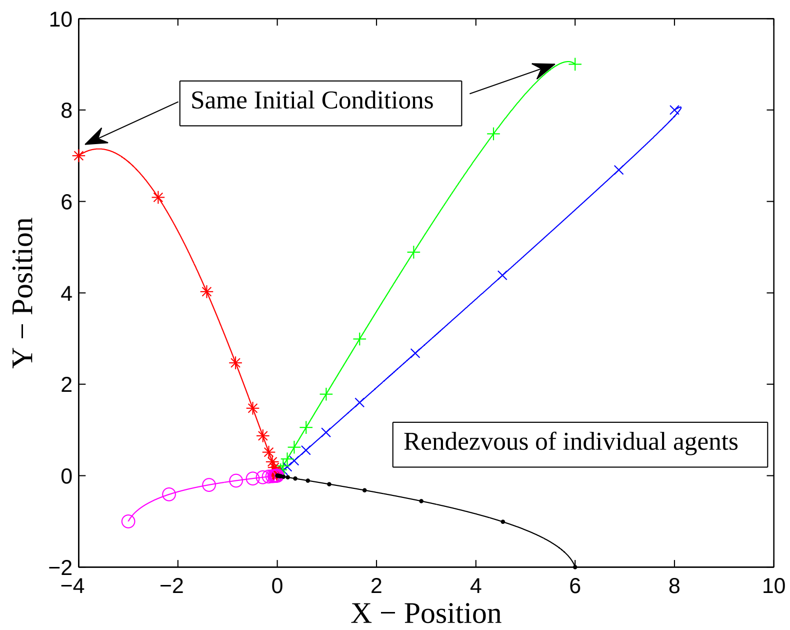

Figure 3.6: Rendezvous with no relative information

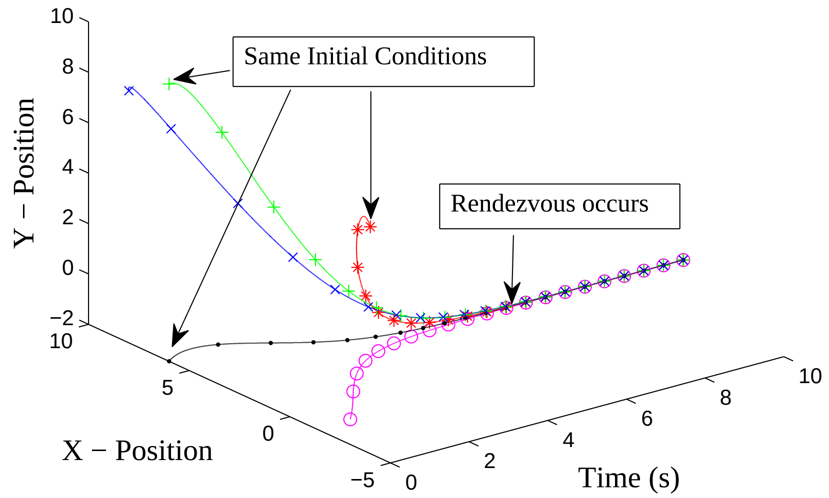

Figure 3.7: Rendezvous with no relative information plotted against time

The control signals U_x and U_y for the agents in the x and y directions when no relative information is present are shown in Figure 3.8 and Figure 3.9.

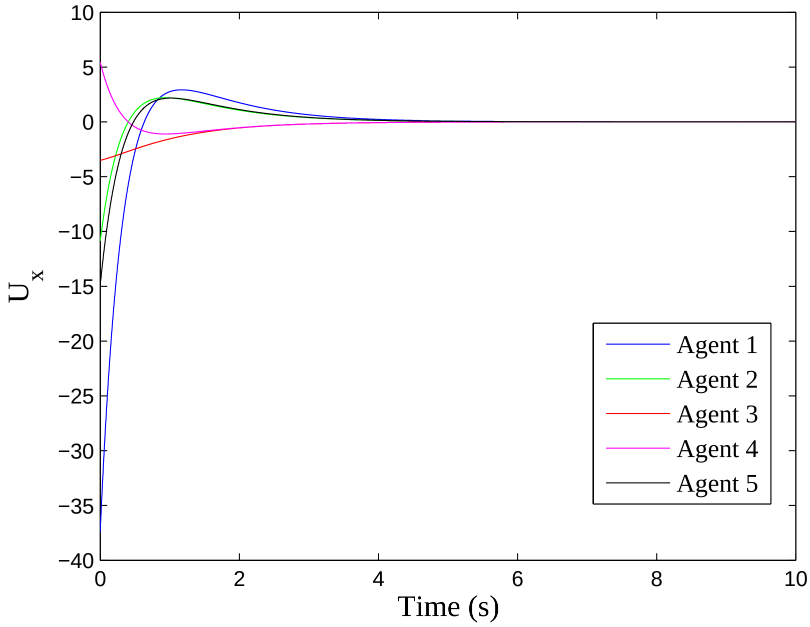

Figure 3.8: Control effort for agents in x direction - No relative information

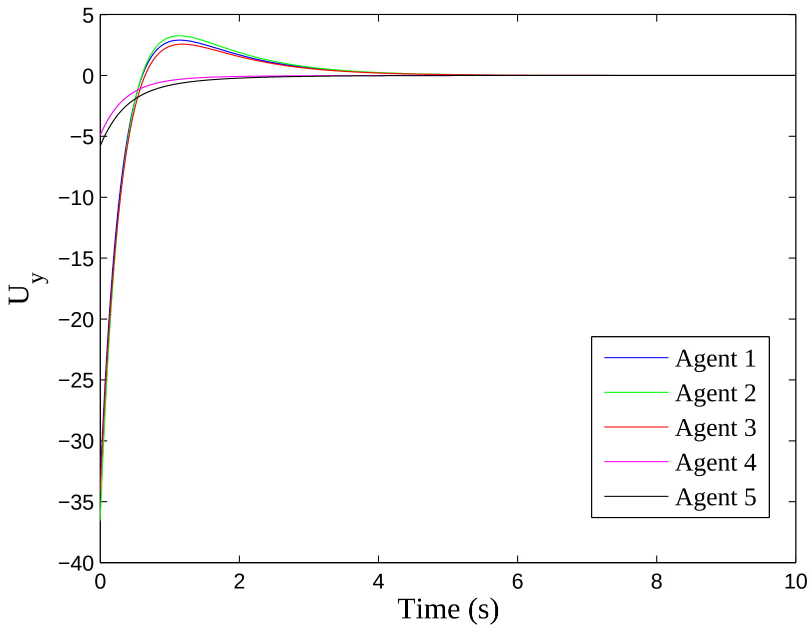

Figure 3.9: Control effort for agents in y direction - No relative information

From Figure 3.2 and Figure 3.6 it can be seen that due to the control law in (3.7) the agents try and reduce the interagent distances while achieving a rendezvous in Figure

3.2 as compared to Figure 3.6. From Figure 3.7 it can be seen that the agents achieve a rendezvous without relative information at $t = 4.2s$.

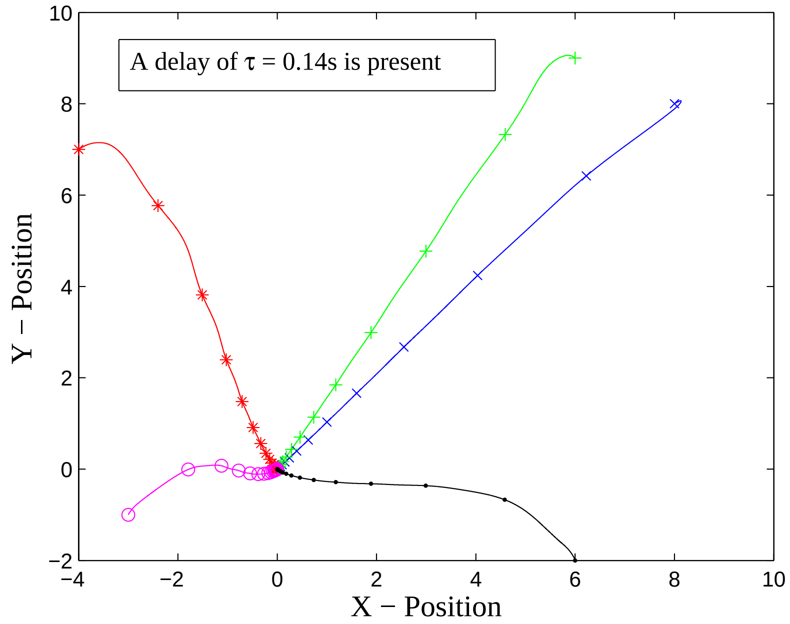

Figure 3.10: Rendezvous with a permissible delay

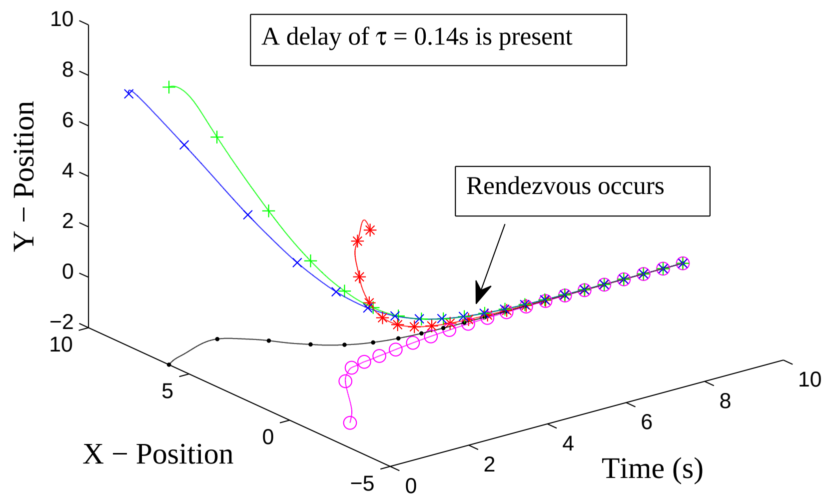

Figure 3.11: Rendezvous with a permissible delay plotted against time

Thus using relative information also has a small advantage with respect to improving the time to achieve the rendezvous. Though it can be seen from the control efforts in Figure

3.4 and Figure 3.5 and Figure 3.8 and Figure 3.9 that using the relative information increases the control effort. In a similar way, Figure 3.10 shows the rendezvous with a delay of $\tau = 0.14s$. Figure 3.11 shows the agents attaining a rendezvous as a function of time.

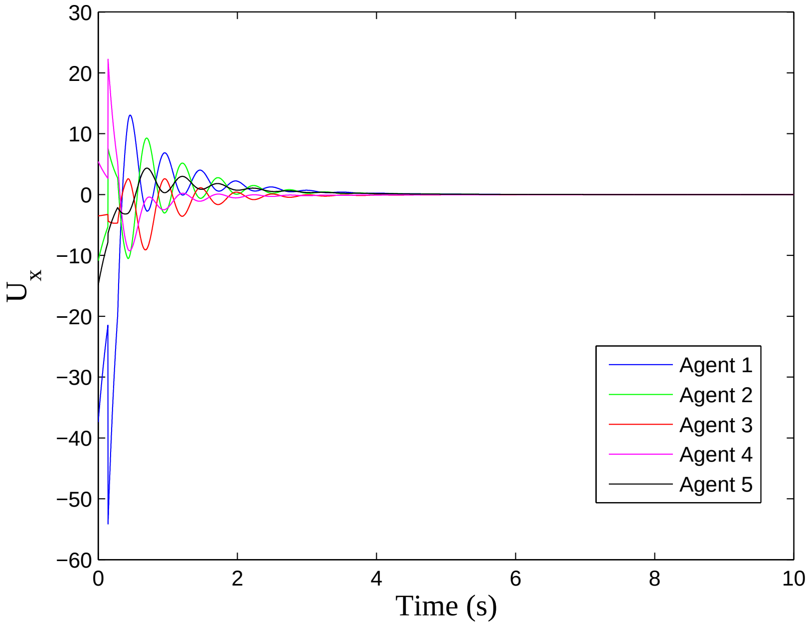

Figure 3.12: Control effort for agents in x direction - Permissible delay

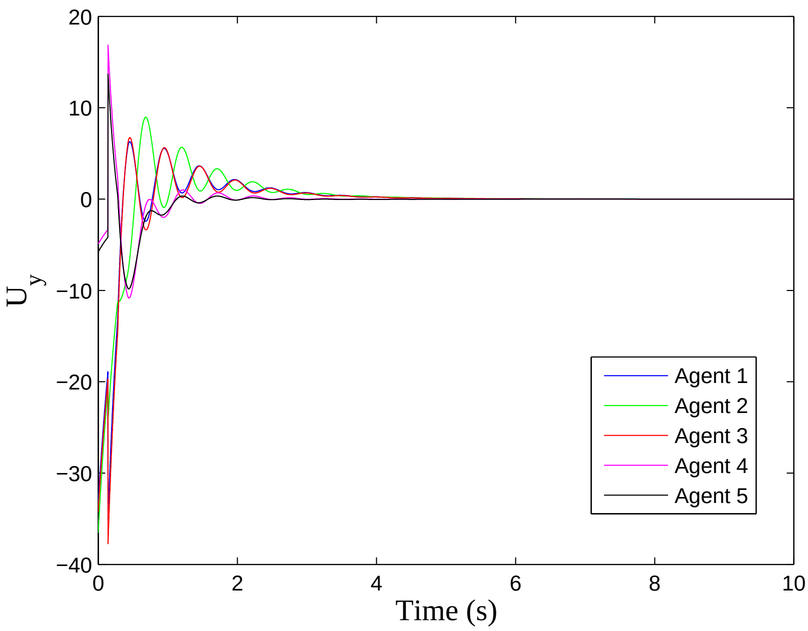

Figure 3.13: Control effort for agents in y direction - Permissible delay

The control signals U_x and U_y for the agents in the x and y directions with permissible delay are shown in Figure 3.12 and Figure 3.13. Figure 3.14 shows the rendezvous with a permissible time-varying delay of $\tau(t) = 0.1 + 0.05cos(t)$.

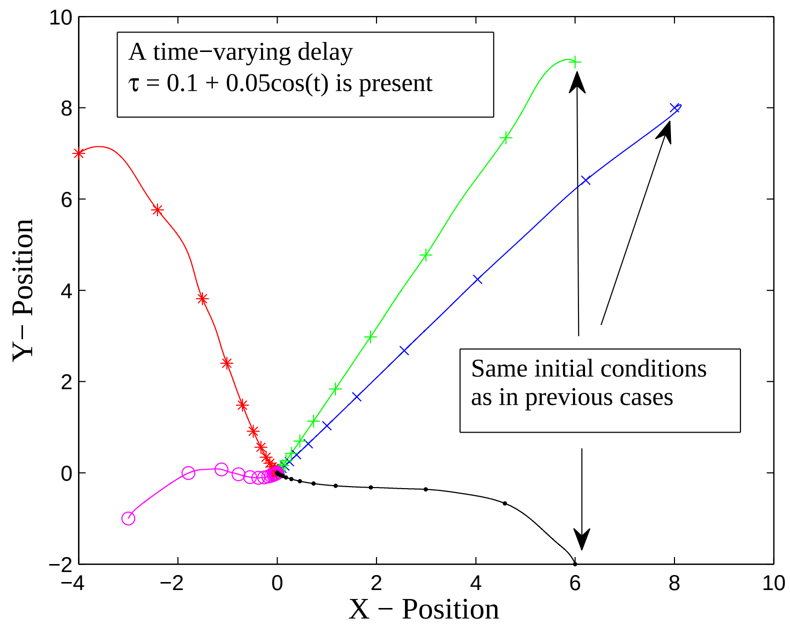

Figure 3.14: Rendezvous with a time-varying delay

Figure 3.15 shows the agents attaining a rendezvous with the time-varying delay.

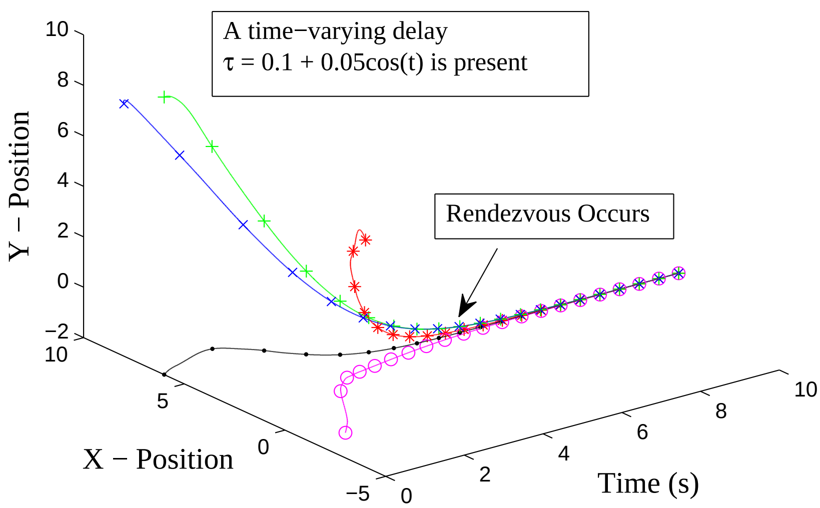

Figure 3.15: Rendezvous with a time-varying delay plotted against time

The control signals U_x and U_y for the agents in the x and y directions with time-varying delay are shown in Figure 3.16 and Figure 3.17.

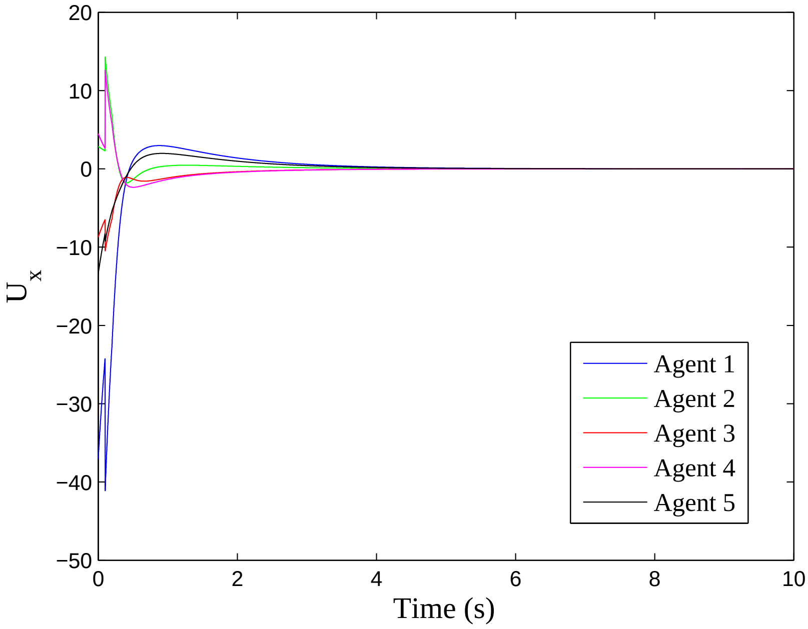

Figure 3.16: Control effort for agents in x direction - Time-varying delay

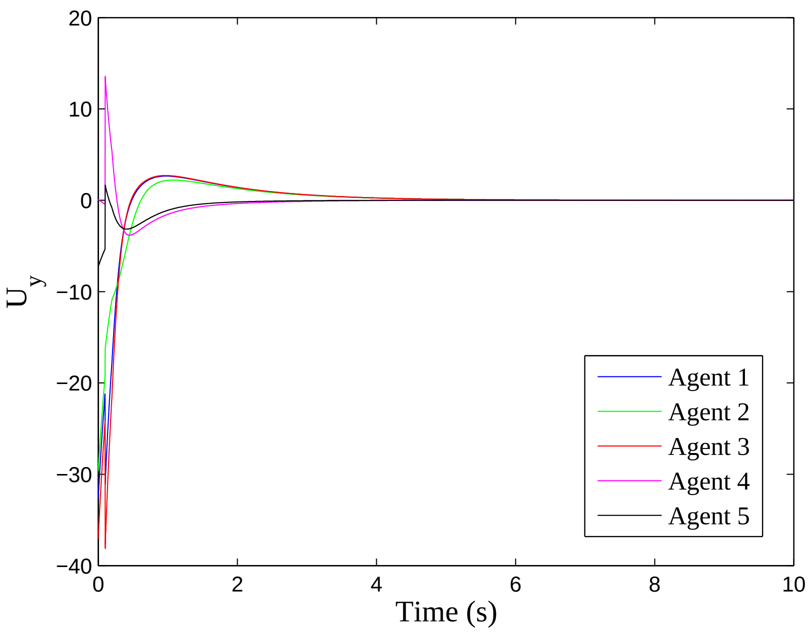

Figure 3.17: Control effort for agents in y direction - Time-varying delay

Figure 3.18 shows that the agents do not attain a rendezvous when the delay is $\tau = 0.22s$ for the same initial conditions.

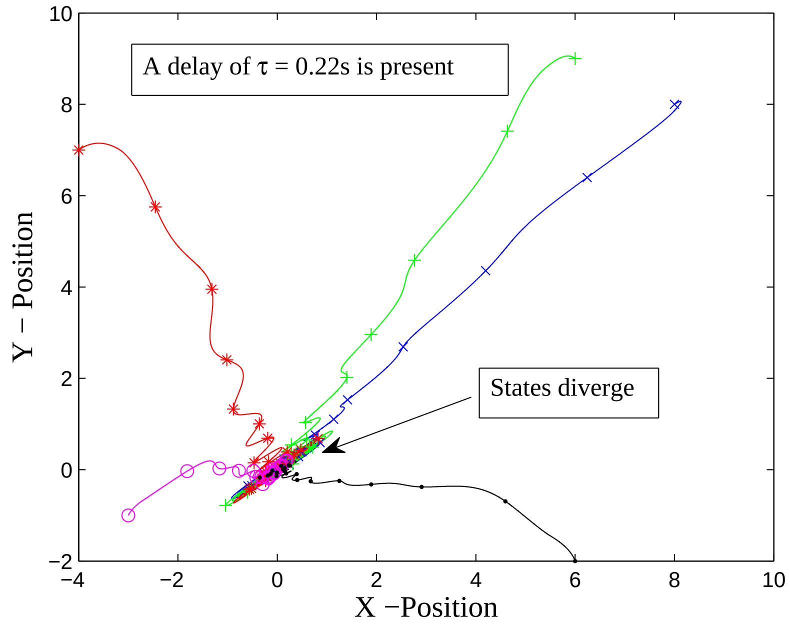

Figure 3.18: No Rendezvous with excess delay

Figure 3.19 shows the results as a function of time. Clearly a rendezvous does not occur among the vehicles when the delay is $\tau = 0.22s$ and so the values for maximum feasible delay τ_{max} obtained from *Proposition 5* and *Proposition 6* in Table 3.3 are reasonably accurate.

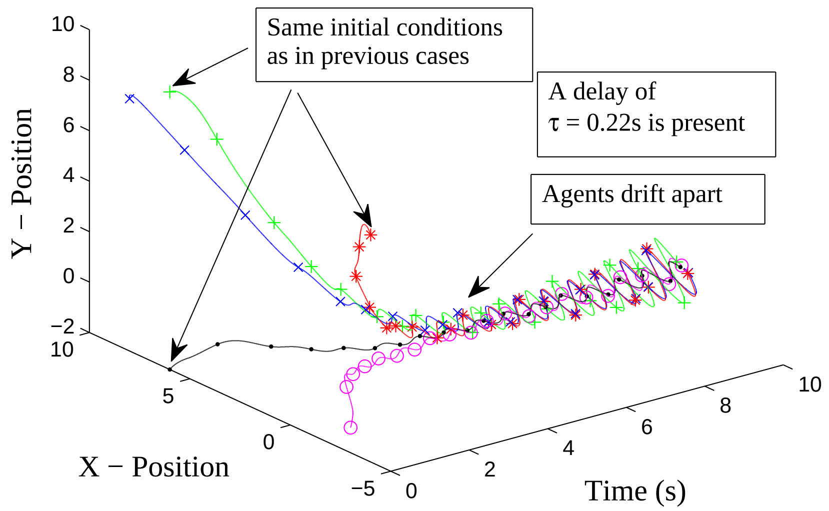

Figure 3.19: No rendezvous with excess delay plotted against time

The control signals U_x and U_y for the agents in the x and y directions with excessive delay are shown in Figure 3.20 and Figure 3.21.

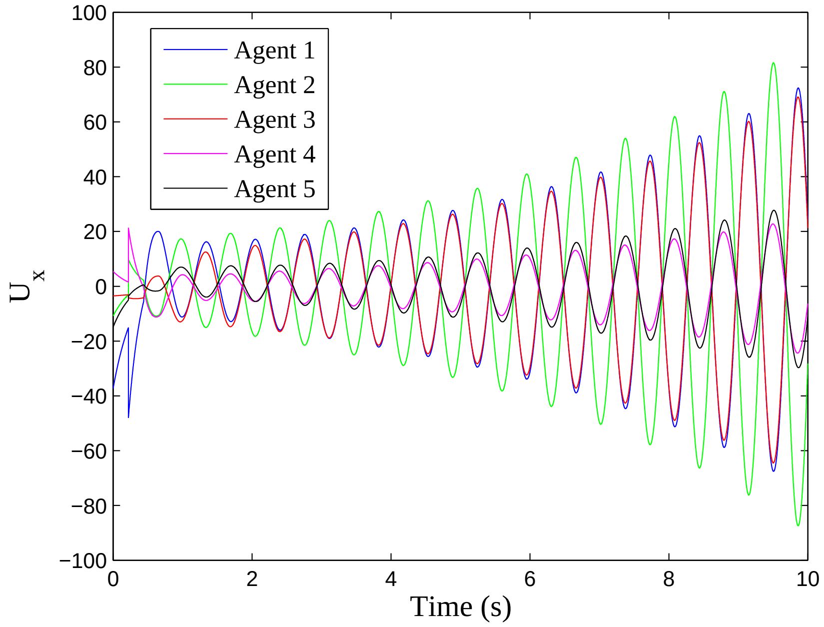

Figure 3.20: Control effort for agents in x direction - Excess delay

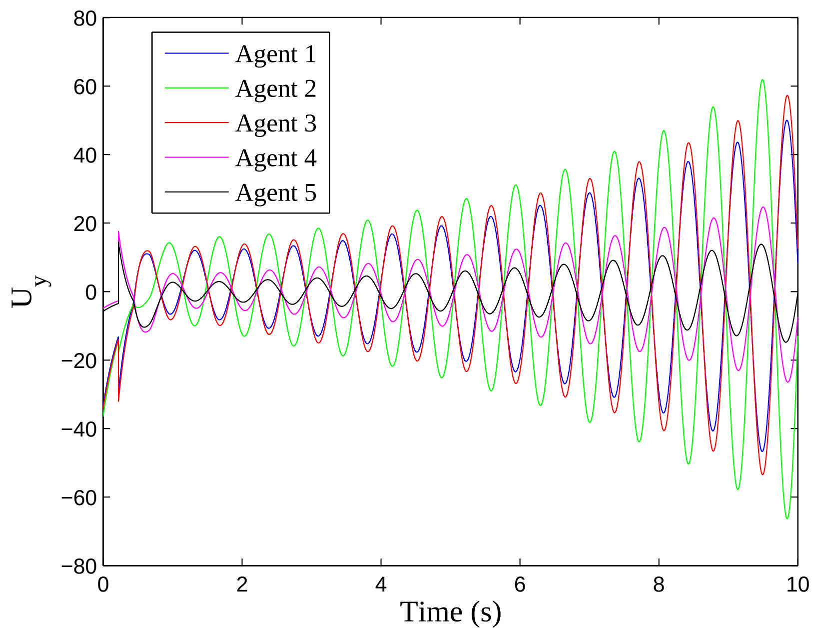

Figure 3.21: Control effort for agents in y direction - Excess delay

With the same initial conditions as considered in these plots the variation of the cost J in (3.8) with increasing delay τ is shown in Figure 3.22.

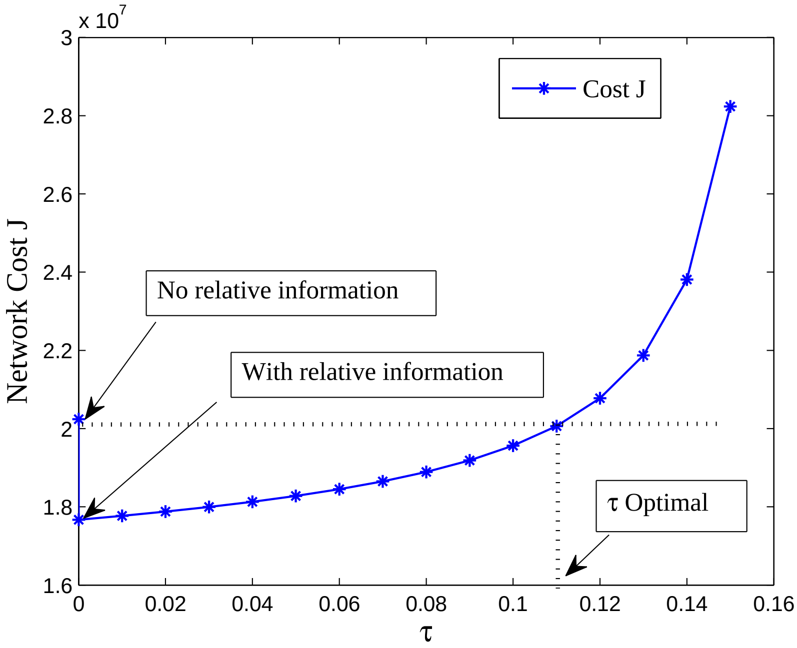

Figure 3.22: The cost J as a function of delay τ

The dotted line in the plot shows the cost J when no network level control is employed, i.e. relative information is not used to stabilize the system. The cost J using relative information is lower than this line in Figure 3.22 for delay $\tau = 0s$. The cost J using relative information remains lower up to the delay of $\tau = 0.11s$ and the cost is greater than the cost without relative information beyond $\tau = 0.11s$. This shows that using delayed relative information in the distributed control law (3.7) is advantageous in the presence of delay less than $\tau = 0.11s$. This also implies that when delays higher than $\tau = 0.11s$ are expected in the system, it is better to use just the node level control to stabilize the system.

3.6.6 Further Analysis

In this section an analysis of the effects of the weighting matrix Q_2 and the effect of the communication topology $\mathscr{L}$ is presented.

3.6.6.1 Effect of Q_2

The cost function J considered in (3.8) is not a standard cost function. The weighting matrix Q_1 is used to solve a standard LQR problem in Step 1 of the control design process in Section 3.3. This matrix must be selected by the user according to required stabilization of a single agent in Step 1. The augmented LQR cost function in (3.8) considers a weighting matrix along with the Laplacian $\mathscr{L}$ to facilitate simultaneous convergence. In this analysis the effect of the weighting matrix Q_2 for a given weighting matrix Q_1 and a given Laplacian $\mathscr{L}$ is studied. The matrix Q_1 is considered to be $Q_1 = 10I_4$ and the network topology is given by (3.121), i.e. the same matrices considered in Section 3.6.4 and Section 3.6.5 while the matrix Q_2 was considered to be $Q_2 = 25I_4$ in these sections. It is obvious to see that increasing the matrix Q_2 will penalize the relative information, as explained in Remark 3.1, to a greater extent. The effects of increasing the magnitude of the weights in Q_2 are give in Table 3.4.

Table 3.4: Effects of Weighting Matrix Q_2

Q_2	Φ	$\hat{\tau}_{max}$(sec)	$Trace(P_s)$ (3.132)
$25I_4$	$0.6736I_2$	0.20	759.2
$50I_4$	$1.0805I_2$	0.10	1319.0
$75I_4$	$1.4126I_2$	0.08	1869.5

Hence it can be observed that with increasing magnitude of the weights in Q_2, the magnitude of the scaling Φ increases thus increasing the contribution from the relative information to the control law (3.7). This also decreases the ability to accept more delays in communication of relative information as shown by the decrease in $\hat{\tau}_{max}$(sec). The cost J increases with Q_2 as demonstrated by the optimal cost associated with $Trace(P_s)$ from (3.132) in Table 3.4.

3.6.6.2 Effect of Communication Topology

In this section the effect of different communication topologies, i.e. different graph Laplacian $\mathscr{L}$, is presented. For this analysis the same weighting matrices Q_1, Q_2 and R as considered in Section 3.6.5 are considered, i.e. $Q_1 = 10I_4$ and $Q_2 = 25I_4$. Increasing the communication links in Figure 3.1, the topology given in Figure 3.23 is considered. The Laplacian $\mathscr{L}$ for the topology in Figure 3.23 is given by

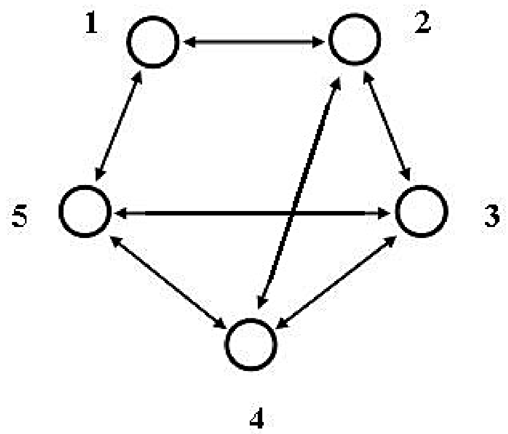

Figure 3.23: Dense communication topology

$$\mathscr{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix} \tag{3.134}$$

Decreasing the communication links in Figure 3.1, the topology given in Figure 3.24 is considered The Laplacian $\mathscr{L}$ for the topology in Figure 3.24 is given by

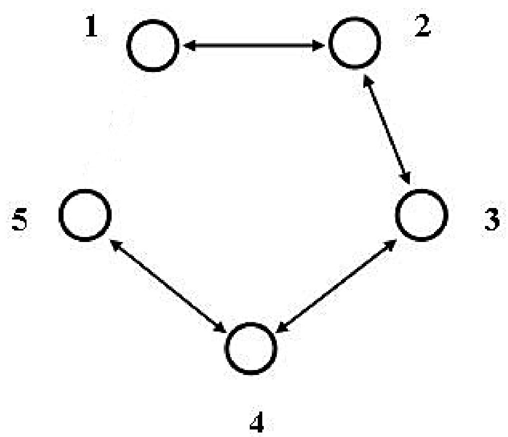

Figure 3.24: Acyclic nearest neighbour topology

$$
\mathscr{L} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3.135}
$$

Note that the topology in Figure 3.24 is still connected. Both the topologies in (3.134) and (3.135) do not yield any significant differences in attaining a rendezvous as compared to the Figure 3.2 in Section 3.6.5 for the topology in (3.121) with same initial conditions but with the changes in the Laplacian $\mathscr{L}$ there are significant implications on the amount of delay that is permissible by each topology. A detailed result is given Table 3.5

Table 3.5: Effects of Communication Topology

Topology	Φ	$\hat{\tau}_{max}$(sec)	$Trace(P_s)$ (3.132)
$\mathscr{L}(3.121)$	$0.6736I_2$	0.20	759.2
$\mathscr{L}(3.134)$	$0.6136I_2$	0.15	985.8
$\mathscr{L}(3.135)$	$0.6591I_2$	0.20	642.7

From Table 3.5 it can be concluded that the method proposed in Section 3.3 yields higher delay carrying capacity and lower cost when the communication topology is connected but has lower communication links, i.e. the Laplacian $\mathscr{L}$ in (3.135), for the same number of agents.

3.7 Concluding Remarks

In this chapter a two steps distributed control design methodology for a network of linear identical dynamical systems was proposed. In this methodology the relative information available to each agent is used to design distributed control laws such that the agents achieve a rendezvous. A certain level of LQR performance at a network level is guaranteed by the proposed control laws. The proposed control law was illustrated with the help of an example where the agents where described by double integrator dynamics describing motion in a two dimensional plane. The method proposed in this chapter was compared with the method proposed in [96]. The two methods are shown to have similar performance.

A bound on the maximum possible communication delay is obtained by an analysis of the proposed control laws in the presence of delays in relative information. In the time-delay analysis, both fixed and time-varying delays are considered. The three methods given by *Proposition 4*, *Proposition 5*, and *Proposition 6* in Appendix B were employed for the time-delay analysis. For the example of agents described by double integrator dynamics considered, the results in Table 3.3 show that *Proposition 4* for the case of fixed delays provides a conservative estimate of the bound on the maximum possible delay. The discretized Lyapunov-Krasovskii functional method from *Proposition 5* provides a more accurate estimate of the bound on the amount of delay. The bound on the amount of delay obtained by *Proposition 6* for time-varying delays is equal to the bound obtained from *Proposition 5*. The number of decision variables employed by *Proposition 4* is considerably less and hence converges faster as compared to the *Proposition 5* and *Proposition 6*. It was also shown that the use of relative information is advantageous in lowering a network level cost

Chapter 4

Delay Based Control of Multi-Agent Systems

4.1 Introduction

In the previous chapter, a distributed control design method for a network of identical agents with sub-optimal LQR performance was presented. The controller was obtained by assuming instantaneous relative information was available at each node, and subsequently time-delay analysis techniques were employed to ascertain the maximum bound that can be accommodated by the network. However in many practical scenarios, communication of relative information is bound to incur a minimum level of delay. This necessitates modelling the network of identical agents as a time-delay system and then employing control design techniques for time-delay systems to design control algorithms for the network. In this chapter the assumption of instantaneous exchange of relative information in the control design process is relaxed, and delay independent and delay dependent control laws for a network of identical agents, guaranteeing an LQR performance, are derived. Lyapunov-Krasovskii functional methods from Appendix B are employed to obtain control laws which make use of the delayed relative state information available at each node.

The organization of this chapter is as follows: In Section 4.2 the problem definition is stated. The linear system model for a network of identical agents, and the control law is explained. In Section 4.3 various control design procedures used to obtain the delay independent and delay dependent gains are explained. In Section 4.4 the controllers designed using the different control design techniques are illustrated with the help of numerical examples. Concluding remarks are given in Section 4.5.

4.2 Problem Definition

In this section a linear system model for a network of identical linear agents is described. The structure of the control law to be designed is then presented. The system is the represented at a network level using Kronecker products. The network level system is then decomposed into node level systems by use of spectral decomposition to aid the control design process.

4.2.1 Linear System Model

As in Chapter 3 a network of N identical linear systems given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{4.1}$$

for $i = 1, \ldots, N$ is considered, where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the states and the control inputs. The constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ and it is assumed that the pair (A, B) is controllable. Each agent is assumed to have knowledge of its local state information along with delayed relative state information. The information exchange between the agents in the network is assumed to be bidirectional. The relative information communicated to each agent (node) is given by

$$z_i(t-\tau) = \sum_{j\in\mathscr{J}_i} (x_i(t-\tau) - x_j(t-\tau)) \tag{4.2}$$

where τ is a delay in communication of relative information. The dynamical systems for which the i^{th} dynamical system has information is denoted by $\mathscr{J}_i \subset \{1, 2, \ldots N\}/\{i\}$.

Using Kronecker products, the system in (4.1) at a network level is given by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{4.3}$$

where the augmented state $X(t) = \mathscr{C}ol(x_1(t), \ldots, x_N(t))$ and $U(t) = \mathscr{C}ol(u_1(t), \ldots, u_N(t))$. The relative information in (4.2) at a network level can be written as

$$Z(t-\tau) = (\mathscr{L} \otimes I_n)X(t-\tau) \tag{4.4}$$

where $\mathscr{L}$ is the Laplacian matrix associated with the sets $\mathscr{J}_i$.

4.2.2 Control Objective

For the exchange of relative information given in (4.4) with respect to the delay τ, two cases are considered. They are stated as follows:

- a known fixed delay
- a bounded time-varying delay with a known maximum bound

The intention is to design control laws of the form

$$u_i(t) = -Kx_i(t) - Hz_i(t-\tau) \tag{4.5}$$

where $K \in \mathbb{R}^{m \times n}$ is designed to achieve consensus and $H \in \mathbb{R}^{m \times n}$, the relative information scaling matrix, is fixed a priori. The matrix H is assumed to be the scaling associated with the measurement of relative information. A possible way of selecting the scaling matrix H is explained with a numerical example in Section 4.4. Substituting (4.5) in (4.1), the closed loop system at a node level is given by

$$\dot{x}_i(t) = (A - BK)x_i(t) - BHz_i(t-\tau) \tag{4.6}$$

The control law in (4.5) is represented at a network level by

$$U(t) = -(I_N \otimes K)X(t) - (\mathscr{L} \otimes H)X(t-\tau) \tag{4.7}$$

Substituting (4.7) into (4.3), the closed loop system at a network level is given by

$$\dot{X}(t) = (I_N \otimes (A - BK))X(t) - (\mathscr{L} \otimes BH)X(t-\tau) \tag{4.8}$$

Since bidirectional communication is assumed, the Laplacian $\mathscr{L}$ for the communication topology is symmetric positive semi-definite. By spectral decomposition $\mathscr{L} = V\Lambda V^T$ where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathscr{L}$ and $\Lambda = \mathscr{D}iag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $\mathscr{L}$. An orthogonal state transformation

$$X \mapsto (V^T \otimes I_n)X = \tilde{X} \tag{4.9}$$

is employed on the system in (4.8). The closed loop system (4.8) in the new coordinates is given by

$$\dot{\tilde{X}}(t) = (I_N \otimes (A - BK))\tilde{X}(t) - (\Lambda \otimes BH)\tilde{X}(t-\tau) \tag{4.10}$$

Because Λ is a diagonal matrix, the system in (4.10) is equivalent to

$$\dot{\tilde{x}}_l(t) = A_0\tilde{x}_l(t) + A_i\tilde{x}_l(t-\tau) \tag{4.11}$$

for $l = 1, \ldots, N$ where

$$A_0 := A - BK \tag{4.12}$$

$$A_l := -\lambda_l BH \tag{4.13}$$

It is assumed the initial condition $\tilde{x}_l(\theta) = \tilde{x}_l(0)$ for $l = 1, \ldots, N$ and $-\tau \leq \theta \leq 0$. The transformed system in (4.11) can be equivalently thought of as

$$\dot{\tilde{x}}_l(t) = A\tilde{x}_l(t) - \lambda_l BH\tilde{x}_l(t-\tau) + Bu_l(t) \tag{4.14}$$

where

$$u_l(t) = -K\tilde{x}_l(t) \tag{4.15}$$

The objective is to design the gain matrix K under the following scenarios

- a delay independent design for a fixed delay τ
- a delay dependent design for a known fixed delay τ
- a delay dependent design for bounded time-varying delays $\tau(t)$

In the above cases a suboptimal level of LQR performance is enforced on the overall system at network level.

Remark 4.1: The stabilization of linear systems with delays, with the structure given in (4.14), has been studied extensively in the control literature. Various stability analysis and control design methods have been proposed. In [120, 121] a descriptor representation along with Lyapunov-Krasovskii functionals are used to obtain stability criteria for linear time-delay systems. In [122] a Lyapunov-Krasovskii functional approach based on the fractioning of the delay is proposed for linear time-delay systems. In [123] bounds on the derivative of delays are considered to derive delay dependent stability criteria. Most recent methods involve establishing LMI feasibility problems (with varying levels of complexity in terms of the number of decision variables). The reader is referred to [124]-[128] for further reading in this area. In this chapter the systems in (4.14) are stabilized simultaneously in the presence of delays while guaranteeing an LQR performance. This is achieved by building on the existing analysis techniques [120, 129]. The techniques in [120, 129], whilst not necessarily the most recent in the literature, have been found to yield tractable LMI representations under certain mild simplifications. This is important because of the large number of decision variables involved, resulting from the multiple agents considered. The results in [120, 129] are shown to provide a good trade-off between unnecessary conservatism and tractability of LMI formulations.

Remark 4.2: In this chapter the same definition of consensus, i.e. convergence to a common value and as in Chapter 3 achieving a rendezvous of the agents by driving the states to the origin, is considered. In this chapter the design of control laws of the form in (4.5) are considered as compared to the controls laws in (3.7) in Chapter 3. In Chapter 3 the control design procedure does not consider delays and an analysis to obtain the maximum possible delay is performed after the control laws are obtained. In this chapter control design methods with LQR performance are obtained when delays are present in the relative information. Though the objective of attaining a rendezvous is the same in this chapter, there is a change in the structure of the control laws in (4.5) as compared to that in (3.7) in Chapter 3. Here the scaling matrix H in (4.5) is assumed to be fixed. From an engineering perspective this can be considered as the scaling provided by the measurement of relative information z_i in (4.2). The mathematical reasons behind this assumptions will be presented after the three design methods in the following section.

4.3 Control Design Methodologies

In this section the three control design methodologies are explained in detail. In the control design methodologies Lyapunov-Krasovskii functional based methods are employed for controller synthesis.

4.3.1 Delay Independent Control Design for a Fixed Delay

Delay independent stability implies that a system is stable for any arbitrary delay. Essentially such a control design provides an insight into the control gains necessary to make the delayed terms in a linear system redundant. In [129] it is argued that when the feedback channel involves delays, delay independent stability criteria are insufficient. For the equivalent system described in (4.14), the feedback channel where the local controller $u_l(t)$ acts, does not involve delays and hence a delay independent control design can be employed for the system in (4.11). In this design methodology the focus is on obtaining the gain matrix K such that the delayed term in (4.14) does not affect the stability of the system for any arbitrary delay. In this section the objective of the delay independent control is the design of the gain matrix K for the system in (4.14) such that the cost functions

$$J_l = \int_0^\infty (\tilde{x}_l^T(t)Q\tilde{x}_l(t) + u_l^T(t)Ru_l(t))dt \tag{4.16}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, are minimized for all $l = 1,\ldots,N$. In (4.16) it is assumed that both Q and R are symmetric positive definite. The justification for using the cost function in (4.16) is stated in Remark 4.8 after Section 4.3.3. Similar cost functions have been used for the design methods in this chapter.

Theorem 4.3.1. *For any arbitrary delay τ, a given scaling matrix $H \in \mathbb{R}^{m\times n}$, selected weighting matrices Q and R, the control laws in (4.15) simultaneously stabilize the transformed systems in (4.14) if there exist symmetric matrices $Z_1 > 0$, $Z_2 > 0$, $S > 0$, and $W > 0 \in \mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{m\times n}$ such the following LMI conditions are satisfied*

$$\begin{pmatrix} -Z_1 & I_n \\ * & -W \end{pmatrix} < 0 \tag{4.17}$$

$$\begin{pmatrix} -Z_2 & I_n \\ * & -S^{-1} \end{pmatrix} < 0 \tag{4.18}$$

$$\begin{pmatrix} \Phi_l & W & WQ^{1/2} & Y^T \\ * & -S^{-1} & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & -R^{-1} \end{pmatrix} < 0 \tag{4.19}$$

where

$$\Phi_l = (AW - BY) + (AW - BY)^T + A_l S^{-1} A_l^T \tag{4.20}$$

for all $l = 1,\dots,N$. The state feedback gain matrix is then given by $K = YW^{-1}$. Furthermore since the cost functions J_l from (4.16) satisfy

$$J_l < \tilde{x}_l^T(0)(Z_1 + \tau Z_2)\tilde{x}_l(0) \tag{4.21}$$

minimizing $Trace(Z_1 + \tau Z_2)$ subject to (4.17)-(4.19) minimizes a bound on the LQR cost.

Proof. In this section the ideas based on *Proposition 2* in Appendix B.2 are used to design delay independent control laws. As in [129], the Lyapunov-Krasovskii functionals given by

$$V_l(\tilde{x}_l(t)) = \tilde{x}_l^T(t)P\tilde{x}_l(t) + \int_{t-\tau}^{t} \tilde{x}_l^T(s)S\tilde{x}_l(s)ds \tag{4.22}$$

for all $i = 1,\dots,N$, where the matrices $P > 0$, $S > 0$ and P and $S \in \mathbb{R}^{n\times n}$ are considered.

Differentiating (4.22) it is required that

$$\dot{\tilde{V}}_l + \frac{d}{dt}\int_0^t (\tilde{x}_l^T(t)Q\tilde{x}_l(t) + u_l^T(t)Ru_l(t))dt < 0 \tag{4.23}$$

Substituting (4.22) in (4.23) yields

$$\begin{aligned}\tilde{x}_l^T(t)(PA_0 + A_0^T P + S)\tilde{x}_l(t) + \tilde{x}_l^T(t)(PA_l)\tilde{x}_l(t-\tau) + \tilde{x}_l^T(t-\tau)(A_l^T P)\tilde{x}_l(t)\\ -\tilde{x}_l^T(t-\tau)S\tilde{x}_l(t-\tau) < -\tilde{x}_l^T(t)Q\tilde{x}_l(t) - u_l^T(t)Ru_l(t)\end{aligned} \tag{4.24}$$

for $l = 1, \ldots, N$ where $u_l(t) = -K\tilde{x}_l(t)$. The inequality (4.24) is equivalent to

$$P > 0 \tag{4.25}$$

$$\begin{pmatrix} PA_0 + A_0^T P + S + Q + K^T RK & PA_l \\ * & -S \end{pmatrix} < 0 \tag{4.26}$$

The matrix inequality in (4.26) is not a convex representation with respect to the matrix variables P, S and K. To develop a convex representation the matrix $W = P^{-1}$ is defined. By pre and post multiplying (4.26) by $\mathscr{D}iag(W, I_n)$, the inequality in (4.26) is equivalent to

$$\begin{pmatrix} \tilde{\Phi} & A_l \\ * & -S \end{pmatrix} < 0 \tag{4.27}$$

for all $l = 1, \ldots, N$ where

$$\tilde{\Phi} = A_0 W + W A_0^T + WSW + WQW + WK^T RKW \tag{4.28}$$

From the Schur complement, inequality (4.27) is equivalent to $S > 0$ and

$$A_0 W + W A_0^T + WSW + WQW + WK^T RKW + A_l S^{-1} A_l^T < 0 \tag{4.29}$$

Subsequently, auxiliary symmetric matrices are defined by Z_1 and $Z_2 \in \mathbb{R}^{n\times n}$ and the change of decision variables $KW = Y$ where $Y \in \mathbb{R}^{m\times n}$ is employed. Using the Schur complement, the inequality (4.29) is equivalent to (4.19). Since $P > 0$, $W = P^{-1}$ implies

$W > 0$. The inequalities given by (4.17) - (4.19) are LMIs in the variables W, Z_1, Z_2, S^{-1} and Y. The required gain matrix is subsequently recovered as $K = YW^{-1}$ after solving the inequalities in (4.17) - (4.19). The inequality in (4.24) is equivalent to

$$\dot{V}_l(\tilde{x}_l(t)) < -\tilde{x}_l^T(t)Q\tilde{x}_l(t) - u_l^T(t)Ru_l(t) \tag{4.30}$$

for $l = 1, \ldots, N$. Integrating both sides of (4.30) from 0 to ∞ yields

$$-\tilde{x}_l^T(0)P\tilde{x}_l(0) - \int_{-\tau}^{0} \tilde{x}_l^T(s)S\tilde{x}_l(s)ds < -J_l \tag{4.31}$$

Here it is assumed that $\tilde{x}_l(\theta) = \tilde{x}_l(0)$ for $l = 1, \ldots, N$ and $-\tau \leq \theta \leq 0$. Consequently the integral in (4.31) can be calculated explicitly and an upper bound on J_l is given by

$$\begin{aligned} J_l &< \tilde{x}_l^T(0)P\tilde{x}_l(0) + \tau\tilde{x}_l^T(0)S\tilde{x}_l(0) \\ &= \tilde{x}_l^T(0)(P + \tau S)\tilde{x}_l(0) \end{aligned} \tag{4.32}$$

Using the Schur complement, inequalities (4.17) and (4.18) imply $Z_1 > P$ and $Z_2 > S$. Hence the inequality (4.21) is satisfied. Since P and S are symmetric positive definite the following minimization problem constitutes minimizing an upper bound on the LQR cost at a network level

$\boxed{\text{Minimize } Trace(Z_1 + \tau Z_2) \text{ subject to (4.17)-(4.19)}}$ $\square$

4.3.2 Delay Dependent Control Design for a Fixed Delay

Prior to stating the objective of delay dependent control design, an explicit model transformation from [129] for the system in (4.11) is first performed. For the system given in (4.11), the following observation holds

$$\begin{aligned} \tilde{x}_l(t-\tau) &= \tilde{x}_l(t) - \int_{t-\tau}^{t} \dot{\tilde{x}}_l(\theta)d\theta \\ &= \tilde{x}_l(t) - \int_{t-\tau}^{t} (A_0\tilde{x}_l(\theta) + A_l\tilde{x}_l(\theta - \tau))\, d\theta \end{aligned} \tag{4.33}$$

for $t \geq \tau$. Using (4.33) the system in (4.11) is represented as

$$\dot{\tilde{x}}_l(t) = (A_0 + A_l)\tilde{x}_l(t) + \int_{t-\tau}^{t} (-A_l A_0 \tilde{x}_l(\theta) - A_l A_l \tilde{x}_l(\theta - \tau))d\theta \tag{4.34}$$

for all $l = 1, \ldots, N$. As argued in [129], the system in (4.34) is transformed, by shifting the time axis and lifting the initial conditions, into the system

$$\dot{y}_l(t) = \bar{A}_{0l} y_l(t) + \int_{t-2\tau}^{t} \bar{A}_l(\theta) y_l(\theta) d\theta \tag{4.35}$$

where

$$\bar{A}_{0l} = A_0 + A_l \tag{4.36}$$

$$\bar{A}_l(\theta) := \begin{cases} -A_l A_0 & \theta \in [t-\tau, t] \\ -A_l A_l & \theta \in [t-2\tau, t-\tau) \end{cases} \tag{4.37}$$

with the new initial condition $y(\theta) = \phi(\theta)$ for $-2\tau \leq \theta \leq 0$. According to [129], stability of (4.35) implies stability of (4.11) but not vice-versa.

Remark 4.3: In [129] it is shown that the transformed system in (4.35) has all the poles of the original system in (4.14) plus additional poles due to the transformation. Consequently stability of the transformed system implies stability of the original system but not vice-versa, due to the lifting of the initial conditions of the original system. In this method, LQR control design has been employed on the transformed system. This will also guarantee a level of performance for the original system in (4.14).

The control design objective for delay dependent control design can now be stated as the design of gain matrix K for the systems in (4.35) such that the cost functions

$$J_l = \int_0^{\infty} (y_l^T(t) Q y_l(t) + u_l^T(t) R u_l(t)) dt \tag{4.38}$$

are minimized for all $l = 1, \ldots, N$, where

$$u_l(t) = -K y_l(t) \tag{4.39}$$

and $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices.

Remark 4.4: Since the systems in (4.35) are obtained form the systems in (4.14) with the control law in (4.15). The gain matrix K, the scaling matrix H and the control law in (4.39) are contained in the systems in (4.35) by the matrices $A_0 = A - BK$ and $A_l = \lambda_l BH$ for $l = 1,\ldots,N$.

Theorem 4.3.2. *For a known fixed delay τ, a given scaling matrix $H \in \mathbb{R}^{m\times n}$, selected weighting matrices Q and R and chosen scalars α_0 and α_1, the control laws in (4.39) simultaneously stabilize the transformed systems in (4.35) if there exist symmetric matrices $\bar{Z} > 0$ and $W > 0 \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ such that the following LMI conditions are satisfied*

$$\begin{pmatrix} -\bar{Z} & I_n \\ * & -W \end{pmatrix} < 0 \tag{4.40}$$

$$\begin{pmatrix} \bar{\Phi}_l & -A_lAW + A_lBY & -A_l^2W & \frac{WQ^{1/2}}{\tau} & \frac{Y^T}{\tau} \\ * & -\alpha_0 W & 0 & 0 & 0 \\ * & * & -\alpha_1 W & 0 & 0 \\ * & * & * & -\frac{1}{\tau} & 0 \\ * & * & * & * & -\frac{R^{-1}}{\tau} \end{pmatrix} < 0 \tag{4.41}$$

where

$$\bar{\Phi}_l = \frac{1}{\tau}(((A + A_l)W - BY) + ((A + A_l)W - BY)^T) + (\alpha_0 + \alpha_1)W \tag{4.42}$$

for all $l = 1,\ldots,N$. The state feedback gain matrix is then given by $K = YW^{-1}$. Furthermore since the J_l from (4.38) satisfy

$$J_l < y_l^T(0)\bar{Z}y_l(0)(1 + \frac{1}{2}\alpha_0\tau^2 + \frac{3}{2}\alpha_1\tau^2) \tag{4.43}$$

minimizing $Trace(\bar{Z})$ subject to (4.40)-(4.41) minimizes a bound on the LQR cost.

Proof. This proof uses a restricted Lyapunov-Krasovskii functional as suggested in *Proposition 3* from Appendix B.3.1 and [129]. For the system in (4.35) consider a Lyapunov-

Krasovskii functional of the form

$$V_l(y(t)) = y_l^T(t)Py_l(t) + \int_{t-2\tau}^{t}\int_{\theta}^{t}\alpha(\theta)y_l^T(s)Py_l(s)dsd\theta \tag{4.44}$$

where $P > 0$ and $P \in \mathbb{R}^{n\times n}$ for all $l = 1,\ldots,N$. The restricted Lyapunov-Krasovskii function in (4.44) is considered so as to facilitate the derivation of LMI constraints by employing the inverse of the Lyapunov matrix P and follow the procedure similar to that in Theorem 4.3.1. The reader is referred to *Proposition 3* from Appendix B.3.1 for the unrestricted form of the Lyapunov-Krasovskii functional. In (4.44) $\alpha(\theta) > 0$ is a positive scalar function defined over the interval $t-2\tau \leq \theta \leq t$. Differentiating (4.44) it is required that

$$\dot{\hat{V}}_l + \frac{d}{dt}\int_{0}^{\infty}(y_l^T(t)Qy_l(t) + u_l^T(t)Ru_l(t))dt < 0 \tag{4.45}$$

Substituting (4.44) in (4.45) yields

$$y_l^T(t)(P\bar{A}_{0i} + \bar{A}_{0i}^T P + \int_{t-2\tau}^{t}\alpha(\theta)Pd\theta)y_l(t) + 2y_l^T(t)\int_{t-2\tau}^{t}P\bar{A}_l(\theta)y_l(\theta)d\theta$$

$$-\int_{t-2\tau}^{t}\alpha(\theta)y_l^T(\theta)Py_l(\theta)d\theta < -y_l^T(t)Qy_l(t) - u_l^T(t)Ru_l(t) \tag{4.46}$$

By adding and subtracting terms $\int_{t-2\tau}^{t} y_l^T(t)M(\theta)y_l(t)d\theta$, involving a symmetric matrix function $M(\theta) \in \mathbb{R}^{n\times n}$, the inequality in (4.46) is equivalent to

$$y_l^T(t)(P\bar{A}_{0l} + A_{0i}^T P + Q + K^T RK + \int_{t-2\tau}^{t}M(\theta)d\theta)y_l(t)$$

$$+\int_{t-2\tau}^{t}\bar{y}_l^T\begin{pmatrix}\alpha(\theta)P - M(\theta) & P\bar{A}_l(\theta)\\ * & -\alpha(\theta)P\end{pmatrix}\bar{y}_l d\theta < 0 \tag{4.47}$$

where

$$\bar{y}_l^T = \begin{pmatrix} y_l^T(t) & y_l^T(\theta)\end{pmatrix} \tag{4.48}$$

Define the symmetric matrix function $M(\theta)$ as

$$M(\theta) := \begin{cases} M_0 & t-\tau \leq \theta \leq t \\ M_1 & t-2\tau \leq \theta < t-\tau \end{cases} \tag{4.49}$$

where M_0 and M_1 are symmetric matrices $\in \mathbb{R}^{n\times n}$ and the scalar function $\alpha(\theta)$ as

$$\alpha(\theta) := \begin{cases} \alpha_0 & t-\tau \leq \theta \leq t \\ \alpha_1 & t-2\tau \leq \theta < t-\tau \end{cases} \tag{4.50}$$

where $\alpha_0 > 0$ and $\alpha_1 > 0$. Then as argued in [129] inequality in (4.47) is satisfied for $P > 0$ and

$$P(A_0+A_l)+(A_0+A_l)^T P+Q+K^T RK+\tau(M_0+M_1) < 0 \tag{4.51}$$

$$\begin{pmatrix} \alpha_0 P - M_0 & -PA_lA_0 \\ * & -\alpha_0 P \end{pmatrix} < 0 \tag{4.52}$$

$$\begin{pmatrix} \alpha_1 P - M_1 & -PA_lA_l \\ * & -\alpha_1 P \end{pmatrix} < 0 \tag{4.53}$$

It can now be stated that the introduction of the matrix function $M(\theta)$ in (4.47) facilitates the representation of the matrix inequalities (4.51)-(4.53). Using the Schur complement and eliminating M_0 and M_1 the inequalities in (4.51)-(4.53) are satisfied if

$$\begin{pmatrix} \Phi_l & -PA_lA_0 & -PA_lA_l \\ * & -\alpha_0 P & 0 \\ * & * & -\alpha_1 P \end{pmatrix} < 0 \tag{4.54}$$

where

$$\Phi_l = \frac{1}{\tau}(P(A_0+A_l)+(A_0+A_l)^T P+Q+K^T RK)+(\alpha_0+\alpha_1)P \tag{4.55}$$

and $A_0 = A-BK$ for all $l = 1,\ldots,N$. To develop a convex representation define $W = P^{-1}$.

Pre and post multiplying (4.54) by $\mathscr{D}iag(W,W,W)$ means (4.54) is equivalent to

$$\begin{pmatrix} \hat{\Phi}_l & -A_lA_0W & -A_lA_lW \\ * & -\alpha_0 W & 0 \\ * & * & -\alpha_1 W \end{pmatrix} < 0 \tag{4.56}$$

where

$$\hat{\Phi}_l = \frac{1}{\tau}((A_0+A_l)W + W(A_0+A_l)^T + WQW) + \frac{1}{\tau}(WK^TRKW) + (\alpha_0+\alpha_1)W \tag{4.57}$$

for all $l = 1,\ldots,N$. Define an auxiliary symmetric matrix $\bar{Z} \in \mathbb{R}^{n\times n}$ and employ the change of decision variables $KW = Y$ where $Y \in \mathbb{R}^{m\times n}$. From applying the Schur complement to (4.56), the inequalities in (4.51)-(4.53) become the LMI stated in the theorem statement in (4.41). Inequality (4.46) is equivalent to

$$\dot{V}_l(y_l(t)) < -y_l^T(t)Qy_l(t) - u_l^T(t)Ru_l(t) \tag{4.58}$$

and integrating both sides of (4.58) from 0 to ∞ yields

$$-y_l^T(0)Py_l(0) - \int_{-2\tau}^{0}\int_{\theta}^{0}\alpha(\theta)y_l^T(s)Py_l(s)dsd\theta < -J_l \tag{4.59}$$

Assuming initial conditions $y_l(s) = y_l(0)$ for $s < 0$, an upper bound for J_l is given by

$$J_l < y_l^T(0)Py_l(0)(1+\frac{1}{2}\alpha_0\tau^2 + \frac{3}{2}\alpha_1\tau^2) \tag{4.60}$$

by explicitly evaluating the integral on the L.H.S of (4.59). Minimization of $Trace(P)$ ensures minimizing an upper bound on the costs J_l for all $l = 1,\ldots,N$. From (4.40) it can be shown by Schur complement that $\bar{Z} > W^{-1} = P$. Hence the condition in (4.43) is satisfied. Consequently the following optimization problem ensures minimization of an upper bound on the LQR cost in (4.60).

$\boxed{\text{Minimize } Trace(\bar{Z}) \text{ subject to (4.40)-(4.41)}}$ $\square$

Remark 4.5: The matrix $A_l = \lambda_l BH$ for $l = 1,\ldots,N$ is contained in the inequality given by (4.41). Note that the matrix H and the scalars α_0 and α_1 are fixed in (4.41) which renders the matrices A_l fixed for all $l = 1, \ldots, N$, and hence (4.41) is an LMI.

4.3.3 Delay Dependent Control design for Time Varying Delays

In the previous sections state feedback control laws were designed based on the assumption of relative information having a fixed delay. Assuming fixed delays in a network is somewhat idealistic, and hence a need arises for control design involving time-varying delays. In [120], a control design methodology has been presented for time varying delay where the delay $\tau(t)$ is a bounded continuous function satisfying $0 \leq \tau(t) \leq \tau_m$ for $t \geq 0$ where τ_m is known. The equivalent systems to (4.14) with time varying delays $\tau(t)$ are given by

$$\dot{\tilde{x}}_l(t) = A\tilde{x}_l(t) - \lambda_l BH\tilde{x}_l(t - \tau(t)) + Bu_l(t) \tag{4.61}$$

for $l = 1, \ldots, N$. Again the objective is to minimize a cost function of the form

$$J_l = \int_0^\infty (\tilde{x}_l^T(t)Q\tilde{x}_l(t) + u_l^T(t)Ru_l(t))dt \tag{4.62}$$

where

$$u_l(t) = -K\tilde{x}_l(t) \tag{4.63}$$

for all $l = 1, \ldots, N$. The control law (4.63) when substituted in (4.61) yields

$$\dot{\tilde{x}}_l(t) = A_0\tilde{x}_l(t) + A_l\tilde{x}_l(t - \tau(t)) \tag{4.64}$$

where

$$A_0 := A - BK \tag{4.65}$$

$$A_l := -\lambda_l BH \tag{4.66}$$

Theorem 4.3.3. *Assume the bound on the delay τ_m is known, then for a given scaling matrix H from (4.61) and given weighting matrices Q and R, the control laws in (4.63) simultaneously stabilize the systems in (4.64) if there exist symmetric matrices $W_1 > 0$,*

$\tilde{Z} > 0$, $\bar{F}_1 > 0$, $\bar{F}_3 > 0$, $\bar{S} > 0 \in \mathbb{R}^{n\times n}$ and the matrices W_2, W_3, $\bar{F}_2 \in \mathbb{R}^{n\times n}$ such that the following LMI conditions are satisfied:

$$\begin{pmatrix} -\tilde{Z} & I_n \\ I_n & -W_1 \end{pmatrix} < 0 \tag{4.67}$$

$$\begin{pmatrix} \bar{S} & 0_n & \bar{S}A_l^T \\ * & \bar{F}_1 & \bar{F}_2 \\ * & * & \bar{F}_3 \end{pmatrix} > 0 \tag{4.68}$$

$$\begin{pmatrix} W_2 + W_2^T + \tau_m \bar{F}_1 & \Phi_l & \tau_m W_2^T & W_1 Q^{1/2} & Y^T \\ * & -W_3 - W_3^T + \tau_m \bar{F}_3 & \tau_m W_3^T & 0 & 0 \\ * & * & -\tau_m \bar{S} & 0 & 0 \\ * & * & * & -I_n & 0 \\ * & * & * & * & R^{-1} \end{pmatrix} < 0 \tag{4.69}$$

where

$$\Phi_l = W_3 - W_2^T + W_1(A + A_l) + \tau_m \bar{F}_2 - Y^T B^T \tag{4.70}$$

for all $l = 1, \dots, N$. The state feedback gain matrix is then given by

$$K = YW_1^{-1} \tag{4.71}$$

Furthermore

$$J_l < \tilde{x}_l^T(0)\tilde{Z}\tilde{x}_l(0) \tag{4.72}$$

and so minimizing $Trace(\tilde{Z})$ subject to (4.67)-(4.69) minimizes a bound on the LQR cost.

Proof. This proof uses concepts from *Proposition 6* from Appendix B.5 to design control laws for the system in (4.61) with a certain level of performance. As in [120] represent the system in (4.64) as a descriptor system[1] given by

$$\begin{aligned}\dot{\tilde{x}}_l(t) &= \tilde{y}_l(t) &(4.73)\\ 0 &= -\tilde{y}_l(t)+(A_0+A_l)\tilde{x}(t)-A_l\int_{t-\tau(t)}^{t}\tilde{y}_l(s)ds &(4.74)\end{aligned}$$

for $l=1,\dots,N$. The system in (4.73)-(4.74) can be written as

$$\begin{aligned}E\dot{\bar{x}}_l(t) &= \begin{pmatrix}\dot{\tilde{x}}_l(t)\\ 0_n\end{pmatrix} &(4.75)\\ &= \tilde{A}_{0l}\bar{x}_l(t)-\begin{pmatrix}0_n\\ A_l\end{pmatrix}\int_{t-\tau(t)}^{t}\tilde{y}_l(s)ds &(4.76)\end{aligned}$$

where

$$\tilde{A}_{0i} = \begin{pmatrix}0_n & I_n\\ (A_0+A_l) & -I_n\end{pmatrix} \qquad (4.77)$$

for all $l=1,\dots,N$. In (4.75) and (4.76) $\bar{x}_l^T(t)=\begin{pmatrix}\tilde{x}_l^T & \tilde{y}_l^T\end{pmatrix}$ and $E=\mathscr{D}iag(I_n,0_n)$. Consider a Lyapunov-Krasovskii functional of the form

$$V_l(t)=V_{1l}(t)+V_{2l}(t) \qquad (4.78)$$

where

$$\begin{aligned}V_{1l}(t) &= \bar{x}_l^T EP\bar{x}_l &(4.79)\\ V_{2l}(t) &= \int_{-\tau_m}^{0}\int_{t+\theta}^{t}\tilde{y}_l^T(s)S\tilde{y}_l(s)dsd\theta &(4.80)\end{aligned}$$

The symmetric matrix $S>0\in\mathbb{R}^{n\times n}$ and

$$P=\begin{pmatrix}P_1 & 0\\ P_2 & P_3\end{pmatrix} \qquad (4.81)$$

[1]A descriptor system has the form $E\dot{x}(t)=Ax(t)+Bu(t)$ where $x(t)\in\mathbb{R}^n$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times n}$, and $E\in\mathbb{R}^{n\times n}$ is singular.

where P_1, P_2 and $P_3 \in \mathbb{R}^{n\times n}$ and P_1 is symmetric positive definite. The cost functions in (4.62) can be represented as

$$J_l = \int_0^{\infty} \bar{x}_l^T(t) \begin{pmatrix} Q+K^TRK & 0_n \\ 0_n & 0_n \end{pmatrix} \bar{x}_l(t)dt \tag{4.82}$$

The objective is to ensure the inequality

$$\dot{V}_l = \dot{V}_{1l}(t) + \dot{V}_{2l}(t) < -\bar{x}_l^T(t) \begin{pmatrix} Q+K^TRK & 0_n \\ 0_n & 0_n \end{pmatrix} \bar{x}_l(t) \tag{4.83}$$

holds. In the left hand side of (4.83)

$$\dot{V}_{1l}(t) = P^T\tilde{A}_{0l} + \tilde{A}_{0l}^T P - 2\int_{t-\tau(t)}^{t} G(\bar{x}_l(t), \tilde{y}_l(s))ds \tag{4.84}$$

and

$$\dot{V}_{2l}(t) = \tau_m \tilde{y}_l^T(t)S\tilde{y}_l(t) - \int_{t-\tau_m}^{t} \tilde{y}_l^T(s)S\tilde{y}_l(s)ds \tag{4.85}$$

where

$$G(\bar{x}_l(t), \tilde{y}_l(s)) = \bar{x}_l^T(t)P^T \begin{pmatrix} 0_n \\ A_l \end{pmatrix} \tilde{y}_l(s) \tag{4.86}$$

Select a matrix $F \in \mathbb{R}^{2n\times 2n}$ such that

$$\tilde{S}_F = \begin{pmatrix} S & \begin{pmatrix} 0_n & A_l^T \end{pmatrix} P \\ P^T \begin{pmatrix} 0_n \\ A_l^T \end{pmatrix} & F \end{pmatrix} > 0 \tag{4.87}$$

for all $l = 1\ldots,N$. Then it follows that

$$\int_{t-\tau}^{t} \begin{pmatrix} \tilde{y}_l(s) \\ \bar{x}_l(t) \end{pmatrix}^T \tilde{S}_F \begin{pmatrix} \tilde{y}_l(s) \\ \bar{x}_l(t) \end{pmatrix} ds > 0 \tag{4.88}$$

By rearranging (4.88), the integral term in (4.84) satisfies

$$\begin{aligned} -2\int_{t-\tau(t)}^{t} G(\bar{x}_l(t),\tilde{y}_l(s))ds &< \int_{t-\tau}^{t} (\tilde{y}_l^T(s)S\tilde{y}_l(s)+\bar{x}_l^T(t)F\bar{x}_l(t))ds \\ &< \int_{t-\tau_m}^{t} \tilde{y}_l^T(s)S\tilde{y}_l(s)ds+\tau_m\bar{x}_l^T(t)F\bar{x}_l(t) \end{aligned} \quad (4.89)$$

Substituting (4.89) in (4.84) and using (4.85), inequality (4.83) is satisfied if

$$0 < \begin{pmatrix} S & [0_n \ A_l^T]P \\ * & F \end{pmatrix} \quad (4.90)$$

and

$$P^T\tilde{A}_{0i}+\tilde{A}_{0i}^TP+\tau_mF+\begin{pmatrix} Q+K^TRK & 0_n \\ 0_n & \tau_mS \end{pmatrix}<0 \quad (4.91)$$

for all $1=1,\ldots,N$. Define $W=P^{-1}$ in conformation with (4.81) as

$$P^{-1}=W=\begin{pmatrix} W_1 & 0_n \\ W_2 & W_3 \end{pmatrix} \quad (4.92)$$

To create convex LMI representations from the matrix inequalities (4.90) and (4.91) define $\tilde{Z}>0\in\mathbb{R}^{n\times n}$. Pre and post multiply (4.90) by $\mathscr{D}iag(S^{-1},W^T)$ and $\mathscr{D}iag(S^{-1},W)$ respectively. Also pre and post multiply (4.91) by W^T and W respectively. Using the linearizations $W^TFW=\bar{F}=\begin{pmatrix} \bar{F}_1 & \bar{F}_2 \\ * & \bar{F}_3 \end{pmatrix}$ and $\bar{S}=S^{-1}$, and $KW_1=Y$, the inequalities in (4.90) and (4.91) can be represented by the LMIs in (4.68) and (4.69). An expression for the maximum bound on the cost J_l can be obtained by integrating (4.83) as

$$J_l<\tilde{x}_l^T(0)P_1\tilde{x}_l(0)+\int_{-\tau_m}^{0}\int_{\theta}^{0}\tilde{y}_l^T(s)S\tilde{y}_l(s)dsd\theta \quad (4.93)$$

for all $l=1,\ldots,N$. Assuming the initial condition $\tilde{x}(\theta)=\tilde{x}(0)$ for $-\tau_m\leq\theta\leq0$, $\tilde{y}_l(\theta)=\dot{\tilde{x}}(\theta)=0$ for $-\tau_m\leq\theta\leq0$. Hence the maximum bound on the cost J_l is given by

$$J_l<\tilde{x}_l^T(0)P_1\tilde{x}_l(0) \quad (4.94)$$

From (4.67) it can be shown by using Schur complement that $\tilde{Z} > W_1^{-1} = P_1$. Hence the inequality in (4.72) is satisfied. Then the following minimization problem constitutes minimizing an upper bound on the LQR cost in (4.62)

$\boxed{\text{Minimize } Trace(\tilde{Z}) \text{ subject to (4.67)-(4.69)}}$

□

Remark 4.6: Note that τ and the matrix H are fixed which renders the matrix A_l in (4.64) fixed for all $l = 1,\dots,N$ and hence (4.68) and (4.69) are LMI representations.

Remark 4.7: With the presentation of the methods in Section 4.3.1, Section 4.3.2 and Section 4.3.3, the justification for considering a constant scaling matrix H in (4.5) is given in this remark. Consider the delay dependent control design method for a fixed delay presented in Theorem 4.3.2. The matrix A_l^2 appears in (4.41). Now if H is not fixed then this term will involve a square term of a design variable and hence (4.41) will not be an LMI. Consider the delay dependent control design method for time-varying delay presented in Theorem 4.3.3. The LMI in (4.68) contains the term $\bar{S}A_l^T$. Again, if H is not fixed one will have to define a new variables $H_s = H\bar{S}$ in (4.68) and $H_w = HW_1$ in (4.69) to obtain an LMI constraint. These variables are not independent and hence it is not a convex constraint. Please note that the delay independent control design method in Theorem 4.3.1 is presented to obtain the gains that would give the user an estimate of the gains required such that system is stable for any arbitrary delay.

Remark 4.8: As explained in Remark 4.7, the term involving the matrix H associated with the delayed relative information needs to be fixed in the control design methods. Hence a two step control design method as considered in Chapter 3 is not feasible for these methods. Consequently the augmented cost function at a network level used in (3.8) is not considered in this chapter. In this chapter, each of the control design process is a single step design and hence a cost function of the form (4.16) is used. Exploring stability methods for time-delay systems which can allow a two step control design process like that of Chapter 3 is a future research direction.

4.4 Numerical Example

To illustrate the design methodologies, a cyclic nearest neighbour configuration of 5 vehicles moving in a 2-Dimensional plane, with each vehicle described by two decoupled double integrators as used in Section 3.6, is considered. The linear system model is given by

$$\dot{\zeta}_i = A\zeta_i + Bu_i \tag{4.95}$$

where ζ_i represents positions and velocities in a 2-Dimensional $x-y$ plane which constitute the states of the i^{th} vehicle. The matrices A and B are given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.96}$$

The Laplacian for the network is given by

$$\mathscr{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.97}$$

The matrix H associated with the relative information exchange in (4.6) is given by

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{4.98}$$

The matrix H is selected such that the scaling for the relative position and velocities is unity.

Remark 4.9: It is fair to assume that scaling matrix H has entries of unity for the relative position and velocity information. For each agent/vehicle a sum of the relative positions and velocities will be available after measurement. Hence the entries in (4.98) will be unity. In a general problem if one assumes that the sum of the difference between the states of an agent and its neighbours is available as relative information the entries for the matrix H will be unity. It is important to note that the theory presented in this chapter does not restrict the entries to unity. If a different scaling is available after measurement the matrix H will have different entries and the possibility of obtaining a controller will depend on the feasibility of the theorems presented in Section 4.3.

The matrices Q and R for the cost functions given in (4.16), (4.38) and (4.82) have been chosen as

$$Q = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.99}$$

The simulations in this section were done using SIMULINK with a fixed-step size of $0.001s$ and *ode4(Runge-Kutta)* solver. For the simulation results shown, the same initial conditions are used to illustrate the different control design methodologies. The gain matrix K obtained by solving the LMIs for delay independent control approach discussed in Section 4.3.1 is given by

$$K = \begin{pmatrix} 11.5156 & 0 & 13.7394 & 0 \\ 0 & 11.5156 & 0 & 13.7394 \end{pmatrix} \tag{4.100}$$

The magnitudes of the gains are very high as compared to the scaling of the delayed relative information in the matrix H. This is perhaps expected as the system is required to be stable for any arbitrary delay τ as the delay independent LMI of Theorem 4.3.1 does not involve τ. Figure 4.1 shows a rendezvous of 5 agents with the delay independent control from (4.100) with a delay of $\tau = 0.1s$. Figure 4.2 shows that the agents attain a rendezvous at around $t = 5s$ with the gain matrix K obtained in (4.100). The control signals U_x and U_y in the x and y directions with delay independent control for the fixed delay are shown in Figure 4.3 and Figure 4.4.

For the method proposed in Section 4.3.2 it is assumed that a fixed communication

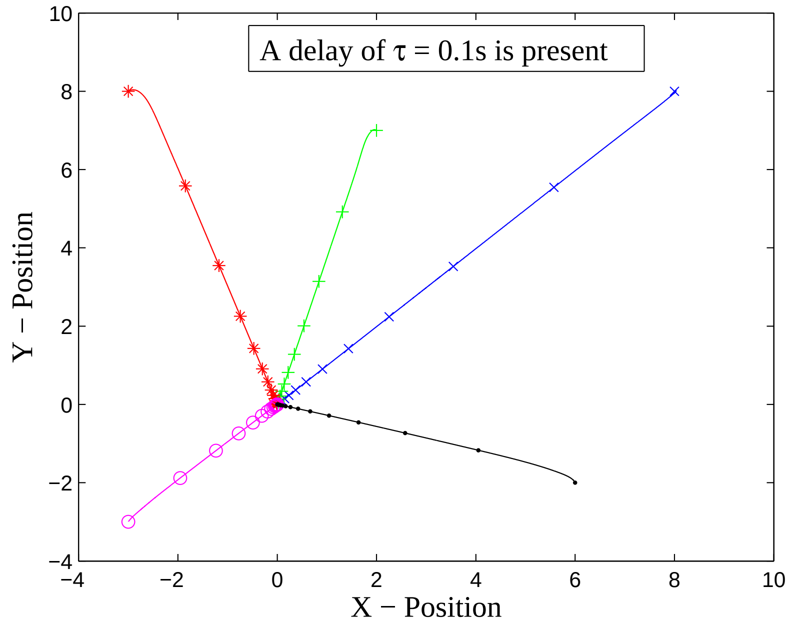

Figure 4.1: Rendezvous with delay independent control for fixed delay

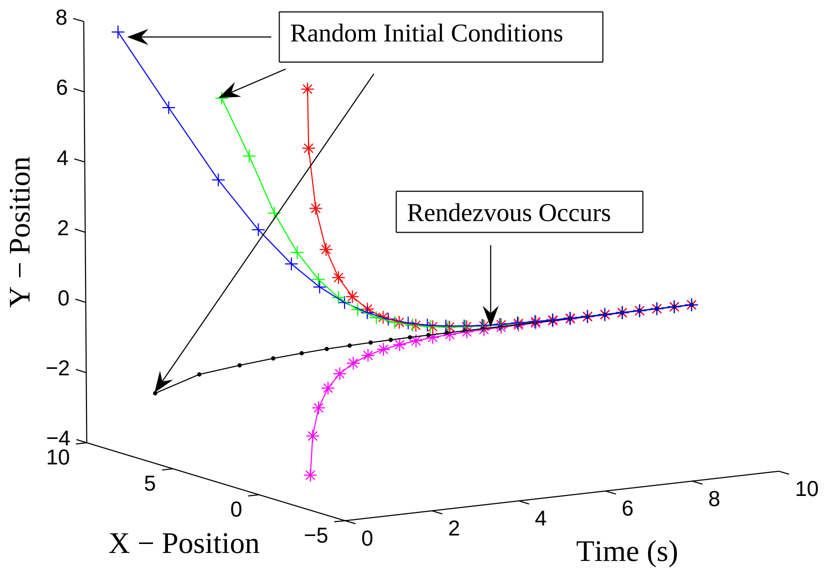

Figure 4.2: Delay independent control with delay of $\tau = 0.1s$ plotted against time

delay of $\tau = 0.1s$ is present in the exchange of relative information. For the delay dependent LMIs in (4.40) and (4.41) the scalars α_0 and α_1 have been chosen as $\alpha_0 = 3$ and $\alpha_1 = 3$ as Theorem 4.3.2 was not feasible for lower integer valus. For $\tau = 0.1s$, the gain

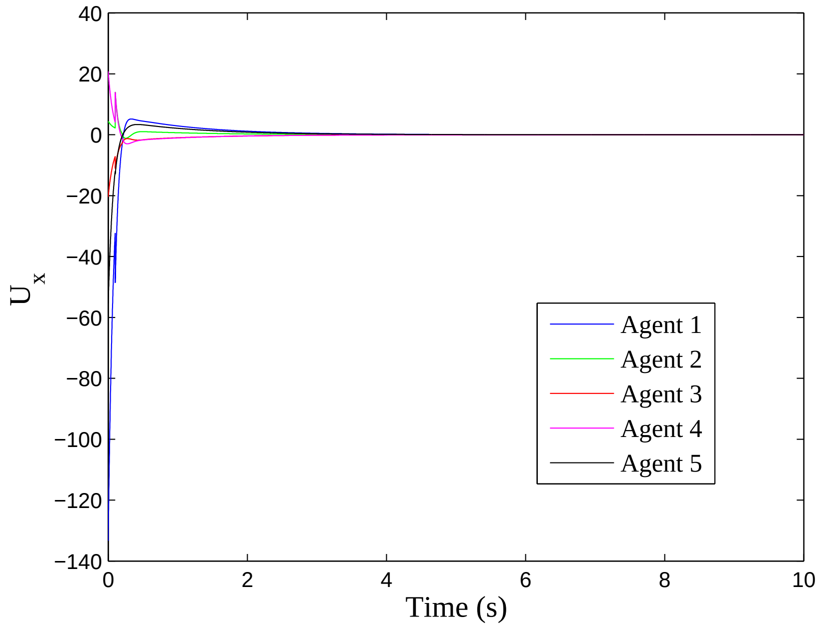

Figure 4.3: Control effort for agents in the x direction - delay independent control

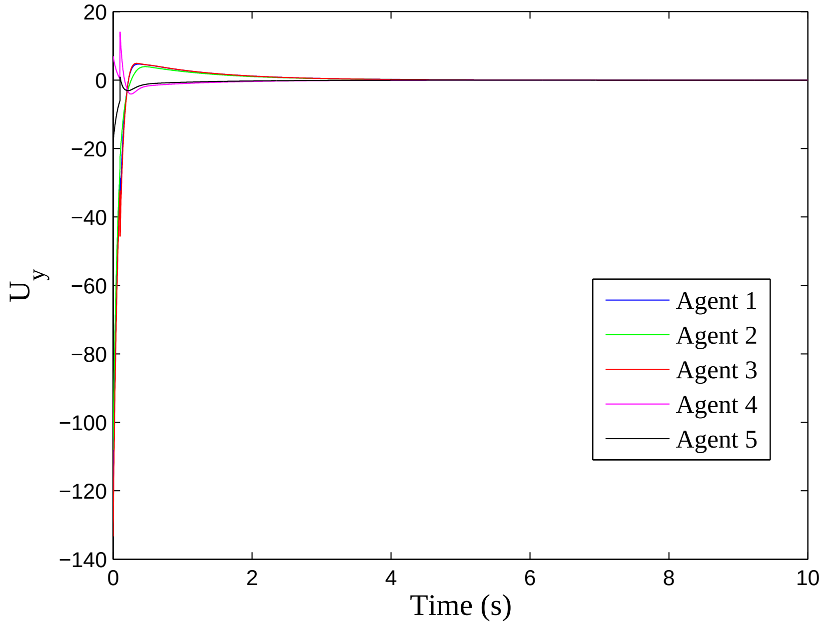

Figure 4.4: Control effort for agents in the y direction - delay independent control

matrix K obtained from this approach is

$$K = \begin{pmatrix} 2.3931 & 0 & 2.8428 & 0 \\ 0 & 2.3931 & 0 & 2.8428 \end{pmatrix} \tag{4.101}$$

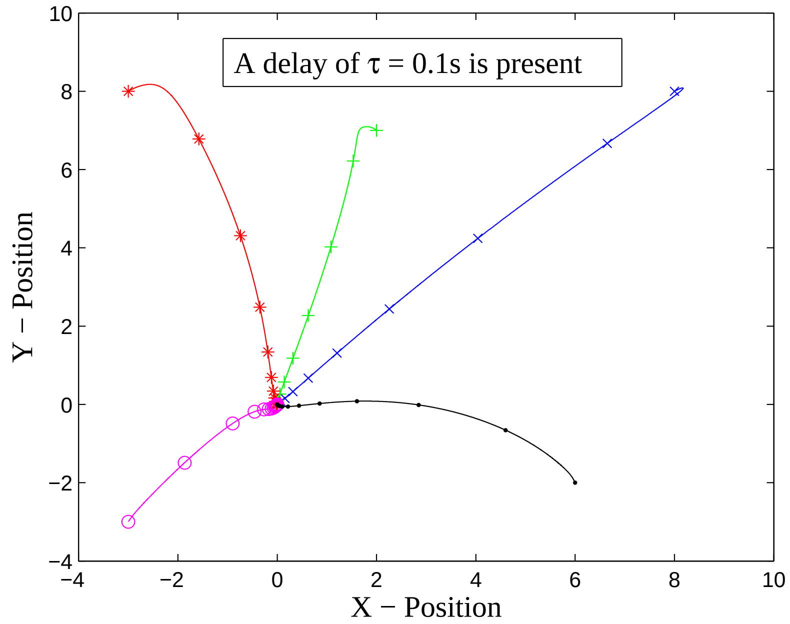

Figure 4.5: Rendezvous with delay dependent control for fixed delay

Figure 4.5 shows a rendezvous of the 5 agents with delay dependent control design for a fixed delay of $\tau = 0.1s$. Figure 4.6 shows that a rendezvous occurs at around $t = 5.2s$ with the gain matrix K obtained in (4.101). The control signals U_x and U_y in the x

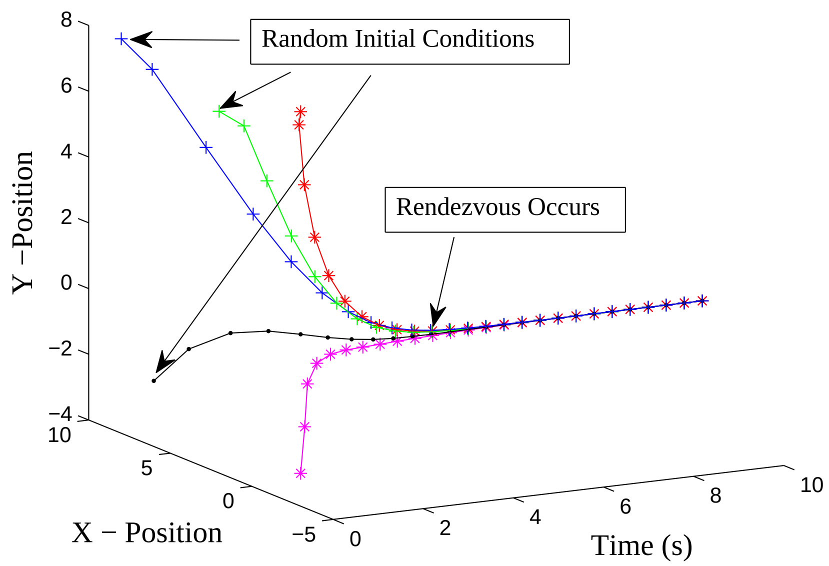

Figure 4.6: Delay dependent control with delay of $\tau = 0.1s$ plotted against time

and y directions with delay dependent control for a fixed delay are shown in Figure 4.7

and Figure 4.8. Comparing the plots in Figure (4.1) and Figure (4.5) it can be seen

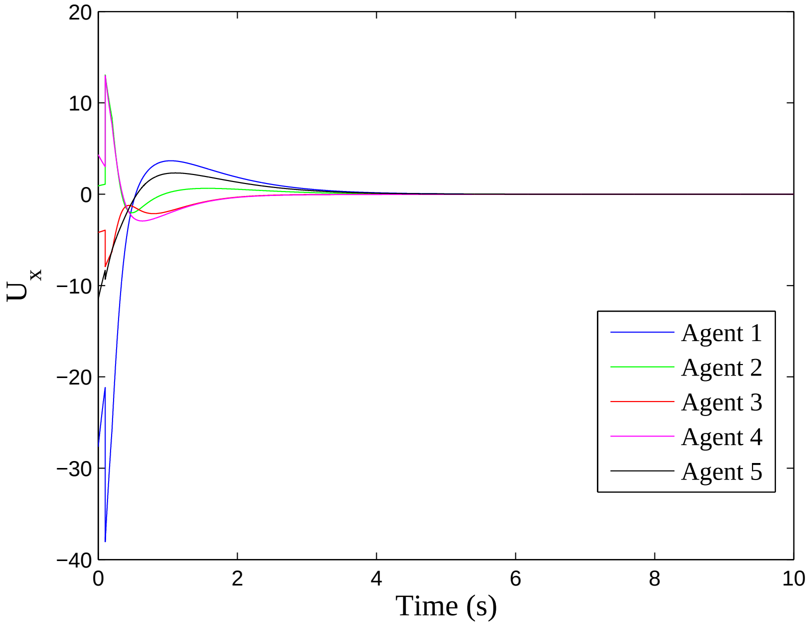

Figure 4.7: Control effort for agents in the x direction - fixed delay

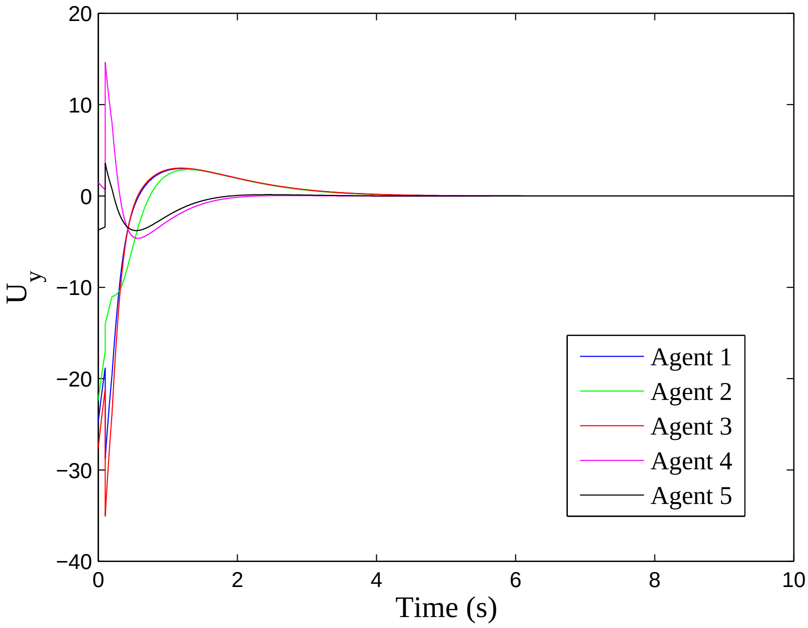

Figure 4.8: Control effort for agents in the y direction - fixed delay

that the performance by delay independent control for a fixed delay is better than the performance of the delay dependent control for a fixed delay. The disadvantage of delay independent control is that it requires significantly higher control effort as seen in Figure

(4.3) and Figure (4.4) as compared to delay dependent control in Figure (4.7) and Figure (4.8). In [129] it is stated that the stability criteria of *Proposition 3* in Appendix B.3.1 is conservative. Hence the gain matrix K in (4.101) should be able to cope with larger delays. Simulations show that the agents do not attain a rendezvous and diverge once τ exceeds $0.6s$. This is shown in Figure 4.9.

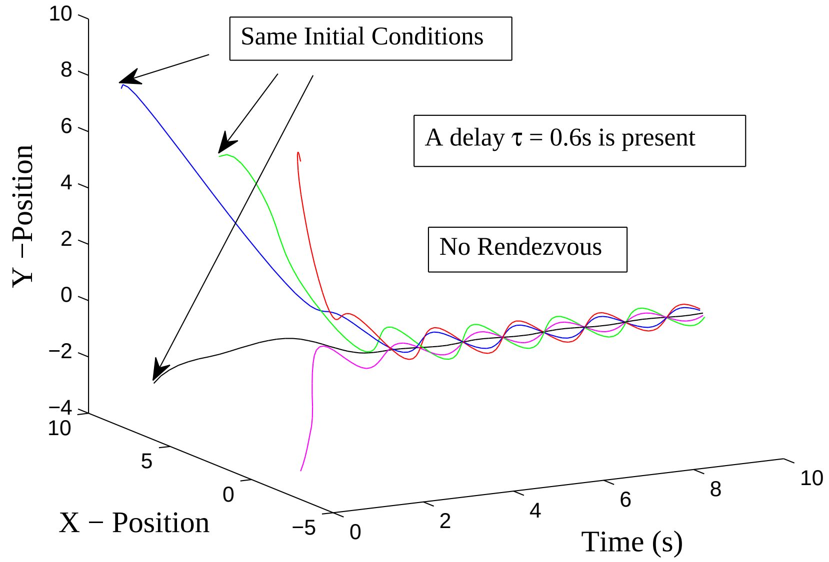

Figure 4.9: No rendezvous after $\tau = 0.6s$

If there is no communication of relative information i.e. the control law in (4.6) is replaced by $u_i(t) = -Kx_i(t)$, for the system in (4.1), a standard LQR problem results for each agent. With the same set of matrices (A, B, Q, R) using the MATLAB command `'lqr'`, the local state feedback gain matrix K obtained is

$$K = \begin{pmatrix} 3.1623 & 0 & 4.0404 & 0 \\ 0 & 3.1623 & 0 & 4.0404 \end{pmatrix} \tag{4.102}$$

Figure 4.10 shows 5 disconnected agents attaining a rendezvous at around $t = 4.8s$. Comparing the plots in Figure 4.6 and Figure 4.10 it can be seen that the performance achieved both with and without the relative information is similar. The rendezvous of individual agents without relative information in a two dimensional plane is shown in Figure 4.11. The control efforts are shown in Figure 4.12 and Figure 4.13.

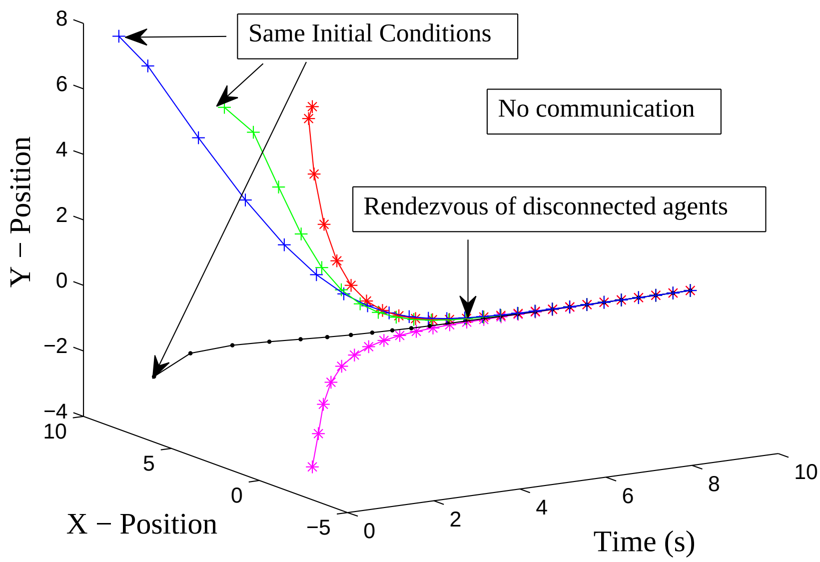

Figure 4.10: Rendezvous of individual agents

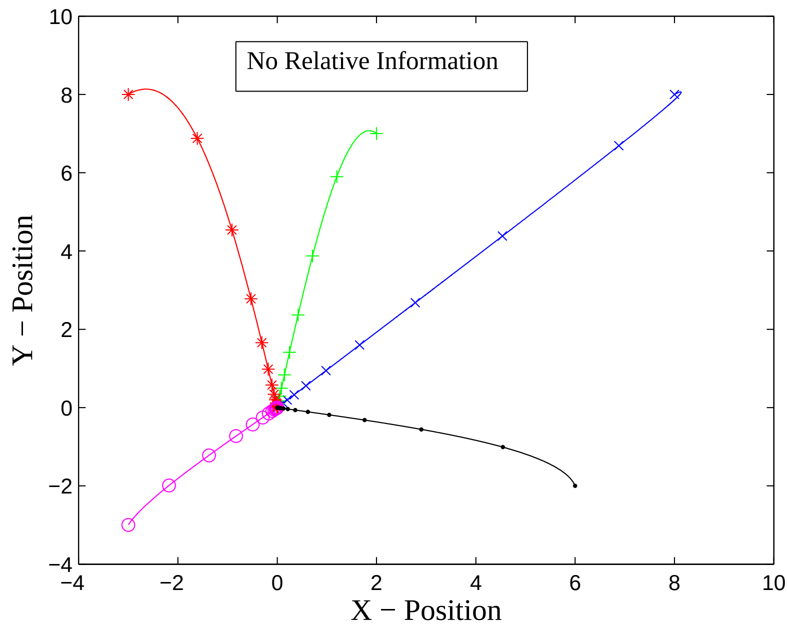

Figure 4.11: Rendezvous without relative information

Remark 4.10: One distinct advantage that can be observed is the reduction of the magnitude of the gains for velocity and position feedback in (4.101) as compared to (4.102). Another advantage can be observed by comparing the control signals in Figure 4.7, Figure 4.8, Figure 4.12 and Figure 4.13. Though the magnitude of the maximum control

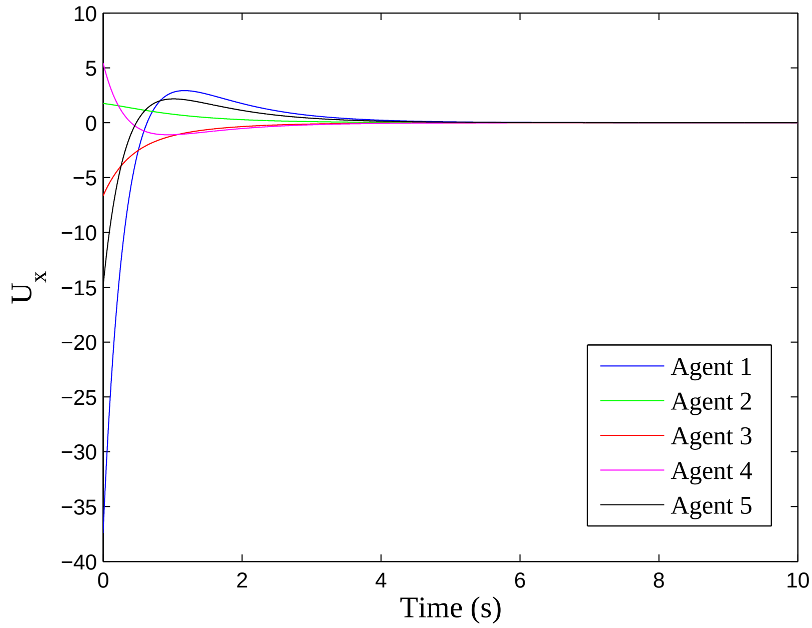

Figure 4.12: Control effort in the x direction - no relative information

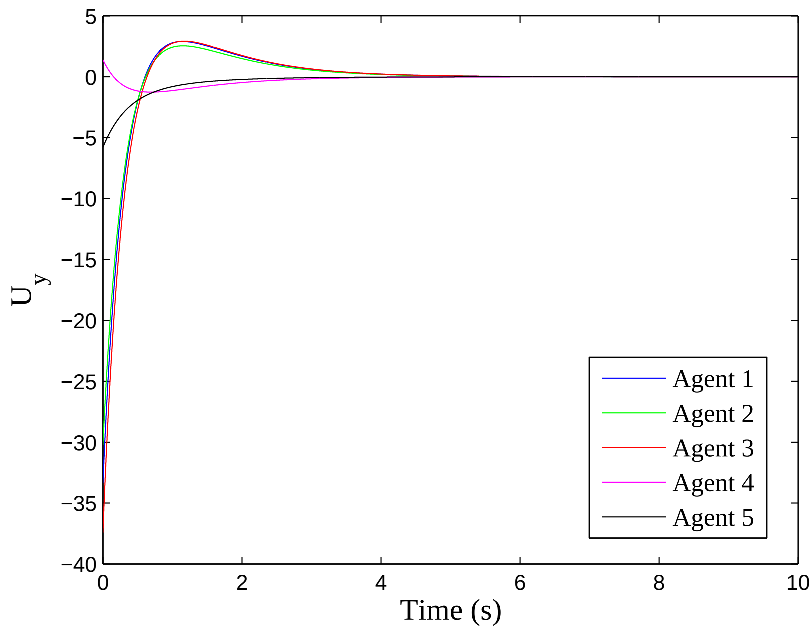

Figure 4.13: Control effort in the y direction - no relative information

effort is almost similar for x direction in Figure 4.7 and Figure 4.12, there is some reduction in the maximum control effort in the y direction in Figure 4.8 as compared to that in Figure 4.13. Thus the use of relative information may also be beneficial to reduce control effort. For the initial conditions considered in this example the LQR cost

J is $4.66 * 10^6$ while that associated with the standard LQR controller without relative information in (4.102) is $4.49 * 10^6$. Thus the use of relative information gives a cost that is almost equal to that of the standard LQR controller without relative information. The control design method in Section 4.3.2 is obtained from a very conservative stability analysis method in [129]. Since the cost obtained from a conservative method is almost equal to that of the standard LQR controller without relative information, it is reasonable to expect that if one is able to obtain the control design method in Section 4.3.2 from less conservative and recent stability analysis methods for fixed delays, a significant cost reduction is possible. This is a future research direction.

For the method proposed in Section 4.3.3 the bounded time-varying delay is given by $0 \leq \tau(t) \leq 0.1s$. The gain matrix obtained from this approach is

$$K = \begin{pmatrix} 2.9779 & 0 & 4.4182 & 0 \\ 0 & 2.9779 & 0 & 4.4182 \end{pmatrix} \tag{4.103}$$

Figure 4.14 shows a rendezvous of the 5 agents with the controller designed for a bounded time-varying delay of $\tau(t) = 0.05 + 0.05cos(t)$. Figure 4.15 shows 5 agents attaining a rendezvous at around $t = 5.5s$ for the bounded time varying delay. The

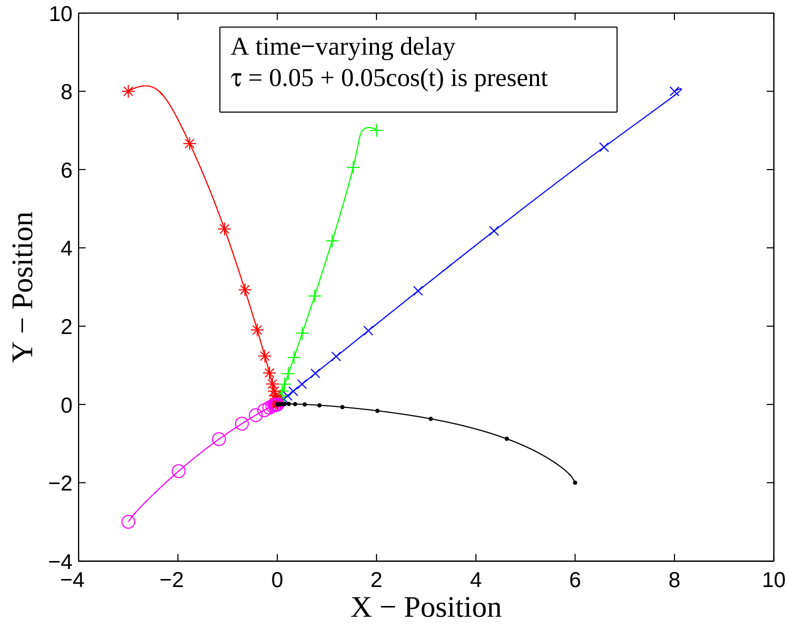

Figure 4.14: Rendezvous with delay dependent control for time-varying delay

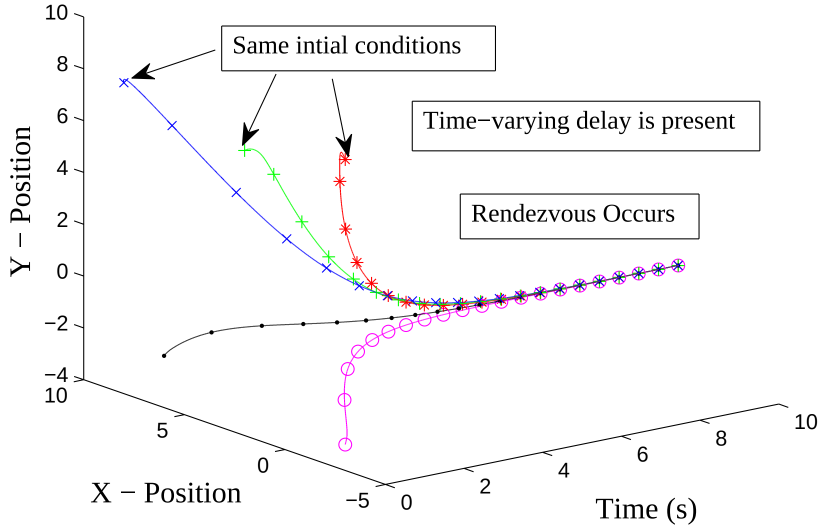

Figure 4.15: Rendezvous with time-varying delay plotted against time

control signals U_x and U_y in the x and y directions with delay dependent control for the case of time-varying delay are shown in Figure 4.16 and Figure 4.17.

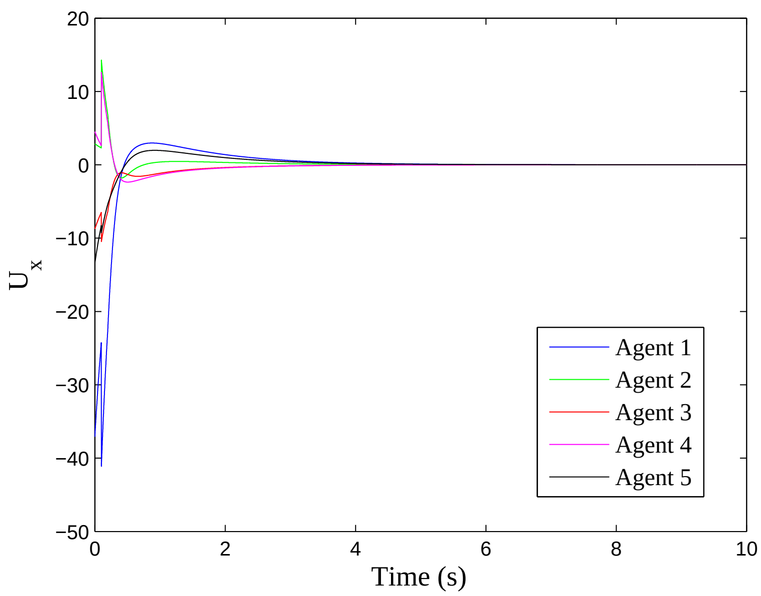

Figure 4.16: Control effort for agents in the x direction - time-varying delay

Remark 4.11: An analysis of the controller obtained from Section 4.3.3 did not provide any advantages in reducing the control effort as compared to the rendezvous of individual agents. This may be due to the the higher gains obtained in (4.103) as compared

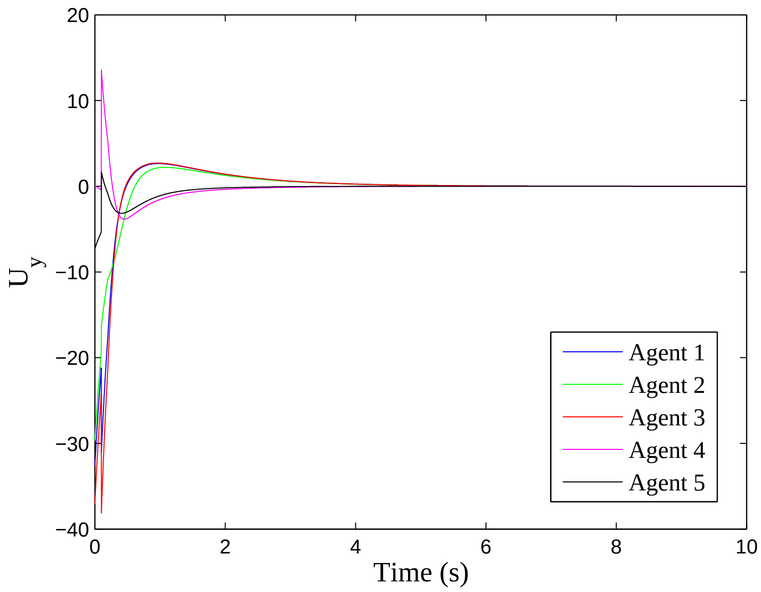

Figure 4.17: Control effort for agents in the y direction - time-varying delay

(4.102). The cost J associated with the control design method for time-varying delay is given by $4.61 * 10^6$. This cost is comparable to the costs obtained from the control design method for a fixed delay given by Theorem 4.3.2 and the standard LQR controller without relative information presented in Remark 4.10. The control design method in Section 4.3.3 is obtained from a very conservative stability analysis method in [120]. Exploration and modification of less conservative methods for the case of time-varying delays can be performed to yield better results. This is also a future research direction.

4.4.1 Effects of Communication Topology

In this section the effects of changes in the communication topology, i.e. the effects of changes in the graph Laplacian $\mathscr{L}$, are presented. For this study the same weighting matrices Q and R in (4.99) for the cost function and the scaling matrix H in (4.98) are considered and the results in the preceding section are repeated for changes in communication topologies. For this study the two communication topologies shown in Figure 3.23 and Figure 3.24 from Section 3.6.6.2 from Chapter 3 are considered.

For the acyclic nearest neighbour interconnection topology described by Figure 3.24

in Section 3.6.6.2 the Laplacian matrix is given by

$$\mathscr{L} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{4.104}$$

For this communication topology, the gain matrix K obtained from the delay independent control design method of Theorem 4.3.1 is given by

$$K = \begin{pmatrix} 11.5146 & 0 & 13.7377 & 0 \\ 0 & 11.5146 & 0 & 13.7377 \end{pmatrix} \tag{4.105}$$

The gain matrix K obtained from the delay dependent control design method for a fixed delay of Theorem 4.3.2 is given by

$$K = \begin{pmatrix} 2.3927 & 0 & 2.8428 & 0 \\ 0 & 2.3927 & 0 & 2.8428 \end{pmatrix} \tag{4.106}$$

The gain matrix K obtained from the delay dependent control design method for time-varying delay of Theorem 4.3.3 is given by

$$K = \begin{pmatrix} 2.9789 & 0 & 4.4182 & 0 \\ 0 & 2.9789 & 0 & 4.4182 \end{pmatrix} \tag{4.107}$$

Comparing the gain matrices for the three methods in (4.100) and (4.105), (4.101) and (4.106), (4.103) and (4.107) one can see that the gain matrices are almost equal for the respective methods. The use of the acyclic communication topology described by (4.104) does not yield any significant differences in the rendezvous of the agents. A possible explanation for this is that the maximum eigenvalue is $\lambda_{max} = 3.618$ for both (4.97) and (4.104).

For the denser communication topology described by Figure 3.23 in Section 3.6.6.2

the Laplacian matrix is given by

$$\mathscr{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix} \tag{4.108}$$

For this communication topology, the gain matrix K obtained from the delay independent control design method of Theorem 4.3.1 is given by

$$K = \begin{pmatrix} 15.5989 & 0 & 18.1368 & 0 \\ 0 & 15.5989 & 0 & 18.1368 \end{pmatrix} \tag{4.109}$$

The delay dependent control design method for a fixed delay of Theorem 4.3.2 is infeasible for this communication topology. A possible explanation for this is that the LMIs in (4.41) are rendered infeasible since they contain A_l^2 where $A_l = -\lambda_l BH$ and the maximum eigenvalue of $\lambda_{max} = 5$ for (4.108) is higher than $\lambda_{max} = 3.618$ for both (4.97) and (4.104). The gain matrix K obtained from the delay dependent control design method for time-varying delay of Theorem 4.3.3 is given by

$$K = \begin{pmatrix} 2.7884 & 0 & 4.6103 & 0 \\ 0 & 2.7884 & 0 & 4.6103 \end{pmatrix} \tag{4.110}$$

Comparing (4.103), (4.107) and (4.110) it can be seen that for this problem the delay dependent gains for the case of time varying delays considered in this example are similar for all three topologies and do not yield any significant differences in the rendezvous of agents.

To summarize, a use of communication topology with high eigenvalues increases the contribution of the delay term A_l in (4.13) thereby increasing the destabilization effect of delays. It is hence advisable to the user of this method that a communication topology with lower eigenvalues should be employed for better results.

4.4.2 Effects of Scaling Matrix H

In this section some effects of the scaling matrix H in (4.98) are presented. For this study the same weighting matrices Q and R in (4.99) are considered. It is considered that the matrix H associated with the relative information exchange in (4.6) is now given by

$$H = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \tag{4.111}$$

Remark 4.12: The delay independent control design method of Theorem 4.3.1 has been presented in this thesis to provide the reader an insight into the gains required such that the delay term A_l in (4.13) has no effects on the stability of the network for any arbitrary delay. For the three network topologies described by the Laplacians in (4.97), (4.104) and (4.108), Theorem 4.3.1 is feasible and obtains gains even when the matrix H is very large. In practical scenarios it will be impossible to provide the high gains resulting from the delay independent control design method and hence in this section some interesting observations of the effects of the matrix H on delay dependent control design methods are presented.

For the matrix H in (4.111), the delay dependent control design method for a fixed delay given by Theorem 4.3.2 does not obtain a solution for any of the three network topologies considered in (4.97), (4.104) and (4.108). The delay dependent control design method for time-varying delays given by Theorem 4.3.3 for the network topologies of (4.97) and (4.104) yields the same gain matrix K given by

$$K = \begin{pmatrix} 3.6071 & 0 & 5.9759 & 0 \\ 0 & 3.6071 & 0 & 5.9759 \end{pmatrix} \tag{4.112}$$

The gains obtained in (4.112) are higher as compared to those in (4.107) and (4.110) to compensate for the increase in contribution of the delay term A_l in (4.13) caused by the increase in H in (4.111). The delay dependent control design method for time-varying delays given by Theorem 4.3.3 is infeasible for the network topology given by (4.108). Thus it can be observed that an increase in the scaling H leads to infeasibility as the contribution of the delay term A_l in (4.13) increases.

In the previous section, i.e. in Section 4.4.1, it was stated that the communication topology described by the Laplacian in (4.108) does not give a solution with the delay dependent control design method of Theorem 4.3.2 when the matrix H is given by (4.98). In this case if the scaling matrix H is considered to be given by

$$H = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{pmatrix} \tag{4.113}$$

Theorem 4.3.2 is then feasible and obtains the gain matrix K as

$$K = \begin{pmatrix} 4.6002 & 0 & 4.6729 & 0 \\ 0 & 4.6002 & 0 & 4.6729 \end{pmatrix} \tag{4.114}$$

To summarize, a higher magnitude of the entries in the scaling matrix H in (4.98) increases the contribution of the delay term A_l in (4.13) thereby increasing the destabilization effect of time delays. It is hence advisable to the user of this method that the magnitude of the scaling matrix H be lower. If the scaling provided by the measurement of relative information results in an infeasible solution after employing the delay dependent control design methods the user may consider using low values for scaling matrix H.

4.5 Concluding Remarks

In this chapter delay independent and delay dependent control design methodologies for fixed delays and delay dependent control design method for time-varying delays are developed to stabilize a network of linear systems by making use of delayed relative state information along with local state information. The delay independent control design for the case of fixed delays results in gains which are high, to compensate for the delayed relative information. The delay dependent control design for fixed delays results in gains which provide similar performance as compared to standard LQR control of the unconnected individual agents. For the double integrator system considered, the gains are lower in magnitude as compared to the gains obtained by solving an individual LQR problem for a single agent. Thus the use of relative information maybe advantageous in terms of distributing the control effort. A control design method to guarantee a certain level of LQR performance with time varying delays was also developed. The control design methodologies for both fixed and time-varying delays are developed for using node level systems obtained from spectral decomposition and are scalable with respect to the number of agents in the multi-agent system.

The stabilization techniques which were used to incorporate LQR performance, whilst not necessarily the most recent in terms of the time-delay literature, were shown to yield tractable LMI representations under certain mild simplifications. This is important because of the large number of decision variables involved resulting from the multiple agents. These techniques provided a good trade-off between unnecessary conservatism and tractability of LMI formulations.

Chapter 5

Output Feedback Control of a Network of Double Integrators

5.1 Introduction

In Chapter 3, a time-delay analysis of the proposed distributed state-feedback controllers for a network of multiple agents was carried out to obtain a maximum bound on the amount of delay that can be accommodated in the exchange of relative information. In Chapter 4, delay independent and delay dependent methods for controller synthesis by state-feedback to achieve network level LQR performance were developed. In both these chapters it was implicitly assumed that time-delays in communication are detrimental to achieving the desired cooperative goals and performance objectives. Though in most cases delays have a negative effect on the stability of a linear system, in some instances delays can have positive effects on a system. In [129] it is illustrated using the example of a vibration absorber, that introduction of time delays can enhance the system performance by reducing the sensitivity of the system to the excitation frequency. Furthermore in some cases the presence of delays can stabilize a system which is unstable when there are no delays in the system [133]. In this chapter the focus is on exploiting the latter property of stabilization by the introduction of delay, to stabilize a network of multiple agents modelled as double integrators.

In many practical cases, the measurement of all the states of a system is not viable.

In such scenarios control by output feedback is necessary. Although (static) output feedback control is a well studied problem, no complete solution has been found [108]. In [133] it was shown that it maybe possible to stabilize a system, which cannot be stabilized by static output feedback, by introducing terms based on a delayed version of the output. This design methodology has been considered by a few researchers in [134]-[138]. In [134], necessary and sufficient conditions for achieving stability of systems which cannot be stabilized without delays are developed. In [135], issues related to robustness with respect to parametric and delay uncertainty are considered. In [136], necessary conditions for the existence of stabilizing static output feedback controllers with multiple delays are developed. In particular, stabilization of a double integrator by using delays in output feedback terms is briefly described. In [137], a new method for control synthesis, based on descriptor discretized Lyapunov-Krasovskii functionals, which can stabilize systems not stabilizable by static output feedback has been developed. In [137], the method is illustrated on double integrator dynamics. In [138], an artificial delay is used to develop a static output feedback sliding mode control law. It is stated by the authors that an advantage of such a method is that it does not increase the order of the system, and is computationally less complex compared to methods involving the use of compensators.

In Chapter 3 and Chapter 4, the state feedback design methodologies were illustrated with the help of an example of a multi-agent system described by double integrator dynamics. Motion described by double integrator dynamics cannot be asymptotically stabilized via static output feedback, i.e. by position information alone. In this chapter exponential stabilization of a network of vehicles described by double integrator dynamics is achieved by introducing delays in relative information. The exponential stabilization method developed guarantees a certain level of performance by ensuring that each agent is stabilized with a required rate of convergence to a desired formation.

The organization of this chapter is as follows: In Section 5.2 the problem definition is stated. The linear system model for a network of double integrators is explained. In Section 5.3 the control design procedure is explained in detail. In Section 5.4 the control design methodology is illustrated with the help of a numerical example. Concluding remarks are given in Section 5.5.

5.2 Problem Formulation

In this chapter, a network of N identical vehicles moving in a 2-Dimensional plane is considered. As in Chapter 3 and 4, the dynamics of each dimension is assumed to be described by a double integrator, and a linear state space representation for each vehicle is given by

$$\begin{aligned} \dot{\xi}_i(t) &= A\xi_i(t) + Bu_i(t) && (5.1) \\ \vartheta_i &= C\xi_i && (5.2) \end{aligned}$$

where $\xi_i = \mathscr{C}ol[x_i, \dot{x}_i, y_i, \dot{y}_i]$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.3}$$

This is equivalent to two decoupled double integrator systems. Here ϑ_i is the measured position (x_i, y_i) of the i^{th} vehicle in the $x - y$ plane. Each vehicle is assumed to have access to its output information and the relative output information from agents which it can interact with (its neighbours). Bidirectional communication is assumed between the agents. This interconnected system is represented by a graph with N vertices (nodes) each representing a vehicle. The signals representing the exchange of relative position information are assumed to have the form

$$z_i(t) = \sum_{j \in \mathscr{J}_i} (\vartheta_i(t) - \vartheta_j(t)) \tag{5.4}$$

for $i = 1 \dots N$. As in the previous chapters, the nonempty set $\mathscr{J}_i \subset \{1, 2, \dots N\}/\{i\}$ denotes the vehicles, for which the i^{th} vehicle has information. The external output measurements relative to the other vehicles sensed by the i^{th} vehicle are represented by the signals $z_i(t)$. At a network level, the system given in (5.1) is represented by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{5.5}$$

where

$$X(t) = \mathscr{C}ol(\xi_1(t), ..., \xi_N(t)) \tag{5.6}$$

$$U(t) = \mathscr{C}ol(u_1(t), ..., u_N(t)) \tag{5.7}$$

At network level, (3.2) can be represented as

$$Z(t) = (\mathscr{L} \otimes C)X(t) \tag{5.8}$$

where $Z(t) = \mathscr{C}ol(z_1(t), \ldots, z_N(t))$. As in Chapters 3 and 4, an assumption is made that each vehicle has information about at least one other vehicle which ensures $rank(\mathscr{L}) = N-1$.

It is common knowledge that the two decoupled double integrators associated with (A,B,C) cannot be stabilized by static output feedback. To circumvent this issue, the existence of distributed static output feedback control laws involving delay terms of the form

$$u_i(t) = -K_1\vartheta_i(t) + K_2\vartheta_i(t-\tau) + K_2(\beta z_i(t-\tau) - d_i) \tag{5.9}$$

where $K_1 = k_1 I_2$ and $K_2 = k_2 I_2$, where k_1 and k_2 are non-zero scalars, is considered. The scalar $\beta > 0$ represents an a priori known scalar weighting for z_i and τ is a fixed chosen delay. The choice if τ is considered as part of the controller design process. In equation (5.9), $z_i(t-\tau)$ is the delayed relative position information given by

$$z_i(t-\tau) = \sum_{j \in \mathscr{J}_i} (\vartheta_i(t-\tau) - \vartheta_j(t-\tau)) \tag{5.10}$$

The $2-D$ vector d_i is the offset in the relative information at each node so that each agent maintains a desired relative distance from its neighbours. For a given β, the gains K_1,

K_2, together with τ must be chosen such that the closed loop network system is stable. Since each agent is described by a two decoupled double integrator systems, (for motion in each planar direction) it can be assumed without loss of generality that $K_1 = k_1 I_2$ and $K_2 = k_2 I_2$ where the scalars k_1 and k_2 represent the gains for a single double integrator with output feedback.

Assumption 5.1: In real engineering systems, relative sensing and communication of information will have delays. Here it is assumed that a minimum delay of $\tau_{min} > 0$ will be present in relative sensing and communication. Since it is assumed that each node has access to its own output information it is also assumed, in addition, that it is possible to store this information and use it in delayed feedback.

The control law given in (5.9) at a network level is given by

$$\begin{aligned} U(t) = -(I_N \otimes K_1 C)X(t) + (I_N \otimes K_2 C)X(t-\tau) + (\beta \mathscr{L} \otimes K_2 C)X(t-\tau) \\ +(I_N \otimes K_2)D \end{aligned} \tag{5.11}$$

where $D = \mathscr{C}ol(d_1, \ldots, d_N)$. Equation (5.11) is represented as

$$U(t) = -(I_N \otimes K_1 C)X(t) + (I_N \otimes K_2)D + ((I_N + \beta \mathscr{L}) \otimes K_2 C)X(t-\tau) \tag{5.12}$$

Substituting (5.12) in (5.5), the closed loop system is given by

$$\dot{X}(t) = A_0 X(t) + A_1 X(t-\tau) + (I_N \otimes BK_2)D \tag{5.13}$$

where the system matrices

$$A_0 = I_N \otimes (A - BK_1 C) \tag{5.14}$$

$$A_1 = (I_N + \beta \mathscr{L}) \otimes BK_2 C \tag{5.15}$$

Since the system (A, B, C) is not stabilizable by static output feedback, the system in (5.13) is not stable for $\tau = 0$.

Remark 5.1: Note that each dynamical system is represented by two independent double integrators as shown in (5.3) to describe motion of a vehicle in a 2-Dimensional plane. A single double integrator could easily have been considered in (5.3) for motion in one dimension and the results employed for motion in the second dimension as it is assumed in this chapter that the motion along each dimension in a 2-Dimensional plane is decoupled and described by double integrator dynamics. In (5.9) $\vartheta_i(t)$ represents the position information for the two dimensions and without loss of generality it is assumed that $K_1 = k_1 I_2$ and $K_2 = k_2 I_2$. An attempt has been made in (5.9) to use the same gain K_2 for the delayed relative position information to exploit the properties of the matrix A_1 in (5.15) in the control design procedure.

5.3 Control Design Procedure

The problem addressed in this section is to design the control law in (5.9) with $\tau > \tau_{min}$ such that the closed loop system (5.13) is stable. In other words, for a given $\beta > 0$ find the triplet (K_1, K_2, τ) with $\tau > \tau_{min}$ such that the system (5.13) is stable.

5.3.1 Model Transformation

First, introduce a linear coordinate transformation of the form

$$\bar{X}(t) = X(t) - X_f \tag{5.16}$$

where $X_f \in \mathbb{R}^{4N}$ is the desired final state of the network. This will have the form

$$X_f = \mathscr{C}ol \left(\begin{array}{cccccccc} x_1^f, & 0, & y_1^f, & 0, & \ldots & x_N^f, & 0, & y_N^f, & 0 \end{array} \right) \tag{5.17}$$

where (x_i^f, y_i^f) is the final desired steady state positions for all $i = 1, \ldots, N$. Using (5.13) the system written in terms of $\bar{X}(t)$ has the form

$$\dot{\bar{X}}(t) = A_0 \bar{X}(t) + A_1 \bar{X}(t-\tau) + (A_0 + A_1) X_f + (I_N \otimes B K_2) D \tag{5.18}$$

For given values of K_1 and K_2 the offset vector D is chosen to satisfy

$$(I_N \otimes K_2)D = (I_N \otimes K_1 C)X_f - ((I_N + \beta \mathscr{L}) \otimes K_2 C)X_f \tag{5.19}$$

Note that $K_2 = k_2 I_2$ then $(I_N \times K_2) = k_2 I_{2N}$ and so provided $k_2 \neq 0$ the equation in (5.19) has an unique solution for D. Furthermore D depends on K_1 and K_2 and

$$D = \frac{1}{k_2}((I_N \otimes K_1 C) - ((I_N + \beta \mathscr{L}) \otimes K_2 C))X_f \tag{5.20}$$

Note that from (5.17) and (5.3)

$$(I_N \otimes A)X_f = 0, \tag{5.21}$$

for any choice of D. Then multiplying both sides of (5.19) on the left by $(I_N \otimes B)$ and adding $(I_N \otimes A)X_f$ to the left hand side yields

$$(I_N \otimes BK_2)D = -(A_0 + A_1)X_f \tag{5.22}$$

when exploiting the definitions of A_0 and A_1 from (5.14) and (5.15). The system in (5.18) is then given by

$$\dot{\bar{X}}(t) = A_0 \bar{X}(t) + A_1 \bar{X}(t - \tau) \tag{5.23}$$

Remark 5.2: Note that the offset D in (5.13) is chosen as in (5.20) once the gains k_1 and k_2 are designed. The offset D also depends on the desired formation encapsulated in X_f. Note that the evaluation of D is centralised as knowledge of the desired final position X_f is required. This renders the controller a partially distributed controller. Also since specific agents are required to maintain a specific position the offset d_i in (5.9) must be communicated to each agent i.

Remark 5.3: In Chapter 3 and Chapter 4 consensus was achieved by attaining a rendezvous of the agents, i.e. by driving all the states of the agents to the origin. In this chapter a method to modify consensus algorithms into formation stabilization algorithms is employed. Though the states X in (5.5) are required to achieve a formation of the agents, the transformed states $\bar{X}$ in (5.23) after employing the transformation in (5.16) and the offset in (5.20) are required to achieve a consensus, i.e. all the states of $\bar{X}$ in (5.23) are driven to the origin.

Because $\mathscr{L}$ is symmetric positive semi-definite, $(I_N + \beta \mathscr{L})$ is symmetric positive definite since $\beta > 0$. Since $(I_N + \beta \mathscr{L})$ is symmetric positive definite, by spectral decomposition $(I_N + \beta \mathscr{L}) = V \Lambda V^T$ where $V \in \mathbb{R}^{N \times N}$ is an orthogonal matrix formed from the eigenvectors of $(I_N + \beta \mathscr{L})$ and $\Lambda = \mathscr{D}iag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $(I_N + \beta \mathscr{L})$. Note that all $\lambda_i \geq 1$ for all $i = 1, \ldots, N$ and that the smallest eigenvalue of $(I_N + \beta \mathscr{L})$ is $\lambda_1 = 1$. As in Chapters 3 and 4, consider an orthogonal transformation

$$\bar{X} \mapsto (V^T \otimes I_4) \bar{X} = \tilde{X} \tag{5.24}$$

In the new coordinates equation (5.23) is given by

$$\dot{\tilde{X}}(t) = \tilde{A}_0 \tilde{X}(t) + \tilde{A}_1 \tilde{X}(t - \tau) \tag{5.25}$$

where

$$\tilde{A}_0 = (I_N \otimes (A - BK_1 C)) \tag{5.26}$$

$$\tilde{A}_1 = (\Lambda \otimes BK_2 C) \tag{5.27}$$

since $V^T (I_N + \beta \mathscr{L}) V = \Lambda$ because V is orthogonal. Equation (5.25) can be represented at node level in transformed coordinates as

$$\dot{\tilde{\xi}}_i(t) = A_0 \tilde{\xi}_i(t) + A_i \tilde{\xi}_i(t - \tau) \tag{5.28}$$

where

$$A_0 = (A - BK_1 C) \tag{5.29}$$

$$A_i = \lambda_i BK_2 C \tag{5.30}$$

for all $i = 1,\dots,N$. It is assumed that the initial condition $\tilde{\xi}_i(t) = \tilde{\xi}_i(0)$ for $t \in [-\tau, 0]$. In order to ensure a level of performance in the closed loop system, as suggested in [138], consider the transformation

$$\tilde{\xi}_{i\alpha}(t) = e^{\alpha t}\tilde{\xi}_i(t) \tag{5.31}$$

for $i = 1,\dots,N$ where $\alpha > 0$. From [138], the system in (5.28) is exponentially stable with a decay rate of α and an exponential gain of $\gamma > 1$ if the following exponential bound holds:

$$\|\tilde{\xi}_i(t)\| < \gamma\|\tilde{\xi}_i(0)\|e^{-\alpha t} \tag{5.32}$$

From (5.31) and (5.32), asymptotic convergence of $\tilde{\xi}_{i\alpha}$ implies exponential convergence of $\tilde{\xi}_i$ at a decay rate α. With this transformation, the system represented in (5.28) becomes

$$\dot{\tilde{\xi}}_{i\alpha}(t) = (A_0 + \alpha I_4)\tilde{\xi}_{i\alpha}(t) + e^{\alpha\tau}A_i\tilde{\xi}_{i\alpha}(t-\tau) \tag{5.33}$$

where A_0 and A_i are as given in (5.29) and (5.30) for all $i = 1,\dots,N$.

5.3.2 Stability Criteria

The stability of system (5.33) will be ascertained using *Proposition 5* in the Appendix B.4. The *Proposition 5* in the Appendix B.4 divides the delay interval $[-\tau, 0]$ into n_p partitions and then employs a discretized Lyapunov functional to test for stability. For completeness the proposition as applicable to the system in (5.33) is stated as:

Proposition 5.3.1. *The system described by (5.33) is asymptotically stable if there exist matrices P_i, $\bar{Q}_{pi}$, S_{pi}, $R_{pqi} \in \mathbb{R}^{4\times 4}$, $P_i = P_i^T$; $S_{pi} = S_{pi}^T > 0$, $R_{pqi} = R_{qpi}^T$, $p = 0,\dots,n_p$, $q = 0,\dots,n_p$; such that*

$$\begin{pmatrix} P_i & \tilde{F}_i \\ * & \tilde{R}_i + \tilde{S}_i \end{pmatrix} < 0 \tag{5.34}$$

$$\begin{pmatrix} \Delta_i & -D_i^s & -D_i^a \\ * & R_{di} + S_{di} & 0 \\ * & * & 3S_{di} \end{pmatrix} < 0 \tag{5.35}$$

where for $p = 1, \dots, n_p$

$$\tilde{F}_i = \begin{pmatrix} \bar{Q}_{0i} & \bar{Q}_{1i} & \dots & \bar{Q}_{n_p i} \end{pmatrix} \tag{5.36}$$

$$\tilde{R}_i = \begin{pmatrix} R_{00i} & R_{01i} & \dots & R_{0n_p i} \\ R_{10i} & R_{11i} & \dots & R_{1n_p i} \\ . & . & \dots & . \\ R_{n_p 0i} & R_{n_p 1i} & \dots & R_{n_p n_p i} \end{pmatrix} \tag{5.37}$$

$$\tilde{S}_i = \begin{pmatrix} \frac{1}{h} S_{0i} & \frac{1}{h} S_{1i} & \dots & \frac{1}{h} S_{n_p i} \end{pmatrix} \tag{5.38}$$

$$\Delta_i = \begin{pmatrix} \Delta_{00i} & \Delta_{01i} \\ * & \Delta_{11i} \end{pmatrix} \tag{5.39}$$

$$\Delta_{00i} = -P(A_0 + \alpha I_4) - (A_0 + \alpha I_4)^T P - \bar{Q}_{0i} - \bar{Q}_{0i}^T - S_{0i} \tag{5.40}$$

$$\Delta_{01i} = \bar{Q}_{n_p i} - P(e^{\alpha\tau} A_i) \tag{5.41}$$

$$\Delta_{11i} = S_{n_p i} \tag{5.42}$$

$$S_{di} = \mathscr{D}iag \begin{pmatrix} S_{d1i} & S_{d2i} & \dots & S_{dn_p i} \end{pmatrix} \tag{5.43}$$

$$S_{dpi} = S_{(p-1)i} - S_{pi} \tag{5.44}$$

$$R_{di} = \begin{pmatrix} R_{d11i} & R_{d12i} & \dots & R_{d1n_p i} \\ R_{d21i} & R_{d22i} & \dots & R_{d2n_p i} \\ . & . & \dots & . \\ R_{dn_p 1i} & R_{dn_p 2i} & \dots & R_{dn_p n_p i} \end{pmatrix} \tag{5.45}$$

$$R_{dpqi} = h(R_{(p-1,q-1)i} - R_{pqi}) \tag{5.46}$$

$$D_i^s = \begin{pmatrix} D_{1i}^s & D_{2i}^s & \dots & D_{n_p i}^s \end{pmatrix} \tag{5.47}$$

$$D_{pi}^s = \begin{pmatrix} D_{0pi}^s \\ D_{1pi}^s \end{pmatrix} \tag{5.48}$$

$$D_{0pi}^s = \frac{h}{2}(A_0 + \alpha I_4)^T (\bar{Q}_{(p-1)i} + \bar{Q}_{pi}) + \frac{h}{2}(R_{(0,p-1)i} + R_{0pi}) - (\bar{Q}_{p-1} - \bar{Q}_p) \tag{5.49}$$

$$D^s_{1pi} = \frac{h}{2}(e^{\alpha\tau}A_i)^T(\bar{Q}_{(p-1)i} + \bar{Q}_{pi}) - \frac{h}{2}(R_{(n_p,p-1)i} + R_{n_p pi}) \tag{5.50}$$

$$D^a_i = \begin{pmatrix} D^a_{1i} & D^a_{2i} & \dots & D^a_{n_p i} \end{pmatrix} \tag{5.51}$$

$$D^a_{pi} = \begin{pmatrix} D^a_{0pi} \\ D^a_{1pi} \end{pmatrix} \tag{5.52}$$

$$D^a_{0pi} = -\frac{h}{2}(A_0 + \alpha I_4)^T(\bar{Q}_{(p-1)i} - \bar{Q}_{pi}) - \frac{h}{2}(R_{(0,p-1)i} - R_{0pi}) \tag{5.53}$$

$$D^a_{1pi} = -\frac{h}{2}(e^{\alpha\tau}A_i)^T(\bar{Q}_{(p-1)i} + \bar{Q}_p) + \frac{h}{2}(R_{(n_p,p-1)i} - R_{n_p pi}) \tag{5.54}$$

and

$$h = \tau / n_p$$

for all $i = 1, \dots, N$

Remark 5.4: In Chapter 3 and Chapter 4 the focus was on minimisation of an LQR cost. A modification of Proposition 5.3.1 into an control design method minimising a quadratic cost was attempted but did not yield tractable LMI constraints due to complexity of the LMI constraints in Proposition 5.3.1 and as explained in Appendix B.6 the other stability methods used in this thesis cannot be used to solve this problem. Hence to incorporate some level of performance an exponential decay rate was introduced in (5.31).

5.3.3 DIRECT Search Optimization Algorithm

In order to have a convex representation in (5.34) and (5.35), the matrices A_0 and A_i in (5.29) and (5.30) must be fixed. This implies that gains K_1 and K_2, and the decay rate α should be fixed. Since the gains K_1 and K_2 are not known a-priori, the matrices A_0 and A_i in (5.29) and (5.30) are dependent on the feedback gains. However, if the gains K_1 and K_2 and the decay rate α are fixed, Proposition 5.3.1, provides a feasibility check for stability for a fixed known delay value τ. The design problem associated here is to identify minimum gain values for k_1 and k_2 and an associated minimum possible delay τ such that Proposition 5.3.1 is satisfied. The solution to such a problem is not straightforward, and often depends on fine gridding of the search space (or similar technique).

However, there is no guarantee of finding the optimal solution, or even a sub-optimal one depending on the type of non-convex surface. In this methodology a solution is sought by making use of a deterministic global optimization algorithm, Dividing Rectangles (DIRECT). The method does not require any derivative information to be supplied, and uses a center point sampling strategy. The method was originally developed in [130] as a modification of the classical one dimensional Lipschitzian optimization algorithm known as the Schubert algorithm [130]. The search space is an n-dimensional hypercube or box, defined as $\mathscr{H} = \{x \in R^n : 0 \leq x_i \leq 1\}$. The algorithm works in the normalized parametric space and transforms to the actual search space when the cost function has to be evaluated. The principle idea can be summarized as: while the algorithm proceeds, the search space is partitioned into smaller hypercubes or boxes and each hypercube is sampled at the center point of the interval. Over many iterations, the algorithm tries to find all the 'potentially optimal' hypercubes or boxes in the search space and then further partitions them, thereby eventually obtaining the global solution. A potentially optimal hypercube either has a low function value compared to the lowest function value obtained till then during the implementation of the DIRECT algorithm or is considerably large to be a good target for the global search. Thus the DIRECT algorithm samples a dense subset of the points in the hypercube and converges to the global optimum as long as the cost function is continuous in the neighborhood of the global minimum. The reader is referred to [130] for details on potentially optimal hypercubes and the division strategies. The algorithm has asymptotic convergence property, and details of the proof are available in [131].

In this section the DIRECT optimization algorithm is employed for two design problems, associated with obtaining minimum gain values for k_1 and k_2 and an associated minimum possible delay τ within specified bounds such that Proposition 5.3.1 is satisfied, described by the following cases:

- Case A: A required (Fixed) rate of decay α
- Case B: Maximum possible rate of decay for given bounds on the gains k_1 and k_2 and τ

Since the transformed systems as in (5.33) are considered the design scaling parameter β in (5.11) is also fixed a-priori.

5.3.3.1 Case A: Fixed Rate of Decay

In this section the problem of finding minimum possible gains k_1 and k_2, for a minimum possible level of delay such that the system in (5.28) is exponentially stabilized with a required rate of decay, is considered. Since there are multiple minimization objectives, a collective optimization objective function is defined with appropriate scaling as follows:

$$J(k_1, k_2, \tau) := W_1 k_1 + W_2 k_2 + W_3 \tau \tag{5.55}$$

subject to feasibility of (5.34) and (5.35) and the side constraints on the optimization variables $k_{1_{min}} \leq k_1 \leq k_{1_{max}}$, $k_{2_{min}} \leq k_2 \leq k_{2_{max}}$ and $\tau_{min} \leq \tau \leq \tau_{max}$. In (5.55) the scalars W_i for $i = 1, 2, 3$ are the weights of the optimization variables k_1, k_2 and τ. The underlying rationale behind this objective function is to obtain the gain set that provides minimum control effort at a minimum possible level of delay. In this section the rate of decay is fixed a priori and there are three design parameters: k_1, k_2 and τ.

5.3.3.2 Case B: Maximum Rate of Decay α

In this section the problem of obtaining the maximum possible rate of decay α, for minimum possible gains k_1 and k_2 and for a minimum possible level of delay such that the system in (5.28) is exponentially stabilized, is considered. The collective optimization objective function for this problem is given by

$$J(k_1, k_2, \tau, \alpha) := \frac{W_1 k_1 + W_2 k_2 + W_3 \tau}{\alpha} \tag{5.56}$$

subject to feasibility of (5.34) and (5.35) and the side constraints on the optimization variables $k_{1_{min}} \leq k_1 \leq k_{1_{max}}$, $k_{2_{min}} \leq k_2 \leq k_{2_{max}}$, $\tau_{min} \leq \tau \leq \tau_{max}$, and $\alpha_{min} \leq \tau \leq \alpha_{max}$. In (5.56) the scalars W_i for $i = 1, 2, 3$ are the weights of the optimization variables k_1, k_2 and τ. The underlying rationale behind this objective function is to obtain the maximum possible decay rate α and the gain set that provides minimum control effort at a minimum possible level of delay.

5.3.3.3 Implementation of DIRECT Algorithm

Initially the bounds for the variables in Section 5.3.3.1 and Section 5.3.3.2 are chosen within desired range and the bounds will be normalized to be in the range $[0,1]$ since it is a requirement for the performance of the DIRECT optimization algorithm. When the feasibility of the LMI constraints in (5.34) and (5.35) for a specific set of the design parameters in the two cases is not satisfied, the cost associated with such a set is penalized by assigning it a large value. The idea is during the iterations, the DIRECT optimization procedure then eliminates that region from the search space. The MATLAB code used for the DIRECT optimization is available from the authors of [131]. The implementation of the DIRECT algorithm is given in Table 5.1 In the DIRECT implementation

Table 5.1: DIRECT implementation

```
bounds = [k1_min     k1_max;
          k2_min     k2_max;
          tau_min    tau_max;
          alpha_min  alpha_max];

[f_min,val_min] = Direct(objective_function,bounds);

function f = objective_function(val)
k1 = val(1); k2 = val(2); tau = val(3); alpha = val(4);
%
W1 = 1; W2 = 1; W3 = 1;

[boolean] = stability_lmi(k1,k2,tau,alpha);
% feasibility of conditions in Proposition 5.3.1

if boolean == 1 % Proposition 5.3.1 feasible
    f = (W1*k1 + W2*k2 + W3*tau)/alpha;
else
    f = 10^6; % Penalize with a large value
end
```

algorithm shown in Table 5.1, the variable '*alpha*' only applies to the implementation of Case B. For the implementation of Case A, the bounds are defined only for the three design parameters (k_1, k_2, τ) and the algorithm is implemented accordingly.

Remark 5.5: The implementation of the DIRECT algorithm along with Proposition 5.3.1 requires a search method along with the use of number of decision variables to check the feasibility of the inequalities in Proposition 5.3.1. This is a complex method to implement and may require considerable amount of time and computational effort depending on the number of divisions used to generate the hypercubes for the DIRECT algorithm, the number of agents and the number of partitions n_p used in Proposition 5.3.1. It is expected that it may take a few minutes to a few days depending on the number of agents and the computational sources available to the user and there is no guarantee of obtaining a solution as this method is non-convex.

5.4 Numerical Example

A network of $N = 4$ agents, described by (5.1) and (5.2) with matrices A, B and C as given in (5.3), connected over a nearest neighbour interconnection topology shown in Figure 5.1 is considered. The desired formation is given by

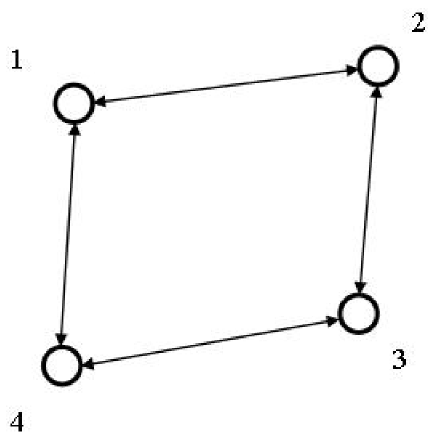

Figure 5.1: Communication topology

$$X_f = \begin{pmatrix} 3 & 0 & 3 & 0 & 3 & 0 & -3 & 0 & -3 & 0 & -3 & 0 & -3 & 0 & 3 & 0 \end{pmatrix}^T \tag{5.57}$$

i.e a square with $(x_i, y_i) = (\pm 3, \pm 3)$. The scaling parameter β is chosen as $\beta = 0.1$ to have a low spectral radius, i.e. the largest eigenvalue, of the matrix $I_N + \beta \mathscr{L}$ in (5.15). The number of partitions of the delay interval for *Proposition 5.3.1* was considered to be $n_p = 1$. After employing the transformations (5.16), (5.24) and (5.31) a system of the form (5.33) is obtained. The DIRECT algorithm has been employed for the two cases discussed in Section 5.3.3.1 and Section 5.3.3.2.

5.4.1 Case A: Fixed Rate of Decay

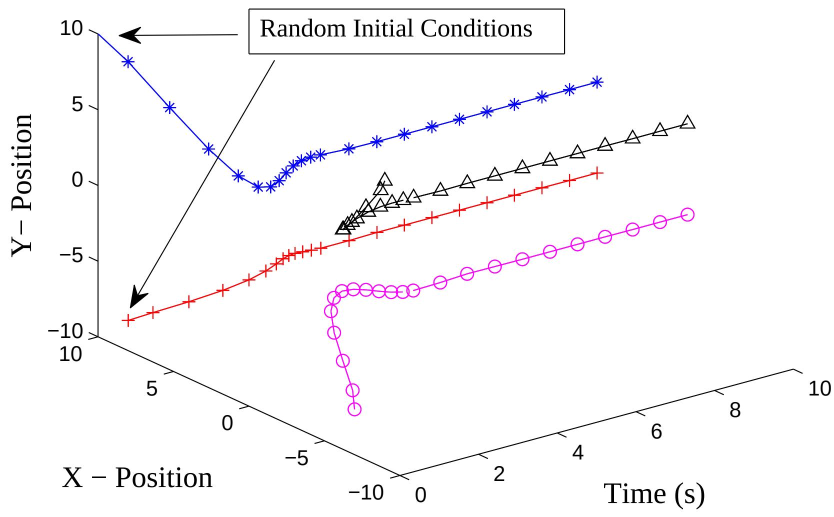

Figure 5.2: Formation of agents plotted against time - Case A

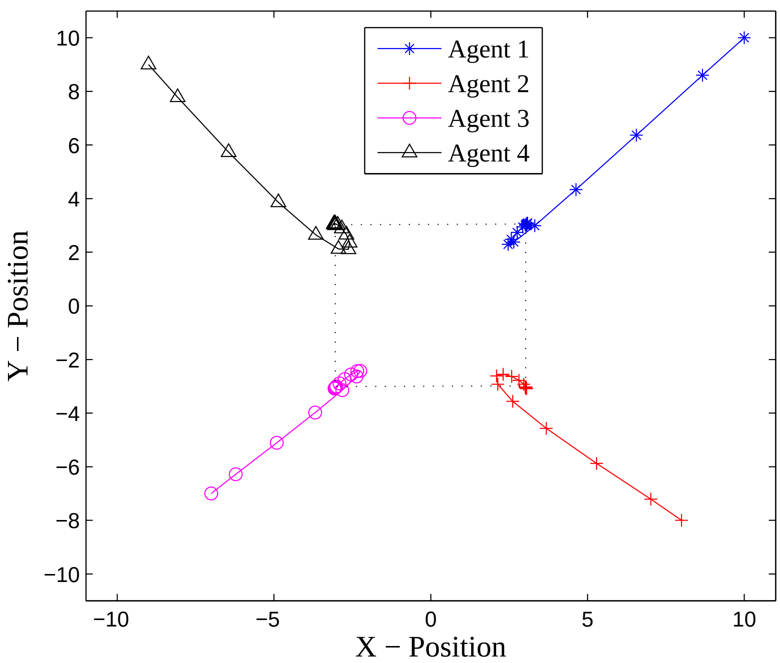

Figure 5.3: Formation of agents - Case A

For this case it is assumed that the required rate of decay is $\alpha = 1$. The weights for the optimization are considered as $W_1 = W_2 = W_3 = 1$. Then the optimization of the cost function in (5.55) is performed using the DIRECT algorithm in the following bounds

$$k_1 \in [10\ 20] \quad k_2 \in [10\ 20] \quad \tau \in [0.1\ 0.5] \tag{5.58}$$

The range of gains is selected so as to minimize oscillatory responses of the agents. The optimal gains k_1 and k_2 and delay τ obtained from within these bounds are

$$k_1 = 18.33.78 \quad k_2 = 11.29 \quad \tau = 0.1667 \tag{5.59}$$

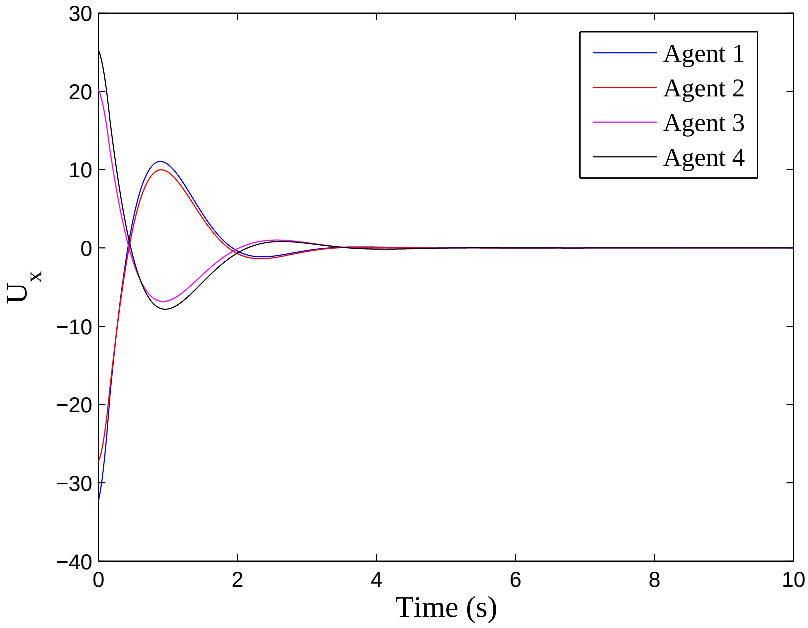

Figure 5.4: Control effort for agents in x-coordinate - Case A

The offset D for this case is calculated using (5.20). The system in (5.13) is simulated with the results obtained in (5.59). The simulations in this section were done using SIMULINK with a fixed-step size of $0.0001s$ and *ode4(Runge-Kutta)* solver. The initial condition for the delayed output in (5.13) is set as $X(t) = X(0)$ for the interval $t \in [-\tau_{opt}, 0]$. Figure 5.2 shows the agent settling into a formation from random initial conditions as a function of time. Figure 5.3 shows the formation of agents in a 2-D

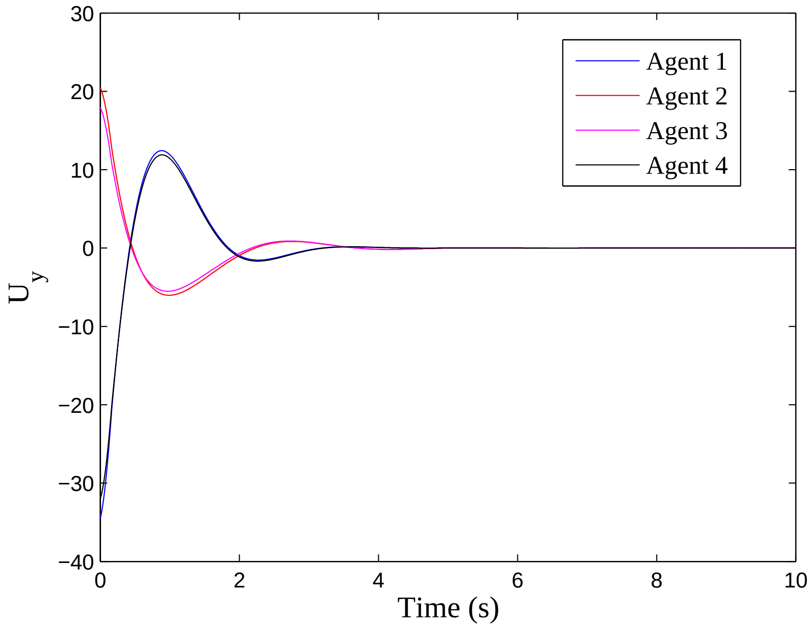

Figure 5.5: Control effort for agents in y-coordinate - Case A

plane. Figure 5.4 and Figure 5.5 show the control efforts U_x and U_y in x and y directions required to achieve the stabilization.

If the same gains k_1 and k_2 as obtained in (5.59) are employed with a very low value of delay $\tau = 0.01$ the system does not stabilize into a formation with the same initial conditions as shown in Figure 5.6 and Figure 5.7. If the same gains k_1 and k_2 as obtained in (5.59) are employed with a high value of delay $\tau = 1s$ the system again does not stabilize into a formation with the same initial conditions as shown in Figure 5.8 and Figure 5.9.

5.4.2 Case B: Maximum Rate of Decay α

In this section maximum possible value for the rate of decay α is obtained. The weights for the optimization are considered as $W_1 = W_2 = 0.1$ and $W_3 = 1$. The DIRECT algorithm has been employed with the following bounds

$$k_1 \in [1\ 15];\quad k_2 \in [1\ 15];\quad \tau \in [0.1\ 0.3];\quad \alpha \in [0.5\ 1.5] \tag{5.60}$$

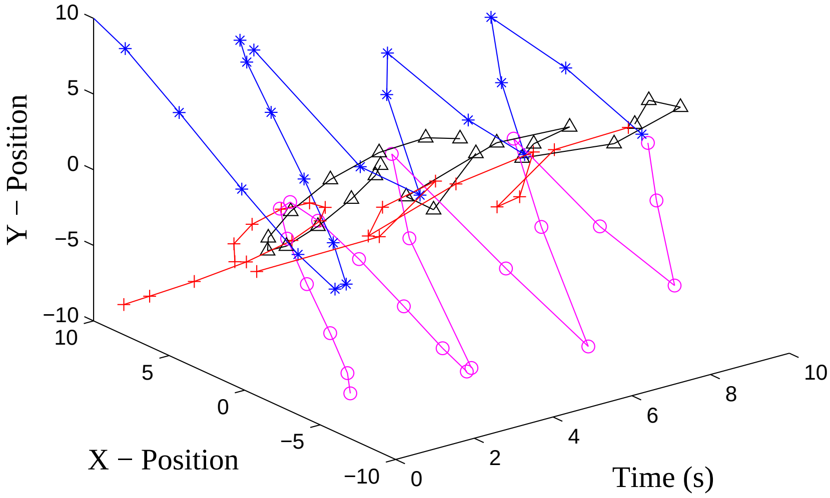

Figure 5.6: No formation of agents with low delay $\tau = 0.01s$ - Case A

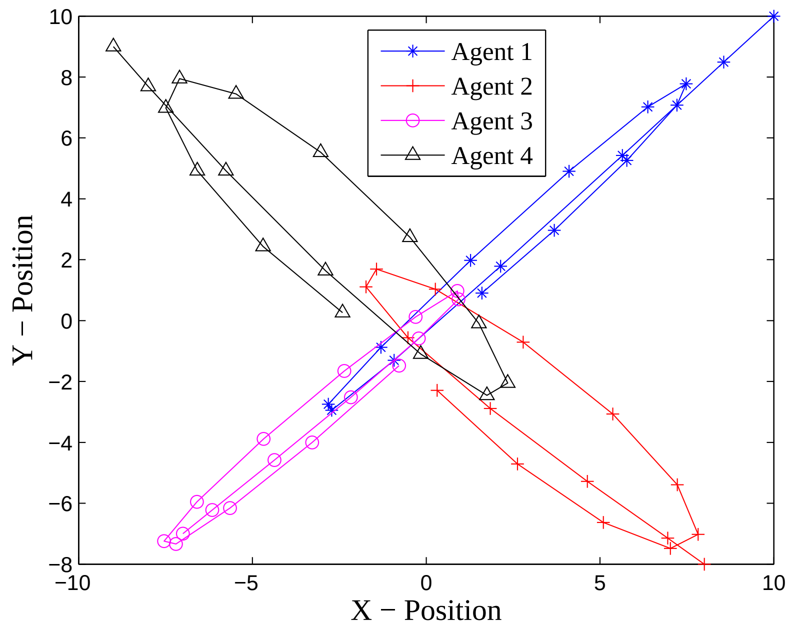

Figure 5.7: No formation of agents with low delay - Case A

The optimal gains k_1 and k_2, delay τ and the maximum possible rate of decay α obtained from within these bounds are

$$k_1 = 7.9936;\ k_2 = 4.8889;\ \tau_{opt} = 0.2913\ \alpha = 0.7915 \tag{5.61}$$

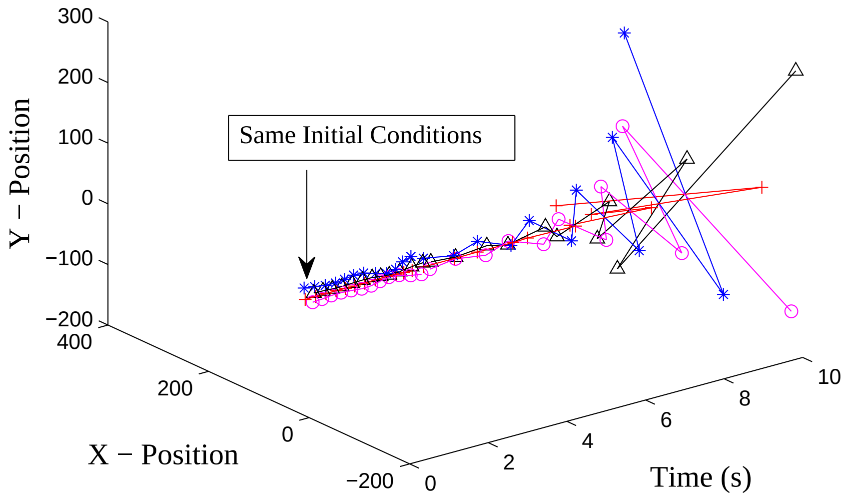

Figure 5.8: No formation of agents with high delay $\tau = 1s$ - Case A

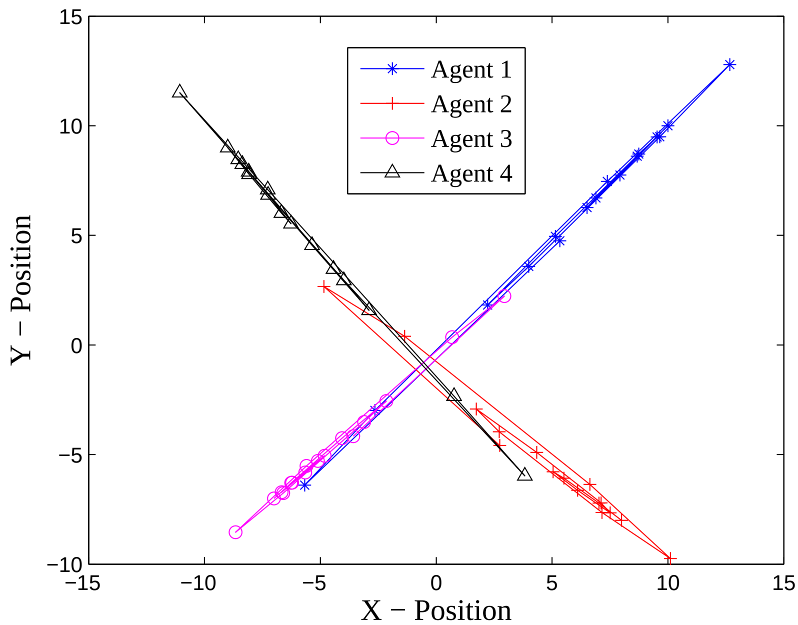

Figure 5.9: No formation of agents with high delay - Case A

The offset D is again calculated using (5.20). The system in (5.5) has been simulated with the control law in (5.12) using the values in (5.61). The simulations in this section were done using SIMULINK with a fixed-step size of $0.0001s$ and *ode4(Runge-Kutta)*

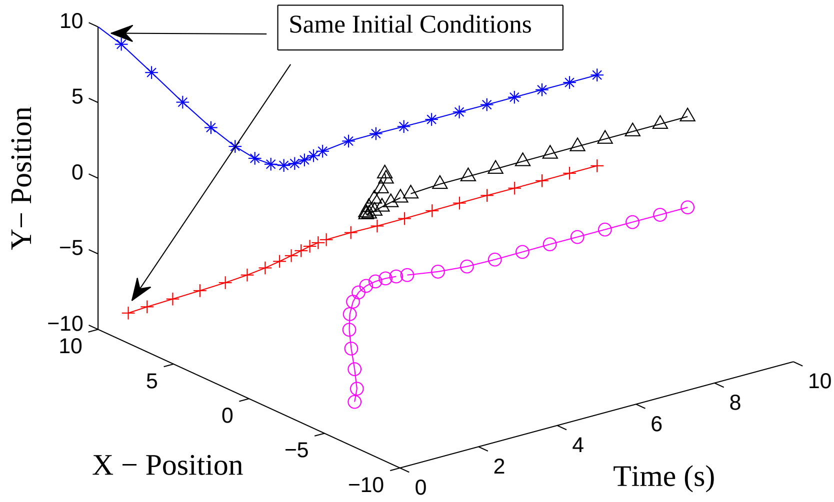

Figure 5.10: Formation of agents plotted against time - Case B

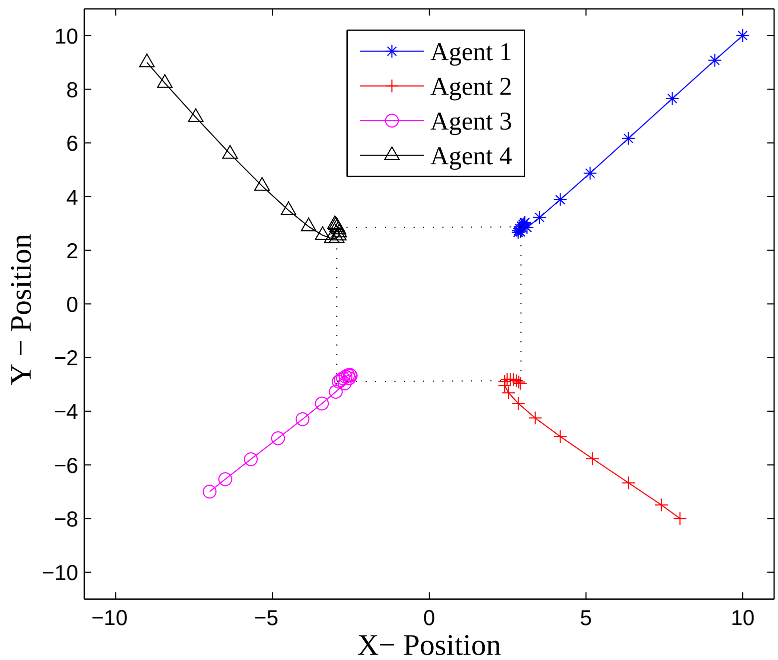

Figure 5.11: Formation of agents - Case B

solver. The initial condition for the delayed output in (5.13) is set as $X(t) = X(0)$ for the interval $t \in [-\tau_{opt}, 0]$ and the same initial conditions as in Case A are used in the simulation. Figure 5.10 shows the agents settling into a formation as a function of time. Figure 5.11 shows the agents form a square in a 2-D plane. The control efforts U_x and

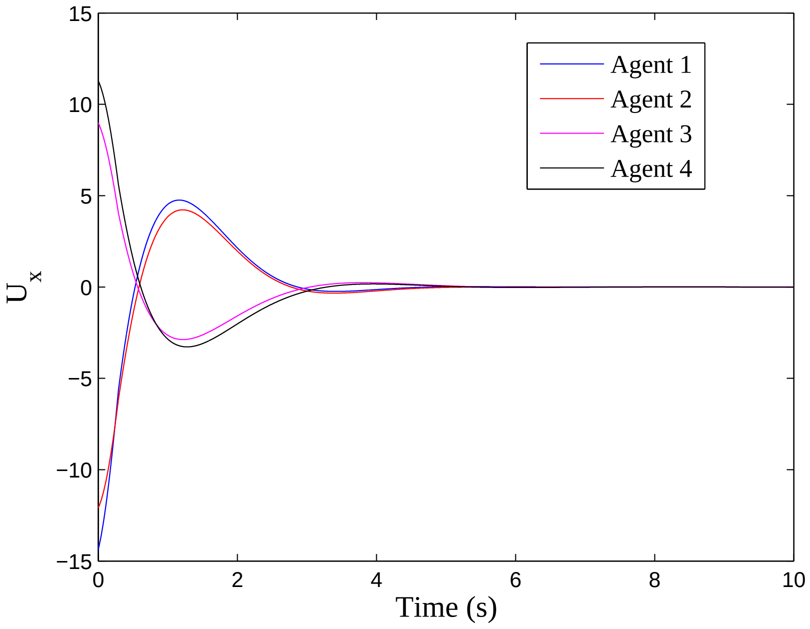

Figure 5.12: Control effort for agents in x direction - Case B

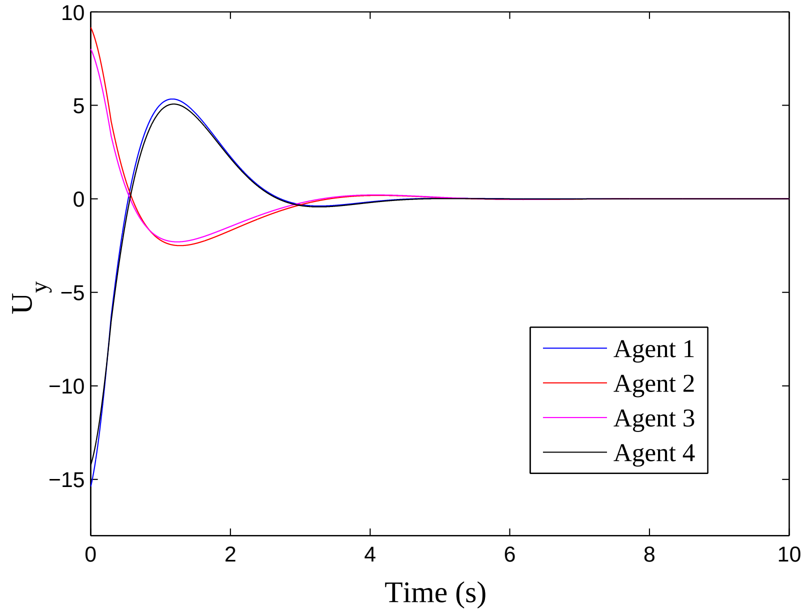

Figure 5.13: Control effort for agents in y direction - Case B

U_y in the x and y directions are shown in Figure 5.12 and Figure 5.13.

If the same gains k_1 and k_2 as obtained in (5.61) are employed with a very low value of delay $\tau = 0.01$ the system does not stabilize into a formation with the same initial conditions as shown in Figure 5.14 and Figure 5.15. If the same gains k_1 and k_2 as

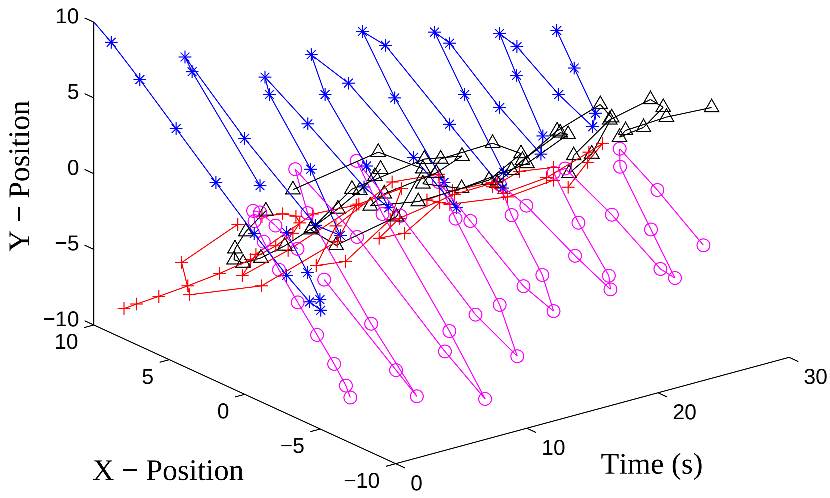

Figure 5.14: No formation of agents with low delay $\tau = 0.01s$ - Case B

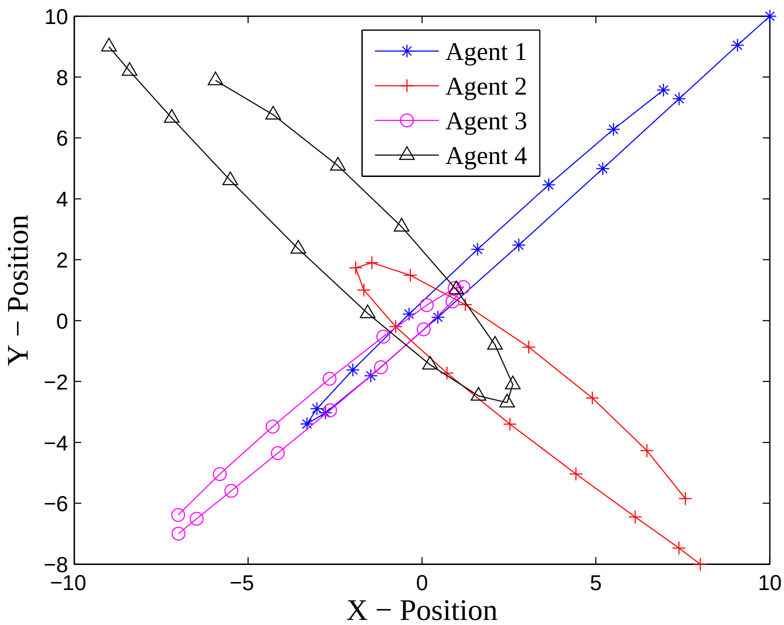

Figure 5.15: No formation of agents with low delay - Case B

obtained in (5.59) are employed with a high value of delay $\tau = 1s$ the system again does not stabilize into a formation with the same initial conditions as shown in Figure 5.16 and Figure 5.17.

Remark 5.6: It is important for the user to note that this method, the implementation of

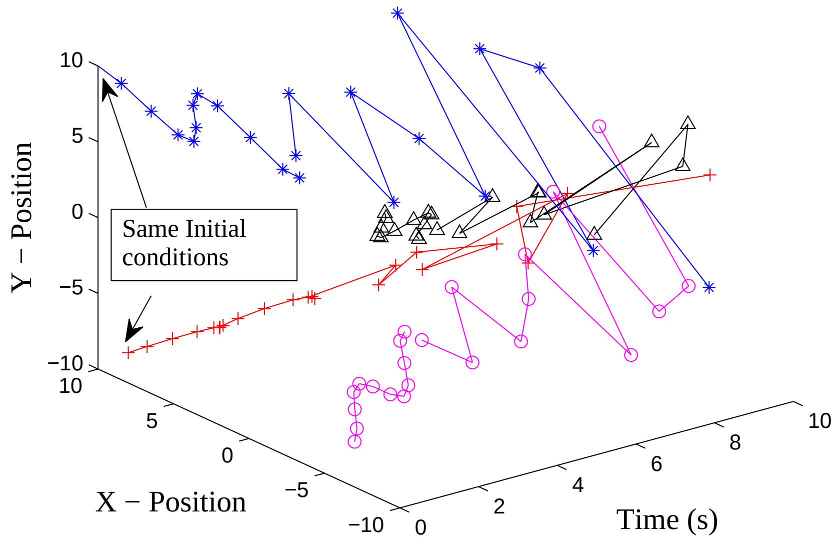

Figure 5.16: No formation of agents with high delay $\tau = 1s$ - Case B

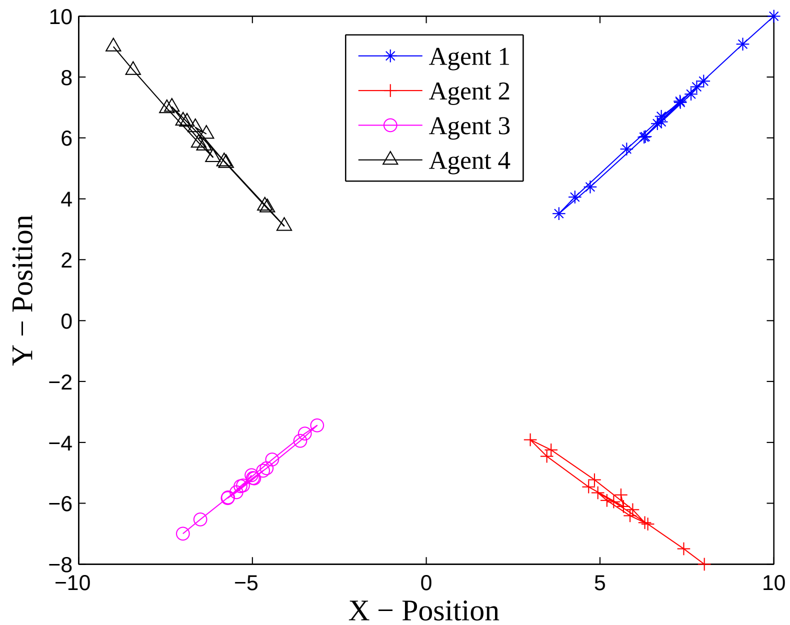

Figure 5.17: No formation of agents with high delay - Case B

DIRECT algorithm along with feasibility of Proposition 5.3.1, is a non-convex problem. The solution of such a problem is not guaranteed and the user will have to implement this method to know if a solution exists. The parameter β was set to a low value of $\beta = 0.1$ to lower the effects of the delayed relative information. The user of this method is required to exercise his freedom in tuning the range for the gains k_1 and k_2, the delay τ and the decay α (Case B only) and also tune the value of β if a solution has not found and a solution may or may not exist.

5.5 Concluding Remarks

In this chapter, the positive effect of delay in communication of relative information to stabilize a multi-agent system is explored. A network of vehicles moving in a plane, each described by double integrator dynamics for each planar direction, is considered. This network of vehicles is not stabilizable by static output feedback, i.e. with position information alone. A novel control law making use of an artificial delay to stabilize such a network is developed. The proposed control law is guaranteed to provide exponential convergence to a desired formation with pre-specified decay rate. The discretized Lyapunov-Krasovskii functional method of Proposition 5.3.1 was used to ascertain the stability of the network. The number of partitions n_p of the delay interval used for discretization was considered to be $n_p = 1$. The control design problem was posed as an optimization problem to obtain the control gains. The DIRECT algorithm was employed for two cases of design objectives. The first case assumed that the rate of decay α for the exponential convergence is fixed or specified a priori and obtains a possible minimum control effort with a minimum level of delay within specified bounds on the gains and the level of delay. The second case obtains maximum possible decay rate α along with minimum control effort and a minimum level of delay within specified bounds on the gains, the level of delay and the decay rate alpha. The rationale behind presenting the two cases can now be explained from the values of the gains and the delay τ obtained from the two cases. In Case A, a higher range of bounds on the gains k_1 and k_2 was necessary to achieve a fixed decay rate of $\alpha = 1$. In Case B lower gains are obtained by compromising the rate of decay from $\alpha = 1$ to $\alpha = 0.7915$. Consequently, the control efforts in Figure 5.12 and Figure 5.13 are considerably lower as compared to Figure 5.4 and Figure 5.5. The tradeoff of Case B is that higher level of delay τ is required. This will place a higher data storage requirement for the use of delayed values.

Chapter 6

Conclusions and Future Research

6.1 Conclusions

This thesis has considered the effects of delays on coordination problems of multi-agent systems. A central theme to this research is incorporating performance characteristics in the analysis and design of control laws to achieve stabilization of multi-agent systems. In a network of multiple agents connected over an information network delays are bound to occur in the communication of relative information. The effects of delays can be categorized into two types: negative and positive effects. In most cases delays have negative effects on the coordination of multiple agents. Delays can cause deviation from the cooperative goal of a multi-agent system, for example, the agents in a multi-agent system, expected to attain a formation, may deviate from their paths due to delays. On the contrary delays can have a positive effect on some systems. For example, introduction of delays may stabilize a system which is otherwise not stabilizable without delays. In this thesis both these effects of delays are explored.

In Chapter 3 distributed control laws are designed for a network of linear identical dynamical systems with sub-optimal LQR performance. The proposed control laws for the dynamical systems incorporate the relative information communicated to each system. The control law design employs an augmented LQR cost function in order to ensure simultaneous stabilization of dynamical systems. The augmented LQR cost function was shown to aid simultaneous convergence by penalizing relative information.

The control law design process employs spectral decomposition of the system represented at a network level and an optimization problem in the form of LMI constraints. In the control design process, the communication of relative information is assumed to be instantaneous. Subsequently an analysis of the closed loop system to ascertain the maximum level of delay that can be accommodated by the network has been performed using methods for stability analysis available in the literature on time-delay systems. The maximum bound on the level of delay was obtained for two types of delays: a) fixed delays and b) time-varying delays. For the case of fixed delays, two methods of stability analysis for linear time-delay systems from [129] were employed. The first method, described by Proposition 4 in Appendix B.3.2, gives a conservative estimate of the delay bound. The second method, described by Proposition 5 in Appendix B.4, partitions the delay interval and employs a discretized Lyapunov-Krasovskii functional. This method gives a more accurate value of the maximum permissible delay. Though a disadvantage of the second method is that it has considerably higher number of decision variables as compared to first and hence has considerably higher computation time. For the case of time-varying delays, an analysis method from [120] is employed to ascertain the maximum bound on the range of delay. Additionally, it was shown that the use of relative information has certain cost benefits up to a certain level of delay. It was also shown that more communication in a network may enhance the destabilizing effects of delays and lower the amount of delay permissible by the network.

In Chapter 3 a time-delay analysis was performed to ascertain the maximum bound on the delay, while the control design process assumed the exchange of instantaneous relative information. Subsequently, in Chapter 4 the assumption of the communication of instantaneous relative information is relaxed. The collection of dynamical systems is then represented as a time-delay system. Some of the existing analysis techniques for time-delay systems are then modified into distributed control design techniques to achieve stabilization of the network of dynamical systems with a certain level of LQR performance. LMI based distributed control design techniques were developed for both fixed and time-varying delays. For the case of fixed delays, delay independent and delay dependent control design methodologies were developed. The delay independent control design methodology obtains controller gains such that the system is stable for any arbitrary delay. The gains obtained by this method were very high as expected. In many practical cases it may not be possible to provide the large gains. When the level of delays are known it is adequate to employ delay dependent control design methods. Delay dependent control design methods ensure stabilization for a given level of delay. For the example considered, the delay dependent control design methodology developed shows that in some cases the use of delayed relative information may decrease the required control effort as compared to the control effort required for stabilizing individual agents without interconnections, while achieving similar performance. The methodology developed for the case of time-varying delays facilitates distributed control design for multi-agent systems with LQR performance for a bounded range of delay. Additionally, it was shown that more communication may increase the destabilizing effects of delays for the control laws proposed in this chapter. The scaling associated with the relative information needs to be lowered in magnitude incase the scaling provide by the measurement of relative information does not yield a feasible result.

In Chapter 3 and Chapter 4 an inherent assumption is that delays in communication of relative information have negative effects on the stabilization of the network of dynamical systems. In Chapter 5 the possibility of delays in relative information having a positive effect on the network is explored. A collection of vehicles moving in a two dimensional plane and each described by double integrator dynamics were considered. It was assumed that the velocity measurements of the vehicle were unavailable. Such a system is not stabilizable by static output feedback. Hence delayed relative output information communicated to each vehicle from its neighbours was exploited to stabilize the system. The stability criteria based on discretized Lyapunov-Krasovskii functionals in Appendix B.4 and DIRECT search optimization algorithm [130] were employed to obtain the control gains. The performance of the system was ensured by maximizing the exponential rate of decay for the system. The network was shown to stabilize into a desired formation by using delayed relative position information. It is important to note that the method proposed in this chapter is non-convex and a solution may or may not exist.

6.2 Future Research Directions

In Chapter 4 the stability analysis techniques that were modified into control design design techniques are not the most recent techniques and are conservative as compared to some of the recent methods. The methods used in this chapter facilitated tractable LMI representations under mild simplifications. Some of the less conservative analysis techniques such as the discretized Lyapunov-Krasovskii Functional method of Appendix B.4 were difficult to be modified into control design techniques with a certain level of LQR performance. As a possible future research direction the vast literature on stability analysis of time-delay systems can be explored and new control design methods with LQR performance can be obtained from suitable analysis methods which have less conservatism. From the complexities encountered during this research, it is believed that the descriptor system based stability analysis techniques may be easier to modify into control design techniques and may also lead to significant cost advantages. The reader should note that the work in this chapter is a preliminary attempt at modifying stability analysis techniques into LQR control design methods for multi-agent systems. More exploration needs to be carried out in this area and descriptor based methods is a good future research direction.

In Chapter 5 the stability criteria based on discretized Lyapunov-Krasovskii functional was used to along with DIRECT search optimization algorithm to obtain the control gains for exponential stabilization. As an alternative to exponential stabilization, an attempt was made to incorporate performance criteria by minimizing a LQ cost functional with insufficient results. As a future research direction the development of alternate methods which guarantee minimization of LQ cost functionals may be attempted. In Chapter 5 the control design process is partially distributed and methods to design fully distributed control laws is a future research direction.

In this thesis two important assumptions were made with respect to the communication of relative information. The communication of relative information is assumed to be bidirectional and the delays in relative information were assumed to be equal to exploit the properties of the resulting symmetric Laplacian. As a possible future research direction an attempt can be made to design control algorithms by considering uni-directional communication and unequal delays in communication of relative information. The design of such control laws will require considerably different approaches as compared to those considered in this thesis as the properties of a symmetric Laplacian cannot be exploited in the case of uni-directional communication. Another assumption made throughout this thesis is that the agents in a multi-agent system are identical. Consideration of non-identical agents in the control design processes presented in this thesis is also a future research direction.

Appendices

Appendix A

LMI Approach to LQR Control

A.1 Linear Matrix Inequalities

A linear matrix inequality (LMI) has the form

$$F(a) \triangleq F_0 + \sum_{i=m}^{l} a_i F_i \tag{A.1}$$

where $a \in \mathbb{R}^m$ is the variable and symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, are given [6]. In many problems the variables in an LMI are matrices. For example, the Lyapunov inequality to be satisfied for stability of the linear system $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, given by

$$A^T P + PA < 0 \tag{A.2}$$

is a LMI inequality where $P = P^T > 0$ is a matrix variable. Nonlinear convex matrix inequalities can be represented as LMIs using Schur complements [6]. For example the set of matrix inequalities

$$F_1(a) - F_2(a)F_3^{-1}(a)F_2^T(a) > 0 \tag{A.3}$$

where $F_i(a) \in \mathbb{R}^{n\times n}$ for $i = 1,2,3$ are matrix variables, can be represented by the following LMI

$$\begin{pmatrix} F_1(a) & F_2(a) \\ F_2^T(a) & F_3(a) \end{pmatrix} > 0 \tag{A.4}$$

A.2 Linear Quadratic Regulator

Consider the LTI system model given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. For the system given in (A.5) the objective of LQR control is to design the control law $u(t) = -Kx(t)$, where $K \in \mathbb{R}^{m\times n}$, such that the cost function

$$J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{A.6}$$

is minimized. Here the weighting matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite. The solution to this problem is given by

$$K = R^{-1}B^T P \tag{A.7}$$

where P is the solution of the continuous time matrix equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{A.8}$$

The matrix equation in (A.8) is called Algebraic Riccati Equation (ARE). The solution to the LQR control problem can also be posed as an LMI problem. Consider a Lyapunov function of the form $V(t) = x^T(t)Px(t)$, where $P \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Differentiating the Lyapunov function it is required that

$$\dot{V}(t) < -\frac{d}{dt}\int_0^t (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{A.9}$$

This implies that for an LQR control law to exist for the system in (A.5) there should exist matrices P and K such that the inequality

$$P(A-BK)+(A-BK)^T P+Q+K^T RK<0 \tag{A.10}$$

is satisfied. The inequality in (A.10) is not linear in the matrix variables P and K. Define $W=P^{-1}$ and pre and post multiply (A.10) by W and define an auxiliary matrix $Y=KW$. The resulting equation can be represented as a matrix inequality by using Schur compliment [6] as

$$\begin{pmatrix} AW+W^T A-BY-Y^T B^T+Q & Y^T \\ Y & -R^{-1} \end{pmatrix}<0 \tag{A.11}$$

where $W>0$ is symmetric. Integrate (A.9) from 0 to ∞ to obtain a maximum bound as $J<x^T(0)Px(0)$. Thus minimizing $Trace(P)$ will result in an optimal LQR performance. Define an auxiliary matrix $Z\in\mathbb{R}^{n\times n}$ such that

$$\begin{pmatrix} -Z & I_{n\times n} \\ I_{n\times n} & -W \end{pmatrix}<0 \tag{A.12}$$

By Schur compliment equation (A.12) yields $Z>W^{-1}=P$. Hence minimizing $Trace(Z)$ minimizes $Trace(P)$. The LMIs (A.11) and (A.12) along with $W>0$ define a set of LMIs to be solved simultaneously. The LQR optimization problem is then defined as minimizing $Trace(Z)$ while satisfying the conditions $W>0$ and (A.11). If the LMI in (A.11) is feasible, then the gain matrix K can be obtained as $K=YW^{-1}$. The reader is referred to standard texts such as [6] for further reading on LMIs.

Appendix B

Stability of Time-Delay Systems

Dynamical systems are often described by representing the evolution of their state variables with respect to time as a differential equation. Those dynamical systems for which the future state evolution depends only on the present state variable can be modelled by ordinary differential equations (ODE) of the form

$$\dot{x}(t) = f(t, x(t)) \tag{B.1}$$

where $x(t) \in \mathbb{R}^n$ are the state variables. However for some systems the future state evolution also depends on the past values of the state variables along with the current state variables so that

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t))) \tag{B.2}$$

where $\tau(t)$ is a fixed or time-varying delay. Such systems are called as *time-delay systems*. In this thesis a linear time-delay system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) \tag{B.3}$$

where $x \in \mathbb{R}^n$ and A_0, $A_1 \in \mathbb{R}^{n \times n}$ occur frequently where the delay $\tau(t)$ can be fixed or time-varying. This class of time-delay systems has been studied in [129], [120]-[123]. The initial condition is given by

$$x_0 = \phi \tag{B.4}$$

where $\phi \in \mathrm{C}[-\tau, 0] \rightarrow \mathbb{R}$. The sections that follow state the different tools which have been used commonly in the literature to analyze the stability of such time-delay systems.

B.1 Lyapunov-Krasovskii Functionals

The stability of the time-delay system in (B.3) is ascertained by the use of Lyapunov-Krasovskii functionals [129]. In this thesis bounded quadratic Lyapunov-Krasovskii functionals have been used to ascertain stability of time-delays systems. An example of a bounded quadratic Lyapunov-Krasovskii functional for the system in (B.3) with fixed delay $\tau(t) = \tau$ is given by

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} \left[\int_{\theta}^{t} x^T(s)S(\theta)x(s)ds \right] d\theta \tag{B.5}$$

where P, $S(\theta) \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices and $x_t = x(t+\theta)$ for $t - \tau \leq \theta \leq t$. A Lyapunov-Krasovskii functional generally consists of quadratic terms which are based on the current state information augmented with quadratic integral terms involving the delay τ as shown in (B.5). The stability of the time-delay system in (B.3) is ascertained by the following theorem:

Proposition 1. *A time-delay system is asymptotically stable if there exists a bounded quadratic Lyapunov-Krasovskii functional $V(x(t))$ such that for some $\varepsilon > 0$, it satisfies*

$$V(x_t) \geq \varepsilon \|x_t\|^2 \tag{B.6}$$

and its derivative along the system trajectory $\dot{V}(x(t))$ satisfies

$$\dot{V}(x_t) \leq -\varepsilon \|x_t\|^2 \tag{B.7}$$

Proof. The reader is referred to Proposition 5.2 [129] for the proof. □

In [129], stability criteria based on bounded Lyapunov - Krasovskii functionals are derived for the system in (B.3) with fixed delays. Two types of stability criteria for fixed delays based on Proposition 1 are presented in the following sections: Delay independent stability criteria and delay dependent stability criteria. The delay independent stability criteria tests if a linear system of the form (B.3) is stable for any arbitrary fixed delay. In reality most delay systems are only stable for some range of delays and become unstable once the delay exceeds the range. For such systems delay dependent stability criteria are obtained to ascertain stability in the case of a permissible delay. In the sections that follow the delay independent and delay dependent stability criteria that have been employed in this thesis are discussed.

B.2 Delay Independent Stability Criteria Based on Lyapunov - Krasovskii Stability Theorem

This criteria uses a quadratic Lyapunov-Krasovskii functional for a fixed delay τ to obtain the following proposition:

Proposition 2. *The system described in (B.3) is asymptotically stable if there exist real symmetric matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$, such that*

$$P > 0 \tag{B.8}$$

$$\begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} < 0 \tag{B.9}$$

is satisfied

Proof. The Lyapunov-Krasovskii functional considered for proof of this theorem is given by

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Sx(s)ds \tag{B.10}$$

The reader is referred to *Proposition 5.14 in [129]* for the proof. □

Remark: The conditions in Proposition 2 are independent of τ.

B.3 Delay Dependent Stability Criteria Based on Lyapunov - Krasovskii Stability Theorem

Two delay dependent stability criteria for fixed delays are presented in this section. These criteria are based on an explicit and an implicit model transformation employed in [129]. The explicit model transformation is obtained from the observation that

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)d\theta \tag{B.11}$$

for $t \geq \tau$ and $-\tau \leq \theta \leq 0$. For $t \geq \tau$, the system in (B.3) can be transformed using (B.11) into

$$\dot{x}(t) = (A_0 + A_1)x(t) + \int_{-\tau}^{0} (-A_1 A_0 x(t+\theta) - A_1 A_1 x(t-\tau+\theta))d\theta \tag{B.12}$$

with the initial condition

$$x(\theta) := \begin{cases} \phi(\theta) & -\tau \leq \theta \leq 0 \\ solution\ of\ (B.3)\ with\ initial\ condition\ (B.4) & 0 \leq \theta \leq \tau \end{cases} \tag{B.13}$$

Implicit model transformation is obtained by using auxiliary matrices in the derivative condition for the Lyapunov-Krasovskii functionals. The derivation of implicit model transformations is omitted from this Appendix. The reader is referred to [129] for further reading on implicit model transformations.

B.3.1 Stability Criteria using Explicit Model Transformation

In [129], Proposition 5.16 derives delay dependent stability by using the explicit model transformation of the system in (B.12). The criteria for a fixed delay is given by the following proposition:

Proposition 3. *The system in (B.3) is asymptotically stable if there exist real symmetric* *matrices P, S_0 and $S_1 \in \mathbb{R}^{n\times n}$ such that*

$$P > 0 \tag{B.14}$$

$$\begin{pmatrix} \hat{M} & -PA_1A_0 & -PA_1^2 \\ * & -S_0 & 0 \\ * & * & -S_1 \end{pmatrix} < 0 \tag{B.15}$$

where

$$\hat{M} = \frac{1}{\tau}(P(A_0+A_1)+(A_0+A_1)^T P)+S_0+S_1 \tag{B.16}$$

Proof. For the system in (B.12), the transformed system is obtained by shifting the initial time to $t = \tau$. The new delay interval is then given by $[-2\tau, 0]$. The Lyapunov-Krasovskii functional for the transformed system is given by

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t}\int_{\vartheta}^{t} x^T(s)S_0x(s)dsd\vartheta + \int_{t-2\tau}^{t-\tau}\int_{\theta}^{t-\tau} x^T(s)S_1x(s)dsd\theta \tag{B.17}$$

where $t-\tau \leq \vartheta \leq t$ and $t-2\tau \leq \theta \leq t-\tau$. The reader is referred to *Proposition 5.16 [129]* for the proof of this proposition. □

Remark: As stated in *Proposition 5.16 [129]* a restricted form of Lyapunov-Krasovskii functional results when

$$S_k = \alpha_k P;\ k = 0,1; \tag{B.18}$$

where α_0 and α_1 are positive scalars. This restricted form of the Lyapunov-Krasovskii functional has been used in the delay dependent control synthesis method for a fixed delay in Section 4.3.2.

B.3.2 Stability Criteria using Implicit Model Transformation

The delay dependent stability criterion obtained in Section B.3.1 is shown to be highly conservative in [129]. A stability criteria for a fixed delay based on implicit model transformation of the system in (B.3) is obtained in [129] and is shown to be less conservative than Proposition 3. The stability criteria is stated by the following proposition:

Proposition 4. *The system in (B.3) is asymptotically stable if there exist real matrices $Y_1^T = Y_1$, $S^T = S$, Y_2, $\in \mathbb{R}^{n\times n}$ and $P^T = P \in \mathbb{R}^{n\times n}$ such that*

$$P > 0 \tag{B.19}$$

$$\begin{pmatrix} \hat{N} & PA_1 - Y_2 & -A_0^T Y_2^T \\ * & -S & -A_1^T Y_2^T \\ * & * & -\frac{1}{\tau_m} Y_1 \end{pmatrix} < 0 \tag{B.20}$$

where

$$\hat{N} = PA_0 + A_0^T P + S + \tau_m Y_1 + Y_2 + Y_2^T \tag{B.21}$$

Proof. The Lyapunov-Krasovskii functional considered for the proof of this theorem is given by

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\theta)Sx(\theta)d\theta + \int_{t-\tau}^{t}\int_{\theta}^{t} g^T(\zeta)\bar{Z}g(\zeta)d\zeta d\theta \tag{B.22}$$

where $t-\tau \le \theta \le t$, $-\tau \le \zeta \le 0$, $\bar{Z} \in \mathbb{R}^{n\times n}$ is symmetric and

$$g(\zeta) = A_0 x(t) + A_1 x(t+\zeta) \tag{B.23}$$

The reader is referred to *Proposition 5.17 in [129]* for the proof. □

B.4 Discretized Lyapunov-Krasovskii Functional Method

The stability criteria obtained in Proposition 4 are still found to be conservative [129]. To overcome this conservatism a method based on discretization of the delay interval $[-\tau\ 0]$, where τ is a fixed delay, into n_p parts, is proposed in [129]. A discretized Lyapunov-Krasovskii functional is then considered over these subintervals. The discretized Lyapunov-Krasovskii functional method is stated by the following proposition:

Proposition 5. *The system described in (B.3) is asymptotically stable if there exist $n \times n$*

matrices $P = P^T$; $\bar{Q}_p$, $S_p = S_p^T$, $p = 0,\ldots,n_p$; $R_{pq} = R_{qp}^T$, $p = 0,\ldots,n_p$, $q = 0,\ldots,n_p$ *and* $h = \tau/n_p$ *such that*

$$\begin{pmatrix} P & \tilde{F} \\ * & \tilde{R}+\tilde{S} \end{pmatrix} > 0 \tag{B.24}$$

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ * & R_d + S_d & 0 \\ * & * & 3S_d \end{pmatrix} > 0 \tag{B.25}$$

where

$$\tilde{F} = \begin{pmatrix} \bar{Q}_0 & \bar{Q}_1 & \ldots & \bar{Q}_{n_p} \end{pmatrix} \tag{B.26}$$

$$\tilde{R} = \begin{pmatrix} R_{00} & R_{01} & \ldots & R_{0n_p} \\ R_{10} & R_{11} & \ldots & R_{1n_p} \\ . & . & \ldots & . \\ R_{n_p0} & R_{n_p1} & \ldots & R_{n_pn_p} \end{pmatrix} \tag{B.27}$$

$$\tilde{S} = \begin{pmatrix} \frac{1}{h}S_0 & \frac{1}{h}S_1 & \ldots & \frac{1}{h}S_{n_p} \end{pmatrix} \tag{B.28}$$

and

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} \\ * & \Delta_{11} \end{pmatrix} \tag{B.29}$$

$$\Delta_{00} = -PA_0 - A_0^T P - \bar{Q}_0 - \bar{Q}_0^T - S_0 \tag{B.30}$$

$$\Delta_{01} = \bar{Q}_{n_p} - PA_1 \tag{B.31}$$

$$\Delta_{11} = S_{n_p} \tag{B.32}$$

$$
\begin{aligned}
S_d &= \mathscr{D}iag\begin{pmatrix} S_{d1} & S_{d2} & \dots & S_{dn_p} \end{pmatrix} && \text{(B.33)}\\
S_{dp} &= S_{(p-1)} - S_p && \text{(B.34)}\\
R_d &= \begin{pmatrix} R_{d11} & R_{d12} & \dots & R_{d1n_p} \\ R_{d21} & R_{d22} & \dots & R_{d2n_p} \\ . & . & \dots & . \\ R_{dn_p1} & R_{dn_p2} & \dots & R_{dn_pn_p} \end{pmatrix} && \text{(B.35)}\\
R_{dpq} &= h(R_{(p-1,q-1)} - R_{pq}) && \text{(B.36)}\\
D^s &= \begin{pmatrix} D_1^s & D_2^s & \dots & D_{n_p}^s \end{pmatrix} && \text{(B.37)}\\
D_p^s &= \begin{pmatrix} D_{0p}^s \\ D_{1p}^s \end{pmatrix} && \text{(B.38)}\\
D_{0p}^s &= \frac{h}{2}A_0^T(\bar{Q}_{(p-1)} + \bar{Q}_p) + \frac{h}{2}(R_{(0,p-1)} + R_{0p}) - (\bar{Q}_{(p-1)} - \bar{Q}_p) && \text{(B.39)}\\
D_{1p}^s &= \frac{h}{2}A_1^T(\bar{Q}_{(p-1)} + \bar{Q}_p) - \frac{h}{2}(R_{(n_p,p-1)} + R_{n_pp}) && \text{(B.40)}\\
D^a &= \begin{pmatrix} D_1^a & D_2^a & \dots & D_{n_p}^a \end{pmatrix} && \text{(B.41)}\\
D_p^a &= \begin{pmatrix} D_{0p}^a \\ D_{1p}^a \end{pmatrix} && \text{(B.42)}\\
D_{0p}^a &= -\frac{h}{2}A_0^T(\bar{Q}_{(p-1)} - \bar{Q}_p) - \frac{h}{2}(R_{(0,p-1)} - R_{0p}) && \text{(B.43)}\\
D_{1p}^a &= -\frac{h}{2}A_1^T(\bar{Q}_{(p-1)} + \bar{Q}_p) + \frac{h}{2}(R_{(n_p,p-1)} - R_{n_pp}) && \text{(B.44)}
\end{aligned}
$$

Proof. The delay interval $[-\tau, 0]$ is divided into n_p segments of equal length $h = \frac{\tau}{n_p}$ given by $[\theta_p, \theta_{p-1}]$, $p = 1, \dots, n_p$, where $\theta_p = -ph$ and $\theta_0 = 0$. This divides the square $\mathscr{S} = [-\tau, 0] \times [-\tau, 0]$ into $n_p \times n_p$ smaller squares $\mathscr{S}_{pq} = [\theta_p, \theta_{p-1}] \times [\theta_p, \theta_{p-1}]$. Each small square is further divided into two smaller rectangles. The reader is referred to [129] for further details.

The discretized Lyapunov-Krasovskii functional employed for this proof is given by

$$
V(x_t) = x^T(t)Px(t) + 2x^T(t)\sum_{p=1}^{n_p} V_{\bar{Q}} + \sum_{p=1}^{n_p}\sum_{q=1}^{n_p} V_R + \sum_{p=1}^{n_p} V_S \tag{B.45}
$$

where

$$V_{\bar{Q}} = \int_0^1 \bar{Q}^{(p)}(\zeta)x(t+\theta_p+\zeta h)hd\zeta \tag{B.46}$$

$$V_R = \int_0^1 (\int_0^1 x^T(t+\theta_p+\zeta h)R^{(pq)}(\zeta,\eta)x(t+\theta_p+\zeta h)hd\eta)hd\zeta \tag{B.47}$$

$$V_S = \int_0^1 x^T(t+\theta_p+\zeta h)S^{(p)}(\zeta)x(t+\theta_p+\zeta h)hd\zeta \tag{B.48}$$

and

$$\bar{Q}^{(p)}(\zeta) = (1-\zeta)\bar{Q}_p+\zeta\bar{Q}_{p-1} \tag{B.49}$$

$$R^{(pq)}(\zeta,\eta) := \begin{cases} (1-\zeta)R_{pq}+\eta R_{p-1,q-1}+(\zeta-\eta)R_{p-1,q} & \zeta \geq \eta \\ (1-\eta)R_{pq}+\zeta R_{p-1,q-1}+(\eta-\zeta)R_{p,q-1} & \zeta < \eta \end{cases} \tag{B.50}$$

$$S^{(p)}(\zeta) = (1-\zeta)S_p+\zeta S_{p-1} \tag{B.51}$$

for $p = 1,\dots,n_p$, $q = 1,\dots,n_p$, $0 \leq \zeta \leq 1$ and $0 \leq \eta \leq 1$. The reader is referred to Proposition 5.22 [129] for the proof of this theorem. □

B.5 Stability Criteria for Time-Varying Delays

In [120] delay dependent stability criteria are derived for a linear time varying delay systems given by

$$\dot{x}(t) = A_0x(t)+A_1x(t-\tau_1(t))+A_2x(t-\tau(t)) \tag{B.52}$$

where $x \in \mathbb{R}^n$, A_0, A_1 and $A_2 \in \mathbb{R}^{n\times n}$. To comply with the structure of (B.3) it is considered that $A_2 \equiv 0$. The linear system is then given by

$$\dot{x}(t) = A_0x(t)+A_1x(t-\tau(t)) \tag{B.53}$$

In this thesis the case of a bounded time-varying delay where $\tau(t)$ are continuous functions satisfying $\forall t \geq 0$, $0 \leq \tau(t) \leq \tau_m$ is considered, where τ_m is the maximum bound on the time-varying delay. The stability criteria for this case is stated in Corollary 1 in

[120]. In [120] the system in B.53 is represented in its equivalent descriptor form given by

$$E\dot{\bar{x}}(t) = \begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \tag{B.54}$$

$$= \begin{pmatrix} 0 & I_n \\ (A_0+A_1) & -I_n \end{pmatrix} \bar{x}(t) - \begin{pmatrix} 0 \\ A_1 \end{pmatrix} \int_{t-\tau(t)}^{t} y(s)ds \tag{B.55}$$

where $\bar{x}(t) = \mathscr{C}ol(x(t), y(t))$, $\dot{x}(t) = y(t)$ and $E = \mathscr{D}iag(I_n, 0)$.

The appropriate corollary as applicable to the structure in (B.55) is given by the following proposition:

Proposition 6. *The system in (B.53) is asymptotically stable if there exist symmetric matrices P_1 and $\bar{R} \in \mathbb{R}^{n\times n}$ and matrices P_2, P_3, Z_1, Z_2 and $Z_3 \in \mathbb{R}^{n\times n}$ such that*

$$P > 0 \tag{B.56}$$

$$\bar{R} > 0 \tag{B.57}$$

$$\begin{pmatrix} \bar{R} & [0\, A_1^T]P^T \\ * & \bar{Z} \end{pmatrix} > 0 \tag{B.58}$$

$$P^T \begin{pmatrix} 0 & I_n \\ A_0+A_1 & -I_n \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ A_0+A_1 & -I_n \end{pmatrix}^T P + \tau_m \bar{Z} + \begin{pmatrix} 0 & 0 \\ 0 & \tau_m \bar{R} \end{pmatrix} < 0 \tag{B.59}$$

where

$$\bar{Z} = \begin{pmatrix} Z_1 & Z_2 \\ * & Z_3 \end{pmatrix}; P = \begin{pmatrix} P_1 & 0 \\ P_2 & P_3 \end{pmatrix} \tag{B.60}$$

Proof. The Lyapunov-Krasovskii functional considered for the proof of this theorem is given by

$$V(t) = V_1(t) + V_2(t) \tag{B.61}$$

where

$$V_1(t) = \bar{x}^T(t) E P \bar{x}(t) \tag{B.62}$$

$$V_2(t) = \int_{-\tau_m}^{0} \int_{t+\theta}^{t} y^T(s) S y(s) ds d\theta \tag{B.63}$$

The reader is referred to Corollary 1 [120] for the proof of this proposition. □

B.6 Additional Comments

In this section a few suggestions to the user of the stability analysis methods presented in this appendix are provided. The delay independent stability analysis method in Section B.2 provides a check whether a system of the form (B.3) is stable for any arbitrary delay τ. The delay dependent stability analysis methods described in Section B.3.1 and Section B.3.2 provide stability checks for a fixed level of delay. The user should note that though the methods in Section B.3.1 and Section B.3.2 provide stability for a fixed level of delay τ for the system in (B.3), these methods imply that the system is stable for any fixed level of delay τ in the interval $[0\ \tau_m]$, where τ_m is the maximum level of delay for which the system in (B.3) is stable. It is obvious that the delay dependent stability analysis method for time-varying delays $\tau(t)$ presented in Section B.5 implies stability for delays in the interval $0 \leq \tau(t) \leq \tau_m$. The discretized Lyapunov-Krasovskii functional method on Section B.4 is different in this sense to the other methods. The stability analysis method in Section B.4 provides a stability check only for the specific level delay τ for which Proposition 5 in Section B.4 is satisfied. This is due to the discretization of the delay interval $[-\tau, 0]$ and subsequent use of the discretized Lyapunov-Krasovskii functional in (B.45). For example, this is the reason why a discretized Lyapunov-Krasovskii functional method needs to be used in Chapter 5 to obtain a level of delay for which a double integrator system describing motion in a plane is stabilized by delayed static output feedback with position information only.

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