Generalised distributivity and the logic of metric spaces

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Abstract

The aim of the thesis is to work towards a many-valued logic over a commutative unital quantale and, at the same time, towards a generalisation of coalgebraic logic enriched over a commutative unital quantale Ω. This is done by noticing that the contravariant powerset adjunction can be generalised to categories enriched over a commutative unital quantale. From here we define categorical algebras for the monad generated by this adjunction. We finish by showing that these categorical algebras are algebras over Set with operations and equations, and show that in some cases we can restrict the arity of those operations to be finite.

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Chapter 1

Introduction

1.1 Motivation

One can answer in a multitude of ways to the question "what is logic?". Logic was for a long time a philosophical subject. It only became a branch of mathematics in 1847, when George Boole defined Boolean algebras in his book "The Mathematical Analysis of Logic". The idea that one can treat a logic as just a set with some operations and equations had a huge impact, as one could create new logical systems by just adding/deleting operations or equations.

This type of logical system, including propositional, intuitionistic, modal logic etc., has been studied ever since. Changing the signature is not the only way one can define new logical systems; another way is to change the set of truth values. This is motivated by the fact that there are statements which are neither true nor false or do not have a clear truth value. A good example of a statement which is neither true nor false is the liar paradox: "this sentence is a lie". On the other hand the statement "this person is tall" has no clear truth value, as we do not have clear boundary for tall persons. Thus a new branch of mathematical logic was needed to formalise statements which had no clear true or false value. This new branch, now called many valued logic, was pioneered at the beginning of the nineteenth century by Łukasiewicz and Post. For a more detailed history on many valued logics see [25].

In all the above logics the number of truth values varies, from two to infinity, but there is always an order relation given by implication. Thus a natural question appears: what would happen if we added more structure on the set of truth values? For example what would happen if we added a metric structure. The first paper in this direction is written by Lawvere [18]. Also we would like to keep the extra structure on algebras.

This relates with the growing interest in coalgebras enriched over posets or, more generally, enriched over a commutative quantale, see Rutten [28] and Worrell [38]. In particular, the question of the existence of a coalgebraic logic in this setting has been asked in [2].

1.2 Coalgebraic logic

In the non-enriched situation we start with a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and ask for a logic that allows us to completely describe T-coalgebras up to bisimilarity.

More specifically, we would like to ensure *strong expressivity* in the sense that for any property $p \subseteq X$ of any T-coalgebra (X, ξ) there is a formula ϕ such that p coincides up to bisimilarity with the semantics $\llbracket \phi \rrbracket_{(X,\xi)}$ of ϕ on (X, ξ).

Moreover, we would like to have *completeness* in the sense that if $\llbracket \phi \rrbracket_{(X,\xi)} \subseteq \llbracket \phi \rrbracket_{(X,\xi)}$ then $\phi \leq \psi$ in the initial algebra of formulas.

To achieve the above (ignoring size problems for the moment), the **first step** is to let $LA = [T([A,2]),2]$ in

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left, "{[-,2]}"] \arrow[loop below, "L^{\mathrm{op}}"] \arrow[r, phantom, "\top"] & \mathsf{Set} \arrow[l, bend left, "{[-,2]}"] \arrow[loop below, "T"]
\end{tikzcd} \tag{1.1}$$

and to treat the initial L-algebra, if it exists, as the “Lindenbaum-algebra” of T. This terminology is justified insofar as the adjoint transpose

$$\delta : L([-,2]) \longrightarrow [T-,2]$$

of the isomorphism $LA \longrightarrow [T([A,2]),2]$ allows us to define the semantics $\llbracket\,\rrbracket_{(X,\xi)}$ with regard to a coalgebra (X, ξ) as the unique arrow from the initial L-algebra $\iota : LI \longrightarrow I$ as in

$$\begin{tikzcd}
LI \arrow[rr, "\iota"] \arrow[d, swap, "L\llbracket\,\rrbracket_{(X,\xi)}"] & & I \arrow[d, "\llbracket\,\rrbracket_{(X,\xi)}"] \\
L([X,2]) \arrow[r, swap, "\delta_X"] & {[TX,2]} \arrow[r, swap, "{[\xi,2]}"] & {[X,2]}
\end{tikzcd} \tag{1.2}$$

But the reason why, at this stage, we cannot truly speak of $\iota : LI \longrightarrow I$ as a Lindenbaum algebra is that it lives in $\mathsf{Set}^{\mathrm{op}}$ and is not (yet) an algebra over Set with elements and operations in the usual sense.

The **second step**, then, consists in using the well-known fact that $[-, 2] : \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is monadic and, therefore, $\mathsf{Set}^{\mathrm{op}}$ is equivalent to a category of algebras defined by operations and equations. In particular, we know that $\mathsf{Set}^{\mathrm{op}}$ is equivalent to the category of complete atomic Boolean algebras, which now allows us to consider (L, ι) as the Lindenbaum algebra of infinitary T-logic.

In a **third step**, based on the adjunction

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathsf{BA} \arrow[l, bend left]
\end{tikzcd} .$$

one investigates finitary logics for coalgebras, as in [16].

The aim of this thesis is to replace Set by the category Ω-Cat of categories enriched over Ω for a commutative quantale Ω. It is based on the Ω-generalisations of the downset monad $\mathcal{D}$ and the upset monad $\mathcal{U}$. We will define algebras for operations $\Sigma_{\mathcal{DU}}$ and equations $E_{\mathcal{DU}}$ and will argue via (1.5), Theorem 4.4.1, Theorem 4.3.6, and Theorem 5.2.5 that $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebras complete the table

$\mathsf{Set}^{\mathrm{op}}$	complete atomic Boolean algebras
Ω-$\mathsf{Cat}^{\mathrm{op}}$	$\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebras

1.3 Basic examples of Ω-categories

Before we talk about methodology let us look at some interesting examples categories enriched over a quantale.

1. $\mathbb{2} = ((2, \leq), 1, \wedge)$. Categories enriched over this are preorders, and $\mathbb{2}$-functors are monotone maps.

2. $[0, \infty] = (([0, \infty], \geq_{\mathbb{R}}), 0, +)$ is a symmetric monoidal closed category. If one denotes by $\dot{-}$ the truncated minus then

$$[0, \infty][r, s] = s \dot{-} r$$

which is due to $t + r \geq_{\mathbb{R}} s \iff t \geq_{\mathbb{R}} s \mathbin{\dot{-}} r$, showing that $\cdot + r$ is left adjoint to $[r, \cdot]$. For more details see [18]. Some examples of generalised metric spaces are:

(a) $[0, \infty]$ itself.

(b) Consider the real numbers R with the metric given by $R(a, b) = \textsf{if } a \leq b \textsf{ then } 0 \textsf{ else } a - b$

(c) Any metric space.

3. Consider $(([0, 1], \geq_{\mathbb{R}}), 0, \max)$. Then

$$[0, 1](x, y) = \textsf{ if } x \geq y \textsf{ then } 0 \textsf{ else } y$$

We call a category enriched over $[0, 1]$ an ultra metric space, see [3]. Some examples are:

(a) $[0, 1]$ itself

(b) $[0, 1]^{\mathrm{op}}$

(c) Let A^{∞} be the finite and infinite words over A. Define $A^{\infty}(v, w) = 0$ if v is a prefix of w and $A^{\infty}(v, w) = 2^{-n}$ otherwise where $n \in \mathbb{N}$ is the largest number such that $v_n = w_n$ (where v_n is the prefix of v consisting of n letters from A).

1.4 Methodology and Thesis Outline

As we said above we take our inspiration from the fact that the complete atomic Boolean algebras are the algebras for the monad given by the following adjunction.

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left, "{[-,2]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,2]}"] \arrow[d, phantom, "\dashv"] & \mathsf{caBA} \arrow[l, bend left, "\mathsf{AT}"] \\
\mathsf{Set} \arrow[u, bend left, "{[-,2]}"] &
\end{tikzcd} \tag{1.3}$$

where the functor $[-,2]$ is the inverse image powerset functor and the functor AT is the functor that for any complete and atomic Boolean algebra A takes its set of atoms, also known as join prime elements.
Using the enriched version of diagram (1.3),

$$\begin{tikzcd}
\mathsf{Pre}^{\mathrm{op}} \arrow[r, bend left, "{[-,\mathbb{2}]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,\mathbb{2}]}"] \arrow[d, phantom, "\dashv"] & \mathsf{caDL} \arrow[l, bend left, "\mathsf{AT}"] \\
\mathsf{Pre} \arrow[u, bend left, "{[-,\mathbb{2}]}"] &
\end{tikzcd}$$

one has a clear way to modify the set of truth values: replace $\mathbb{2}$ with a new set Ω. Furthermore, using enriched category theory, one keeps the same structure on models as the one on the set of truth values, for example if one chooses as $\Omega = [0,\infty]$ then one has a metric structure on the algebra.

According to [14], if Ω is small and has a symmetric monoidal closed structure, and is complete and cocomplete then one has a dual adjunction:

$$D = [-,\Omega] \dashv U = [-,\Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}, \tag{1.4}$$

Observation 1.4.1. *If a small category is complete and cocomplete then it is a preorder. Thus one has the above adjunction if only if Ω is a quantale. So from the start one needs to be in the context of categories enriched over quantales.*

We want to consider $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ as the category of algebras of a 'Ω-Cat-logic'.

Since $[-,\Omega] : \Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \Omega\text{-}\mathsf{Cat}$ need not be monadic itself, we are going to study instead its monadic closure. That is, we work with the category $\Omega\text{-}\mathsf{Cat}^{DU}$ of algebras for the monad DU. And as $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is complete and cocomplete then the comparison functor $K : \Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \Omega\text{-}\mathsf{Cat}^{DU}$ has a left adjoint, call it AT.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "{[-,\Omega]}"] \arrow[d, phantom, "\dashv"] & \Omega\text{-}\mathsf{Cat}^{DU} \arrow[l, bend left, "\mathsf{AT}"] \\
\Omega\text{-}\mathsf{Cat} \arrow[u, bend left, "{[-,\Omega]}"] &
\end{tikzcd} \tag{1.5}$$

The whole process is described in the following diagram.

$$\begin{tikzcd}
 & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat} \arrow[dl, bend left=10, "\mathsf{AT}"'] & & \\
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[ur, bend left=30, "{[-,\Omega]}"] \arrow[ur, phantom, bend left=10, "\vdash"] \arrow[rr, bend left=15, "K"] \arrow[rr, phantom, "\top"] \arrow[d, bend left, "D={[-,\Omega]}"] & & \Omega\text{-}\mathsf{Cat}^{DU} \arrow[ul, hook] \arrow[ll, bend left=15, "\mathsf{AT}"] \arrow[d, bend left, "\cong"'] & \langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle\text{-alg} \arrow[d, bend left, "\cong"'] \\
\Omega\text{-}\mathsf{Cat} \arrow[u, bend left, "U={[-,\Omega]}"] \arrow[u, phantom, "\dashv"] \arrow[loop below, "{\mathcal{D},\mathcal{U}}"] & & \Omega\text{-}\mathsf{Cat}^{\mathcal{DU}} \arrow[u, bend left] \arrow[r, bend left=10] \arrow[r, phantom, "\cong"] & \mathsf{CCD} \arrow[u, bend left] \arrow[l, bend left=10]
\end{tikzcd} \tag{1.6}$$

where:

- c-Ω-Cat is the category with objects complete and cocomplete Ω-categories and arrows limits and colimits preserving Ω-functors.

- $\mathcal{D}$ and $\mathcal{U}$ are the generalization of the upset and downset monads. On objects $\mathcal{D}X = [X^{\mathrm{op}}, \Omega]$ and, $\mathcal{U}X = [X, \Omega]^{\mathrm{op}}$ respectively. On an arrow $f : X \longrightarrow Y \in \Omega\text{-}\mathsf{Cat}$ the monads $\mathcal{D}$ and $\mathcal{U}$ are given by the left Kan extension $Lan_{dX}(dY \circ f)$ and the right Kan extension $Ran_{uX}(uY \circ f)$, respectively.

- Both $\Omega\text{-}\mathsf{Cat}^{DU}$ and $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ are the Eilenberg-Moore category of algebras for the monad DU and the monad $\mathcal{DU}$, respectively.

- CCD is the category whose objects are ccd algebras and arrows are Ω-functors preserving all limits and colimits. An algebra (A, α) is ccd if it is a $\mathcal{D}$-algebra and if the structure map α has a left adjoint.

- $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-alg is the category whose objects are set algebras with operations $\Sigma_{\mathcal{DU}}$ and equations $E_{\mathcal{DU}}$.

- $\mathsf{AT} : \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is defined on objects as $\mathsf{AT}(A) = \mathsf{At}(A)^{\mathrm{op}}$, where $\mathsf{At}(A)$ is the full subcategory of A whose objects are atoms, where an object $a \in A$ is an atom if and only if the functor $A(a, -)$ preserves all colimits. On arrows see Section 3.2.2.

- $\mathsf{AT} : \Omega\text{-}\mathsf{Cat}^{DU} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is defined on an algebra (A, α) as $\mathsf{AT}(A, \alpha) = \mathsf{At}(A)^{\mathrm{op}}$

Let us give a brief descrpition of this thesis. First we show that if Ω is a commutative quantale, then one has the adjunction $\mathsf{AT} \dashv [-, \Omega] : \Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}.$, and then give sufficient and necessary conditions to restrict this adjunction to an equivalence: $\bar{\mathsf{AT}} \simeq \overline{[-, \Omega]} : \mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{o}p$, where $\mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat}$ is the full subcategory of atomic complete and cocomplete Ω-categories, and $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}$ is the subcategory of Cauchy complete Ω-categories. Furthermore we show that the image of $U = [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is in $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}$. Thinking of $\mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat}$ as a category of algebras we observe, that for an algebra $A \in \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$, one has two kinds of operations: weighted limits and weighted colimits, which is analogous to what happens in the case of lattices, where $\bigvee$ is a colimit and $\bigwedge$ is a limit. Obtaining the distributive law between these two kinds of operations directly from this monad proved to be too difficult, so for that we split the problem in two and we defined two monads, $(\mathcal{D}, d, \mu), (\mathcal{U}, u, \nu) : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}$ and a distributive law between them, and shown that the composite monad $\mathcal{D}\mathcal{U}$ is equal to the monad $DU = [[-, \Omega], \Omega]$. Furthermore, we give necessary and sufficient conditions in order for the comparison functor to be full and faithful and calculate its image. This approach, to break the monad DU into two separate monads with a distributive law and show that their composite is isomorphic to it, was used in the case $\Omega = \mathbb{2}$ in [23] and it was also extended for quantaloid enriched categories in [32], and also used to show some topological results in [12] and [36].

After this, we define algebras in the usual sense as a set with operations and equations and show that these algebras are in fact the algebras for the adjunction $U \dashv D$. We end by showing that one can define finitary versions of the monads $\mathcal{D}$ and $\mathcal{U}$, and show that for some classes of quantales there exists a distributive law between them. We conclude by defining finitary set algebras for these monads but not for their composite.

In the next paragraph we shall give the outline of the thesis.

In **Chapter 2** we introduce the context and foundations of our work and discuss related literature.

In **Chapter 3** we want to generalise the equivalence between $\mathsf{Set}^{\mathrm{op}}$ and caBA

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left, "{[-,2]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,2]}"] \arrow[d, phantom, "\dashv"] & \mathsf{caBA} \arrow[l, bend left, "\mathsf{AT}"] \\
\mathsf{Set} \arrow[u, bend left, "{[-,2]}"] &
\end{tikzcd}$$

Thus we replace the adjunction $[-,2] \dashv [-,2] : \mathsf{Set} \longrightarrow \mathsf{Set}^{\mathrm{op}}$ by $[-,\Omega] \dashv [-,\Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ and obtain the adjunction

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend left, "{[-,\Omega]}"] \arrow[r, phantom, "\top"] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}, \arrow[l, bend left, "\mathsf{AT}"]
\end{tikzcd}$$

where $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ is the subcategory of $\Omega\text{-}\mathsf{Cat}$ whose objects are complete and cocomplete Ω-categories and arrows are continuous and cocontinuous Ω-functors. We then characterise the subcategories on which this adjunction restricts to an equivalence

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}_{cc}^{\mathrm{op}} \arrow[r, bend left, "{[-,\Omega]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,\Omega]}"] \arrow[d, phantom, "\dashv"] & \mathsf{ca}\text{-}\Omega\text{-}\mathsf{Cat} \arrow[l, bend left, "\mathsf{AT}"] \\
\Omega\text{-}\mathsf{Cat} \arrow[u, bend left, "{[-,\Omega]}"] &
\end{tikzcd}$$

where $\Omega\text{-}\mathsf{Cat}_{cc}$ is the full subcategory of Cauchy complete Ω-categories, see [18], and $\mathsf{ca}\text{-}\Omega\text{-}\mathsf{Cat}$ is the full subcategory of atomic complete and cocomplete Ω-categories. Thus we obtain a generalised version of $\mathsf{Set}^{\mathrm{op}} \simeq \mathsf{caBA}$. We end the chapter with an instantiation of these results to different Ω's, and show that in the case of $\Omega = \mathbb{2}$ we obtain the same results as in the literature.

In **Chapter 4**, we start from the same diagram

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left, "{[-,2]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,2]}"] \arrow[d, phantom, "\dashv"] & \mathsf{caBA} \arrow[l, bend left, "\mathsf{AT}"] \\
\mathsf{Set} \arrow[u, bend left, "{[-,2]}"] &
\end{tikzcd}$$

but start from the fact that caBa is the category of algebras for the monad $[[-,2],2]$. Note that presenting caBA with operations and equations gives us (infinitary) propositional logic.

We want to obtain a presentation by, operations and equations of, $[[-,\Omega],\Omega]$-algebras. For that, following [23], we introduce two monads

$$(\mathcal{D},d,\mu),(\mathcal{U},u.\nu):\Omega\text{-}\mathsf{Cat}\longrightarrow\Omega\text{-}\mathsf{Cat},$$

where the functor $\mathcal{D}$ and $\mathcal{U}$ on objects is given by $\mathcal{D}X = [X^{\mathrm{op}},\Omega]$ and $\mathcal{U}X = [X,\Omega]^{\mathrm{op}}$, and on arrows is given by a left Kan and a right Kan extension, respectively. The units d,u are the two Yoneda embeddings, and multiplications are adjoints to units: $\mu\dashv d\mathcal{D}$ and $u\mathcal{U}\dashv\nu$.

We then show that $\mathcal{D}\mathcal{U} = [[-,\Omega],\Omega]$, not only as functors but also as monads. This relies again on the fact that we are in the context of enriched categories of quantales, and the fact that on arrows $\mathcal{D}$ and $\mathcal{U}$ are given by Kan extensions thus they adjoints to D and U on arrows, see Proposition 2.4.2. The advantage of spiting DU is that the algebras for $\mathcal{D}$ and $\mathcal{U}$ have a canonical form: their structure map is an adjoints to their monad's unit, and calculate a weighted colimits and a weighted limits, respectively. We then show that the category of $\Omega\text{-}\mathsf{Cat}^{\mathcal{D}\mathcal{U}}$ is isomorphic to the category of completely distributive Ω-categories, CCD, a generalization of constructive completely distributive lattices of [8].

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "{[-,\Omega]}"] \arrow[d, phantom, "\dashv"] & \mathsf{CCD} \arrow[l, bend left, "\mathsf{AT}"] \\
\Omega\text{-}\mathsf{Cat} \arrow[u, bend left, "{[-,\Omega]}"] \arrow[loop below, "{\mathcal{D},\mathcal{U}}"] &
\end{tikzcd}$$

We also connect the image of the comparison functor $K = [-,\Omega]$ with the duality from the previous chapter. As in the previous chapter we will instantiate these results to different Ω's and show that in the case of $\Omega = \mathbb{2}$ we obtain same results as in the literature.

As the result of this section we know that ccds are algebras which have weighted colimits and limits as operations and a distributive law connecting them. This still does not give us a logic for Ω-categories given by operations and equations, a question we address in the next chapters.

In **Chapter 5** In this chapter we have the main result of this thesis: a description, of DU-algebras, by operations $\Sigma_{\mathcal{DU}}$ and equations $E_{\mathcal{DU}}$. We start by defining two type of algebras $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$ and $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$ corresponding to $\mathcal{D}$ and $\mathcal{U}$, respectively.

The innovative aspect of this chapter is that a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra

$$(A, (v \star -)_{(v \in \Omega)}, (\bigsqcup_J)_{(\text{J cardinal})})$$

is defined as Set-algebra, and from the equations we obtain an order relation $\leq_{\mathcal{D}}$, given by $a \leq_A b$ iff $a \sqcup a' = a'$ and a Ω-category structure, given by

$$A(a, a') = \bigvee \{ v \in \Omega \mid v \star a \leq a' \}.$$

Similarly a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra

$$(B, (v \rhd -)_{(v \in \Omega)}, (\bigsqcap_J)_{(\text{J cardinal})})$$

has an order structure $\leq_B$ given by $b \leq_{\mathcal{U}} b'$ iff $b \sqcap b' = b$ and a Ω-category structure, given by

$$B(b, b') = \bigvee \{ v \in \Omega \mid b \leq_B v \rhd b' \}.$$

Now to have simultaneously a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$ and a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$ algebra all we need to do is to ensure that:

- the two structures are compatible, that is that the order relation and the Ω-category structure generated by $E_{\mathcal{D}}$ and $E_{\mathcal{U}}$ are the equivalent.
- there exists a normal form, that is that there exists a way to "distribute" operations from $\Sigma_{\mathcal{U}}$ over operations from $\Sigma_{\mathcal{D}}$. The desired distributivity equation is deduced from the fact that $\mathcal{DU}$-algebras are ccd

for any set J and any functions $\varphi : J \longrightarrow \Omega$ and $G : J \times A \longrightarrow \Omega$

$$\prod_J \varphi(j) \triangleright \Big(\bigsqcup_A G(j)(a) \star a\Big) = \bigsqcup_A \{\varphi, \downarrow G(-,a)\} \star a, \tag{1.7}$$

where $\{\varphi, \downarrow G(-,a)\}$ is a limit computed in Ω with $\downarrow G(j) : A^{\mathrm{op}} \longrightarrow \Omega$ given by $\downarrow G(j) = \mathrm{Lan}_i G = \int^{b \in A} A(-, i(b)) \otimes G(j)(b)$ for $i : |A| \longrightarrow A^{\mathrm{op}}$ the object inclusion functor.

In **Chapter 6**, we introduce a finitary version of the monads $\mathcal{D}$ and $\mathcal{U}$, and show that for some cases of quantales there exist a distributive law between them, thus allowing us to pursue the quest of a finitary logic.

In **Chapter 7**, we discuss future work, such as adding contravariant operations, like implication, and finishing the work on finitary monads. We would also like to connect this framework with **MV**-algebras. Finally, we want to apply the results of the thesis to coalgebraic logic over Ω-Cat.

1.5 Related work

In the following we will outline related work and how it interacts with the present thesis. Before we continue we should say that all the results in this thesis are a generalization of results known for preorders.

The results of Chapter 3 generalize the next two equivalences:

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathsf{caBA} \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Pre}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathsf{CDL} \arrow[l, bend left]
\end{tikzcd}$$

The results in Chapter 4 about the composite power monads are a generalization of the work in Marmolejo et. al. [23]. The closest generalization of this to ours is the work of Stubbe [32], where he proves that the double composite monad is equal to the double power monad for quantaloid enriched categories. In some aspects this result is slightly more general than the first result in Chapter 4.

Another closely related work, but towards topology is the work of Hofmann [12]. He is interested in generalising the know duality between topologic spaces and CCD, and generalise to approach spaces. He defines the category of approach spaces, see [19] for more on approach spaces, and the categories of cocomplete topological/approach spaces as those spaces X for which the Yoneda embedding $d_x : X \longrightarrow [X^{\mathrm{op}}, \Omega]$ has a left adjoint, and the categories of $\mathsf{CDTop}, \mathsf{CDApp}$ as those spaces for which the left adjoint of the Yoneda embedding has a further left adjoint as well. With those he proves the following two adjunctions.

$$\begin{tikzcd} \mathsf{Top}^{\mathrm{op}} \arrow[r, shift left] & \mathsf{CDTop} \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{App}^{\mathrm{op}} \arrow[r, shift left] & \mathsf{CDApp} \arrow[l, shift left] \end{tikzcd} \tag{1.8}$$

This adjunction is interesting, and is closely related to our work, via Cauchy completeness, but it is in a direction orthogonal to ours.

The notion of a ccd category has been defined, as a category who's Yoneda embedding has a left adjoint and this left adjoint has a left adjoint as well, in [8]. As this definition encapsulates the distributivity of colimits over limits, the left adjoint preserves both limits and colimits, it makes sense to be used in many interesting works, such as [23], [32], and [17].

The category of distributive complete Ω-lattices of Lai and Zhang [17] coincides with what we denote CCD in Definition 4.3.2. Compared to their work, we add the argument of how to obtain CCD from the monad $[[-, \Omega], \Omega]$ and we show that the CCD is isomorphic to the category of (ordinary, set-based) $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebras.

In Pu and Zhang [24] it is shown, amongst other things, that the category of anti-symmetric CCD's is monadic over Set, but the proof proceeds by Beck's monadicity theorem whereas we give the operations and equations $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ explicitly.

The double powerset monad $\mathcal{DU}$ is investigated in detail, in the case $\Omega = \mathbb{2}$, by Vickers in [36, 33, 34, 35].

1.6 Publications

From the present work, the content, of chapters 4 and 5, is in the course of being published, and has been presented at CMCS 2016 under the title "On the logic of generalised metric spaces" authors Octavian Babus and Alexander Kurz.

Chapter 2

Background

In this chapter we shall discuss the context in which we work and related work.

2.1 Quantales and monoidal categories

We use commutative quantales because they are both examples of monoidal categories and complete lattices, and as one would like an order relation on the set of truth values, commutative unital quantales are the best candidates.

Definition 2.1.1. By a commutative unital quantale we understand a tuple $\Omega = (\Omega_0, 1, \otimes)$, where Ω_0 is a complete lattice, $\otimes : \Omega \times \Omega \longrightarrow \Omega$ is an associative and commutative binary operation , which preserves all colimits in both arguments, and and element $1 \in \Omega_0$ such that $1 \otimes x = x$ for all $x \in \Omega$.

Definition 2.1.2. By a monoidal category we understand a tuple $\Omega = (\Omega_0, I, \otimes, (l_x)_{x \in \mathrm{ob}\Omega_0}, (r_x)_{x \in \mathrm{ob}\Omega_0}, (a_{xyz})_{x,y,z \in \mathrm{ob}\Omega_0})$ where Ω_0 is a category, I is an object of Ω_0, $\otimes : \Omega_0 \times \Omega_0 \longrightarrow \Omega_0$ is a bifunctor, and

- $l_x : x \longrightarrow I \otimes x$ and $r_x : x \longrightarrow x \otimes I$
- $a_{xyz} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$

are natural isomorphisms such that the following diagrams commute

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a"] \arrow[dr, "r\otimes \mathrm{id}"'] & & x \otimes (I \otimes y) \arrow[dl, "\mathrm{id}\otimes l"] \\
& x \otimes y &
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
((w \otimes x) \otimes y) \otimes z \arrow[r, "a"] \arrow[d, "a\otimes \mathrm{id}"'] & (w \otimes x) \otimes (y \otimes z) \arrow[r, "a"] & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "a"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}\otimes a"']
\end{tikzcd} \tag{2.2}$$

Definition 2.1.3. A monoidal category is called *symmetric* if there exits a natural transformation $s_{xy} : x \otimes y \longrightarrow y \otimes x$ such that the following diagrams commute

$$\begin{tikzcd}
x \otimes y \arrow[r, "s_{xy}"] \arrow[dr, "\mathrm{id}_{x\otimes y}"'] & y \otimes x \arrow[d, "s_{yx}"] \\
& x \otimes y
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
I \otimes x \arrow[rr, "s_{Ix}"] \arrow[dr, "l_x"'] & & x \otimes I \arrow[dl, "r_x"] \\
& X &
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{xyz}"] \arrow[d, "s_{xy}\otimes \mathrm{id}_z"'] & x \otimes (y \otimes z) \arrow[r, "s_{x(y\otimes z)}"] & (y \otimes z) \otimes x \arrow[d, "a_yzx"] \\
(y \otimes x) \otimes z \arrow[r, "a_{yxz}"'] & y \otimes (x \otimes z) \arrow[r, "\mathrm{id}_y\otimes s_{xz}"'] & y \otimes (z \otimes x)
\end{tikzcd} \tag{2.5}$$

Definition 2.1.4. A monoidal category Ω is called *closed* if the functor $- \otimes v$ has a right adjoint for any $v \in \Omega$.

Proposition 2.1.5. *Any quantale is a symmetric monoidal closed category.*

Proof. Let Q be a quantale. We have to show is that $\otimes : Q \times Q \longrightarrow Q$ is a functor, that is a monotone map.

So let $p, q, r \in Q$ such that $p \leq q$ we have to show that $r \otimes p \leq r \otimes q$ and that $p \otimes r \leq q \otimes r$, and as we are in a lattice this is equivalent to

$$(r \otimes p) \vee (r \otimes q) = r \otimes q$$

and

$$(p \otimes r) \vee (q \otimes r) = q \otimes r,$$

but as we know that $\otimes$ preserves all colimits in both arguments and that $p \vee q = q$, both are true.

As $\otimes$ has a right adjoint Q is closed. Indeed, this is true since $\otimes$ preserves colimits in both arguments and Q is cocomplete. $\square$

Some example of commutative unital quantales and monoidal categories are:

Example 2.1.6.

1. $\mathbb{2} = (\{0 \leq 1\}, \wedge, 1)$ is a unital commutative quantale.
2. $[0, \infty] = (([0, \infty], \geq_{\mathbb{R}}), +, 0)$ is a unital commutative quantale.
3. $\mathsf{Set} = (\mathsf{Set}, \times, \{*\})$ is a monoidal category. **But is not a quantale.**
4. Consider $(([0,1], \leq), 1, \lambda x, y \,.\, \max(0, x + y - 1))$. Then
$$[0,1](x,y) \;=\; \mathsf{if}\ x \leq y\ \mathsf{then}\ 1\ \mathsf{else}\ 1 - x + y$$
5. Consider $(([0,1], \leq), 1, \min)$. Then
$$[0,1](x,y) \;=\; \mathsf{if}\ x \leq y\ \mathsf{then}\ 1\ \mathsf{else}\ y$$
6. Consider $(([0,1], \leq), 1, \cdot)$ where $x \cdot y$ is the usual multiplication. Then
$$[0,1](x,y) \;=\; \mathsf{if}\ x \leq y\ \mathsf{then}\ 1\ \mathsf{else}\ \frac{y}{x}$$

For the rest of the thesis, whenever is appropriate, by a quantale we understand a commutative unital quantale.

2.2 Enriched category theory

One could say that enriched category theory is a natural generalization of category theory, in the sense that instead of having Hom-sets one has Hom-objects. In the following $\Omega = (\Omega_0, I, \otimes)$ is a monoidal category, with I the unit of $\otimes$.

Definition 2.2.1. A Ω-category A consists of a set $ob\,A$, a hom-object $A(a,b) \in \Omega_0$, a composition law $M = M_{a,b,c} : A(b,c) \otimes A(a,b) \longrightarrow A(a,c)$, for each triple of objects, and an identity element $j_a : I \longrightarrow A(a,a)$ for each objects; subject to the associativity and unit axioms expressed by the commutativity of

$$\begin{tikzcd}
(A(c,d) \otimes A(b,c)) \otimes A(a,b) \arrow[rr, "a"] \arrow[d, "M\otimes 1"'] & & A(c,d) \otimes (A(b,c) \otimes A(a,b)) \arrow[d, "1\otimes M"] \\
A(b,d) \otimes A(a,b) \arrow[dr, "M"'] & & A(c,d) \otimes A(a,c) \arrow[dl, "M"] \\
& A(a,d) &
\end{tikzcd}$$

$$\begin{tikzcd}
A(b,b) \otimes A(a,b) \arrow[r, "M"] & A(a,b) & A(a,b) \otimes A(a,a) \arrow[l, "M"'] \\
I \otimes A(a,b) \arrow[u, "j_b\otimes 1"] \arrow[ur, "l"'] & & A(a,b) \otimes I \arrow[u, "1\otimes j_a"'] \arrow[ul, "r"]
\end{tikzcd} \tag{2.6}$$

If there is no source of confusion for any Ω-category A we will write the set of objects with just A.

Let us give some examples of enriched categories. **All of these example, but the first one, are quantale enriched categories. We put the Set example in here to show that enriched categories theory is a generalisation of "normal" category theory.** Most of these were also stated in the introduction, also let us mention that if Ω is monoidal closed then Ω is a self enriched category. Indeed, for any two objects v_1, v_2 of Ω we have an object $\Omega(v_1, v_2)$ given by the right adjoint of $- \otimes v_1$.

Example 2.2.2. 1. $\mathsf{Set} = (\mathsf{Set}, \times, I)$ where I is the one element set. A Set-category $\mathcal{C}$ is then formed by:

- a set called $ob\mathcal{C}$

- for any two elements $a, b \in ob\mathcal{C}$, a set of arrows $\mathcal{C}(a,b) \in \mathsf{Set}$,

such that for all $a, b, c, d \in ob\mathcal{C}$ we have

- an identity arrow $\mathrm{id}_a : I \longrightarrow A(a,a)\ \forall a \in ob\mathcal{C}$
- composition of arrows $\circ : \mathcal{C}(b,c) \times \mathcal{C}(a.b) \longrightarrow \mathcal{C}(a,c)$

such that for any $f \in \mathcal{C}(a,b)$, $g \in \mathcal{C}(b,c)$ and $h \in \mathcal{C}(c,d)$ we have

$$f \circ \mathrm{id}_a = f = \mathrm{id}_b \circ f$$

and

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Thus a Set-enriched category is an ordinary (small) category in the sense of [20].

2. $\mathbb{2} = (2 = \{0 \leq 1\}, 1, \wedge)$. A $\mathbb{2}$-enriched category P is then formed by

   - a set called P, and
   - for any two elements $a, b \in P$ an element $P(a,b)$ of $\{0,1\}$.

   If we write $a \leq b$ if $P(a,b) = 1$, then for all $a, b, c, d \in P$ we have

   - $a \leq a$, and
   - if $a \leq b$ and $b \leq c$ then $a \leq c$.

   As the two diagrams in Definition 2.2.1 do not introduce any more equations, a $\mathbb{2}$-enriched category is a peorder.

3. $[0,\infty] = (([0,\infty], \geq_{\mathbb{R}}), 0, +)$ is a symmetric monoidal closed category. A $[0,\infty]$-category G is a

   - set G, where
   - for any two elements a, b of G we have a number $G(a,b) \in [0,\infty]$.

   Furthermore, for all $a, b, c \in G$ we have

   - $0 \geq G(a,a)$, thus $G(a,a) = 0$, and
   - $G(b,c) + G(a,b) \geq G(a,c)$

As the two diagrams in Definition 2.2.1 do not introduce any more equations a $[0,\infty]$-enriched category is a generalized metric space. For more details see [18].

4. Let $\Omega = (\Omega_0, 1, \otimes)$ be a unital commutative quantale, then a category A enriched over Ω consists of a set A and together with a a function $A(-,-) : A \times A \longrightarrow A$ such that for any $a, b, c \in A$ we have $1 \leq A(a,a)$ and $A(a,b) \otimes A(b,c) \leq A(a,c)$.

5. For any monoidal category $\Omega = (\Omega_0, I, \otimes)$ if Ω_0 has an initial object $\bot \in \Omega_0$ then any set A becomes an enriched category over Ω if we take $A(a,b) = \bot$ for all $\mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat}, b \in obA$. We call such enriched categories discrete.

Definition 2.2.3. For any two Ω-categories A and B a Ω-functor is any map $F : A \longrightarrow B$ such that for any two objects a, b of A there exists an arrow in Ω $F_{a,b} : A(a,b) \longrightarrow B(Fa, Fb)$, such that the following diagrams commute

$$\begin{tikzcd}
A(b,c) \otimes A(a,b) \arrow[r, "M"] \arrow[d, "F \otimes F"] & A(a,c) \arrow[d, "F"] \\
B(Fb,Fc) \otimes B(Fa,Fb) \arrow[r, "M"'] & B(Fa,Fc)
\end{tikzcd} \tag{2.7}$$

$$\begin{tikzcd}
& A(a,a) \arrow[dd, "F"] \\
I \arrow[ur, "J"] \arrow[dr, "J"'] & \\
& B(Fa,Fa)
\end{tikzcd} \tag{2.8}$$

Before we calculate what this means in the examples we gave above, let us also define what a natural transformation is.

Definition 2.2.4. Let $F, G : A \longrightarrow B$ be two Ω-functors, then a Ω-natural transformation $\eta : F \longrightarrow G$ is an A-indexed family of arrows in Ω $I \mapsto B(F(A), G(a))$ satisfying the following diagram:

$$\begin{tikzcd}
 & I \otimes A(a,b) \arrow[r] & B(Fa,Ga) \otimes B(Fa,Fb) \arrow[dr] & \\
A(a,b) \arrow[ur] \arrow[dr] & & & B(Fa,Fb) \\
 & A(a,b) \otimes I \arrow[r] & B(Ga,Gb) \otimes B(Ta,Sa) \arrow[ur] &
\end{tikzcd}$$

(2.9)

In general Ω-naturality is not equivalent to "normal" naturality, but if the functor $V = \Omega_0(I,-) : \Omega_0 \longrightarrow \mathsf{Set}$ is faithful then the two notions are equivalent. For more details see [14, Chapter 1.3].

Now let us go with these definitions through our examples. We shall also compare the notion of Ω-natural transformation with the usual one.

Example 2.2.5.

1. Let $\mathcal{C}, \mathcal{D}$ be two (small) categories. Then a Set-functor $F : A \longrightarrow B$ is a map $F : ob\mathcal{C} \longrightarrow ob\mathcal{D}$ such that for any two objects $a, b \in ob\mathcal{C}$ we have a function $F : \mathcal{C}(a,b) \longrightarrow \mathcal{D}(Fa, Fb)$ such that for any $f \in A(a,b)$ and $g \in A(b,c)$ we have

   $$F(g \circ f) = F(g) \circ F(f)$$

   and

   $$F(\mathrm{id}_a) = \mathrm{id}_{Fa}.$$

   Thus a Set-functor is functor in the usual sense.

   Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. Then a Set-enriched natural transformation, $\eta : F \longrightarrow G$ is a $ob\mathcal{C}$-indexed family of arrows in Set, $\eta_a : I \longrightarrow \mathcal{D}(F(a), G(a))$ for all $a \in ob\mathcal{C}$. As for Set, $I = \{*\}$, we have that η_a is equivalent to an arrow $\eta_a : F(a) \longrightarrow G(a)$. The diagram 2.9 tells us that for every two objects a, b of $\mathcal{C}$ and any $h : a \longrightarrow b$ one has

   $$\eta_b \circ F(h) = G(h) \circ \eta_a.$$

   Thus, η is a natural transformation in the usual sense.

2. Let P, Q be two preorders. Then a $\mathbb{2}$-functor is a map $F : P \longrightarrow Q$ such that for any $p, q \in P$ such that $p \leq_P q$ we have that $F(p) \leq_Q F(q)$. Thus a $\mathbb{2}$-functor is a monotone map.

Let $F, G : P \longrightarrow Q$ be two monotone maps then a $\mathbb{2}$-enriched natural transformation $\eta : F \longrightarrow G$ is an $ob\, P$-indexed family of arrows in $\{0 \leq 1\}$, $\eta_a : 1 \longrightarrow Q(F(p), G(p))$. That means that there is a $\mathbb{2}$-enriched natural transformation between F and G if and only if for all $p \in P$ we have

$$F(p) \leq G(p).$$

Thus a natural transformation between two monotone maps, is a pointwise order between the two maps.

3. Let A, B be two generalized metric spaces then a $[0, \infty]$-functor is a map $F : A \longrightarrow B$ such that $A(a, b) \geq_{\mathbb{R}} B(F(a), F(b))$ for all $a, b \in A$. For the rest of this thesis these kind of maps will be called non-expanding.

   Let $F, G : A \longrightarrow B$ be two non-expanding maps, then there is a $[0, \infty]$-enriched natural transformation between them if and only for all $a \in A$ we have $B(F(a), G(a)) = 0$.

4. Let $\Omega = (Q, 1, \otimes)$ be a unital commutative quantale, and let A, B be two Ω-categories then a Ω-functor $F : A \longrightarrow B$ is a map such that $A(a, b) \leq B(F(a), F(b))$ for all objects a, b of A.

   Let $F, G : A \longrightarrow B$ be two Ω-functors then $\eta : F \longrightarrow G$ is a natural transformation if for any $a \in A$ there exits $\eta_a : I \longrightarrow B(F(a), G(a))$, that is $I \leq B(F(a), G(a))$.

Remark 2.2.6. Every quantale-enriched category A has also a preorder structure given by

$$a \leq b \Leftrightarrow I \leq A(a, b).$$

Indeed, this relation is reflexive as we have

$$I \leq A(a, a),$$

and it is transitive as we have

$$I = I \otimes I \leq A(b, c) \otimes A(a, b) \leq A(a, c).$$

Thus, a natural transformation between two quantale-enriched functors exists if an only if one of the two functors is pointwise bigger than the other one in the order given above.

2.3 Constructions on enriched categories

For the rest of this thesis we will suppose that Ω is a commutative unital quantale, and all enriched categories are small.

2.3.1 Functor categories

Let A, B be two Ω-categories. We denote with $[A, B]$ the category of functors from A to B. Then, following [14, Chapter 2], $[A, B]$ is a Ω-category with

$$[A,B](F,G) = \int_{a \in A} B(F(a), G(a)),$$

where $\int_{a \in A}$ is the end of $B(F-, G-) : A^{\mathrm{op}} \times A \longrightarrow \Omega$. As Ω is a quantale then this end is simply a meet in Ω. Thus

$$[A,B](F,G) = \bigwedge_{a \in A} B(F(a), G(a)).$$

For any Ω-category C we can define the contravariant functor $[-, C] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ given on objects by

$$A \mapsto [A, C],$$

and on an arrows $F : A \longrightarrow B$ by

$$[F, C](G) = G \circ F$$

for any $G : B \longrightarrow C$.

2.3.2 Limits and Colimits

In this section we will discuss everything we need about weighted limits and colimits, including formulas for them in Ω and how they look in categories enriched over quantales. For the remainder of the thesis, by a limit or a colimit, we understand a weighted one. Most of these notions are in [14, Chapter 3]. Also note that because Ω is a quantale all isomorphisms are equalities, thus instead of $\cong$ we will write $=$ in all of the following equations.

Definition 2.3.1. Let K, A be two Ω-categories, and let $\psi : K \longrightarrow \Omega$ and $F : K \longrightarrow A$ be two Ω-functors, then we call the limit of F weighted by ψ the representing object $\{\psi, F\}$ of

$$A(a, \{\psi, F\}) = [K, \Omega](\psi, A(a, F-)) \tag{2.10}$$

with counit $\mu : \psi \longrightarrow A(\{\psi, F\}, F-)$. Dually, a colimit for $\varphi : K^{\mathrm{op}} \longrightarrow \Omega$ and $F : K \longrightarrow V$ is the representing object $\varphi \star F$ of

$$A(\varphi \star F, a) = [K^{\mathrm{op}}, \Omega](\varphi, A(F-, a)) \tag{2.11}$$

with counit $\mu : \varphi \longrightarrow A(F-, \varphi \star F)$.

Let us give some examples of limits and colimits.

Example 2.3.2. 1. Let $\Omega = \mathbb{2}$, and let $K = \{*\}$ be a category with one object, and as $K = K^{\mathrm{op}}$ we can do both limits and colimits for all the examples below. $\psi : K \longrightarrow \mathbb{2}$ and $F : K \longrightarrow \mathbb{2}$ given by

- $\psi(*) = 0$ and $F(*) = 1$. Then we have that $\{\psi, F\}$ is the object of $\mathbb{2}$ such that

  $$\mathbb{2}(a, \{\psi, F\}) = \mathbb{2}(0, \mathbb{2}(a, 1)) = 1, \quad \forall a \in \mathbb{2}$$

  thus $\{\psi, F\} = 1$.

  On the other hand the colimit is given by

  $$\mathbb{2}(\psi \star F, a) = \mathbb{2}(0, \mathbb{2}(1, a)) = 1, \quad \forall a \in \mathbb{2}$$

  thus $\psi \star F = 0$.
- $\psi(*) = 0$ and $F(*) = 0$. Once again we have

  $$\mathbb{2}(a, \{\psi, F\}) = \mathbb{2}(0, \mathbb{2}(a, 1)) = 1, \quad \forall a \in \mathbb{2}$$

  with $\{\psi, F\}$ again being 1. The colimit is again given by $\psi \star F = 0$.
- $\psi(*) = 1$ and $F(*) = 1$ Using the adjunction $- \wedge a \dashv \mathbb{2}(a, -)$ we have

  $$\mathbb{2}(a, \{\psi, F\}) = \mathbb{2}(1, \mathbb{2}(a, 1)) = \mathbb{2}(1 \wedge a, 1) = \mathbb{2}(a, 1),$$

  thus we have $\{\psi, F\} = 1$. On the other hand the colimit is given by

  $$\mathbb{2}(\psi \star F, a) = \mathbb{2}(\psi, \mathbb{2}(F, a)) = \mathbb{2}(1 \wedge 1, a) = \mathbb{2}(1, a),$$

  thus $\psi \star F = 1$.

- $\psi(*) = 1$ and $F(*) = 0$. In the same way as above we have

  $$\mathbb{2}(a, \{\psi, F\}) = \mathbb{2}(\psi, \mathbb{2}(a, F)) = \mathbb{2}(\psi \wedge a, F) = \mathbb{2}(a, F)$$

  thus $\{\psi, F\} = F = 0$, and

  $$\mathbb{2}(\psi \star F, a) = \mathbb{2}(\psi, \mathbb{2}(F, a)) = \mathbb{2}(\psi \wedge F, a) = \mathbb{2}(0, a)$$

  thus $\psi \star F = 0$.

Thus, if $\Omega = \mathbb{2}$ and K is a singleton set, we have that $\psi \star F = \psi \wedge F$ and $\{\psi, F\} = \mathbb{2}(\psi, F)$.

2. As in the previous example let $\Omega = \mathbb{2}$ and let $K = \{*\}$, but let A be any poset. Let $F : K \longrightarrow A$ be any monotone map. Abusing notation let us write $F(*)$ as F. Now let $\psi : K \longrightarrow \Omega$ be any map given by:

   - $\psi(*) = 0$.

     $$A(a, \{\psi, F\}) = \mathbb{2}(\psi, A(a, F)) = 1,$$

     and

     $$A(\psi \star F, a) = \mathbb{2}(\psi, A(F, a)) = 1.$$

     Thus $\{\psi, F\} = 1$, and respectively $\psi \star F = 0$.

   - $\psi(*) = 1$.

     $$A(a, \{\psi, F\}) = \mathbb{2}(\psi, A(a, F)) = A(a, F),$$

     and

     $$A(\psi \star F, a) = \mathbb{2}(\psi, A(F, a)) = A(F, a).$$

     Thus we have $\psi \star F = \psi \star F = F$.

3. Again let $\Omega = \mathbb{2}$, and A and K be any two posets. Also let $\psi : K \longrightarrow \Omega$, $\varphi : K^{\mathrm{op}} \longrightarrow \Omega$ and $F : K \longrightarrow A$ be three monotone maps. We then have

   $$A(a, \{\psi, F\}) = [K, \mathbb{2}](\psi, A(a, F-)) = \bigwedge_{k \in K} \mathbb{2}(\psi(k), A(a, F(k))).$$

We want to calculate the right hand side, so we make a case distinction according to the value of $\psi(k)$. Thus we partition K as $K = K_1 \cup K_2$ where $K_1 = \{k \in K \mid \psi(k) = 0\}$ and $K_2 = \{k \in K \mid \psi(k) = 1\}$. With this notation the right hand side becomes

$$\begin{aligned}
&\bigwedge_{k\in K} \mathbb{2}(\psi(k), A(a, F(k))) \\
&= \bigwedge_{k\in K_1} \mathbb{2}(\psi(k), A(a, F(k))) \wedge \bigwedge_{k\in K_2} \mathbb{2}(\psi(k), A(a, F(k))) \\
&= \bigwedge_{k\in K_1} \mathbb{2}(0, A(a, F(k))) \wedge \bigwedge_{k\in K_2} \mathbb{2}(1, A(a, F(k))) \\
&= \bigwedge_{k\in K_1} 1 \wedge \bigwedge_{k\in K_2} A(a, F(k)) \\
&= \bigwedge_{k\in K_2} A(a, F(k)) \\
&= A(a, \bigwedge_{k\in K_2} F(k)).
\end{aligned}$$

Thus $\{\psi, F\} = \bigwedge_{k\in K}\{F(k) \mid \psi(k) = 1\}$.

In a similar way one has that $\varphi \star F = \bigvee_{k\in K}\{F(k) \mid \varphi(k) = 1\}$.

So in the case of posets the weight only chooses which elements should be taken into consideration and which should be ignored.

4. Let $\Omega = [0, \infty]$. The limits and colimits of general metric spaces have been studied in a series of articles by Rutten, see [29] and [27]. We shall give some examples.

   - Firstly let us look at $K = \{*\}$ and $\psi, F : K \longrightarrow \Omega$. Using the fact that $\otimes$ is commutative and that it is a left adjoint to $\Omega(a, -)$, we obtain:

     $$\Omega(a, \{\psi, F\}) = \Omega(\psi, \Omega(a, F)) = \Omega(\psi \otimes a, F) = \Omega(a, \Omega(\psi, F)).$$

     thus $\{\psi, F\} = \Omega(\psi, F) = F \mathbin{\dot{-}} \psi$.

     On the other hand we have

     $$\Omega(\psi \star F, a) = \Omega(\psi, \Omega(F, a)) = \Omega(\psi \otimes a).$$

     Thus $\psi \star F = \psi \otimes F = \psi + F$.

- Let K be a discrete Ω-category, as in 5, and let $\psi : K \longrightarrow \Omega$ be given by $\psi(k) = I$ for all $k \in K$. Also let $G : K \longrightarrow \Omega$ be any Ω-functor. Then as K is discrete we have both $\{\psi, G\}$ and $\psi \star G$. Let us look at what these are.

  $$\begin{aligned}
  \Omega(v, \{\psi, G\}) &= [K, \Omega]\Omega(\psi, \Omega(v, G)) \\
  &= \bigwedge_{k \in K} \Omega(\psi(k) \otimes a, G(k) \\
  &= \bigwedge_{k \in K} \Omega(I \otimes a, G(k)) \\
  &= \bigwedge_{k \in K} \Omega(v, G(k)) \\
  &= \sup_{k \in K}(G(k) \dot{-} v) \\
  &= (\sup_{k \in K} G(k)) \dot{-} v \\
  &= \Omega(v, \sup_{k \in K} G(k)) \\
  &= \Omega(v, \bigwedge_{k \in K} G)
  \end{aligned}$$

  and

  $$\begin{aligned}
  \Omega(\psi \star G, v) &= [K, \Omega]\Omega(\psi, \Omega(G, v)) \\
  &= \bigwedge_{k \in K} \Omega(\psi \otimes GK, v) \\
  &= \sup_{k \in K} v \dot{-} Gk \\
  &= v \dot{-} \inf_{k \in K} Gk \\
  &= \Omega(\inf_{k \in K} Gk, v) \\
  &= \Omega(\bigvee_{k \in K} Gk, v)
  \end{aligned}$$

  Thus unweighted limits and colimits have the same value as the non enriched limits and colimits.

5. Now let Ω be any symmetric monoidal closed category, and let A be any Ω-category. For any functor $F : A \longrightarrow A$ and any object a of A the special colimit $A(-, a) \star F$ is equal to $F(a)$. Indeed, using Yoneda lemmma, see [14, Chapter 1.7], we have

   $$A(A(-, a) \star F, b) = [A^{\text{op}}, \Omega](A(-, a), A(F-, b)) = A(F(a), b).$$

We also have the dual property for limits $\{A(a,-),F\} = F(a)$. Indeed we have

$$A(b,\{A(a,-),F\}) = [A,V](A(a,-),A(b,F-)) = A(b,F(a)).$$

Let K be the one object Ω-category. Colimits over it are called tensor product and limits over it are called cotensor product. The naming comes from the value of these limits in Ω. Limits and colimits where the weight is constant I, the tensor's unit, are called ends and, respectively, coends. Following [14, Chapter 3.10] if a Ω-cateogory has both cotensor products and ends, and tensor products and coends, then it has all small limits, and respectively all small colimits.

As we have seen above limits and colimits in Ω have a special form, and are given by:

$$\{\psi,F\} = [K,\Omega](\psi,F) = \bigwedge_{k\in K} \Omega(\psi(k),F(k)), \tag{2.12}$$

and

$$\varphi \star G = \bigvee_{k\in K} \varphi(k) \otimes G(k) \tag{2.13}$$

for any Ω-category K and any Ω-functors, $\psi,F,G : K \longrightarrow \Omega$ and $\varphi : K^{\mathrm{op}} \longrightarrow \Omega$.

Indeed we have

$$\begin{aligned}
\Omega(v,\{\psi,F\}) &= [K,\Omega](\psi,\Omega(v,F)) \\
&= \bigwedge_{k\in K} \Omega(\psi(k),\Omega(v(,F(k)) \\
&= \bigwedge_{k\in K} \Omega(v\otimes\psi(k),F(k)) \\
&= \bigwedge_{k\in K} \Omega(v,\Omega(\psi(k),F(k))) \\
&= \Omega(v,\bigwedge_{k\in K} \Omega(\psi(k),F(k))) \\
&= \Omega(v,[K,\Omega](\psi,F))
\end{aligned}$$

and

$$\begin{aligned}\Omega(\varphi \star G, v) &= [K,\Omega](\varphi, \Omega(G,v))\\ &= \bigwedge_{k\in K} \Omega(\varphi(k) \otimes G(k), v)\\ &= \bigwedge_{k\in K} \Omega^{\mathrm{op}}(v, \varphi(k) \otimes G(k))\\ &= \Omega^{\mathrm{op}}(v, \bigvee_{k\in K} \varphi(k) \otimes G(k))\\ &= \Omega(\bigvee_{k\in K} \varphi(k) \otimes G(k), v)\end{aligned}$$

where by $\bigvee_{k\in K}$ we understand $\bigwedge_{k\in K}$ in Ω^{op}.

2.3.3 The indexing category can be discrete

These results are inspired by limits and colimits in posets. It is a known fact that for posets all limits and all colimits are generated by discrete indexing categories. We will show that this is true also for categories enriched over quantales. Intuitively this happens because limits and colimits are given in terms of limits in the quantale, and the quantale itself is a poset.

Before we continue let us recall what we mean by a discrete Ω-category and discrete Ω-functors.

Definition 2.3.3. A Ω-category A is called discrete if $A(a,b) = \perp$ for any two objects a, b of A, where $\perp$ is the initial object of Ω. For any two Ω-categories A, B a Ω-functor $F : A \longrightarrow B$ is called discrete if A is discrete.

One can see that any set A is a discrete Ω-category, and if A is discrete then every map $F : A \longrightarrow B$ is a Ω-functor.

With this definition one has

Proposition 2.3.4. *Let K, A be any two Ω-categories and let $\psi : K \longrightarrow \Omega$ and $F : K \longrightarrow A$ be any two Ω functors. Then the limit $\{\psi, F\}$ is equal to a limit $\{\psi', F'\}$, where $\psi' : K' \longrightarrow \Omega$ and $F' : K' \longrightarrow A$ are two discrete Ω-functors. The same statement holds for colimits.*

Proof. Define $K' = obK$. By k we will understand an object of K and the same object of K'. Define $\psi' : K' \longrightarrow \Omega$ and $F : K' \longrightarrow \Omega$ as

$$\psi'(k) = \psi(k) \text{ and } F'(k) = F(k).$$

Let us show that $\{\psi', F'\} = \{\psi, F\}$.

$$\begin{aligned} A(a, \{\psi', F'\}) &= [K', \Omega](\psi', A(a, F'-)) \\ &= \bigwedge_{k \in K'} \Omega(\psi'(k), A(a, F'(k)) \\ &= \bigwedge_{k \in K} \Omega(\psi(k), A(a, F(k)) \\ &= [K, \Omega](\psi, A(a, F-)) \\ &= A(a, \{\psi, F\}) \end{aligned}$$

Also because Ω is a quantale, thus a poset, the counit $\mu : \psi \longrightarrow A(\{\psi, F\}, F-)$ is equivalent to $\psi(k) \leq A(\{\psi, F\}, F(k))$ for all $k \in K$ thus we have the "same" counit $\mu' : \psi' \longrightarrow A(\{\psi', F'\}, F'-)$. ☐

Thus for the rest of the thesis, whenever convenient, we will assume that limits and colimits are discrete.

Remark 2.3.5. The fact that weighted limits and colimits can be discrete means that we can treat them as operations, this will be relevant in Chapter 5.

2.3.4 Iteration of finite limits

Again looking at posets, if one wants to calculate a finite limit or colimit then one only needs to to know how to calculate the limit/colimit of two elements. Again this property is due to limits/colimits being calculated via limits in Ω, which is a poset.

Definition 2.3.6. We call a limit $\{\psi, F\}$ binary if the discrete index category has two objects.

So let us assume we know how to calculate the binary limit $\{\psi, F\}$ for any $\psi : 2 \longrightarrow \Omega$ and $F : 2 \longrightarrow A$, where 2 is the discrete Ω-category with two objects and A is any Ω-category. Now

Proposition 2.3.7. *Let K be any finite discrete Ω-category, whose cardinality is greater or equal to two, and let $\alpha : K \longrightarrow \Omega$ and $G : K \longrightarrow A$ be any two discrete Ω-functors. Then the limit $\{\alpha, G\}$ can be calculated with binary limits.*

Proof. Let us do induction on the cardinality of K. If the $K's$ cardinality is 2then we have nothing to show. So let us suppose we have this property for cardinality smaller than n and let us prove it for n. So let K be a set such that $|K| = n$.

Choose an object k of K, and define $K' = K \smallsetminus \{k\}$ that is the set K without the element k. Define $\alpha' : K' \longrightarrow \Omega$ and $G' : K' \longrightarrow A$ as

$$\alpha'(k') = \alpha(k') \text{ and } G'(k') = G(k') \forall k' \in K'.$$

Also define $\psi : \{1, 2\} \longrightarrow \Omega$ and $F : \{1, 2\} \longrightarrow A$ as

$$\psi(1) = \alpha(k), \psi(2) = I, F(1) = G(k), F(2) = \{\alpha', G'\}.$$

If we show that

$$\{\psi, F\} = \{\alpha, G\},$$

using the induction principle we have finished.

$$\begin{aligned}
A(a, \{\psi, F\}) &= [2, \Omega](\psi, A(a, F-)) \\
&= \Omega(\psi(1), A(a, F(1))) \wedge \Omega(\psi(2), A(a, F(2))) \\
&= \Omega(\alpha(k), A(a, F(k))) \wedge \Omega(I, A(a, \{\alpha'.G'\})) \\
&= \Omega(\alpha(k), A(a, F(k))) \wedge A(a, \{\alpha'.G'\}) \\
&= \Omega(\alpha(k), A(a, F(k))) \wedge [K', \Omega](\alpha', A(a, G'-)) \\
&= \Omega(\alpha(k), A(a, F(k))) \wedge \bigwedge_{k' \in K'} \Omega(\alpha'(k'), A(a, G'(k'))) \\
&= \Omega(\alpha(k), A(a, F(k))) \wedge \bigwedge_{k' \in K'} \Omega(\alpha'(k'), A(a, G'(k')))) \\
&= \bigwedge_{k \in K} \Omega(\alpha(k), A(a, F(k))) \\
&= \bigwedge_{k \in K} \Omega(\alpha(k), A(a, F(k))) \\
&= A(a, \{\alpha, G\}).
\end{aligned}$$

□

2.3.5 Distribution of finite limits over finite colimits in the quantale Ω

In the previous two subsection we have showed some results regarding limits and colimits in categories enriched over commutative unital quantales, now we will show a result about limits and colimits in the quantale Ω, viewed as a self enriched category.

Let $\Omega = (\Omega_0, I, \otimes)$ be a commutative unital quantale. Let K be a finite set, and for each $k \in K$ let K_k be a finite set as well. Now let $\psi : K \longrightarrow \Omega$ be any Ω-functor, and for each $k \in K$ let $\varphi_k, G_k : K_k \longrightarrow \Omega$ be two Ω-functors. Let $G : K \longrightarrow V$ be the Ω-functor given by $G(k) = \varphi_k \star G_k$. Then one can construct the following limit

$$\{\psi, G\}.$$

We want to express this limit as a colimit.

Remark 2.3.8. One can define such a limit only because we assume that K and each K_k are sets, so our reduction of limits and colimits to discrete Ω-categories is important and necessary.

Before that let us fix some concepts.

Definition 2.3.9. For any set K and any K-tuple of sets K_k we call a choice function, f any function $f : K \longrightarrow \oplus_{k \in K} K_k$ such that $f(k) \in K_k$. The set of choice functions is called $\Sigma = \{f : K \longrightarrow \oplus_{k \in K} K_k \mid \text{f choice function}\}$.

Remark 2.3.10. Let us note that the choice functions are needed in order to keep the colimit finite.

Now define $\phi, F : \Sigma \longrightarrow \Omega$ given by $\phi(f) = I$ and $F(f) = \{\psi, F_f\}$ where $F_f : K \longrightarrow \Omega$ is given by $F_f(k) = \varphi_k(f(k)) \otimes G_k(f(k))$.

Proposition 2.3.11. *With the above notations, if Ω has a total order we have*

$$\{\psi, G\} = \phi \star F$$

Proof. We have to show that

$$\bigwedge_{k \in K} \psi(k) \pitchfork \bigvee_{k' \in K_k} \varphi_k(k') \otimes G_k(k') = \bigvee_{f \in \Sigma} \bigwedge_{k \in K} \psi(k) \pitchfork (\varphi_k(f(k)) \otimes G_k(f(k)))$$

As each K_k is finite and the order on Ω is total then the colimit $\bigvee_{k' \in K_k} \varphi_k(k') \otimes G_k(k')$ is reached, in the sense that there exists an element, denoted by κ_k, of K_k such that

$$\varphi_k(\kappa_k) \otimes G_k(\kappa_k) = \bigvee_{k' \in K_k} \varphi_k(k') \otimes G_k(k').$$

Define the choice function $f_\kappa : K \longrightarrow \oplus_{k\in K} K_k$ as $f_\kappa(k) = \kappa_k$. Now obviously we have

$$\begin{aligned}\bigwedge_{k\in K} \psi(k) \pitchfork \bigvee_{k'\in K_k} \varphi(k) \otimes G_k(k') &= \bigwedge_{k\in K} \psi(k) \pitchfork \varphi_k(\kappa_k) \otimes G_k(\kappa_k)\\ &= \bigwedge_{k\in K} \psi(k) \pitchfork \varphi_k(f_\kappa(k)) \otimes G_k(f_\kappa(k))\\ &\leq \bigvee_{f\in\Sigma} \bigwedge_{k\in K} \psi(k) \pitchfork (\varphi_k(f(k)) \otimes G_k(f(k)))\end{aligned}$$

For the other direction we have to show that for any choice function f we have

$$\bigwedge_{k\in K} \psi(k) \pitchfork (\varphi_k(f(k)) \otimes G_k(f(k))) \leq \bigwedge_{k\in K} \psi(k) \pitchfork \varphi_k(f_\kappa(k)) \otimes G_k(f_\kappa(k)).$$

From our definition of f_κ we have that for every choice function f and every $k \in K$ we have $\varphi_k(f(k)) \otimes G_k(f(k)) \leq \varphi_k(f_\kappa(k)) \otimes G_k(f_\kappa(k))$, and as $\pitchfork$ is monotone on the right argument we have

$$\psi(k) \pitchfork \varphi_k(f(k)) \otimes G_k(f(k)) \leq \psi(k) \pitchfork \varphi_k(f_\kappa(k)) \otimes G_k(f_\kappa(k)).$$

□

2.4 Kan extensions

Complete atomic boolean algebras are the algebras for the double powerset monad. The contravariant powerset functor is a special case of the functor $[-, C] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ defined in Subsection 2.3.1. In this section we will define and state some properties of the left and right adjoints, if they exist, of the functor $[F, C]$ for any V-categories A, B, C and any Ω-functor $F : A \longrightarrow B$.

Let A, B, C be any two Ω-categories and let $F : A \longrightarrow C$ be any functor. Then we have the following two functors: $\tilde{F} : A \longrightarrow [C^{\mathrm{op}}, \Omega]$ and $\hat{F} : A^{\mathrm{op}} \longrightarrow [C, \Omega]$ given by $\tilde{F}(a) = C(-, F(a))$ and $\hat{F}(a) = C(F(a), -)$.

Definition 2.4.1. For any three Ω-categories A, B, C and any two Ω-functors $F : A \longrightarrow C$ and $G : A \longrightarrow B$ we define the left Kan extension of G along F, denoted by $\mathrm{Lan}_F G$, to be the colimit $\hat{F} \star G = C(F-, -) \star G$, if it exists. Dually we define the right Kan extension, to be the limit $\{\tilde{F}, G\} = \{C(-, F-), G\}$, if it exists.

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Lan}_F G"] & B \\
A \arrow[u, "F"] \arrow[ur, "G"'] \arrow[ur, phantom, "\Uparrow", shift left=3]
\end{tikzcd} \tag{2.14}$$

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Ran}_F G"] \arrow[r, phantom, "\Downarrow"', shift right=2] & B \\
A \arrow[u, "F"] \arrow[ur, "G"']
\end{tikzcd} \tag{2.15}$$

So if B is complete then the right Kan extension exists and if B is cocomplete then the left one exists. Now let us show that, if they exist, these extensions are adjoints to the precomposition.

Proposition 2.4.2. *For any three Ω-categories A, B, C and any Ω-functor $F : A \longrightarrow C$ then the left Kan extension along F is the left adjoint of $[F, B]$, if it exists. Dually the right Kan extension along F is the right adjoint of $[F, B]$, if it exists.*

Proof. Let us look at the following diagram

$$\begin{tikzcd}
{[C,B]} \arrow[r, "- \circ F"] \arrow[r, phantom, "\top", shift left=4] \arrow[r, phantom, "\top"', shift right=3] & {[A,B]} \arrow[l, bend right=40, "\mathrm{Ran}_F -"'] \arrow[l, bend left=40, "\mathrm{Lan}_F -"]
\end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd}
A \arrow[rr, "F"'] & & C
\end{tikzcd}$$

Let $G : A \longrightarrow B$ and $H : C \longrightarrow B$. Then we have

$$\begin{aligned}
[C,B](\mathrm{Lan}_F G, H) &= [C,B](C(F-,-) \star G, H) \\
&= \bigwedge_{c\in C} B(C(F-,c) \star G, H(c)) \\
&= \bigwedge_{c\in C} \{C(F-,c), B(G-, H(c))\} \\
&= \bigwedge_{c\in C} \bigwedge_{a\in A} C(F(a),c) \pitchfork B(G(a),H(c)) \\
&= \bigwedge_{a\in A} \bigwedge_{c\in A} C(F(a),c) \pitchfork B(G(a),H(c)) \\
&= \bigwedge_{a\in A} [C,\Omega](C(F(a),-), B(G(a),H-)) \\
&= \bigwedge_{a\in A} B(G(a), H(F(a))) \\
&= [A,B](G, H \circ F)
\end{aligned}$$

Thus $\mathrm{Lan}_F - \dashv - \circ F$.

On the other hand we have

$$\begin{aligned}
[C,B](H, \mathrm{Ran}_F G) &= [C,B](H, \{C(-,F-), G\}) \\
&= \bigwedge_{c\in C} B(H(c), \{C(c,F-), G\}) \\
&= \bigwedge_{c\in C} \bigwedge_{a\in A} C(c,F(a)) \pitchfork B(H(c),G(a)) \\
&= \bigwedge_{a\in A} \bigwedge_{c\in C} C(c,F(a)) \pitchfork B(H(c),G(a)) \\
&= \bigwedge_{a\in A} [C^{\mathrm{op}},\Omega](C(-,F(a)), B(H-,G(a))) \\
&= \bigwedge_{a\in A} B(H(F(a)), G(a)) \\
&= [A,B](H \circ F, G)
\end{aligned}$$

□

Proposition 2.4.3. *For any two functors $F : A \longrightarrow C$ and $G : A \longrightarrow B$, if the left or right Kan extension exists, the following statements are true*

1. *One has natural transformations $\alpha : G \longrightarrow (\mathrm{Lan}_F G) \circ F$ and $\beta : (\mathrm{Ran}_F G) \circ F \longrightarrow G$, and furthermore these are isomorphisms if and only if F is full and faithful.*

2. *For any other functor $H : C \longrightarrow B$ such that there exists $\eta : G \longrightarrow H \circ F$ there exists a unique $\eta' : \mathrm{Lan}_F G \longrightarrow H$ such that $\eta' \circ \alpha = \eta$*

Proof. Let us prove the first statement.

Let $F : A \longrightarrow C$ and $G : A \longrightarrow B$ be two Ω-functors and let us suppose the left Kan extension of G along F exists. Then

$$\begin{aligned}(\mathrm{Lan}_F G) \circ F &= (C(F-,-) \star G) \circ F \\ &= C(F-,F) \star G.\end{aligned}$$

Now let us show that for every a in A we have an arrow in Ω

$$\alpha_a : I \longrightarrow B(G(a), C(F-,F(a)) \star G)).$$

But that is just the following composition

$$I \longrightarrow C(F,F) \longrightarrow B(G, C(F,F) \star G)$$

where the left arrow is the identity arrow and the right arrow is the counit of the colimit $C(F-,F) \star G$.

We still have to show that α is an isomorphism if and only if F is full and faithful.

The map $f : d_A \longrightarrow C(F-,F) : A \longrightarrow [A^{\mathrm{op}}, \Omega]$ is an isomorphism if and only if F is fully faithful. Now using the fact that $G \cong (d_A-) \star G$ and that $\alpha = (f-) \star G$ we have that α is an isomorphism if F is fully faithful. To show the converse take $B = \Omega$ and $G = A(a,-)$ then we have $(\mathrm{Lan}_F G) \circ F = C(F-,F) \star A(a,-) = C(F-,F(a))$ thus α is an isomorphism.

Let us prove the second statement.
Let us assume that there exists $H : C \longrightarrow B$ such that there exists $\eta : G \longrightarrow H \circ F$.

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Lan}_F G"'] \arrow[r, bend left=50, "H"] \arrow[r, phantom, bend left=25, "\Uparrow"] & B \\
A \arrow[u, "F"] \arrow[ur, "G"'] \arrow[ur, phantom, shift left=3ex, "\Uparrow"] &
\end{tikzcd} \tag{2.17}$$

We have

$$[A,B](G,H\circ F)=[C,B](\mathrm{Lan}_F G,H)$$

□

In the following let us give some example of Kan extensions.

Example 2.4.4. 1. Let A,B be any two Ω-categories, and let $d_A : A \longrightarrow [A^{\mathrm{op}},\Omega]$ be the Yoneda embedding, then for any map $F : A \longrightarrow B$ we have

$$\mathrm{Lan}_{d_A}F = - \star F : [A^{\mathrm{op}},\Omega] \longrightarrow B,$$

and

$$\mathrm{Lan}_F d_A = B(F-,-) : B \longrightarrow [A^{\mathrm{op}},\Omega].$$

Indeed, for any $\varphi : A^{\mathrm{op}} \longrightarrow \Omega$, we have

$$(\mathrm{Lan}_{d_A}F)(\varphi)=\bigvee_{a\in A}[A^{\mathrm{op}},\Omega](A(-,a),\varphi)\otimes F(a)=\bigvee_{a\in A}\varphi(a)\otimes F(a),$$

and for any $b\in B$, we have

$$(\mathrm{Lan}_F d_A)(b)=\bigvee_{a\in A}B(F(a),b)\otimes A(-,a)=B(F-,b).$$

2.5 Density

Definition 2.5.1. Let $\mathcal{C},\mathcal{D}$ be two Ω- categories and let $F:\mathcal{C}\longrightarrow\mathcal{D}$ be a Ω-functor. We say that F is dense if every object of d of $\mathcal{D}$ is exhibited as

$$D=\mathcal{D}(F-,D)\star F,$$

by $\mathrm{id}:\mathcal{D}(F-,d)\longrightarrow\mathcal{D}(F-,d)$.

Following [14, Chapter 5] there are other equivalent, definitions, of a dense functor.

Proposition 2.5.2. *Let $F:\mathcal{C}\longrightarrow\mathcal{D}$ be a functor. The next statements are equivalent.*

1. *The functor F is dense.*

2. *The functor* $\tilde{F} : \mathcal{D} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \Omega]$ *given by* $\tilde{F}(d) = \mathcal{D}(F-, d)$ *is full and faithful.*

3. *For any* Ω*-category* $\mathcal{B}$*, the restriction to* $\mathcal{A} - Cocts[\mathcal{D}, \mathcal{B}]$ *of the functor* $[F, 1] : [\mathcal{D}, \mathcal{B}] \longrightarrow [\mathcal{C}, \mathcal{B}]$ *is fully faithful, where by* $\mathcal{A} - Cocts[\mathcal{D}, \mathcal{B}]$ *we understand the full subcategory of* $[\mathcal{D}, \mathcal{B}]$ *of those functors who preserve all* $\mathcal{A}$*-indexed colimits.*

4. *For any two objects* C, D *of* $\mathcal{D}$ *the map*

$$\tilde{F} : \mathcal{D}(C, D) \longrightarrow [\mathcal{C}^{\mathrm{op}}, \Omega](\mathcal{D}(F-, C), \mathcal{D}(F-, D))$$

   is an isomorphism.

5. *The identity* $\mathrm{id} : F \longrightarrow \mathrm{id}_{\mathcal{D}} \circ F$ *exhibits* $\mathrm{id}_{\mathcal{D}}$ *as* $\mathrm{Lan}_F F$.

6. *Some isomorphism* $\phi : F \longrightarrow \mathrm{id}_{\mathcal{D}} \circ F$ *exhibits* $\mathrm{id}_{\mathcal{D}}$ *as* $\mathrm{Lan}_F F$.

We shall not prove it here. One can find a proof in [14, Chapter 5].

Example 2.5.3. Let X be any Ω-category then the Yoneda embedding $dX : X \longrightarrow [X^{\mathrm{op}}, \Omega]$ is dense. In oder to show that d_X is dense $\varphi : X^{\mathrm{op}} \longrightarrow \Omega$ and $x \in X$ we have to show that $\mathrm{Lan}_{d_X} d_X(\varphi)(x) \cong \varphi(X)$. Using example 2.4.4 we have that $\mathrm{Lan}_{dX}\, dX = [X^{\mathrm{op}}, \Omega](d_X, -)$. Thus $(\mathrm{Lan}_{d_X} d_X)(\varphi)(x) = [X^{\mathrm{op}}, \Omega](X(-, x), \varphi) = \varphi(x)$.

Proposition 2.5.4. *Let* A, B *be two categories and let* $G : A \longrightarrow C$ *be any* Ω*-functor, such that the left Kan extension* $\mathrm{Lan}_{d_A} G$ *exists. Then* $\mathrm{Lan}_{d_A} G$ *preserves all colimits in* $[A^{\mathrm{op}}, \Omega]$.

Proof. As $\mathrm{Lan}_{d_A} G = - \star G$ for any other colimit $\varphi \star F : A^{\mathrm{op}} \longrightarrow \Omega$ Using Fubini theorem, see [14, 3.23], we have that

$$(\mathrm{Lan}_{d_A} G)(\varphi \star F) = (\varphi \star F) \star G = \varphi \star (F - \star G).$$

□

2.6 Monads and KZ-doctrines

In the following we will give some definitions about monads, and some results about a special class of monads called KZ-doctrines. For more on monads see [20], and for KZ-doctrines see [15] and [22].

Definition 2.6.1. A *monad* (M, η, μ) in a category X consists of an endofunctor $M : X \longrightarrow X$ and two natural transformations $\eta : 1 \longrightarrow M$ and $\mu : MM \longrightarrow M$ which make the next two diagrams commute:

$$\begin{tikzcd}
DDD \arrow[r, "\mu_D"] \arrow[d, "D\mu"'] & DD \arrow[d, "\mu"] \\
DD \arrow[r, "\mu"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "\eta_D"] \arrow[dr, "1"'] & DD \arrow[d, "\mu"] & D \arrow[l, "D\eta"'] \arrow[dl, "1"] \\
 & D &
\end{tikzcd}
\qquad (2.18)$$

Now let $U = (U, u, n)$, $D = (D, d, m)$ be two monads. What does one need for DU to be a monad as well? First thing one would need is a multiplication: $\mu_{DU} : DUDU \longrightarrow DU$. If one has a natural transformation $\delta : UD \longrightarrow DU$ then we could define μ_{DU} as $n \circ mUU \circ D\delta U$. Of course this has to satisfy a couple of commutative diagrams. And it has been shown in [1] that the composite of two monads is again a monad if there exists a distributive law between them, like below.

Definition 2.6.2. Let $U = (U, u, n)$, $D = (D, d, m)$ be two monads, then a *distributive law* of U over D is any natural transformation $r : UD \Rightarrow DU$ satisfying Beck's axioms, as presented in [1]

$$\begin{tikzcd}
 & U \arrow[dl, "Ud"'] \arrow[dr, "dU"] & \\
UD \arrow[rr, "r"'] & & DU \\
 & D \arrow[ul, "uD"] \arrow[ur, "Du"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"'] & DUD \arrow[r, "Dr"] & DDU \arrow[d, "mU"] \\
UD \arrow[rr, "r"'] & & DU \\
UUD \arrow[u, "nD"] \arrow[r, "Ur"'] & UDU \arrow[r, "rU"'] & DUU \arrow[u, "Dn"']
\end{tikzcd}
\qquad (2.19)$$

The notion of KZ-doctrines dates all the way back to mid sixties, and is due to Kock and Zoberlein. The following proposition gives three different descriptions the first one is due to [15] and the other ones are due to [22]. We would like to emphasise that the two notions are equivalent because we work in a quantale enriched setting, otherwise some of these concepst would hold up to isomorphism.

Proposition 2.6.3. *For any 2-category $\mathcal{C}$, a functor $D : \mathcal{C} \longrightarrow \mathcal{C}$ is a KZ-doctrine if it satisfies any of the following equivalent conditions:*

1. *Natural transformations $d : 1 \longrightarrow D, m : DD \longrightarrow D$ and for each $C \in \mathcal{C}$, a 2-cell $\lambda_C : Dd \longrightarrow dD$, natural in C, satisfying the following four equations:*

   D0 *d is two sided unit for m, that is $m \circ Dd = m \circ dD = id$*

   D1 *$\lambda_C \circ d_C$ is an identity 2-cell*

   D2 *$m_C \circ \lambda_C$ is an identity 2-cell*

   D3 *$m_C \circ Dm_C \circ \lambda_{TC}$ is an identity 2-cell*

2. *For the functor D there exists natural transformations $d : 1 \Rightarrow D$ and $m : DD \longrightarrow D$ such that the following forms a fully faithful adjoint string $Dd \dashv m \dashv dD$. By a fully faithful adjoint string we understand an adjunction string $F \dashv G \dashv H$ where the unit of the first adjunction and the counit of the second one are isomorphisms.*

3. *The functor D is a monad $D = (D, d, m)$ which stisfies the following adjunctions $Dd \dashv m \dashv dD$.*

Proof. A complete proof of the equivalence between the first two properties can be found in [15] and in [22]. In here we will only give a sketch.

$1 \Rightarrow 2$

We have to construct $\eta_{Dd \dashv m} : Id \longrightarrow m \circ Dd, \varepsilon_{Dd \dashv m} : Dd \circ m \longrightarrow 1, \eta_{m \dashv dD} : 1 \longrightarrow dD \circ m$, and $\varepsilon_{m \dashv dD} : m \circ dD \longrightarrow 1$. From **D** we get both $\eta_{Dd \dashv m}$ and $\varepsilon_{m \dashv dD}$, so we only have to construct the other two. Let us look at the next diagram which follows from the naturality of d:

$$\begin{tikzcd}
DD \arrow[r, "dDD"] \arrow[d, "m"'] & DDD \arrow[d, "Dm"] \\
D \arrow[r, "dD"'] & DD
\end{tikzcd} \tag{2.20}$$

thus $Dm \circ dDD = dD \circ m$ and has $\lambda_D : DdD \longrightarrow dDD$ one has $Dm\lambda_D : Dm \circ DdD \longrightarrow Dm \circ dDD$, and as $Dm \circ DdD = Id$ then one has

$$Dm\lambda_D : Id \longrightarrow Dd \circ m$$

Now if we apply the naturality of m one has the next diagram:

$$\begin{tikzcd}
DD \arrow[r, "m"] \arrow[d, "DdD"'] & D \arrow[d, "dD"] \\
DDD \arrow[r, "mD"'] & DD
\end{tikzcd} \tag{2.21}$$

thus $dD \circ m = mD \circ DdD$ and as $mD\lambda_D : mD \circ DdD \longrightarrow mD \circ dDD$ and $mD \circ dDD = Id$ one has

$$mD\lambda_D : Dd \circ m \longrightarrow Id.$$

$2 \Rightarrow 1$

Uses diagram pasting and can be found in [22]

$2 \Leftrightarrow 3$

We have to show the commutativity of diagrams (2.18). The triangular diagrams are valid from the fully faithful adjoint string condition. To obtain the square we have to expand it.

$$\begin{tikzcd}
DDD \arrow[rr, "Id"] \arrow[dr, "mD"'] & & DDD \arrow[rr, "Dm"] & & DD \arrow[dr, "m"] & \\
& DD \arrow[ur, "dDD"'] \arrow[rr, "m"'] & & D \arrow[ur, "dD"] \arrow[rr, "Id"'] & & D
\end{tikzcd} \tag{2.22}$$

$\square$

The following proposition is due to [15], and it holds in the full generality of two categories. This result can be said to be one of the defining property of KZ-doctrines.

Proposition 2.6.4. *Let $M = (M, \eta, \mu)$ be any KZ-doctrine, then $A = (A, \alpha)$ is a M-algebra if and only if the structure map α is a left adjoint of η_M. Dually for any co-KZ-doctrine $N = (N, \zeta, \nu)$ a tuple $B = (B, \beta)$ is a N-algebra if and only if β is a right adjoint of ζ.*

As a corollary we have that a distributive law between KZ-doctrines, if exist, is unique. This result can be found in [23] where it is proved in the context of $\Omega = \mathbb{2}$.

Corollary 2.6.5. *If D or U is either KZ(co)-doctrines then there is at most one distributive laws $r : UD \longrightarrow DU$.*

The next result also comes from [23] where it is proved for $\Omega = \mathbb{2}$, but the proof remains the same as they work in any 3-category where 3-cells form a poset. Also one has to note that to prove this result one needs exactly that isomorphism of natural transformations implies equality.

Proposition 2.6.6. *For monads D and U and a natural transformation $r : UD \longrightarrow DU$:*

1. *If (D, d, μ) is KZ and (U, u, ν) is either KZ or co-KZ then $r : UD \longrightarrow DU$ is a distributive law if it satisfies $r \circ Ud = dU$ and $r \circ uD \leq Du$;*
2. *If (U, u, ν) is co-KZ and (D, d, μ) is either KZ or co-KZ then $r : UD \longrightarrow DU$ is a distributive law if it satisfies $r \circ uD = Du$ and $r \circ Ud \leq dU$.*

Proof. In order for r to be a distributive law we need to show the comutativity of diagrams (2.19).

$$\begin{tikzcd}
& U \arrow[dl, "Ud"'] \arrow[dr, "dU"] & \\
UD \arrow[rr, "r"'] & & DU \\
& D \arrow[ul, "uD"] \arrow[ur, "Du"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"'] & DUD \arrow[r, "Dr"] & DDU \arrow[d, "mU"] \\
UD \arrow[rr, "r"'] & & DU \\
UUD \arrow[u, "nD"] \arrow[r, "Ur"'] & UDU \arrow[r, "rU"'] & DUU \arrow[u, "Dn"']
\end{tikzcd}$$

From the assumption $r \circ Ud = dU$ and the naturality of d, and respectively r, the outside of the next two diagrams commute:

$$\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"] & DUD \arrow[r, "Dr"] & DDU \arrow[d, "mU"'] \\
UD \arrow[u, bend left=40, "UdD", "\vdash"'] \arrow[ur, "dUD"'] \arrow[rr, "r"'] & & DU \arrow[u, bend right=40, "dDU"']
\end{tikzcd} \tag{2.23}$$

and

$$\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"] & DUD \arrow[r, "Dr"] & DDU \arrow[d, "mU"'] \\
UD \arrow[u, bend left=40, "UDd"] \arrow[rr, "r"'] & & DU. \arrow[ul, "DUd"] \arrow[u, bend right=40, "DdU"', "\dashv"]
\end{tikzcd} \tag{2.24}$$

Thus we have

$$dDU \circ r = Dr \circ rD \circ UdD, \tag{2.25}$$

and

$$DdU \circ r = Dr \circ rD \circ UDd. \tag{2.26}$$

Now post-composing (2.25) with mU and that the counit of $m \dashv dD$ is an isomorphism we have

$$mU \circ Dr \circ rD \circ UdD = mU \circ dDU \circ r = r,$$

and now pre-composing this equation with Um we obtain

$$mU \circ Dr \circ rD \circ UdD \circ Um = r \circ Um.$$

From here using the unit of $m \dashv dD$ we obtain

$$mU \circ Dr \circ rD \leq r \circ Um \tag{2.27}$$

Now post-composing (2.26) with mU, and using that the unit of $Dd \dashv m$ is an isomorphism we have

$$mU \circ Dr \circ rD \circ UDd = mU \circ DdU \circ r = r,$$

and now pre-composing this equation with Um we obtain

$$mU \circ Dr \circ rD \circ UDd \circ Um = r \circ Um.$$

From here using the counit of $Dd \dashv m$ we obtain

$$r \circ Um \leq mU \circ Dr \circ rD \tag{2.28}$$

Thus from D a KZ-doctrine and $r \circ Ud = dU$ one obtains $r \circ Um = mU \circ Dr \circ rD$. Now let us show that assuming $r \circ uD \leq Du$ we have $r \circ uD = Du$.

$$\begin{tikzcd}
 & & DU \arrow[d, "DUd"'] \arrow[dr, "DdU"] \arrow[drr, "1_{DU}"] & & \\
D \arrow[urr, bend left, "Du"] \arrow[r, bend left, "Dd"] \arrow[r, bend right, "dD"'] \arrow[r, phantom, "\Downarrow"] \arrow[drr, bend right, "uD"'] & DD \arrow[r, "DuD"] & DUD \arrow[r, "Dr"] & DDu \arrow[r, "mU"] & DU \\
 & & UD \arrow[u, "dUD"] \arrow[r, "r"'] & DU \arrow[u, "dDU"'] \arrow[ur, "1_{DU}"'] &
\end{tikzcd} \tag{2.29}$$

From the naturality of u we have

$$DuD \circ Dd = DUd \circ Du,$$

and post-composing with Dr one gets

$$Dr \circ DuD \circ Dd = Dr \circ DUd \circ Du = DdU \circ Du,$$

and post-composing with mU, one obtains

$$mU \circ Dr \circ DuD \circ Dd = mU \circ DdU \circ Du = Du.$$

From the naturality of d we have

$$DuD \circ dD = dUD \circ uD,$$

and now post-composing with Dr and using the naturality of r one has

$$Dr \circ DuD \circ dD = Dr \circ dUD \circ uD = dDU \circ r \circ uD,$$

and post-composing with mU, one obtains

$$mU \circ Dr \circ DuD \circ dD = mU \circ dDU \circ r \circ uD = r \circ uD.$$

Now as D is a KZ-doctrine we have $Dd \leq dD$ thus we have $Du \leq r \circ uD$.

To conclude the proof we still have to show that $r \circ Dm = mD \circ Ur \circ rU$, which follows from U being a KZ or co-KZ-doctrine and $r \circ uD = Du$. ☐

Chapter 3

Duality and logic

As we said in the introduction we are interested in a many valued logic which has extra structure on the set of truth values, for example a metric distance. Before we continue let us remind the reader that clasical propositional logic is a boolean algebra. Thus we want to generalize the notion of Boolean algebras to a many valued setting. To do that we are using a categorical approach. First we will restate the known fact that the category of complete atomic boolean algebras is equivalent to the cagegory of algebras for the double powerset monad, see [21]. Second we will show that this adjunction can be generalised.

3.1 Double powerset monad and boolean algebras

We want to characterise Boolean algebras, or at least complete atomic boolean algebras, as the algebras for a monad.

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "{[-,2]}"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathrm{op}} \arrow[l, bend left, "{[-.2]}"] \end{tikzcd} \tag{3.1}$$

This is a monadic adjunction so we know that $\mathsf{Set}^{\mathrm{op}}$ is isomorphic to the category of algebras, so if we show that $\mathsf{Set}^{\mathrm{op}} \simeq \mathsf{caBA}$ then we are done.

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[r, bend left, "{\mathcal{P}=[-,2]}"] \arrow[r, phantom, "\simeq"] \arrow[d, bend left, "{[-,2]}"] \arrow[d, phantom, "\dashv"] & \mathsf{caBA} \arrow[l, bend left, "\mathsf{AT}"] \\
\mathsf{Set} \arrow[u, bend left, "{[-,2]}"] &
\end{tikzcd} \tag{3.2}$$

We will show this equivalence in four steps.

The **first** step will be to show that a complete atomic Boolean algebra B is isomorphic to the powerset of some set X. Let us recall what we understand by a complete atomic boolean algebra.

Definition 3.1.1. For a boolean algebra B we call $a \in B$ an *atom* if a is join prime, that is (if $a \leq b \vee c$ then either $a \leq b$ or $a \leq c$). Another way to express that a is join prime is to say that $B(a, -)$ preserves colimits.

Definition 3.1.2. A Boolean algebra $(B, \vee, \wedge, \neg, 0, 1)$ is *complete* if the lattice $(B, \vee, \wedge, 0, 1)$ is complete, and is *atomic* if every element of B is a join of atoms.

Let B be a complete atomic boolean algebra, and let X be the set of atoms. Let us show that B is isomorphic to $\mathcal{P}(X)$.

We will show that the function $F : B \longrightarrow \mathcal{P}(X)$ given by

$$F(b) = \{a \in B \mid a \leq b\}$$

is an isomorphism.
We have $F(b) = \{b\}$ if and only if b is an atom. Indeed as any element b of B is a join of atoms, we have that b is above all the atoms that generate it, thus $F(b) = \{a \in X \mid a \leq b\} = \{b\}$ if and only if b is an atom.
Let us show that F is a morphism of boolean algebras.

- $\neg$: We have to show that $F(\neg b) = \neg F(b)$ for all $b \in B$, that is equivalent to $F(b)$ and $F(\neg b)$ being complements in $\mathcal{P}(X)$. Let us show that

  $$F(\neg b) \cup F(b) = X \text{ and } F(\neg b) \cap F(b) = \phi.$$

  As we have $F(b) \cup F(\neg b) \subseteq X$ we only have to show that $X \subseteq F(b) \cup F(\neg b)$. For any atom a we have $a \leq 1 = a \leq (b \vee \neg b)$. Therefore either $a \leq b$ or $a \leq \neg b$ thus $a \in F(b) \cup F(\neg b)$.

Now suppose that there exists $a \in F(b)$ and $a \in F(\neg b)$ then $a \leq b$ and $a \leq \neg b$ thus $a \leq (b \wedge \neg b) = 0$ which is impossible as a is an atom. Thus we have

$$F(\neg b) = X - F(b) = \neg F(b).$$

$\vee$: Let b, c be any two elements of B then we have

$$F(b \vee c) = \{a \in X \mid a \leq (b \vee c)\} = \{a \in X \mid a \leq b \text{ or a } \leq c\} = F(b) \cup F(c).$$

1: As we have $1 = \bigvee\{a \in X\}$ then

$$F(1) = F(\vee\{a \in x\}) = \cup_{a \in X} F(a) = \cup_{a \in X}\{a\} = X.$$

0: This is obvious as we have $0 = \neg 1$

$\wedge$: This is also obvious as we have $a \wedge b = \neg(\neg a \vee \neg b)$.

We still have to show that F is an isomorphism, so that that F is injective and surjective. Let $b, c \in B$ such that $F(b) = F(c)$, but B being atomic means that every element is the join of all the atoms below it, thus we have

$$b = \bigvee\{a \in F(b)\} = \bigvee\{a \in F(c)\} = c.$$

Thus F is injective.

Now let us show that F is surjective. Let Y be any subset of X and let us show that there exists $y \in B$ such that $F(y) = Y$. As B is complete then the join $\bigvee\{a \in Y\}$ exists, and let us take $y = \bigvee\{a \in Y\}$. Now we have

$$F(y) = F(\bigvee\{a \in Y\}) = \bigcup\{F(a) \mid a \in Y\} = \{\{a\} \mid a \in Y\} = Y.$$

Thus we have shown the following proposition:

Proposition 3.1.3. *Any complete atomic boolean algebra is isomorphic to the powerset of its set of atoms.*

The **second** step is to show that for any set X the set of atoms of $\mathcal{P}(X)$ is isomorphic to X.

So let X be any set and let $S \in \mathcal{P}(X)$ be an atom. Then for every other two subsets S_1, S_2 of X such that $S \subseteq S_1 \cup S_2$ we have $S \subseteq S_1$ or $S \subseteq S_2$, now obviously S has to be a singleton, otherwise choose S_1, S_2 to be a partition of S. Thus the set of atoms of $\mathcal{P}(X)$ is indeed isomorphic to X.

The **third** and final step is to show that this equivalence is functorial we have to show that for every morphism of boolean algebras $f : A \longrightarrow B$ we have a map $\mathsf{AT}(f) : \mathsf{At}(B) \longrightarrow \mathsf{At}(a)$ and furthermore $\mathcal{P}(\mathsf{AT}(f)) = f$, and for every map of sets $g : X \longrightarrow Y$ we have $\mathsf{AT}(\mathcal{P}(g)) = g$.

Let $f : A \longrightarrow B$ be a boolean algebra morphism, then f has a left adjoint $g : B \longrightarrow A$ given by

$$g(b) = \wedge\{a \mid b \leq f(a)\}.$$

Let us show that if $b \in B$ is an atom then so is $g(b)$, so let $a_1, a_2 \in A$ such that

$$g(b) \leq a_1 \vee a_2,$$

then using the adjunction and that f is a boolean algebras morphism we have

$$b \leq f(a_1) \vee f(a_2).$$

As b is an atom then we have

$$b \leq f(a_1) \text{ or } b \leq f(a_2),$$

but that means that $g(b) \leq a_1$ or $g(b) \leq a_2$. Thus we define $\mathsf{AT}(f) = g\mid_{\mathsf{At}(B)}$ Let us check that $\mathcal{P}(\mathsf{AT}(f)) = f$. Let us look at the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathsf{AT}"'] & B \arrow[d, "\mathsf{AT}"] \\
\mathsf{At}(A) \arrow[d, "\mathcal{P}"'] & \mathsf{At}(B) \arrow[l, "\mathsf{AT}(f)"'] \arrow[d, "\mathcal{P}"] \\
\mathcal{P}(\mathsf{At}(A)) \arrow[r, "\mathcal{P}(\mathsf{AT}(f))"'] & \mathcal{P}(\mathsf{At}(B))
\end{tikzcd} \tag{3.3}$$

Let $S \in \mathcal{P}(\mathsf{At}(A))$ and let us take $s = \bigvee\{a \in S\}$. As we have shown that $\mathcal{P}(\mathsf{AT}(A))(s) = S$, in order to show that $f = \mathcal{P}(\mathsf{AT}(f))$ all we have to prove is that $\mathcal{P}(\mathsf{AT}(f))(S) = f(s) = \vee\{f(a) \mid a \in S\}$.

We have $\mathcal{P}(\mathsf{AT}(f))(S) = S \circ \mathsf{AT}(f) = \{b \in \mathsf{AT}(B) \mid \exists a \in S, b \leq f(a)\}$.
Now, as B is atomic, we have $\vee\{b \in \mathsf{AT}(B) \mid \exists a \in S, b \leq f(a)\} = f(s)$.

Let X, Y be two sets and let $g : X \longrightarrow Y$ be any function. Let us show that $\mathsf{AT}(\mathcal{P}(g)) = g$. Let $S \in \mathcal{P}(Y)$ then $\mathcal{P}(g)(S) = S \circ g = \{A \subseteq X \mid \exists x \in A. g(x) \in S\}$ we have $\mathsf{AT}(\mathcal{P}(g))\{x\}$ as the intersection of all those sets containing $\{g(x)\}$ thus $\mathsf{AT}(\mathcal{P}(g))\{x\} = \{g(x)\}$ thus $g = \mathsf{AT}(\mathcal{P}(g))$.

So we have shown that complete atomic boolean algebras are the algebras for the double powerset monad. So one can say that propositional logic is given by the double powerset monad. If we move to enriched category theory we have a similar situation, in the sense, that again we have the adjunction $[-, \mathbb{2}] \dashv [-, \mathbb{2}] : \mathsf{Pre} \longrightarrow \mathsf{Pre}^{\mathrm{op}}$ and we also have a nice representation for $\mathsf{Pre}^{\mathrm{op}}$ as the category of completely distributive atomic lattices. Thus one can say that in order to define a new logic we can change this monad, in the sense that instead of functoring into $\mathbb{2}$ we can functor into a commutative unital quantale. Of course this will not cover all many-valued logics, but only those for which the set of truth values is a quantale.

3.2 Enriched adjuction and duality

Now following [14, Chapter 1.5], for Ω a quantale, we have the following adjunction

Proposition 3.2.1. *$U = [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is a left adjoint of $D = [-, \Omega] : \Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \Omega\text{-}\mathsf{Cat}$.*

Proof. $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}([X, \Omega], Y) \cong \Omega\text{-}\mathsf{Cat}(Y, [X, \Omega]) \cong \Omega\text{-}\mathsf{Cat}(X, [Y, \Omega])$. □

So the first step in describing the logic for Ω is to find a category $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ equivalent or at least adjoint to $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$. Why adjoint? Because every adjunction can be restricted to an equivalence.

Definition 3.2.2. We call $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ the subcategory of $\Omega\text{-}\mathsf{Cat}$ whose objects are complete and cocomplete Ω-categories and whose arrows are limits and colimits preserving Ω-functors.

Now let us define the two functors $P : \Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ and $\mathsf{AT} : \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$.

3.2.1 The right adjoint $P = [-, \Omega]$

Proposition 3.2.3. *The functor* Ω*-category* $[X, \Omega]$ *is complete and cocomplete and* $[f, \Omega] : [X, \Omega] \longrightarrow [Y, \Omega]$ *preserves all weighted limits and colimits, for all* $X, Y \in \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ *and for all* $f \in \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}(X, Y)$.

Proof. In functor categories weighted limits and colimits are defined pointwise, see [14, Chapter 3.3]. Since Ω is complete and cocomplete then so is $[X, \Omega]$.

One still has to prove that $[f, \Omega] : [X, \Omega] \longrightarrow [Y, \Omega]$ preserves limits. Consider the following limit $\{\phi, G\} \in [X, \Omega]$, given by: $G : K \longrightarrow [X, \Omega]$ and $\phi : K \longrightarrow \Omega$. G gives rise to $G' : K \otimes X \longrightarrow \Omega$ thus $\{\phi, G\}A = \{\phi, G'(-, A)\}$.

Now

$$\begin{aligned} [f, \Omega](\{\phi, G\})(y) &\overset{(1)}{=} \{\phi, G\}(f(y)) \\ &\overset{(2)}{=} \{\phi, G'(-, f(y))\} \\ &\overset{(3)}{=} \{\phi, [f, \Omega] \circ G(-)(y)\} \\ &\overset{(4)}{=} \{\phi, [f, \Omega] \circ G\}(y), \end{aligned}$$

where step (1) is the definition of $[f, \Omega]$, step (2) is the definition of a limit in a functor category, see [14, Chapter 3.3], step (3) is moving back from G' to G and step (4) is again the definition.

The functor $[f, \Omega] : [X, \Omega] \longrightarrow [Y, \Omega]$ preserves colimits. Consider a colimit in $[X, \Omega]$. Let $G : K \longrightarrow [X, \Omega]$ and $\phi : K^{\mathrm{op}} \longrightarrow \Omega$. G gives rise to $G' : K \otimes X \longrightarrow \dot{\Omega}$Then $(\phi \star G)(A) = \phi \star G'(-, A)$.

$$\begin{aligned} [f, \Omega](\phi \star G)(y) &\overset{(1)}{=} (\phi \star G)(f(y)) \\ &\overset{(2)}{=} \phi \star G'(-, f(y)) \\ &\overset{(3)}{=} \phi \star ([f, \Omega] \circ G(-))(y) \\ &\overset{(4)}{=} (\phi \star [f, \Omega] \circ G)(y), \end{aligned}$$

where step (1) is the definition of $[f, \Omega]$, step (2) is the definition of a colimit in a functor category, step (3) is moving back from G' to G and step (4) is again the definition. □

3.2.2 The left adjoint AT

As in the case of complete atomic Boolean algebras the functor AT will take a complete and cocomplete Ω-category A to the full subcategory of atoms.

Definition 3.2.4. An *atom* in a category $\mathcal{C}$ is an object $C \in \mathcal{C}$ such that $\mathcal{C}(C, -)$ preserves all weighted colimits. Then $\mathsf{At}(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ whose objects are atoms.

Example 3.2.5.

1. In posets, atoms are known as completely prime elements. In a completely distributive lattice, being an atom is equivalent to being join-irreducible.
2. The category $[0, \infty]$ seen as a generalized metric space has only one atom 0. Indeed, suppose we have another atom $a \in [0, \infty]$ such that $a \geq 0$, then choose $v, b \in [0, \infty]$ such that $v + b > a$, and $v < v + b - a$, and $b < a$. Then we have $[0, \infty](a, v + b) = v + b - a > v = v + [0, \infty](a, b)$.
3. Let $[X^{\mathrm{op}}, \Omega]$ be a functor category, then using the Yoneda lemma and the definition of a colimit in a functor category, see [14, Chapter 3.3], one has that any representable is an atom. Moreover (see [14, Chapter 5.5]) one has that $[X^{\mathrm{op}}, \Omega] \simeq [\mathsf{At}(X)^{\mathrm{op}}, \Omega]$. In general one has $X \subseteq \mathsf{At}([X^{\mathrm{op}}, \Omega])$.
4. For any category X, rewriting the definition of an atom in terms of distributors and using that $[X^{\mathrm{op}}, \Omega]$ is complete and cocomplete, an element $f \in [X^{\mathrm{op}}, \Omega]$ is an atom if the distributor $[X^{\mathrm{op}}, \Omega](f, -)$ has a right adjoint. This connects atoms and Cauchy completeness [18].

Let $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ be the category defined in Definition 3.2.2. Now for all objects $A \in \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ define

$$\mathsf{AT}(A) = \mathsf{At}(A)^{\mathrm{op}}.$$

In order to define AT on arrows we need some additional results.

Lemma 3.2.6. *For any $H : A \longrightarrow B$ in $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$, there exists a left adjoint $L : B \longrightarrow A$ in $\Omega\text{-}\mathsf{Cat}$.*

Proof. Using the result : " A functor $T : \mathcal{C} \longrightarrow \mathcal{B}$ has a left adjoint if and only if $Ran_T \, \mathrm{id}_{\mathcal{C}}$ exists and is preserved by T. Then the left adjoint S is

$Ran_T\,\mathrm{id}$, and the counit $\epsilon : ST \longrightarrow \mathrm{id}$ of the Kan extension is the counit of the adjunction", found in [14, Chapter 4.8], we have that H has a left adjoint.

Indeed, as A is complete, the right Kan extension $Ran_H 1_A = \{B(b, H-), \mathrm{id}_A\}$ exists, and as H preserves all weighted limits, it also preserves this limit. ☐

Lemma 3.2.7. *For all* $A, B \in \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$*,* $H : A \longrightarrow B$*,* L *the left adjoint of* H *and* $i_A : At(A) \longrightarrow A$ *and* $i_B : At(B) \longrightarrow B$ *the atom inclusion functors, there exists* $f : At(B) \longrightarrow At(A)$ *such that* $L \circ i_B = i_A \circ f$*.*

Proof. Define $f = L \circ id_B$. We have to show that the codomain of $L \circ id_B$ is $At(A)$.

Let $b \in At(B)$. Then $L(i_B(b)) \in At(A)$ means that $A(L(i_B(b)), -)$ preserves all colimits, so let $\varphi \star G$ be a colimit in A.

$$\begin{aligned} A(L(i_B(b)), \varphi \star G) &\cong B(i_B(b), H(\varphi \star G)) \\ &\cong B(i_B(b), \varphi \star HG) \\ &\cong \varphi \star B(i_B(b), HG) \\ &\cong \varphi \star A(L(i_B(b)), G). \end{aligned}$$

☐

If $\mathcal{C}$ and $\mathcal{D}$ are categories then for every functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ we also have a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$ such that $F^{\mathrm{op}}(C) = F(C)$ for all $C \in \mathcal{C}$.

So $\mathsf{AT}(H) = f^{\mathrm{op}}$ as defined in the previous lemma. Also this defines a functor because composition of adjoints is a again an adjoint.

3.2.3 The adjunction $\mathsf{AT} \dashv [-, \Omega]$

Theorem 3.2.8. *For any* $X \in \Omega\text{-}\mathsf{Cat}$ *and* $A \in \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$*, we have the isomorphism of categories* $\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, At(A))^{\mathrm{op}} \cong \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [X, \Omega])$*, furthermore it is natural in both arguments.*

Proof. Let $X \in \Omega\text{-}\mathsf{Cat}$ and $A \in \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$.

We have to define

$$\phi_{XA} : \Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, At(A))^{\mathrm{op}} \longrightarrow \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [X, \Omega])$$

and

$$\phi^{-1}_{XA} : \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [X, \Omega]) \longrightarrow \Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, At(A))^{\mathrm{op}}$$

and after that we have to show that ϕ_{XA} and ϕ^{-1}_{XA} form an isomorphism of categories.

First define ϕ_{XA} on objects. For all $h : X^{\circ} \longrightarrow At(A)$ define

$$\phi_{XA}(h) = A(h-, -) : A \longrightarrow [X, \Omega].$$

We have to show that $\phi_{XA}(h)$ is in $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$, which is equivalent to saying that $A(h-, -)$ preserves all weighted limits and colimits. Preservation of limits is shown in [14, Chapter3.2], and as $h-$ is an atom and atoms preserve colimits (see Definition 3.2.4), $A(h-, -)$ also preserves weighted colimits.

Now define ϕ^{-1}_{XA} on objects. Let $H : A \longrightarrow [X, \Omega]$ and let $L : [X, \Omega] \longrightarrow A$ be its left adjoint, and also let $(d_X)^{\mathrm{op}} : X^{\mathrm{op}} \longrightarrow [X, \Omega]$, $(d_X)^{\mathrm{op}}(x) = X(x, -)$ be the Yoneda embedding.

$$\begin{tikzcd}
A \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & {[X,\Omega]} \arrow[l, bend right, "L"'] \\
\mathsf{At}(A) \arrow[u, "i_A"] & X^{\mathrm{op}} \arrow[l, "h"] \arrow[u, "(d_X)^{\mathrm{op}}"']
\end{tikzcd} \tag{3.4}$$

We have to define $h : X^{\mathrm{op}} \longrightarrow At(A)$ such that $i_A \circ h = L \circ Y$. Then we will take $\phi^{-1}_{XA}(H) = h$.

Define $h = L \circ Y$. Now we have to show that $(L \circ Y)(x) \in At(A)$ for all $x \in X$. So let $x \in X$. We have to show that $A(L(X(x, -), -)$ preserves all colimits.

Thus let $\varphi \star G$ be a colimit in A. Then

$$\begin{aligned} A(L(X(x,-),\varphi \star G) &\overset{(1)}{\cong} [X,\Omega](X(x,-),H(\varphi \star G)) \\ &\overset{(2)}{\cong} [X,\Omega](X(x,-),\varphi \star HG) \\ &\overset{(3)}{\cong} (\varphi \star HG)(x) \\ &\overset{(4)}{=} \varphi \star HG(-)(x) \\ &\overset{(5)}{\cong} \varphi \star [X,\Omega](X(x,-),HG) \\ &\overset{(6)}{\cong} \varphi \star A((L \circ Y)(x),G), \end{aligned}$$

where step (1) is due to $L \dashv H$ adjunction, step (2) is due to H preserving weighted colimits, step (3) is due to Yoneda lemma, step (4) is due to the definition of a weighted colimit in functor category, step (5) is again due to Yoneda lemma and step (6) is again due to $L \dashv H$ adjunction.

Now we will show that ϕ_{XA} and ϕ_{XA}^{-1} are inverse functions on objects.

First we have to show $(\phi_{XA} \circ \phi_{XA}^{-1})(H) = H$ and as $\phi_{XA}(\phi_{XA}^{-1}(H)) = A(h-,-)$ we just have to prove $A(h-,-) = H$.

$$\begin{aligned} A(h(x),a) &\overset{(1)}{=} A(i_A(h(x)),a) \\ &\overset{(2)}{=} A(L(X(x,-),a) \\ &\overset{(3)}{=} [X,\Omega](X(x,-),H(a)) \\ &\overset{(4)}{\cong} H(a)(x), \end{aligned}$$

where step (1) follows from $i_A(a') = a'$ for all $a' \in A$, step (2) is due to the commutativity of diagram (3.4), step (3) is due to the $L \dashv H$ adjunction and final step (4) is due to Yoneda lemma.

Now we have to show $(\phi_{XA}^{-1} \circ \phi_{XA})(h) = h$, and with L the left adjoint of $A(h-,-)$ we just have to show $L \circ Y = h$. Using the fact that $L \dashv A(h-,-)$ and again Yoneda lemma, we get the following isomorphism:

$$A(L(X(x,-)),a) \cong [X,\Omega](X(x,-),A(h-,a))) \cong A(h(x),a).$$

As the above isomorphism is natural in its second argument we can apply Yoneda lemma to $[A,\Omega](A(L(X(x,-)),-),A(h(x),-)$ and get

$$[A,\Omega](A(L(X(x,-)),-),A(h(x),-) \cong A(h(x),L(X(x,-))).$$

As on the left hand side we only have isomorphisms, on the right hand side we also have only isomorphisms. Also as we are working in quantale enriched categories we have that $L \circ Y = h$.

Until now we have defined ϕ_{XA} and ϕ_{XA}^{-1} on objects, and in order to have functors we one still has to define them on arrows, and for that one could use the concept of conjugate natural transformation, defined in [20, Chapter 4.7] .

Definition 3.2.9. Given two adjunctions,

$$(L_1, R_1, \varphi_1, \eta_1, \epsilon_1), \text{ and } (L_2, R_2, \varphi_2, \eta_2, \epsilon_2) : X \longrightarrow A$$

between the same two categories, two natural transformations

$$\alpha : L_1 \Rightarrow L_2, \qquad \beta : R_2 \Rightarrow R_1$$

are said to be *conjugate* (for the given adjunctions) when the diagram

$$\begin{tikzcd}
A(L_2x, a) \arrow[r, "\cong", "\varphi_2"'] \arrow[d, "(\alpha_x)^* = A(\alpha_x, a)"'] & X(x, R_2a) \arrow[d, "(\beta_a)_* = X(x, \beta_a)"] \\
A(L_1x, a) \arrow[r, "\cong", "\varphi_1"'] & X(x, R_1a)
\end{tikzcd}$$

commutes for every pair of objects $x \in X, a \in A$.

Why should one use conjugate natural transformations here? Because given $\alpha : L_1 \Rightarrow L_2$ there exists a unique $\beta : R_2 \Rightarrow R_1$ such that the pair (α, β) is conjugate, and dually given β we have a unique α (see [20, Chapter 4.7, Theorem 2]).

As we use the adjunction from (3.4) to define ϕ_{XA}^{-1} on objects we will use conjugate natural transformation to define it on arrows. So let $H_1, H_2 : A \longrightarrow [X, \Omega]$ and $\beta : H_1 \Rightarrow H_2$, and let $L_1 \dashv H_1$, and $L_2 \dashv H_2$. Then we have a unique $\alpha : L_2 \Rightarrow L_1$ which is conjugate to β. So $\phi_{XA}^{-1}(\beta) = \alpha \circ Y$.

Now for ϕ_{XA} let $h_1, h_2 : X^\circ \longrightarrow At(A)$ and $\alpha : h_1 \Rightarrow h_2$, and let $L_1 \dashv A(h_1-, -)$ and $L_2 \dashv A(h_2-, -)$. As we proved above, $L_1 \circ Y = h_1$, and $L_2 \circ Y = h_2$ thus $i_A \circ \alpha : L_1 \circ Y \Rightarrow L_2 \circ Y$ is a natural transformation. We want to find $\bar{\alpha} : L_1 \Rightarrow L_2$

and, as $(i_A \circ \alpha)_x : I \longrightarrow A(L_1(X(x,-)), L_2(X(x,-)))$, let $\bar{\alpha}_{X(x,-)} = (i_A \circ \alpha)_x$. We still have to define it on every $\varphi \in [X, \Omega]$. Now

$$\begin{aligned}
A(L_1\varphi(x), L_2\varphi(x)) &\overset{(1)}{\cong} A(L_1(\varphi \star Y)(x), L_2(\varphi \star Y)(x)) \\
&\overset{(2)}{\cong} A(\varphi \star L_1 Y(x), \varphi \star L_2 Y(x)) \\
&\overset{(3)}{\cong} \{\varphi, A(L_1 Y(x), \varphi \star L_2 Y(x))\} \\
&\overset{(4)}{\cong} \{\varphi, \varphi \star A(L_1 Y(x), L_2 Y(x))\},
\end{aligned}$$

where step (1) is due to $\varphi \cong \varphi \star Y$ see [14, Chapter 3.3], step (2) is due to preservation of colimits by left adjoints, step (3) follows from the fact that the hom functor changes colimits on first position into limits outside and step (4) follows from the fact that atoms preserves colimits and LY is an atom. As we require $\Omega(I,-) : \Omega \longrightarrow \mathsf{Set}$ to be faithful, a Ω-natural transformation is equivalent to a "normal" natural transformation, so now we can define $\bar{\alpha}$ as a "normal" natural transformation and thus as we have defined a natural transformation from L_2 to L_1, we have completely defined a Ω-natural transformation from L_2 to L_1. Let β be the conjugate natural transformation for $\bar{\alpha}$, then $\phi_{XA}(\alpha) = \beta$. Now obviously ϕ_{XA} and ϕ_{XA}^{-1} are also inverse one to another on arrows.

We still have to show that ϕ_{XA} is natural in first argument and pseudonatural in the second one.

The naturality in the first argument is equivalent to the commutativity of diagram (3.5).

Given $g : X^{\mathrm{op}} \longrightarrow Y^{\mathrm{op}}$ in Ω-Cat, there exists a unique $g^{\mathrm{op}} : X \longrightarrow Y$ in Ω-Cat such that $g^{\mathrm{op}}(x) = g(x)$.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}(Y^{\mathrm{op}}, At(A))^{\mathrm{op}} \arrow[r, "\phi_{YA}"] \arrow[d, "\Omega\text{-}\mathsf{Cat}(g, At(A))"'] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [Y, \Omega]) \arrow[d, "\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [g^{\mathrm{op}}, \Omega])"'] \\
\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, At(A))^{\mathrm{op}} \arrow[r, "\phi_{XA}"'] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [X, \Omega])
\end{tikzcd} \tag{3.5}$$

Let $h : Y^{\mathrm{op}} \longrightarrow At(A)$ then $h \circ g : X^{\mathrm{op}} \longrightarrow At(A)$, so we get

$$(\phi_{YA} \circ \Omega\text{-}\mathsf{Cat}(g, At(A)))(h) = A(h \circ g-, -)$$

and also

$$\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A,[g^{\mathrm{op}},\Omega])\circ\phi_{XA}(h)=A(h-,-)\circ g=A(h\circ g-,-).$$

Thus the diagram is commutative.

The naturality in second argument is equivalent to the commutativity up to isomorphism of the diagram (3.6).

Let $f:A\longrightarrow C$ in $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ and $\bar{f}$ its left adjoint, thus according to 3.2.7 we have $\bar{f}\circ i_C:At(C)\longrightarrow At(A)$.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}},At(C))^{\mathrm{op}} \arrow[r, "\phi_X C"] \arrow[d, "\Omega\text{-}\mathsf{Cat}(X{,}\bar{f}\circ i_C)"'] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(C,[X,\Omega]) \arrow[d, "\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(f{,}[X{,}\Omega])"] \\
\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}},At(A))^{\mathrm{op}} \arrow[r, "\phi_{XA}"'] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A,[X,\Omega])
\end{tikzcd} \tag{3.6}$$

Let $h:X^{\mathrm{op}}\longrightarrow At(C)$. Then on the bottom side we get

$$A(\bar{f}\circ i_C\circ h-,-)$$

and on the top side

$$C(h-,f-),$$

which are isomorphic under the adjunction of $\bar{f}\dashv f$ and i_C being just an inclusion functor. And as we are in a quantale enriched category, isomorphism of functors is equality. ☐

Now, as a corollary, we obtain the following adjunction.

Theorem 3.2.10. *The functor* $\mathsf{AT}:\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}\longrightarrow\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ *is a left adjoint of* $P:\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$.

Proof. Let $X\in\Omega\text{-}\mathsf{Cat}$ and $A\in\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$. We have to show that $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}(\mathsf{AT}A,X)\cong\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A,PX)$ which is equivalent to $\Omega\text{-}\mathsf{Cat}(X,\mathsf{At}(A)^{\mathrm{op}})\cong\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A,[X,\Omega])$, and as $\Omega\text{-}\mathsf{Cat}(X,\mathsf{At}(A)^{\mathrm{op}})\cong\Omega\text{-}\mathsf{Cat}(X^{\circ},\mathsf{At}(A))^{\mathrm{op}}$ see [14, 2.28], we just have to prove that there is a natural isomorphism in both arguments, between the next categories

$$\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}},\mathsf{At}(A))^{\mathrm{op}}\cong\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A,[X,\Omega]).$$

☐

3.3 Reducing $\mathsf{AT} \dashv P$ to a duality

We want to reduce the adjunction $\mathsf{AT} \dashv P$ to an equivalence, by restricting the objects in $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ and $\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$. And we also want the reduction to be correct, in the sense that the subcategory of $\Omega\text{-}\mathsf{Cat}$ is in the image of $[-,\Omega]$.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend right, "P"'] \arrow[r, phantom, "\perp"] & \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat} \arrow[l, bend right, "\mathsf{AT}"'] \\
\Omega\text{-}\mathsf{Cat_{cc}}^{\mathrm{op}} \arrow[u, "i_{\Omega\text{-}\mathsf{Cat_{cc}}}"'] \arrow[r, bend right, "\bar{P}"'] \arrow[r, phantom, "\simeq"] & \mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat} \arrow[l, bend right, "\bar{\mathsf{AT}}"'] \arrow[u, "i_{\mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat}}"']
\end{tikzcd} \tag{3.7}$$

Following [14, Chapter 5] we define:

Definition 3.3.1. We say that a category $X \in \Omega\text{-}\mathsf{Cat}$ is *Cauchy Complete* if $X \cong At([X,\Omega])^{\mathrm{op}}$. Furthermore we call $\Omega\text{-}\mathsf{Cat_{cc}}$ the full subcategory of $\Omega\text{-}\mathsf{Cat}$ of those Cauchy complete Ω-categories.

Remark 3.3.2.

1. Let $\Omega = [0,\infty]$ and let Q and R be the rational and real numbers, respectively, with the usual Euclidean metric. Then the map in $H : [Q,\Omega] \longrightarrow [R,\Omega]$ given by $H(f)(r) = \lim_n f(q_n)$ where (q_n) is a Cauchy sequence with limit r, is in $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ and cannot be restricted to a map $\mathsf{At}(H) : R \longrightarrow Q$. So Cauchy completeness is necessary.

2. Any poset is Cauchy complete, see [26].

3. In a functor category $[X^{\mathrm{op}},\Omega]$, an element $f \in [X,\Omega]$ is an atom if the distributor $f^* : 1 \longrightarrow [X,\Omega]$ given by $[X,\Omega](f,-)$ is a left adjoint. Thus one has that the category of atoms $\mathsf{At}([X^{\mathrm{op}},\Omega])$ is the Cauchy closure of X.

4. As shown in [18], a generalised metric space X is isomorphic to $\mathsf{At}([X^{\mathrm{op}},\Omega])$ if it is Cauchy complete in the usual sense of metric spaces.

5. For any category X we have that $[X^{\mathrm{op}},\Omega] \simeq [(\mathsf{At}([X^{\mathrm{op}},\Omega]^{\mathrm{op}},\Omega]$, thus K is full if and only if $\mathsf{At}([X^{\mathrm{op}},\Omega]) \cong X$

Cauchy completeness has been studied intensely in the last decades, so I apologise for any reference I have forgotten. For a survey on Cauchy completeness see [5] and for more on it see [30], [37], [13] and [26].

Definition 3.3.3. We call a Ω-category X atomic if the atom's inclusion functor, $\imath : \mathsf{At}(X) \longrightarrow X$, is dense. The full subcategory of c-Ω-Cat whose objects are atomic complete and cocomplete Ω-categories is called ac-Ω-Cat.

Now let us show that, with these restrictions, the functors AT and P are correctly defined and that they indeed form an equivalence.

First step. Let us show that for every Ω-category X the functor category $[X, \Omega]$ is atomic. But that follows from the fact that the Yoneda embedding is a dense functor, see Example 2.5.3, and that the representables are atoms.

The rest follows from the next two results.

Lemma 3.3.4. *If A is cocomplete and the atom-inclusion functor i_A : $\mathsf{At}(A) \longrightarrow A$ is dense then $A \cong [\mathsf{At}(A)^{\mathrm{op}}, \Omega]$.*

Proof. Let $A \in \mathsf{c}$-Ω-Cat be such that $i : \mathsf{At}(A) \longrightarrow A$ is dense. According to Proposition 2.5.2 if i is dense then $\tilde{i} : A \longrightarrow [\mathsf{At}(A)^{\mathrm{op}}, \Omega]$, defined by $\tilde{i}a = A(i-, a)$, is fully faithful. So we just have to show that it is essentially surjective. Let $H : \mathsf{At}(A)^{\mathrm{op}} \longrightarrow \Omega$. As A is cocomplete, $H \star i$ exists. Then

$$\tilde{i}(H \star i) \cong H \star \tilde{i}i \cong H \star d_{\mathsf{At}(A)} \cong H.$$

Thus $\tilde{i}$ is essentially surjective and so $A \cong [\mathsf{At}(A)^{\mathrm{op}}, \Omega]$. □

Thus we have $\mathsf{AT}(A) \cong \mathsf{AT}P\mathsf{AT}(A)$ for all $A \in A$.

Thus $\bar{P} = P|_{\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}}$ and $\bar{\mathsf{AT}} = \mathsf{AT}|_{\mathsf{ac}\text{-}\Omega\text{-}\mathsf{Cat}}$ are correctly defined, and they form an equivalence.

As, for any Ω-category X we have that $[X, \Omega]$ is complete, thus Cauchy complete, the image of $U = [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is in $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{\mathrm{op}}$. Thus the adjunction $D = [-, \Omega] \dashv U = [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ can be restricted to $D = [-, \Omega] \dashv U = [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{\mathrm{op}}$. Thus the above restriction is useful and ac-Ω-Cat is still a category of algebras.

3.4 Applications and conclusions

First let us show that our results instantiate to known results in the case $\Omega = \mathbb{2}$.

First of all, following [26] any poset is Cauchy complete, so $\mathbb{2}-\mathsf{Cat_{cc}}^{\text{op}} = \mathbb{2}$-$\text{Cat}^{\text{op}}$. The objects of ac-Ω-Cat are complete and cocomplete atomic posets. We have showed (in item 3 of Example 2.3.2) that, for posets, weighted limits and colimits are meets, and respectively joins. For a poset P an atom $a \in P$ is any element such that

$$a \leq (p \vee q) \Rightarrow a \leq p \text{ or } a \leq q.$$

Thus the objects of ac-Ω-Cat are atomic complete lattices. Now, if every element in an atomic lattice is a join of atoms then the lattice is completely distributive. For finite lattices we obtain Birkhoff's theorem of representation of finite lattices. Which states "Any finite distributive lattice L is isomorphic to the lattice of lower sets of the partial order of the join-irreducible elements of L", see [10].

For $\Omega = [0, \infty]$ we obtain a duality between cauchy complete generalised metric spaces and atomic complete and cocomplete generalised metric spaces. Unlike the situation for posets, not all generalised metric spaces are Cauchy Complete, see [18]. The objects of ac-Ω-Cat are atomic complete and cocomplete generalised metric spaces, but unlike the case of posets one still needs the weights, thus the algebras for $[0, \infty]$ are some sort of generalised lattices, where the operations are all weighted limits and weighted colimits. So the form of our algebras complicates drastically and it will be presented in chapter 5.

So in this chapter we have showed that Ω-Cat^{op} is an adjunct category to the category c-Ω-Cat, whose objects are complete and cocomplete Ω-categories and arrows are continuous and cocontinous Ω-functors, and that, if we restrict to Cauchy complete Ω-categories on one side, and to atomic complete and cocomplete Ω-categories on the other side, we have an equivalence. Still our work is not done, we still have to check whether this adjunction is monadic, and express the algebras for $[[-, \Omega], \Omega]$ with operations and equations.

This will be done in the next chapters.

Chapter 4

Monads and algebras

As we said at the end of the previous chapter, obtaining a description with operations of equations for the $DU = [[-,\Omega],\Omega]$-algebras is still our main interest. One could see that these algebras are have limits and colimits as operations, but obtaining a "syntactic" equation for distributivity is not easy. Thus we decided to apply "divide et impera" in the sense that we will define two monads, and show that their composite is $[[-,\Omega],\Omega]$. This technique is inspired by [23], [32],[33].

4.1 Monads

The aim of this section is to describe two monads $\mathcal{D},\mathcal{U} : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}$ such that $DU = \mathcal{D}\mathcal{U}$, where $U = [-,\Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}{\mathsf{Cat}_{\mathsf{cc}}}^{\mathrm{op}}$ and $D = [-,\Omega] : \Omega\text{-}{\mathsf{Cat}_{\mathsf{cc}}}^{\mathrm{op}} \longrightarrow \Omega\text{-}\mathsf{Cat}$ are the two adjoints we defined in 3.2. Furthermore, $(\mathcal{D},d,\mu)$ will be a KZ-doctrine, and $(\mathcal{U},u,\nu)$ will be a co-KZ-doctrine, which in turn will help us to describe the distributive law relating them. This section is inspired from [23] and [32].

Recall that for any category X, one has two Yoneda embeddings $dX : X \longrightarrow [X^{\mathrm{op}},\Omega]$ given by $x \mapsto X(-,x)$ and $uX : X \longrightarrow [X,\Omega]^{\mathrm{op}}$ given by $x \mapsto X(x,-)$.

On objects, $\mathcal{D}$ maps X to $[X^{\mathrm{op}},\Omega]$ and on arrows it constructs the left Kan extension along Yoneda, while $\mathcal{U}$ maps an object X to $[X,\Omega]^{\mathrm{op}}$ and on an arrow to the right Kan extension along Yoneda. Thus for any $f : X \longrightarrow Y$

in Ω-Cat, let $\mathcal{D}f$ be defined as $\mathrm{Lan}_{dX} dY \circ f = \mathrm{Lan}_{dX} Y(-, f)$, and let $\mathcal{U}f$ be defined as $\mathrm{Ran}_{dX} uY \circ f = \mathrm{Ran}_{dX} Y(f, -)$ as in:

$$\begin{tikzcd}
\mathcal{D}X \arrow[r, "\mathcal{D}f=", "Lan_{d_X}(d_Y \circ f)"'] & \mathcal{D}Y & & \mathcal{U}X \arrow[r, "\mathcal{U}f=", "Ran_{u_X}(u_Y \circ f)"'] & \mathcal{U}Y \\
X \arrow[u, "d_x"] \arrow[r, "f"'] & Y \arrow[u, "d_Y"'] & & X \arrow[u, "u_X"] \arrow[r, "f"'] & Y \arrow[u, "u_Y"']
\end{tikzcd} \tag{4.1}$$

Writing down the formula for left and right Kan extensions, see [14, Chapter 4.2], we obtain for $\varphi : X^{\mathrm{op}} \longrightarrow V$ and $\psi : X \longrightarrow \Omega$

$$\begin{aligned}
\mathcal{D}f(\varphi) &= Lan_{dX}(dY \circ f)(\varphi) \\
&= \bigvee_{x \in X} [X^{\mathrm{op}}, \Omega](X(-, x), \varphi) \otimes Y(-, f(x)) \\
&= \bigvee_{x \in X} \varphi(x) \otimes Y(-, f(x)) = \varphi \star (d_Y \circ f),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{U}f(\psi) &= Ran_{uX}(uY \circ f)(\psi) \\
&= \bigwedge_{x \in X} \mathcal{U}X(\psi, X(x, -)) \pitchfork Y(f(x), -) \\
&= \bigwedge_{x \in X} \psi(x) \pitchfork Y(f(x), -).
\end{aligned}$$

But considering that we calculate this end in $[Y, \Omega]^{\mathrm{op}}$, in $[Y, \Omega]$ it becomes

$$\mathcal{U}f(\psi) = \bigvee_{x \in X} \psi(x) \otimes Y(f(x), -) = \psi \star (uY \circ f)$$

From the universal property of Kan extensions one obtains

Proposition 4.1.1. *There exist natural transformations* $\lambda : \mathcal{D}d \longrightarrow d\mathcal{D}$ *and* $\delta : u\mathcal{U} \longrightarrow \mathcal{U}u$ *.*

Proof. The map $\mathcal{D}d$ is a left Kan extension of $d\mathcal{D} \circ d$ along d and as we have $d\mathcal{D} \circ d = d\mathcal{D} \circ d$, then from the universality property of the left Kan extension one has a unique natural transformation $\lambda : \mathcal{D}d \longrightarrow d\mathcal{D}$.

$$\begin{tikzcd}
\mathcal{D}X \arrow[r, bend left, "d\mathcal{D}X"] \arrow[r, bend right, "\mathcal{D}dX"'] \arrow[r, phantom, "\lambda\uparrow"] & \mathcal{D}X \\
X \arrow[u, "dX"] \arrow[r, "dX"'] & \mathcal{D}X \arrow[u, "d\mathcal{D}X"']
\end{tikzcd} \tag{4.2}$$

Same argument holds for δ but for right Kan extensions. $\square$

We want $\mathcal{D}$ to be a KZ-doctrine, so the multiplication $\mu : \mathcal{D}\mathcal{D} \longrightarrow \mathcal{D}$ has to be a left adjoint of $d\mathcal{D}$. As $d\mathcal{D}$ preserves all limits and the right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along $d\mathcal{D}$ exists, using [14, Theorem 4.81], we know that the left adjoint of $d\mathcal{D}$ exists and is expressed by $Ran_{d\mathcal{D}}\,\mathrm{id}_{\mathcal{D}}$. Dually, the right adjoint of $u\mathcal{U}$ exists and is expressed by $Lan_{u\mathcal{U}}\,\mathrm{id}_{\mathcal{U}}$.

$$\begin{tikzcd}
\mathcal{D}\mathcal{D} \arrow[r, "\mu", "Ran_{d\mathcal{D}}\,\mathrm{id}"'] & \mathcal{D} \\
\mathcal{D} \arrow[u, "d\mathcal{D}"] \arrow[ur, "\mathrm{id}_{\mathcal{D}}"'] &
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{U}\mathcal{U} \arrow[r, "\nu=", "Lan_{uU}\,\mathrm{id}_{\mathcal{U}}"'] & \mathcal{U} \\
\mathcal{U} \arrow[u, "u\mathcal{U}"] \arrow[ur, "\mathrm{id}_{\mathcal{U}}"'] &
\end{tikzcd} \tag{4.3}$$

$$\mu G = \bigwedge_{\varphi \in \mathcal{D}X} \mathcal{D}\mathcal{D}X(G, \mathcal{D}X(-,\varphi)) \pitchfork \varphi \qquad \nu F = \bigwedge_{\psi \in \mathcal{U}X} [\mathcal{U}X, \Omega](G, u\mathcal{U}(\psi)) \pitchfork \psi \tag{4.4}$$

Furthermore as $d\mathcal{D}$ and $u\mathcal{U}$ are full and faithful, thus one has $\mu \circ d\mathcal{D} = id_{\mathcal{D}}$, and ,respectively $\nu \circ u\mathcal{U} = id_{\mathcal{U}}$. Following Proposition 2.6.3 to show that $\mathcal{D}$ is a KZ-doctrine we just have to prove that $\mu \circ \mathcal{D}d = \mathrm{id}_{\mathcal{D}}$ as well. To prove that we know that μ is a left adjoint so it preserves left Kan extensions: so

$$\begin{aligned}
\mu X \circ \mathcal{D}dX &= \mu X \circ Lan_{dX}(d\mathcal{D}X \circ dX) \\
&= Lan_{dX}(\mu X \circ \mathcal{D}dX \circ dX) \\
&= Lan_{dX}(\mathrm{id}_{\mathcal{D}X} \circ d) \\
&= Lan_{dX} dX \\
&= \mathrm{id}_{\mathcal{D}X}.
\end{aligned}$$

$$\begin{tikzcd}
\mathcal{D}X \arrow[r, "\mathcal{D}d_X"] & \mathcal{D}\mathcal{D}X \arrow[r, "\mu"] & \mathcal{D}X \\
X \arrow[u, "d_X"'] \arrow[r, "d_X"'] & \mathcal{D}X \arrow[u, "d_{\mathcal{D}X}"] \arrow[ur, "1"'] &
\end{tikzcd} \tag{4.5}$$

In a similar way one shows that $\nu \circ \mathcal{U}u = \mathrm{id}_{\mathcal{U}}$, so $\mathcal{U}$ is a co-KZ-doctrine. Thus we have provn the following:

Proposition 4.1.2. *$(\mathcal{D}, d, \mu)$ is a KZ-doctrine and $(\mathcal{U}, u, \nu)$ is a co-KZ doctrine.*

4.2 Distributive laws

In the previous section we have constructed two monads, but for their composite to be a monad one needs a distributive law between them.
Verifying that a natural transformation is indeed a distributive law may not be easy, but thanks to [23], for KZ-doctrines, we just have to check the conditions of Proposition 2.6.6. To construct $\mathcal{D}$ and $\mathcal{U}$, we have used Kan extensions, thus it make sense that a distributive law between them is a Kan extension as well. Looking at the left hand side of Diagram 2.19, and as both $u\mathcal{D}$ and $\mathcal{U}d$ are full and faithful, a Kan extension along any of them would make that triangle commute, so intuitively, it should make no difference from which triangle one starts. So if one calculates all four Kan extensions one obtains that:

1. $r^r_{\mathcal{D}} = Ran_{u\mathcal{D}}\,\mathcal{D}u = \mathcal{U}\mathcal{D}(\mathcal{D}u, -)$
2. $r^l_{\mathcal{U}} = Lan_{\mathcal{U}d}\, d\mathcal{U} = \mathcal{U}\mathcal{D}(\mathcal{U}d, -)$
3. $r^l_{\mathcal{D}} = Lan_{u\mathcal{D}}\,\mathcal{D}u = \mathcal{U}\mathcal{D}(d\mathcal{D} \circ d, -) \star d\mathcal{U} \circ u$
4. $r^r_{\mathcal{U}} = Ran_{\mathcal{U}d} d\mathcal{U} = \{\mathcal{U}\mathcal{D}(-, \mathcal{U}d), d\mathcal{U}\}$

Now as for any X and any $\varphi \in \mathcal{D}X$ and any $\psi \in \mathcal{U}X$ one has $\mathcal{D}uX(\varphi)(\psi) = \mathcal{U}dX(\varphi)(\psi)$ it follows $Ran_{u\mathcal{D}}\,\mathcal{D}u = Lan_{\mathcal{U}d}\, d\mathcal{U}$.

Proposition 4.2.1. *The natural transformation $r = Ran_{u\mathcal{D}}\,\mathcal{D}u = Lan_{\mathcal{U}d}\, d\mathcal{U} : \mathcal{U}\mathcal{D} \longrightarrow \mathcal{D}\mathcal{U}$ is a distributive law between $\mathcal{D}$ and $\mathcal{U}$.*

Proof. First let us calculate all of the Kan extensions we defined above, and then show that two equal ones satisfy the conditions of Proposition 2.6.6.

$$\begin{aligned}
r^l_D G &= (Lan_{u\mathcal{D}X}\mathcal{D}uX)G \\
&= (Lan_{u\mathcal{D}X}(Lan_d X(d\mathcal{U}X \circ uX)))G \\
&= Lan_{u\mathcal{D}X\circ dX}(d\mathcal{U}X \circ uX)G \\
&= \bigvee_{x\in X} \mathcal{U}\mathcal{D}X(u\mathcal{D}X \circ dX(x), G) \otimes d\mathcal{U}X \circ uX(x) \\
&= \bigvee_{x\in X} [\mathcal{D}X, \Omega]^{\mathrm{op}}(\mathcal{D}X(X(-,x),-), G) \otimes \mathcal{U}X(-, X(x,-)) \\
&= \bigvee_{x\in X} [\mathcal{D}X, \Omega](G, \mathcal{D}X(X(-,x),-)) \otimes [X, \Omega](X(x,-),-)
\end{aligned}$$

$$\begin{aligned}
r^r_D G &= Ran_{u\mathcal{D}}\mathcal{D}uG = \\
&= \bigwedge_{\varphi\in\mathcal{D}X} \mathcal{U}\mathcal{D}X(G, u\mathcal{D}X\varphi) \pitchfork \mathcal{D}uX\varphi \\
&= \bigwedge_{\varphi\in\mathcal{D}X} [\mathcal{D}X, \Omega](\mathcal{D}X(\varphi,-), G) \pitchfork \mathcal{D}uX\varphi \\
&= \bigwedge_{\varphi\in\mathcal{D}X} G\varphi \pitchfork \mathcal{D}uX\varphi \\
&= [\mathcal{D}X, \Omega](G, \mathcal{D}uX) \\
&= \mathcal{U}\mathcal{D}X(\mathcal{D}uX, G) \\
r^l_U G &= (Lan_{\mathcal{U}dX} d\mathcal{U}X)G \\
&= \bigvee_{\psi\in\mathcal{U}X} \mathcal{U}\mathcal{D}X(\mathcal{U}dX\psi, G) \otimes \mathcal{U}X(-,\psi) \\
&= \mathcal{U}\mathcal{D}X(\mathcal{U}dX, G) \\
r^r_U G &= (Ran_{\mathcal{U}dX} d\mathcal{U}X)G \\
&= \bigwedge_{\psi\in\mathcal{U}X} \mathcal{U}\mathcal{D}X(G, \mathcal{U}dX\psi) \pitchfork \mathcal{U}X(-,\psi) \\
&= \bigwedge_{\psi\in\mathcal{U}X} [\mathcal{D}X, \Omega](\mathcal{U}dX\psi, G) \pitchfork [X, \Omega](\psi, -)
\end{aligned}$$

$\mathcal{D}uX : \mathcal{D}X \longrightarrow \mathcal{D}\mathcal{U}X$, let $\varphi \in \mathcal{D}X$

$$\begin{tikzcd}
\varphi \ni \mathcal{D}X \arrow[r, "\mathcal{D}uX"] & \mathcal{D}\mathcal{U}X \\
X \arrow[u, "dX"] \arrow[r, "uX"'] & \mathcal{U}X \arrow[u, "d\mathcal{U}X"']
\end{tikzcd}$$

$$\begin{aligned}\mathcal{D}uX\varphi &= \bigvee_{x\in X} \mathcal{D}X(X(-,x),\varphi) \otimes d\mathcal{U}X \circ uX(x)\\ &= \bigvee_{x\in X} \varphi(x) \otimes [X,\Omega](X(x,-),-)\end{aligned}$$

$\mathcal{U}dX : \mathcal{U}X \longrightarrow \mathcal{U}\mathcal{D}X$, let $\psi \in \mathcal{U}X$

$$\begin{tikzcd}\psi \ni \mathcal{U}X \arrow[r, "\mathcal{U}dX"] & \mathcal{U}\mathcal{D}X \\ X \arrow[u, "uX"] \arrow[r, "dX"'] & \mathcal{D}X \arrow[u, "u\mathcal{D}X"']\end{tikzcd}$$

$$\begin{aligned}\mathcal{U}dX\psi &= \bigwedge_{x\in X} \mathcal{U}X(\psi, X(x,-)) \pitchfork u\mathcal{D}X \circ dX(x)\\ &= \bigwedge_{x\in X} \psi(x) \pitchfork \mathcal{D}X(X(-,x),-)\\ &\overset{\text{op}}{=} \bigvee_{x\in X} \psi(x) \otimes \mathcal{D}X(X(-,x),-)\end{aligned}$$

Thus the two Kan extensions are equal. As $u\mathcal{D}$ and $\mathcal{U}d$ are full and faithful we obtain the commutativity of the two triangles, so we only need to show the inequalities from Proposition 2.6.6.
Let $\varphi \in \mathcal{D}X$, then we have

$$\begin{aligned}r_U^l \circ u\mathcal{D}X(\varphi) &= r_U^l(\mathcal{D}X(\varphi,-)\\ &= \mathcal{U}\mathcal{D}X(\mathcal{U}dX, \mathcal{D}X(\varphi,-))\\ &= [\mathcal{D}X,\Omega](\mathcal{D}X(\varphi,-), \mathcal{U}dX)\\ &= \mathcal{U}dX(\varphi)\\ &= \bigvee_{x\in X} \varphi(x) \otimes \mathcal{D}X(X(-,x),-)\\ &= \mathcal{D}uX(\varphi).\end{aligned}$$

Let ψ, ψ' be objects of $\mathcal{U}X$, then we have to show that

$$r_D^r \circ \mathcal{U}dx(\psi)(\psi') \le d\mathcal{U}X(\psi)(\psi').$$

$$\begin{aligned}
r^r_D \circ \mathcal{U}dX(\psi) &= \mathcal{U}\mathcal{D}X(\mathcal{D}uX, \mathcal{U}dX(\psi)) \\
&= [\mathcal{D}X, \Omega](\mathcal{U}dX(\psi), \mathcal{D}uX) \\
&= [\mathcal{D}X, \Omega](\bigvee_{x \in X} \psi(x) \otimes \mathcal{D}X(X(-,x),-), \mathcal{D}uX) \\
&= \bigwedge_{x \in X} \psi(x) \pitchfork [\mathcal{D}X, \Omega](\mathcal{D}X(X(-,x),-), \mathcal{D}uX) \\
&= \bigwedge_{x \in X} \psi(x) \pitchfork \mathcal{D}uX(X(-,x)) \\
&= [X, \Omega](\psi, \mathcal{D}uX \circ dX) \\
&= [X, \Omega](\psi, d\mathcal{U}X \circ uX) \\
&= \mathcal{U}X(d\mathcal{U}X \circ uX, \psi),
\end{aligned}$$

thus applied to ψ' we have

$$\begin{aligned}
r^r_D \circ \mathcal{U}dX(\psi)(\psi') &= \mathcal{U}X(d\mathcal{U}X \circ uX(\psi'), \psi) \\
&= [X, \Omega](\psi, \mathcal{U}X(\psi', uX)) \\
&= [X, \Omega](\psi, [X, \Omega](uX, \psi')) \\
&= [X, \Omega](\psi, \psi') \\
&= \mathcal{U}X(\psi', \psi) \\
&= d\mathcal{U}x(\psi)(\psi').
\end{aligned}$$

So r is indeed a distributive law. □

In a similar way one has a distributive law

$$l = Ran_{\mathcal{D}u}\, u\mathcal{D} = Lan_{d\mathcal{U}}\, \mathcal{U}d : \mathcal{D}\mathcal{U} \longrightarrow \mathcal{U}\mathcal{D},$$

given by $l = \mathcal{D}\mathcal{U}(-, \mathcal{D}u)$.

Proposition 4.2.2. *The natural transformation l defined above is a left adjoint to r.*

Proof. Let X be any Ω-category, and let G be any object in $\mathcal{U}\mathcal{D}X$ and F any object in $\mathcal{D}\mathcal{U}X$ then we have to show

$$[\mathcal{D}X, V](G, \mathcal{D}\mathcal{U}X(F, \mathcal{D}uX,)) \cong [\mathcal{U}X^{\mathrm{op}}, \Omega](F, \mathcal{U}\mathcal{D}X(\mathcal{U}dX, G)) \tag{4.6}$$

We have:

$$
\begin{aligned}
[\mathcal{D}X,\Omega](G,\mathcal{D}\mathcal{U}X(F,\mathcal{D}ux) &= \bigwedge_{\varphi} G(\varphi) \pitchfork \mathcal{D}\mathcal{U}X(F,\mathcal{D}uX(\varphi)) \\
&= \bigwedge_{\varphi} G(\varphi) \pitchfork \bigwedge_{\psi} F(\psi) \pitchfork \mathcal{D}uX(\varphi)(\psi),
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{U}X^{\mathrm{op}},\Omega](F,\mathcal{U}\mathcal{D}X(\mathcal{U}dX,G)) &= [\mathcal{U}X^{\mathrm{op}},\Omega](F,[\mathcal{D}X,\Omega](G,\mathcal{U}dX)) \\
&= [[X,\Omega],V](F,\bigwedge_{\varphi} G(\varphi) \pitchfork \mathcal{U}dX(\varphi)) \\
&= \bigwedge_{\varphi} G(\varphi) \pitchfork [[X,\Omega],V](F,\mathcal{U}dX(\varphi)) \\
&= \bigwedge_{\varphi} G(\varphi) \pitchfork \bigwedge_{\psi} F(\psi) \pitchfork \mathcal{U}dX(\varphi)(\psi).
\end{aligned}
$$

And as $\mathcal{U}dX(\varphi)(\psi) = \mathcal{D}uX(\varphi)(\psi)$ we obtain what we wanted. $\square$

4.3 Algebras

In this section we will discuss the algebras generated by the two monads defined above and their composites. As $\mathcal{D}$ is a KZ-doctrine, following [15], a $\mathcal{D}$-algebra A is a tuple $A = (A,\alpha)$ such that $\alpha : \mathcal{D}A \longrightarrow A$ is a left adjoint to d_A, and since $\mathcal{U}$ is a co-KZ-doctrine a $\mathcal{U}$-algebra B is a tuple $B = (B,\beta)$ such that $\beta : \mathcal{U}B \longrightarrow B$ is a right adjoint to u_B.

Proposition 4.3.1. *The carrier A of a $\mathcal{D}$-algebra $A = (A,\alpha_A)$ is co-complete, and the carrier C of an $\mathcal{U}$-algebra $C = (C,\beta_C)$ is complete. Moreover, $f : (A,\alpha_A) \longrightarrow (B,\alpha_B)$ is $\mathcal{D}$-morphism if and only if f preserves all weighted colimits. Dually a map $g : (C,\beta_C) \longrightarrow (D,\beta_D)$ is a $\mathcal{U}$-morphism if and only if it preserves all weighted limits.*

Proof. We will only prove it for $\mathcal{U}$, the proof for $\mathcal{D}$ is similar. Let (C,β_C) be a $\mathcal{U}$-algebra. To prove that C is complete let $\psi : K \longrightarrow \Omega$ and $F : K \longrightarrow C$ be any two functors. We have to show that the limit $\{\psi, F\}$ exists in C. As $\mathcal{U}C$ is complete, the limit $\{\psi, u_C \circ F\}$ exists and as as α preserves limits so does $\alpha(\{\psi, uC \circ F\}$ in C. And as we have $\alpha \circ uC = \mathrm{id}_C$, the limit $\{\psi, F\} = \{\psi, \alpha \circ uC \circ F\} = \alpha\{\psi, uC \circ F\}$ exists in C.
As $f : (C,\beta_C) \longrightarrow (D,\beta_D)$ is a U-morphism the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}C \arrow[r, bend left, "\mathcal{U}f"] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\beta_C"'] \arrow[d, phantom, "\vdash"] & \mathcal{U}D \arrow[l, bend left, "-\circ f"] \arrow[d, bend right, "\beta_D"'] \arrow[d, phantom, "\vdash"] \\
C \arrow[r, "f"'] \arrow[u, bend right, "uC"'] & D \arrow[u, bend right, "uD"']
\end{tikzcd} \tag{4.7}$$

As the functors Uf, β_A, and β_B are right adjoints they preserve all limits, and, as every element of $\mathcal{U}A$ is a canonical limit of representables the continuity of f is equivalent to the commutativity of the diagram. $\square$

The following transfers the notion of constructive complete distributivity of [8] from $\mathbb{2}$ to a commutative quantale Ω.

Definition 4.3.2. A $\mathcal{D}$ algebra (A, α) is called ccd if the structure map α has a left adjoint. We denote with CCD the subcategory of $\mathcal{D}$-alg whose objects are ccd and the arrows are limits and colimits preserving functors. Dually, a $\mathcal{U}$-algebra for which the structure map has a right adjoint is called ${}^{\mathrm{op}}\mathsf{ccd}$, we denote with ${}^{\mathrm{op}}\mathsf{CCD}$ the subcategory of $\mathcal{U}$-alg whose objects are ${}^{\mathrm{op}}\mathsf{ccd}$ algebras and arrows are limit and colimit preserving functors.

The terminology is justified by the following:

Example 4.3.3. In the case $\Omega = \mathbb{2}$, a poset A equipped with a $\mathcal{D}$-algebra structure α is a join semi-lattice. Moreover, A is ccd in the sense of the definition above iff it is constructive completely distributive in the usual order-theoretic sense.

Remark 4.3.4. Note that this definition for complete distributivity is used in many other places such as [12] and in the context of quantoloid enriched categories in [31], and in the setting of quantale enriched categories in [17].

Definition 4.3.5. A $\mathcal{DU}$-algebra is a $\mathcal{U}$-algebra (A, β) which has a $\mathcal{D}$-structure $\alpha : \mathcal{D}A \longrightarrow A$ such that α is a $\mathcal{U}$-homomorphism, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{UD}A \arrow[r, "r_A"] \arrow[dr, "\mathcal{U}\alpha"'] & DUA \arrow[r, "\mathcal{D}\beta"'] & \mathcal{D}A \arrow[d, "\alpha"] \\
& \mathcal{U}A \arrow[r, "\beta"'] & A
\end{tikzcd} \tag{4.8}$$

For any two $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ (A, α_A, β_A) and (B, α_B, β_B) a $\mathcal{DU}$-morphism from A to B is a map $f : A \longrightarrow B$ such that it is simultaneously $\mathcal{D}$ and $\mathcal{U}$ morphism.

As the structure map α of a ccd-algebra (A, α) is both a left and a right adjoint then the carrier A is complete and cocomplete.

Theorem 4.3.6. *$\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}} \cong \mathsf{CCD}$, and $\Omega\text{-}\mathsf{Cat}^{\mathcal{UD}} \cong {}^{\mathrm{op}}\mathsf{CCD}$.*

Proof. We shall only prove the first isomorphism, the second one being similar. Let (A, α, β) be a $\mathcal{DU}$-algebra.

The functor α has a left adjoint if and only if the right Kan extension of $1_{\mathcal{D}A}$ along α exists, and is preserved by α. As $\mathcal{D}A$ is complete the right Kan extension exists, so we just have to show that α preserves it. Let us take

$$\delta(a) \coloneqq Ran_\alpha \mathrm{id}_{\mathcal{D}A} = \{A(a, \alpha -),\ \mathrm{id}_{\mathcal{D}A}\}.$$

As (A, α, β) is a $\mathcal{DU}$-algebra, and $lA \dashv rA$ and the functor $\mathcal{U}$ is a right Kan extension on arrows, diagram (4.8) becomes:

$$\begin{tikzcd}
\mathcal{UD}A \arrow[r, bend right=15, "rA"'] \arrow[r, phantom, "\perp"] \arrow[dr, bend right=15, "\mathcal{U}\alpha"'] \arrow[dr, phantom, "\perp"] & \mathcal{DU}A \arrow[l, bend right=15, "lA"'] \arrow[r, bend right=15, "\mathcal{D}\beta"'] \arrow[r, phantom, "\perp"] & \mathcal{D}A \arrow[l, bend right=15, "\mathcal{D}uA"'] \arrow[d, "\dashv\alpha"', "\dashv"] \\
 & \mathcal{U}A \arrow[ul, bend right=15, "-\circ\alpha"'] \arrow[r, bend right=15, "\beta"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right=15, "uA"'] \arrow[u, bend left=40, "\delta"] \arrow[u, bend right=40, "dA"']
\end{tikzcd} \tag{4.9}$$

Now for each $a \in A$ let us look at the following limit in $\mathcal{UD}A$, given by $A(a, \alpha -) : \mathcal{D}A \longrightarrow \Omega$ and $lA \circ \mathcal{D}uA : \mathcal{D}A \longrightarrow \mathcal{UD}A$:

$$\{A(a, \alpha -), lA \circ \mathcal{D}uA\}.$$

The natural transformations r and l are distributive laws so they satisfy diagram (2.19), thus we have

$$rA \circ u\mathcal{D}A = \mathcal{D}uA \text{ and } rA \circ u\mathcal{D}A = \mathcal{D}uA,$$

and respectively

$$lA \circ \mathcal{D}uA = u\mathcal{D}A, \text{ and } lA \circ d\mathcal{U}A = \mathcal{U}dA.$$

Thus we obtain:

$$\mathcal{D}\beta \circ rA \circ lA \circ \mathcal{D}uA = \mathrm{id}_{\mathcal{D}A}.$$

So we have:

$$\begin{aligned}
\alpha\{A(-,\alpha-), \mathrm{id}_{\mathcal{D}A}\} &= \alpha\{A(-,\alpha-), \mathcal{D}\beta \circ rA \circ lA \circ \mathcal{D}uA\} \\
&= \alpha \circ \mathcal{D}\beta \circ rA\{A(-,\alpha-), lA \circ \mathcal{D}uA\} \\
&= \beta \circ \mathcal{U}\alpha\{A(-,\alpha-), lA \circ \mathcal{D}uA\} \\
&= \{A(-,\alpha-), \beta \circ \mathcal{U}\alpha \circ lA \circ \mathcal{D}uA\} \\
&= \{A(-,\alpha-), \alpha \circ \mathcal{D}\beta \circ rA \circ lA \circ \mathcal{D}uA\} \\
&= \{A(-,\alpha-), \alpha\}
\end{aligned}$$

Thus α has a left adjoint.

Now let (A,α) be, ccd. Then A is complete, so the left Kan extension of id_A along uA exists. Call $\beta \coloneqq Lan_{uA}\mathrm{id}_A$. As $uA : A \longrightarrow \mathcal{U}A$ preserves colimits, β is a right adjoint of uA.
We still need to show the commutativity of diagram (4.8), so let us look at the following diagram:

$$\begin{tikzcd}
\mathcal{U}\mathcal{D}A \arrow[r, bend right=15, "rA"'] \arrow[r, phantom, "\perp"] \arrow[dr, bend right, "\mathcal{U}\alpha"'] \arrow[dr, phantom, "\perp"] & \mathcal{D}\mathcal{U}A \arrow[l, bend right=15, "lA"'] \arrow[r, bend right=15, "\mathcal{D}\beta"'] \arrow[r, phantom, "\perp"] & \mathcal{D}A \arrow[ll, bend right=30, "u\mathcal{D}A"'] \arrow[l, bend right=15, "\mathcal{D}uA"'] \arrow[d, bend right=15, "\alpha"'] \arrow[d, phantom, "\vdash"] \\
& \mathcal{U}A \arrow[u, "d\mathcal{U}A"] \arrow[ul, bend right, "\mathcal{U}\delta"'] \arrow[r, bend right=15, "\beta"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right=15, "uA"'] \arrow[u, bend right=15, "\delta"']
\end{tikzcd} \tag{4.10}$$

Then using the naturality of u and that $lA \circ \mathcal{D}uA = u\mathcal{D}A$, one has $\mathcal{U}\delta \circ uA = lA \circ \mathcal{D}uA \circ \delta$. And considering that $\mathcal{U}\delta \circ uA \dashv \beta \circ \mathcal{U}\alpha$ and that $lA \circ \mathcal{D}uA \circ \delta \dashv \alpha \circ \mathcal{D}\beta \circ rA$, we conclude that $\beta \circ \mathcal{U}\alpha = \alpha \circ \mathcal{D}\beta \circ rA$

To finish the proof we have to show that (A,β) is a $\mathcal{U}$-algebra, which is equivalent to the commutativity of the following diagrams, where ν is the multiplication of $\mathcal{U}$.

$$\begin{tikzcd}
\mathcal{U}\mathcal{U}A \arrow[r, bend right=15, "\nu A"'] \arrow[r, phantom, "\top"] \arrow[d, bend right=15, "\mathcal{U}\beta"'] \arrow[d, phantom, "\dashv"] & \mathcal{U}A \arrow[l, bend right=15, "\mathcal{U}uA"'] \arrow[d, bend right=15, "\beta"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{U}A \arrow[u, bend right=15, "\mathcal{U}uA"'] \arrow[r, bend right=15, "\beta"'] \arrow[r, phantom, "\top"] & A \arrow[l, bend right=15, "uA"'] \arrow[u, bend right=15, "uA"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "uA"] \arrow[dr, "\mathrm{id}_A"] & \mathcal{U}A \arrow[d, "\beta"] \\
& A
\end{tikzcd} \tag{4.11}$$

As both $\beta \circ \nu A$ and $\beta \circ \mathcal{U}\beta$ have the same right adjoint, we have $\beta \circ \nu A \cong \beta \circ \mathcal{U}\beta$, and as we are quantale enriched setting, isomorphism of arrows means equality. Thus we have shown that (A, β) is indeed a $\mathcal{U}$-algebra.

As we require maps in CCD to preserve limits and colimits, and $\mathcal{D}\mathcal{U}$-morphisms also preserve limits and colimits, we have nothing left to show. □

Whereas naturally occurring metric spaces, such as Euclidean spaces, are typically not ccd, it is the case that spaces of many-valued *predicates* over metric spaces are ccd:

Example 4.3.7. For any X in Ω-Cat,

1. $(\mathcal{D}X, \mu X)$ is ccd.
2. $(\mathcal{U}X, \nu X)$ is ${}^{\mathrm{op}}\mathsf{ccd}$.

In the following let us show that the structure map for $\mathcal{D}$-algebras calculates colimits and the structure map for U-algebras calculates limits.

Proposition 4.3.8. *Let A be a Ω-category and $\alpha : \mathcal{D}A \longrightarrow A$ with $\alpha \dashv dA$. Then $\alpha(G) = G \star \mathrm{id}_A$ and $\alpha = \mathrm{Lan}_{dA}\,\mathrm{id}_A$. Moreover, $\alpha \circ dA = \mathrm{id}_A$. Let B be a Ω-category and $\beta : \mathcal{U}B \longrightarrow B$ with $uB \dashv \beta$. Then $\beta(F) = \{F, \mathrm{id}_B\}$ and $\beta = \mathrm{Ran}_{uB}\,\mathrm{id}_B$. Moreover, $\beta \circ uB = \mathrm{id}_B$.*

Proof. To show $\alpha(G) = G \star \mathrm{id}_A$ we calculate

$$A(G \star 1_A, a) = \{G, A(-, a)\} = \mathcal{D}A(G, A(-.a))) = \mathcal{D}A(G, dA(a)) = A(\alpha(G), a)$$

for all $a \in A$ and $G \in \mathcal{D}A$. Moreover

$$\mathrm{Lan}_{dA}\,\mathrm{id}_A(G) = \bigvee_{a \in A} DA(A(-, a), G) \otimes a = \bigvee_{a \in A} Ga \otimes a = G \star \mathrm{id}_A \tag{4.12}$$

Finally, as dA is fully faithful we have $\mathrm{Lan}_{dA}\,\mathrm{id}_A \circ dA = \mathrm{id}_A$ thus $\alpha \circ dA = \mathrm{id}_A$. □

4.4 $\mathcal{DU}$ is equivalent to $[[-,\Omega],\Omega]$

In this section we will connect $\mathcal{DU}$ with the double dualisation monad DU, and construct a left adjoint AT of the comparison functor K.

Theorem 4.4.1. *For any commutative quantale Ω, the composite monad $\mathcal{DU}$ is equivalent to the monad generated by the adjunction $[-,\Omega] \dashv [-,\Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$.*

Proof. This theorem is proved in the case $\Omega = \mathbb{2}$ in [23], and in the case of quanloid enriched categories in [32]. We will give a proof in the setting of quantale enriched categories.
The first step is to prove that $\mathcal{DU} = DU$ as functors. Let X, Y be Ω-categories and let $f : X \longrightarrow Y$ be a Ω-functor.
Then: $\mathcal{DU}X = [([X,\Omega]^{\mathrm{op}})^{\mathrm{op}},\Omega] = [[X,\Omega],\Omega] = DUX$. Now on arrows, as $\mathcal{U}$ is given by a right Kan extension, we have that $Uf \dashv \mathcal{U}f$ and as D is a 2-functor we have

$$DUf \dashv D(\mathcal{U}f).$$

As $\mathcal{D}$ on arrows is given by a left Kan extension we have that

$$\mathcal{D}(\mathcal{U}f) \dashv D(\mathcal{U}f).$$

Thus as both have the same right adjoint we have

$$\mathcal{DU}f = DUF.$$

To show that the two monads are equivalent we have to show that the unit and the multiplication of both monads are the same. We have $\eta_X : X \longrightarrow [[X,\Omega],\Omega]$ given by $\eta_X(x)(T) = T(x)$. But we also have

$$d\mathcal{U} \circ u(x)(T) = UX(T, u(x)) = [X,\Omega](X(x,-),T) = T(x)$$

and as $dU \circ u$ is the unit of $\mathcal{DU}$ we have shown the two units are equal.
The multiplication of DU is given by $\mu_{DU} = D\epsilon U$. Now again one has $\epsilon = \mathcal{D}u \circ d$ so $\epsilon U = u\mathcal{D} \circ D$ so we have $\mathcal{D}u\mathcal{DU} \circ \mathcal{D}d\mathcal{U} = \mathcal{D}\epsilon\mathcal{U} \dashv D\epsilon U$.
Following [1] the multiplication of $\mathcal{DU}$ is given by

$$\mu\nu \circ \mathcal{D}r\mathcal{U} = \mathcal{D}\nu \circ \mu\mathcal{UU} \circ \mathcal{D}r\mathcal{U}.$$

We have that

$$\mathcal{D}\nu \circ \mu\mathcal{UU} \circ \mathcal{D}r\mathcal{U}$$

has a left adjoint given by

$$\mathcal{D}l\mathcal{U}\circ\mathcal{D}d\mathcal{U}\mathcal{U}\circ\mathcal{D}u\mathcal{U}.$$

So to finish we have to show that

$$\mathcal{D}u\mathcal{D}\mathcal{U}\circ\mathcal{D}d\mathcal{U}=\mathcal{D}l\mathcal{U}\circ\mathcal{D}d\mathcal{U}\mathcal{U}\circ\mathcal{D}u\mathcal{U}.$$

But that is true from the naturality of u and $l\circ d\mathcal{U}=\mathcal{U}d$. □

4.5 The comparison functor $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}\to\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$

Following [20], let the comparison functor $K:\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}\longrightarrow\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ be given by $KX=(X,D\epsilon X)$, for the adjunction $U\dashv D$. As $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is cocomplete, K has a left adjoint. Let us recall that a $\mathcal{DU}$-algebra is a complete and cocomplete category, and a $\mathcal{DU}$-algebra morphism is continuous and cocontinuous functor, thus it makes sense for its left adjoint to be a version of the functor AT defined in the previous chapter. First recall how we defined AT there.

We define a functor $\mathsf{AT}:\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}\longrightarrow\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ on objects as $\mathsf{AT}(A,\alpha,\beta)=(\mathsf{At}(A))^{\mathrm{op}}$. In order to define AT on maps we needed the following results.

Lemma 4.5.1. *For any $H:A\longrightarrow B$ in $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$, there exists a left adjoint $L:B\longrightarrow A$ in $\Omega\text{-}\mathsf{Cat}$.*

Lemma 4.5.2. *For all $A,B\in\mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}$ and $H:A\longrightarrow B$ with left adjoint L, there exists $f:\mathsf{At}(B)\longrightarrow\mathsf{At}(A)$ such that $L\circ i_B=i_A\circ f$, where $i_A:\mathsf{At}(A)\longrightarrow A$ and $i_B:\mathsf{At}(B)\longrightarrow B$ are the atom-inclusion maps.*

We can now define $\mathsf{AT}(H)=f^{\mathrm{op}}$ with f as in the lemma. This defines a functor because composition of adjoints is again an adjoint. With these we can show that

Theorem 4.5.3. *The functor $\mathsf{AT}:\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}\longrightarrow\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is a left adjoint of $K:\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}\longrightarrow\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$.*

Proof. Let $X\in\Omega\text{-}\mathsf{Cat}$ and $A\in\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$. We have to show that

$$\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}(\mathsf{AT}(A),X)\cong\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}(A,KX)$$

which is equivalent to

$$\Omega\text{-}\mathsf{Cat}(X,\mathsf{At}(A)^{\mathrm{op}})\cong\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}(A,[X,\Omega]),$$

and as

$$\Omega\text{-}\mathsf{Cat}(X, \mathsf{At}(A)^{\mathrm{op}}) \cong \Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, \mathsf{At}(A))^{\mathrm{op}}$$

(see [14, 2.28]) we have to prove that there is a natural isomorphism

$$\Omega\text{-}\mathsf{Cat}(X^{\mathrm{op}}, \mathsf{At}(A))^{\mathrm{op}} \cong \Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}(A, [X, \Omega]) \cong \mathsf{c}\text{-}\Omega\text{-}\mathsf{Cat}(A, [X, \Omega]),$$

which is Theorem 3.2.8. □

After having constructed a left adjoint AT of K, we next ask when $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is a full reflective subcategory of $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$, that is, we ask when K is fully faithful. We also want to characterise the image of K.

4.5.1 A fully faithfulness of the comparison and its image

In the case of $\Omega = \mathbb{2}$ the comparison K is fully faithful, but this is not true for all commutative quantales Ω. In this subsection, we give necessary and sufficient conditions for K to be fully faithful and describe its image.

For that let us prove the following.

Proposition 4.5.4. *1. The order $x \leq y \Leftrightarrow \Omega(x, y) \geq e$ is the order of Ω.*

2. $[X, \Omega]$ is anti-symmetric for any Ω-category X.

3. $[X, Y]$ is anti-symmetric iff Y is anti-symmetric.

Proof. First statement follows from the definition of $\Omega(v, -)$ as the right adjoint of $- \otimes v$ for all $v \in \Omega$. Let $v, w \in \Omega$ such that $e \leq \Omega(v, w)$ then using the adjunction we have $e \otimes v \leq w$ thus $v \leq w$ in the poset Ω.

The second statement follows from the third one, so we shall only prove the third statement. " $\Leftarrow$ " Suppose that $[X, Y]$ is not anti-symmetric, then there exists $g_1 \neq g_2 : X \longrightarrow Y$ such that

$$e \leq [X, Y](g_1, g_2) \text{ and } e \leq [X, Y](g_2, g_1).$$

Because $e \leq \bigwedge_{x \in X} Y(g_1(x), g_2(x))$ and $e \leq \bigwedge_{x \in X} Y(g_2(x), g_1(x))$ is equivalent to $e \leq Y(g_1(x), g_2(x))$ and $e \leq Y(g_2(x), g_1(x))$ for all $x \in X$ and because Y is anti-symmetric we obtain that

$$g_1 = g_2.$$

" $\Rightarrow$ " Take X to be a single element Ω-category.

$\square$

Using Proposition 4.5.4 we notice that K is faithful on $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}(X, Y)$ if and only if X is anti-symmetric. Indeed, if X is not anti-symmetric let $g_1, g_2 : Y \longrightarrow X$ be two distinct equivalent maps. Then as Ω is we have that $Kg_1 = Kg_2$.

For K to be full we need that for any two categories $X, Y \in \Omega\text{-}\mathsf{Cat}$ and every map $H : KX \longrightarrow KY$ there exists a map $h : Y \longrightarrow X$ such that $Kh = H$. Using the adjunction, we have $K \circ \mathsf{AT}(H) = H$ so if one can make sure that $\mathsf{At}(KX) \cong X$ and $\mathsf{At}(KY) \cong Y$ then the functor K will be full. For that we need the following definition [18, 13]. means.

Definition 4.5.5. We say that $X \in \Omega\text{-}\mathsf{Cat}$ is Cauchy complete if $X \simeq \mathsf{At}([X, \Omega])^{\mathrm{op}}$. We denote by $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}$ the full subcategory of $\Omega\text{-}\mathsf{Cat}$ spanned by the antisymmetric Cauchy complete categories.

As Ω is complete and cocomplete, and anti-symmetric the image of $U : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ is in $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{\mathrm{op}}$, so in fact we only need that K is full and faithful on $\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{\mathrm{op}}$, as seen in the next theorem.

Theorem 4.5.6. *The comparison functor for the adjunction* $[-, \Omega] \dashv [-, \Omega] : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}{}^{\mathrm{op}}$ *is full and faithful.*

To characterise what the image of K is, we use the description of full reflective subcategories by orthogonality, see [4, Chapter 5.4]. As in the previous chapter, see 3.3, we will show that the condition a $\mathcal{DU}$-algebra has to satisfy to be in the image of K is to be atomic.

Theorem 4.5.7. *An algebra A in $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ is isomorphic to an algebra in the image of K if and only if it is atomic.*

Proof. We shall use orthogonality [4, Chapter 5.4]. First let us take X in $\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$ and show that it is atomic. Let us denote by $\theta : \mathrm{id} \longrightarrow K\mathsf{AT}$ the unit of the adjunction $\mathsf{AT} \dashv K$. From orthogonality we obtain that for every $B \in \Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ and any $f : B \longrightarrow X$ we have a unique factorisation through θ_B, so let us take $B = X$ and $f = \mathrm{id}_X$. There exists a $g : [\mathsf{At}^{\mathrm{op}}(X), \Omega] \longrightarrow X$ such that g preserves limits and colimits and such that $g \circ \theta_X = \mathrm{id}_X$. Thus, for every $x \in X$ one has

$$g(\theta_X(x)) = x.$$

Now $\theta_X(x) = X(-,x) : \mathsf{At}^{\mathrm{op}}(X) \longrightarrow \Omega$ and as every presheaf is a colimit of representables one has

$$X(-,x) = X(-,x) \star d_{\mathsf{At}(X)}.$$

Thus

$$\begin{aligned}
x &= g(X(-,x)) = g(X(-,x) \star d_{\mathsf{At}(X)}) \\
&= g(\bigvee_{x' \in \mathsf{At}(X)} X(x',x) \otimes \mathsf{At}(X)(-,x')) = \bigvee_{x' \in \mathsf{At}(X)} X(x',x) \otimes g(\mathsf{At}(X)(-,x'))) \\
&= \bigvee_{x' \in \mathsf{At}(X)} X(x',x) \otimes g(X(-,x'))) = \bigvee_{x' \in \mathsf{At}(X)} X(x',x) \otimes x')) \\
&= X(i_X -,x) \star i_X
\end{aligned}$$

Thus as $i_X : \mathsf{At}(X) \longrightarrow X$ is dense, X is atomic.

The converse follows from Lemma 3.3.4 as $X \cong [\mathsf{At}^{\mathrm{op}}(X), \Omega] = \mathcal{D}(\mathsf{At}(X))$, which is ccd. $\square$

4.6 The special case of $\Omega \cong \Omega^{\mathrm{op}}$

In the case $\Omega = \mathbb{2}$, we have ${}^{\mathrm{op}}\mathsf{CCD} = \mathsf{CCD}$ (since the dual of a completely distributive lattice is a completely distributive lattice). But this is not true for general Ω. Here we show that $\mathcal{DU}$-algebras and $\mathcal{UD}$-algebras can be identified if $\Omega \cong \Omega^{\mathrm{op}}$ in Ω-Cat.

The following two propositions are due to [8] in the case $\Omega = \mathbb{2}$.

Proposition 4.6.1. *Let A, B be any two complete and cocomplete categories, then B is* ccd *if A is* ccd *and we have one of the following:*

$$\begin{tikzcd}
B \arrow[r, tail, bend left=40, "i_1", "\perp"'] \arrow[r, tail, bend right=40, "i_2"', "\perp"] & A \arrow[l, "p" description] & & B \arrow[r, tail, "i" description] & A \arrow[l, bend right=40, "p_1"', "\perp"] \arrow[l, bend left=40, "p_2", "\perp"']
\end{tikzcd} \tag{4.13}$$

Proof. Let us show that B is CCD using the first property.

Let us define $\alpha_B : \mathcal{D}B \longrightarrow B$ with $\alpha_B = p \circ \alpha_A \circ \mathcal{D}i_2$, and $\delta_B : B \longrightarrow \mathcal{D}B$ with $\delta_B = p \circ \delta_A \circ \mathcal{D}i_1$ as in the following diagram

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{D}B \arrow[r, bend left, "\mathcal{D}i_1"] \arrow[r, bend right, "\mathcal{D}i_2" description] \arrow[d, "\alpha_B" description] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, phantom, shift right=3, "\dashv"] & \mathcal{D}A \arrow[l, "\mathcal{D}p" description] \arrow[l, bend left=50, "-\circ i_2"] \arrow[l, phantom, shift left=2, "\perp"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=6, "\perp"] \arrow[d, "\alpha_A" description] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, phantom, shift right=3, "\dashv"] \\
B \arrow[u, bend left=30, "\delta_B"] \arrow[u, bend right=30, "dB"'] \arrow[r, bend left, "i_1"] \arrow[r, bend right, "i_2"'] & A \arrow[l, "p" description] \arrow[l, phantom, shift left=2, "\perp"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[u, bend left=30, "\delta_A"] \arrow[u, bend right=30, "dA"']
\end{tikzcd} \tag{4.14}$$

then we have $\delta_B \dashv \alpha_B \dashv (-\circ i_2) \circ dA \circ i_2$. Thus all we have to show is that

$$dB = (-\circ i_2) \circ dA \circ i_2.$$

As i_2 is an embedding for any two objects of B, b and b' we have

$$\begin{aligned}
((-\circ i_2) \circ dA \circ i_2)(b)(b') &= ((-\circ i_2) \circ A(-, i_2(b))(b') \\
&= A(i_2(b'), i_2(b)) \\
&= B(b', b) \\
&= dB(b)(b').
\end{aligned}$$

□

Proposition 4.6.2. *If* $\Omega \cong \Omega^{\mathrm{op}}$ *for any* Ω*-category* A*, if* A *is* ccd *then* A^{op} *is also* ccd.

Proof. As $\Omega \cong \Omega^{\mathrm{op}}$, for any X we have $[X^{\mathrm{op}}, \Omega] \cong [X, \Omega]^{\mathrm{op}}$, see [14, 2.28]. Now as A is CCD we have the following:

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left, "\delta_A"] \arrow[r, bend right, "d_A"'] & DA \arrow[l, "\alpha_A" description] \arrow[l, phantom, shift left=2, "\perp"] \arrow[l, phantom, shift right=2, "\perp"]
\end{tikzcd}$$

and if we apply the 2-functor $()^{\mathrm{op}} : \Omega\text{-}\mathsf{Cat} \longrightarrow \Omega\text{-}\mathsf{Cat}$ on it we obtain

$$\begin{tikzcd}
A^{\mathrm{op}} \arrow[r, hook, bend left=40, "\delta_A^{\mathrm{op}}"] \arrow[r, hook, bend right=40, "d_A^{\mathrm{op}}"'] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] & (DA)^{\mathrm{op}} \arrow[l, "\alpha_A^{\mathrm{op}}" description]
\end{tikzcd}$$

and as $(DA) \cong D(A^{\mathrm{op}})$ and $D(A^{\mathrm{op}})$ is CCD from Proposition 4.6.1 we deduce A^{op} is also CCD. ☐

Corollary 4.6.3. *If $\Omega \cong \Omega^{\mathrm{op}}$ as Ω-categories then* $\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}} \cong \Omega\text{-}\mathsf{Cat}^{\mathcal{UD}}$.

Proof. If A is ccd then A^{op} is ${}^{\mathrm{op}}\mathsf{ccd}$, so A^{op} is simultaneously ccd and ${}^{\mathrm{op}}\mathsf{ccd}$. ☐

Remark 4.6.4. That a $\mathcal{DU}$-algebra is a $\mathcal{UD}$-algebra means that the distributive laws $l : \mathcal{DU} \longrightarrow \mathcal{UD}$ and $r : \mathcal{UD} \longrightarrow \mathcal{DU}$ imply each other, which is well-known in lattice theory.

Example 4.6.5.

1. $[0,1] \cong [0,1]^{\mathrm{op}}$ means that $[0,1]$ has a $\mathcal{DU}$ structure as well an $\mathcal{UD}$ structure.

2. $[0,\infty] \ncong [0,\infty]^{\mathrm{op}}$. Indeed, suppose for a contradiction that we have such an isomorphism $\varphi : \Omega^{\mathrm{op}} \longrightarrow \Omega$ in $\Omega\text{-}\mathsf{Cat}$. Then for all $a, b \in \Omega$ one has $\Omega(a,b) = \Omega(\varphi(b), \varphi(a))$. Taking $a = 0$, it follows from $\varphi(0) = \infty$ that we have $b = \infty \dot{-} \varphi(b)$ for all $b \in \Omega$, but this is not possible.

4.7 Conclusions

In this chapter we have a categorical answer for the question " what are the algebras for the monad $[[-,\Omega],\Omega]$".

$\mathsf{Set}^{\mathrm{op}}$	complete atomic Boolean algebra
$\Omega\text{-}\mathsf{Cat}^{\mathrm{op}}$	CCD

More precisely, we have seen that the subcategory ${\Omega\text{-}\mathsf{Cat}_{\mathsf{cc}}}^{\mathrm{op}}$ of Cauchy complete Ω-categories is equationally definable in the abstract categorical sense that it is an orthogonal subcategory of the monadic category CCD.

Recall that a $\mathcal{D}$-algebra (A, α) is ccd if α has a left adjoint, or, equivalently, if it preserves all weighted limits. So let J be an indexing category and let

$\psi : J \longrightarrow \Omega$ and $F : J \longrightarrow \mathcal{D}A$ be two maps, and let $\{\psi, F\}$ be their limit. That α preserves it means:

$$\alpha(\{\psi, F\}) = \{\psi, \alpha(F)\} \tag{4.15}$$

Written with ends and coends and taking in consideration what it means to be a limit in a functor category, see [14, Chapter 3], one has:

$$\bigvee_{a\in A}(\bigwedge_{j\in J}\psi(j) \pitchfork F(j)(a)) \otimes a = \bigwedge_{j\in J}\psi(j) \pitchfork (\bigvee_{a\in A} F(j)(a) \otimes a) \tag{4.16}$$

Dually a $\mathcal{U}$-algebra (B, β) is ${}^{\mathrm{op}}\mathsf{ccd}$ if its structure map β is a left adjoint, that is, if β preserves all weighted colimits:

Proposition 4.7.1. *The $\mathcal{U}$-algebra (B, β) is ${}^{\mathrm{op}}\mathsf{ccd}$ if and only if for any K and $\varphi : J^{\mathrm{op}} \longrightarrow \Omega$ and $C : J \longrightarrow \mathcal{U}B$ we have*

$$\{\{\varphi, C^{\mathrm{op}}\}, \mathrm{id}_B\} = \{\varphi, \{C^{\mathrm{op}}, \mathrm{id}_B\}\} \tag{4.17}$$

In the example below, Ω in items 1, 2 and 4 is completely distributive as a Ω-category, because $\Omega = \mathcal{D}(1)$. Item 3 shows a metric space which is complete and cocomplete as a Ω-category but is not completely distributive as a Ω-category.

Still this does not define a logic in the usual sense, as an algebra on a set given by operations and equations. So in the following chapters we will give a description of $\mathcal{D}$,$\mathcal{U}$, and $\mathcal{DU}$-algebras as sets with operations and equations.

Chapter 5

Variety of algebras

In this chapter we will take the algebras for the monads: $\mathcal{D}$,$\mathcal{U}$, and $\mathcal{DU}$ and construct algebras over sets with operations and equations.

In the next section by a representative of any cardinal $\mathbf{K}$ we understand any set K whose number of elements is $\mathbf{K}$. For any finite cardinal $\mathbf{N}$ one can think of the representative N as the subset $N = \{1, 2, 3, .., \mathbf{N}\} \subset \mathbb{N}$. Let us denote by $\mathcal{K}$ the class of all cardinals. For any two cardinals $\mathbf{K_1}, \mathbf{K_2} \in \mathcal{K}$ by the direct sum of $\mathbf{K_1}$ and $\mathbf{K_2}$, denoted by $\mathbf{K_1} \oplus \mathbf{K_2}$ we understand any set whose number of elements is equal to $\mathbf{K_1} + \mathbf{K_2}$. For example if the set K_1 is a representative for $\mathbf{K_1}$ and the set K_2 is a representative for $\mathbf{K_2}$ then the direct sum $K_1 \oplus K_2$ is a representative for $\mathbf{K_1} \oplus \mathbf{K_2}$. For any set A and any cardinal $\mathbf{K}$ by $A^{\mathbf{K}}$ we understand the direct product of $\Pi_{\mathcal{K}} A$ times, or equivalently if K is a representative for $\mathbf{K}$ we understand the set of all functions $[K, A]$. In the context of this chapter we will not distinguish from a cardinal and its representative set. Also if the cardinal $\mathbf{K_1}, \mathbf{K_2}$ are finite then the representative set for $\mathbf{K_1} \oplus \mathbf{K_2}$ is the set $\{1, 2, ..., K_1, K_1 + 1, K_1 + 2, ..., K_1 + K_2\} \subset \mathbb{N}$.

5.1 Syntactic $\mathcal{D}$-algebras and $\mathcal{U}$-algebras

Definition 5.1.1. By a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra we understand a set A together with a family of unary operations $(v \star \_)_{v \in \Omega} : A \longrightarrow A$ indexed by Ω, and a family of operations $\bigsqcup_K : A^K \longrightarrow A$, where K ranges over all cardinals, satisfying the following 6 axioms. Dually the notions of a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra is given by a set B together with a family of unary operations $(v \rhd \_)_{v \in \Omega} : B \longrightarrow B$ and for each cardinal K an operation $\bigsqcap_K : B^K \longrightarrow B$ satisfying the following

6 axioms:

1. $I \star - = \mathrm{id}_A$ $\qquad$ $I \triangleright \_ = \mathrm{id}_A$

2. For all $a \in A$, $b \in B$ and $v, w \in \Omega$

   $v \star (w \star a) = (v \otimes w) \star a$ $\qquad$ $v \triangleright (w \triangleright b) = (v \otimes w) \triangleright b$

$$\begin{tikzcd}
\Omega \times \Omega \times A \arrow[r, "\mathrm{id}_\Omega \times \_\star\_"] \arrow[d, "\otimes \times \mathrm{id}_A"'] & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega \times A \arrow[r, "\_\star\_"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times \Omega \times B \arrow[r, "\mathrm{id}_\Omega \times \_\triangleright\_"] \arrow[d, "\otimes \times \mathrm{id}_B"'] & \Omega \times B \arrow[d, "\_\triangleright\_"] \\
\Omega \times B \arrow[r, "\_\triangleright\_"'] & B
\end{tikzcd}$$

3. For all $v \in \Omega$ and $a_k \in [K, A]$, $b_k \in [K, B]$

   $v \star \bigsqcup_K a_k = \bigsqcup_K (v \star a_k)$ $\qquad$ $v \triangleright \bigsqcap_K b_k = \bigsqcap_K (v \triangleright b_k)$

$$\begin{tikzcd}
\Omega \times A^K \arrow[rr, "\mathrm{id}_\Omega \times \bigsqcup_K"] \arrow[d, "\Delta \times \mathrm{id}_A^K"'] & & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega^K \times A^K \arrow[r, "(\_\star\_)^K"'] & A^K \arrow[r, "\bigsqcup_K"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times B^K \arrow[rr, "\mathrm{id}_\Omega \times \bigsqcap_K"] \arrow[d, "\Delta \times \mathrm{id}_B^K"'] & & \Omega \times B \arrow[d, "\_\triangleright\_"] \\
\Omega^K \times B^K \arrow[r, "(\_\triangleright\_)^K"'] & B^K \arrow[r, "\bigsqcap_K"'] & B
\end{tikzcd}$$

4. For all $a \in A$, $b \in B$ and $v_k \in [K, \Omega]$

   $(\bigvee_K v_k) \star a = \bigsqcup_K (v_K \star a)$ $\qquad$ $(\bigvee_K v_k) \triangleright b = \bigsqcap_K (v_K \triangleright b)$

$$\begin{tikzcd}
\Omega^K \times A \arrow[rr, "(\bigvee_K) \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A^K \times \Delta"'] & & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega^K \times A^K \arrow[r, "(\_\star\_)^K"'] & A^K \arrow[r, "\bigsqcup_K"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^K \times B \arrow[rr, "(\bigvee_K) \times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_B^K \times \Delta"'] & & \Omega \times B \arrow[d, "\_\triangleright\_"] \\
\Omega^K \times B^K \arrow[r, "(\_\triangleright\_)^K"'] & B^K \arrow[r, "\bigsqcap_K"'] & B
\end{tikzcd}$$

5. For a set K and function $J : K \longrightarrow \mathsf{Set}$ let us denote with $\bar{J} = \coprod_{k \in K} Jk$. For each $k \in K$ let $a_k : J(k) \longrightarrow A$ and let $a : \bar{J} \longrightarrow A$ be the map induced by the coproduct. For each $k \in K$ let $b_k : J(k) \longrightarrow B$ and let $b : \bar{J} \longrightarrow B$ be the map induced by the coproduct.

$$\bigsqcup_K \Big(\bigsqcup_{Jk} a_k \Big) = \bigsqcup_{\bar{J}} a \qquad\qquad \bigsqcap_K \Big(\bigsqcap_{Jk} b_k \Big) = \bigsqcap_{\bar{J}} b$$

$$\begin{tikzcd}
\Pi_{k \in K} A^{J(k)} \arrow[r, "\Pi_{k \in K} \bigsqcup_{J(k)}"] \arrow[dr, "\bigsqcup_{\bar{J}K}"'] & A^K \arrow[d, "\bigsqcup_K"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{k \in K} B^{J(k)} \arrow[r, "\Pi_{k \in K} \bigsqcap_{J(k)}"] \arrow[dr, "\bigsqcap_{\bar{J}K}"'] & B^K \arrow[d, "\bigsqcap_K"] \\
& B
\end{tikzcd}$$

6. For all $a \in A$ and $b \in B$

$$\bigsqcup_K \Delta a = a \qquad\qquad \bigsqcap_K \Delta b = b$$

$$\begin{tikzcd}
A^K \arrow[r, "\bigsqcup_K"] & A \\
A \arrow[u, "\Delta"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B^K \arrow[r, "\bigsqcap_K"] & B \\
B \arrow[u, "\Delta"] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

7. For any two cardinals J, K and any bijective function $f : J \longrightarrow K$ one has

$$\bigsqcup_J \circ A^f = \bigsqcup_K \qquad\qquad \bigsqcap_J \circ A^f = \bigsqcap_K$$

$$\begin{tikzcd}
J \arrow[d, "f"'] & A^J \arrow[r, "\bigsqcup_J"] & A \\
K1 & A^K \arrow[u, "A^f"] \arrow[ur, "\bigsqcup_K"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B^J \arrow[r, "\bigsqcap_J"] & B \\
B^K \arrow[u, "B^f"] \arrow[ur, "\bigsqcap_K"'] &
\end{tikzcd}$$

Before we continue let us fix some notations. If the cardinal K is 2 then we will denote $\bigsqcup_K = \sqcup$ and $\bigsqcap_K = \sqcap$, furthermore these will be used in infix position. For any cardinal K by an element a_K of A^K we understand any function $a_K : K \longrightarrow A$. If K is finite a_K can be represented as a tuple $a_K = (a_1, a_2, ..., a_k)$ where $k = |K|$.

Example 5.1.2. 1. For any quantale Ω, the Ω-category Ω is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra, with $\bigsqcup$ given by $\bigvee$ and $v \star -$ given by $v \otimes -$. The fact that this satisfies all the axioms is trivial. In a similar way Ω is also a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra with $\bigsqcap$ given by $\bigwedge$ and $v \triangleright -$ given by $\Omega(v, -)$.

2. Any cococmplete Ω-category A is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra. For any $v \in \Omega$ and $a \in A$ we define $v \star a$ as the colimit of a weighted by v. And for every cardinal K and any $a_K \in A^K$ we define $\bigsqcup_K a_K$ as the colimit of a_K weighted by constant Ω-functor $I_K : K \longrightarrow \Omega$ given by $I_K(k) = I$ for all $k \in K$. That is equivalent to saying that $\bigsqcup_K$ is a coend.

3. Any complete Ω-category A is a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$–algebra

4. For any quantale Ω and any Ω-category X the functor category $[X, \Omega]$ is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra and the functor category $[X, \Omega]^{\mathrm{op}}$ is a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra. If $\Omega \cong \Omega^{\mathrm{op}}$ then any functor category is both a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$ and a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra.

Let A be a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra, then if we only look at axioms 5, 6, 7, $(A, (\bigsqcup_K)_{(K \in \mathcal{K})})$ is a $\bigvee$-semi-lattice. Then it makes sense that A has an order relation given by $a \leq b \Leftrightarrow a \sqcup b = b$. For more details on this one can check any book on lattice theory, for example [10]. With this order one has the following proposition:

Proposition 5.1.3. *Let J, K be any two cardinals and let $f : J \longrightarrow K$ be any function, then one has:*

$$\bigsqcup_J \circ A^f \leq \bigsqcup_K \qquad\qquad \bigsqcap_J \circ A^f \geq \bigsqcap_K$$

$$\begin{tikzcd}
J \arrow[dd, "f"'] & A^J \arrow[rr, "\sqcup_J"] & & A & B^J \arrow[rr, "\sqcap_J"] & & B \\
& & & & & & \\
K & A^K \arrow[uu, "A^f"] \arrow[uurr, "\sqcup_K"', ""{name=U}] & & & B^K \arrow[uu, "B^f"] \arrow[uurr, "\sqcap_K"', ""{name=P}] & &
\arrow[from=1-2, to=U, Rightarrow]
\arrow[from=P, to=1-5, Rightarrow]
\end{tikzcd}$$

Proof. Before we start the proof let us prove some preliminaries results.
For each cardinal K and each $a_K \in [K, A]$ the value of $\bigsqcup_K a_K$ does not depend on duplicates in a_K. That is if there exists $k_1, k_2 \in K$ such that $a_K(k_1) = a_K(k_2)$ then if we note with $K' = K - \{k_1\}$, and $K'' = K - \{k_2\}$, and $a_{K'} : K' \longrightarrow A$, and $a_{K''} : K'' \longrightarrow A$ given by $a_{K'}(k) = a_K(k)$ for all $k \in K'$ and $a_{K''}(k) = a_K(k)$ for all $k \in K''$ then one has

$$\bigsqcup_K a_K = \bigsqcup_{K'} a_{K'} = \bigsqcup_{K''} a_{K''}.$$

For simplicity let us suppose that those are the only duplicates in a_K. There exists a bijection on K, call it $s : K \longrightarrow K$, which sorts a_K in the sense that if we think K to be ordered then $A^s(a_K)$ is a monotone map. The way one can construct this bijection is simple: first we sort a_K keeping track of the initial position, define $s(k)$ as the initial place of $a_K(k)$. This is indeed a bijection, as the sorting just permutes the element of a_K. We have

$$\bigsqcup_K a_K = \bigsqcup_K A^s(a_K),$$

where the duplicate elements are "next" to each other, thus we can apply Axioms 5 to "split" K into the partition $K = J \oplus 2 \oplus J'$ such that 2 represents the two equal values of a_K. Then we have

$$\begin{aligned}\bigsqcup_K a_K &= \bigsqcup_K A^s(a_K)\\ &= \bigsqcup_3(\bigsqcup_J a_J, a_K(k_1) \sqcup a_K(k_2), \bigsqcup_{J'})\\ &= \bigsqcup_3(\bigsqcup_J a_J, a_K(k_1), \bigsqcup_{J'} a_{J'})\\ &= \bigsqcup_3(\bigsqcup_J a_J, a_K(k_2), \bigsqcup_{J'} a_{J'})\\ &= \bigsqcup_{K''} A^{s''} a_{K''}\\ &= \bigsqcup_{K'} A^{s'} a_{K'},\end{aligned}$$

where $s' : K' \longrightarrow K'$ and $s'' : K'' \longrightarrow K''$ are two sorting bijections on $a_{K'}$, and respectively, on $a_{K''}$.
Let a_K be any tuple in A^K such that a_K has no duplicates. We have to show that

$$\bigsqcup_J A^f(a_K) \sqcup \bigsqcup_K a_K = \bigsqcup_K a_K.$$

Now let us calculate the left hand side:

$$\begin{aligned}\bigsqcup_J A^f(a_K) \sqcup \bigsqcup_K a_K &\overset{A5}{=} \bigsqcup_{J\oplus K} A^f(a_K) \oplus a_K\\ &\overset{g}{=} \bigsqcup_{J\oplus K} A^g(A^f(a_K) \oplus a_K)\\ &\overset{A5}{=} \bigsqcup_K(\bigsqcup_{K_1} a_{K_1}, \bigsqcup_{K_2} a_{K_2}, ..)\\ &\overset{A6}{=} \bigsqcup_K(a_{k_1}, a_{k_2}, ..)\\ &= \bigsqcup_K a_K\end{aligned}$$

Where $g : J \oplus K \longrightarrow J \oplus K$ is a bijective function which sorts the tuple $A^f(a_K) \oplus a_K$. Now we choose to split $J \oplus K$ into a partition $J \oplus K = \oplus_{k\in K} K_k$, where each $A^g(A^f(a_K) \oplus a_K)$ on K_k is constant, furthermore it is maximally constant, in the sense that we cannot add any more elements to K_k. $\square$

As any $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra A has a preorder structure on it, given by $a \leq b \Leftrightarrow a \sqcup b = b$, one can ask if A also has a Ω-category structure. The same question can be asked for $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebras. The answer for both questions is yes and is given by the following.

Proposition 5.1.4. *Any $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra A has a Ω category structure given by*

$$A(a,b) = \bigvee \{ v \in \Omega \mid (v \star a) \leq b \},$$

for any $a, b \in A$. Also any $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra B has a Ω-category structure given by

$$B(b,b') = \bigvee \{ v \in \Omega \mid b \leq (v \rhd b') \},$$

for any $b, b' \in B$.

Proof. In order to show that A is a Ω-category then we have to show that for any $a, b, c \in A$ one has $I \leq A(a,a)$ and $A(a,b) \otimes A(b,c) \leq A(a,c)$.

Now using axiom 1 and axiom 6 one has $I \star a \sqcup a = a \sqcup a = a$, which means that $I \star a \leq a$ and as we take a join in Ω we have

$$I \leq \bigvee \{ v \in \Omega \mid v \star a \leq a \} = A(a,a).$$

Now let v and v' be such that $(v \star a) \leq b$ and $(v' \star b) \leq c$, which by definition is equivalent to $(v \star a) \sqcup b = b$ and $(v' \star b) \sqcup c = c$. Replacing b in last equation with $(v \star a) \sqcup b$ one obtains:

$$\begin{aligned} c &= (v' \star (v \star a \sqcup b)) \sqcup c \\ &= ((v' \star (v \star a)) \sqcup (v' \star b)) \sqcup c \\ &= (((v' \otimes v) \star a) \sqcup (v' \star b)) \sqcup c \\ &= ((v' \otimes v) \star a) \sqcup ((v' \star b) \sqcup c) \\ &= ((v \otimes v') \star a) \sqcup c \end{aligned}$$

Thus one has $((v \otimes v') \star a) \leq c$. Using that $\otimes$ preserves colimits, that is $\bigvee$, in both arguments one obtains:

$$\begin{aligned} A(a,b) \otimes A(b,c) &= \bigvee \{ v \in \Omega \mid (v \star a) \leq b \} \otimes \bigvee \{ v' \in \Omega \mid (v' \star b) \leq c \} \\ &= \bigvee \bigvee \{ v \in \Omega \mid (v \star a) \leq b \} \otimes \{ v' \in \Omega \mid (v' \star b) \leq c \} \\ &= \bigvee \bigvee \{ v, v' \in \Omega \mid ((v \otimes v') \star a) \leq c \} \\ &\leq \bigvee \{ v \in \Omega \mid (v \star a) \leq c \} \\ &= A(a,c) \end{aligned}$$

The proof for B is similar, and relies again on the fact that $\otimes$ preserves colimits. $\square$

One could ask why we do not define $A(a,b)$ as that $v \in \Omega$ such that $v \star a = b$, and the answer is because $\star$ is not injective in general. For that let us look again at $\Omega = [0,\infty]$, and let us note that $w \star \infty = \infty$ for all $w \in \Omega$, thus there is no unique $w \in \Omega$ to define $[0,\infty](\infty,\infty)$.

Example 5.1.5. Let us look at $\Omega = (([0,\infty] \geq), 0, +)$. Define $v \star a = v + a$ and $\bigsqcup_K(v_1,...,v_k) = \inf_{\mathbb{R}}(v_1,..v_k)$, thus Ω is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra. Let us check that the Ω-category structure given by Proposition 5.1.4 is the usual one. Let $a,b \in [0,\infty]$, then one has

$$\{v \in \Omega \mid v + a \geq_{\mathbb{R}} b\} = \{v \in \Omega \mid v \geq_{\mathbb{R}} b - a\}$$

Now obviously $[0,\infty](a,b) = b \dot{-} a = \inf\{v \in \Omega \mid v \geq_{\mathbb{R}} b-a\} = \bigvee\{v \in \Omega \mid v \geq_{\mathbb{R}} b - a\}$. Also let us note that $\bigwedge\{v \in \Omega \mid v \geq_{\mathbb{R}} b-a\} = \infty$

One has two equivalent definitions of a semi-lattice, one using operations and equations, and one saying that a semi-lattice is a complete/cocomplete poset. One could ask if this is true for $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle / \langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$- algebras. The answer is yes and it is given by the next theorem.

Theorem 5.1.6. *Let A be a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra and B a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra, then one has the following statements:*

1. *For any $v \in \Omega$ and $a,b \in A$ we have $A(v \star a, b) = \Omega(v, A(a,b))$. Thus $v \star a$ is the colimit of a weighted by v.*
2. *The operation $\bigsqcup_K$ is a coend, in the sense that for any cardinal K one has $A(\bigsqcup_K a_k, b) = \bigwedge_{k \in K} A(a_k, b)$.*
3. *For any $v \in \Omega$ and $a,b \in B$ we have $B(a, v \rhd b) = \Omega(v, A(a,b))$. Thus $v \rhd b$ is the limit of b weighted by v.*
4. *The operation $\bigsqcap_K$ is an end, in the sense that for any cardinal K one has $B(a, \bigsqcap_K b_k) = \bigwedge_{k \in K} B(a, b_k)$.*

Proof. 1. For any $a,b \in A$ and $v \in \Omega$ we have

$$\Omega(v, A(a,b)) = \bigvee\{w \in \Omega \mid v \otimes w \leq \bigvee\{v' \in \Omega \mid v' \star a \leq b\}\},$$

and using Axiom 2,

$$A(v \star a, b) = \bigvee\{v' \in \Omega \mid v' \star (v \star a) \leq b\} = \bigvee\{v' \in \Omega \mid (v' \otimes v) \star a \leq b\}.$$

We have to prove that every element in the first join is smaller than one element in the second one and that every element of the second join is also smaller than an element in the first one.

So let $v' \in \Omega$ such that $(v' \otimes v) \star a \leq b$ then obviously $v \otimes v' \leq \bigvee\{w \in \Omega \mid w \star a \leq b\}$.

For the other direction let $w \in \Omega$ such that $v \otimes w \leq \bigvee\{v' \in \Omega \mid v' \star a \leq b\}$, and let us note $\bar{v} = \bigvee\{v' \in \Omega \mid v' \star a \leq b\}$. As Ω is cocomplete then $\bar{v} \star a \leq b$ and using that $\star$ is monotone we have

$$(v \otimes w) \star a \leq \bar{v} \star a \leq b$$

thus $v \otimes w \in \{v' \in \Omega \mid v' \star a \leq b\}$.

2. To show that $A(\bigsqcup_K a_k, b) = \bigwedge_{k \in K} A(a_k, b)$ one has to prove that

$$\bigvee\{v \in \Omega \mid v \star (\bigsqcup_K a_k) \leq b\} = \bigwedge_{k \in K} \bigvee\{v \in \Omega \mid v \star a_k \leq b\},$$

for any cardinal K and any tuple $(a_1, .., a_k)$ in A^K.

For direction " $\leq$ " follows from, for all $k \in K$, and for all $v \in \Omega$ such that $v \star (\bigsqcup_K a_k) \leq b$ we have $v \star a_k \leq b$. Indeed using axioms 5 and 6 one has:

$$\begin{aligned}(v \star a_k) \sqcup b &= (v \star a_k) \sqcup ((v \star \bigsqcup_K a_k) \sqcup b) \\ &= (v \star a_k) \sqcup ((\bigsqcup_K (v \star a_k)) \sqcup b) \\ &= \bigsqcup_K (v \star a) \sqcup b \\ &= b\end{aligned}$$

So one has $\bigvee\{v \in \Omega \mid v \star (\bigsqcup_K a_k) \leq b\} \leq \bigwedge_{k \in K} \bigvee\{v \in \Omega \mid v \star a_k \leq b\}$.

Now to show the other inequality let us observe that for any $v \in \Omega$ such that $v \star a_k \leq b$ for all $k \in K$ then one also has $v \star (\bigsqcup_K a_k) \leq b$.

□

Thus any $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra is co-complete as a Ω-category, and any $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra is complete as a Ω-category.

Definition 5.1.7. For any two universal $\mathcal{D}$-algebras $(A, (v \star^A \_)_{v \in \Omega}, \bigsqcup_K^A)$ and $(B, (v \star^B \_)_{v \in \Omega}, \bigsqcup_K^B)$, a map $f : A \longrightarrow B$ is a morphism if f preserves all operations, that is if the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "v \star^A \_"] \arrow[d, "f"'] & A \arrow[d, "f"] & & A^K \arrow[r, "\bigsqcup_K^A"] \arrow[d, "f^K"'] & A \arrow[d, "f"] \\
B \arrow[r, "v \star^B \_"'] & B & & B^K \arrow[r, "\bigsqcup_K^B"'] & B
\end{tikzcd} \tag{5.1}$$

Theorem 5.1.8. *The category $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-alg of $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebras and their morphisms is isomorphic to the category of $\mathcal{D}$-algebras, and the category of $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebras and their morphisms is isomorphic to the category of $\mathcal{U}$-algebras.*

Proof. In order to prove the first statement let us define two functors, and show they are inverse to each other.

$$\begin{tikzcd}
\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle\text{-alg} \arrow[rr, bend left, "P"] & & \mathcal{D}\text{-alg} \arrow[ll, bend left, "Q", "\cong"']
\end{tikzcd} \tag{5.2}$$

Let $(A, (v \star \_)_{v \in \Omega}, (\bigsqcup_K)_K)$ be a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra, define $\alpha : \mathcal{D}A \longrightarrow A$ by

$$\alpha(\varphi) = \bigsqcup_{|A|} \varphi(a) \star a, \qquad \forall \varphi \in \mathcal{D}A.$$

We have to show that α is a Ω-functor and that it is a left adjoint to d_A. Indeed let $\psi, \varphi \in \mathcal{D}A$, we have to check that

$$\mathcal{D}A(\varphi, \psi) \leq A(\alpha(\varphi), \alpha(\psi)) = \bigvee \{ v \in \Omega \mid v \star (\bigsqcup_{|A|} \varphi(a) \star a) \leq \bigsqcup_{|A|} \psi(b) \star b \}.$$

It is enough to show that $\mathcal{D}A(\varphi, \psi) \star (\bigsqcup_{|A|} \varphi(a) \star a) \leq \bigsqcup_{|A|} \psi(b) \star b$. Using Axioms 3 and 2 one rewrites the left hand as

$$\bigsqcup_{|A|} ((\mathcal{D}A(\varphi, \psi) \otimes \varphi(a)) \star a) \leq \bigsqcup_{|A|} \psi(a) \star a.$$

And as $\forall a \in A$ we have $\mathcal{D}A(\varphi, \psi) \otimes \varphi(a) \leq \psi(a)$ the above inequality is true. Thus α is a Ω-functor.

In order for (A, α) to be a $\mathcal{D}$-algebra we need to show is that $\alpha \dashv d_A$, but that follows from Theorem 5.1.6 items 1 and 2.
Let $f : A \longrightarrow B$ be a morphism of $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebras, then as f preserves all operations f preserves colimits, thus f is also a morphism of $\mathcal{D}$-algebras.

Define

$$P((A, (v \star \_)_{v \in \Omega}, (\bigsqcup_K)_K) = (A, \alpha),$$

and

$$P(f) = f.$$

Let (A, α) be a $\mathcal{D}$-algebra then A is cocomplete. Let $K = \{*\}$ be a singleton set. For any $v \in \Omega$ and $a \in A$ let $\varphi : K^{\mathrm{op}} \longrightarrow \Omega$ and $F : K \longrightarrow \Omega$ be two Ω-functors such that $\varphi(*) = v$ and $F(*) = a$ then define

$$v \star a = \varphi \star F.$$

Now for any cardinal $\mathbf{K}$ let K be a discrete Ω-category of cardinal $\mathbf{K}$. For any element a_K of A^K define

$$\bigsqcup_K a_K = \mathbf{I_K} \star a_K,$$

where $\mathbf{I_K} : K^{\mathrm{op}} \longrightarrow \Omega$ is the constant Ω-functor given by $\mathbf{I_K}(k) = I$ for all $k \in K$.

In order for $Q(A, \alpha) = (A, (v \star -)_{v \in \Omega}, (\bigsqcup_K) K \textit{ cardinal})$ to be a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra, one needs to check the axioms of Definition 5.1.1.

1. Let $a, b \in A$ then one has $A(I \star a, b) = \Omega(I.A(a.b)) = A(a, b)$ thus $I \star a = a$.

2. Let $a, b \in A$ and $v, w \in \Omega$ then one has

$$\begin{aligned} A(v \star (w \star a), b) &= \Omega(v, A(w \star a, b)) \\ &= \Omega(v, \Omega(w, A(a, b))) \\ &= \Omega(v \otimes w, A(a, b)) \\ &= A((v \otimes w) \star a, b). \end{aligned}$$

   thus $v \star (w \star a) = (v \otimes w) \star a$.

3. Let K be any cardinal and $a_k \in A^K$, and $b \in A$ and $v \in \Omega$. Then $A(v \star \bigsqcup_K a_k, b) = \Omega(v, A(\bigsqcup_K a_k, b)) = \Omega(v, \bigwedge_{k\in K} A(a_k, b)) = \bigwedge_{k\in K} \Omega(v, A(a_k, b)) = \bigwedge_{k\in K} A(v \star a_k, b) = A(\bigsqcup_K (v \star a_k), b)$.

4. Let K be any cardinal and $a, b \in A$ and $v_k \in \Omega^K$. Then $A((\bigvee_K v_k) \star a, b) = \Omega(\bigvee_K v_k, A(a, b)) = \bigvee_K A(a_k \star a, b) = A(\bigsqcup_K (a_k \star a), b)$.

5. Follows from the fact that in Ω conical limits are associative. Again this is true because Ω is a quantale.

6. Let K be any cardinal and a, b be elements of A, then one has:

$$\begin{aligned} A(\bigsqcup_K \Delta a, b) &= A(\mathbf{I_K} \star \Delta a, b) \\ &= [K^{\mathrm{op}}, \Omega](\mathbf{I_K}, A(\Delta a, b)) \\ &= \bigwedge_{k\in K} \Omega(\mathbf{I_K}(k), A(\Delta a(k), b)) \\ &= \bigwedge_{k\in K} \Omega(I, A(a, b)) \\ &= \bigwedge_{k\in K} A(a, b) \\ &= A(a, b). \end{aligned}$$

   Thus $\bigsqcup_K \Delta a = a$.

7. Let J, K be two cardinals and $f : J \longrightarrow K$ a bijective function, and $a_K \in A^K$, and $b \in A$. One has:

$$\begin{aligned} A(\bigsqcup_J A^f(a_K), b) &= A(\mathbf{I_J} \star A^f(a_K), b) \\ &= \bigwedge_{j\in J} A(A^f(a_K)(j), b) \end{aligned}$$

   and

$$A(\bigsqcup_K a_K, b) = \bigwedge_{k\in K} A(a_K(k), b).$$

   All we have to show is that

$$\bigwedge_{j\in J} A(A^f(a_K)(j), b) = \bigwedge_{k\in K} A(a_K(k), b).$$

Now we have to show that this is indeed an isomorphism. If we start with a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra A then as we define colimits via it's operations and then we define the operations via colimits we have nothing to show, but if we start with a $\mathcal{D}$-algebra B we have to show that the Ω-category structure given by B being a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra is the same as the original Ω-category structure. So let (B, α_B) be a $\mathcal{D}$-algebra and let $(B, (v \star -)_{v \in \Omega}, (\bigsqcup_K = \bigvee_K)_{K\ cardinal})$ be it's corresponding $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra. Let us note with $B_{\sqcup}(a,b) = \bigvee \{ v \in \Omega \mid v \star a \leq b \}$. We have to show that $B(a,b) = B(a,b)_{\sqcup}$, which this is equivalent to

$$v \star a \leq b \Leftrightarrow v \leq B(a,b).$$

But that is the definition of a colimit.
The fact that the structure map is the same follows from direct calculation. □

5.2 Syntactic $\mathcal{DU}$-algebras

In order to make the definition of a $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra more readable we need some preliminary results.

Lemma 5.2.1. *Let* $(A, (v \star -)_{(v \in \Omega)}, (\bigsqcup_K)_K)$ *be a* $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$*-algebra and* $(A, (v \triangleright -)_{(v \in \Omega)}, (\bigsqcap_K)_K)$ *be a* $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$*-algebra. In particular* A *is a meet-semi lattice and join semi-lattice, so the order given by these is compatible if and only if we have the following two absorption axioms:*

1. $a \sqcap (a \sqcup b) = a$ *for all* $a, b \in A$
2. $a \sqcup (a \sqcap b) = a$ *for all* $a, b \in A$

Definition 5.2.2. By a distributive $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra we understand, a set A together with two unary family of operations $(v \star -)_{(v \in \Omega)} : A \longrightarrow A$ and $(v \triangleright -)_{(v \in \Omega)} : A \longrightarrow A$, and for each cardinal K two K-arity operations $\bigsqcup_K : A^K \longrightarrow A$ and $\bigsqcap_K : A^K \longrightarrow A$, such that $(A, (v \star -)_{(v \in \Omega)}, (\bigsqcup_K)_K)$ is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra and $(A, (v \triangleright -)_{(v \in \Omega)}, (\bigsqcap_K)_K)$ is a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra satisfying the following equations:

1. $a \sqcap (a \sqcup b) = a$ for all $a, b \in A$
2. $a \sqcup (a \sqcap b) = a$ for all $a, b \in A$
3. for any $v \in \Omega$ and any $a, b \in A$ one has $(v \star a) \leq b \Leftrightarrow a \leq (v \triangleright b)$

4. for any cardinal K and any $\varphi : K \longrightarrow \Omega$ and $G : K \times A \longrightarrow \Omega$

$$\bigsqcap_K \varphi(k) \rhd (\bigsqcup_A G(k)(a) \star a) = \bigsqcup_A f(a) \star a, \tag{5.3}$$

where $f : A \longrightarrow \Omega$ is given by $f(a) = \bigwedge_{k \in K} \varphi(k) \pitchfork\downarrow G(k)(a)$ where $\downarrow G(k) : A^{\mathrm{op}} \longrightarrow \Omega$ is given by $\downarrow G(k) = \mathrm{Lan}_i G = \int^{b \in A} A(-, b) \otimes G(k)(b)$ where $i :\mid A \mid \longrightarrow A^{\mathrm{op}}$ is the object inclusion functor.

Remark 5.2.3. 1. One can say that the distributive axiom still needs some work, in the sense that it is not entirely syntactic but the same is true about the complete distributivity axiom which uses choice functions. Furthermore the distribuive law written like that is a normal form axiom, and it comes from the fact that $\mathcal{DU}$-algebras are CCD, thus they are $\mathcal{D}$-algebras. So it makes sense that all terms to be written only using operations from $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$.

2. Let us not that the equivalence $\mathsf{CCD} \cong \Omega\text{-}\mathsf{Cat}^{\mathcal{DU}}$ plays a crucial role in the above definition, equation (5.3) is exactly the ccd property.

3. One needs to calculate f using the closure of G. Indeed, let us look at the following example:

   Let P be the following poset $P = \{0 \leq 1 \leq 2 \leq 3\}$, and let $K = \{1, 2\}$ be a two elements set. Then let $\psi : K \longrightarrow \mathbb{2}$ and $G : K \times P \longrightarrow \mathbb{2}$ be given by $\psi(1) = \psi(2) = 1$ and $G(1)(0, 1, 2, 3) = (0, 1, 0, 1)$ and $G(2)(0, 1, 2, 3) = (1, 0, 1, 0)$.

   Using infix notation we have

$$\begin{aligned}\bigsqcap_K \varphi(k) \rhd (\bigsqcup_A G(k)(a) \star a) &= (1 \rhd \bigsqcup_P (0, 1, 0, 1)) \sqcap (1 \rhd \bigsqcup (1, 0, 1, 0)) \\ &= (1 \rhd 3) \sqcup (1 \rhd 2) \\ &= 2\end{aligned}$$

   and on the other side we have

$$f(0) = \psi(0) \pitchfork G(0)(0) \wedge \psi(1) \pitchfork G(1)(0) = 0,$$

$$f(1) = \psi(0) \pitchfork G(0)(1) \wedge \psi(1) \pitchfork G(1)(1)) = 0,$$

$$f(2) = \psi(0) \pitchfork G(0)(2) \wedge \psi(1) \pitchfork G(1)(2) = 0,$$
$$f(3) = \psi(0) \pitchfork G(0)(3) \wedge \psi(1) \pitchfork G(1)(3) = 0.$$

Thus we have $\bigsqcup_{a \in P} f(a) \star a = 0$ which is clearly different than the other side.

Now if we would have taken the up closure, $\downarrow G$, of G in the calculation of f then $f(1) = f(2) = 1$ thus we would have had $\bigsqcup_{a \in P} f(a) \star a = 2$.

4. As it was shown in [8], and we will reprove it below, in the case $\Omega = \mathbb{2}$ our distributivity law is equivalent to the usual distributive law using choice functions.

Let $\Omega = \mathbb{2}$, then any completely distributive lattice is a distributive $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra and any distributive $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra is a completely distributive lattice.

Let L be a completely distributive lattice, then for any $\mathcal{L} \in \mathcal{P}(L)$ we have

$$\{\bigwedge\{\bigvee S \mid S \in \mathcal{L}\}\} = \bigvee\{\bigwedge\{T(s) \mid S \in \mathcal{L}\} \mid T \text{ choice function}\}. \tag{5.4}$$

Let $\mathcal{L} \in \mathcal{D}L$ then we have

$$\bigcap \mathcal{L} = \{\bigwedge\{T(s) \mid S \in \mathcal{L}\} \mid T \text{ choice function}\}.$$

Indeed, let $x \in \bigcap \mathcal{L}$ then x is in S for all S in $\mathcal{L}$. Now let T be the choice function that for any S takes x, then $\bigwedge T(S) = x$ thus $x \in \{\bigwedge\{T(s) \mid S \in \mathcal{L}\} \mid T \text{ choice function}\}$. For the other direction let $x \in \{\bigwedge\{T(s) \mid S \in \mathcal{L}\} \mid T \text{ choice function}\}$, then there exists a choice function T such that $x = \bigwedge\{T(S) \mid S \in \mathcal{L}\}$ and as each set S is down closed then $x \in S$ thus $x \in \cap \mathcal{L}$,

For every $\mathcal{L} \in \mathcal{P}L$ we have

$$\cap\{\downarrow S \mid S \in \mathcal{L}\} = \downarrow \{\wedge\{T(S) \mid T \mid S \in \mathcal{L}\} \text{ choice function}\}$$

Let $x \in \{\wedge\{T(S) \mid T \mid S \in \mathcal{L}\} \text{ choice function}\}$ then for each $S \in \mathcal{L}$ we have $x \leq T(S)$ thus $x \in \downarrow S$ for each S so $x \in \cap\{\downarrow S \mid S \in \mathcal{L}\}$. Now le $x \in \cap\{\downarrow S \mid S \in \mathcal{L}\}$ thus for each S there exists $y_s \in S$ such that $x \leq y_S$. Now define $T(S) = y_S$. Thus $x \in \downarrow \{\wedge\{T(S) \mid T \mid S \in \mathcal{L}\} \text{ choice function}\}$.

Proposition 5.2.4. *The* Ω*-category structure given by* A *being a* $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$*-algebra and a* $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$*-algebra are compatible, that is for all* $a, b \in A$ *we have* $\bigvee \{ v \in \Omega \mid v \star a \leq_{\sqcup} b \} = \bigvee \{ v \in \Omega \mid a \leq_{\sqcap} v \rhd b \}$

Proof. With the absorption rules defined above one shows that the order given by $\sqcup$ is equivalent with the order given by $\sqcap$.
Let a, b be elements of A, then one has $A_{\sqcup}(a, b) = \bigvee \{ v \in \Omega \mid v \star a \leq_{\sqcup} b \}$ and $A_{\sqcap}(a, b) = \bigvee \{ v \mid a \leq_{\sqcap} v \rhd b \}$, but as the order is the same and using Axiom 3, then one has $A_{\sqcup}(a, b) = A_{\sqcap}(a, b)$. ☐

Now let us show that $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ algebras are indeed $\mathcal{DU}$-algebras.

Theorem 5.2.5. *The category of distributive* $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$*-algebras is isomorphic to the category of* $\mathcal{DU}$*-algebras.*

Proof. Let (A, α, β) be a $\mathcal{DU}$-algebra, let us show that $(A, (v \star -)_{v \in \Omega}, (v \rhd -)_{v \in \Omega}, (\bigsqcup_K = \bigvee_K)_K, (\bigsqcap_K = \bigwedge_K)_K)$ is a distributive $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra.

We have to check the axioms of definitions 5.2.2, anything else has been checked for $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$ and $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebras.

The axiom $v \star \leq b \Leftrightarrow a \leq v \rhd b$ follows from $A(v \star a, b) = \Omega(v, A(a, b)) = A(a, v \rhd b)$.

First the idempotention rules. Let a, b in A and let us show that

$$A(-, a) \sqcup ((A(-, a) \sqcap A(-, b)) = A(-, a)$$

and

$$A(-, a) \sqcap ((A(-, a) \sqcup A(-, b)) = A(-, a).$$

That means that for any $c \in A$ one has $A(c, a) \sqcup ((A(c, a) \sqcap A(c, b)) = A(c, a)$, and its counterpart, but this is true as Ω is a lattice. Now as α preserves all weighted limits and colimits, in particular the special ones from above, one has $a = \alpha(A(-, a)) = \alpha(A(-, a) \sqcup ((A(-, a) \sqcap A(-, b))) = a \sqcup (a \sqcap b)$ and its counterpart. Thus A satisfies the idempotence axioms.

As α has a left, we have the following distributivity rule : for any cardinal K and any two Ω-functors $\varphi : K \longrightarrow \Omega$ and $G : K \times A^{\mathrm{op}} \longrightarrow \Omega$ we have

$$\bigsqcup_A f(a) \star a = \bigsqcap_K \varphi(k) \rhd (\bigsqcup_A G(k)(a) \star a), \tag{5.5}$$

where $f : A \longrightarrow \Omega$ is given by $f(a) = \bigwedge_{k \in K} \varphi(k) \pitchfork G(k)(a)$ But to have a $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra one needs to have the above distributive law for any map $F : K \times |A| \longrightarrow \Omega$ not only for Ω-functors. So let $F : K \times |A| \longrightarrow \Omega$ be such a map and let $\varphi : K \longrightarrow \Omega$ be a weight map.

For any $k \in K$ one has the $\downarrow F(k) : A^{\mathrm{op}} \longrightarrow \Omega$ given by

$$\downarrow F(k) = Lan_i F(k) = \bigvee_{b \in A} A(-, b) \otimes F(k)(b) \tag{5.6}$$

where $i : |A| \longrightarrow A^{\mathrm{op}}$ is the object inclusion functor.

Let us show that $\bigvee_{a \in A} F(k)(a) \otimes a = \bigvee_{a \in A} \downarrow F(k)(a) \otimes a$.

$$\begin{aligned} \bigvee_{a \in A} \downarrow F(k)(a) \otimes a &= \bigvee_{a \in A} (\bigvee_{b \in A} A(a, b) \otimes F(k)(b)) \otimes a \\ &= \bigvee_{a \in A} \bigvee_{b \in A} A(a, b) \otimes F(k)(b) \otimes a \\ &= \bigvee_{b \in A} F(k)(b) \otimes \bigvee_{a \in A} A(a, b) \otimes a \\ &= \bigvee_{b \in A} F(k)(b) \otimes b \end{aligned}$$

Now $\downarrow F$ is a Ω functor then we can apply the above distributive law:

$$\begin{aligned} \bigsqcap_K \varphi(k) \triangleright (\bigsqcup_A F(k)(a) \star a) &= \bigsqcap_K \varphi(k) \triangleright (\bigsqcup_A \downarrow F(k)(a) \star a) \\ &= \bigsqcup_A f(a) \star a \end{aligned}$$

where $f(a) = \bigwedge_{k \in K} \varphi(k) \pitchfork \downarrow F(k)(a)$.
Thus A satisfies axiom (5.3).

Let A be a distributive $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra, then A is a $\mathcal{DU}$-algebra. Indeed using axiom (5.3) one obtains that the structure map $\alpha : \mathcal{D}A \longrightarrow A$ given by $\alpha\varphi = \bigsqcup_{a \in A} \varphi(a) \star a$ has a left adjoint. $\square$

5.3 Applications and conclusion

First of all let us recall that in the case of $\Omega = \mathbb{2}$, $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebras are $\vee$-semi-lattices. Indeed, that is true as we have showed that $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebras are $\mathcal{D}$-algebras.

For any other quantale Ω, we have the the propositional Ω-logic is given by $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebras. That means that our logic language is given by:

$$\mathcal{L} : p \mid \ v \star - \mid \ v \triangleright - \mid \bigsqcup \mid \bigsqcap .$$

Where p are atomic propositions and $v \star -, v \triangleright -, \bigsqcup$, and $\bigsqcap$ are the operations defined in 5.1.1 satisfying the axioms from 5.1.1 and 5.2.2.

For example let us look at the final stream coalgebra on the metric space $\mathbb{R}$, given by the functor $F : \mathsf{Set} \longrightarrow \mathsf{Set}$ given by $F(X) = \mathbb{R} \times X$. Then the elements of the final coalgebra are infinite streams $a : \mathbb{N} \longrightarrow \mathbb{R}$. So one can ask: what is the distance between two streams, or if one stream is smaller than another one or if the join of two streams is smaller than another one, or even if a stream is a finite join of streams.

Now let us instantiate these syntactic algebras to different quantales. Also let us recall that for any Ω-category A one has operations $v \star -$, and $v \triangleright -$, satisfying equations 1-3 from Definition 5.1.1 if an only if A has singleton weighted colimits, and respectively, weighted limits, where by singleton we mean that the index category has just one object. Thus if we recall Example 2.3.2 we have:

Example 5.3.1.

First of all $\Omega = \mathbb{2}$. Let P be a set such that for any cardinal K we have operations $\bigsqcup_K$ and $\bigsqcap$, and for each $p \in P$ we have $0 \star p = \perp$, $1 \star p = p$, $0 \triangleright p = \top$, $1 \triangleright p = p$, where $\perp = \bigsqcap_P \mathrm{id}_P$ and $\top = \bigsqcup \mathrm{id}_P$, where id_P is the identity function on P. Furthermore if these operations satisfy the equations from Definition 5.1.1 then P is a $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$-algebra and a $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebra. In order for P to be a $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebra then P has to satisfy equation (5.3), which in item 4 of remark 5.2.3 was shown to be equivalent to P being completely distributive.

Example 5.3.2. In the case of $\Omega = [0, \infty]$ weights play a crucial role [29, Section 4]. Instead of only being allowed to take joins, we now also have operations $v \star -$, that is, $v + -)$, which allow us to add a constant $v \in \Omega$ to each element one takes the join over. Without this additional expressive power it would be impossible to express the notion of Cauchy limit as given in [29, Section 4]. For example, let us take a Cauchy sequence $s : \mathbb{N} \longrightarrow [0, \infty]$ such that $\lim_{n \to \infty} s_n = 1$ and such that there exists $n \in \mathbb{N}$ such that $s_n = 0$. Then without having the weights, that is, without the operations $v + -$, the colimit, as it was just an infinum, would be 0.

Example 5.3.3.

1. For $\Omega = \mathbb{2}$, a $\mathcal{D}$-algebra is a $\mathcal{DU}$-algebra if for all K, $\psi : K \longrightarrow \mathbb{2}$, and $S : K \longrightarrow \mathcal{D}A$ we have (4.16) becomes
$$\bigvee\{a \mid a \in \cap S\} = \cap\{\bigvee\{a \in F | F \in S\}\},$$
which is equivalent to complete distributivity [8].

2. Let $\Omega = [0, \infty]$ and consider (Ω, α, β) as a $\mathcal{DU}$-algebra. For any $G : \Omega^{\mathrm{op}} \longrightarrow \Omega$ and any $F : \Omega \longrightarrow \Omega$ we have
$$\alpha(G) = G \star \mathrm{id}_\Omega \ = \inf_{v \in \Omega} G(v) + v,$$
$$\beta(F) = \{F, \mathrm{id}_\Omega\} \ = \sup_{v \in \Omega} a \mathbin{\dot{-}} F(a).$$
The distributive law on $[0, \infty]$ is instantiated to: For any category K and any $\psi : K \longrightarrow \Omega$, $F : K \longrightarrow [\Omega^{\mathrm{op}}, \Omega]$ and $\varphi : \Omega^{\mathrm{op}} \longrightarrow \Omega$ we have
$$\inf_{v \in \Omega}(\sup_{k \in K} F(k)(v) \mathbin{\dot{-}} \psi(k)) + v \ = \ \sup_{k \in K}(\inf_{v \in \Omega} F(k)(v) + v) \mathbin{\dot{-}} \psi(k) \tag{5.7}$$

3. Let $\Omega = [0, \infty]$ and $R = \mathbb{R} \cup \{-\infty, \infty\}$ be the real numbers with the metric $R(a, b) = \ \mathsf{if}\ a \leq b\ \mathsf{then}\ 0\ \mathsf{else}\ a - b$. $R(-\infty, a) = 0$, $R(a, -\infty) = \infty$, $R(\infty, a) = \infty$, $R(a, \infty) = 0$, $R(\infty, \infty) = 0$, $R(\infty, -\infty) = \infty$, $R(-\infty, \infty) = 0$. Tensor and cotensor are defined by $R(v \otimes r, s) = \Omega(v, R(r, s))$ and by $R(r, v \pitchfork s) = \Omega(v, R(r, s))$. It follows $v \otimes r = r - v$ and $v \pitchfork r = v + r$ for any $v \in [0, \infty]$ and $r \in R$. But is it not a $\mathcal{DU}$-algebra, since in
$$\sup_{r \in R} r - (\inf_{k \in K} \psi(k) + F(k)(r)) = \inf_{k \in K} \psi(k) + (\sup_{r \in R} r - F(k)(r)) \tag{5.8}$$
with $K = \{*\}$ and $\psi(*) = \infty$, the left hand side is 0 while the right hand side is ∞.

4. For $[0, 1]$, we will look again only at $[0, 1]$ as an algebra, so let α and β be again the structure maps. And let $G : [0, 1]^{\mathrm{op}} \longrightarrow [0, 1]$ and $F : [0, 1] \longrightarrow [0, 1]$ be two maps then.
$$\alpha(G) = G \star \mathrm{id}_\Omega = \inf_{r \in [0,1]} \max(G(r), r),$$

$$\beta(F) = \{F, \mathrm{id}_\Omega\} = \sup_{r \in [0,1]} [0,1](F(r), r) = \sup_{r \in [0,1]} \{r \geq F(r)\}$$

The distributive law on $[0,1]$ is instantiated to: For any category K and any $\psi : K \longrightarrow [0,1]$, $F : K \longrightarrow [[0,1]^{\mathrm{op}}, [0,1]]$, and $\varphi : [0,1]^{\mathrm{op}} \longrightarrow [0,1]$ we have

$$\inf_{v \in \Omega} \max((\sup_{k \in K} [0,1](\psi(k), F(k)(v)), v) = \sup_{k \in K} [0,1](\psi(k), \inf_{v \in \Omega} \max(F(k)(v), v)) \tag{5.9}$$

Chapter 6

Finitary $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$ and $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebras

In this section we will inspect the subclass of $\langle \Sigma_{\mathcal{D}}, E_{\mathcal{D}} \rangle$, respectively $\langle \Sigma_{\mathcal{U}}, E_{\mathcal{U}} \rangle$-algebras, where the arity of $\bigsqcup$ and $\bigsqcap$ is finite. We will define the finitary monads, and show that in some cases there exists a distributive law between them.

6.1 Finitary monads

Let X be any Ω-category and let us define with $\mathcal{D}_{\mathsf{f}}X$ the full subcategory of $\mathcal{D}X$ whose objects are finite colimits of representables, and with $\mathcal{U}_{\mathsf{f}}X$ the full subcategory of $\mathcal{U}X$ whose objects are finite limits of representables. Thus $(\mathcal{U}_{\mathsf{f}}X)^{\mathrm{op}}$ is the full subcategory of $[X, \Omega]$ whose objects are finite colimits of representables.

We define $\mathcal{D}_{\mathsf{f}}$ on arrows as the restriction and corestriction of $\mathcal{D}$ on arrows.

$$\begin{tikzcd}
\mathcal{D}X \arrow[r, "\mathcal{D}h", "Lan_{d_X}(d_Y \circ h)"'] & \mathcal{D}Y \\
\mathcal{D}_{\mathsf{f}}X \arrow[u, hook] \arrow[r, "\mathcal{D}_{\mathsf{f}}h", "(\mathcal{D}f)|_{\mathcal{D}_{\mathsf{f}}X}"'] & \mathcal{D}_{\mathsf{f}}Y \arrow[u, hook] \\
X \arrow[u, "d_X"] \arrow[r, "h"'] & Y \arrow[u, "d_Y"']
\end{tikzcd} \tag{6.1}$$

Following 2.5.4 we have that $Lan_{d_X} d_Y \circ h$ preserves all colimits, thus $\mathcal{D}_{\mathsf{f}}$ is defined correctly.

Proposition 6.1.1. *There exists a natural transformation* $\lambda : \mathcal{D}_{\mathsf{f}} d \longrightarrow d\mathcal{D}_{\mathsf{f}}$

Proof. This follows from the definition of $\mathcal{D}_{\mathsf{f}}$ on arrows. Let X be any Ω-category then we have

$$\begin{tikzcd}
\mathcal{D}_{\mathsf{f}}X \arrow[r, bend left=10, "d\mathcal{D}_{\mathsf{f}}X"] \arrow[r, bend right=10, "\mathcal{D}_{\mathsf{f}}dX"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X \\
X \arrow[u, "dX"] \arrow[r, "dX"'] & \mathcal{D}_{\mathsf{f}}X \arrow[u, "d\mathcal{D}_{\mathsf{f}}"']
\end{tikzcd} \tag{6.2}$$

Indeed as $\mathcal{D}_{\mathsf{f}}dX = \mathrm{Lan}_{dX}\left(d\mathcal{D}_{\mathsf{f}}X \circ dX\right)$ and $d\mathcal{D}_{\mathsf{f}}X \circ dX = d\mathcal{D}_{\mathsf{f}}X \circ dX$ we have a unique natural transformation $\lambda : \mathcal{D}_{\mathsf{f}}dX \longrightarrow d\mathcal{D}_{\mathsf{f}}X$. ☐

Proposition 6.1.2. *The functor $\mathcal{D}_{\mathsf{f}}$ is a KZ-doctrine.*

Proof. As we already have $\lambda : \mathcal{D}_{\mathsf{f}}d \longrightarrow d\mathcal{D}_{\mathsf{f}}$, using 2.6.3 all we have to do is define a natural transformation $\mu : \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}} \longrightarrow \mathcal{D}_{\mathsf{f}}$ such that $\mu \circ d\mathcal{D}_{\mathsf{f}} = \mu \circ \mathcal{D}_{\mathsf{f}}d = \mathrm{id}_{\mathcal{D}_{\mathsf{f}}}$.

Let X be any Ω-category and let us define $\mu X : \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X \longrightarrow \mathcal{D}_{\mathsf{f}}X$ as

$$\mu X(F) = F \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}. \tag{6.3}$$

We have to show that this is correctly defined, in the sense that $\mu X(F)$ is a finite colimit of representables for any $F \in \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X$. For that let $F \in \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X$, then F is a finite colimit of reperesentables so let $\varphi \star G$ be its representation, where $\varphi : K^{\mathrm{op}} \longrightarrow \Omega$ is a weight and $G : K \longrightarrow \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X$ if given by $Gk =$

$\mathcal{D}_{\mathsf{f}}X(-, g_k)$ where g_k is an object of $\mathcal{D}_{\mathsf{f}}X$. Every g_k is a finite colimit of representables $\varphi_k \star D_k$, where for every $k \in K$ we have $\varphi_k : K_k \longrightarrow \Omega$ and $D_k : K_k \longrightarrow \mathcal{D}_{\mathsf{f}}X$ given by $D_k(k') = X(-, x_{kk'})$ for some $x_{kk'} \in X$. Thus μF is given by

$$\begin{aligned}
\mu X(F) &= F \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X} \\
&= (\varphi \star G) \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X} \\
&= \varphi \star (G - \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}) \\
&= \bigvee_{k \in K} \varphi(k) \otimes (G(k) \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}) \\
&= \bigvee_{k \in K} \varphi(k) \otimes (\mathcal{D}_{\mathsf{f}}X(-, g_k) \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}) \\
&\overset{(1)}{=} \bigvee_{k \in K} \varphi(k) \otimes g_k \\
&= \bigvee_{k \in K} \varphi(k) \otimes \Big(\bigvee_{k' \in Kk} \varphi_k(k') \otimes X(-, x_{kk'}) \Big) \\
&= \bigvee_{k \in K} \bigvee_{k' \in K_k} \varphi(k) \otimes \varphi_k(k') \otimes X(-, x_{kk'})
\end{aligned}$$

where step (1) follows from item 5 of Example 2.3.2.

And as K and each K_k is finite the above colimit is finite, thus μ is correctly defined.

Using again item 5 of Example 2.3.2 one obtains $\mu X \circ d\mathcal{D}_{\mathsf{f}}X = \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}$.

In order to prove $\mu X \circ \mathcal{D}_{\mathsf{f}} dX = \mathrm{id}_{\mathcal{D}_{\mathsf{f}}x}$ it is enough to show that $\mu \dashv d\mathcal{D}_{\mathsf{f}}X$. Indeed as dX is a dense functor and μ preserves left Kan extensions, we have

$$\begin{aligned}
\mathrm{id}_{\mathcal{D}_{\mathsf{f}}X} &= \mathrm{Lan}_{dX}\, dX \\
&= \mathrm{Lan}_{dX}\, (\mu X \circ d\mathcal{D}_{\mathsf{f}}X \circ dX) \\
&= \mu X \circ \mathrm{Lan}_{dX}\, (d\mathcal{D}_{\mathsf{f}}X \circ dX) \\
&= \mu X \circ \mathcal{D}_{\mathsf{f}} dX
\end{aligned}$$

$$\begin{tikzcd}
\mathcal{D}_{\mathsf{f}} \arrow[r, "\mathcal{D}_{\mathsf{f}} dX"] \arrow[rr, bend left, "\mathrm{Lan}_{dX}\, dX = \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}"] & \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X \arrow[r, "\mu X"] & \mathcal{D}_{\mathsf{f}}X \\
X \arrow[u, "d_X"] \arrow[r, "dX"'] & \mathcal{D}_{\mathsf{f}}X \arrow[u, "d\mathcal{D}_{\mathsf{f}}X"'] \arrow[ur, "\mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}"'] &
\end{tikzcd} \tag{6.4}$$

To prove $\mu X \dashv d\mathcal{D}_{\mathsf{f}}X$, let $F \in \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X$ and $\varphi \in \mathcal{D}_{\mathsf{f}}X$.

$$\begin{aligned}
\mathcal{D}_{\mathsf{f}}X(\mu X F, \varphi) &= \mathcal{D}_{\mathsf{f}}X(F \star \mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}, \varphi) \\
&= \{F, \mathcal{D}_{\mathsf{f}}X(\mathrm{id}_{\mathcal{D}_{\mathsf{f}}X}, \varphi)\} \\
&= \mathcal{D}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X(F, \mathcal{D}_{\mathsf{f}}X(-, \varphi))
\end{aligned}$$

Thus $(\mathcal{D}_{\mathsf{f}}, \mu, d)$ is a KZ-doctrine. $\square$

In a similar way one shows that $\mathcal{U}_{\mathsf{f}}$ is a co-KZ-doctrine.

Remark 6.1.3. Unlike the functor category $\mathcal{D}X$, its subcategory $\mathcal{D}_{\mathsf{f}}X$ is not complete in general, not even for finite limits. Indeed, let us look at the following examples.

Example 6.1.4.

1. First let us show that even for preorders, for some Ω-categories X, $\mathcal{D}_{\mathsf{f}}X$ is not complete. So let $\Omega = \mathbb{2}$, and let X be any infinite discrete poset, then $\mathcal{D}_{\mathsf{f}}X$ has no top element. Indeed, if it existed the top element of $\mathcal{D}_{\mathsf{f}}X$ would be X, but all elements of $\mathcal{D}_{\mathsf{f}}X$ are finite colimits of representables that is all elements of $\mathcal{D}_{\mathsf{f}}X$ are finite subsets of X.

2. Now let $\Omega = (([0,\infty] \geq), 0, +)$

   Let $\mathbb{N}^*$ be the set of positive natural numbers $\{1, 2, ...\}$ with the following metric

   $$\mathbb{N}^*(n, m) = \frac{1}{m} \dot{-} \frac{1}{n}.$$

   Let us show that in general $v \pitchfork \mathbb{N}^*(-, n)$ is not in $\mathcal{D}_{\mathsf{f}}\mathbb{N}^*$. Let n be any natural number, and let $v \in [0, \infty]$ such that $v < \frac{1}{n}$. We will show that $v \pitchfork \mathbb{N}^*(-, n)$ cannot be represented by a finite colimit of representables.

   As we have $v < n$ then there exists $m > n$ such that $v \leq \mathbb{N}^*(m, n)$ thus for all $n' > n$ we have

   $$v \pitchfork \mathbb{N}^*(n', n) > 0$$

   indeed let $m = \wedge\{a \in \mathbb{N}^* \mid a \geq \frac{n}{1-nv}\}$ that is the smallest natural number greater than $\frac{n}{1-nv}$. Now let $K' = \mathbb{N}/\{1, 2, 3, .., m-1\}$ that is the set of natural numbers greater than m, and for each $k \in K'$ define $a_k = v \pitchfork \mathbb{N}^*(k,n)$ then obviously we have

$$v \pitchfork \mathbb{N}^*(-,n) = \inf_{k \in K'} a_k \otimes \mathbb{N}^*(-,k).$$

We have to show that for each $k' > k$ we have

$$a_k + \mathbb{N}^*(k',k) > a_{k'}$$

and that

$$a_{k'} > a_k.$$

The first follows from

$$(v \pitchfork \mathbb{N}^*(k,n)) + \mathbb{N}^*(k',k) > v \pitchfork \mathbb{N}^*(k',n)$$

but as $v < \mathbb{N}^*(k,n)$ we have

$$(v \pitchfork \mathbb{N}^*(k,n)) + \mathbb{N}^*(k',k) = v \pitchfork (\mathbb{N}^*(k,n) + \mathbb{N}^*(k',k)) > v \pitchfork \mathbb{N}^*(k',n).$$

The second follows from the monotonicity of $a_k = \frac{k-n-nkv}{nk}$.

Let us show that one cannot express $v \pitchfork \mathbb{N}^*(-,n)$ as a finite colimit of representables. Let us suppose that there exists a finite set K and $\varphi : K \longrightarrow \Omega$ and $g : K \longrightarrow \mathbb{N}^*$ such that

$$v \pitchfork \mathbb{N}^*(-,n) = \inf_{k \in K} \varphi(k) \otimes \mathbb{N}^*(-,g(k)).$$

Using the results from above we have that for each $k \in K$ we have $\varphi(k) = a_{g(k)}$ and again using the results from above we have that if for all $m \geq \vee\{g(k) \mid k \in K\}$ we have

$$\inf_{k \in K} \varphi(k) \otimes \mathbb{N}^*(m, g(k)) > v \pitchfork \mathbb{N}^*(m,n).$$

Thus for every natural number n the limits of the form $v \pitchfork \mathbb{N}^*(-,n)$ where $v < \frac{1}{n}$ cannot be represented by a finite colimit of representables. Thus $\mathcal{D}_{\mathsf{f}}\, \mathbb{N}^*$ is not complete under finite limits.

We know that for some quantales, like $\mathbb{2}$, if the poset X is finitely complete then $\mathcal{D}_{\mathsf{f}}X$ is also finitely complete. So let us give some sufficient conditions for $\mathcal{D}_{\mathsf{f}}X$ to be finitely complete.

Proposition 6.1.5. *If $\Omega = (\Omega_0, \top, \wedge)$ is a quantale where $\otimes$ is $\wedge$ and the order is total and X is finitely complete then $\mathcal{D}_{\mathsf{f}}X$ is finitely complete as well.*

Proof. Because the tensor is a limit, and for any $a \in \Omega$ the co-tensor $(a \pitchfork -)$ is a right adjoint, the co-tensor $a \pitchfork -$ preserves the tensor. That is for any $a, b, c \in \Omega$ we have

$$a \pitchfork (b \wedge c) = (a \pitchfork b) \wedge (a \pitchfork c).$$

Now let K be any finite set and let $\psi : K \longrightarrow \Omega$ and $G : K \longrightarrow \mathcal{D}_{\mathsf{f}}X$ be any two Ω-functors. As, for each $k \in K$, $G(k)$ is a finite colimit of representables, let it be represented by $G(k) = \varphi_k \star X(-, g_k)$, where K_k is a finite set and $\varphi_k : K_k \longrightarrow \Omega$ and $g_k : K_k \longrightarrow X$ are Ω-functors.

In the following let us write $\kappa = \oplus_{k \in K} K_k$. A choice function f is any any function $f : K \longrightarrow \kappa$, such that $f(k) \in K_k$. Let $\Sigma = \{f : K \longrightarrow \kappa \mid \text{choice function}\}$ be the set of all choice functions.

Let x be any element of X. With the notations from above we have:

$$\begin{aligned}
\{\psi, G\}(x) &= \bigwedge_{k \in K} \psi(k) \pitchfork \bigvee_{k' \in K_k} \varphi_k(k') \wedge X(x, g_k(k')) \\
&\overset{1}{=} \bigvee_{f \in \Sigma} \bigwedge_{k \in K} \psi(k) \pitchfork (\varphi_k(f(k)) \wedge X(x, g_k(f(k)))) \\
&\overset{2}{=} \bigvee_{f \in \Sigma} \bigwedge_{k \in K} (\psi(k) \pitchfork \varphi_k(f(k))) \wedge (\psi(k) \pitchfork X(x, x_k(f(k)))) \\
&\overset{3}{=} \bigvee_{f \in \Sigma} \bigwedge_{k \in K} (\psi(k) \pitchfork \varphi_k(f(k))) \wedge X(x, \psi(k) \pitchfork x_k(f(k))) \\
&\overset{4}{=} \bigvee_{f \in \Sigma} \bigwedge_{k \in K} (\psi(k) \pitchfork \varphi_k(f(k))) \wedge \bigwedge_{k \in K} X(x, \psi(k) \pitchfork x_k(f(k))) \\
&\overset{5}{=} \bigvee_{f \in \Sigma} (\bigwedge_{k \in K} \psi(k) \pitchfork \varphi_k(f(k))) \wedge X(x, \bigwedge_{k \in K} \psi(k) \pitchfork x_k(f(k)))
\end{aligned}$$

where step 1 follows from Subsection 2.3.5, step 2 follows from $\psi(k) \pitchfork -$ being a right adjoint and thus preserving limits, steps 3 and 5 follow from the fact that Yoneda embedding preserves limits and X is finitely complete, step 4 represents the fact that $\wedge$ is commutative and associative.

Thus we have

$$\{\psi, G\} = \bigvee_{f \in \Sigma} \Big(\bigwedge_{k \in K} \psi(k) \pitchfork \varphi_k(f(k)) \Big) \wedge X(-, \bigwedge_{k \in K} \psi(k) \pitchfork x_k(f(k))).$$

Now, as K is finite, and each K_k is finite, then Σ is also finite and thus $\{\psi, G\}$ is a finite colimit of representables. Thus $\mathcal{D}_{\mathsf{f}}X$ is finitely complete. $\square$

6.1.1 The distributive law

We have two monads $\mathcal{D}_{\mathsf{f}}$ and $\mathcal{U}_{\mathsf{f}}$ in order for their composite to be again a monad we need a distributive law between them.

Proposition 6.1.6. *If for any Ω-category X, $\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}X$ is complete under all finite limits then the natural transformation $\delta : \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}} \longrightarrow \mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}$ given by $\delta = \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}(\mathcal{D}_{\mathsf{f}}\, u, -)$ is correctly defined.*

Proof. $\mathcal{D}_{\mathsf{f}}uX : \mathcal{D}_{\mathsf{f}}X \longrightarrow \mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}X$ is given by $\mathcal{D}_{\mathsf{f}}uX(\varphi) = \varphi \star [X, \Omega](X(?, -), -) = \bigvee_{x \in X} \varphi(x) \otimes [X, \Omega](X(x, -), -)$.
Let X be any Ω-category and let F be an object in $\mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X$. Then F is a finite limit of representables, so let K be a finite discrete Ω-category and $\psi : K \longrightarrow \Omega$ and $G : K \longrightarrow \mathcal{D}_{\mathsf{f}}X$ such that $F = \{\psi, \mathcal{D}_{\mathsf{f}}X(G, -)\}$.

$$\begin{aligned}
\delta X(F) &= \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X(\mathcal{D}_{\mathsf{f}}\, uX, F) \\
&= \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X(\mathcal{D}_{\mathsf{f}}\, uX, \{\psi, \mathcal{D}_{\mathsf{f}}X(G, -)\}) \\
&= \{\psi, \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X(\mathcal{D}_{\mathsf{f}}\, uX, \mathcal{D}_{\mathsf{f}}X(G, -))\} \\
&= \bigwedge_{k \in K} \psi(k) \pitchfork \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}X(\mathcal{D}_{\mathsf{f}}uX, \mathcal{D}_{\mathsf{f}}X(G(k), -)) \\
&= \bigwedge_{k \in K} \psi(k) \pitchfork [\mathcal{D}_{\mathsf{f}}X, \Omega](\mathcal{D}_{\mathsf{f}}X(G(k), -), \mathcal{D}_{\mathsf{f}}uX) \\
&= \bigwedge_{k \in K} \psi(k) \pitchfork \mathcal{D}_{\mathsf{f}}\, uX(G(k)) \\
&= \bigwedge_{k \in K} \psi(k) \pitchfork (G(k) \otimes [X, \Omega](X(?, -), -))
\end{aligned}$$

But this is a finite limit, and as $\mathcal{D}_{\mathsf{f}}X$ is complete under finite limits it is in $\mathcal{D}_{\mathsf{f}}X$. Thus δ is correctly defined. $\square$

As we have seen above in general one cannot expect $\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}X$ to be finitely complete, but Proposition 6.1.5 gives us examples of quantales where this happens.

Corollary 6.1.7. *If $\Omega = (\Omega_0, \top, \wedge)$ is a quantale where the tensor is the meet and the order is total we have the distributive law $\delta = \mathcal{U}_{\mathsf{f}}\mathcal{D}_{\mathsf{f}}(\mathcal{D}_{\mathsf{f}}\, u, -)$.*

Proof. Indeed following Proposition 6.1.5 and using the fact that for any Ω-category X, $\mathcal{U}_{\mathsf{f}}X$ is finitely complete we have that $\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}X$ is finitely complete. □

6.1.2 Algebras for the finitary monads

As $\mathcal{D}_{\mathsf{f}}$ is a KZ-doctrine and and $\mathcal{U}_{\mathsf{f}}$ is a co-KZ-doctrine, using Proposition 2.6.4, the structure map of these monads' algebras is an adjoint to their respective units. Thus one can calculate easily the structure map. As always for any Ω-category A we we will write d_A and u_A for the two Yoneda embedings. That is $d_A(a) = A(-,a)$ and $u_A(a) = A(a,-)$ for all $a \in A$.

So let (A, α) be a $\mathcal{D}_{\mathsf{f}}$ algebra, then $\alpha \dashv d_A$.

Proposition 6.1.8. *For any finitely Ω-category A, if the map $\alpha : \mathcal{D}_{\mathsf{f}}A \longrightarrow A$ given by $\alpha(\varphi) = \varphi \star \mathrm{id}_A$ exists, then it is a left adjoint to $d_A : A \longrightarrow \mathcal{D}_{\mathsf{f}}A$.*

Proof. We have to show that for any $a \in A$ and $\varphi \in \mathcal{D}_{\mathsf{f}}A$ we have

$$A(\alpha(\varphi), a) = \mathcal{D}_{\mathsf{f}}A(\varphi, d_A a). \tag{6.5}$$

If we start with the left hand side we have

$$\begin{aligned} A(\alpha(\varphi), a) &\overset{1}{=} A(\varphi \star \mathrm{id}_a, a) \\ &\overset{2}{=} \{\varphi, A(\mathrm{id}_A, a)\} \\ &\overset{3}{=} \mathcal{D}A(\varphi, A(\mathrm{id}_A, a)) \\ &\overset{4}{=} \mathcal{D}_{\mathsf{f}}A(\varphi, A(-,a)) \\ &\overset{5}{=} \mathcal{D}_{\mathsf{f}}A(\varphi, d_A a) \end{aligned}$$

where step (1) is the definition of α, step (2) is the preservation of limits of the Yoneda embedding, step (3) is the end formula of a limit in Ω, step (4) follows from the fact that $\mathcal{D}_{\mathsf{f}}A$ is a full subcategory of $\mathcal{D}A$, and that φ and $A(-,a)$ are objects of $\mathcal{D}_{\mathsf{f}}A$, and step (5) is a rewriting. □

Now we know that the objects of $\mathcal{D}_\mathsf{f}$ are finite colimits of representables, and that colimits involving representables have a special form. So let (A,α) be a $\mathcal{D}_\mathsf{f}$-algebra and let φ be any object of $\mathcal{D}_\mathsf{f}A$. Then $\varphi = \epsilon \star A(-,G)$ where K is a finite set and $\epsilon : K^{\mathrm{op}} \longrightarrow \Omega$ and $G : K \longrightarrow A$ are Ω-functors.

$$\begin{aligned}\alpha(\varphi) &= \varphi \star \mathrm{id}_A\\ &= (\epsilon \star A(-,G)) \star \mathrm{id}_A\\ &= \epsilon \star (A(-,G) \star \mathrm{id}_A)\\ &= \epsilon \star G\end{aligned}$$

Thus one can see that any $\mathcal{D}_\mathsf{f}$- algebra (A,α) is finitely cocomplete. Furthermore, any finitely coccomplete Ω-category is a $\mathcal{D}_\mathsf{f}$-algebra. Indeed, if A is finitely cocomplete then $\alpha : \mathcal{D}_\mathsf{f}A \longrightarrow A$ given by $\alpha(\varphi) = \varphi \star \mathrm{id}_A$ exists thus it is a left adjoint to d_A, and as such (A,α) is a $\mathcal{D}_\mathsf{f}$-algebra.

Now let (B,β) be a $\mathcal{U}_\mathsf{f}$ algebra. Then $u_A \dashv \beta$. In the same way like above we have the following description of β.

Proposition 6.1.9. *For any Ω-category B, if the map $\beta : \mathcal{U}_\mathsf{f}B \longrightarrow B$ given by $\beta(\psi) = \{\psi, \mathrm{id}_B\}$ exists, then it is a right adjoint to $u_A : A \longrightarrow \mathcal{U}_\mathsf{f}A$.*

Proof. We need to prove that for any $b \in B$ and $\psi \in \mathcal{U}_\mathsf{f}B$ we have

$$B(b,\beta(\psi)) = \mathcal{U}_\mathsf{f}B(u_B(b),\psi). \tag{6.6}$$

Again with some calculations we obtain

$$\begin{aligned}B(b,\beta(\psi)) &= B(b,\{\psi,\mathrm{id}_B\})\\ &= \{\psi, B(b,\mathrm{id}_B)\}\\ &= [B,\Omega](\psi, u_B(b))\\ &= \mathcal{U}B(u_B(b),\psi)\\ &= \mathcal{U}_\mathsf{f}B(u_B(b),\psi)\end{aligned}$$

□

Again as, for any $\mathcal{U}_\mathsf{f}$-algebra (B,β), the objects of $\mathcal{U}_\mathsf{f}B$ are finite limits of representables then β can be refined even more. So let ψ be represented by

$\{\epsilon, B(G,-)\}$, where K is a finite set and $\epsilon : K \longrightarrow \Omega$, and $G : K \longrightarrow B$ are two Ω-functors. We have

$$\begin{aligned}\beta(\psi) &= \{\{\epsilon, B(G,-)\}, \mathrm{id}_B\} \\ &= \{\epsilon, B(G,-) \star \mathrm{id}_B\} \\ &= \{\epsilon, G\}\end{aligned}$$

Thus as in the case of $\mathcal{D}_\mathsf{f}$-algebras, a $\mathcal{U}_\mathsf{f}$-algebra is finitely complete, and a Ω-category B is a $\mathcal{U}_\mathsf{f}$-algebra if and only if it is finitely complete.

Definition 6.1.10. A morphism between two $\mathcal{D}_\mathsf{f}$-algbebras $(A_1, \alpha_1), (A_2, \alpha_2)$ is any Ω-functor $h : A_1 \longrightarrow A_2$ such that $\alpha_2 \circ \mathcal{D}_\mathsf{f} h = h \circ \alpha_1$.

Unfortunately in the case of $\mathcal{D}_\mathsf{f}\mathcal{U}_\mathsf{f}$-algebras we could not show that they are ccd. This is mainly due to the fact that $\mathcal{D}_\mathsf{f} A$ is not complete in general, and thus one could not show easily the existence of the left adjoint from being a $\mathcal{D}_\mathsf{f}\mathcal{U}_\mathsf{f}$-algebra.

6.2 Syntactic $\mathcal{D}_\mathsf{f}$ and $\mathcal{U}_\mathsf{f}$-algebras

In the following we want to give a description with operations and equations of $\mathcal{D}_\mathsf{f}$ and $\mathcal{U}_\mathsf{f}$-algebras. As $\mathcal{D}_\mathsf{f}$ and $\mathcal{U}_\mathsf{f}$ are finite version of $\mathcal{D}$, and respectively $\mathcal{D}$, then one can think of these algebras as $\langle \Sigma_\mathcal{D}, E_\mathcal{D} \rangle$ and $\langle \Sigma_\mathcal{U}, E_\mathcal{U} \rangle$-algebras where we restrict the cardinal of $\bigsqcup$ and $\bigsqcap$ to finite. Also because of the iteration of finite limits, see Subsection 2.3.4, it is enough to restrict the cardinal of $\bigsqcup$, and respectively $\bigsqcap$, to two, thus making them binary operations.

Definition 6.2.1. By a $\langle \Sigma_{\mathcal{D}_\mathsf{f}}, E_{\mathcal{D}_\mathsf{f}} \rangle$-algebra we understand a set A together with a family of unary operations $(v \star \_)_{v \in \Omega} : A \longrightarrow A$ indexed by Ω, and a binary operation $\sqcup : A \times A \longrightarrow A$, satisfying the following 6 axioms. Dually the notions of a $\langle \Sigma_{\mathcal{U}_\mathsf{f}}, E_{\mathcal{U}\mathsf{f}} \rangle$-algebra is given by a set B together with a family of unary operations $(v \triangleright \_)_{v \in \Omega} : B \longrightarrow B$ and a binary operation $\sqcap_K : B \times B \longrightarrow B$ satisfying the following 6 axioms. In the following both $\sqcup$ and $\sqcap$ will be written in infix notation.

1. $I \star - = \mathrm{id}_A$ $\qquad$ $I \triangleright \_ = \mathrm{id}_A$

2. For all $a \in A$, $b \in B$ and $v, w \in \Omega$

$$v \star (w \star a) = (v \otimes w) \star a \qquad\qquad v \rhd (w \rhd b) = (v \otimes w) \rhd b$$

$$\begin{tikzcd}
\Omega \times \Omega \times A \arrow[r, "\mathrm{id}_\Omega \times \_\star\_"] \arrow[d, "\otimes \times \mathrm{id}_A"'] & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega \times A \arrow[r, "\_\star\_"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times \Omega \times B \arrow[r, "\mathrm{id}_\Omega \times \_\rhd\_"] \arrow[d, "\otimes \times \mathrm{id}_B"'] & \Omega \times B \arrow[d, "\_\rhd\_"] \\
\Omega \times B \arrow[r, "\_\rhd\_"'] & B
\end{tikzcd}$$

3. For all $v \in \Omega$ and $a_1, a_2 \in A$, $b_1, b_2 \in B$

$$v \star (a_1 \sqcup a_2) = (v \star a_1) \sqcup (v \star a_2) \qquad\qquad v \rhd (b_1 \sqcap b_2) = (v \rhd b_1) \sqcap (v \rhd b_2)$$

$$\begin{tikzcd}
\Omega \times A \times A \arrow[rr, "\mathrm{id}_\Omega \times \sqcup"] \arrow[d, "\Delta \times \mathrm{id}_A \times \mathrm{id}_A"] & & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega^2 \times A^2 \arrow[r, "(\_\star\_) \times (\_\star\_)"'] & A \times A \arrow[r, "\sqcup"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times B \times B \arrow[rr, "\mathrm{id}_\Omega \times \sqcap"] \arrow[d, "\Delta \times \mathrm{id}_B \times \mathrm{id}_B"] & & \Omega \times B \arrow[d, "\_\rhd\_"] \\
\Omega^2 \times B^2 \arrow[r, "(\_\rhd\_) \times (\_\rhd\_)"'] & B \times B \arrow[r, "\sqcap"'] & B
\end{tikzcd}$$

4. For all $a \in A$, $b \in B$ and $v_k \in [K, \Omega]$

$$(\textstyle\bigvee_K v_k) \star a = \sqcup(v_K \star a) \qquad\qquad (\textstyle\bigvee_K v_k) \rhd b = \sqcap(v_K \rhd b)$$

$$\begin{tikzcd}
\Omega^K \times A \arrow[rr, "(\bigvee_K) \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A^K \times \Delta"'] & & \Omega \times A \arrow[d, "\_\star\_"] \\
\Omega^K \times A^K \arrow[r, "(\_\star\_)^K"'] & A^K \arrow[r, "\sqcup"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^K \times B \arrow[rr, "(\bigvee_K) \times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_B^K \times \Delta"'] & & \Omega \times B \arrow[d, "\_\rhd\_"] \\
\Omega^K \times B^K \arrow[r, "(\_\rhd\_)^K"'] & B^K \arrow[r, "\sqcap"'] & B
\end{tikzcd}$$

5. For all $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$ we have

$$(a_1 \sqcup a_2) \sqcup a_3 = a_1 \sqcup (a_2 \sqcup a_3) \qquad\qquad (b_1 \sqcup b_2) \sqcup b_3 = b_1 \sqcup (b_2 \sqcup b_3)$$

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\sqcup \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_B \times \sqcup"] & A \times A \arrow[d, "\sqcup"] \\
A \times A \arrow[r, "\sqcup"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times B \times B \arrow[r, "\sqcap \times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_B \times \sqcap"] & B \times B \arrow[d, "\sqcap"] \\
B \times B \arrow[r, "\sqcap"'] & B
\end{tikzcd}$$

6. For all $a \in A$ and $b \in B$

$$a \sqcup a = a \qquad b \sqcap b = b$$

$$\begin{tikzcd}
A \times A \arrow[r, "\sqcup"] & A \\
A \arrow[u, "\Delta"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times B \arrow[r, "\sqcap"] & B \\
B \arrow[u, "\Delta"] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

7. For all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have

$$a_1 \sqcup a_2 = a_2 \sqcup a_1 \qquad b_1 \sqcap b_2 = b_2 \sqcap b_1$$

$$\begin{tikzcd}
A \times A \arrow[r, "T_A"] \arrow[dr, "\sqcup"'] & A \times A \arrow[d, "\sqcup"'] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times B \arrow[r, "T_B"] \arrow[dr, "\sqcap"'] & B \times B \arrow[d, "\sqcap"'] \\
& B
\end{tikzcd}$$

where $T_A : A \times A \longrightarrow A \times A$ and $T_B : B \times B \longrightarrow B \times B$ are twists isomorphism.

As in all the proofs of Section 5.1 we never used any cardinality arguments all the proofs we have done there transport immediately here. Thus we will not restate them. We have that $\langle \Sigma_{\mathcal{D}_\mathsf{f}}, E_{\mathcal{D}_\mathsf{f}} \rangle$-algebras and $\langle \Sigma_{\mathcal{U}_\mathsf{f}}, E_{\mathcal{U}\mathsf{f}} \rangle$-algebras are Ω-categories, and they are finitely cocomplete, and respectively finitely complete. Furthemore they are $\mathcal{D}_\mathsf{f}$, and respectively $\mathcal{U}_\mathsf{f}$-algebras.

Also let us note that finitary does not necessarily mean finite, for example the poset $\mathbb{Q}$ of rational numbers is an example of a finitary lattice, but it is not a lattice which admits arbitrarily large joins. Indeed, we have that every real number is a colimit of all the rational numbers smaller than it, so if $\mathbb{Q}$ would admit arbitrarily large joins then all irrational numbers should be part of $\mathbb{Q}$.

6.2.1 Conclusion

So in this Chapter we have showed that the monads $\mathcal{D}$ and $\mathcal{U}$ restrict to finitary versions $\mathcal{D}_\mathsf{f}$, and respectively $\mathcal{U}_\mathsf{f}$, and that in some cases there exists a distributive law between them. We have also defined finitary version of the algebras $\langle \Sigma_\mathcal{D}, E_\mathcal{D} \rangle$, and respectively $\langle \Sigma_\mathcal{U}, E_\mathcal{U} \rangle$, but not of $\langle \Sigma_{\mathcal{D}\mathcal{U}}, E_{\mathcal{D}\mathcal{U}} \rangle$-algebras. That is because in the definition of $\langle \Sigma_{\mathcal{D}\mathcal{U}}, E_{\mathcal{D}\mathcal{U}} \rangle$-algebras we defined the distributive law (5.3) using the fact that $\mathcal{D}\mathcal{U}$-algebras are ccd. But we do not have the same description of $\mathcal{D}_\mathsf{f}\mathcal{U}_\mathsf{f}$-algebras. Thus the definition $\mathcal{D}_\mathsf{f}\mathcal{U}_\mathsf{f}$-algebras as algebras given on a set with operations and equations still needs work.

The reason why at this moment we have not defined a finitary version of $\langle \Sigma_{\mathcal{D}\mathcal{U}}, E_{\mathcal{D}\mathcal{U}} \rangle$-algebras is because, $\mathcal{D}_\mathsf{f}\mathcal{U}_\mathsf{f}$-algebras are not ccd, and thus we could not find an equation like (5.3).

Chapter 7

Conclusions and future work

We have shown in Theorem 5.2.5 that for any commutative quantale Ω the category Ω-Cat of Ω-categories, or, in other words, the category of Ω-valued generalised metric spaces, is isomorphic to a category of algebras for operations and equation in the usual sense, if we admit operations of infinite unbounded arity.

Moreover, due to the duality underlying our approach, these operations have a logical interpretation and the equations can be seen as logical axioms.

The value of Theorem 5.2.5 resides not only in its statement but even more so in how we proved it: We didn't guess $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ and then proved the theorem, but we derived $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ in a systematic fashion from the functor $[-, \Omega]$. We started from the aim to derive the logic of Ω-valued predicates, that is, the logic given implicitly by the structure of the categories $[X, \Omega]$. To extract this logical structure, we considered $[X, \Omega]$ as algebras for the monad induced by $[-, \Omega]$. We then employed a result linking that monad to the 'semi-lattice' monads $\mathcal{D}$ and $\mathcal{U}$. The algebraic structure of these monads computes limits and colimits and an equational description of these was given as $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$.

It lies in the nature of this method that the logic $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ we derived from Ω is not purely syntactic but still depends on Ω. The operations are infinitary and the laws contain side conditions depending on Ω. We can think of Ω as an oracle that we need to consult in our reasoning. Restricting to particular, syntactically given Ω and then describing $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$ fully syntactically, so that consulting the oracle can be replaced by asking an automated theorem prover, is a task of future research.

We have defined algebras on a set with operations and equations that are equivalent to the algebras for the composite monad $\mathcal{DU}$, and even restricted them to a finitary version, so we have an answer to the question: what is the logic of a quantale Ω. Now there are many ways we can continue this research. Let us walk through some of them.

As we mentioned in the conclusion of the previous chapter until now we could not define algebras on a set with operations and equations for the monad $\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}$, so this is the first thing we want to investigate.

We defined algebras on a set with operations and equations so one would like to build some algebraic constructions on it, like subalgebras, and maybe some generalization of filters and ideals. Of course from a categorical point of view filters and ideals are just objects of the sheaf and presheaf categories, but still would be nice to define them syntactically.

If we have a distributive law between the monads $\mathcal{D}_{\mathsf{f}}$ and $\mathcal{U}_{\mathsf{f}}$ that means that $\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}$ is a monad and even more for any Ω-category X we have

$$\mathcal{D}_{\mathsf{f}}\mathcal{U}_{\mathsf{f}}X = [[X,\Omega]_f,\Omega]_f,$$

where by $[X,\Omega]_f$ we mean the subcategory of $[X,\Omega]$ whose objects are a finite colimit of representables. Now one can ask if

$$[-,\Omega]_f \dashv [-,\Omega]_f.$$

Having syntactic algebras we can add contravariant operations, such as implication. In this sense we can try using frames, like in the work of [11], [9] and then define operations like in [7]. For this we could use that every bimodule $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ generates a closure operator.

$$\begin{tikzcd}
{[X^{\mathrm{op}},\Omega]} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & {[Y,\Omega]^{\mathrm{op}}} \arrow[ll, bend left] \\
 & & \\
X \arrow[uu, "d_X"] \arrow[uurr, "\alpha"] & & Y \arrow[uu, "u_Y"'] \arrow[uull]
\end{tikzcd} \tag{7.1}$$

Also we want to research the connection with MV-algebras, especially for $[0,1]$-algebras. For more on MV-algebras, see [6]. One has to notice that the signatures of our algebras and MV-algebras are different, in the sense that MV-algebras have contravariant operations, this is one of the reason we want to investigate the addition of contravariant operations to our setting.

On top of the equational logic given by $\langle \Sigma_{\mathcal{DU}}, E_{\mathcal{DU}} \rangle$-algebras and the equational calculus for the finitary algebras one would also like a proof calculus like Gentzen systems, or Hilbert's natural style deduction. For that I suppose one could notice that the at the basis of all these systems lies an entailment relation and one could replace this relation by a bimodule.

In [2] the authors studied the coalgebraic logic for enriched categories. That is in (1.1) they replaced Set by Ω-Cat, and showed that , under some conditions, modalities can still be defined.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[loop below, "L^{\mathrm{op}}"] & & \Omega\text{-}\mathsf{Cat} \arrow[ll, bend right=20, "{[-,\Omega]}"'] \arrow[ll, bend left=20, "{[-,\Omega]}"] \arrow[ll, phantom, shift right=2, "\top"] \arrow[loop below, "T"]
\end{tikzcd}$$

Now we can improve these results by replacing Ω-$\mathsf{Cat}^{\mathrm{op}}$ with Ω-$\mathsf{Cat}^{\mathcal{DU}}$, and thus adding modal operators on top of our propositional logic.

$$\begin{tikzcd}
\Omega\text{-}\mathsf{Cat}^{\mathcal{DU}} \arrow[loop below, "L^{\mathrm{op}}"] & & \Omega\text{-}\mathsf{Cat} \arrow[ll, bend right=20, "{[-,\Omega]}"'] \arrow[ll, bend left=20, "{[-,\Omega]}"] \arrow[ll, phantom, shift right=2, "\top"] \arrow[loop below, "T"]
\end{tikzcd}$$

Thus there is still a lot of work to be done in this area, and we hope this is just the beginning.

Bibliography

[1] J. Beck. Distributive laws. In B. Eckmann, editor, *Seminar on Triples and Categorical Homology Theory*, volume 80 of *Lecture Notes in Mathematics*, pages 119–140. Springer Berlin Heidelberg, 1969.

[2] M. Bílková, A. Kurz, D. Petrisan, and J. Velebil. Relation lifting, with an application to the many-valued cover modality. *Logical Methods in Computer Science*, 9(4), 2013.

[3] M. M. Bonsangue, F. van Breugel, and J. J. M. M. Rutten. Generalized metric spaces: Completion, topology, and power domains via the yoneda embedding. *Theor. Comput. Sci.*, 193(1-2):1–51, Feb. 1998.

[4] F. Borceaux. *Handbook of Categorical Algebra 1*, volume 1. Cambridge University Press, 1994.

[5] D. D. Borceux, Francis. Cauchy completion in category theory. *Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques*, 27(2):133–146, 1986.

[6] R. Cignoli, I. d'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*. Trends in Logic. Springer Netherlands, 1999.

[7] J. Dunn. Gaggle theory: An abstraction of galois connections and residuation, with applications to negation, implication, and various logical operators. In J. van Eijck, editor, *Logics in AI*, volume 478 of *Lecture Notes in Computer Science*, pages 31–51. Springer Berlin Heidelberg, 1991.

[8] B. W. Fawcett and R. J. Wood. Constructive complete distributivity i. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1990.

[9] M. Gehrke. Generalized kripke frames. 2005.

[10] G. Grätzer and B. Davey. *General Lattice Theory.* Springer, 2003.

[11] C. Hartonas and J. M. Dunn. Stone duality for lattices. *Algebra Universalis*, 37:391–401, 1997.

[12] D. Hofmann. Duality for distributive space, 2010. arXiv:1009.3892v1.

[13] G. M. Kelly and V. Schmitt. Notes on enriched categories with colimits of some class. *Theory Appl. Categ.*

[14] M. Kelly. *Basic Concepts of Enriched Category Theory.*

[15] A. Kock. Monads for which structures are adjoint to units. *Journal of Pure and Applied Algebra*, 104(1):41 – 59, 1995.

[16] A. Kurz and J. Rosický. Strongly complete logics for coalgebras. July 2006.

[17] H. Lai and D. Zhang. Many-Valued Complete Distributivity. 2006.

[18] F. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, XLIII, 1973. Republished in *Reprints in Theory Appl. Categ.*

[19] R. Lowen. *Approach Spaces: The Missing Link in the Topology-uniformity-metric Triad.* Oxford mathematical monographs. Clarendon Press, 1997.

[20] S. Maclane. *Categories for the Working Mathematician.* Springer, 1971.

[21] E. G. Manes. *Algebraic Theories.* Springer, 1976.

[22] F. Marmolejo. Doctrines whose structure forms a fully faithful adjoint string. *Theory Appl. Categ*, 3:24–44, 1997.

[23] F. Marmolejo, R. Rosebrugh, and R. J. Wood. A basic distributive law. *Journal of Pure and Applied Algebra*, 168(2):209–226, 2002.

[24] Q. Pu and D. Zhang. Categories Enriched Over a Quantaloid: Algebras. *Theory and Applications of Categories*, 2015.

[25] N. Rescher. *Many-Valued Logic.* 1969.

[26] G. Rosolini. A note on cauchy completeness for preorders.

[27] J. J. M. M. Rutten. Elements of generalized ultrametric domain theory. *Theor. Comput. Sci.*, 170(1-2):349–381, 1996.

[28] J. J. M. M. Rutten. Relators and metric bisimulations (extended abstract). In *CMCS'98*, volume 11, 1998.

[29] J. J. M. M. Rutten. Weighted colimits and formal balls in generalized metric spaces. *Topology and its Applications*, 89, 1998.

[30] R. Street. Cauchy characterization of enriched categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 51(1):217–233, 1981.

[31] I. Stubbe. Towards "dynamic domains": Totally continuous cocomplete q-categories. *Theor. Comput. Sci.*, 373(1-2):142–160, Mar. 2007.

[32] I. Stubbe. The double power monad is the composite power monad. Technical report, LMPA, Université du Littoral-Côte d'Opale, Sept. 2013.

[33] S. Vickers. The double powerlocale and exponentiation: A case study in geometric reasoning. *Theory and Applications of Categories*, 12:372–422, 2004.

[34] S. Vickers. Localic completion of generalized metric spaces I. *Theory and Applications of Categories*, 14:328–356, 2005.

[35] S. Vickers. Localic completion of generalized metric spaces II: Powerlocales. *Journal of Logic and Analysis*, 1(11):1–48, 2009.

[36] S. Vickers and C. Townsend. A universal characterization of the double powerlocale. *Theoretical Computer Science*, 316:297–321, 2004.

[37] R. Walters. Sheaves on sites as cauchy-complete categories. *Journal of Pure and Applied Algebra*, 24(1):95 – 102, 1982.

[38] J. Worrell. Coinduction for recursive data types: partial order, metric spaces and Omega-categories. In *CMCS'00*, volume 33 of *ENTCS*, 2000.