



## **Essays in Game Theory**

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To my mother and father, who taught me to play C

# Essays in Game Theory

by

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## Abstract

This thesis explores interactions among agents whose rationality is bounded in distinct ways. It consists of three self-contained chapters/papers. Chapters 2 and 3 consider myopic and hard-wired strategy revisions based on evolutionary game dynamics, while Chapter 4 deals with rationally inattentive agents who acquire costly information in a flexible manner. The thesis, thus, spans two extremes of the range of models with boundedly rational agents.

The first paper proposes a novel way to formalize matching mechanisms in evolutionary games. The proposed formalization nests group selection models such as the haystack (Maynard Smith, 1964) and trait-group models (Wilson, 1975). It is shown that evolutionary optima can be obtained as Nash equilibria under appropriately defined matching rules.

In the second paper matching rules are endogenized and the co-evolution of cooperation and matching is studied in social dilemma situations. It turns out that only full-or-null assortativity levels are evolutionarily stable. The extent to which efficient outcomes are achieved by this endogenization process is evaluated, which crucially depends on the structure of the particular interaction considered.

The third paper extends recent models of flexible information acquisition to an uncountable-action-space setting: a beauty contest coordination game. Necessary conditions for the existence of equilibria with well-behaved strategies are derived. It is established that affine equilibria exist only if the fundamental is normally distributed. A higher coordination motive, a more concentrated prior distribution of the fundamental and higher information costs lead to less attention being paid to the fundamental. Moreover, flexible information acquisition technology is shown to result in equilibrium multiplicity under certain parameter combinations.

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# Declaration

Chapter 2 is joint work with Martin Kaae Jensen. A version of this chapter has appeared under the title “Evolutionary Games with Group Selection” in the *Discussion Papers in Economics* series (number 14/09), Department of Economics, University of Leicester. Versions of this chapter were also presented at the following conferences:

- UECE Lisbon Meetings 2012: Game Theory and Applications, ISEG/Technical University of Lisbon (under the title “Evolutionary Games and Matching Rules”)
- Norms, Actions, Games 2014, King’s College (poster presentation)
- Controversies in Game Theory: Homo Oeconomicus vs. Homo Socialis (2014), ETH Zürich
- PhD Conference in Economics (2014), University of Leicester (poster presentation)
- Royal Economic Society PhD Meetings & Job Market (2015), UCL.

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- Controversies in Game Theory II: Social Preference Theory and Mechanism Design (2015), ETH Zürich.

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# List of Abbreviations

- AAE** Aggregately Affine Equilibrium
- BRC** Best Reply Correspondence
- ESS** Evolutionarily Stable Strategy
- ESSGS** Evolutionarily Stable Strategy with Group Selection
- HD** Hawk-Dove
- MHD** Missing Hero Dilemma
- NEGS** Nash Equilibrium with Group Selection
- PD** Prisoners' Dilemma
- PDF** Probability Density Function
- SH** Stag Hunt
- SMFE** Smooth, Monotone, Full-support Equilibrium
- UD** Underprovision Dilemma
- VD** Volunteer's Dilemma

# Chapter 1

## Introduction

Perfect reasoning of agents is among the most usual assumptions in economics. Agents are commonly assumed to be making optimal decisions after careful deliberation while fully using any information to which they have access. This thesis examines settings with agents whose rationality is bounded in two distinct ways: the sophistication of their decision-making process and their information-processing capacity. The different models are suitable to analyze situations that differ in their informational intensity and time scale.

On the one hand, Chapters 2 and 3 study evolutionary environments where agents are hard-wired to behave as their genes dictate. In evolutionary models, unsuccessful behavior is rooted out by the forces of natural selection. They are designed to explore what types of behavior can survive under evolutionary competition over long time periods. They can also serve in providing evolutionary underpinnings for observed behavior.

On the other hand, Chapter 4 is about agents who are able to make sophisticated decisions and always use best responses to the actions of others. The environment is one of incomplete information where agents have limitations in their information-processing capacity. They can acquire more information but this comes at a cost. These models are designed to analyze short-term and information-heavy situations such as asset markets where, even though the fundamental value of the asset is not accurately known to market participants, they can spend resources to gather information about it.

**Evolutionary models** In evolutionary environments, agents are considered to be hard-wired to behave as their genes dictate. Following the spirit of Maynard Smith and Price (1973), the evolutionary process is modeled in a game-theoretic manner. The different behaviors encoded in the genes are viewed as strategies in a normal-form game whereas the biological fitness (expected number of offspring) of the a-

gents is represented by a (symmetric) payoff matrix. Instead of being best-responders, as in conventional game theory, players do not carefully consider their decisions. The focus is on which gene would survive in the long run and what behavior it codes for, as opposed to finding mutual best responses.

It is well-known (see for example Weibull, 1995) that if agents meet uniformly randomly, the outcomes of such processes – termed Evolutionarily Stable Strategies – are the strategies followed by the players in symmetric Nash equilibria of the underlying normal-form game. In this way, the notion of Nash equilibrium (and with it self-serving behavior) gains evolutionary support. Hence, it seems impossible to offer prosocial behavior – encountered in humans and other animals – an evolutionary underpinning. Several authors have proposed mechanisms that can give rise to cooperative behavior. These include kin selection (Hamilton, 1964a,b), local interactions (Eshel, Samuelson, and Shaked, 1998; Nowak and May, 1992), and homophily (Alger and Weibull, 2012, 2013). What all these mechanisms have in common is that they match individuals in a manner that is non-(uniformly) random.

Chapter 2 proposes a novel way to formalize nonrandom matching in evolutionary settings. This formalization extends nonrandom matching (see for example Bergström, 2003, 2013) to any  $n$ -player  $m$ -strategy underlying normal-form game. Based on this, new concepts for the study of evolutionary models under nonrandom matching are introduced. These concepts, namely Nash Equilibrium with Group Selection and Evolutionarily Stable Strategy with Group Selection are shown to be generalizations of the concepts of Nash Equilibrium and Evolutionarily Stable Strategy, respectively. They extend the scope of their counterparts to situations where individuals do not meet in a uniformly random manner but by following arbitrary matching rules that match them in groups according to their types.

It is shown that group selection models such as the haystack model of Maynard Smith (1964) and the trait-group model of Wilson (1975) lead to special cases of matching rules and can therefore be studied using the newly introduced concepts. Moreover, a welfare theorem is provided that states that any evolutionary optimum can become a Nash Equilibrium with Group Selection if an appropriate matching rule is in place.

Chapter 3 builds on the results of Chapter 2. It asks the questions of how particular matching rules can come to existence and how different social-dilemma environments can lead to different matching rules evolving. In this fashion, the match-

ing rules that were assumed to be exogenous in Chapter 2 are endogenized. Thus, the matching process itself is co-evolving along with the cooperation level of the population. The matching rule follows a constant index of assortativity form as described in Bergström (2003) with the index of assortativity being determined dynamically. Players vote on whether to increase or decrease the assortativity level of the matching process based on previously obtained payoffs.

It turns out that only full or null assortativity levels are evolutionarily stable, depending on the strategic nature of the dilemma at hand. The resulting evolutionarily stable levels of pro-social behavior are accordingly determined. The degree to which the social dilemma – and the resulting tragedy of the commons – can be overcome is evaluated using an invasion-barrier approach. The cases of social dilemmas where democratic consensus is able to significantly improve long-run efficiency are underprovision dilemmas and those volunteer's dilemmas in which two volunteers can generate close-to-efficient outcomes. At the other end of the spectrum, in missing hero dilemmas and volunteer's dilemmas where two volunteers meeting is far from efficient, voting is not able to lift the population away from zero cooperation levels, thus leading to inefficient outcomes. Prisoner's dilemmas rank in-between the two aforementioned cases.

**Rational inattention/ information acquisition** The rational inattention literature has considered situations where individuals are constrained in their ability to process information. Sims (1998, p. 320), who pioneered this field, found that prices can be sticky in highly heterogeneous markets if “individuals have many things to think about and limited time” and therefore “can devote only limited intellectual resources to [...] tasks of datagathering and analysis.” It is, thus, apparent that such “weaknesses” can affect economic outcomes. In a recent paper, Yang (2015) studies an investment game where players make a binary decision – to invest in an asset or not. Yang relaxes Sims's assumption of a fixed informational constraint. Introducing a “flexible information acquisition” technology, he allows players to endogenously decide *how much* information (which is costly in time and resources) they want to acquire. As in the work of Sims, they can also choose *what kind* of information they want to acquire; depending on how relevant they assess each event to be.

In a closely related literature, economists have been exploring the effects of different information structures in the way markets aggregate information.<sup>1</sup> This is of high importance as there may be cases where prices cease to reflect the fundamental value of assets. One example of this are stock market bubbles. The model that has been mostly used for these analyses is that of a beauty contest. In beauty contests players aim to take actions close to (i) the realization of a random variable (the fundamental) and, (ii) to the average action of the population. Their payoff is decreasing in the squared distances from these two values, which makes beauty contests games with strategic complementarities. The only public type of information is a common prior about the fundamental. Subsequently, players may privately receive one or more signals (depending on the information structure of the considered model), update their prior and take actions. Information structures that have been analyzed are usually rigid in the sense that the signals follow particular functional forms that may not necessarily be optimal from the players' point of view. Thus, they may be transmitting (partially) redundant information.

Chapter 4 brings the two strands of literature together. It explores how economies aggregate information when it is difficult for market participants to process information but they are given the freedom to choose on which events they want to concentrate. It studies a beauty contest in which players use Yang's (2015) flexible information acquisition technology which is generalized to continuous action spaces.

Results show that higher coordination motives, a more concentrated prior of the fundamental, or higher information costs lead to players paying less attention and thus acquiring less information. In addition, the structure of flexible information acquisition settings is found to be the source of the existence of multiple equilibria. This can happen even when there is a unique equilibrium in the full-information version of the game at hand, as is the case in beauty contests.

In the following three chapters the ideas that were laid out here are described in detail. Chapter 5 concludes and provides suggestions for future research.

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<sup>1</sup>See for example Allen, Morris, and Shin (2006), Angeletos and Pavan (2004, 2007), Morris and Shin (2002), and Myatt and Wallace (2012) among others.

## Chapter 2

# Evolutionary Games with Group Selection

### Chapter Abstract

This paper introduces two new concepts in evolutionary game theory: Nash equilibrium with Group Selection (NEGS) and Evolutionarily Stable Strategy with Group Selection (ESSGS). These concepts generalize Maynard Smith and Price (1973) to settings with arbitrary matching rules, in particular they reduce, respectively, to Nash equilibrium and ESS when matching is random. NEGS and ESSGS are to group selection models (Kerr and Godfrey-Smith, 2002; Bergström, 2002) what Nash Equilibrium and ESS are to the standard replicator dynamics: any NEGS is a steady state, any stable steady state is a NEGS, and any ESSGS is asymptotically stable. We also show that all steady states of any haystack/trait-group model (Maynard Smith, 1964; Wilson, 1977; Cooper and Wallace, 2004) are steady states of a group selection model under an appropriately defined matching rule. We proceed to prove what may be called “the second welfare theorem of evolution:” any evolutionary optimum will be a NEGS under some matching rule. Our results are illustrated in a range of Prisoners’ Dilemma games.

## 2.1 Introduction

The canonical evolutionary game theory model of Maynard Smith and Price (1973) plays an important role in biology, economics, political science, and other fields. Its equilibrium concept, an *evolutionarily stable strategy* (ESS) describes evolutionary outcomes in environments where populations are *large* and matching is *random*.<sup>1</sup> Since an ESS is a refinement of Nash equilibrium, it obviously *cannot* explain any behavioral departure from purely self-serving behavior in the one-shot Nash sense. In particular it cannot account for cooperative behavior in say, a prisoners' dilemma, or shed light on altruism more generally, nor can it account for any other non-Nash behaviors such as spite (Hamilton, 1970; Alger and Weibull, 2012) or costly punishment (Fehr and Gächter, 2000).

In order to explain such deviations from Nash behavior, evolutionary game theory turned to models with a finite number of agents hence departing from the first of the mentioned conditions of Maynard Smith and Price (1973). Thus in Schaffer (1988), the finite set of individuals have “market power” and can influence average fitness while making simultaneous decisions (playing the field). In the model preferred by Maynard Smith (1982) – namely repeated games – a few agents, usually just two, can perfectly monitor and record each others' past actions and condition their strategies hereupon (in evolutionary theory, the repeated games approach is usually referred to as *direct reciprocity*). Both of these frameworks have led to an enormous body of research in economics and game theory (see *e.g.* Alós-Ferrer and Ania, 2005; Leininger, 2006; Samuelson, 2002; Vega-Redondo, 1997, and references therein).

While evolutionary *game theorists* turned to finite populations, evolutionary biologists more broadly devoted as much – if not more – attention to a departure from the second basic condition of Maynard Smith and Price (1973), namely the assumption that matching is random. When matching is non-random – possibly indirectly so due to prolonged interaction of individuals in separated groups (Maynard Smith, 1964) – the fitness of an individual will depend on the group he is assigned to, and so different groups will on average meet with varying reproductive success (Kerr and Godfrey-Smith, 2002; see also Bergström, 2002). Thus non-random match-

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<sup>1</sup>Intuitively, random matching means that an individual's type has no influence on what type of individual he is likely to be matched to.

ing invariably leads to *group selection* whereby one can trace the evolutionary success of certain types of groups and not just their constituent individuals (we return to this topic in a moment). Take the prisoners' dilemma. Matching is assortative if after each round of play cooperators have higher probability of being matched to other cooperators than to defectors. This is often a highly realistic assumption corresponding for example to situations where a large group of individuals cannot perfectly monitor each others' past behaviors but receive some "revealing signals" about opponents' types and exert some influence on with whom they are matched (Wilson and Dugatkin, 1997; Bergström, 2003) or if they are matched according to a "meritocratic matching" process in the sense of Nax, Murphy, and Helbing (2014b). Non-random matching also results if matching depends on the geographical location of individuals (Eshel, Samuelson, and Shaked, 1998; Nowak and May, 1992; Skyrms, 2004); or if (genetically) similar individuals match assortatively as in models of kin selection (Hamilton, 1964b; Alger and Weibull, 2010). When matching is non-random a variety of different *groups* will generally coexist at any given moment in time. For example in the prisoners' dilemma, some groups will consist of defectors only, some of cooperators only, and some will be mixed. Thus the average fitness will differ across groups, as will the fitness a specific type of individual obtains if he is placed into different groups. It follows that evolutionary pressure takes place not just at the individual level but also at the group level even though individuals are ultimately the fitness bearing entities.<sup>2</sup>

Now, the existing literature on non-random matching is usually informal and/or deals only with special cases (typically two types who are matched pairwise and assortatively). As a basis for this paper's main results, we begin in section 2.2 by laying out a unified model in a general and self-contained manner. Compared to existing literature, we add value by setting up a model that allows for arbitrary matching rules (ways to match populations into groups), any number of strategies, arbitrary group sizes, and any possible payoff structure in the group stages (*i.e.*, any possible underlying symmetric normal form game, see section 2.2.1). The most substantial

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<sup>2</sup>Kerr and Godfrey-Smith (2002) show that one may, with equal formal correctness, think of selection taking place at the individual or the group level. This difference in perspective has been (and is) the topic of a heated debate in evolutionary biology, a key reference here being the book "Unto Others" by Sober and Wilson (1999). As explained in section 2.2 we are going to take a so-called "individualist" perspective in this paper, and will not go into the more philosophical aspects of the levels of selection controversy.

of the mentioned generalizations is that we allow for arbitrary matching rules. Indeed, this is what allows us to show (section 2.2.4) that group selection models based on prolonged interaction in what Maynard Smith (1964) calls “haystacks” (see also Wilson, 1975 and Cooper and Wallace, 2004), can be recast as models of group selection based directly on non-random matching (e.g. Bergström, 2003; Kerr and Godfrey-Smith, 2002). Intuitively, this is not all that surprising from the individualist perspective described above: What matters at the end of the day is whether individuals are matched randomly with each other or not. Precisely how any departure from random matching comes about is secondary to the fact that as soon as matching is non-random, different groups with different average fitness levels will exist and this is ultimately what group selection is all about. In fact, we shall from now on make this viewpoint explicit by not separating models of non-random matching from other models of group selection.

The first substantial contribution of the present paper is to fill the resulting gap in the literature. Specifically, we are going to ask what game theoretic equilibrium concepts form group selection’s natural parallels to Nash equilibrium and ESS. This leads to two new equilibrium concepts, namely a *Nash equilibrium with group selection* (NEGS) and an *evolutionarily stable strategy with group selection* (ESSGS). These concepts turn out to be intuitive once the underlying evolutionary game, which we call a *group selection game*, is understood. Interestingly, this game turns out to be novel even from a game theoretic perspective: as in standard imperfect information games, agents make decisions without knowing with certainty the strategies pursued by opponents – all they know is the distribution of the opponents’ strategies, or to put it in the evolutionary terminology, the probabilities of ending up in any of the different kinds of groups. Crucially, these probabilities depend on the actual strategies pursued by the agents. For simplicity, imagine a large group of individuals, each of whom has a choice between two strategies, “honesty” (H) or “deception” (D). Agents must commit to a strategy before being allocated into equal-sized groups where they execute these strategies (equivalently, they choose their actions with imperfect knowledge about opponents’ actions). Given a specific *matching rule* (a way to divide a population with a given fraction of *H*- and *D*-types into groups of equal size) and given that agents *know* the population-wide composition into *H* and *D* types, each agent can calculate the probabilities of ending up in any specific kind of group as a function of the specific strategy chosen (*H* or

$D$ ).<sup>3</sup> In a NEGS, individuals' optimal choices precisely lead to the population-wide composition into  $H$  and  $D$  types which formed the basis for their decisions in the first place. Intuitively, in a NEGS the deceptive individuals' purpose is to keep the honest individuals in check (and vice versa): without a sufficiently large population of deceivers, the benefit of choosing to deceive will outweigh that of being honest because deceptive individuals will face a relatively small chance of being matched to another deceiver *even though* matching is assortative. The concept of an ESSGS simply adds a "non-invasion" criterion to this Nash/fixed point criterion precisely as is the case with random matching (Maynard Smith and Price, 1973).

After defining group selection games and proving equilibrium existence, we turn to the relationship with the dynamic evolutionary model of section 2.2. Thus in theorem 2.4 we prove that any NEGS is a steady state for the replicator dynamics, that any (Lyapunov) stable steady state for the replicator dynamics is a NEGS, and that any ESSGS is an asymptotically stable state of the replicator dynamics (see Ritzberger and Weibull, 1995). These results extend existing results on Nash equilibrium and ESS (Hofbauer and Sigmund, 1998; Maynard Smith and Price, 1973; Weibull, 1995) to settings with non-random matching, and show that NEGS and ESSGS are important new evolutionary game theory concepts. Immediately, a long list of research questions report themselves in that one could attempt to "transfer" over to group selection models all of the existing results from evolutionary game theory. We shall leave the bulk of this for future research, for example we are not going to go into topics related to neutrally stable strategies, asymptotically stable sets, doubly symmetric games or the fundamental theorem of natural selection (for these "text-book" issues see the monographs of Hofbauer and Sigmund, 1998 or Weibull, 1995). Instead we are going to focus in section 2.4 on a question which in some sense "ignited" this whole literature. The point of the prisoners' dilemma is that Nash equilibrium – and with it evolutionary models based on random matching – may easily fail to produce outcomes that maximize average payoff/welfare in the population.<sup>4</sup> The question from the group selection point of view then becomes: What types of

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<sup>3</sup>Obviously, the number of possible group compositions depends on the group size as well as the number of strategies. With two strategies and groups of size two, any individual can end up in precisely two different kinds of groups – one where the opponent is of the same type and one where he is not.

<sup>4</sup>In the language of welfare analysis, the outcome does not maximize *utilitarian* social welfare. This, of course, also implies a break-down of *Pareto optimality*.

(non-random) matching *will*, if any, lead to optimality? Our main result in this regard (theorem 2.5) might be called the “second welfare theorem of evolutionary theory” telling us that *any* outcome that is optimal will in fact be a NEGS under *some* matching rule.

The structure of the paper is as follows: section 2.2 describes the general group selection model and section 2.3 defines group selection games, NEGS and ESSGS. Section 2.3 also contains our main theoretical results. Section 2.4 discusses the fitness/welfare issues with basis in the aforementioned “second welfare theorem” and 2.5 contains a number of applications. Finally, section 2.6 concludes.

## 2.2 Group Selection in Evolutionary Theory

In this section we present a unified model of non-random matching based group selection. The model is closely related to Kerr and Godfrey-Smith (2002) and Bergström (2003) both of which consider non-random matching as the primus motor of group selection, and both of which adopt an “individualist perspective” that assigns fitness to individuals rather than the groups they form.<sup>5</sup> While their analysis restricts attention to two strategies/types and certain relatively restrictive types of closed form group formation rules, we allow for any number of strategies and, more importantly, arbitrary rules of group formation (called *matching rules* in what follows). The latter is crucial when in section 2.2.4 we go on to show that any trait-group model – where matching is random but groups are isolated for prolonged spells in what Maynard-Smith calls “haystacks” – can be recast within our setting with non-random matching in such a way that the equilibria/steady states and dynamics remain the same. This observation substantially extends the scope of our general results, and it also dispels the notion that group selection models based on non-random matching are somehow not “true” models of group selection (certainly, any difference will be at most a question of interpretation and terminology – any result about observables, *i.e.*, dynamics and equilibria will remain the same).<sup>6</sup>

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<sup>5</sup>As shown by Kerr and Godfrey-Smith (2002), one can formally recast such models so that groups become the fitness bearing entities (so the two frameworks are formally equivalent). The group-based fitness perspective is strongly advocated in the famous book “Unto Others” by Sober and Wilson (1999). See also Maynard Smith (1998) and Okasha (2005) for more on this issue.

<sup>6</sup>It is also worth mentioning that from a more technical perspective, our model is crafted so that the main structure of the traditional evolutionary game theory model (*e.g.* Weibull, 1995) is retained.

Briefly, the model can be summarized as follows: At each date there is a large set of individuals, formally the continuum  $I = [0, 1]$ . At the beginning of each period, the agents are allocated into groups of the same finite size  $n \in \mathbb{N}$ . This happens in accordance with what we call a *matching rule* (formally defined in subsection 2.2.2) which is a function that maps the type frequency of the set of agents into the distribution of group types.<sup>7</sup> After the  $n$ -sized groups are formed, the individuals in each group face an  $n$ -player symmetric normal-form game (section 2.2.1). In accordance with the basic premise of evolutionary game theory, agents are hard-wired to follow the same strategy as the parent ('like begets like'). Thus an individual who is fathered by a parent who executed strategy  $j$ , say, in the previous round will mechanically execute strategy  $j$  in his group game, regardless of the resulting payoff/composition of individuals in the specific group he is drawn into. The payoff determines the *fitness*, *i.e.*, the (expected) number of children the agent will send on to the next round.<sup>8</sup> Finally, after the group game stage, a new generation is born with the relative proportion of each type determined by the success (fitness) this type's strategy enjoyed across the different groups. The above process then repeats itself leading to a new generation and so on. The evolutionary outcome of this group selection process is a steady state of the resulting replicator dynamical system as described in section 2.2.3.

### 2.2.1 The underlying normal form group games

Our description begins with the underlying normal form game that agents face in the group stages. Although in evolutionary models, individuals act purely mechanically and play the strategy inherited from the parent, they nonetheless participate in a standard normal form game and receive payoffs/fitnesses accordingly. We need to make this game theoretic aspect clear to set the stage for this paper's main results.

Let  $n \in \{2, 3, \dots\}$  denote the *group size* so that  $N = \{1, \dots, n\}$  is the *set of play-*

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This both makes the connection with standard replicator dynamics transparent and paves the way for our analysis in subsequent sections.

<sup>7</sup>Our concept of a matching rule is closely related to a construction due to Kerr and Godfrey-Smith (2002, p.484) who, however, consider only the case of two strategies (the extension to any number of strategies is non-trivial as will become clear).

<sup>8</sup>A different explanation of fitness that is more plausible in economic contexts is to think of it as the number of agents copying one's behavior because it is more successful: More successful behaviors will have more followers in the next round of play.

ers in a group. A group game is a symmetric normal form game  $G = \langle N, M, A \rangle$  where  $M = \{1, \dots, m\}$  is the set of pure strategies and  $A : M \times M^{n-1} \rightarrow \mathbb{R}$  is the payoff function over pure strategies that all players share. Following Dasgupta and Maskin (1986), a symmetric game has the property that  $A(y^i, y^{-i}) = A(y^i, \tilde{y}^{-i})$  where  $y^i \in M$  is any pure strategy for player  $i$ , and  $y^{-i}, \tilde{y}^{-i} \in M^{n-1}$  are pure strategy profiles of  $i$ 's opponents where  $\tilde{y}^{-i}$  is any permutation of  $y^{-i}$ . A (symmetric) *Nash equilibrium* for  $G$  is defined in the usual way as a vector  $\sigma^* \in S_m \equiv \{\sigma \in \mathbb{R}_+^m : \sum_{j=1}^m \sigma_j = 1\}$  such that  $A(\sigma^*, (\sigma^*, \dots, \sigma^*)) \geq A(\sigma, (\sigma^*, \dots, \sigma^*))$  for all  $\sigma \in S_m$ .<sup>9</sup>

It is convenient to write the previous payoff structure in a way that makes explicit reference to the group structure. Call an individual who executes pure strategy  $j \in M$  a *type  $j$  individual*. Due to symmetry, the payoff to such a type  $j$  individual depends only on the *number* of opponents in his group who play each of the  $m$  strategies (as opposed to *which* opponents follow what strategies). Next imagine that this type  $j$  individual finds himself in a *group*, group  $i$  say, consisting of  $n_1^i$  individuals of type 1,  $n_2^i$  individuals of type 2, and so on up to  $n_m^i$ .<sup>10</sup> In this situation, the individual's payoff will be equal to  $A(j, j^{\text{opp}})$  where  $j^{\text{opp}} \in M^{n-1}$  is any vector of opponents' strategies which contains  $n_1^i$  strategy 1 entries,  $\dots$ ,  $n_{j-1}^i$  strategy  $j-1$  entries,  $n_j^i - 1$  strategy  $j$  entries,  $n_{j+1}^i$  strategy  $j+1$  entries,  $\dots$ ,  $n_m^i$  strategy  $m$  entries. Crucially, we can write the payoff  $A(j, j^{\text{opp}})$  simply as  $A_j^{(n_1^i, \dots, n_m^i)}$  or even as  $A_j^i$  where  $i$  is the index of the specific group the individual finds himself in (as long as we keep record of the group composition  $n^i = (n_1^i, \dots, n_m^i)$  of group  $i$ ).

In this way, we can capture all of the information we need about the normal form game in a sequence  $(A_j^i)$  where  $j = 1, \dots, m$  and  $i = 1, \dots, \gamma_{n,m}$ . Here  $\gamma_{n,m}$  is the *number of different  $n$ -sized groups that can be formed with  $m$  different pure strategies*.<sup>11</sup> From combinatorics we know that  $\gamma_{n,m}$  precisely equals the number of multisets of cardinality  $n$  with elements taken from a set with cardinality  $m$  (see Aigner, 2007, p.

<sup>9</sup>Letting  $\sigma^i \in S_m \equiv \{\sigma \in \mathbb{R}_+^m : \sum_{j=1}^m \sigma_j = 1\}$  denote a *mixed strategy* for player  $i$  and  $\sigma^{-i} \in S_m^{n-1}$  denote a mixed strategy profile of player  $i$ 's opponents, it is easy to see that  $A(\sigma^i, \sigma^{-i}) = \sum_{y \in M^n} A(y^i, y^{-i}) \prod_{k \in N} \sigma_{y^k}^k$ .

<sup>10</sup>Note that since the individual himself is counted here, we necessarily have  $n_j^i \geq 1$  (there is at least one of the individual's own type). Of course we must also have  $\sum_k n_k^i = n$  and each  $n_k^i$  must be non-negative.

<sup>11</sup>Of course, we must be a little careful here because some of these are not really properly defined. Specifically,  $A_j^i$  is not well-defined unless  $n_j^i \geq 1$ . But building this explicitly into the notation leads to unwarranted complications.

15), *i.e.*

$$\gamma_{n,m} = \frac{(n+m-1)!}{n!(m-1)!} = \binom{n+m-1}{m-1}. \quad (2.1)$$

For example,  $\gamma_{2,2} = 3$  since three different groups can be formed if the group size equals 2 and there are 2 possible strategies (these groups are, respectively, one where both are of type 1, one where both are of type 2, and one where the individuals follow different strategies).

## 2.2.2 Group formation

We now turn to the question of how groups are formed out of each generation's individuals. The key concept is that of a *matching rule* which generalizes what Kerr and Godfrey-Smith (2002, p. 484) call a “rule of group assembly” to more than 2 pure strategies (the concept is also related to Bergström, 2003, as returned to below).

A *population strategy* is the frequency distribution of the different types in the population, *i.e.*, a vector  $\mathbf{x} = (x_1, \dots, x_m) \in S_m$  where the typical element  $x_j$  is the fraction of  $j$ -strategists in the population. A *group state* similarly represents the group frequencies and so is a vector  $\mathbf{g} = (g_1, g_2, \dots, g_{\gamma_{n,m}}) \in S_{\gamma_{n,m}}$  where the typical element  $g_i$  is the fraction of all groups that is of type  $i$ .

A *matching rule* is simply a function that maps a population strategy  $\mathbf{x} \in S_m$  into a group state  $\mathbf{g} \in S_{\gamma_{n,m}}$ . So, intuitively, a matching rule describes how any given population is allocated into groups.

**Definition 2.1. (Matching Rules)** A *matching rule* is a function  $\mathbf{f} : S_m \rightarrow S_{\gamma_{n,m}}$  that maps any population strategy  $\mathbf{x} \in S_m$  into a group state  $\mathbf{f}(\mathbf{x}) \in S_{\gamma_{n,m}}$ .

It is natural – but not necessary for any of our results – to impose consistency on matching rules by demanding that the fraction of  $j$ -type individuals allocated into the different groups equals the fraction  $x_j$  of individuals of type  $j$  that are actually present in the population. Since the proportion of  $j$ -type individuals in an  $i$ -type group by definition is  $n_j^i/n$ , the fraction of individuals that are of type  $j$  and in  $i$ -type groups will, for any given group state  $\mathbf{g}$ , be  $n_j^i g_i/n$ . Hence, *across all groups* the fraction of the population that is of type  $j$  is  $\sum_{i=1}^{\gamma_{n,m}} n_j^i g_i/n$ . This number must then equal  $x_j$  for every individual to be allocated to one (and only one) group:

$$\sum_{i=1}^{\gamma_{n,m}} \frac{n_j^i}{n} f_i(\mathbf{x}) = \sum_{i \in \text{supp}(j)} \frac{n_j^i}{n} f_i(\mathbf{x}) = x_j, \text{ for } j = 1, \dots, m \quad (2.2)$$

where  $\text{supp}(j)$  is the set of group types that contain at least one  $j$ -strategist.

When a matching rule satisfies (2.2), we say that it is *consistent*. While our main examples of matching rules below are consistent, a very important special case of our setting, namely haystack/trait-group models (Cooper and Wallace, 2004; Maynard Smith, 1964; Wilson, 1975) generally does not lead to consistent matching rules (see section 2.2.4).<sup>12</sup>

Recall that  $n_j^i f_i(\mathbf{x})/n$  is the fraction of the total population that is of type  $j$  and is allocated to a group of type  $i$  under the matching rule  $\mathbf{f}$ . When  $x_j > 0$  we may divide this by the fraction  $x_j$  of the population that is of type  $j$  in order to get the fraction of  $j$ -type individuals that is allocated to a group of type  $i$ :

$$w_j^i(\mathbf{x}) \equiv \frac{n_j^i}{n x_j} f_i(\mathbf{x}) \quad (2.3)$$

This may be compared with Bergström (2003) who studies group selection (again in the special case  $n = m = 2$ ), and who takes the  $w_j^i$ s as primitives instead of the matching rule. More specifically, Bergström (2003) considers the difference  $w_1^1 - w_2^2$  and calls this difference the ‘index of assortativity’. We return to the index of assortativity in example 3 below where we also show how one gets from a model based on a constant index of assortativity to our formulation with matching rules.

We finish this subsection by presenting a number of concrete examples of matching rules. We shall be calling on these repeatedly throughout the rest of this paper.

### Examples of Matching Rules

1. **Complete segregation.** Different strategies *do not mix*. All individuals are allocated into groups with only individuals of the same type and thus all groups contain a single type of individual each ( $n$  individuals that follow the same strategy). The group types that have  $n$  individuals of the same type get a non-negative frequency whereas all other kinds of groups get a frequency of zero.

<sup>12</sup>This is because the proportion of, say, cooperators in the population at the beginning of the dispersion phase will not necessarily coincide with the proportion of cooperators at the end of the  $T - 1$ -th generation

Due to the consistency requirements for matching rules, we get that the group type that contains  $n$   $j$ -types should get a frequency of  $x_j$ . So, formally, the matching rule for complete segregation is the following.

$$\begin{aligned} f_i(\mathbf{x}) &= x_j \quad , \quad \text{if } n_j^i = n \\ f_i(\mathbf{x}) &= 0 \quad , \quad \text{otherwise.} \end{aligned} \quad (2.4)$$

e.g. When  $n = m = 2$  the matching rule for complete segregation take the form:

$$f_1(x_1, x_2) = x_1 \quad f_2(x_1, x_2) = 0 \quad f_3(x_1, x_2) = x_2.$$

2. **Random matching.** Let us define the *opponent profile* of a type  $j$  individual in a type  $i$  group to be the vector  $v_j^i = (v_1^i, \dots, v_j^i, \dots, v_m^i) \equiv (n_1^i, \dots, n_j^i - 1, \dots, n_m^i)$  that shows how many opponents of each type a type  $j$  individual faces when she is drawn into a group of type  $i$ . Obviously, individuals of different types that face the same opponent profile will be in groups of different types. We will say that *matching is random* when the (ex ante) probability of an individual (conditional on her type) to end up facing a specific opponent profile is independent of her type. If this is the case, then the frequencies of group types will follow a multinomial distribution (see for example Lefebvre, 2007, p. 22):<sup>13</sup>

$$f_i(\mathbf{x}) = \frac{n!}{\prod_{j \in M} n_j^i!} \prod_{j \in M} x_j^{n_j^i}. \quad (2.5)$$

Notice that for  $m = 2$ , the random matching rule becomes

$$f_i(x_1, x_2) = \frac{n!}{n_1^i!(n - n_1^i)!} x_1^{n_1^i} x_2^{n - n_1^i}.$$

That is, it boils down to the binomial distribution (see Kerr and Godfrey-Smith,

<sup>13</sup>To show that the property described above holds for the matching rule of equation (2.5), let us consider a group of type  $i$  with  $n_j^i \geq 1$  for some  $j \in M$ . Notice that a  $j$ -type in that group has  $n_1^i$  type 1 opponents, ...,  $n_j^i - 1$  type  $j$  opponents, ...,  $n_m^i$  type  $m$  opponents. So the opponent profile for a  $j$  strategist in a type  $i$  group will be  $v = (n_1^i, \dots, n_j^i - 1, \dots, n_m^i)$ . Indeed, the probability of a type  $j$  individual (conditional on her type) to end up in group with opponent profile  $v = (v_1, \dots, v_m)$  is given by:  $w_j^i(\mathbf{x}) = \frac{n_j^i}{n x_j} \frac{n!}{\prod_{k \in M} n_k^i!} \prod_{k \in M} x_k^{n_k^i} = \frac{(n-1)!}{\prod_{k \in M} v_k!} \prod_{k \in M} x_k^{v_k}$ . i.e. it is independent of the individual's strategy  $j$ .

2002, p. 484).

### 3. Constant Index of Assortativity.

Bergström (2003) studies 2-person prisoner's dilemma population games by using the 'index of assortativity' which he defines as "the difference between the probability that a C-strategist meets a C-strategist and the probability that a D-strategist meets a C-strategist". In terms of notation used in this paper (with  $x_1$  and  $x_2$  denoting the proportion of cooperators and defectors in the population respectively), this means that the index of assortativity when the population strategy is  $(x_1, x_2)$  will be:

$$\alpha(x_1, x_2) = w_1^1(x_1, x_2) - w_2^2(x_1, x_2) = \frac{f_1(x_1, x_2)}{x_1} - \frac{f_2(x_1, x_2)}{2x_2}.$$

Bergström goes on to analyze prisoners' dilemma games under "assortative matching" rules that have a constant index of assortativity  $\alpha$  for all values of  $\mathbf{x}$ . As one easily verifies, the matching rule corresponding to a constant index of assortativity  $\alpha$  is:

$$\begin{aligned} f_1(\mathbf{x}) &= x_1(1 - (1 - \alpha)x_2) \\ f_2(\mathbf{x}) &= 2(1 - \alpha)x_1x_2 \\ f_3(\mathbf{x}) &= x_2(1 - (1 - \alpha)x_1). \end{aligned}$$

In the case of  $\alpha = 0$  the rule coincides with the random matching rule and in the case of  $\alpha = 1$  it coincides with the complete segregation rule (for both of these statements we of course need  $n = m = 2$ , *i.e.*, two players and two strategies).

### 2.2.3 Steady states

At this point we have defined all of the key ingredients of a group selection model: A set of agents  $I = [0, 1]$ , the normal form group game  $G = \langle N, M, A \rangle$  (here  $N = \{1, \dots, n\}$  and  $M = \{1, \dots, m\}$  where  $n$  is the group size and  $m$  the number of strategies/types), and the matching rule  $\mathbf{f}: S_m \rightarrow S_{\gamma_{n,m}}$  which in each period allocates the newborn generation into groups (recall from section 2.2.1 that  $\gamma_{n,m}$  is the number of different  $n$ -sized groups that can be formed from  $m$  different strategies).

Given the tuple  $\langle I, G, \mathbf{f} \rangle$ , we are now in a position to describe the dynamical

system that constitutes the evolutionary model of group selection. The standard solution concept in group selection models as defined is that of a steady state which we now proceed to discuss.

Let  $\mathbf{x}^t \in S_m$  denote the population strategy at date  $t$  (the vector of frequencies of the different types at the given date, see subsection 2.2.2). At date  $t$ , the population is allocated into groups according to the matching rule  $\mathbf{f}$ , hence  $\mathbf{f}(\mathbf{x}^t) \in S_{\gamma_{n,m}}$  is the resulting group frequency distribution. Regardless of which group an individual of type  $j$  ends up in, he will mechanically follow the strategy of his type (as inherited from the parent) and fitness will be distributed accordingly. Now recall from equation (2.3) of section 2.2.2 that  $w_j^i(\mathbf{x}) = n_j^i f_i(\mathbf{x}) / n x_j$  is the fraction of  $j$ -type individuals that is allocated to groups of type  $i$  under the matching rule  $\mathbf{f}$  when the population strategy is  $\mathbf{x}$  and  $x_j > 0$ . From section 2.2.1 we know that the payoff/fitness of a  $j$ -type who finds himself in a group of type  $i$  is  $A_j^i$ . The *average fitness of a type  $j$  individual at date  $t$*  is consequently  $\sum_{i \in \text{supp}(j)} w_j^i(\mathbf{x}^t) A_j^i$ . This average fitness will be denoted by  $\pi_j(\mathbf{x}^t)$ , and if we substitute for  $w_j^i(\mathbf{x}^t)$  it is clear that this is given by:

$$\pi_j(\mathbf{x}) \equiv \sum_{i \in \text{supp}(j)} \frac{n_j^i}{n x_j} f_i(\mathbf{x}) A_j^i \quad (2.6)$$

Since  $\pi_j(\mathbf{x})$  is the average fitness of a  $j$ -type, the *average fitness of all types in the population* will be:

$$\bar{\pi}(\mathbf{x}) = \sum_{j=1}^m x_j \pi_j(\mathbf{x}) \quad (2.7)$$

All that remains now is to describe how these fitnesses determine the next generation. At this point we have deliberately avoided saying whether time is to be thought of as discrete or continuous. In fact, we are going to describe both, since both play important roles in the existing literature.

Beginning with the discrete time version, the well-known replicator dynamics equations (Hammerstein and Selten, 1994; Taylor and Jonker, 1978; Weibull, 1995, pp. 122-4), formalize the (sensible) notion that at time  $t + 1$  the proportion of the population that is of type  $j$  must equal the proportion of type  $j$  individuals at date  $t$  times the *relative fitness* of a type  $j$  individual.

**Definition 2.2.** *The discrete time replicator dynamics of the group selection model  $\langle I, G, \mathbf{f} \rangle$  is given by the equations:*

$$x_j^{t+1} = x_j^t \frac{\pi_j(\mathbf{x}^t)}{\bar{\pi}(\mathbf{x}^t)} \quad \text{for all } j \in M. \quad (2.8)$$

where  $\pi_j$  and  $\bar{\pi}$  were defined in equations (2.6) and (2.7), respectively.

Turning next to the continuous-time case, the definition becomes (see Hofbauer and Sigmund, 1998, p. 67; Weibull, 1995, p. 72):

**Definition 2.3.** *The continuous time replicator dynamics of the group selection model  $\langle I, G, \mathbf{f} \rangle$  is given by the equations:*

$$\dot{x}_j = x_j(\pi_j(\mathbf{x}) - \bar{\pi}(\mathbf{x})) \quad \text{for all } j \in M. \quad (2.9)$$

where  $\pi_j$  and  $\bar{\pi}$  were defined in equations (2.6) and (2.7), respectively.

**Definition 2.4.** *A steady state of a group selection model  $\langle I, G, \mathbf{f} \rangle$  is a rest point of any of the dynamical systems (2.8) or (2.9)*

Clearly the steady states are the same whether time is continuous or discrete. Different notions of stability such as Lyapunov and asymptotic stability are defined as usual in either case, and the associated steady states (if any) are said to be Lyapunov stable, asymptotically stable, and so on. Since any uniform population strategy – *i.e.*, any strategy where all individuals are of the same type – will be a steady state, it is clear that stability must be considered or else the model will have no predictive power.

## 2.2.4 Relationship with trait-group models

In this section, we briefly consider the group selection models in Maynard Smith (1964), Wilson (1975) and Cooper and Wallace (2004). As in the model described in the previous sections, the population of individuals is split into groups in these models and interaction in each group determines the number of offspring of different types that will enter the population of individuals in the next period, etc. The difference is that assortativity does not stem from the matching process, but from prolonged interaction *within* groups (called “haystacks” in Maynard Smith, 1964, and trait-groups in Wilson, 1975). Following Cooper and Wallace (2004) closely from now on, consider a population of individuals who can be of two types. To facilitate

comparison with this paper's main model, we assume the population to be a continuum, so at any moment in time a proportion  $x_1 \in [0, 1]$  will be of type 1 and a proportion  $x_2 = 1 - x_1 \in [0, 1]$  of type 2. At the "dispersion phase", the population is split into *trait-groups* consisting of two individuals each. These groups are formed randomly (using the random matching rule of example 2). The two individuals in a group proceed to execute the strategy that their type dictates and get payoffs in fitness terms according to a symmetric normal-form matrix  $(A_j^i)$  (see section 2.2.1). The fitness of each individual determines the number of children it will send to the next generation. The trait-group's offspring is at this point "pooled" and dispersed but, crucially, the offspring are not pooled with the offspring of all the other groups as in the model of section 2.2.3. Instead, the trait-groups remain separated for  $T > 1$  generations, so the second generation of a specific trait group is split into subgroups consisting of offspring of that trait group only (again the dispersion is pairwise and random). This second generation of subgroups proceed precisely as before to execute their type strategies, produce offspring according to the matrix  $(A_j^i)$ , and in this way the trait-group's *third* generation is born. The process repeats itself until, after  $T$  generations the trait-group's combined offspring is finally returned to the aggregate population. The aggregate population is then again randomly matched into new trait-groups, and so on.

The described model is called a *T-period trait-group model*. A *steady state* is defined in the usual way as a population strategy  $\mathbf{x} = (x_1, x_2)$  with the property that if the initial proportions of the types are  $x_1$  and  $x_2 = 1 - x_1$ , respectively, then at any future date these will be the proportions of the two types in the aggregate population also.

As we now proceed to show, whether assortativity stems from matching (as in our preferred model), or from prolonged interaction in groups (as in the models of Maynard Smith, 1964, Wilson, 1975, and Cooper and Wallace, 2004) is of no consequence in the sense that for any model of the second variety one can reconstruct the steady states and dynamics with a model from the former.

**Theorem 2.1.** *Consider a T-period trait-group model with group size n, number of pure strategies m and a symmetric payoff matrix  $(A_j^i)$ . Consider also the symmetric n-player, m-strategy normal form game G with payoff matrix  $(A_j^i)$ . Then there is a matching rule  $\mathbf{f}$  such that the dynamics and steady states of the group selection model*

$\langle I, G, \mathbf{f} \rangle$  coincide with the dynamics and steady states of the trait-group model.

A detailed proof is provided in Appendix A.3.1. Here we provide a sketch in the 2-player, 2-strategy case stressing what the matching rule associated with a specific trait-group model actually looks like.

Consider a  $T$ -period trait-group model with payoff matrix  $(A_j^i)$ . We remind the reader that  $A_j^i$  indicates the payoff that a  $j$ -type individual receives when found in an  $i$ -type group. As this is a 2-player, 2-strategy model, there are three group-types: group-type 1 which contains two individuals of type 1, group-type 2 which contains one individual of type 1 and one individual of type 2, and group-type 3 which contains two individuals of type 2.

We are tracking the evolution of the population between two consecutive dispersion phases. In order to do that we need to calculate the expected fitness (number of descendants) that an individual of each type will get at the end of the  $T$  periods whereby the trait-groups remain separated from each other. To that effect, we use a law-of-large-numbers argument and we calculate the distribution of groups across all trait-groups at the  $T$ -th period.<sup>14</sup> This makes us able to calculate the expected fitness for starting individuals (individuals in the original population at the dispersion phase) of each of the types in the trait-group model. We also calculate the expected payoffs for the group selection model with a matching rule given by

$$f_i(\mathbf{x}) = \frac{\sum_{k=1}^3 r_k(\mathbf{x}) g_i^k}{\sum_{l=1}^3 \sum_{k=1}^3 r_k(\mathbf{x}) g_l^k} \quad (2.10)$$

where  $r_1(\mathbf{x}) = x_1^2$ ,  $r_2(\mathbf{x}) = 2x_1x_2$  and  $r_3(\mathbf{x}) = x_2^2$  are the components of the random matching rule  $\mathbf{r}(\mathbf{x})$  (see example 2). The various  $g_i^k$  are only dependent on the particular payoff matrix  $(A_j^i)$  and the number of generations  $T$  that the individuals spend in their respective trait-groups isolated from the rest of the population and are, thus, given for any trait-group model. They express the (expected) proportion of  $i$ -type groups that are found in a trait-group whose first-generation parents were a pair of type  $j$  at the pair-matching stage of generation  $T$ .

We then show that the expected fitness of  $j$ -types under the trait-group model  $\pi_j^{\text{TG}}$  and under the matching rule given by equation (2.10)  $\pi_j^{\mathbf{f}}$  satisfy the following

<sup>14</sup>Note that applying the law of large numbers on a continuum is not entirely innocent (see e.g. the Introduction in Al-Najjar, 1995). As is well known, however, the resulting difficulties can be overcome in several different ways (see Appendix III in Acemoglu and Jensen, 2012, for an overview).

condition:

$$\frac{\pi_j^{\text{TG}}(\mathbf{x})}{\pi_k^{\text{TG}}(\mathbf{x})} = \frac{\pi_j^{\mathbf{f}}(\mathbf{x})}{\pi_k^{\mathbf{f}}(\mathbf{x})} \text{ for all } \mathbf{x} \in S_m \text{ and } j, k \in M$$

which is a necessary and sufficient condition for the two models to yield the same dynamics (and therefore steady states) due to a symmetry of the replicator dynamics (see equations (2.8) and (2.9)).

It follows that the dynamics of the trait-group model will be the same as those of the group selection model  $\langle I, G, \mathbf{f} \rangle$ . And, so, the steady states of the two models will coincide as well.

## 2.3 Group Selection Games

In the previous section we described in full detail what we think is a natural unified model of group selection capturing both group selection based directly on non-random matching (e.g. Bergström, 2003; Kerr and Godfrey-Smith, 2002), and haystack/trait-group models (e.g. Cooper and Wallace, 2004; Maynard Smith, 1964; Wilson, 1975). Except when we described the group games in section 2.2.1, we made no mention of game theory – in fact the only reason we did mention this was because we need it in this section. In this section we are going to shift the perspective entirely to a game theoretic one. The basic underlying object of study will remain the same: We have a continuum  $I = [0, 1]$  that is now referred to as *the set of players*, we have an underlying  $n$ -player  $m$ -strategy symmetric normal form game  $G$  as described in section 2.2.1, and we have a matching rule  $\mathbf{f}$  as described in section 2.2.2. But the “story” will be very different. All three together will define a game which we call a *group selection game*:

**Definition 2.5. (Group Selection Games)** *A group selection game is a tuple  $\langle I, G, \mathbf{f} \rangle$  where  $I$  is a continuum of players,  $G$  is a symmetric  $n$ -player,  $m$ -strategy normal form game, and  $\mathbf{f}$  is a matching rule.*

Here is the structure of the game: As mentioned, there is a continuum  $I = [0, 1]$  of agents. These are identical, in particular they have the same finite set of pure strategies  $M = \{1, \dots, m\}$  given from the normal form game  $G$ . The game is symmetric,

so we can conveniently summarize a (*pure*) *strategy profile* by its frequency distribution  $\mathbf{x} = (x_1, \dots, x_m) \in S_m$  where the  $j$ 'th coordinate is the *fraction* of the players whose strategy is  $j \in \{1, \dots, m\}$ . These frequency distributions (each of them representing a class of strategy profiles) are the *population strategies* of the previous section. The individual player takes  $\mathbf{x}$  as given and being infinitesimally small, his own choice of strategy will not affect the relative proportions expressed in  $\mathbf{x}$ . Now, in one description, the game has two *stages*: In the *first* stage, players choose their strategies and in the *second* stage they are allocated into groups of the same finite size  $n \in \{2, 3, \dots\}$  where they execute their strategies.<sup>15</sup> What is crucial here is that agents do not know with certainty which group they will end up in when they choose their strategies. However, because the structure of the game is known (common knowledge), an agent *will* know the *rule* according to which agents are allocated into groups, and so will be able to calculate the probability of ending up in any particular type of group after a specific strategy is chosen. This brings us back to (2.3) of section 2.2.2. Recall from that section that if  $x_j > 0$  then  $w_j^i(\mathbf{x}) = n_j^i f_i(\mathbf{x}) / (n x_j)$  is the fraction of type  $j$  individuals that are allocated into groups of type  $i$  under the matching rule  $\mathbf{f}$  (and the population strategy  $\mathbf{x}$ ). The case where  $x_j = 0$  is returned to in a moment. It is clear that from an expected payoff point-of-view,  $w_j^i(\mathbf{x})$  is the *ex-ante probability a  $j$ -strategist has of being "drawn" into group  $i$* . It follows that the *expected payoff* to strategy  $j$  will equal

$$\pi_j(\mathbf{x}) = \sum_{i \in \text{supp}(j)} w_j^i(\mathbf{x}) A_j^i, \quad (2.11)$$

where we remind the reader that  $A_j^i$  is the payoff received from playing strategy  $j$  in a group of type  $i$  (section 2.2.1); and  $\text{supp}(j)$  is the set of groups that contain at least one  $j$ -strategist (section 2.2.2). Comparing with section 2.2.3, this expected payoff precisely coincides with the average fitness to a type  $j$  individual in the (deterministic) evolutionary group selection model.

Now, for the previous two definitions it is required that  $x_j > 0$ . The definition of the  $w_j^i$ 's in (2.3) and so the definition of the  $\pi_j$ 's in (2.11) are extended to the boundary of  $S_m$  ( $\text{bd}_j(S_m) = \{x \in S_m : x_j = 0\}$ ) by taking  $w_j^i(\mathbf{x}) = \lim_{\tilde{x}_j \downarrow 0} n_j^i f_i(\tilde{\mathbf{x}}) / n \tilde{x}_j$

<sup>15</sup>From a game theoretic perspective, it is much more natural to think of this as a situation involving uncertainty (a type of Bayesian game). But the imperfect information perspective actually turns out to be non-standard because probabilities are endogenously determined.

whenever  $\mathbf{x} \in \text{bd}_j(S_m)$ . Evidently, we need to assume that these limits exist. Note that the limit  $\lim_{x_j \downarrow 0} f_i(\tilde{\mathbf{x}})/\tilde{x}_j$ , if it exists, is precisely the (upper) partial derivative of  $f_i$  with respect to  $x_j$ ,  $\partial_j^+ f_i(\mathbf{x})$ . Hence  $w_j^i(\mathbf{x}) = n_j^i \partial_j^+ f_i(\mathbf{x})/n$  when  $x_j = 0$ .

**Definition 2.6. (NEGS)** *Let  $\langle I, G, \mathbf{f} \rangle$  be a group selection game. A population strategy  $\mathbf{x}^* \in S_m$  is a Nash Equilibrium with Group Selection (NEGS) if for all  $j$  with  $x_j^* > 0$ :*

$$\pi_j(\mathbf{x}^*) \geq \pi_k(\mathbf{x}^*) \text{ for all } k \in M. \quad (2.12)$$

*The average payoff (the welfare) at a NEGS  $\mathbf{x}^*$  is denoted by  $\bar{\pi}(\mathbf{x}^*) = \mathbf{x}^* \cdot \pi(\mathbf{x}^*)$ .*

Here is an intuitive description. In a NEGS, each agent takes the population strategy  $x^*$  as given due to his infinitesimal size. Based on group payoff and matching rules, the agent can then calculate the expected payoff (11) resulting from each of the strategies. Note that, as explained above, this calculation involves calculating the probability of ending up in any of the different groups as a function of the strategy chosen. If the agent belongs to a non-negligible set of players who choose the same strategy as him, strategy  $j$  say, then condition (2.12) says that he will not be strictly better off by switching to a different strategy. When the resulting population strategy  $x^*$  implies the probabilities (2.3) that formed the basis of the agents' calculations in the first place, the game is at an equilibrium. Note that in a NEGS, the probabilities of being drawn into the groups are "self-fulfilling" in the sense that agents' ex-post decisions lead to the ex-ante probabilities upon which the decisions are based.

In the following sections we shall see that this concept has a very close relationship with the steady states of the canonical group selection model (section 2.2.3). The remainder of this section is devoted to showing that the NEGS concept is well-founded, and strengthening the equilibrium concept.

Our first result states that any group selection game has an equilibrium when certain regularity conditions are satisfied by the matching rule. Note that the differentiability requirement will be trivially satisfied if the matching rule is differentiable at the boundary of  $S_m$  (we remind the reader that  $\text{supp}(j)$  is the set of group types that contain at least one individual of type  $j$ , see equation (2.2)).

**Theorem 2.2.** *Let  $\langle I, G, \mathbf{f} \rangle$  be a group selection game and assume that  $\mathbf{f}$  is continuous and that the (upper) partial derivatives  $\partial_j^+ f_i(\mathbf{x})$  exist whenever  $x_j = 0$  (for all  $j \in M$  and  $i \in \text{supp}(j)$ ). Then  $\langle I, G, \mathbf{f} \rangle$  has a NEGS.*

**Proof.** See Appendix A.3.2.

Note that the assumptions on  $f$  imply continuity of the payoff function  $\pi$ . Also notice that all matching rules of section 2.2.2 satisfy the conditions of Theorem 2.2.

In the literature, ESS is usually defined in games with random matching and in the special case when  $n = 2$  (see Hofbauer and Sigmund, 1998, p. 63). The appropriate generalization of the ESS concept to include non-random matching and any number of strategies is the following.

**Definition 2.7. (ESSGS)** *Let  $\langle I, G, \mathbf{f} \rangle$  be a group selection game. A population strategy  $\hat{\mathbf{x}} \in S_m$  is an Evolutionarily Stable Strategy with Group Selection (ESSGS) if for each  $\mathbf{y} \in S_m \setminus \{\hat{\mathbf{x}}\}$ , there exists  $\bar{\varepsilon}_{\mathbf{y}} > 0$  such that*

$$\hat{\mathbf{x}} \cdot \pi(\varepsilon \mathbf{y} + (1 - \varepsilon)\hat{\mathbf{x}}) > \mathbf{y} \cdot \pi(\varepsilon \mathbf{y} + (1 - \varepsilon)\hat{\mathbf{x}}) \quad \text{for all } \varepsilon \in (0, \bar{\varepsilon}_{\mathbf{y}}). \quad (2.13)$$

The ESSGS concept is a strengthening of the NEGS concept, just as the traditional notion of an ESS is a strengthening of Nash equilibrium:

**Theorem 2.3.** *Let  $\langle I, G, \mathbf{f} \rangle$  be a group selection game with  $\mathbf{f}$  satisfying the assumptions of Theorem 2.2. Then any ESSGS is a NEGS.*

**Proof.** By way of contradiction, let us assume that some  $\hat{\mathbf{x}} \in S_m$  is an ESSGS but *not* a NEGS. Then, there exists some  $\mathbf{y} \in S_m$  such that  $(\mathbf{y} - \hat{\mathbf{x}}) \cdot \pi(\hat{\mathbf{x}}) > 0$ . But from the definition of an ESSGS, there must exist some  $\bar{\varepsilon}_{\mathbf{y}} \in (0, 1)$  such that for all  $\varepsilon \in (0, \bar{\varepsilon}_{\mathbf{y}})$ ,  $(\mathbf{y} - \hat{\mathbf{x}}) \cdot \pi(\varepsilon \mathbf{y} + (1 - \varepsilon)\hat{\mathbf{x}}) < 0$ . As explained after theorem 2.2, the differentiability at the boundary and continuity in the interior of  $S_m$  of the matching rule, imply continuity of  $\pi$ . By continuity therefore  $(\mathbf{y} - \hat{\mathbf{x}}) \cdot \pi(\hat{\mathbf{x}}) \leq 0$ , a contradiction.  $\square$

In evolutionary models with random matching, there is a clear and well-known connection between dynamic models of the *replicator* type and game theoretic concepts such as Nash equilibrium and evolutionarily stable strategies (Hofbauer and Sigmund, 1998). We next show that any NEGS is a steady state of the replicator dynamics, and that any *stable* steady state of the replicator dynamics is a NEGS.

Furthermore, we are going to prove that any *stable steady state* of the replicator dynamics (be it Lyapunov or in the  $\omega$ -limit sense) will be a NEGS. These results directly parallel known results on models with random matching with underlying replicator-dynamics processes (e.g. Hofbauer and Sigmund, 1998, Theorem 7.2.1; Weibull, 1995, Proposition 3.10) or even in broader classes of dynamics (Ritzberger and Weibull, 1995). Finally, we will prove that any ESSGS is asymptotically stable for the associated replicator dynamics.

Note that the following result also applies to trait-group models since (by Theorem 2.1) such models can be recast as non-random matching models *with the same dynamics* and therefore, of course, the same set of (stable) steady states.

**Theorem 2.4.** *Let  $\langle I, G, \mathbf{f} \rangle$  be a group selection game and assume that  $\mathbf{f}$  satisfies the assumptions of Theorem 2.2 and consider the evolutionary steady states of the associated dynamical systems (2.8)-(2.9). Then,*

1. *Any NEGS is a steady state of the discrete time replicator dynamics (2.8) as well as the continuous time replicator dynamics (2.9).*
2. *If  $\mathbf{x}^*$  is the  $\omega$ -limit of an orbit  $x(t)$  of the replicator dynamics (2.9) that lies everywhere in the interior of  $S_m$ , then  $\mathbf{x}^*$  is a NEGS.*
3. *If  $\mathbf{x}^*$  is Lyapunov stable for the replicator dynamics (2.9), then  $\mathbf{x}^*$  is a NEGS.*
4. *Assume that  $\mathbf{f}$  is of class  $C^1$ . Then if  $\mathbf{x}^*$  is an ESSGS, it is asymptotically stable under the replicator dynamics (2.9).*

**Proof.** See Appendix A.3.3.

In particular, under the result of theorem 2.1, all ESSs of a trait-group model (section 2.2.4) will be NEGS of the group selection game under the appropriately defined matching rule.

**Remark 2.4.1.** *It is easy to see that if  $\langle I, G, \mathbf{f} \rangle$  is a group selection game under random matching, then the set of Nash equilibria with group selection coincides with the set of symmetric Nash equilibria in the underlying normal form game  $G$ . Likewise, when matching is random the set of evolutionarily stable strategies with group selection coincides with the set of evolutionarily stable strategies. For a formal proof, see Appendix A.3.4.*

In the next section we will see the relationship between these results and welfare in equilibrium.

## 2.4 Group Selection and the Fitness of Populations

Group selection can explain behavioral traits such as altruism or cooperation which cannot arise in Nash equilibrium and so cannot be favored by natural selection if matching is random (see remark 2.4.1). Importantly, such departures from egoism may be superior to the outcomes under random matching in the sense that the *average fitness* may be higher. The classical example here is of course the prisoners' dilemma where the outcome of random matching yields lower average fitness than outcomes with assortative matching (see section 2.5 and also Bergström, 2002). In this section we are going to discuss these issues drawing on both the abstract results and the concrete examples of the previous sections. As will become clear, our new concepts (NEGS and ESSGS) allow us to push the discussion substantially forward in comparison with existing literature.

First, we need to define the concepts involved. Recall from section 2.2 that the *average fitness*  $\bar{\pi}(\mathbf{x})$  at a population strategy  $\mathbf{x} \in S_m$  is given by  $\bar{\pi}(\mathbf{x}) = \sum_{j=1}^m x_j \pi_j(\mathbf{x})$  (equation (2.7)). In the context of a group selection game  $\langle I, G, \mathbf{f} \rangle$ , we referred instead to this as the *average payoff* or the *welfare* (see equation (2.11)). Since average fitness in the evolutionary model is obviously equal to average payoff in the (evolutionary) game theory model, and since by theorem 2.4 we know how the various equilibrium/steady state concepts relate to each other, we need not differentiate between them in what follows. Accordingly, we use the term *average fitness* exclusively from now on. Average fitness at a population strategy  $\mathbf{x}$  will from now on be denoted by  $\bar{\pi}^{\mathbf{f}}(\mathbf{x})$  so as to explicitly mention the matching rule. This allows us to easily compare average fitnesses under different matching rules for a fixed underlying payoff structure/normal form game  $G$  (e.g., prisoners' dilemma or hawk-dove).

Now as was already mentioned, random matching – or for that matter any other specifically given matching rule  $\mathbf{f}$  – may not maximize average fitness in a NEGS  $\mathbf{x}^*$ . This observation also remains valid if instead of NEGS we focus on ESSGS. Thus, evolution under non-random matching certainly does not imply fitness maximization. The interesting next question therefore is whether for a *fixed* underlying nor-

mal form game there exists *some* matching rule under which average fitness will be maximized at an NEGS; and if the answer is yes, to characterize these matching rules in concrete situations. Thus in the prisoners' dilemma, random matching is inferior in average fitness terms as, from what we will see in section 2.5, a rule such as complete segregation will lead to equilibria where everybody cooperates *i.e.* to average fitness maximization. When discussing this topic it is important to understand that when  $\mathbf{f}$  is varied, not only does the set of NEGS (and ESSGS and also, the set of steady states of the replicator dynamics) change – the average fitness  $\bar{\pi}^{\mathbf{f}}(\mathbf{x})$  will also change at any given population strategy  $\mathbf{x}$ . So if some population strategy maximizes welfare but is not a NEGS at some matching rule  $\mathbf{f}^1$ , it could be a NEGS at another matching rule  $\mathbf{f}^2$  but no longer maximize welfare! Any sensible discussion must therefore consider the *joint* selection of a population strategy and matching rule as captured by the following definition.

**Definition 2.8. (Evolutionary Optimum)** *Let  $G$  be a symmetric  $n$ -player,  $m$ -strategy normal form game. A population strategy  $\mathbf{x}^* \in S_m$  together with a matching rule  $\mathbf{f}^*$  is said to be an evolutionary optimum if  $\bar{\pi}^{\mathbf{f}^*}(\mathbf{x}^*) \geq \bar{\pi}^{\mathbf{f}}(\mathbf{x})$  for all  $(\mathbf{x}, \mathbf{f}) \in \mathcal{E} = \{(\mathbf{x}, \mathbf{f}) : \mathbf{x} \text{ is a steady state of } \langle I, G, \mathbf{f} \rangle\}$ .*

Intuitively, a population strategy  $\mathbf{x}^*$  and a matching rule  $\mathbf{f}^*$  form an optimum if they lead to maximum average *fitness of the population* among all population strategy/matching rule combinations that satisfy the steady state restriction. Note that the restriction to steady states is entirely natural here: Any population strategy that is *not* a steady state under some matching rule would immediately be “destroyed” by natural selection.<sup>16</sup> Given these definitions, we can now answer the previous question:

**Theorem 2.5.** *Let  $(\mathbf{x}^*, \mathbf{f}^*)$  be an evolutionary optimum. Then there exists a matching rule  $\mathbf{h}$  which satisfies the assumptions of theorem 2.2, such that  $\mathbf{x}^*$  is a NEGS under  $\mathbf{h}$ , and such that  $(\mathbf{x}^*, \mathbf{h})$  is an evolutionary optimum (in particular,  $\bar{\pi}^{\mathbf{h}}(\mathbf{x}^*) = \bar{\pi}^{\mathbf{f}^*}(\mathbf{x}^*)$ ).*

**Proof.** See Appendix A.3.5.

<sup>16</sup>Note in this connection that *any* uniform population strategy is a steady state (in fact, any uniform population strategy is a steady state under *any* matching rule).

Theorem 2.5 can be thought of as the “second welfare theorem of evolution” telling us that *any* evolutionary optimum can be “decentralized” in the evolutionary environment through *some* matching rule.<sup>17</sup> That this should be so is easy to see in simple cases, but it is in general a surprising result. In most standard games (including the ones considered in this paper), there is a premium on coordination/uniformity, and so what is needed in order to reach an evolutionary optimum is a sufficiently high level of assortativity. In games where there is a premium on agents in a group being *different* – *e.g.*, due to specialization – it will instead be a sufficiently high degree of dissociation that leads to evolutionary optimality. It is not obvious that Theorem 2.5 should hold in the latter case, to say nothing of cases that are neither assortative nor dissociative.

## 2.5 Group Selection and the Prisoners' Dilemma Game

In this section, we look into what is arguably the most analyzed game in the literature and which has served as the canonical way to model altruistic behavior: the Prisoners' Dilemma (PD).

We analyze a number of Prisoners' Dilemma group selection games under different matching rules: rules that are derived from trait-group models (as described in section 2.2.4) and those that belong to the class of constant index of assortativity (see example 3 of section 2.2.2). We also provide comparative statics results for the class of matching rules with a constant index of assortativity. Interesting results arise in different families of PD games as equilibrium dynamics change. This renders us capable to discuss risk dominance and how this is affected under various levels of assortative matching.

Finally, one may want to make comparisons of the type: “Say  $x^*$  is an equilibrium of a population game under some matching rule  $\mathbf{f}$ . If the players were using strategy  $x^*$  in the associated normal form game, would their expected payoff be higher or lower than the population-wide average payoff in the population game?”. Such comparisons are carried out for the three classes of PD games. Our results suggest that equilibrium welfare is monotonically increasing as the equilibrium ‘level of cooperation’ increases. Despite that, it may be the case that in order to estab-

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<sup>17</sup>If in Definition 2.8 matching rules are required to be consistent, one can show that the “decentralizing” matching rule of Theorem 2.5 can be chosen to be consistent also.

lish a population strategy with high level of cooperation  $x^*$  as an equilibrium one has to do that in expense of welfare. This is because if the players were using  $x^*$  in the normal form game, they would get a higher expected payoff than they get in equilibrium.

### The prisoners' dilemma

The two players involved in the game have two possible (pure) strategies each: Cooperate (C) or Defect (D). In our notation, a PD game is a game with  $A_2^2 > A_1^1 > A_2^3 > A_1^2$ .<sup>18</sup> The payoff matrices of four PD games are shown in Table 2.1. In any PD game, there exists a unique pure strategy Nash equilibrium (D,D) as defection strictly dominates cooperation. The outcome is far from optimal as there is an obvious Pareto improvement if we move to (C,C).

<table border="1" style="border-collapse: collapse; margin: auto;"> <thead> <tr><th></th><th>C</th><th>D</th></tr> </thead> <tbody> <tr><th>C</th><td>11, 11</td><td>2, 16</td></tr> <tr><th>D</th><td>14, 2</td><td>5, 5</td></tr> </tbody> </table> <p style="text-align: center;">(a) <math>A_1^2 + A_2^2 &lt; A_1^1 + A_2^3</math></p>		C	D	C	11, 11	2, 16	D	14, 2	5, 5	<table border="1" style="border-collapse: collapse; margin: auto;"> <thead> <tr><th></th><th>C</th><th>D</th></tr> </thead> <tbody> <tr><th>C</th><td>55, 55</td><td>0, 70</td></tr> <tr><th>D</th><td>70, 0</td><td>40, 40</td></tr> </tbody> </table> <p style="text-align: center;">(b) <math>A_1^2 + A_2^2 = A_1^1 + A_2^3</math></p>		C	D	C	55, 55	0, 70	D	70, 0	40, 40
	C	D																	
C	11, 11	2, 16																	
D	14, 2	5, 5																	
	C	D																	
C	55, 55	0, 70																	
D	70, 0	40, 40																	
<table border="1" style="border-collapse: collapse; margin: auto;"> <thead> <tr><th></th><th>C</th><th>D</th></tr> </thead> <tbody> <tr><th>C</th><td>11, 11</td><td>2, 18</td></tr> <tr><th>D</th><td>18, 2</td><td>5, 5</td></tr> </tbody> </table> <p style="text-align: center;">(c) <math>A_1^2 + A_2^2 &gt; A_1^1 + A_2^3</math> I</p>		C	D	C	11, 11	2, 18	D	18, 2	5, 5	<table border="1" style="border-collapse: collapse; margin: auto;"> <thead> <tr><th></th><th>C</th><th>D</th></tr> </thead> <tbody> <tr><th>C</th><td>55, 55</td><td>38, 64</td></tr> <tr><th>D</th><td>64, 38</td><td>40, 40</td></tr> </tbody> </table> <p style="text-align: center;">(d) <math>A_1^2 + A_2^2 &gt; A_1^1 + A_2^3</math> II</p>		C	D	C	55, 55	38, 64	D	64, 38	40, 40
	C	D																	
C	11, 11	2, 18																	
D	18, 2	5, 5																	
	C	D																	
C	55, 55	38, 64																	
D	64, 38	40, 40																	

Table 2.1: The payoff matrices of four Prisoners' Dilemma games.

Bergström (2003) distinguishes among three types of PD games: those that have the structure of linear, superadditive and subadditive games of shared output. In our notation, the conditions for each of the three types are  $A_1^2 + A_2^2 = A_1^1 + A_2^3$ ,  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  and  $A_1^2 + A_2^2 > A_1^1 + A_2^3$  respectively. The intuition behind this categorization is the following. Imagine there are four individuals: two following C and two following D. If they are aggregately better off (in the sense of their sum of payoffs) by being matched into mixed pairs rather than into segregated pairs (*i.e.* if  $A_1^2 + A_2^2 > A_1^1 + A_2^3$ ), then the game is of the “subadditive” type whereas if they are better off being segregated rather than being mixed (*i.e.* if  $A_1^1 + A_2^3 > A_1^2 + A_2^2$ ) the game is of the “superaddi-

<sup>18</sup>Strategy 1 is Cooperation and strategy 2 is Defection. Also, groups of type 1 contain two cooperators, groups of type 2 contain one cooperator and one defector and groups of type 3 contain two defectors.

itive" type. The "linear" case where  $A_1^2 + A_2^2 = A_1^1 + A_2^3$  is a special transitional situation between the two more generic ones. We keep this categorization as the three different types of games yield different results not only when we look into their dynamic behavior under non-random matching but also when equilibrium welfare is taken into account.

### 2.5.1 Trait-group matching rules

Here we consider examples of trait-group models with a PD payoff structure. We calculate the matching rules for the PD games of Table 2.1 and for the number of generations spent within the trait-groups varying between 1, 2, and 3 using the methodology described in 2.2.4. We then derive the dynamics of the corresponding trait-group model given by the replicator equation (2.9). The results can be seen in Figure 2.1. In the diagrams  $x$  denotes the proportion of cooperators in the population whereas  $\dot{x}$  denotes the rate of change of  $x$  (continuous time is assumed).

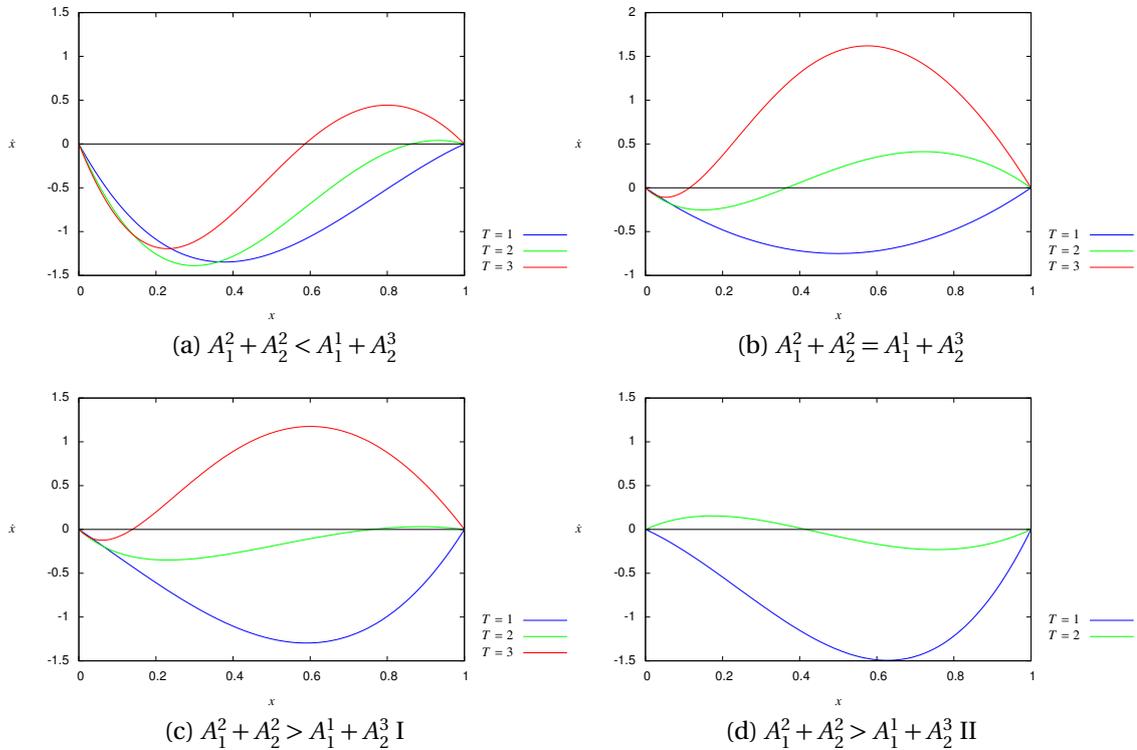


Figure 2.1: Dynamics for the PD games of Table 2.1 under trait-group matching rules for different values of  $T$ .

Generally, our results demonstrate that the trait-group structure leads to results similar to the ones we get for the respective games under matching rules with some positive assortativity (for comparison see 2.5.2 below). So, we can see that as the number of generations that individuals spend in their respective trait-groups  $T$  increases, the Nash equilibrium of the normal form game ( $x = 0$ ) may no longer be unique or even a NEGS – although it is always a steady state of the replicator dynamics. Interestingly, our analysis of PD games of the “subadditive” type yields novel results: depending on the specific game at hand, the trait-group structure can lead to different patterns of dynamics. Of course, in all cases, the results for  $T = 1$  are exactly the same as the results we get for random matching:  $x = 0$  is the unique NEGS and it is also an ESSGS.

More specifically, as can be seen in Figure 2.1a, when  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  as  $T$  increases new equilibria (in the NEGS sense) emerge: one at  $x = 1$  and one with  $0 < x < 1$ . The extreme equilibria ( $x = 0$  and  $x = 1$ ) are evolutionarily stable (ESSGSs) whereas the intermediate one is not. This suggests that there are two basins of attraction, one for each of the extreme equilibria. Furthermore, as the value of  $T$  increases, the basin of attraction of the full cooperation equilibrium ( $x = 1$ ) increases. These results are similar to the ones we get for the respective class of games under constant index of assortativity matching rules (see case 1 of section 2.5.2 and the discussion there).

For PD games with  $A_1^2 + A_2^2 = A_1^1 + A_2^3$  (see Figure 2.1b) we observe the same pattern as for the ones with  $A_1^2 + A_2^2 < A_1^1 + A_2^3$ . These results are in line with the analysis of Cooper and Wallace (2004) who focus only in this family of games. This is in contrast with the results that we get for these games under constant index of assortativity matching rules (see case 2 of section 2.5.2) as we do not get the “switch” from the population strategy  $x = 0$  being the unique equilibrium to  $x = 1$  being the unique equilibrium. This suggests that the trait-group structure does not introduce a uniform assortativity for all values of  $x$ . This is also corroborated by our results for the last family of PD games that we discuss straight away.

When  $A_1^2 + A_2^2 > A_1^1 + A_2^3$ , our results show that we can get two distinct cases. For example, in the game shown in table 2.1c, our results follow the ones for the cases where  $A_1^2 + A_2^2 \leq A_1^1 + A_2^3$ : increased values of  $T$  lead to two ESSGSs and the basin of attraction of the ESSGS at  $x = 1$  where the whole population cooperates increases. On the other hand, for the game depicted in Table 2.1d, we observe that as the value

of  $T$  increases from 1 to 2, we get a *unique interior* NEGS  $x^* \in (0, 1)$  which is also an ESSGS. This result is similar to the ones we get for the same family of PD games under a constant index of assortativity matching rule (see case 3 of section 2.5.2).

As mentioned before, Cooper and Wallace (2004) discuss only the linear case of PD games ( $A_1^2 + A_2^2 = A_1^1 + A_2^3$ ). It turns out that the results are richer when we drop the linear restriction. In the superadditive case, we see that we can get two different kinds of dynamic behavior as  $T$  increases – depending on the particular game at hand. It is clear that the trait-group structure adds assortativity to the process. This assortativity, however, is not uniform for all population strategies  $x$  nor is it necessarily the same for C and D types.<sup>19</sup>

### 2.5.2 Constant index of assortativity matching rules

We study the different types of PD games under constant index of assortativity matching rules as described in example 3 of section 2.2.2.<sup>20</sup> Our equilibrium analysis of Prisoners' Dilemma games, confirms the results of Bergström (2003). More than that, risk dominance results apply to Prisoners' Dilemma games when  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  and suggest that as the index of assortativity increases, the cooperative strategy becomes risk dominant.

#### Equilibria of the group selection game

Our analysis applied to PD games under constant index of assortativity matching rules (see example 3 of section 2.2.2) confirms the results of Bergström (2003) which are summarized below.

1. If  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  then
  - (a) if  $\alpha < (A_2^2 - A_1^1)/(A_2^2 - A_2^3)$ , there is a *unique equilibrium* at  $x^* = 0$  (all play D),
  - (b) if  $\alpha > (A_2^3 - A_1^1)/(A_1^1 - A_1^2)$ , there is a *unique equilibrium* at  $x^* = 1$  (all play C),
  - (c) if  $\alpha = (A_2^2 - A_1^1)/(A_2^2 - A_2^3)$  or  $\alpha = (A_2^3 - A_1^1)/(A_1^1 - A_1^2)$ , there are *two equilibria*: one at  $x_1^* = 0$  and one at  $x_2^* = 1$  and

<sup>19</sup>See Bergström (2013) for more examples of non-random matching with this property.

<sup>20</sup>This also captures the random matching ( $\alpha = 0$ ) and complete segregation ( $\alpha = 1$ ) rules.

(d) if  $(A_2^2 - A_1^1)/(A_2^2 - A_2^3) < \alpha < (A_2^3 - A_1^1)/(A_1^1 - A_1^2)$ , there are *three equilibria*: one at  $x_1^* = 0$ , one at  $x_2^* = ((A_2^3 - A_1^1)/(1 - \alpha) + A_1^1 - A_1^2)/(A_1^1 + A_2^3 - A_1^2 - A_2^2)$  and one at  $x_3^* = 1$ .

2. If  $A_2^2 + A_1^2 = A_2^3 + A_1^1$  then

(a) if  $\alpha < (A_2^2 - A_1^1)/(A_2^2 - A_2^3)$ , there is a *unique equilibrium* at  $x^* = 0$  (all play D),

(b) if  $\alpha > (A_2^2 - A_1^1)/(A_2^2 - A_2^3)$ , there is a *unique equilibrium* at  $x^* = 1$  (all play C) and

(c) if  $\alpha = (A_2^2 - A_1^1)/(A_2^2 - A_2^3)$ , there is a *continuum of equilibria*. Actually, *any*  $x \in [0, 1]$  is an equilibrium.

3. If  $A_1^2 + A_2^2 > A_1^1 + A_2^3$ , then there is a *unique equilibrium* given by:

$$x^*(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq \frac{A_2^3 - A_1^2}{A_1^1 - A_1^2} \\ \left( \frac{A_1^1 - A_2^3}{1 - \alpha} + A_1^1 - A_1^2 \right) / (A_2^2 - A_2^3 + A_1^1 - A_1^2) & \text{if } \frac{A_2^3 - A_1^2}{A_1^1 - A_1^2} < \alpha < \frac{A_2^2 - A_1^1}{A_2^2 - A_2^3} \\ 1 & \text{if } \alpha \geq \frac{A_2^2 - A_1^1}{A_2^2 - A_2^3} \end{cases} .$$

The dynamics diagrams for all three cases are shown in Figure 2.2 for constant index of assortativity rules with different values of  $\alpha$ . The equilibrium analysis above is summarized in Figure 2.3.

It is interesting to see that the comparative statics results when  $A_1^2 + A_2^2 > A_1^1 + A_2^3$  are very similar to the results we get for a Hawk-Dove game whereas when  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  the comparative statics results are similar to those for a Stag Hunt game.<sup>21</sup> It is not coincidental that the conditions hold for the respective games.

### Risk Dominance

Notice that in the case where  $A_1^2 + A_2^2 < A_1^1 + A_2^3$  (where two ESSGs exist for certain values of  $\alpha$ ), there is a value  $\alpha^* = ((A_2^2 - A_1^1) - (A_1^1 - A_2^3)) / ((A_2^2 - A_1^1) + (A_1^1 - A_2^3))$  for which the basin of attraction of the ESSG at  $x = 1$  is greater than that of the ESSG at  $x = 0$  iff  $\alpha \in (\alpha^*, 1]$ . We can interpret that as follows: Assume that players in the

<sup>21</sup>For more details on these comparisons see Appendix A.2.

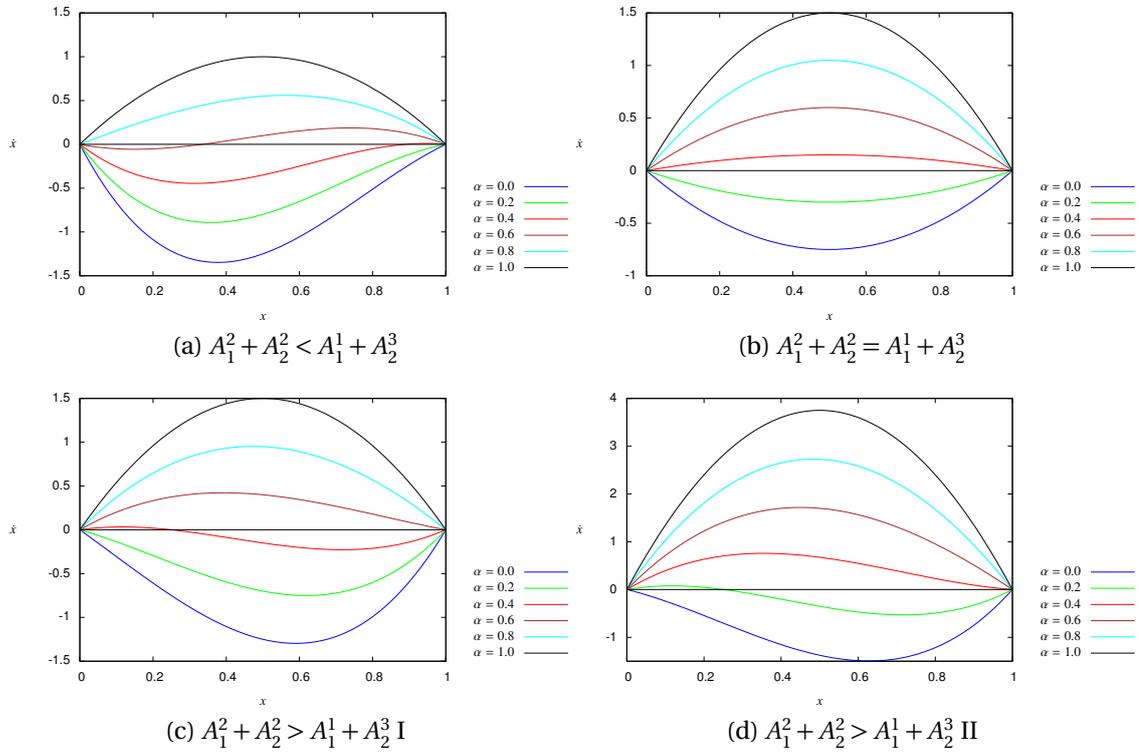


Figure 2.2: Dynamics for the PD games of Table 2.1 under constant index of assortativity matching rules.

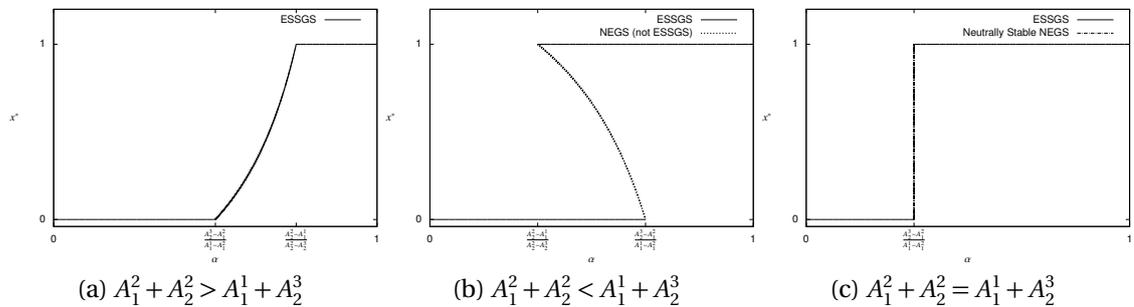


Figure 2.3: Comparative statics: NEGS as a function of the index of assortativity for the three different cases of PD games.

population do not know whether each of the other players is going to play C or D and so, using the principle of insufficient reason, they ascribe equal probabilities (equal to 0.5 each) to each other player following C and D.<sup>22</sup> Then, if  $\alpha \in (\alpha^*, 1]$  the expected payoff for a player following C is higher than his expected payoff when he follows D and so, given the aforementioned beliefs, it is a best response for all of them to follow C, leading to the population strategy being  $x = 1$ . Conversely when  $\alpha \in [0, \alpha^*)$ . So, in the terms described above, we can have a notion of *risk dominance* in the group selection game.

### 2.5.3 Welfare

In Appendix A.1.3 we show how welfare for equilibrium population strategies under the appropriate matching rules can be calculated. We use this methodology in order to conduct welfare analysis for the various PD games discussed in this section. We carry out our analysis restricting ourselves to consistent matching rules as defined by condition (2.2). The comparison of equilibrium welfare in the group selection game and welfare in the normal form game when both players use the corresponding mixed strategy is shown in Figure 2.4 for each of the three cases.

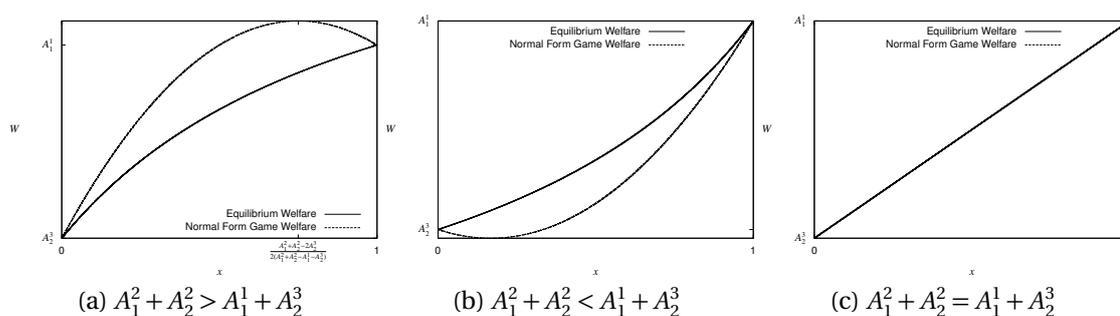


Figure 2.4: Equilibrium welfare and normal form payoff in three PD games.

The maximum level of welfare is obtained when the equilibrium population strategy is the one where all cooperate ( $x = 1$ ) and it coincides with the maximum expected payoff players using symmetric strategies can get in the normal form game when  $A_1^2 + A_2^2 > 2A_1^1 > A_1^1 + A_2^3$ . In the case where  $A_1^2 + A_2^2 > 2A_1^1$  the maximum value of welfare in the normal form game is obtained when both players play C with

<sup>22</sup>See also Carlsson and Van Damme (1993).

probability  $p_C = (A_1^2 + A_2^2 - 2A_2^3) / (2(A_1^2 + A_2^2 - A_1^1 - A_2^3))$ . However, when this population strategy is implemented as an equilibrium in the group selection game, it does not grant the players such high expected payoffs as the frequency of (C,D) or (D,C) pairs is not high enough. The implementation of an assortative matching rule can make the population strategy an equilibrium but this happens at the expense of obtained payoff at that population strategy. Also, if we restrict ourselves to equilibrium payoffs, then the payoff obtained at  $x = (A_1^2 + A_2^2 - 2A_2^3) / (2(A_1^2 + A_2^2 - A_1^1 - A_2^3))$  is no longer the optimal payoff. Once again, utilitarian optimality is achieved when  $x = 1$  (all cooperate) is implemented as an equilibrium.

## 2.6 Conclusion

This paper had two main purposes. Firstly, to extend the existing machinery of evolutionary game theory to include models of group selection; and secondly, to use the new concepts developed to discuss the relationship between different kinds of selection and the fitness of populations. Two new equilibrium concepts were proposed, *Nash equilibrium with group selection* (NEGS) and *evolutionarily stable strategy with group selection* (ESSGS). These equilibrium concepts contain as special cases the standard ones; indeed when matching is random, the set of NEGS is just the symmetric Nash equilibria and the set of ESSGS is the evolutionarily stable strategies. We proceeded to show in our main theoretical result (theorem 2.4) that NEGS and ESSGS are for models with arbitrary matching rules what Nash equilibrium and ESS are for models with random matching. In particular, any stable steady state of the replicator dynamics is a NEGS and any ESSGS is an asymptotically stable steady state. As in the standard random matching setting, these results form the theoretical foundation upon which evolutionary game theory rests; hence our concepts extend the traditional game theoretic framework to models with group selection. As for the fitness of populations, our main result is the “second welfare theorem” of evolution (theorem 2.5) which states that *any* evolutionary optimum will be a NEGS under some matching rule.

We also showed (theorem 2.1) that models with structured populations, such as the haystack and trait-group models, can be captured by appropriately defined matching rules. This makes the dynamics and equilibrium analysis of such com-

plicated models considerably easier as one can then simply apply the concepts of NEGS and ESSGS in a straightforward manner.

From an applied point of view, the great advantage of the game theoretic approach is the additional structure it imposes compared to dynamic models of the replicator type. In particular, the analysis becomes simpler and the results become more powerful. Recall that *all* uniform population strategies (all individuals employing the same strategy) are steady states for the replicator dynamics. In fact, the set of steady states includes *everything* that is “evolutionarily feasible” (and a good way to think of this set is in fact as evolutionary models’ parallel to the feasible set of an exchange economy). This of course makes stability analysis absolutely critical in the dynamic setting – the problem being that such stability analysis is *not* straightforward in group selection models where the replicator dynamics forms a complex non-linear dynamical system.<sup>23</sup> In contrast, we saw in section 2.5 that the set of NEGS and ESSGS can be computed with great ease in group selection games, and equally importantly, the game theoretic formulation allows for abstract analysis and the derivation of general results. An example of such a general result is that under random matching, the set of NEGS coincides with the symmetric Nash equilibria in the underlying normal form game which intuitively means that random matching precisely corresponds to “self-serving” behavior in general. Such a result would be impossible to establish within the traditional group selection framework of section 2.2. The “second welfare theorem” of evolution (theorem 2.5) is another example of this.

Often, matching is a geographical phenomenon (think of viruses, neighborhood imitation amongst humans, or trait-group models as studied in section 2.2.4), or a reflection of individuals’ limited ability to monitor other individuals (see the introduction for further details). But when matching rules correspond to institutions or conventions, not explaining how they come about misses half the story. A clear weakness of existing group selection models – including the results in this paper – is in this connection that the matching rules are taken as given. An obvious topic for future research would be to model the evolution of the matching rules (*i.e.*, to en-

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<sup>23</sup>Thus, consider for example the discrete time replicator dynamics of equation (2.8) in the often-studied case with two strategies. Unlike in models with random matching where the  $\pi$ s are linear, in models with non-random matching these coefficients will depend on the population strategy through the matching rules in an often very complicated way. This of course makes even local stability analysis a daunting task.

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dogenize them). Consider monitoring: If individuals gain an advantage by increasing their ability to monitor (by increasing their intelligence and memory), we can see how matching rules will over time evolve to be less and less random (typically more and more assortative). This then would be a true endogenous description of matching (institutions, conventions). The simplicity of the game theoretic framework presented in this paper should definitely put such a theory of matching rules within reach.

# Chapter 3

## Assortativity Evolving From Social Dilemmas

### Chapter Abstract

Assortative mechanisms can overcome tragedies of the commons that otherwise result in dilemma situations. Assortativity criteria include genetics (e.g. kin selection), preferences (e.g. homophily), locations (e.g. spatial interaction) and actions (e.g. meritocracy), usually presuming an exogenously fixed matching mechanism. Here, we endogenize the matching process with the aim of investigating how assortativity itself, jointly with cooperation, is driven by evolution. Our main finding is that only full-or-null assortativities turn out to be long-run stable, their relative stabilities depending on the exact incentive structure of the underlying social dilemma. The resulting social loss is evaluated for general classes of dilemma games, thus quantifying to what extent tragedy of the commons may be endogenously overcome.

## 3.1 Introduction

The ‘puzzle of cooperation’ is as old as the theory of evolution (Darwin, 1871). When the incentives of the population and of its individuals are misaligned (‘mixed-motive situations’/‘social dilemmas’), two evolutionary forces collide; the individual and the collective. As a result, collective action (Olson, 1965) may fail and tragedy of the commons (Hardin, 1968) may result. However, many mechanisms exist through which cooperative behaviors are achieved against the evolutionary tendency towards noncooperative behaviors that would prevail in the absence of a suitable mechanism (see West, Mouden, and Gardner, 2011, for a recent review of mechanisms and of common misconceptions). Hence, the ‘puzzle of cooperation’ is that nature, involving humans and animals alike, provides us with many examples of social dilemma situations that are successfully resolved by suitable mechanisms, but also with many other examples resulting in tragedy of the commons.

Perhaps the best methodology to study the evolution of cooperation is provided by game theory (von Neumann and Morgenstern, 1944). Without suitable mechanisms, the game-theoretic predictions in mixed-motive situations are associated with non-cooperative strategizing (Nash, 1951) which contradicts collective interests. In that case, no stable equilibrium exists that reconciles the population and individual interests (Aumann, 1974, 1987). The game-theoretic literature has addressed this issue at length (beginning with Axelrod, 1984; Hamilton, 1963, 1964a,b). This literature shows that, absent suitable mechanisms, societies evolve to socially undesirable outcomes if interactions in the population are well-mixed/random (Nash, 1950; Young, 2011).

The class of mechanisms that we shall focus on in this paper share the common feature of non-random interactions, that is, they are assortative mechanisms. The first formulations of assortative mechanisms date back to Wright (1921, 1922, 1965). Well-known processes that lead to assortativity are genetics (‘kin selection’; Domingue et al., 2014; Hamilton, 1964a,b), locality (‘spatial interactions’; Abdellaoui, Verweij, and Zietsch, 2014; Eshel, Samuelson, and Shaked, 1998; Nowak and May, 1992; Skyrms, 2004), preferences (‘homophily’; Alger and Weibull, 2012, 2013; Xie, Cheng, and Zhou, 2015), and actions which, in the context of mixed-motive situations, is a ‘meritocratic’ criterion that matches players according to their decision to cooperate (the decision associated with ‘merit’) or to defect (Gunnthorsdottir et

al., 2010; Nax, Murphy, and Helbing, 2014a; Rabanal and Rabanal, 2014). Under sufficiently assortative mechanisms, the game-theoretic predictions improve (e.g. Bergström, 2003; Hamilton and Taborsky, 2005a,b; Jensen and Rigos, 2014; Nax, Murphy, and Helbing, 2014a). Indeed, social dilemmas can be resolved via assortativity, and better, cooperative outcomes may be reachable.

The effects of assortativity on the underlying strategic nature of a mixed-motive situation are substantial. It is unlikely, however, that assortativity fell from the sky. More likely, it evolved driven by evolutionary dynamics within the population and across populations. In this paper, we contribute to the assortativity literature by endogenizing the evolution of assortativity, in particular of actions. Our results also apply to the contexts of kin selection, local interaction or homophily. In our model, the evolution of assortativity is assumed to be driven by democratic consensus based on one of the most basic models also for social forces or biological auctions, featuring diffusion and intensity of preferences in either direction.<sup>1</sup> To the best of our knowledge there exists no comparable prior study of evolving assortativity based on democratic consensus dynamics, which permits an interpretation in terms of human interactions too.<sup>2</sup> Related is, for example, Newton (2014) who studies evolving assortativity, when assortativity is allowed to depend on types, in the context of the preference evolution model by Alger and Weibull (2012, 2013, 2014, 2015). Other ways of endogenizing the matching rule may lead to different results, and these are avenues for further research we shall sketch in our concluding discussion.

In terms of underlying games, we focus on the most general class of symmetric two-by-two social dilemmas that nests the standard prisoners' dilemma (PD) (Rapoport and Chammah, 1965) but also includes other games. All agents are of one type, one whose behavior is driven by his own material self-interest alone. All our social dilemmas considered, not just the PD, are important situations that often occur with costly effects in reality. The PD is the best-known example of social dilemmas, that is, of situations with the common characteristic that individuals have an

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<sup>1</sup>Democratic consensus of this kind is an aggregation rule used also by many animal species (Chatterjee, Reiter, and Nowak, 2012; Couzin et al., 2011) such as bees selecting hive-locations (Seeley and Visscher, 2004) or ants choosing nest sites (Franks et al., 2002). In voting theory, such rules are known as range, average, cardinal, utility or score voting; numerous proto-democratic human collectives use this rule (Staveley, 1972), similar to tug of war or voting by clapping/shouting.

<sup>2</sup>In biology, models have been proposed based on different factors such as invasion by mutants (Dieckmann and Doebeli, 1999; Jiang, Bolnick, and Kirkpatrick, 2013) or other, indirect factors (Bearhop et al., 2005; Dyson-Hudson and Smith, 1978).

incentive to defect when facing cooperators. The evolution of cooperation amongst humans and animals in social dilemma situations has received enormous attention, and the PD in particular has been studied widely in this context beginning with Axelrod and Hamilton (1981). Beyond the PD, there are related, less well-known social dilemmas of comparable practical importance. All our social dilemmas share the public goods character, but games differ with respect to their (*i*) Nash equilibria and (*ii*) socially desirable outcomes. Our social dilemma situations include the prisoners' dilemma, the volunteer's dilemma (Diekmann, 1985; Myatt and Wallace, 2008; Raihani and Bshary, 2011), the missing hero dilemma (Schelling, 1971) and the underprovision dilemma.<sup>3</sup>

In terms of dynamics, we use standard evolutionary replicator equations (Taylor, 1979; Taylor and Jonker, 1978). In the standard mathematical formulation of such a dynamic (e.g. Eshel, 1983; Eshel, Motro, and Sansone, 1997; Helbing, 1992; Weibull, 1995), we would assume a well-mixed population, that is, pairs would be drawn uniformly at random from the population. Here, we shall focus on action-assortative matching instead, using recently introduced methods (Bergström, 2003; Jensen and Rigos, 2014). In our dilemma games, such a rule is 'meritocratic' as it 'rewards' ('punishes') cooperators (defectors) by matching them with other cooperators (defectors). Assortativity itself evolves by democratic consensus. In the PD game, for example, cooperators prefer more assortativity in order to be matched less often with defectors, while defectors prefer less assortativity for the opposite reason. In which direction this struggle evolves depends on how many people stand on either side, and by how much they benefit from either change.

Our analysis proceeds in three steps. First, we study the stability of equilibria given an exogenous level of assortativity. Second, we endogenize the evolution of assortativity and investigate the stability of regimes under our voting dynamic. Finally, we evaluate which outcome is more stable in the long run. Our main findings summarize as follows. Only null-or-full assortativities are long-run stable, providing evolutionary support for models making either assumption depending on context as in Wright (1921, 1922, 1965). We can thus identify how long-run performances in terms of assortativity and cooperativeness depend on the exact incentive structure

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<sup>3</sup>As a byproduct of our operationalization, we introduce the 'underprovision dilemma', a variant of the volunteer's dilemma, which to the best of our knowledge has not previously been considered but certainly also represents an important class of games deserving investigation.

of the underlying social dilemma. Seemingly small differences between classes of social dilemmas, and even within the same social dilemma class, matter crucially for convergence properties. Our analysis quantifies system-level endogenous efficiencies relative to social and assortative optima, in particular to what degree tragedy of the commons is overcome.

## 3.2 The Model

### 3.2.1 Social dilemmas

We start by setting out the general setup. Here, we have a continuum population in the closed interval  $[0, 1]$  that can follow one of two strategies, either ‘cooperate’ ( $C$ ) or ‘defect’ ( $D$ ). (Alternative labels could be ‘contribute’ and ‘free-ride’.) Denote by  $x$  the proportion of players playing  $C$ . Individuals in the population follow one of the two strategies, get matched to one other individual in the population, and then carry out their strategy in their pair. The exact process by which they get selected in pairs will be discussed in the next section.

**Social dilemma** Underlying all our interactions is a social dilemma game represented by a matrix of the following kind:

	C	D
C	$r, r$	$a, 1$
D	$1, a$	$0, 0$

Hence a social dilemma is defined by  $G = (r, a)$ . To ensure that C-C is not an equilibrium under random matching, we impose  $0 < r < 1$  for all  $G$ , which defines the common ‘public goods character’: defection is always a best response against cooperation. Moreover, we restrict  $a \in (-1, r)$ , so that C-D outcomes can either be more or less efficient than C-C, while D-D remains the least efficient in all cases and cooperation by one player always leads to the other player obtaining a higher payoff. We therefore investigate the following four different types of (well-known) social dilemma games:

**Prisoners’ dilemma** (Rapoport and Chammah, 1965) The PD game is obtained by setting  $2r > 1 + a$  and  $a < 0$ . Defection is a strictly dominant strategy, and total payoffs are highest in C-C. The unique Nash equilibrium is D-D.

**Volunteer's dilemma (Diekmann, 1985)** The VD game is obtained by setting  $2r < 1 + a$  and  $a > 0$ .<sup>4</sup> Cooperation is a best response against defection, and the outcome where exactly one player contributes maximizes total payoffs. The game is a symmetric anti-coordination game with two (efficient) C-D Nash equilibria in pure strategies and one in mixed strategies. The mixed-strategy equilibrium is the unique symmetric one.

**Underprovision dilemma** The UD game is obtained by setting  $2r > 1 + a$  and  $a > 0$ . It is a natural variant of the VD, but, to the best of our knowledge, has not been treated formally previously. Like in the VD, cooperation is a best response against defection, but now the synergies to mutual cooperation (beyond cost sharing) are so high that C-C maximizes total payoffs. The game is a symmetric anti-coordination game with two (inefficient) C-D Nash equilibria in pure strategies and one in mixed strategies. As in VD, the mixed-strategy equilibrium is the unique symmetric one.

**Missing hero dilemma** The MHD game is obtained by setting  $2r < 1 + a$  and  $a < 0$ .<sup>5</sup> Defection is a strictly dominant strategy and the unique Nash equilibrium is D-D, but –other than in the standard PD– the C-D outcome maximizes total payoffs.

Figure 3.1 illustrates how the different social dilemmas live in  $r$ - $a$  space. The differences in the nature of the dilemmas are summarized as follows:

		Efficient Outcome	
		C-D ( $a > 2r - 1$ )	C-C ( $a < 2r - 1$ )
Best Reply versus D	C ( $a > 0$ )	VD	UD
	D ( $a < 0$ )	MHD	PD

Note that only in the VD game it is the case that the pure-strategy Nash equilibria of the baseline normal-form game and the efficient outcomes coincide (the 'baseline loss' of the Nash equilibrium prediction is 0). In all other cases, there is a positive 'baseline loss': either C-C is efficient but D-D (prisoners')/ C-D (underprovision) is equilibrium, or C-D is efficient but D-D is the equilibrium (missing hero dilemma). Since we consider a one-population matching protocol, the asymmetric

<sup>4</sup>In the basic formulation (Diekmann, 1985),  $r = a$ . However, (Diekmann, 1985) mentions the possibility of synergies through cost sharing ( $a < r$ ) considered in (Weesie and Franzen, 1998).

<sup>5</sup>A less-studied variant of the VD (Diekmann, 1985), recently studied in (Diekmann and Przepiorka, 2015), with a different equilibrium structure; see also (Schelling, 1971).

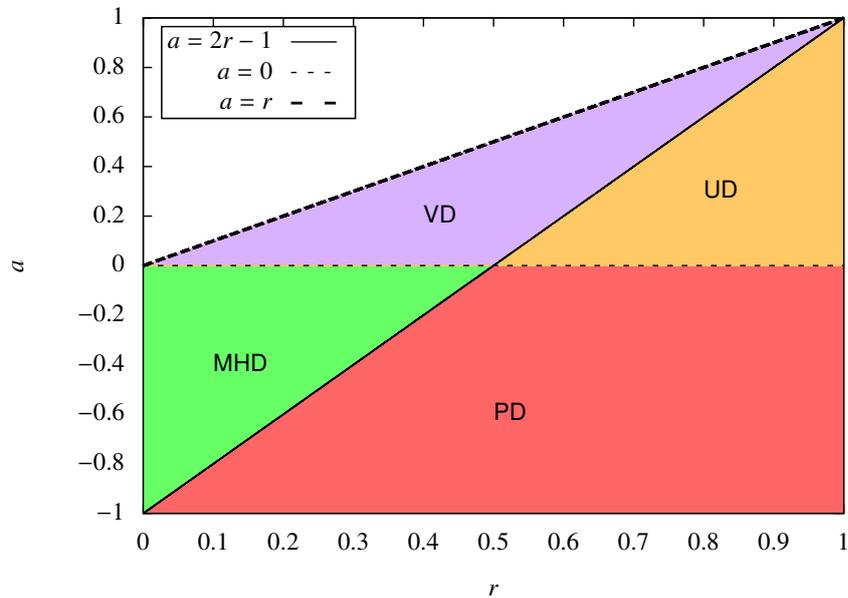


Figure 3.1: Types of social dilemmas.

pure-strategy Nash equilibria C-D cannot be achieved but only the mixed-strategy equilibria.

### 3.2.2 Action Assortativity

We follow Nax, Murphy, and Helbing (2014a) in the definition of an action-assortative matching rule represented by a *constant index of assortativity*  $\alpha \in [0, 1]$  (Bergström, 2003). At one extreme ( $\alpha = 1$ ) is full assortativity, where cooperators are matched with cooperators (and defectors with defectors) with probability one. The other extreme ( $\alpha = 0$ ) is random matching—the standard assumption in the literature—where players are randomly matched with each other independently of their actions.

**Environment** The environment  $E$  is defined by a social dilemma,  $G = (r, a)$ , together with a given level of assortativity,  $\alpha$ ;  $E = (G, \alpha)$ .

We use notation from Jensen and Rigos (2014) so that, for any social dilemma  $G$ , we can fully describe a (*consistent*) *matching rule* by only describing how one variable changes with respect to  $x$  (the proportion of cooperators in the population). This variable, denoted by  $\phi$ , expresses the proportion of pairs formed (ac-

ording to the matching rule) that are of the mixed type (*i.e.* contain one cooperator and one defector, also  $f_2$ ). The expression for  $\phi$  to represent a constant-index-of-assortativity matching rule will then be

$$\phi(x) = f_2(x) = 2(1-\alpha)x(1-x),$$

where  $\alpha$  is the index of assortativity. The proportions of homogeneous pairs ( $f_1$  for two cooperators, and  $f_3$  for two defectors) will be given by

$$f_1(x) = x - \frac{\phi(x)}{2} \quad f_3(x) = 1 - x - \frac{\phi(x)}{2}.$$

**Efficiency** For a given  $E = (G, \alpha)$  and a proportion of cooperators  $x$ , efficiency (the average payoff in the population) is given by

$$W(x, \alpha) = r f_1 + \frac{1+a}{2} f_2 + 0 \cdot f_3 = r x + (1-\alpha)(1-x)x(1+a-r).$$

The average payoff of a cooperator is  $\pi_C = (r f_1 + a f_2/2)/x$ , and that of a defector is  $\pi_D = (f_2/2)/(1-x)$ . Thus, the dynamics of  $x$  become

$$\dot{x} = x(1-x)(\pi_C - \pi_D). \quad (3.1)$$

**Definition 3.1.** *Given environment  $E = (G, \alpha)$ ,  $x^* \in [0, 1]$  is an environment equilibrium if, in some arbitrarily small neighbourhood around  $x^*$ ,  $\dot{x} \geq 0$  when  $x \leq x^*$  and  $\dot{x} \leq 0$  when  $x \geq x^*$ .*

**Lemma 3.1.** *For any environment  $E$ , there exists an environment equilibrium.*

**Proof.** See Appendix B.1.1.

**Lemma 3.2.** *For all our social dilemmas  $G$ , maximum environment equilibrium efficiency is increasing in the assortativity,  $\alpha$ , of the environment.*

**Proof.** See Appendix B.1.2.

### 3.2.3 Full dynamics

The dynamics on assortativity  $\alpha$  that we consider are motivated by utility voting. We assume that the tendency for  $\alpha$  to increase/decrease is driven by relative size

and growth of the two populations that would benefit from an  $\alpha$ -increase/decrease. In particular,  $\alpha$  is governed by the following dynamics: each player gets one vote to cast; either for higher or for lower  $\alpha$ . The probability for a player  $i$ , currently matched into a homogeneous group (C-C or D-D), to vote for an increase of  $\alpha$  is increasing in  $i$ 's payoff. Similarly, the probability for a player  $j$  who is currently matched into a heterogeneous group (C-D) to vote for an increase of  $\alpha$  is decreasing in  $j$ 's payoff. In particular, players use a logit-based rule to decide whether to vote for higher or lower meritocracy.

More specifically, let us denote by  $M$  and  $m$  respectively the highest and lowest payoff that can be attained by any player in a given social dilemma. If a player gets  $M$  ( $m$ ), then, with probability one, he votes for an increase (decrease) of  $\alpha$  if in a homogeneous (heterogeneous) group and for a decrease of  $\alpha$  if in a heterogeneous (homogeneous) group. If receiving a payoff of  $u \in (m, M)$ , a player in a homogeneous group votes for an increase of  $\alpha$  with a probability given by

$$p_{\text{homog}}^+(u) = \frac{e^{g(u)}}{1 + e^{g(u)}}, \quad (3.2)$$

where  $u$  is the payoff the player received and the normalizing function  $g(\cdot)$  is given by

$$g(u) = \frac{1}{M - u} - \frac{1}{u - m}. \quad (3.3)$$

The function  $g$  is used so that comparisons between different types of social dilemmas can be made.

Consequently this means that, for  $u \in (m, M)$ , the probability that the player votes for a decrease of  $\alpha$  is

$$p_{\text{homog}}^-(u) = \frac{1}{1 + e^{g(u)}}. \quad (3.4)$$

Hence, the ‘‘excess’’ probability ( $p^+ - p^-$ ) for a player matched in a homogeneous group to vote for an increase of  $\alpha$  is

$$z_{\text{homog}}(u) = \frac{e^{g(u)} - 1}{e^{g(u)} + 1} \quad (3.5)$$

Similarly, for players in heterogeneous groups, the excess probability for them

to vote for an increase in  $\alpha$  will be

$$z_{\text{heter}}(u) = \frac{1 - e^{g(u)}}{e^{g(u)} + 1}. \quad (3.6)$$

Obviously,  $z_{\text{heter}}(u) = -z_{\text{homog}}(u)$ . In our calculations, we will use the function  $z(\cdot)$  given by

$$z(u) = \begin{cases} -1 & \text{if } u = m \\ \frac{\exp(g(u))-1}{\exp(g(u))+1} & \text{if } u \in (m, M) \\ 1 & \text{if } u = M \end{cases} \quad (3.7)$$

Let  $v^+$  and  $v^-$  denote the number of votes for an increase and for a decrease of the level of assortativity respectively. Aggregating the votes, these are  $v^+ = f_1(x, \alpha)z(r) + f_3(x, \alpha)z(0)$  and  $v^- = f_2(x, \alpha)(z(a) + z(1))/2$ .

Now the exact form of the dynamics takes a replicator-style form:

$$\dot{\alpha} = \alpha(1 - \alpha)(v^+ - v^-) \quad (3.8)$$

The main patterns that arise under these dynamics are depicted in Figure 3.2.

**Full equilibrium** We are interested in identifying states that are stable under the full dynamics. We define a *full equilibrium* as follows.

**Definition 3.2.** *For any social dilemma  $G = (r, a)$ , a pair  $(x^*, \alpha^*)$  will be called a full equilibrium if it is a stable node of the full dynamics (equations 3.1 and 3.8).*

Obviously, for  $(x^*, \alpha^*)$  to be a full equilibrium, it is necessary for  $x^*$  to be an environment equilibrium of  $E = (G, \alpha^*)$ , and  $\alpha^*$  to be an evolutionarily stable state of the voting dynamics given  $x^*$ .

**Observation 1.** *All full equilibria of any social dilemma  $G$  have either  $\alpha^* = 1$  or  $\alpha^* = 0$ .*

For a proof and discussion see Appendix B.1.3.

Whether full assortativity is ‘more robust’ than null assortativity depends on the type and/or exact parameter values of the underlying social dilemma  $G$ .<sup>6</sup> In search

<sup>6</sup>Note that Bergström (2013) studies games where populations with varying assortativity levels (beyond full-or-null) compete.

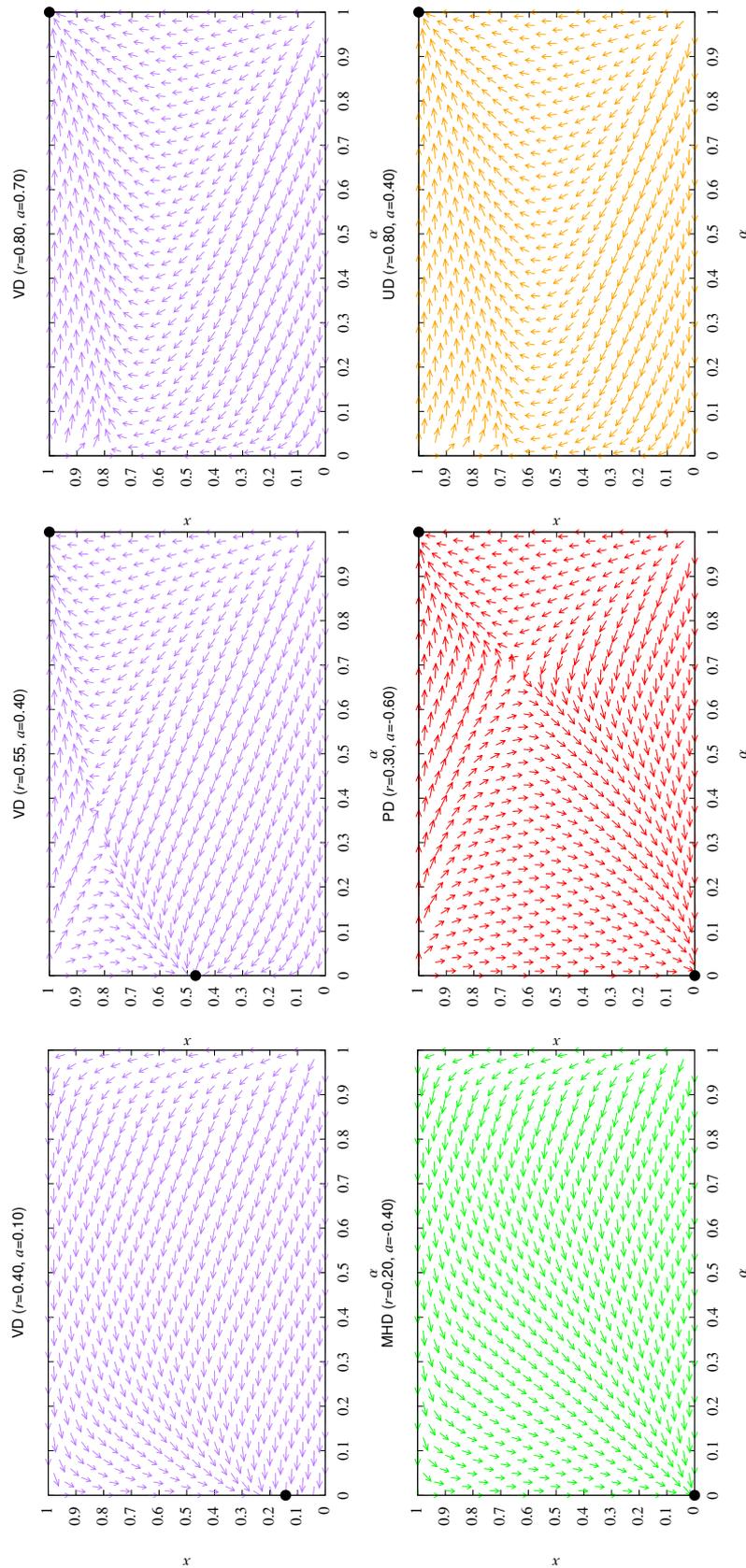


Figure 3.2: The principal patterns that result from the full dynamics for the different types of social dilemmas. Black dots indicate the full equilibria.

of a refinement akin to ‘stochastic stability’ (Foster and Young, 1990) for the case of continuous populations, we shall formalize this comparison as follows.

**Definition 3.3.** *Assortativity robustness* Let  $(x_0^*, 0)$  and  $(x_1^*, 1)$  denote the two candidate full equilibria given  $G$ . Then, full assortativity is  $\varrho$ -robust for some  $\varrho \in [0, 1]$  if, for every  $\varrho' \in (\varrho, 1]$ , all convex combinations  $\varrho'(x_1^*, 1) + (1 - \varrho')(x_0^*, 0)$  lie in the basin of attraction of  $(x_1^*, 1)$  and, for every  $\varrho'' \in [0, \varrho)$ , all convex combinations  $\varrho''(x_1^*, 1) + (1 - \varrho'')(x_0^*, 0)$  lie in the basin of attraction of  $(x_0^*, 0)$ .

Note that assortativity robustness  $\varrho$  can be seen as a measure of the expected full equilibrium efficiency relative to full assortativity (the assortative optimum).<sup>7</sup>

Figure 3.3 summarizes the robustness analysis in  $r$ - $a$  space.

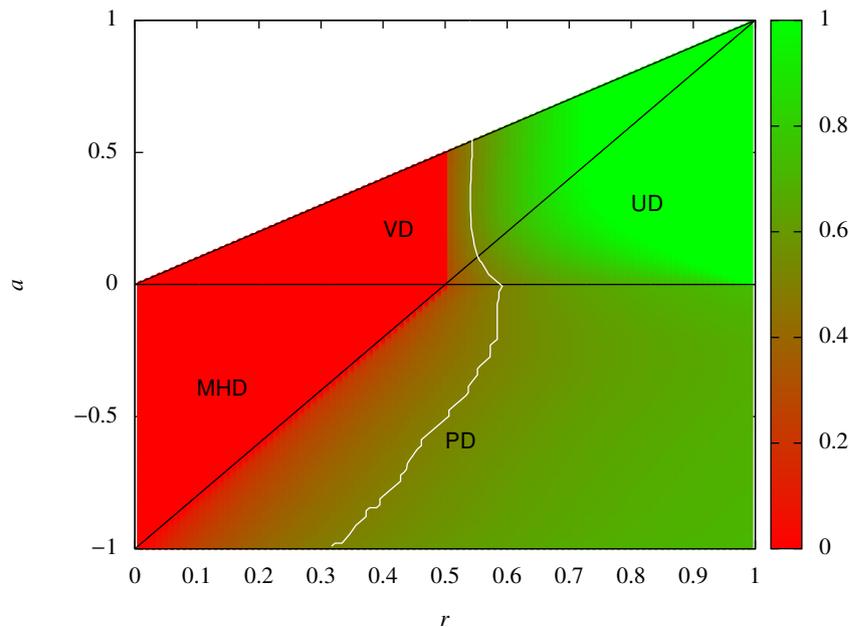


Figure 3.3: Robustness of full assortativity. The white line separates the cases where full/null assortativity is more robust.

### 3.3 Discussion

The ‘puzzle of cooperation’ in the sense of how and why cooperation amongst animals or humans emerges and survives in some social dilemmas but not in others,

<sup>7</sup>The expected full equilibrium efficiency is expressed by  $\varrho r + (1 - \varrho)(a / (1 - r + a))$  when  $a > 0$  and  $\varrho r$  when  $a \leq 0$  which is compared to  $r$ , the efficiency under full assortativity, yielding an expression that is linear in  $\varrho$  in both cases.

has kept scientists busy for many years. One strand of research in this area has been to understand the role of assortativity, through various mechanisms, in overcoming the inherent social dilemma. Indeed, some of the best-known social dilemma mechanisms such as kin selection, homophily, spatial interaction and action assortativity belong to this family.

In this paper, we focus on action assortativity and break with the assumption of a pre-existent, fixed level of assortativity. Instead, we propose a dynamic by which assortativity co-evolves with cooperation through democratic consensus. That way, we are able to study what assortativity-cooperation pairs are evolutionarily co-stable. When endogenizing matching through democratic consensus, we find that only full-or-null assortativities are stable. Depending on the nature of the social dilemma, cooperation above baseline Nash predictions may thus emerge, in some cases (as in most underprovision dilemmas and in volunteer's dilemmas with  $r > 0.5$ ) up to the socially optimal level. The emergent levels of assortativity mitigate the tragedy of the commons in the prisoners' dilemma, substantially in most of them with  $r > 0.5$ . Only in missing-hero dilemmas and in volunteer's dilemmas with  $r < 0.5$ , the population completely fails to lift itself above the worst possible outcome by becoming more assortative. Our findings explain why and how the co-stability of assortativity and cooperation depends crucially on the incentive structure of the underlying social dilemma.

Avenues for future include exploring different ways of endogenizing matching tailored, not to action assortativity, but to kin selection, homophily and spatial interaction. For example, when interactions are spatial, then individuals' choices to relocate would determine evolving assortativity.

## Chapter 4

# A Beauty Contest with Flexible Information Acquisition

### Chapter Abstract

This paper studies beauty-contest coordination games in a flexible information acquisition setting à la Yang (2015). A continuum of players receive payoffs based on the squared distances of their actions from an unobserved fundamental state of the world and the average action among all players. Each player receives a signal whose probability distribution conditional on the value of the fundamental is part of their strategy. Thus, she chooses not only how precise but also what kind of information she wants to get about the fundamental, while paying a cost linear to the reduction of entropy. Necessary conditions are derived for well-behaved equilibria. The case of aggregately affine equilibria (AAE) where the average action is an affine function of the fundamental is examined in detail. AAE exist only if the fundamental is normally distributed. Higher information costs, a stronger coordination motive or a more concentrated distribution of the fundamental lead to less attention being paid to the fundamental. When information costs are high, there is a unique equilibrium where players do not acquire information. For a large region of the parameter space, there exists a unique equilibrium within the classes of AAE and equilibria without information acquisition. Interestingly, when the coordination motive is high and for relatively low (but bounded away from zero) information costs, there is a multiplicity of equilibria within the classes considered, suggesting that flexible information acquisition technology can be the source of multiple equilibria.

## 4.1 Introduction

This paper studies a beauty-contest coordination game (as formulated by Morris and Shin, 2002) under a “flexible information acquisition” technology à la Yang (2015). Beauty contests capture situations where “players wish to do the right thing[...] and do it together” (Myatt and Wallace, 2012). A continuum of agents have two motives: a “fundamental” motive in the sense that they want to take actions “close” to the value of some unobserved but “real” random variable, and a “coordination” motive in the sense that they want their action to be “close” to the average action of the population. Information on the unobserved random variable (henceforth *the fundamental*) can be extremely valuable to the players. It can help them to be close to the realized value of the fundamental and also serve as a coordination device. If acquiring such information is costly, then players face a trade-off between the benefit of the extra piece of information weighed against its cost.

Players are assumed to be (rationally) inattentive (Sims, 1998, 2003, 2006): they are constrained in the amount of information they can process, and so they need to focus their attention on the most important events for them. Unlike in the work of Sims, though, instead of this constraint being exogenously given, it is determined endogenously in a “flexible” manner following Yang (2013, 2015) and Yang and Zeng (2015) who use this technology in binary choice models. Each player is allowed to relax her constraint by paying a cost. In this way, the information acquisition technology is flexible from two perspectives: players decide on the *amount of information* they wish to purchase as well as on the *best way* to use this limited information (to which events they want to pay more attention). They do so by choosing the functional form of the distribution that their signals are going to follow.

Several questions arise. Firstly, can flexible information acquisition technology be extended to continuous choice models? Moreover, how do information costs, the coordination motive, and the distribution of the fundamental affect the way players acquire information? Additionally, issues of equilibrium multiplicity that appear in Yang (2015) are addressed: do multiple equilibria arise because of the game structure or because of the information acquisition technology? In short, it is shown that the concept of flexible information acquisition can be successfully extended to environments of continuous choice. Also, a higher coordination motive, higher information costs and more concentrated fundamental lead to less attention being paid

to the fundamental. Finally, multiple equilibria can arise because of the information acquisition technology, even though the full-information version of the game being played has a unique equilibrium.

Beauty contest games have been extensively studied in the literature. Motivating examples in economics for such interactions come from industrial organization where players are assumed to be competing in settings with strategic complementarities (see Myatt and Wallace, 2012, 2015), financial markets where players are traders and try to forecast the value of the fundamental while competing with each other (Allen, Morris, and Shin, 2006) or investment games (as in Angeletos and Pavan, 2004). Such games have also been used to model policy choices of political party members (Dewan and Myatt, 2008). The study of beauty contests gained so much attention because of the profound intuitions it can offer on the way that economies aggregate information as well as on the value that information has for economies.

Since the seminal work of Morris and Shin (2002), the literature that studies the value of information in these coordination games has been growing constantly, with the social value of information being in focus. This literature studies under which conditions more “public” or more “private” information is socially optimal. Most authors consider exogenous information structures where players cannot affect the information they get and only make decisions based on the signals they receive. Exceptions include Myatt and Wallace (2012, 2015), Dewan and Myatt (2008) and Hellwig and Veldkamp (2009). In their models, the way players obtain information is endogenized as they can affect the information they get by purchasing more signals from different sources or by bearing a cost in order to increase their signal precision. Despite endogenizing the information acquisition process, previous research has assumed specific functional forms for the distribution of the fundamental as well as of the signals that the players observe, typically taken to be Gaussian. In these cases, they identify linear equilibria *i.e.* equilibria where the players’ actions are linear or affine functions of the signals they receive.

This paper builds on the recent article by Yang (2015) who introduced the notion of *flexible information acquisition* based on the work on rational inattention (Sims, 1998, 2003; Woodford, 2009). Yang (2015) studied a two-player coordination game with two strategies (invest or not) that has a global-game structure. Players want to always invest if the value of the fundamental is high, to never invest when the fun-

damental is low, and to invest only if the other player invests as well for fundamental values between two thresholds. Each of the two players observes a signal and takes an action conditional on it. Crucially, they decide the distribution that their signal will follow conditional on the value of the fundamental. This comes at a cost that is linear in Shannon's measure of mutual information between the distribution of the fundamental and the player's chosen distribution over signals.<sup>1</sup> The result is that players want to obtain information about the fundamental only if its prior distribution is concentrated enough between the two "threshold" values described earlier and information costs are low enough. When costs are relatively high, he identifies a unique equilibrium of the game whereas for low information costs there is multiplicity of equilibria.

Yang (2015) and the model presented here share two main features: players in both models (i) use a flexible information acquisition technology and (ii) play a coordination game. On the other hand, there are three main differences. Firstly, Yang's players face a binary decision problem whereas a continuous action set is studied here. Secondly, the equilibrium structure of the full-information version of the two games is different. Multiple equilibria can arise in Yang's investment game whereas in beauty contests under full information there exists a unique equilibrium. Lastly, instead of using the cost of miscoordination as a parameter, the coordination motive of individuals is explicitly modeled in beauty contests and enters players' objective functions as a parameter. Given these differences, the questions laid out earlier can be addressed.

As a general result, an extension of Yang's (2015) apparatus to settings with continuous choice can be achieved. If the action space is unbounded and the prior has a full support over an unbounded space, it is possible to identify well-behaved equilibria in which players acquire information. Additionally, it is shown that affine equilibria (where players' average action is an affine function of the fundamental) exist *only if* the fundamental is normally distributed. Focusing on these equilibria the following results are established.

As in beauty contests the coordination motive is explicitly modeled as a parameter, it allows one to study how it affects equilibrium outcomes. When used in con-

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<sup>1</sup>Mutual information (see for example Kolmogorov, 1956; Shannon, 1948) has to do with the reduction of entropy. It measures the reduction of the "randomness" of one random variable once the value of another random variable is known.

junction with the other parameters of the model (the cost of information and the variance of the distribution of the fundamental), one can obtain rich comparative statics results. Thus, the exact way in which these variables affect the (equilibrium) level of attention being paid to the fundamental can be explored. In line with intuition, higher costs lead to less attention being paid to the fundamental as information about its value becomes more expensive. Similar results are obtained for a more concentrated fundamental: If players know approximately where the fundamental lives (when the prior has low variance), the marginal benefit from acquiring information is quite low and thus, less of it is obtained.

Possibly against intuition, a higher coordination motive leads to less attention being paid to the fundamental – even to the point of no information acquisition. The mechanism through which this works is that if players have a strong coordination motive, knowing the actual value of the fundamental is not of particular help to them. This is because coordinating close to the actual value of the fundamental is expensive as it requires more information. So, it is cheaper for them to obtain less information and focus their actions around the ex-ante expected value of the fundamental. In this way, they cannot be too far away from efficient coordination while still paying some attention to the fundamental.

Finally, as discussed above, in the full-information version of Yang's game many equilibria appear. It is therefore unclear whether the multiplicity of equilibria stems from the equilibrium structure of the game studied or whether it emerges from the flexible information acquisition technology. In a beauty contest game with full information there is a unique equilibrium where all players take actions equal to the (perfectly known) value of the fundamental. Thus, studying a beauty contest with flexible information acquisition, as is done here, can shed light on this issue. It is found that, indeed, in the case of a normally distributed prior multiple affine equilibria arise when the coordination motive is high and for relatively low (but bounded away from zero for any given value of the coordination motive) information costs.<sup>2</sup> A unique equilibrium is present in all other parameter combinations. So, it seems that the information acquisition technology is (at least partially) responsible for many equilibria appearing under some conditions, therefore making coordination harder for players to achieve.

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<sup>2</sup>Similar results are also found by Myatt and Wallace (2012) when entropy-reduction costs are being assumed.

The layout of the paper is as follows: Section 4.2 sets up the model while in Section 4.3 best responses that follow a smoothness condition are being calculated. Necessary conditions for “well-behaved” equilibria and equilibria where no information acquisition takes place are derived in Section 4.4. Section 4.5 takes an exhaustive look into the case of aggregately affine equilibria where the average action of the population is an affine function of the realization of the fundamental. Section 4.6 concludes.

## 4.2 Model Setup

This section introduces the beauty contest game as is set up by Myatt and Wallace (2012) (based on Morris and Shin, 2002) and the information acquisition technology of Yang (2015).

A continuum of identical expected utility-maximizing players, who are indexed by  $i \in [0, 1]$ , are playing a coordination game. Each of them obtains payoff given by

$$u_i = \bar{u} - (1 - \gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a})^2 - C((S_i, q_i))$$

where  $a_i \in \mathbb{R}$  is player  $i$ 's action and  $\bar{a} = \int_0^1 a_i di$  represents the agents' average action.<sup>3</sup> The variable  $\theta$  (the fundamental motive) is a random variable that has a continuous probability density function (PDF)  $p$  over  $\Theta = \mathbb{R}$  with a well-defined mean  $\bar{\theta}$  and variance  $\sigma_\theta^2$ .<sup>4</sup> The distribution  $p$  is assumed to be common knowledge or a common prior that the players have on the fundamental. Players are incentivized in two different ways: a) they want to coordinate and b) they want to get close to the realized value of  $\theta$ . The parameter  $\gamma \in (0, 1)$  determines how strong the coordination motive is. Finally,  $\bar{u}$  is a variable that may depend on the action profile of the whole population (*e.g.* it may depend on the variance of the players' actions) but cannot be affected by a single player's action as there is a continuum of players and a single player's action has zero contribution to any aggregate variable. In any case,  $\bar{u}$  is strategically irrelevant but may be relevant for welfare. Since the focus is on the strategic interaction between players, the expression that determines  $\bar{u}$  does

<sup>3</sup>It is assumed that agents act in a way such that the value of  $\bar{a}$  is well-defined. For a discussion on this point see Myatt and Wallace (2012, footnotes 3 and 6).

<sup>4</sup>It is also assumed that  $p$  is continuously differentiable ( $p \in \mathcal{C}^1$ ).

not affect the results reported here.

Agents can obtain information on the value of  $\theta$ . They do so by choosing an information structure that consists of a set of signals  $S_i$  and a selection of probability distributions over  $S_i$  conditional on the value of  $\theta$  i.e.  $q_i : \Theta \rightarrow \Delta(S_i)$ .<sup>5</sup> The signal that player  $i$  receives is drawn independently of the signals of the rest of the players and will follow the conditional distribution  $q_i(\theta)$ . The function  $C((S_i, q_i))$  represents the cost of obtaining this information.

The cost of information acquisition is linear in Shannon's mutual information measure between  $q_i$  and  $p$ . That is, the more informative the information acquisition strategy  $(S_i, q_i)$  is, the higher the cost of information. More explicitly, the cost of information is given by

$$C((S_i, q_i)) = \mu \cdot I((S_i, q_i)) = \mu \left( \int_{\Theta} \int_{S_i} q_i(s_i|\theta) \log q_i(s_i|\theta) ds_i p(\theta) d\theta - \int_{S_i} \int_{\Theta} q_i(s_i|\theta) p(\theta) d\theta \log \left( \int_{\Theta} q_i(s_i|\theta) p(\theta) d\theta \right) ds_i \right) \quad (4.1)$$

where  $\mu \geq 0$  is the cost per unit of information.<sup>6</sup> So, a beauty contest with flexible information acquisition can be summarized by a tuple  $(p, \gamma, \mu)$ .

**Timing:** The timing is as follows:

1. All players decide upon their information acquisition strategy.
2. The value of  $\theta$  is realized.
3. Players receive a signal according to their information acquisition strategy.
4. Players take their actions  $a \in \Theta$  contingent on the signal they received.

Player  $i$  has to decide upon a strategy that consists of three parts: (a) a signal space  $S_i \in 2^{\mathbb{R}}$  – a subset of  $\mathbb{R}$ ; (b) a ( $p$ -measurable) mapping  $q_i : \Theta \rightarrow \Delta(S_i)$  that gives a probability distribution on  $S_i$  conditional on the value of  $\theta$ ;<sup>7</sup> and (c) a mapping

<sup>5</sup>Throughout the paper,  $\Delta(X)$  will represent the space of probability distributions over  $X$ .

<sup>6</sup>Throughout the paper,  $\log$  denotes the natural logarithm and so the unit of measurement of information is the *nat*. If the logarithms were taken with a base 2, the unit of measurement of information would be the *bit*. As Yang (2015) points out, the choice of the unit of measurement does not change the results as 1 bit equals  $\log 2$  nats.

<sup>7</sup>In what follows,  $q_i(s_i|\theta)$  denotes the PDF of  $s_i$  conditional on the value of  $\theta$ .

$\lambda_i : S_i \rightarrow \Delta(A_i)$  from the signal space to the space of probability distributions over the action space  $A_i = \mathbb{R}$ . The mapping  $\lambda_i$  is required to be measurable and integrable with respect to the measure over the signals induced by  $q_i$  on  $p$ .<sup>8</sup> The probability density with which player  $i$  follows action  $a_i$  conditional on receiving signal  $s_i$  will be denoted by  $\lambda_i(a_i | s_i)$ .

Let player  $i$ 's strategy be denoted by  $m_i = (S_i, q_i, \lambda_i)$ . The whole population's strategy profile will be denoted by  $\mathbf{m}$  and the strategy profile of the population excluding player  $i$  by  $\mathbf{m}_{-i}$ . In what follows, the average action of the population conditional on the realization of the random variable  $\theta$  will be denoted by

$$\bar{a}(\theta) \equiv \int_0^1 \int_{A_i} \int_{S_j} a_j \lambda_j(a_j | s_j) q_j(s_j | \theta) ds_j da_j dj.$$

### 4.3 Best Responses

In this section the best response correspondence for a player  $i$  is calculated. The steps taken for this calculation are the following: It is firstly shown that an optimal strategy for a player  $i$  should use (a) a signal space that has the same cardinality as the action space and (b) a probability distribution over actions conditional on the signal received that is degenerate (*i.e.* assigns all probability mass to a single action).<sup>9</sup> Given the result from the first step, player  $i$ 's best response can be summarized by a function  $r_i : \Theta \rightarrow \Delta(A_i)$  that gives a probability distribution over the action space conditional on the value of  $\theta$ . As a second step, necessary conditions that  $r_i$  has to satisfy if it is to be a best response are derived. This is done by considering (local) variations of  $r_i$  and demanding that an optimal  $r_i$  should do at least as well as any of these variations. By taking first order conditions, a unique form of  $r_i$  when  $\bar{a}(\theta)$  (the expected average action conditional on the value of the fundamental) is strictly increasing is identified.

<sup>8</sup>Formally  $\lambda_i : S_i \rightarrow \Delta(A_i)$  and  $\lambda_i \in L^2(S_i, q_i dp)$ .

<sup>9</sup>This result is well known for cases where the number of choices is finite (*e.g.* Woodford, 2008; Yang, 2015).

### 4.3.1 Determining the signal space and the signal-to-action function

In order to determine the optimal signal space that a player will use let  $\mathbf{m}_{-i}$  be any strategy profile that player  $i$ 's opponents are using. Let  $s_i$  be the signal that player  $i$  obtained by following the information acquisition strategy  $(S_i, q_i)$ . Given  $s_i$  and  $\mathbf{m}_{-i}$ , player  $i$  will form a belief on the value of  $\theta$  and from that (by pushing forward) on the value of  $\bar{a}$ .<sup>10</sup> From these beliefs, player  $i$  will form expectations  $\theta^i$  and  $\bar{a}^i$  on the respective variables.

**Lemma 4.1.** *For any strategy profile of player  $i$ 's opponents  $\mathbf{m}_{-i}$ , any information acquisition strategy  $(S_i, q_i)$  of player  $i$ , and any signal  $s_i$  that player  $i$  may have received, player  $i$  has a unique optimal action given by*

$$a_i = (1 - \gamma)\theta^i(s_i; (S_i, q_i)) + \gamma\bar{a}^i(s_i; (S_i, q_i), \mathbf{m}_{-i}).$$

**Proof.**

Let  $p^i(\theta|s_i; m_i)$  denote the distribution of  $\theta$  conditional on the event of player  $i$  receiving signal  $s_i$  while using information acquisition strategy  $(S_i, q_i)$ . This distribution has to satisfy Bayes's law:

$$p^i(\theta|s_i; (S_i, q_i)) = \frac{q_i(s_i|\theta)p(\theta)}{\int_{\Theta} q_i(s_i|\theta)p(\theta)d\theta}. \quad (4.2)$$

If player  $i$  knows player  $j$ 's strategy, she can also infer the distribution of player  $j$ 's signal (conditional on  $i$  receiving signal  $s_i$ ) by using Bayes's rule. That would be:

$$q_j^i(s_j|s_i; (S_i, q_i), m_j) = \int_{\Theta} q_j(s_j|\theta)p^i(\theta|s_i; (S_i, q_i))d\theta. \quad (4.3)$$

So, player  $j$ 's action distribution from player  $i$ 's viewpoint will be

$$\nu_j^i(a_j|s_i; (S_i, q_i), m_j) = \int_{S_j} \lambda_j(a_j|s_j)q_j^i(s_j|s_i; (S_i, q_i), m_j)ds_j \quad (4.4)$$

<sup>10</sup>Formally it should be  $\bar{a}_{-i}$  but as the contribution of a single player to the mean action of a continuum of players is zero,  $\bar{a}_{-i} = \bar{a}$ .

and player  $j$ 's expected action from  $i$ 's viewpoint will be

$$a_j^i(s_i; (S_i, q_i), m_j) = \int_{A_j} a_j v_j^i(a_j | s_i; (S_i, q_i), m_j) da_j. \quad (4.5)$$

So, from player  $i$ 's viewpoint, the expected average action of her opponents (which is equal to the expected average action over the whole population as player  $i$ 's action cannot affect the mean action in a continuum population) will be

$$\bar{a}^i(s_i; (S_i, q_i), \mathbf{m}_{-i}) = \bar{a}_{-i}^i(s_i; (S_i, q_i), \mathbf{m}_{-i}) = \int_0^1 \int_{A_j} a_j v_j^i(a_j | s_i; (S_i, q_i), m_j) da_j dj \quad (4.6)$$

and her expected value (estimation) of the fundamental will be

$$\bar{\theta}^i(s_i; (S_i, q_i)) = \int_{\Theta} \theta p^i(\theta | s_i; (S_i, q_i)) d\theta. \quad (4.7)$$

From  $i$ 's viewpoint, given that she received signal  $s_i$ ,  $i$ 's expected utility is

$$\mathbb{E}_i(u_i | s_i) = \int_{\Theta} (\bar{u} - (1-\gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a})^2 - C(S_i, q_i)) p^i(\theta | s_i; (S_i, q_i)) d\theta. \quad (4.8)$$

Now assume that player  $i$  has already fixed her information acquisition strategy and all other players have decided upon their strategy. Player  $i$  then receives her signal  $s_i$ . Given that she is maximizing expected utility, her action has to satisfy the following first order condition.

$$\begin{aligned} a_i^*(\mathbf{m}_{-i} | s_i; (S_i, q_i)) &= (1-\gamma) \int_{\Theta} \theta p^i(\theta | s_i; (S_i, q_i)) d\theta + \\ &+ \gamma \int_{\Theta} \bar{a}(\theta) p^i(\theta | s_i; (S_i, q_i)) d\theta \\ &= (1-\gamma) \int_{\Theta} \theta p^i(\theta | s_i; (S_i, q_i)) d\theta + \gamma \bar{a}^i(s_i; (S_i, q_i), \mathbf{m}_{-i}) \end{aligned} \quad (4.9)$$

From the equation above it is seen that – as long as the integrals appearing in the right-hand side of equation (4.9) are well-defined – there is a unique value of  $a_i^*$  that satisfies the above condition.  $\square$

In light of the above result, in an optimal strategy (best response) of player  $i$  each signal should map to a unique action  $a_i \in A_i$  rather than a distribution over actions.

More than that, there should be a unique signal that maps to each action. Consider two strategies:  $m_i = (S_i, q_i, \lambda_i)$  where each action has a unique signal that maps to it (through  $\lambda_i$ ) and  $m' = (S'_i, q'_i, \lambda'_i)$  where some actions (measurable under the measure induced on  $A_i$  by  $q'_i$  and  $\lambda'_i$ ) have multiple signals that map to them. Denote by  $S'(a_i)$  the set of signals that map to action  $a_i$  under  $\lambda'_i$  i.e.  $S'(a_i) = \{s'_i \in S'_i : \Pr_{\lambda'_i}(a_i | s'_i) = 1\}$  and by  $s^{a_i}$  the (unique) signal that maps to action  $a_i$  under  $\lambda_i$  for all  $a_i \in A_i$ . Let also  $q(s^{a_i} | \theta) = \sum_{s' \in S'(a_i)} q'(s | \theta)$  for  $S'(a_i)$ . It is clear that the expected value of  $u_i$  from the two strategies will be the same as they give the same probability density on actions for the same values of  $\theta$ . It is also true that  $I(S'_i, q'_i) > I(S_i, q_i)$  due to the convexity of mutual information in  $q$  (see Fozunbal, McLaughlin, and Schafer, 2005). Therefore,  $m'_i$  is more expensive to player  $i$  than  $m_i$  and thus not an optimal choice.<sup>11</sup>

From Lemma 4.1 and the result of the previous paragraph, the signal-to-action function  $\lambda_i$  has to be a bijection from  $S_i$  to  $A_i$ .<sup>12</sup> Thus, the signal space should have the same cardinality as the action space.<sup>13</sup> So, player  $i$ 's signal space can be reduced to be a space isomorphic to  $A_i = \mathbb{R}$ . Since signals are important only as far as they prescribe probabilities over actions, the exact choice of the signal space will not change players' actions as long as it is isomorphic to  $\mathbb{R}$  and  $\lambda_i$  is a bijection. One such example is that the signal space is  $\mathbb{R}$  and  $\lambda(a_i | s_i) = \delta(a_i - s_i)$ : Dirac's delta function. For ease of exposition, in what follows it is assumed that players use  $\mathbb{R}$  as their signal space and Dirac's delta as their signal-to-action mapping i.e. a player  $i$ , upon receiving signal  $s \in \mathbb{R}$ , will follow action  $a_i = s$ .

Players are going to use the whole of  $\mathbb{R}$  as a signal space but the notation  $S_i$  to denote player  $i$ 's signal space will be kept. It is also the case that  $A_i = \Theta = \mathbb{R}$  for all  $i$  but the action space will be explicitly mentioned in order to avoid confusion.

<sup>11</sup>For similar arguments see Woodford (2008, 2009) and Yang (2015).

<sup>12</sup>In the sense that  $\lambda_i$  gives probability one to a unique action  $a_i$  for each signal  $s_i \in S_i$  and for each action  $a_i \in \mathbb{R}$  there exists a unique signal for which  $\Pr_{\lambda_i}(a_i | s_i) = 1$ .

<sup>13</sup>This should happen even if some of these signals are never used. Of course, if any of the signals is not to be used, this would immediately mean that the corresponding action would never be used by player  $i$ .

### 4.3.2 Determining the form of the information acquisition strategy

What remains is to determine how players decide upon their information acquisition strategy.

Keeping in mind the result of subsection 4.3.1, player  $i$ 's strategy can be summarized by a function  $r_i : \theta \rightarrow \Delta(A_i)$  which gives a probability distribution on the action space of player  $i$  conditional on the value of  $\theta$ . So, by writing that the information acquisition strategy of player  $i$  is  $r_i(a_i|\theta)$ , it is meant that player  $i$  is using strategy  $m_i = (\mathbb{R}, q_i(s_i|\theta), \delta(a_i - s_i))$  with  $q_i(s_i|\theta) = r_i(a_i|\theta)$ .

Now, observe that from player  $i$ 's point of view, the only way that the other players are affecting her payoff is through the effect of their strategies on the average action  $\bar{a}$ . Thus, player  $i$  is not affected by *the way* that the particular  $\bar{a}(\theta)$  comes about. This means that the object to which she is best-responding is the function  $\bar{a}(\theta)$  which summarizes all of her opponents' strategies.

So, with a slight abuse of notation, the decision problem of player  $i$  is to maximize the following:

$$V(r_i, \mathbf{r}_{-i}) = U(r_i, \mathbf{r}_{-i}) - \mu I(r_i). \quad (4.10)$$

Where

$$\begin{aligned} U(r_i, \mathbf{r}_{-i}) &= \bar{u} - (1-\gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2 r_i(a_i|\theta) p(\theta) da_i d\theta - \\ &\quad - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2 r_i(a_i|\theta) da_i d\theta \end{aligned} \quad (4.11)$$

with  $\bar{a}(\theta)$  given by

$$\bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r_j(a_j|\theta) da_j dj. \quad (4.12)$$

The mutual information between  $r_i$  and  $p$  is given by

$$\begin{aligned} I(r_i) &= \int_{\Theta} \int_{A_i} \log \left( \frac{r_i(a_i|\theta)p(\theta)}{p(\theta) \int_{\Theta} r_i(a_i|\theta)p(\theta) d\theta} \right) r_i(a_i|\theta)p(\theta) da_i d\theta = \\ &= \int_{\Theta} \int_{A_i} \log(r_i(a_i|\theta)) r_i(a_i|\theta)p(\theta) da_i d\theta - \int_{A_i} \log(R_i(a_i)) R_i(a_i) da_i \end{aligned} \quad (4.13)$$

where  $R_i(a_i) = \int_{\Theta} r_i(a_i|\theta)p(\theta) d\theta$  is the marginal of action  $a_i$ .

As a first result, it is easy to show that if information is costless, player  $i$  has a unique best response to any  $\bar{a}(\theta)$ .

**Proposition 4.2.** *Let  $(p, \gamma, \mu)$  be a beauty contest with flexible information acquisition. If  $\mu = 0$ , then for any  $\bar{a}(\theta)$ , player  $i$  has a unique best response that gives a probability mass of 1 to the action  $a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta)$ .*

**Proof.** As  $\mu = 0$ , player  $i$  can obtain full information on the value of  $\theta$  without paying any costs. So, her optimization problem becomes

$$\max_{a^*(\theta)} -(1 - \gamma)(a^*(\theta) - \theta)^2 - \gamma(a^*(\theta) - \bar{a}(\theta))^2$$

Taking a first order condition, one obtains that

$$a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta).$$

So, given any  $\bar{a}(\theta)$ , player  $i$  has a unique best action  $a^*(\theta)$ . Thus, her best response would be to give a probability mass of 1 to that action (conditional on  $\theta$ ). That is, her best response would be to use  $r_i(a_i|\theta) = \delta(a_i - a^*(\theta))$  with  $\delta$  being Dirac's delta function.  $\square$

The analysis from now on will focus on strategy profiles that satisfy some continuity and smoothness conditions. In particular the following are defined:

**Definition 4.1** (Monotone full-support profile). *A strategy profile  $\mathbf{r}$  will be called a monotone, full-support profile if it satisfies the following conditions:*

1. *for (almost) all  $i \in [0, 1]$  player  $i$ 's action  $a_i$  has a well-defined mean and variance*

2. the average action function  $\bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r_j(a_j|\theta) da_j dj$  is twice continuously differentiable ( $\bar{a} \in \mathcal{C}^2$ )
3. the “best action” function  $a^* : \Theta \rightarrow A$ , defined as  $a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta)$ , is bijective with  $a^{*\prime}(\theta) > 0$  for all  $\theta \in \Theta$ .

The inverse of  $a^*$  will be denoted by  $\phi(x)$ . i.e.  $a^*(\theta) = x \Leftrightarrow \phi(x) = \theta$ .

Notice that condition 3, which implies that  $\lim_{\theta \rightarrow \pm\infty} a^*(\theta) = \pm\infty$ , is not too restrictive as it is satisfied for all weakly increasing  $\bar{a}$ . It is also satisfied for decreasing  $\bar{a}$  as long as  $\bar{a}'(\theta) > -(1 - \gamma)/\gamma$ , even though such behavior would not make intuitive sense. More than that, condition 1 implies that  $\bar{a}(\theta)$ , and therefore  $a^*(\theta)$ , are well-defined for all  $\theta$ .

Any given monotone full-support strategy profile summarized by an  $\bar{a}$ , along with the distribution  $p$  of the fundamental, induces a probability distribution of the best action (see Definition 4.1 condition 3). The PDF of this probability distribution will be denoted by  $g$  and is given by

$$g(x) = p(\phi(x))\phi'(x).$$

**Definition 4.2** (Smooth strategy). *A strategy  $r_i$  of player  $i$  will be called smooth if  $r_i(a_i|\theta)$  is continuous with respect to the Lebesgue measure (contains no atoms) for all  $\theta \in \Theta$  and  $a_i$  has a well-defined mean and variance.*

As two limiting cases of smooth strategies are the one where  $r(a_i|\theta) = \delta(a_i - f(\theta))$  for some function  $f : \Theta \rightarrow A_i$  and the one where  $r(a_i|\theta) = \delta(a_i - c)$  for some constant  $c$ . In the first case, player  $i$ 's action is *deterministic*. For each value of  $\theta$  she will use action  $f(\theta)$  with probability one. This means that she gets an *infinite amount of information* and therefore has to pay an infinite cost. This, of course, would not be optimal unless  $\mu = 0$ , as was seen in Proposition 4.2. On the other extreme, when  $r(a_i|\theta) = \delta(a_i - c)$ , player  $i$  plays action  $c$  with probability one *independently of the value of  $\theta$* . This conveys no information about  $\theta$  to her. This would be optimal when information costs are too high. The following proposition provides necessary and sufficient conditions for a smooth best response (a strategy in-between these two extremes) to exist.

**Proposition 4.3.** *Let  $(p, \gamma, \mu)$  with  $\mu > 0$  be a beauty contest with flexible information acquisition. Let also  $\mathbf{r}_{-i}$  be a monotone, full-support strategy profile of player  $i$ 's opponents for which the variance of the best action is  $\sigma_{a^*}^2$ .*

*Player  $i$  has a smooth best response to  $\mathbf{r}_i$  if and only if  $\mu \in (0, 2\sigma_{a^*}^2)$ . This smooth best response is unique and is given by*

$$r(a_i|\theta) = R(a_i) \frac{a^{*\prime}(\theta)}{p(\theta)} \frac{1}{\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - a^*(\theta))^2}{\mu}\right)$$

where

$$R(a_i) = \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[g(x)](\xi)](a_i).$$

**Proof.** See Appendix C.2.1.

In the above,  $\mathcal{F}_x$  and  $\mathcal{F}_\xi^{-1}$  denote the Fourier and inverse Fourier transforms defined as

$$\mathcal{F}_x[f(x)](\xi) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i x \xi) dx \quad (4.14)$$

$$\text{and } \mathcal{F}_\xi^{-1}[F(\xi)](x) = \int_{-\infty}^{+\infty} F(\xi) \exp(2\pi i x \xi) d\xi \quad (4.15)$$

respectively ( $i$  is the imaginary unit). Notice that  $\mathcal{F}_x[g(x)]$  is closely related to the characteristic function of the random variable  $x$ .<sup>14</sup> In the rest of the paper, the terms “best response to  $\mathbf{r}_{-i}$ ” and “best response to  $\bar{a}(\theta)$ ” will be used interchangeably as all monotone, full-support profiles  $\mathbf{r}_{-i}$  that yield the same  $\bar{a}(\theta)$  will lead to the same best response from player  $i$ .<sup>15</sup> In the Appendix some examples are provided to demonstrate what solutions to the decision problem look like in simple situations where  $\theta$  is normally distributed and  $\bar{a}$  is affine.

In any strategy player  $i$  might use, she receives a signal  $s_i$  which prescribes a unique action  $a_i$  for her to play (see Lemma 4.1). So, given this signal, she forms a posterior about belief about the best action. One would expect that the prescribed action  $a_i$  should coincide with the expectation of the best response. This intuition is

<sup>14</sup>In particular, the characteristic function of random variable which is distributed according to a probability density function  $p$  is  $\psi(\omega) = \int_{-\infty}^{+\infty} \exp(i x \omega) p(x) dx$  whereas  $\mathcal{F}_x[p(x)](\xi) = \int_{-\infty}^{+\infty} \exp(-2\pi i x \xi) p(x) dx$ .

<sup>15</sup>Recall that  $\bar{a}(\theta) = \int_{[0,1]^i} \int_{A_j} a_j r_j(a_j|\theta) da_j dj$ .

confirmed by the following proposition. The proposition also shows what the effect of the information cost to the player's best response is.

**Proposition 4.4.** *Let  $(p, \gamma, \mu)$  be a beauty contest with flexible information acquisition and  $\mathbf{r}_{-i}$  be a monotone, full-support strategy profile of player  $i$ 's opponents for which  $\text{Var}(a^*) = \sigma_{a^*}^2 > \mu/2$ . In player  $i$ 's smooth best response, her posterior on the best action  $a^*$  has a PDF given by*

$$\varrho(a^*|a_i) = \frac{1}{\sqrt{2\pi\mu}} \exp\left(-\frac{(a_i - a^*)^2}{\mu}\right).$$

**Proof.** From Bayes's law, one gets:

$$\varrho(x|a_i) = \frac{\tau(a_i|x)g(x)}{R(a_i)} \quad (4.16)$$

where  $\tau$  is the PDF of  $a_i$  conditional on the best action  $x$ . As  $a^*(\cdot)$  is bijective with inverse  $\phi(\cdot)$ , one can derive  $\tau$  from the result of Proposition 4.3 with a change of variable:

$$\tau(a_i|x) = \frac{R(a_i)}{p(\phi(x))\phi'(x)} \frac{1}{\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - x)^2}{\mu}\right) = R(a_i) \frac{1}{g(x)} \frac{1}{\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - x)^2}{\mu}\right)$$

and comparing with (4.16), one obtains

$$\varrho(x|a_i) = \frac{1}{\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - x)^2}{\mu}\right)$$

where  $x$  is the best action. □

The above result states that, when best responding, players are becoming more uncertain about what action they should follow as information becomes more expensive. It also states that a player's posterior belief about the best action is always normally distributed, *independently* of the functional form of the prior  $p$ .

Taking a closer look at the result of Proposition 4.3 in light of Proposition 4.4, the effect of an increasing cost  $\mu$  becomes clear. A higher  $\mu$  forces the player to have a less accurate posterior on what the best action is (as a more accurate belief would be more costly). The only other part where  $\mu$  enters the player's strategy is in  $R(a_i)$ . For  $r(a_i|\theta)$  to be a (conditional) probability distribution  $\int_{-\infty}^{+\infty} r(a_i|\theta) da_i = 1$

must hold for all  $\theta$ . As higher  $\mu$  induces more dispersed  $\varrho(x|a_i)$ , in order for this condition to be satisfied,  $R$  needs to become more concentrated. When the value of  $\mu$  reaches  $2\sigma_{a^*}^2$ ,  $R$  needs to have zero variance which means that player  $i$  stops acquiring information.

## 4.4 Equilibrium

### 4.4.1 Smooth, monotone, full-support equilibria

Having established conditions for the existence of a smooth best response to monotone, full-support strategy profiles, attention is now shifted towards equilibria. In particular, the focus will be on the class of (Nash) equilibria whereby individual strategies are smooth and the whole strategy profile of the population is monotone and has full-support. Such equilibria are defined formally in the following definition.

**Definition 4.3** (Smooth, monotone, full-support Equilibrium). *An action profile  $\mathbf{r}$  is called a smooth, monotone, full-support equilibrium (SMFE) if*

1.  $\mathbf{r}$  is a monotone, full-support profile,
2.  $r_i$  is smooth for all  $i \in [0, 1]$  and
3.  $r_i$  is a best response to  $\mathbf{r}$  for all  $i \in [0, 1]$ .

The next result follows from the definition.

**Corollary 4.5.** *Consider a beauty contest with flexible information acquisition. Then all SMFE are symmetric i.e. in equilibrium all players use strategies that are equal to the same strategy  $r$  almost everywhere.*

**Proof.** As there is a continuum of agents, any single player  $i$  cannot influence the average action taken by the population for any value of  $\theta$ . This means that all players face the same decision problem. Recall that each player has a unique best reply (up to deviations of measure zero, see Proposition 4.3) to a monotone, full-support profile. Thus, in equilibrium, the strategies that the players are using should be equal to the same strategy  $r$  almost everywhere.  $\square$

The following necessary condition for an SMFE is established.

**Proposition 4.6.** *Let  $B = (p, \gamma, \mu)$  be a beauty contest with flexible information acquisition. Let also  $\bar{a}(\theta) \in \mathcal{C}^2$  be the average action function of an SMFE of  $B$ . Then*

$$\bar{a}(\theta) = \theta + \frac{\mu}{2(1-\gamma)(1+\gamma(\bar{a}'(\theta)-1))} \left( \frac{p'(\theta)}{p(\theta)} - \frac{\gamma \bar{a}''(\theta)}{1+\gamma(\bar{a}'(\theta)-1)} \right). \quad (4.17)$$

Equivalently, the best action function  $a^*(\theta)$  should satisfy

$$a^*(\theta) = \theta + \frac{\mu\gamma}{2(1-\gamma)} \frac{1}{a^{*\prime}(\theta)} \frac{d}{d\theta} \left( \log \left( \frac{p(\theta)}{a^{*\prime}(\theta)} \right) \right) \quad (4.18)$$

and its inverse  $\phi(x)$  should satisfy

$$\phi(x) = x - \frac{\mu\gamma}{2(1-\gamma)} \frac{d}{dx} \log(p(\phi(x))\phi'(x)). \quad (4.19)$$

**Proof.** See Appendix C.2.2.

#### 4.4.2 Equilibria without information acquisition

It might be possible for equilibria where no information acquisition takes place to exist. Such an equilibrium would be one where players' actions do not vary conditional on  $\theta$ . In this case, players obtain no information ( $I(S_i, q_i) = 0$ ) and the cost they have to incur is zero. Thus, from player  $i$ 's point of view, the average action does not vary with  $\theta$  ( $\bar{a}(\theta) = \bar{a}$ ). A necessary condition for such an equilibrium to exist is derived in what follows.

**Lemma 4.7.** *In an equilibrium where no information acquisition takes place, all players play  $\bar{\theta} = \int_{\Theta} \theta p(\theta) d\theta$  with probability 1 independently of the value of  $\theta$ .*

**Proof.**

Assuming that the rest of the population is using strategies without information acquisition, player  $i$  is facing a population where  $\bar{a}(\theta) = \bar{a}$ . Player  $i$ 's expected payoff from not acquiring information is

$$\tilde{V}(a_i) = \bar{u} + \int_{\Theta} (-(1-\gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a})^2) p(\theta) d\theta$$

and so, if she maximizes, she has to choose

$$a_i^* = \gamma \bar{a} + (1 - \gamma) \bar{\theta}.$$

Now for a strategy like that to be an equilibrium strategy, it has to be that  $a_i^* = \bar{a}$  and thus,  $\bar{a} = \bar{\theta}$ .  $\square$

Notice that for the above result the assumption that  $\gamma < 1$  is crucial. This is because if the players have no fundamental motive ( $\gamma = 1$ ), then the condition  $a_i^* = \bar{a}$  is not enough to pin down the equilibrium as the usual coordination problems arise.

**Proposition 4.8.** *A beauty contest with flexible information acquisition  $(p, \gamma, \mu)$  admits a Nash equilibrium without information acquisition if and only if*

$$\mu \geq 2(1 - \gamma)^2 \sigma_\theta^2.$$

**Proof.** See Appendix C.2.3.

### 4.4.3 Equilibrium properties

In the previous section necessary conditions that SMFE and equilibria without information acquisition should satisfy were derived. Although it is not possible to give a solution to equation (4.18) in closed form, it is possible to describe some properties that any SMFE should have. These have to do with the behavior of  $a^*$  close to  $\pm\infty$  as well as its ex-ante expected value.

It is firstly shown that in any equilibrium (either SMFE or equilibrium without information acquisition) the ex-ante expected values of  $\bar{a}$ ,  $a^*$  and  $\theta$  are equal to one another. This is obvious in the case of equilibria without information acquisition as the whole population plays  $\bar{\theta} = \int_{-\infty}^{+\infty} \theta p(\theta) d\theta$  with probability one. In the case of SMFE, the following proposition is provided.

**Proposition 4.9.** *Let  $\mathbf{r}$  be an SMFE with average action function  $\bar{a}$  and best action function  $a^*$ . Then*

$$\int_{-\infty}^{+\infty} a^*(\theta) p(\theta) d\theta = \int_{-\infty}^{+\infty} \bar{a}(\theta) p(\theta) d\theta = \int_{-\infty}^{+\infty} \theta p(\theta) d\theta = \bar{\theta}.$$

**Proof.** See Appendix C.2.4.

Proposition 4.9 says that the “mean error” that players make is zero. They will miss their target  $\theta$  most of the time but on average they should be correct.

Another result is that when  $\theta$  goes to  $+\infty$  ( $-\infty$ ) the difference  $a^*(\theta) - \theta$  is negative (positive).

**Proposition 4.10.** *Let  $r$  be an SMFE with best action function  $a^*$ . Then  $\lim_{\theta \rightarrow +\infty} a^*(\theta) - \theta < 0$  and  $\lim_{\theta \rightarrow +\infty} \bar{a}(\theta) - \theta < 0$ . Also  $\lim_{\theta \rightarrow -\infty} a^*(\theta) - \theta > 0$  and  $\lim_{\theta \rightarrow -\infty} \bar{a}(\theta) - \theta > 0$ .*

**Proof.** By solving condition (4.18) for  $p$  one gets

$$p(\theta) = \frac{p(\theta')}{a^{*\prime}(\theta')} a^{*\prime}(\theta) \exp\left(\frac{2(1-\gamma)}{\mu\gamma} \int_{\theta'}^{\theta} a^{*\prime}(t)(a^*(t) - t) dt\right) \quad (4.20)$$

for any  $\theta' \in \mathbb{R}$ . And so

$$\frac{p(\theta)}{a^{*\prime}(\theta)} = \frac{p(\theta')}{a^{*\prime}(\theta')} \exp\left(\frac{2(1-\gamma)}{\mu\gamma} \int_{\theta'}^{\theta} a^{*\prime}(t)(a^*(t) - t) dt\right).$$

Now, taking the limit for  $\theta \rightarrow +\infty$ :

$$\lim_{\theta \rightarrow +\infty} \frac{p(\theta)}{a^{*\prime}(\theta)} = \frac{p(\theta')}{a^{*\prime}(\theta')} \exp\left(\frac{2(1-\gamma)}{\mu\gamma} \int_{\theta'}^{+\infty} a^{*\prime}(t)(a^*(t) - t) dt\right).$$

As  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = 0$  (see proof of proposition 4.9) and  $p(\theta')/a^{*\prime}(\theta')$  for any  $\theta'$ , it has to be that

$$\int_{\theta'}^{+\infty} a^{*\prime}(t)(a^*(t) - t) dt = -\infty$$

for all  $\theta' \in \mathbb{R}$ . Clearly, as  $a^{*\prime}(\theta) > 0$  for all  $\theta$  this can happen only if  $\lim_{\theta \rightarrow +\infty} a^*(\theta) - \theta < 0$ . A similar argument can be given for the case where  $\theta \rightarrow -\infty$ . The same arguments for  $\bar{a}$  can be given if one takes into account the definition of  $a^*(\theta) = (1-\gamma)\theta + \gamma\bar{a}(\theta)$ .  $\square$

The above proposition shows that in equilibrium players are biased towards the center of the distribution and they take actions with extreme values (compared to the ex-ante mean of the distribution) less often. An immediate implication of the above proposition is that there exists  $\theta_0$  for which  $\bar{a}(\theta_0) = a^*(\theta_0) = \theta_0$ .

Finally, some results on the relationship between the cost  $\mu$  and the variances of the fundamental and the best action are given.

**Proposition 4.11.** *Let  $(p, \gamma, \mu)$  be a beauty contest and  $\sigma_{a^*}^2$  be the variance of the best action  $a^*$  in an SMFE. Then  $\sigma_\theta^2 \geq \sigma_{a^*}^2$ . In particular*

$$\sigma_\theta^2 - \sigma_{a^*}^2 = \frac{\mu\gamma}{2(1-\gamma)} + \frac{\gamma}{1-\gamma} \mathbb{E}(\text{Var}(a_i|\theta)) + \frac{1}{\gamma} \text{Var}(\theta - a^*).$$

**Proof.** See Appendix C.2.5.

In conjunction with Proposition 4.3, the following can be shown.

**Corollary 4.12.** *Let  $(p, \gamma, \mu)$  be a beauty contest with flexible information acquisition that has an SMFE. Then  $\mu < 2(1-\gamma)\sigma_\theta^2$ .*

**Proof.** As in an SMFE  $r$  is the best response to a monotone, full-support strategy profile that induces a best action function  $a^*(\theta)$ , it has to be that  $\text{Var}(a^*) < \mu/2$ . Using this along with the result of Proposition 4.11, one gets that

$$\mu < 2(1-\gamma)\sigma_\theta^2 - 2\gamma \mathbb{E}(\text{Var}(a_i|\theta)) - \frac{2(1-\gamma)}{\gamma} \text{Var}(\theta - a^*) < 2(1-\gamma)\sigma_\theta^2. \quad \square$$

The boundary  $2(1-\gamma)\sigma_\theta^2$  is an upper bound to the value of  $\mu$  and is not tight. What one can tell for sure is that if  $\mu$  exceeds this value, then  $(p, \gamma, \mu)$  has no SMFE.

## 4.5 Application: Aggregately Affine Equilibria

In the literature, the notion of a linear equilibrium is often encountered (see for example Angeletos and Pavan, 2007; Morris and Shin, 2002; Myatt and Wallace, 2012). This concept usually means that the action of a player is some linear or affine function of the signal she receives. Here, the concept of an *aggregately affine equilibrium* (AAE) is introduced. In an AAE, players act in such a way that the average action across the population given the realization of the fundamental – and hence the best action too – are linear functions of the realization of the fundamental *i.e.*  $a^*(\theta) = \kappa\theta + b$  for some  $\kappa > 0$  and  $b \in \mathbb{R}$ . This section studies exhaustively the case of AAE. This is made possible by the result of the following proposition which shows that AAE are possible to occur only if the fundamental is normally distributed.

**Proposition 4.13.** *Let  $(p, \gamma, \mu)$  be a beauty contest with flexible information acquisition that admits an aggregately affine equilibrium, then  $p$  is a normal distribution.*

**Proof.** In an AAE the best action function is given by  $a^*(\theta) = \kappa\theta + b$ . So,  $a^{*'} = \kappa$  and  $a^{*''} = 0$ . Using that in equation (4.18) one obtains:

$$\kappa\theta + b = \theta + \frac{\mu\gamma}{2(1-\gamma)} \frac{1}{\kappa} \frac{d}{d\theta} \log p(\theta).$$

And thus,

$$\log p(\theta) = \int \frac{2(1-\gamma)\kappa}{\mu\gamma} ((\kappa-1)\theta + b) d\theta + C$$

where  $C \in \mathbb{R}$  is an integrating constant. It will have to be chosen so that the condition  $\int_{-\infty}^{+\infty} p(\theta) d\theta = 1$  is satisfied. From the previous equation:

$$\log p(\theta) = \frac{(1-\gamma)\kappa}{\mu\gamma} ((\kappa-1)\theta^2 + 2b\theta) + C.$$

Completing the square in the brackets and taking the exponential of both sides one obtains:

$$p(\theta) = \exp(C') \exp\left(\frac{(1-\gamma)\kappa(\kappa-1)}{\mu\gamma} \left(\theta - \frac{b}{1-\kappa}\right)^2\right)$$

for some other constant  $C'$ . Now, for  $\int_{-\infty}^{+\infty} p(\theta) d\theta = 1$  to be satisfied, it has to be that  $\kappa \in (0, 1)$ , otherwise the resulting  $p$  will not be integrable. It is clear that – for an appropriate selection of  $C'$  – the previous expression is a normal distribution with a mean  $\theta_0 = b/(1-\kappa)$  and variance  $\sigma_\theta^2 = \mu\gamma/(2(1-\gamma)\kappa(1-\kappa))$ .  $\square$

In what follows, a normal prior is assumed and the resulting AAE are calculated.

Assume that the fundamental motive of a beauty contest game follows a normal distribution with mean  $\theta_0$  and variance  $\sigma^2$  i.e.  $p(\theta) = (\sqrt{2\pi}\sigma)^{-1} \exp(-(\theta-\theta_0)^2/(2\sigma^2))$ . Using the result of Proposition 4.6, in any SMFE the average action of the population conditional on the value of  $\theta$  has to satisfy the following differential equation:

$$\bar{a}(\theta) = \theta - \frac{\mu(\theta - \theta_0)}{2\sigma^2(1-\gamma)(1 + \gamma(\bar{a}'(\theta) - 1))} + \frac{\mu\gamma}{2(1-\gamma)(1 + \gamma(\bar{a}'(\theta) - 1))^2} \bar{a}''(\theta)$$

or equivalently, the best action  $a^*$  has to satisfy

$$a^*(\theta) = \theta + \frac{\mu\gamma}{2(1-\gamma)} \frac{1}{a^{*\prime}(\theta)} \left( -\frac{\theta - \theta_0}{\sigma^2} - \frac{d}{d\theta} (\log a^{*\prime}(\theta)) \right). \quad (4.21)$$

In order to identify AAE, by substituting  $a^*(\theta) = \kappa\theta + b$  into equation (4.21), one obtains two solutions for the value of  $\kappa$ :

$$\kappa_+ = \frac{1}{2} + \frac{\sqrt{\sigma^2 - \frac{2\mu\gamma}{1-\gamma}}}{2\sigma} \quad \kappa_- = \frac{1}{2} - \frac{\sqrt{\sigma^2 - \frac{2\mu\gamma}{1-\gamma}}}{2\sigma}$$

whereas the corresponding values for  $b$  are

$$b_+ = \frac{\mu\gamma}{2(1-\gamma)\sigma^2\kappa_+} \theta_0 \quad b_- = \frac{\mu\gamma}{2(1-\gamma)\sigma^2\kappa_-} \theta_0.$$

Of course, for these solutions to exist, it is required that  $\sigma^2 - \frac{2\mu\gamma}{1-\gamma} \geq 0$  which is equivalent to  $\mu \leq \frac{(1-\gamma)}{2\gamma} \sigma^2$  for  $\gamma \in (0, 1)$ . It is easy to confirm that  $a^*(\theta) = \theta \Leftrightarrow \theta = \theta_0$ .

Now recall that for an SMFE it is required that  $\text{Var}(a^*) > \mu/2$ . This condition in the case of AAE boils down to  $2\kappa^2\sigma^2 - \mu > 0$ . Solving this condition for  $\mu$  for the two cases of  $\kappa_+$  and  $\kappa_-$  one gets the following:

1. For an AAE with  $a^{*\prime}(\theta) = \kappa_+$  to exist, it has to be that either

$$\gamma \leq \frac{1}{2} \text{ and } \mu < 2(1-\gamma)^2\sigma^2 \quad \text{or} \quad \gamma > \frac{1}{2} \text{ and } \mu < \frac{1-\gamma}{2\gamma}\sigma^2.$$

2. For an AAE with  $a^{*\prime}(\theta) = \kappa_-$  to exist, it has to be that

$$\gamma > \frac{1}{2} \quad \text{and} \quad \mu > 2(1-\gamma)^2\sigma^2.$$

All the above parameter combinations are compatible with  $\mu \leq \frac{(1-\gamma)}{2\gamma}\sigma^2$ .

Finally, the equilibrium where no information acquisition is taking place is taken into consideration. As shown in Proposition 4.8) such an equilibrium exists if and only if  $\mu \geq 2(1-\gamma)^2\sigma^2$ . Notice that in such equilibria  $a^{*\prime}(\theta) = 1-\gamma$  for all values of  $\theta$ . So, the first derivative of  $a^*$  is constant *even though this is not an SMFE*, as the strategy that the players use is not smooth.

Interestingly, if  $\gamma \geq \frac{1}{2}$  and  $\mu \in \left( 2(1-\gamma)^2\sigma^2, \frac{(1-\gamma)}{2\gamma}\sigma^2 \right)$  there is *multiplicity of equilibria* i.e. there exist three equilibria of the classes that are being considered: two SMFE

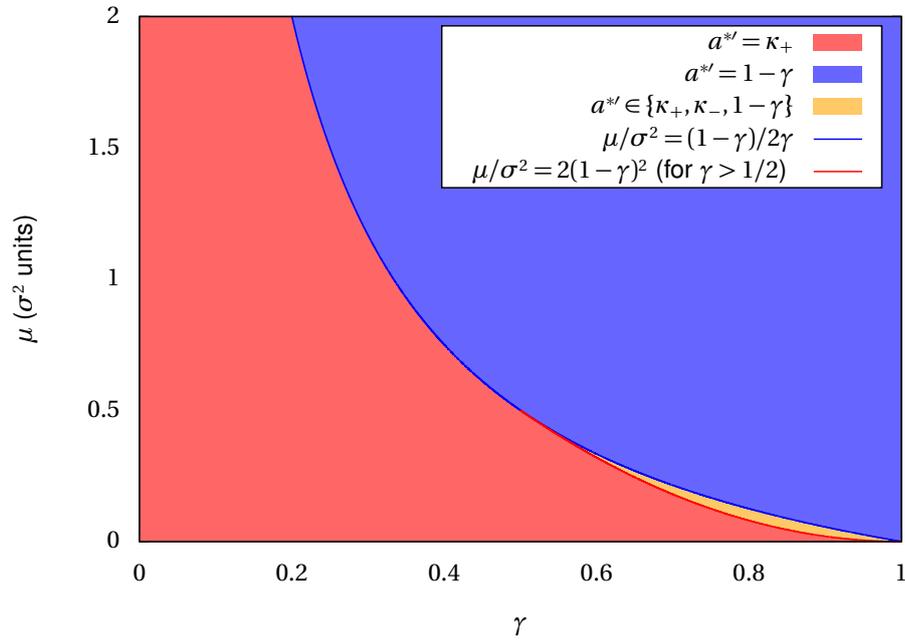


Figure 4.1: Aggregately affine SMFE and equilibria without information acquisition in different regions of  $\mu$  and  $\gamma$  when  $\theta$  is normally distributed with variance  $\sigma^2$ .

(one with  $a^{*l} = \kappa_+$  and one with  $a^{*l} = \kappa_-$ ) and an equilibrium without information acquisition ( $a^{*l} = 1 - \gamma$ ). In all other cases, there is a unique equilibrium within the classes under consideration. These results are summarized in Figure 4.1.

The multiplicity arises in the case of *intermediate* values of  $\mu$ . This comes in sharp contrast to the result of Yang (2015) who observes multiplicity of equilibria for low values of the information cost – despite the game studied here and the one studied in Yang (2015) both being coordination games.

When information is cheap, players obtain more of it and the game gets closer to a full-information one. In Yang's setting the full information game has many equilibria for a range of values of the random variable, this multiplicity is recovered when information costs are low. It is therefore unclear whether the multiplicity of equilibria is present in his model because of the underlying form of the game or because of the information acquisition technology employed.

Similar to Yang's result, when costs are low the equilibrium structure of the underlying game is recovered: there is a unique equilibrium where information acquisition takes place. This is also true for the full information case of a beauty contest (all players play  $\theta$ ).

Surprisingly, though, multiple equilibria arise in this setting when the coordination motive is high ( $\gamma > 1/2$ ). Obviously, this cannot stem from the underlying structure of the game. An intuitive explanation for this result is as follows.

When  $\gamma$  is small, players are able to coordinate to a unique equilibrium where they acquire information. As  $\gamma$  increases, less information is acquired – as the importance of getting close to the realized value of  $\theta$  is decreasing and the motive to coordinate increases. When  $\gamma$  reaches the critical value for which  $\mu = 2(1 - \gamma)^2 \sigma^2$ , the coordination motive becomes so strong that an equilibrium where no player acquires any information is established: if no player acquires information, they are sure to perfectly coordinate at  $\bar{\theta}$  – saving the costs of acquiring information. When costs are high, there is no overlapping region where an AAE and an equilibrium without information acquisition coexist. However, when costs are lower, players can still be acquiring information even when the conditions are there for an equilibrium without information acquisition to exist, as information is cheap enough.

## 4.6 Conclusion

This paper extended the concept of flexible information acquisition technology (Yang, 2015) to a setting with continuous action sets: a beauty contest. Necessary conditions for Smooth, Monotone, Full-support Equilibria and equilibria without information acquisition were derived. Aggregately affine equilibria, where the players' average action is an affine function of the fundamental, were found to exist only in cases where the fundamental is normally distributed. Higher information costs, higher coordination motives and a more concentrated fundamental lead to the players paying less attention to the fundamental. For a large set of parameter combinations, there exists a unique equilibrium within the classes of AAE and equilibria without information acquisition. When the coordination motive is high and information costs are relatively low but bounded away from zero, there is multiplicity of equilibria. This shows that the flexible information acquisition technology is partially responsible for the appearance of multiple equilibria, even in situations where the full-information version of the interaction studied has a unique equilibrium.

# Chapter 5

## Conclusion

This thesis studied individuals whose rationality is bounded in distinct ways. In Chapter 2 individuals were not deliberating their decisions, they were just following the behavior for which their genes coded. Therefore, the long-run behavior of the population followed the replicator dynamics. From a more economic perspective, individuals were viewed as imitating relatively successful behavior as observed in the population. Even though they were not actually considering the strategic nature of their interactions, they were informed about how well different behaviors fared on average. In Chapter 3 they were also using behavioral heuristics to make decisions. They were voting on the way in which they want to be matched while being reinforced by previously obtained payoffs. It was seen that appropriately chosen matching processes can lead to efficient outcomes and that under favourable conditions democratic consensus can help populations overcome the tragedy of the commons.

At the other end of the spectrum, in Chapter 4 individuals were able to make sophisticated, strategic decisions but they were uninformed about a payoff-relevant element, termed “the fundamental.” The acquisition of information was crucial for them but their information-processing power was limited. As a result, they had to focus their attention on the events they considered more important. When playing a beauty contest coordination game, such inattentive individuals were found to be paying more attention to the payoff-relevant random variable as (i) their coordination motive or (ii) the cost of information decreased or when (iii) the distribution of the random variable was highly variant. Importantly, the particular information acquisition technology employed was the source for the appearance of multiple equilibria under particular parameter combinations.

As is always the case with scientific research, as questions are answered new questions appear. Therefore, in a similar sense as Chapter 2, one could ask what kind of matching rules can make efficient outcomes evolutionarily stable and how

one could maximize the basin of attraction of such evolutionarily stable states.

In the path of Chapter 3, different ways to endogenize matching rules could be explored. Such endogenization processes could be resulting from collective decision making (as in the model studied in Chapter 3) but need not be restricted to that. In one such formulation, the matching process could be decided by a central planner who is somehow constrained *e.g.* in not knowing the exact distribution over strategies of the population or who needs to pay a cost to decrease the entropy of the matching process. A different way could be that the driver of the evolution of assortativity is some natural process *e.g.* if individuals' tendency to match assortatively is hard-coded and subject to evolutionary pressures.

Finally, in a direction similar to the one of Chapter 4, one could begin by relaxing the assumption that individuals are identical in their coordination motives and informational costs. It is natural to think that some people are better at processing information than others. Do they do better than less attentive individuals when they play coordination games or are they unable to exploit their seemingly advantageous position? What is the welfare loss associated with this heterogeneity? A different agenda could explore how flexible information acquisition changes results in games with strategic substitutabilities *e.g.* where players have anti-coordination motives. Such research would also need to resolve technical issues that include finding a way to deal with situations where players' best responses are not uniquely defined.

# **Appendices**

# Appendix A

## Appendix to Chapter 2

### A.1 Finding all equilibria in $2 \times 2$ games

In this section we provide a tool that makes it easy for one to find and visualize NEGSs and ESSGs in the  $2 \times 2$  case. By use of our method, we can easily identify equilibria of such games by looking for intersections between two lines: one that depends on the payoffs (the *equilibrium curve*) and one that depends on the matching rule in effect (the *matching rule curve*). An example is shown in Figure A.1; the equilibrium state is at the intersection of the two lines.

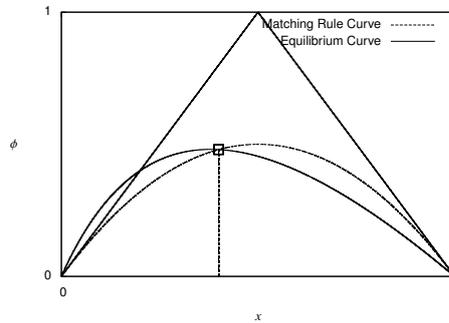


Figure A.1: Example of finding an equilibrium.

In what follows, we analyze games that have a payoff bimatrix of the general form presented in Table A.1. Without loss of generality, we will assume that  $A_1^1 \geq A_2^3$ .

	Strategy 1	Strategy 2
Strategy 1	$A_1^1, A_1^1$	$A_1^2, A_2^2$
Strategy 2	$A_2^2, A_1^2$	$A_2^3, A_2^3$

Table A.1: The general form of a  $2 \times 2$  game.  $A_1^1 \geq A_2^3$ .

### A.1.1 The matching rule curve

A matching rule for the  $2 \times 2$  case, will be of the form  $\mathbf{f}(\mathbf{x}) = (f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2))$ . Now notice that under consistency, it can be easily described by only defining one of the three coordinates  $f_i(\mathbf{x})$ . This is because in order for  $\mathbf{f}$  to satisfy the equations in (2.2) (two linearly independent equations in our example of 2 strategies), only one degree of freedom remains.<sup>1</sup> We pick the value of  $f_2(\mathbf{x})$  – that expresses the extent to which the two strategies get mixed with one another – to describe the matching rule. Of course, because there are only two strategies available, the state can be summarized by the proportion of individuals using Strategy 1 (the remaining individuals are clearly using Strategy 2). We will use  $x$  to denote this proportion and thus to express the state.<sup>2</sup> So any matching rule will be described by a function  $\phi : [0, 1] \rightarrow [0, 1]$ . Under the consistency requirement in (2.2), the three coordinates of  $\mathbf{f}$  can be calculated to be:

$$f_1(x) = x - \frac{1}{2}\phi(x) \quad f_2(x) = \phi(x) \quad f_3(x) = 1 - x - \frac{1}{2}\phi(x). \quad (\text{A.1})$$

More than that, the conditions  $0 \leq f_1(x)$ ,  $0 \leq f_2(x)$  and  $0 \leq f_3(x)$  must be satisfied for all  $x \in (0, 1)$ . From these, we get that the values  $\phi$  can take are restricted by:

$$0 \leq \phi(x) \leq 2x \quad \text{for } x \in \left[0, \frac{1}{2}\right], \quad 0 \leq \phi(x) \leq 2(1-x) \quad \text{for } x \in \left(\frac{1}{2}, 1\right]. \quad (\text{A.2})$$

So any consistent matching rule in the case of 2-strategy, 2-person normal form games can be summarized by a function  $\phi$  that satisfies (A.2).

It is now possible for us to draw diagrams that show what matching rules look like. Examples of graphs of matching rules are given in Figure A.2. A matching rule is summarized by a line that begins at  $(0,0)$ , assumes values ‘within’ the triangle bounded by (A.2) and ends at  $(1,0)$ .

Under this formalization, the random matching rule will be given by

$$\phi(x) = 2x - 2x^2$$

<sup>1</sup>Equations (2.2) are in essence ‘balancing conditions’ similar to condition (2) in Alger and Weibull (2012). *i.e.* They ensure that the number of 1-strategists that are matched to 2-strategists is equal to the number of 2-strategists that are matched to 1-strategists.

<sup>2</sup>Obviously,  $x_1 = x$  and  $x_2 = 1 - x$ .

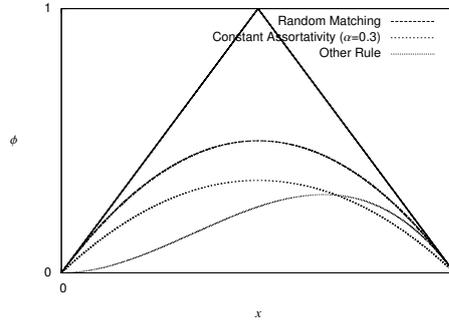


Figure A.2: Examples of Matching Rule Curves.

whereas the complete segregation rule is simply

$$\phi(x) = 0.$$

Another example would be the constant index of assortativity rule (Bergström, 2003) which can be summarized by

$$\phi(x) = 2(1 - \alpha)x(1 - x)$$

where  $\alpha \in [0, 1]$  is the index of assortativity.

### A.1.2 The equilibrium curve

Under any matching rule, it is easy to show that an interior state  $x^*$  is an equilibrium iff:

$$\pi_1(x^*) = \pi_2(x^*) \Leftrightarrow$$

$$[(A_2^2 - A_2^3)x^* + (A_1^1 - A_1^2)(1 - x^*)]\phi(x^*) = 2(A_1^1 - A_2^3)x^*(1 - x^*) \quad (\text{A.3})$$

and, looking for boundary equilibria, if  $\phi$  is differentiable at 0 and at 1, for  $x = 0$  to be an equilibrium, it must be the case that:

$$\pi_1(0) \leq \pi_2(0) \Rightarrow (A_1^1 - A_1^2) \frac{\partial \phi}{\partial x}(0) \geq 2(A_1^1 - A_2^3) \quad (\text{A.4})$$

and for  $x = 1$  to be an equilibrium, it must be the case that:

$$\pi_1(1) \geq \pi_2(1) \Rightarrow (A_2^3 - A_2^2) \frac{\partial \phi}{\partial x}(1) \leq 2(A_1^1 - A_2^3). \quad (\text{A.5})$$

Now, provided that there is actually some strategic interaction occurring between the two players, *i.e.* either  $A_2^2 \neq A_2^3$  or  $A_1^1 \neq A_1^2$  (or both), then from condition (A.3) we get two cases:

- If  $A_1^1 \neq A_2^3$ , then an interior state will be an equilibrium iff the value of  $\phi$  for that state is equal to the value of a function  $E$  for that given state. We will call this function the *equilibrium curve* of the game and it is given by:

$$E(x) = \frac{2(A_1^1 - A_2^3)x(1-x)}{(A_2^2 - A_2^3)x + (A_1^1 - A_1^2)(1-x)}. \quad (\text{A.6})$$

- In the case where  $A_1^1 = A_2^3$ , then the condition for an interior state to be an NEGS is:

$$\begin{cases} \phi(x) = 0 & \text{or} \\ x = \frac{A_1^2 - A_1^1}{A_2^2 - A_2^3 + A_1^2 - A_1^1} \end{cases} \quad (\text{A.7})$$

Condition (A.7) says that any state for which the two strategies do not mix at all will be an equilibrium state (obviously, as no strategy gets an advantage over the other) and, more importantly, that the state  $\frac{A_1^2 - A_1^1}{A_2^2 - A_2^3 + A_1^2 - A_1^1}$  will be an equilibrium *for all matching rules* (as long as this value is within the boundaries  $(0,1)$ ).

**Stability and the Equilibrium Curve** If we assume that the matching rule is  $\mathcal{C}^1$ , then we can easily check that a state  $x$  will be an ESSGS iff

$$\begin{cases} \phi(x) = E(x) & \text{and} \\ \frac{\partial \phi}{\partial x}(x) > \frac{\partial E}{\partial x}(x) \end{cases} \quad (\text{A.8})$$

Using the above analysis in conjunction with diagrams like the one in Figure A.2 can help us spot NEGS and ESSGSs very easily. All one has to do is to plot the matching rule  $\phi$  and the equilibrium curve  $E$  on the same diagram. If the two lines meet at an interior state, then this state is a NEGS. If along with that the equilibrium curve is above the matching rule to the left of the state and below it to the right of the

state, then the state is an ESSGS as well. Finally, for the states 0 and 1, one can say that in order for one of these states to be a NEGS (ESSGS), then it has to be that the slope of the matching rule is greater than (or equal to) the slope of the equilibrium curve at that state.

### A.1.3 Welfare in $2 \times 2$ games

In the case of  $2 \times 2$  games, by using the formalization introduced above, we can make equilibrium welfare considerations. What we are interested in is to see how the different equilibria fare in terms of welfare. For a  $2 \times 2$  game, the welfare at state  $x$  when the value of the matching rule at  $x$  is  $\phi$  is given by:

$$W(x, \phi) = A_2^3 + (A_1^1 - A_2^3)x + \frac{(A_1^2 + A_2^2 - A_1^1 - A_2^3)\phi}{2} \quad (\text{A.9})$$

And as long as  $A_1^2 + A_2^2 \neq A_1^1 + A_2^3$ , solving for  $\phi$ , we get:

$$\phi = \frac{2(W - A_2^3)}{A_1^2 + A_2^2 - A_1^1 - A_2^3} - \frac{2(A_1^1 - A_2^3)x}{A_1^2 + A_2^2 - A_1^1 - A_2^3} \quad (\text{A.10})$$

For any value of  $W$ , the above equation gives the set of points on the  $(x, \phi)$  plane that yield an average payoff of  $W$  for the population. We will call such lines *isogrowth lines* as all points on each of these lines leads to the same growth rate of the population (which is the same as the average payoff). Drawing such lines can help us visualize what is really happening in terms of welfare under the various matching rules. More than that, by combining the isogrowth lines with the equilibrium curves of different games, we can see which matching rules can lead to some (utilitarian) optimality. An example of an isogrowth diagram is depicted in Figure A.3. Finally, using the welfare function (A.9) along with the equilibrium curve (A.6) we can calculate the equilibrium welfare in the group selection game and then compare that to the expected payoff of a player in the normal form game. Such comparisons are carried out in Section A.2 for two classes of  $2 \times 2$  games.

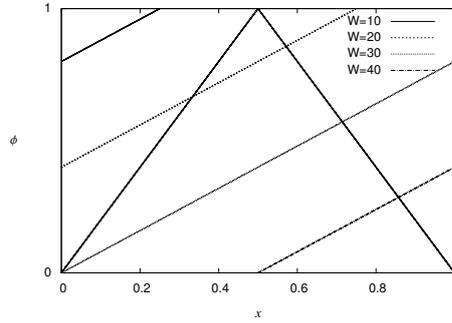


Figure A.3: An example isogrowth diagram.

## A.2 Comparative Statics and Welfare analysis for HD and SH Games

In this section we analyze a number of group selection games with 2 players and 2 strategies under different matching rules. We apply a method that allows us to graphically portray matching rules and makes the process of finding NEGSs and ESSGs as simple as finding the intersections of two curves.<sup>3</sup> We also provide comparative statics results for the class of matching rules with a constant index of assortativity. We restrict ourselves to analysis of consistent matching rules (see section 2.2.2) throughout this section.

### A.2.1 Hawk-dove/chicken

A game often analyzed in the literature of both economics and biology is the Hawk-Dove (HD) game.<sup>4</sup> Players in this game have two available pure strategies: Hawk (H) and Dove (D). In our formalization, a Hawk-Dove game is a  $2 \times 2$  game with  $A_2^2 > A_1^2 > A_1^1 > A_2^3$ .<sup>5</sup> The payoff matrices of three Hawk-Dove games are depicted in Table A.2.

In this game, there are three Nash Equilibria: Two asymmetric ones in pure strategies  $(H, D)$  and  $(D, H)$  and a symmetric one in mixed strategies where both players play Dove with probability  $p_D = \frac{A_1^2 - A_2^3}{A_1^2 + A_2^2 - A_1^1 - A_2^3}$  and Hawk with probability  $p_H = \frac{A_2^2 - A_1^1}{A_1^2 + A_2^2 - A_1^1 - A_2^3}$ . In the group selection game the state will be summarized by  $x$  which indicates the proportion of the population that follows D.

<sup>3</sup>The method is described in Section A.1 of the Appendix in detail.

<sup>4</sup>Economists usually refer to this game as Chicken rather than Hawk-Dove.

<sup>5</sup>As a convention in what follows and without loss of generality we will assume that  $A_1^1 \geq A_2^3$ .

	D	H
D	50, 50	40, 80
H	80, 40	0, 0

(a)  $A_1^2 + A_2^2 > 2A_1^1$

	D	H
D	50, 50	40, 60
H	60, 40	0, 0

(b)  $A_1^2 + A_2^2 = 2A_1^1$

	D	H
D	50, 50	20, 60
H	60, 20	0, 0

(c)  $A_1^2 + A_2^2 < 2A_1^1$

Table A.2: The payoff matrices of three Hawk-Dove games.

### Equilibria of the group selection game

Now, in order to find the NEGS and ESSGs of the PD game, we follow the methodology proposed described in Section A.1 of the Appendix. The equilibrium curves of the games in Table A.2 are shown in Figure A.4.

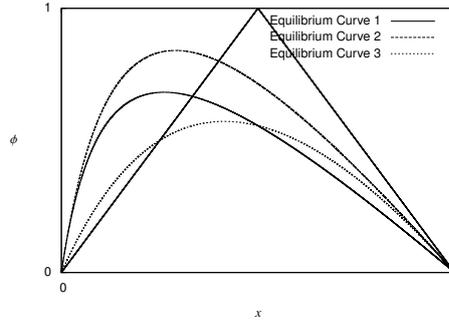


Figure A.4: Equilibrium curves of the HD games in Table A.2.

**Random Matching** As expected, the unique equilibrium of the group selection game under the Random Matching rule yields the unique symmetric Nash equilibrium of the game where a proportion  $x^* = \frac{A_1^2 - A_2^3}{A_1^2 + A_2^2 - A_1^1 - A_2^3}$  of the population play  $D$ .

**Complete Segregation** Under complete segregation, there is a unique equilibrium of the group selection game  $x^* = 1$  where the whole population follows  $D$ .

**Constant Index of Assortativity** Under a constant index of assortativity rule, the group selection game has a unique equilibrium given by:

$$x^* = \begin{cases} \frac{\frac{A_1^1 - A_2^3}{1 - \alpha} + A_1^2 - A_1^1}{A_1^2 + A_2^2 - A_1^1 - A_2^3} & \text{if } 0 \leq \alpha < \frac{A_2^2 - A_1^1}{A_2^2 - A_2^3} \\ 1 & \text{if } \frac{A_2^2 - A_1^1}{A_2^2 - A_2^3} \leq \alpha \leq 1 \end{cases} \quad (\text{A.11})$$

The equilibrium-finding process is shown in Figure A.5 for constant index of assortativity rules for different values of  $\alpha$ . The comparative statics results are summarized in Figure A.6.

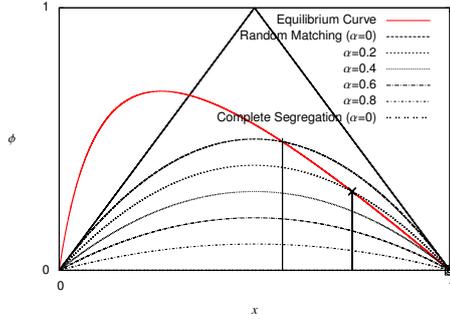


Figure A.5: NEGS with a constant index of assortativity in a Hawk-Dove game.

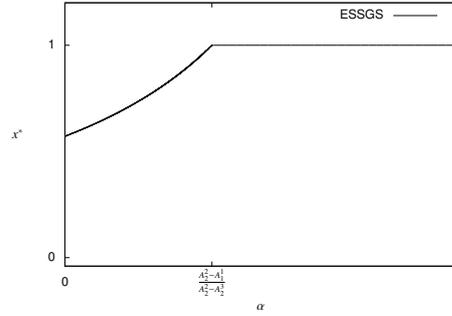


Figure A.6: NEGS as a function of the index of assortativity in a Hawk-Dove game.

In the HD game, strategies  $x \in [0, \frac{A_1^2 - A_2^3}{A_2^2 + A_1^2 - 2A_2^3})$  cannot be equilibria of the group selection game *under any (consistent) matching rule* due to constraint (A.2) on  $\phi$ .

## Welfare

In order to conduct welfare analysis, we use the methodology described in Section A.1.3 of the Appendix. The isogrowth diagram of a Hawk-Dove game is shown in Figure A.7. The comparison of equilibrium welfare in the group selection game and the normal form game is shown in Figure A.8. Notice that the equilibrium welfare curve is not defined for  $x \in [0, \frac{A_1^2 - A_2^3}{A_2^2 + A_1^2 - 2A_2^3})$  as these states can never be attained as equilibria of the group selection game. In all HD games, the level of equilibrium welfare is strictly increasing with the proportion of Doves in the population and thus, maximum equilibrium welfare is obtained when the equilibrium state is  $x = 1$  *i.e.* when the whole population follows D.

Now, in the case where  $A_1^2 + A_2^2 \leq 2A_1^1$ , maximum equilibrium welfare coincides with the maximum expected payoff players using symmetric strategies can get in the normal form game (which is attained when both players play D with certainty).

In the case where  $A_1^2 + A_2^2 > 2A_1^1$ , the normal form game maximum expected payoff (under symmetric strategies) is obtained if both players play D with probability  $p_D^* = \frac{A_1^2 + A_2^2 - 2A_2^3}{2(A_1^2 + A_2^2 - A_1^1 - A_2^3)}$ . However, when a matching rule that makes  $x = p_D^*$  an equilibrium is implemented, equilibrium welfare is reduced below  $A_1^1$ . This is because the

proportion of Hawk-Dove pairs – which are efficient in the utilitarian sense – is reduced in favor of more Hawk-Hawk and Dove-Dove pairs which are not as efficient.

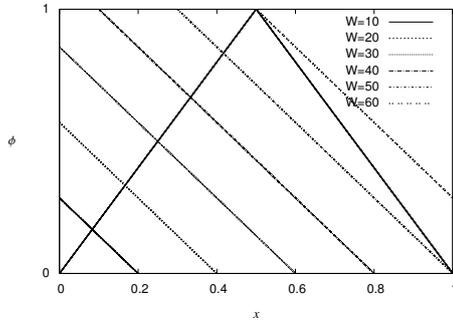


Figure A.7: Isogrowth diagram for a HD game.

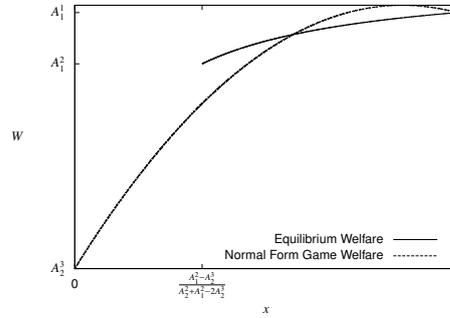


Figure A.8: Equilibrium welfare and normal form payoff for a HD game.

### A.2.2 Stag hunt

Another game with interesting insights on social behavior is the Stag Hunt.<sup>6</sup> In our notation a SH game will have values  $A_1^1 > A_2^2 \geq A_2^3 > A_1^2$ . The payoff matrices of three SH game are depicted in Table A.3.

	S	H
S	100,100	0,70
H	70,0	60,60

	S	H
S	100,100	0,70
H	70,0	70,70

	S	H
S	100,100	0,80
H	80,0	70,70

Table A.3: The payoff matrices of three SH games.

The game has three Nash equilibria, all symmetric. Two of them are in pure strategies (S,S) and (H,H) and one in mixed strategies where both players play S with probability  $p_S = \frac{A_2^3 - A_1^2}{A_1^1 + A_2^3 - A_1^2 - A_2^2}$  and H with probability  $p_H = \frac{A_1^1 - A_2^2}{A_1^1 + A_2^3 - A_1^2 - A_2^2}$ . Also, we require that  $A_2^2 + A_2^3 > A_1^1 + A_1^2$  so that even though the pure strategy equilibrium (S,S) is payoff dominant (*i.e.* it yields higher payoffs for both players), the pure strategy equilibrium (H,H) is risk dominant (*i.e.* if we assume that players are not sure which strategy their opponent will follow and assign equal probabilities to the two strategies, then the expected payoff from playing H exceeds the expected payoff from playing S).<sup>7</sup>

<sup>6</sup>For an extensive analysis see Skyrms (2004).

<sup>7</sup>See Carlsson and Van Damme (1993).

The importance of the Stag Hunt is that it shows that although the efficient outcome (S,S) is a Nash equilibrium, it may not always be selected. More than that, it has been shown that in some stochastic evolutionary models the risk dominant outcome occurs with probability 1 (Young, 1993) and that in global games, the risk dominant outcome is the only one that survives iterative elimination of dominated strategies when noise tends to vanish (Carlsson and Van Damme, 1993). So the literature suggests that in several environments it is the risk dominant rather than the payoff dominant outcome that prevails. We show that in our model this inefficiency can be amended under matching rules with high enough assortativity.

### Equilibria of the group selection game

The equilibrium curves of the games in Table A.3 are shown in Figure A.9.

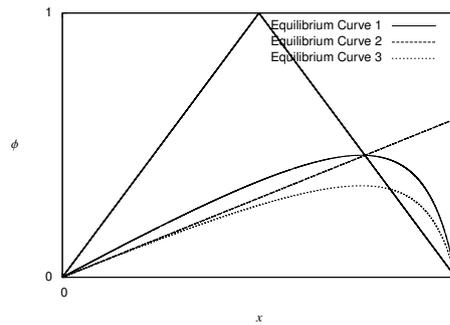


Figure A.9: Equilibrium curves of the SH games in Table A.3.

**Random Matching** As before, under the Random Matching rule, as expected, we get that there are three NEGS in the group selection game that coincide with the three Nash equilibria of the normal form game: two stable ones (ESSGSs) at  $x = 0$  and  $x = 1$  (where the whole population follows H and S respectively) and a NEGS which is not an ESSGS where a fraction of the population  $x = \frac{A_2^3 - A_1^2}{A_1^1 - A_1^2 + A_2^3 - A_2^2}$  follows S.

**Complete Segregation** Under the complete segregation matching rule, there is only one NEGS where the whole population follows S ( $x = 1$ ) and it is also an ESSGS.

**Constant Index of Assortativity** Under a matching rule with a constant index of assortativity  $\alpha$  we have two cases depending on the value of  $\alpha$ :

- if  $\alpha \leq \frac{A_2^3 - A_1^2}{A_1^3 - A_1^2}$  we have three NEGS: two NEGS that are also ESSGs where everybody follows H ( $x = 0$ ) or S ( $x = 1$ ) and a NEGS which is not an ESSG where a proportion of the population  $x = \frac{A_2^3 - A_1^1 + A_1^1 - A_1^2}{A_1^1 + A_2^3 - A_1^2 - A_2^2}$  follows S
- if  $\alpha > \frac{A_2^3 - A_1^2}{A_1^3 - A_1^2}$  there is only one NEGS that is also an ESSG where the whole population follows S ( $x = 1$ ).

The equilibrium-finding process is shown in Figure A.10 for constant index of assortativity rules for different values of  $\alpha$ . The comparative statics results are summarized in Figure A.11.

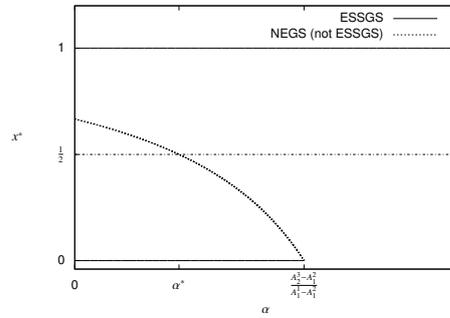
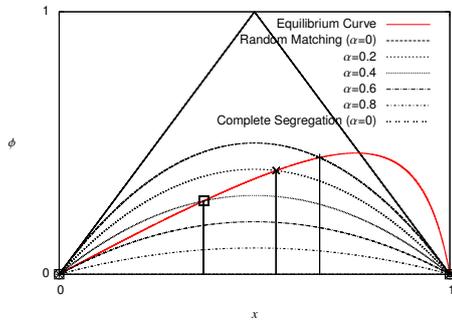


Figure A.10: NEGS with a constant index of assortativity in a SH game.

Figure A.11: NEGS as a function of the index of assortativity.

As in the case of the Hawk-Dove game, in the Stag Hunt there are some states that cannot be attained as equilibria under any matching rule. At these states, namely  $x \in \left( \frac{A_1^1 - A_1^2}{2A_1^1 - A_1^2 - A_2^2}, 1 \right)$ , the dynamics will tend to lead the population towards  $x = 1$  where they all follow S under any matching rule. So, if it happens that the system reaches one of these states, then it will be eventually brought to the state where the whole population uses the efficient strategy S.

**Risk Dominance** Notice that there is a value  $\alpha^* = \frac{(A_2^2 - A_1^2) - (A_1^1 - A_2^3)}{(A_2^2 - A_1^2) + (A_1^1 - A_2^3)}$  for which the basin of attraction of the ESSG at  $x = 1$  is greater than that of the ESSG at  $x = 0$  iff  $\alpha \in (\alpha^*, 1]$ . We can interpret that as follows: Assume that players in the population do not know whether each of the other players is going to play S or H and so, using the principle of insufficient reason, they ascribe equal probabilities (equal to 0.5 each) to each other player following S and H.<sup>8</sup> Then, if  $\alpha \in (\alpha^*, 1]$  the expected payoff for a player following S is higher than his expected payoff when he follows H and so,

<sup>8</sup>See also Carlsson and Van Damme (1993).

given the aforementioned beliefs, it is a best response for all of them to follow H, leading to the state being  $x = 1$ . Conversely when  $\alpha \in [0, \alpha^*)$ .

So, in the terms described above, we can have a notion of *risk dominance* in the group selection game. Of course – having assumed that  $A_2^2 + A_2^3 > A_1^1 + A_1^2$  as is usually done in Stag Hunt games – in the case where  $\alpha = 0$ , it is always the case that the risk dominant equilibrium is the one where the whole population follows H ( $x = 0$ ).

### Welfare

The isogrowth diagram of a Stag Hunt game is shown in Figure A.12. The comparison of equilibrium welfare in the group selection game and the normal form game is shown in Figure A.13. Notice that the equilibrium welfare curve is not defined for  $x \in \left(\frac{A_1^1 - A_1^2}{2A_1^1 - A_1^2 - A_2^2}, 1\right)$  as these states can never be attained as equilibria of the group selection game. The maximum level of welfare is obtained when the equilibrium state is the one where everybody follows S ( $x = 1$ ) and it coincides with the maximum expected payoff players using symmetric strategies can get in the normal form game.

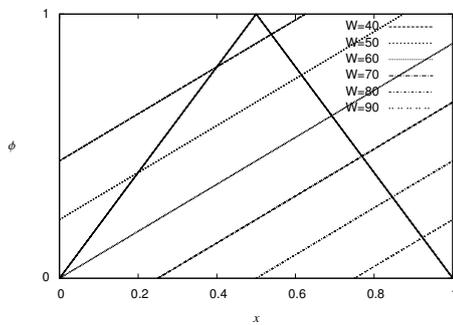


Figure A.12: Isogrowth diagram for a SH game.

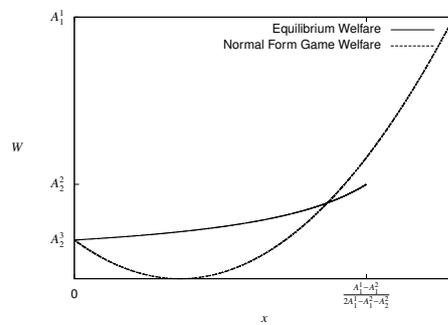


Figure A.13: Equilibrium welfare and normal form payoff for a SH game.

## A.3 Omitted Proofs

### A.3.1 Proof of Theorem 2.1

**Proof.** We will restrict our attention to  $2 \times 2$  games but similar extensions will hold for games with more strategies and/or players.

As the population is evolving in two different time modes (one related to dispersion phases and another one related to generations within trait-groups), we choose to use  $t$  to denote dispersion phases and  $\tau$  to denote generations within the trait-groups. Our aim is to identify how the population evolves from one dispersion phase to the next. Intuitively, that would relate more to discrete-time dynamics but one can extend that to continuous time if the time scale of that evolutionary changes need to occur is large enough. In what follows, we calculate the fitness of each type of individual. The relevant dynamic equations (either discrete- or continuous-time) will determine the evolution of the population thereafter.

At dispersion phase  $t$ , the original population which comprises of a proportion of  $x_1$  1-type individuals and  $x_2$  2-type individuals is being randomly drawn to form trait-groups of initial size 2. The outcome is that there will be a proportion  $x_1^2$  of type-1 trait-groups ( $\{11\}$ ), a proportion  $2x_1x_2$  of type-2 trait-groups ( $\{12\}$ ) and a proportion  $x_2^2$  of type-3 trait-groups ( $\{22\}$ ). Each of these trait-groups will evolve independently and in isolation of the rest of the trait-groups for  $T$  generations.

Now at each generation  $\tau$  a (isolated) trait-group will have a population that comprises of  $N_1$  1-type and  $N_2$  2-type individuals. These individuals are going to be drawn into pairs at each generation where they will act according to their types and get payoffs. Obviously, out of  $N_1 + N_2$  individuals  $\frac{N_1 + N_2}{2}$  groups (pairs) can be formed. Let's call  $\kappa$  the random variable that indicates how many of these groups are of type 2. Then the number of groups of type 1 will be given by  $\frac{N_1 - \kappa}{2}$  and the number of groups of type 2 will be given by  $\frac{N_2 - \kappa}{2}$ . The probability that  $\kappa$  type 2 groups will be formed by a population of  $N_1$  1-type and  $N_2$  2-type individuals is given by

$$F(\kappa; N_1, N_2) = \begin{cases} \frac{N_1!N_2!(N_1 - \kappa - 1)!!(N_2 - \kappa - 1)!!}{(N_1 - \kappa)!(N_2 - \kappa)!\kappa!(N_1 + N_2 - 1)!!} & \kappa \in \{0, 1, \dots, \min\{N_1, N_2\}\} \\ 0 & \text{otherwise} \end{cases}$$

where

$$(2l - 1)!! = \begin{cases} \prod_{i=1}^l (2i - 1) & \text{if } l \in \mathbb{N}_+ \\ 1 & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

is an appropriate extension of the odd factorial.

Each of the pairs formed in the first stage (when drawn from the population) will be the first generation of a *trait-group* that will evolve separately from all other groups (pairs) for  $T$  generations.

The evolution process of each of the *trait-groups* follows a stationary Markov chain. The state of the trait-group at time  $t$  is a vector  $\omega^t = (\omega_1^t, \omega_2^t, \omega_3^t)$  that represents the number of pairs of each type in the given *trait-group*. Let  $A_{\max}$  be the maximum number of children that can be obtained by a matched pair in a trait-group ( $A_{\max} = \max_{i \in \{1,2,3\}} \sum_j n_j^i A_j^i$ ). Then, after  $T$  generations, a trait-group that began with 2 individuals cannot exceed a population of  $K = \left(\frac{A_{\max}}{2}\right)^T$ . So, the number of groups after  $T$  periods cannot exceed  $\frac{K}{2}$ . This means that the state space is finite and is  $\Omega = \{(\omega_1, \omega_2, \omega_3) \in \mathbb{N}^3 : 0 < \sum_i \omega_i \leq \frac{K}{2}\}$ . We will let  $\mu$  denote the cardinality of  $\Phi$ . We can also impose an ordering  $\succ$  such that:

$$\omega \succ \omega' \Leftrightarrow \begin{cases} \sum_i \omega_i < \sum_j \omega'_j & \text{or} \\ \sum_i \omega_i = \sum_j \omega'_j \text{ and } \omega_1 > \omega'_1 & \text{or} \\ \sum_i \omega_i = \sum_j \omega'_j \text{ and } \omega_1 = \omega'_1 \text{ and } \omega_2 > \omega'_2 & \end{cases}$$

This is a *total order* over  $\Omega$  and thus it induces a ranking  $\#$  of the elements of  $\Omega$ . By  $\#\omega$  we will denote the rank of state  $\omega$  under  $\succ$ . Likewise  $\#^{-1}n$  will denote the  $n$ -th state according to the ranking induced by  $\succ$ .

The only element of the Markov chain that we need to determine is the transition probabilities. So, the probability that state  $\omega'$  will occur at period  $t + 1$  when we know that the trait-group was at state  $\omega$  at period  $t$  is given by:

$$P(\omega'|\omega) = \begin{cases} F(\omega'_2; N_1(\omega), N_2(\omega)) & \text{if } (\omega'_1, \omega'_3) = \left(\frac{N_1(\omega) - \omega'_2}{2}, \frac{N_2(\omega) - \omega'_2}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

where  $N_1(\omega) = 2\omega_1 A_1^1 + \omega_2 A_1^2$  and  $N_2(\omega) = 2\omega_3 A_2^3 + \omega_2 A_2^2$  give the population of 1-type and 2-type individuals in the trait-group at state  $\omega$  respectively. We will call  $\mathcal{P}$  the matrix that is defined as follows:  $\mathcal{P}_{ij} = P(\#^{-1}j | \#^{-1}i)$ .

At each period  $\tau$ , let  $\mathbf{P}(\tau) \in S_\mu$  denote the vector whose  $i$ -th entry gives the probability that the trait-group is in state  $\#^{-1}i$ . As the Markov process is stationary,  $\mathbf{P}(\tau)$  will be given by  $\mathbf{P}(\tau) = \mathcal{P}^\tau \mathbf{P}(0)$ . Where  $\mathbf{P}(0)$  is the initial state of the trait-group *i.e.*  $\mathbf{P}(0) \in \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, 1, 0, \dots, 0)\}$  as there's exactly one pair of individuals in each of the trait-groups in the beginning. In the interest of brevity, we will

call these vectors  $\mathbf{P}^1(0)$ ,  $\mathbf{P}^2(0)$  and  $\mathbf{P}^3(0)$  respectively. So, at the end of the  $T$  periods we will have  $\mathbf{P}^i(T) = \mathcal{D}^T \mathbf{P}^i(0)$  for  $i = 1, 2, 3$ . So after  $T$  periods have gone by, the expected number of type- $i$  groups that will be at a trait-group that contained one type- $k$  group at time 0 will be:

$$g_i^k = \sum_{l=1}^{\mu} P_l^k(T) (\#^{-1} l)_i$$

Actually, as we have a continuum of trait-groups, by using a law of large numbers we can say that the distribution of group types in trait-groups will be (almost surely) exactly the one given by the above formula.

We will calculate the average fitness that each starting  $j$ -type individual will get (*i.e.* the number of descendants a  $j$ -type is expected to have) after  $T$  periods. A  $j$ -type individual that is drawn into a  $k$  type trait-group is expected to have

$$\sum_{i \in \text{gsupp}(k)} g_i^k \frac{n_j^i}{n_j^k} A_j^i$$

descendants. As the distribution of trait-groups is given by the random matching rule  $\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x}), r_3(\mathbf{x})) = (x_1^2, 2x_1x_2, x_2^2)$  and as a  $k$ -type trait-group contains  $n_j^k$  first-generation  $i$ -type individuals, we can calculate the average fitness of a first-generation  $j$ -type individual by:

$$\pi_j(\mathbf{x}) = \frac{\sum_{k \in \text{supp}(j)} r_k(\mathbf{x}) n_j^k \sum_{i \in \text{gsupp}(k)} g_i^k \frac{n_j^i}{n_j^k} A_j^i}{\sum_{k \in \text{supp}(j)} r_k(\mathbf{x}) n_j^k}$$

Where  $\text{gsupp}(k) = \{i \in \{1, 2, 3\} | g_i^k > 0\}$ . Explicitly, for type-1 individuals, the average fitness is

$$x_1 g_1^1 A_1^1 + x_2 (2g_1^2 A_1^1 + g_2^2 A_1^2) \quad (\text{A.12})$$

whereas for type-2 individuals, average fitness is

$$x_2 g_3^3 A_2^3 + x_1 (2g_3^2 A_2^3 + g_2^2 A_2^2). \quad (\text{A.13})$$

The system will follow the replicator dynamics (either the discrete-time version of equation (2.8) or the continuous-time version of equation (2.9)) with fitness functions given by (A.12) and (A.13). We will show that a group selection model with a

matching rule given by

$$f_i(\mathbf{x}) = \frac{\sum_{k \in \text{gsupp}^{-1}(i)} r_k(\mathbf{x}) g_i^k}{\sum_{l=1}^3 \sum_{k \in \text{gsupp}^{-1}(l)} r_k(\mathbf{x}) g_l^k} \quad (\text{A.14})$$

has exactly the same dynamic behavior as the trait-group model. Actually, we can rewrite the above matching rule as

$$f_i(\mathbf{x}) = \frac{\sum_{k=1}^3 r_k(\mathbf{x}) g_i^k}{\sum_{l=1}^3 \sum_{k=1}^3 r_k(\mathbf{x}) g_l^k} \quad (\text{A.15})$$

as  $g_l^k = 0$  for all  $k \notin \text{gsupp}^{-1}(l)$ .

In the group selection model  $\langle I, \mathbf{f}, G \rangle$ , the payoffs for type-1 individuals is given by (see equation (2.11))

$$\frac{f_1(\mathbf{x})}{x_1} A_1^1 + \frac{f_2(\mathbf{x})}{2x_1} A_1^2$$

while the fitness for type-2 individuals is given by

$$\frac{f_2(\mathbf{x})}{x_2} A_2^3 + \frac{f_2(\mathbf{x})}{2x_2} A_2^2.$$

The key observation that makes it easy to show the result is that two models have identical dynamics if they have identical fractions  $\frac{\pi_1(\mathbf{x})}{\pi_2(\mathbf{x})}$  for all  $\mathbf{x} \in S_2$ . So, in order for the trait-group model to have the same dynamics as the group selection model, it is sufficient for  $\mathbf{f}(\mathbf{x})$  to satisfy:

$$\frac{x_1 g_1^1 A_1^1 + x_2 (2g_1^2 A_1^1 + g_2^2 A_1^2)}{x_2 g_3^3 A_2^3 + x_1 (2g_3^2 A_2^3 + g_2^2 A_2^2)} = \frac{\frac{f_1(\mathbf{x})}{x_1} A_1^1 + \frac{f_2(\mathbf{x})}{2x_1} A_1^2}{\frac{f_2(\mathbf{x})}{x_2} A_2^3 + \frac{f_2(\mathbf{x})}{2x_2} A_2^2}$$

It is easy to confirm that the above condition is satisfied for the matching rule given by (A.14). Also notice that  $\mathbf{f}$  satisfies  $f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) = 1$  and  $f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}) \geq 0$  as well as the conditions of theorem 2.2.

Notice that  $\mathbf{f}$  as calculated above would not necessarily be consistent as it may fail to satisfy condition (2.2). As the trait-group model and the group selection model under  $\mathbf{f}$  share the same dynamics, they will also have the same steady states. It is also interesting to point out that the matching rule  $\mathbf{f}$  reduces to the random matching rule when  $T = 1$  (in this case  $g_k^k = 1$  and  $g_i^k = 0$  for  $k \neq i$ ).  $\square$

### A.3.2 Proof of Theorem 2.2

**Best reply correspondence** The best reply correspondence (BRC) is a correspondence  $B : S_m \rightrightarrows S_m$  defined by:

$$B(\mathbf{x}) = \{\mathbf{y} \in S_m : (\forall \tilde{\mathbf{y}} \in S_m) \Pi(\mathbf{y}, \mathbf{x}) \geq \Pi(\tilde{\mathbf{y}}, \mathbf{x})\}$$

and gives the mixed strategies an agent can follow so as to maximize his/her expected payoff given that the state is  $\mathbf{x}$ .

We also define the *value function*  $V : S_m \rightarrow \mathbb{R}$  that gives the maximum payoff an agent can achieve at any given state. Formally:  $V(\mathbf{x}) = \max_{\mathbf{y} \in S_m} \Pi(\mathbf{y}, \mathbf{x})$ .

#### Equilibrium

We intend to show that under some assumptions on  $\mathbf{f}$ , an equilibrium state always exists. We will prove the existence result by using Kakutani's fixed point theorem. In order to do that, we need to show that the BRC is convex-valued, nonempty-valued and upper hemicontinuous. These prerequisites are proven in Lemmata A.1 and A.2.

**Lemma A.1** (Convex-valued BRC). *For any group selection game under a matching rule*

$\mathcal{G} = \langle I, G, \mathbf{f} \rangle$  *the best reply correspondence  $B$  is convex-valued.*

**Proof.** We can identify three different cases for  $B(\mathbf{x})$ :

- $B(\mathbf{x}) = \emptyset$  and thus  $B$  is convex-valued at  $\mathbf{x}$ .
- $B(\mathbf{x}) = \{\mathbf{y}^*\}$  *i.e.* the best reply correspondence contains only one element at  $\mathbf{x}$  and thus  $B$  is convex-valued at  $\mathbf{x}$ .
- $B(\mathbf{x})$  contains at least two elements at  $\mathbf{x}$  *i.e.* there exist  $\mathbf{y}_1^*, \mathbf{y}_2^* \in S_m$  such that

$$\Pi(\mathbf{y}_1^*, \mathbf{x}) \geq \Pi(\mathbf{y}, \mathbf{x}) \text{ for all } \mathbf{y} \in S_m$$

$$\Pi(\mathbf{y}_2^*, \mathbf{x}) \geq \Pi(\mathbf{y}, \mathbf{x}) \text{ for all } \mathbf{y} \in S_m$$

which is possible only if  $\Pi(\mathbf{y}_1^*, \mathbf{x}) = \Pi(\mathbf{y}_2^*, \mathbf{x}) = L$ . Now, for all  $\lambda \in [0, 1]$  we have the following series of equalities:

$$\begin{aligned} \Pi(\lambda \mathbf{y}_1^* + (1 - \lambda) \mathbf{y}_2^*, \mathbf{x}) &= (\lambda \mathbf{y}_1^* + (1 - \lambda) \mathbf{y}_2^*) \cdot \pi(\mathbf{x}) = \\ &= \lambda \mathbf{y}_1^* \cdot \pi(\mathbf{x}) + (1 - \lambda) \mathbf{y}_2^* \cdot \pi(\mathbf{x}) = \lambda \Pi(\mathbf{y}_1^*, \mathbf{x}) + (1 - \lambda) \Pi(\mathbf{y}_2^*, \mathbf{x}) = \\ &= \Pi(\mathbf{y}_1^*, \mathbf{x}) = \Pi(\mathbf{y}_2^*, \mathbf{x}) = L \end{aligned}$$

So, for any  $\mathbf{y}_1^*, \mathbf{y}_2^* \in B(\mathbf{x})$  we get that  $\lambda \mathbf{y}_1^* + (1 - \lambda) \mathbf{y}_2^* \in B(\mathbf{x})$  for all  $\lambda \in [0, 1]$  and thus  $B$  is convex-valued at  $\mathbf{x}$ .

Since these are the only possible cases, we can conclude that  $B$  is convex-valued in  $S_m$ .  $\square$

**Lemma A.2** (BRC: Nonempty-valued and upper hemicontinuous). *For a group selection game under a matching rule  $\mathcal{G} = \langle I, G, \mathbf{f} \rangle$ , if*

1.  $\mathbf{f}$  is continuous on  $S_m$  and
2. the partial derivatives  $\partial_j f_i$  for all  $j \in M$  and all  $i \in \text{supp}(j)$  exist on  $\text{bd}_j(S_m)$

then the best reply correspondence  $B$  is non-empty valued and upper hemicontinuous.

**Proof.** From assumption 2 of the lemma, we get that the limits  $\lim_{\tilde{\mathbf{x}} \rightarrow \mathbf{x}} \frac{f_i(\tilde{\mathbf{x}})}{\tilde{x}_j} = \partial_j f_i$  for all  $j \in M$  and all  $i \in \text{supp}(j)$  exist on  $\text{bd}_j(S_m)$  and from the definition of  $\pi_j$  (2.6), we get that

$$\lim_{\tilde{\mathbf{x}} \rightarrow \mathbf{x}} \pi_j(\tilde{\mathbf{x}}) = \pi_j(\mathbf{x}) \quad \text{on } \text{bd}_j(S_m).$$

So,  $\pi_j$  are continuous on  $\text{bd}_j(S_m)$  and since all  $f_i$  are continuous on  $S_m$ ,  $\pi_j$  are continuous on  $S_m \setminus \text{bd}_j(S_m)$  as sums of quotients of continuous functions. So,  $\pi$  is continuous on  $S_m$  and therefore,  $\Pi$  is continuous on  $S_m^2$ .

Now we can see that the conditions for Berge's maximum theorem are satisfied: (i)  $S_m$  is compact and (ii)  $\Pi$  is continuous. So, using Berge's theorem, we get that the value function  $V$  is continuous on  $S_m$  and that the best reply correspondence  $B$  is nonempty-valued, compact-valued, upper hemicontinuous and has a closed graph on  $S_m$ .

The results needed are the nonempty-valuedness and upper hemicontinuity of  $B$ .  $\square$

Now we have all that is needed in prove the theorem. From the results of Lemmata A.1 and A.2, we know that  $B : S_m \rightarrow S_m$  is a nonempty-valued, convex-valued, upper hemicontinuous correspondence defined on the nonempty, compact and convex set  $S_m$ . So, the conditions for the application of Kakutani's fixed point theorem are satisfied. From Kakutani's fixed point theorem, we get that there exists a  $\mathbf{x}^* \in S_m$  such that  $\mathbf{x}^* \in B(\mathbf{x}^*)$  which means that there exists a  $\mathbf{x}^* \in S_m$  such that

$$\Pi(\mathbf{x}^*, \mathbf{x}^*) \geq \Pi(\mathbf{x}, \mathbf{x}^*) \quad \text{for all } \mathbf{x} \in S_m.$$

That is,  $\mathcal{G}$  has an equilibrium. □

### A.3.3 Proof of Theorem 2.4

Firstly, note that definition 2.6 implies that a NEGS  $\mathbf{x}^*$  satisfies

$$\mathbf{x}^* \cdot \pi(\mathbf{x}^*) \geq \mathbf{y} \cdot \pi(\mathbf{x}^*) \quad \text{for all } \mathbf{y} \in S_m. \quad (\text{A.16})$$

1. Let  $\mathbf{x}^* \in S_m$  be a NEGS,  $I(\mathbf{x}) \equiv \{j \in M \mid x_j > 0\}$  and  $O(\mathbf{x}) \equiv \{j \in M \mid x_j = 0\}$ . Then, from (A.16) we get for all  $\mathbf{y} \in S_m$ :  $\sum_{j \in M} y_j \pi_j(\mathbf{x}^*) \leq \sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) + \sum_{q \in O(\mathbf{x}^*)} x_q^* \pi_q(\mathbf{x}^*)$ . Hence:

$$\sum_{j \in M} y_j \pi_j(\mathbf{x}^*) \leq \sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) \quad (\text{A.17})$$

Now let  $p = \arg \max_{j \in M} \pi_j(\mathbf{x}^*)$  and  $r = \arg \max_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*)$ . Clearly,  $\sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) \leq \pi_r(\mathbf{x}^*) \leq \pi_p(\mathbf{x}^*)$  where the second inequality holds because  $I(\mathbf{x}^*) \subseteq M$ . Hence for all  $\mathbf{y} \in S_m$ :

$$\sum_{j \in M} y_j \pi_j(\mathbf{x}^*) \leq \sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) \leq \pi_r(\mathbf{x}^*) \leq \pi_p(\mathbf{x}^*) \quad (\text{A.18})$$

Taking  $\mathbf{y} = (0, \dots, 0, \underbrace{1}_{p\text{-th}}, 0, \dots, 0)$ , we get  $\pi_p(\mathbf{x}^*) \leq \sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) \leq \pi_r(\mathbf{x}^*) \leq \pi_p(\mathbf{x}^*)$  which obviously implies that  $\sum_{l \in I(\mathbf{x}^*)} x_l^* \pi_l(\mathbf{x}^*) = \pi_p(\mathbf{x}^*)$ . But this is only possible if  $\pi_j(\mathbf{x}^*) = \pi_k(\mathbf{x}^*)$  for all  $j, k \in I(\mathbf{x}^*)$ , and this in turn implies that  $\pi_j(\mathbf{x}^*) = \mathbf{x}^* \cdot \pi(\mathbf{x}^*)$  for all  $j \in I(\mathbf{x}^*)$ . From equation (2.9), we therefore get  $\dot{x}_j^* = 0$  for all  $j \in M$ , *i.e.*,  $\mathbf{x}^*$  is a steady state.

2. Assume that  $x(t) \in \text{int } S_m$  converges to  $\mathbf{x}^*$  and that  $\mathbf{x}^*$  is not a NEGS. That  $\mathbf{x}^*$  is not a NEGS means that there exists a  $j$  with  $\mathbf{e}_j \cdot \pi(\mathbf{x}^*) = \pi_j(\mathbf{x}^*) > \mathbf{x}^* \cdot \pi(\mathbf{x}^*)$ . Hence  $(\pi_j(\mathbf{x}^*) - \mathbf{x}^* \cdot \pi(\mathbf{x}^*)) \geq \varepsilon > 0$  for some  $\varepsilon > 0$ . Since  $x(t)$  converges and  $\pi_j$  is continuous on the interior,  $(\pi_j(x(t)) - \mathbf{x}^* \cdot \pi(x(t))) \rightarrow 0$  as  $t \rightarrow \infty$ . This is a contradiction. Note that at the boundary, this holds because we have defined the  $\pi_j$ 's so that they are continuous onto the boundary. If we had not done that, the claim would in general be false for a vector  $\mathbf{x}^*$  on the boundary. This same problem does not arise with random matching/in the usual replicator dynamics setting because the payoff functions trivially are continuous. This shows exactly why our “continuous extension to the boundary” is the right thing to do.

3. Precisely as in the previous proof and by continuity of the  $\pi$ 's we get that if  $\mathbf{x}^*$  is not an NEGS then there exists an  $\varepsilon > 0$  such that for all  $x$  in a neighborhood of  $\mathbf{x}^*$ :  $(\pi_j(x) - x \cdot \pi(x)) \geq \varepsilon > 0$ . This contradicts Lyapunov stability.

4. Following Weibull (1995, pp. 95–100), we will use Lyapunov's direct method to prove the proposition. What we need is to find a scalar function  $H$  that is defined on a neighborhood  $Q$  of  $\mathbf{x}^*$  which has the following properties: (i)  $H$  is continuously differentiable on  $Q$ , (ii)  $H(\mathbf{x}^*) = 0$ , (iii)  $H(\mathbf{y}) > 0$  for all  $\mathbf{y} \in Q \setminus \{\mathbf{x}^*\}$  and (iv)  $\dot{H}(\mathbf{y}) = \frac{d}{dt} H(\mathbf{y}) < 0$  for all  $\mathbf{y} \in Q \setminus \{\mathbf{x}^*\}$ .

Let us consider the set  $Q_{\mathbf{x}^*} \equiv \{\mathbf{y} \in S_m \mid I(\mathbf{x}^*) \subseteq I(\mathbf{y})\}$  i.e. the set of all population strategies that assign positive weights to all the pure strategies that  $\mathbf{x}^*$  assigns positive weights. Obviously,  $\mathbf{x}^* \in Q_{\mathbf{x}^*}$  and  $Q_{\mathbf{x}^*}$  is an open set (in the topology induced from  $\mathbb{R}^m$ ). So,  $Q_{\mathbf{x}^*}$  is a neighborhood of  $\mathbf{x}^*$ . We will show that the function  $H_{\mathbf{x}^*} : Q_{\mathbf{x}^*} \rightarrow \mathbb{R}$  defined by

$$H_{\mathbf{x}^*}(\mathbf{y}) = \sum_{j \in I(\mathbf{x}^*)} x_j^* \log \left( \frac{x_j^*}{y_j} \right)$$

satisfies all of the above conditions (i–iv) i.e. that it is a strict local Lyapunov function on  $Q_{\mathbf{x}^*}$ .

First of all, it is easy to verify that (i)  $H_{\mathbf{x}^*}$  is continuously differentiable on  $Q_{\mathbf{x}^*}$  and that (ii)  $H_{\mathbf{x}^*}(\mathbf{x}^*) = 0$ . Now, as  $\mathbf{x}^*$  is an ESSGS, we know that there exists a neighborhood  $U$  of  $\mathbf{x}^*$  such that condition (2.13) holds for all  $\mathbf{y} \in U$ . We will consider the restriction of  $H_{\mathbf{x}^*}$  on the set  $U \cap Q_{\mathbf{x}^*}$ , a neighborhood of  $\mathbf{x}$ . The next step is to show that  $H_{\mathbf{x}^*}$  is strictly positive on  $U \cap Q_{\mathbf{x}^*}$ . As the function  $-\log(\cdot)$  is convex, we get from Jensen's

inequality:

$$H_{\mathbf{x}^*}(\mathbf{y}) = \sum_{j \in I(\mathbf{x}^*)} x_j^* \left( -\log \left( \frac{y_j}{x_j^*} \right) \right) \geq -\log \left( \sum_{j \in I(\mathbf{x}^*)} x_j^* \left( \frac{y_j}{x_j^*} \right) \right) \geq -\log \left( \sum_{j \in M} y_j \right) = 0$$

Now, in the case where  $I(\mathbf{x}^*) = I(\mathbf{y})$ , the first inequality is strict (because of the log's strict concavity) and in the case where  $I(\mathbf{x}^*) \subsetneq I(\mathbf{y})$ , the second inequality is strict. In any case, we will always have that (iii)  $H_{\mathbf{x}^*}(\mathbf{y}) > 0$  for all  $\mathbf{y} \in U \cap Q_{\mathbf{x}^*} \setminus \{\mathbf{x}^*\}$ .

The last step is to show that  $\dot{H}_{\mathbf{x}^*}$  is negative for all  $\mathbf{y} \in U \cap Q_{\mathbf{x}^*} \setminus \{\mathbf{x}^*\}$ . Indeed:

$$\dot{H}_{\mathbf{x}^*}(\mathbf{y}) = \sum_{j \in I(\mathbf{x}^*)} \partial_j H_{\mathbf{x}^*}(\mathbf{y}) \dot{y}_j = - \sum_{j \in I(\mathbf{x}^*)} \frac{x_j^*}{y_j} \dot{y}_j$$

and using equation (2.9), we get:

$$\dot{H}_{\mathbf{x}^*}(\mathbf{y}) = -\mathbf{x}^* \cdot \pi(\mathbf{y}) + \bar{\pi}(\mathbf{y}) = (\mathbf{y} - \mathbf{x}^*) \pi(\mathbf{y})$$

which we know is negative because of (2.13). So, the final condition (iv) is satisfied.  $\square$

### A.3.4 Proof of Remark 2.4.1

Let us denote by  $\mathbf{y}^i \in S_m$  the (mixed) strategy used by player  $i$  in the normal-form game  $G$  and by  $\mathbf{x}^{-i} \in S_m$  the strategy used in the normal-form game  $G$  by *all* player  $i$ 's opponents. Let also  $P_i(\mathbf{y}^i | \mathbf{x}^{-i})$  be the expected payoff of player  $i$  in the normal-form game when he/she is using strategy  $\mathbf{y}^i$  and *all* of his opponents use strategy  $\mathbf{x}^{-i}$ . Since  $G$  is symmetric, we have  $P_i(\mathbf{y}^i | \mathbf{x}^{-i}) = P_j(\mathbf{y}^j | \mathbf{x}^{-j})$  for all  $i, j \in N$ . So we can write  $P(\mathbf{y} | \mathbf{x})$  to express the expected payoff in the normal-form game of any player using strategy  $\mathbf{y}$  when all his opponents use the same strategy  $\mathbf{x}$ .

A symmetric Nash equilibrium of game  $G$  is a strategy  $\mathbf{x}^* \in S_m$  such that:

$$P(\mathbf{x}^* | \mathbf{x}^*) \geq P(\mathbf{y} | \mathbf{x}^*) \quad \text{for all } \mathbf{y} \in S_m.$$

So, for  $\mathbf{x}^*$  to be a symmetric Nash equilibrium, if every opponent of any given player  $i$  is using strategy  $\mathbf{x}^*$ , it must be a best response for player  $i$  to use the same strategy  $\mathbf{x}^*$  as well.

On the other hand, a strategy  $\mathbf{x}^*$  will be an equilibrium in game  $\langle I, G, \mathbf{f} \rangle$  iff:

$$\Pi(\mathbf{x}^*, \mathbf{x}^*) \geq \Pi(\mathbf{y}, \mathbf{x}^*) \quad \text{for all } \mathbf{y} \in S_m$$

Where  $\Pi(\mathbf{y}, \mathbf{x}^*)$  expresses the expected payoff of an individual using strategy  $\mathbf{y}$  while the rest of the population is using strategy  $\mathbf{x}^*$ . In order to prove the proposition, all we need to show is that

$$\Pi(\mathbf{y}, \mathbf{x}) = P(\mathbf{y}|\mathbf{x}) \quad \text{for all } \mathbf{y} \in S_m \quad (\text{A.19})$$

under the random matching rule. If we let  $\mathbf{e}_j$  be the probability vector that corresponds to pure strategy  $j$ , then (A.19) boils down to

$$\pi_j(\mathbf{x}) = P(\mathbf{e}_j|\mathbf{x}) \quad \text{for all } j \in M. \quad (\text{A.20})$$

**Calculating  $\pi_j(\mathbf{x})$ .** Let us denote by  $M_{-j}^i$  the set of all strategies other than  $j$  represented in group  $i$  and by  $\Gamma_j^1$  the set of all groups that contain exactly one individual following strategy  $j$ . Formally  $M_{-j}^i = \{k \in M \setminus \{j\} | i \in \text{supp}(j)\}$  and  $\Gamma_j^1 = \{i \in \Gamma_{n,m} | n_j^i = 1\}$ . Calculating  $\pi_j(\mathbf{x})$  under  $\mathbf{r}^{n,m}$  yields:

$$\pi_j(\mathbf{x}) = \sum_{i \in \text{supp}(j)} \frac{(n-1)! x_j^{n_j^i - 1}}{(n_j^i - 1)!} \prod_{k \in M_{-j}^i} \frac{x_k^{n_k^i}}{n_k^i!} A_j^i, \quad \mathbf{x} \in S_m \setminus \text{bd}_j(S_m) \quad (\text{A.21})$$

$$\pi_j(\mathbf{x}) = \sum_{i \in \Gamma_j^1} (n-1)! \prod_{k \in M_{-j}^i} \frac{x_k^{n_k^i}}{n_k^i!} A_j^i, \quad \mathbf{x} \in \text{bd}_j(S_m) \quad (\text{A.22})$$

**Calculating  $P(\mathbf{e}_j|\mathbf{x})$ .** In general, all players use mixed strategies *i.e.* a randomization over the set of pure strategies  $M$ . We will denote the pure strategy a player  $l$  ends up using after the randomization process has taken place – *i.e.* the realization of player  $l$ 's mixed strategy – as  $\mathbf{s}^l$ . The probability of a player ending up in a situation where his/her opponents follow (pure) strategies  $\mathbf{s}^{-l} \in M^{n-1}$  with  $\mathbf{s}^{-l} = (s^1, \dots, s^{l-1}, s^{l+1}, \dots, s^n)$  will be denoted by  $p(\mathbf{s}^{-l})$ . When all player  $l$ 's oppo-

nents use the same strategy  $\mathbf{x}$ , those probabilities can be calculated to be:

$$p(\mathbf{s}^{-l}) = \prod_{k \in M} (x_k)^{\nu_k(\mathbf{s}^{-l})}$$

where  $\nu_k(\mathbf{s}^{-l}) \in \{0, 1, \dots, n-1\}$  is the number of player  $l$ 's opponents using strategy  $k$  in the ordered set  $\mathbf{s}_{-l}$ .

Let us fix player  $l$ 's strategy (realization) to be  $\mathbf{s}^l = \mathbf{e}_j$ . Since the game  $G$  is symmetric, the payoff of player  $l$  will not depend on the exact ordering in  $\mathbf{s}^{-l}$  but on the vector  $\nu(\mathbf{s}^{-l}) = (\nu_1(\mathbf{s}^{-l}), \dots, \nu_m(\mathbf{s}^{-l}))$ . This means that different  $\mathbf{s}^{-l}$ 's with the same  $\nu(\mathbf{s}^{-l})$  will yield the same payoff for player  $l$ . The number of the different  $\nu$  outcomes are elements is  $\gamma_{n-1, m}$ . Let us use  $\kappa \in \Gamma_{n-1, m}$  to index the different  $\nu$ . By abusing notation, we can calculate the probability of a specific  $\nu^\kappa$  to occur as

$$p(\nu^\kappa) = \frac{(n-1)!}{\prod_{k \in M} \nu_k^\kappa!} \prod_{k \in M} (x_k)^{\nu_k^\kappa}. \quad (\text{A.23})$$

As player  $l$  is using strategy  $j$ , if he ends up in a situation where his/her opponents' realizations are  $\kappa$ , it is as if he ends up in a group  $i$  where  $n_k^i = \nu_k^\kappa$  for  $k \neq j$  and  $n_j^i = \nu_j^\kappa + 1$ . This group will be in  $\text{supp}(j)$  and we will write  $i = j \triangleright \kappa$  and read: " $i$  is the group that we get if we add an individual who uses strategy  $j$  to a set of opponents whose realizations are  $\kappa$ ". Notice that the probabilities in (A.23) are *independent* of player  $l$ 's choice of strategy. So, the probability of player  $l$  ending up in situation  $i$  conditional on him using strategy  $j$  will be the same as the probability realization  $\kappa$  occurring. Using the  $i$ - rather than the  $\kappa$ - indexing, we can rewrite (A.23) (abusing the notation once again) as:

$$p(i|j) = p(j \triangleright \kappa | j) = p(\nu^\kappa) = \frac{(n-1)! x_j^{n_j^i - 1}}{(n_j^i - 1)!} \prod_{k \in M_{-j}^i} \frac{(x_k)^{n_k^i}}{n_k^i!}.$$

Now, in each of these cases  $i$ , player  $l$  gets a payoff of  $A_j^i$  and his expected payoff is:

$$P(\mathbf{e}_j | \mathbf{x}) = \sum_{i \in \text{supp}(j)} p(i|j) A_j^i = \sum_{i \in \text{supp}(j)} \frac{(n-1)! x_j^{n_j^i - 1}}{(n_j^i - 1)!} \prod_{k \in M_{-j}^i} \frac{(x_k)^{n_k^i}}{n_k^i!} A_j^i. \quad (\text{A.24})$$

In the special case where  $\mathbf{x} \in \text{bd}_j S_m$ , player  $l$  can be sure that he is the only one using strategy  $j$  and thus, the only groups that get positive probability are the ones in  $\Gamma_j^1$  which have  $n_j^i = 1$ . So his/her expected payoff is:

$$P(\mathbf{e}_j|\mathbf{x}) = \sum_{i \in \Gamma_j^1} p(i|j) A_j^i = \sum_{i \in \Gamma_j^1} (n-1)! \prod_{k \in M_{-j}^i} \frac{(x_k)^{n_k^i}}{n_k^i!} A_j^i. \quad (\text{A.25})$$

By comparing equation (A.21) to (A.24) and equation (A.22) to (A.25), we can see that

$$\pi_j(\mathbf{x}) = P(\mathbf{e}_j|\mathbf{x})$$

and as we showed that for an arbitrary  $j$ , it holds for all  $j \in M$ .  $\square$

### A.3.5 Proof of Theorem 2.5

Let us define the following sets of group types:

$$\mathcal{E}(\mathbf{x}^*) = \{i \in \Gamma_{n,m} : \text{supp}^{-1}(i) \subseteq I(\mathbf{x}^*)\}$$

$$[M] = \{i \in \Gamma_{n,m} : \text{supp}^{-1}(i) = \{j\} \text{ for some } j \in M\}$$

$\mathcal{E}(\mathbf{x}^*)$  consists of the group types that contain only individuals of types that are present in the population at  $\mathbf{x}^*$ .  $\mathcal{E}'(\mathbf{x}^*)$  will denote its complement *i.e.* group types that contain at least one individual of one of the types that are not present at  $\mathbf{x}^*$ .  $[M]$  consists of the groups types that contain only one type of individuals. We will denote the group type that contains only individuals of type  $j$  by  $[j]$ . Now we can separate all group types in the following four categories:

- $SP(\mathbf{x}^*) = \mathcal{E}(\mathbf{x}^*) \cap [M]$  is the set of all group types that contain a single type of individuals that are present at  $\mathbf{x}^*$ .
- $SA(\mathbf{x}^*) = \mathcal{E}'(\mathbf{x}^*) \cap [M]$  is the set of all group types that contain a single type of individuals that are absent at  $\mathbf{x}^*$ .
- $MP(\mathbf{x}^*) = \mathcal{E}(\mathbf{x}^*) \setminus [M]$  is the set of all group types that contain more than one types of individuals that are present at  $\mathbf{x}^*$ .

- $MA(\mathbf{x}^*) = \mathcal{E}'(\mathbf{x}^*) \setminus [M]$  is the set of all group types that contain more than one types of individuals and at least one of them is absent at  $\mathbf{x}^*$ .

Let us define for any  $\mathbf{x} \in S_m$  the following:

$$\mu = \operatorname{argmin}_{j \in I(\mathbf{x}^*)} \frac{x_j}{x_j^*}$$

We construct  $\mathbf{h}$  as follows:

- For all  $i \in MA(\mathbf{x}^*)$  we define  $h_i(\mathbf{x}) = 0$ .
- For all  $i \in SA(\mathbf{x}^*)$  we define  $h_{[j]}(\mathbf{x}) = x_j$ .
- For all  $i \in MP(\mathbf{x}^*)$  we define  $h_i(\mathbf{x}) = \frac{x_\mu}{x_\mu^*} f_i^*(\mathbf{x}^*)$ .
- For all  $i \in SP(\mathbf{x}^*)$  we define  $h_{[j]}(\mathbf{x}) = \frac{x_\mu}{x_\mu^*} f_{[j]}^*(\mathbf{x}^*) + x_j - \frac{x_\mu}{x_\mu^*} x_j^*$ .

It is easy to check that  $\mathbf{h}$  is a matching rule as it satisfies definition 2.1 i.e. it is a function from  $S_m$  to  $S_{\gamma_{n,m}}$ . More than that it is also easy to see that  $\mathbf{h}(\mathbf{x}^*) = \mathbf{f}(\mathbf{x}^*)$  and so  $(\mathbf{x}^*, \mathbf{h})$  is an evolutionary optimum. All we have to do is to show that  $\mathbf{x}^*$  is a NEGS under  $\mathbf{h}$ .

Now let us define  $A^* = \max_{(\mathbf{x}, \mathbf{f}) \in \mathcal{E}} \bar{\pi}_{\mathbf{f}}(\mathbf{x})$ . As  $(\mathbf{x}^*, \mathbf{h})$  is an evolutionary optimum, it has to be that  $\mathbf{x}^*$  is a steady state of the replicator dynamics under  $\mathbf{h}$ . So:

1. For all  $j \in I(\mathbf{x}^*)$  it has to be that  $\pi_{\mathbf{h}}(\mathbf{x}^*) = A^*$  which is ensured by the fact that  $\mathbf{h}(\mathbf{x}^*) = \mathbf{f}(\mathbf{x}^*)$  and
2. there is no restriction for all  $j \in O(\mathbf{x}^*)$ .

For  $\mathbf{x}^*$  to be a NEGS it must hold that:

$$\mathbf{x}^* \cdot \pi_{\mathbf{h}}(\mathbf{x}^*) \geq \mathbf{y} \cdot \pi_{\mathbf{h}}(\mathbf{x}^*) \quad \text{for all } \mathbf{y} \in S_m.$$

Notice that from point 1. above, if  $\mathbf{x}^* \in \operatorname{int} S_m$ , it is a NEGS as  $\mathbf{y} \cdot \pi_{\mathbf{h}}(\mathbf{x}^*) = A^*$  for all  $\mathbf{y} \in S_m$  and the proposition holds.

If  $\mathbf{x}^* \in \text{bd } S_m$ , then all we need to do is show that  $\pi_{\mathbf{h}j}(\mathbf{x}^*) \leq A^*$  for all  $j \in O(\mathbf{x}^*)$ . By definition

$$\pi_{\mathbf{h}j}(\mathbf{x}^*) = \sum_{i \in \text{supp}(j)} \frac{n_j^i}{n} \partial_j^+ h_i(\mathbf{x}^*) A_j^i = \partial_j^+ h_{[j]}(\mathbf{x}^*) A_j^{[j]} + \sum_{i \in M \setminus \text{supp}(j)} \frac{n_j^i}{n} \partial_j^+ h_i(\mathbf{x}^*) A_j^i = A_j^{[j]}.$$

Finally, notice that under any matching rule the states  $e_j = (0, \dots, \underbrace{0, 1}_{j\text{-th}}, 0, \dots, 0)$  are steady states and the payoff of all individuals on these states is simply:  $\bar{\pi}_{\mathbf{h}}(\mathbf{e}_j) = A_j^{[j]}$ . But as  $(\mathbf{x}^*, \mathbf{h})$  is an evolutionary optimum, we know that  $A_j^{[j]} \leq A^*$  for all  $j \in M$ . So,  $\pi_{\mathbf{h}j}(\mathbf{x}^*) \leq A^*$  for all  $j \in M$ .  $\square$

# Appendix B

## Appendix to Chapter 3

### B.1 Omitted Proofs

#### B.1.1 Proof of Lemma 3.1

In order for  $x^* \in (0, 1)$  to be an environment equilibrium, we need

$$\pi_C(x^*, \alpha) = \pi_D(x^*, \alpha). \quad (\text{B.1})$$

When  $a - r + 1 \neq 0$  the above condition yields

$$x^* = \frac{a - r + \frac{r}{1-\alpha}}{a - r + 1}. \quad (\text{B.2})$$

Notice that the above condition is only necessary as we haven't verified that the stability condition is satisfied. Taking stability into account, we get:

$$\begin{aligned} \dot{x} > 0 &\Rightarrow \pi_C(x, \alpha) > \pi_D(x, \alpha) \Rightarrow \\ x(a - r + 1) &< a - r + \frac{r}{1-\alpha} \end{aligned} \quad (\text{B.3})$$

So  $x^* \in (0, 1)$  given by (B.2) will be an environment equilibrium for social dilemmas with  $a - r + 1 \geq 0$  as in that case (B.3) yields  $\dot{x} > 0 \Rightarrow x < x^*$ .

Also,  $x^* = 0$  will be an equilibrium if

$$\alpha \leq \frac{a}{a - r} \quad (\text{B.4})$$

and  $x^* = 1$  will be an equilibrium if

$$\alpha \geq 1 - r. \quad (\text{B.5})$$

Following Jensen and Rigos (2014) we can separate the set of social dilemmas in three classes depending on their equilibrium behavior.

Case A:  $a - r + 1 > 0$ . This class contains most of the social dilemmas considered here and includes all MHDs, VDs, UDs and some PDs. All environments of social dilemmas of this class always have a unique equilibrium given by

$$x^* = \begin{cases} 0 & \text{if } \alpha \leq \frac{a}{a-r} \\ \frac{a-r+\frac{r}{1-\alpha}}{a-r+1} & \text{if } \frac{a}{a-r} < \alpha < 1-r \\ 1 & \text{if } \alpha \geq 1-r \end{cases} \quad (\text{B.6})$$

Case B:  $a - r + 1 < 0$ . This class contains a subset of PDs. In this type of social dilemmas the only possible environment equilibria are at  $x^* = 0$  and  $x^* = 1$ . More specifically, for  $\alpha \leq 1 - r$  we have a unique environment equilibrium at  $x^* = 0$ . For  $\alpha \in (1 - r, \frac{a}{a-r})$  we have two environment equilibria: one at  $x^* = 0$  and one at  $x^* = 1$ . Finally, for  $\alpha \geq \frac{a}{a-r}$  we have a unique environment equilibrium at  $x^* = 1$ .

Case C:  $a - r + 1 = 0$ . This class contains a subset of PDs. In this type of social dilemmas we have a unique environment equilibrium at  $x^* = 0$  for  $\alpha < 1 - r (= -a)$ , a unique environment equilibrium at  $x^* = 1$  for  $\alpha > 1 - r$  and a continuum of environment equilibria for  $\alpha = 1 - r$ , actually in this case any  $x \in [0, 1]$  is an environment equilibrium.  $\square$

### B.1.2 Proof of Lemma 3.2

Consider a social dilemma  $G = (a, r)$  and its corresponding environments  $E = (G, \alpha)$  with  $\alpha \in [0, 1]$ .

If  $x^* \in (0, 1)$  can be an environment equilibrium only for social dilemmas with  $a - r + 1 \geq 0$  and it has to satisfy

$$x^* = \frac{a - r + \frac{r}{1-\alpha}}{a - r + 1}$$

(see equation B.2).

The uniform population without any cooperators ( $x = 0$ ) will be an environment equilibrium if

$$\alpha \leq -\frac{a}{r - a}$$

and the uniform population consisting solely of cooperators ( $x = 1$ ) will be an environment equilibrium if

$$\alpha \geq 1 - r.$$

Now using the efficiency formula

$$W(x, \alpha) = rx + (1 - \alpha)(1 - x)x(1 + a - r)$$

we can calculate efficiency at an interior environment equilibrium. After calculation, this gives

$$W_G^{\text{int}}(\alpha) = \frac{\alpha r + (1 - \alpha)a}{1 + a - r}. \quad (\text{B.7})$$

It is clear that efficiency when the environment equilibrium is  $x^* = 1$  will be

$$W_G^1(\alpha) = r$$

and when the environment equilibrium is  $x^* = 0$  efficiency is

$$W_G^0(\alpha) = 0.$$

Notice that  $0 \leq W_G^{\text{int}}(\alpha) \leq r$ .

We look into efficiency for the three classes of social dilemmas mentioned in B.1.2.

Case A:  $a - r + 1 > 0$ . As described in (B.3), the interior point will be an environment equilibrium for some environments if  $1 + a - r > 0$  in which case it is the unique environment equilibrium and it is clear from (B.7) that efficiency in this case is increasing in  $\alpha$ . More specifically:

$$\max W^*(\alpha) = W^*(\alpha) = \begin{cases} \frac{a}{1+a-r} & \text{if } \alpha \leq \frac{a}{a-r} \\ \frac{\alpha r + (1-\alpha)a}{1+a-r} & \text{if } \frac{a}{a-r} < \alpha < 1-r \\ r & \text{if } \alpha \geq 1-r \end{cases} \quad (\text{B.8})$$

Case B and C:  $a - r + 1 \leq 0$ . In such environments, for low values of  $\alpha$  ( $\alpha \leq -a/(r-a)$ ) the unique environment equilibrium is  $x^* = 0$  and equilibrium efficiency is 0. For high values of  $\alpha$  ( $\alpha > 1 - r$ ), the unique equilibrium is at  $x^* = 1$ . For intermediate values of  $\alpha$  there are two environment equilibria at  $x^* = 0$  and  $x^* = 1$  in

Case B and a continuum of equilibria (all  $x^* \in [0, 1]$ ) in Case C. In both of these types of environments, maximum equilibrium efficiency in these cases is achieved for the environment equilibrium at  $x^* = 1$  and from the analysis here it is clear that in these social dilemmas maximum equilibrium efficiency is increasing in  $\alpha$ . More specifically:

$$\max W^*(\alpha) = \begin{cases} 0 & \text{if } \alpha < 1 - r \\ r & \text{if } \alpha \geq 1 - r \end{cases}$$

□

### B.1.3 Proof of Observation 1

We formally prove the statement of Observation 1 for cases where either  $z(0) < 0$  (VDs with  $r < 0.5$  and all MHDs) or  $z(0) + z(1) + z(a) + z(r) > 0$  (PDs with  $a - r + 1 < 0$ , and VDs and UDs with  $a > 1 - r$ ). For the rest of the cases we provide computational results to support the statement.

Clearly, in order for the pair  $(x^*, \alpha^*)$  to be a full equilibrium (*i.e.* an asymptotically stable state of the full dynamics), we need  $x^*$  to be an environment equilibrium for  $E = (G, \alpha^*)$ , and  $\alpha^*$  to be an evolutionarily stable state of the voting dynamics given  $x^*$ . We begin with the following observation.

**Observation 2.** *Consider a social dilemma  $G$ . If  $(x^*, \alpha^*)$  with  $\alpha^* \in (0, 1)$  is a full equilibrium, then  $x^* \in (0, 1)$ .*

**Proof.**

By way of contradiction, say  $(1, \alpha^*)$  with  $\alpha^* \in (0, 1)$  is a full equilibrium. Then the  $\alpha$  dynamic (3.8) for  $x = 1$  yields

$$\dot{\alpha} = \alpha(1 - \alpha)z(r).$$

So, for all points  $(1, \alpha)$  with  $\alpha \in (0, 1)$ ,  $\dot{\alpha}$  retains its sign. That is, if  $z(r) > 0$  the only possible full equilibrium with  $x^* = 1$  will be the one at  $(x^*, \alpha^*) = (1, 1)$  and if  $z(r) < 0$  the only possible full equilibrium with  $x^* = 1$  will be the one at  $(x^*, \alpha^*) = (1, 0)$ . Finally, if  $z(r) = 0$ , there exists no full equilibrium with  $x^* = 1$  as  $\dot{\alpha} = 0$  for all  $(x, \alpha)$  with  $x = 1$ . This is a contradiction.

Similarly for  $x^* = 0$ . Say  $(0, \alpha^*)$  is a full equilibrium. Then the  $\alpha$  dynamic for  $x = 0$

yields

$$\dot{\alpha} = \alpha(1 - \alpha)z(0).$$

As  $z(0) < 0$  for all social dilemmas,  $\dot{\alpha} < 0$  for all  $\alpha \in (0, 1)$  and thus, the only possible full equilibrium with  $x^* = 0$  is  $(x^*, \alpha^*) = (0, 0)$ . This is a contradiction.  $\square$

In light of Observation 2, any potential full equilibrium  $(x^*, \alpha^*)$  that contradicts the statement of Observation 1 needs to have  $x^* \in (0, 1)$  and  $\alpha^* \in (0, 1)$ .

For the rest of the proof, we define the quantity  $Z = z(1) + z(a) + z(r) + z(0)$  which turns out to be crucial for the behavior of the dynamic system.

**Case 1:**  $Z > 0$ . For  $\alpha \in (0, 1)$ , the  $\alpha$  dynamics equation (3.8) gives:

$$\dot{\alpha} > 0 \Rightarrow \alpha x(1 - x)Z > x(1 - x)(z(a) + z(1)) + x^2 z(r) + (1 - x)^2 z(0)$$

So, for social dilemmas with  $Z > 0$  any interior steady state  $(x^*, \alpha^*)$  (with  $\dot{\alpha} = \dot{x} = 0$ ) will not be stable as a small deviation to  $(x^*, \alpha^* + \varepsilon)$  will drive the system away from  $(x^*, \alpha^*)$ .

**Case 2:**  $z(r) < 0$ . So, we now focus on social dilemmas with  $Z < 0$ . In this case, the sign of  $z(r)$  is important.

We proceed with another observation.

**Observation 3.** For any social dilemma  $G = (r, a)$  with  $Z < 0$  and  $z(r) < 0$ , and for all  $\alpha \in (0, 1)$  and  $x \in (0, 1)$ ,  $\dot{\alpha} < 0$ .

**Proof.** Define  $V = v^+ - v^-$ . For  $(x, \alpha) \in (0, 1)^2$ , we have the following:

$$\dot{\alpha} > 0 \Leftrightarrow V > 0$$

$$\frac{\partial V}{\partial \alpha} = x(1 - x)Z < 0$$

So,  $V$  would obtain its maximum value for  $\alpha = 0$ . This value is

$$V_0 = (x^2 z(r) + (1 - x)^2 z(0)) + (-x(1 - x)(z(a) + z(1)))$$

As  $z$  is increasing, the maximum value the first term of the right-hand side of the above equation is  $z(r)$ , which is negative in the social dilemmas under consideration. Also, notice that as  $z(1) = 1$  for all social dilemmas and  $z(a) > -1$ , the second term in the equation will also be negative.  $\square$

So, any full equilibrium would need to have  $\alpha^* = 0$ .

**Case 3:  $Z < 0$  and  $z(r) > 0$ .** For the rest of the cases, we note that there is always at most one interior rest point of the full dynamics.

We numerically calculate the Jacobian matrix of the dynamical system and take the real parts of its two eigenvalues at the interior rest point. Figure B.1 provides the plot of the maximum real part of the aforementioned eigenvalues. Since there is always at least one eigenvalue with a positive real part, we can conclude that the interior rest point cannot be stable and thus cannot be an environment equilibrium.

Furthermore, as the interior rest point (if it exists) is unique and a saddle point, there can be no closed trajectories in the interior of the state space. From this, we conclude that there must exist stable points (sinks) at the boundary of the state space which are the full equilibria.

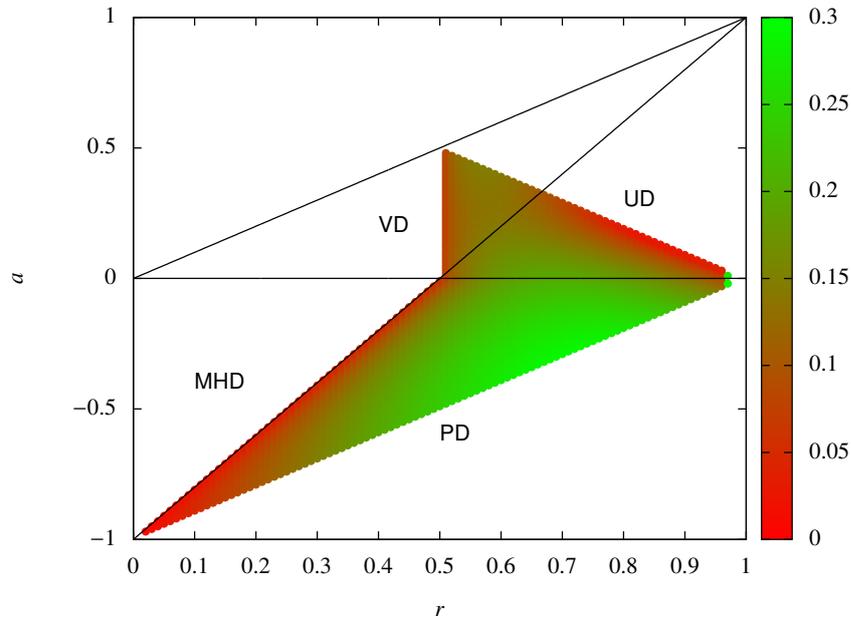


Figure B.1: Maximum eigenvalue of the Jacobian matrix calculated at the interior rest point for social dilemmas with  $z(r) > 0$

# Appendix C

## Appendix to Chapter 4

### C.1 Examples of smooth best responses

**Example 1.** Let  $p(\theta) = \exp(-\pi\theta^2)$  and  $\bar{a}(\theta) = \theta$ . Then  $a^*(\theta) = \theta$  and  $\phi(x) = x$ . Thus  $\phi'(x) = 1$ . So,

$$\begin{aligned} R_i(x) &= \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[\exp(-\pi x^2)](\xi)](x) = \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \exp(-\pi\xi^2)](x) = \\ &= \mathcal{F}_x^{-1}[\exp(-\pi(1-\mu\pi)\xi^2)](x) = \frac{1}{\sqrt{1-\pi\mu}} \exp\left(-\frac{\pi x^2}{1-\pi\mu}\right) \end{aligned}$$

For the above to be integrable, it has to be that  $\mu < 1/\pi$ . Notice that  $\int_{-\infty}^{+\infty} R_i(x) dx = 1$  which is a condition that is required. So

$$r(a_i|\theta) = \frac{1}{\sqrt{1-\pi\mu}} \exp\left(-\frac{\pi a_i^2}{1-\pi\mu}\right) \exp(\pi\theta^2) \frac{1}{\sqrt{\pi\mu}} \exp\left(-\frac{(a_i-\theta)^2}{\mu}\right)$$

The expected action of player  $i$  conditional on  $\theta$  can be calculated to be:

$$\mathbb{E}a_i(\theta) = \int_{-\infty}^{+\infty} a_i r(a_i|\theta) da_i = (1-\pi\mu)\theta.$$

**Example 2.** Let  $p(\theta) = (\sigma\sqrt{2\pi})^{-1} \cdot \exp(-(\theta - \theta_0)^2/2\sigma^2)$  and  $\bar{a}(\theta) = a_0 + \tilde{\kappa}\theta$ . Then,  $\phi(x) = (x - \gamma a_0)/(1 + \gamma(\tilde{\kappa} - 1))$  and  $\phi'(x) = (1 + \gamma(\tilde{\kappa} - 1))^{-1}$ . The marginal  $R_i(\cdot)$  can be calculated:

$$R_i(a_i) = \frac{1}{\sqrt{2\pi(1 + \gamma(\tilde{\kappa} - 1))^2\sigma^2 - \pi\mu}} \exp\left(-\frac{(a_i - (\gamma a_0 + (1-\gamma)\theta_0 + \gamma\tilde{\kappa}\theta_0))^2}{2(1 + \gamma(\tilde{\kappa} - 1))^2\sigma^2 - \mu}\right)$$

It is required that  $\mu < 2(\sigma(1 + \gamma(\tilde{\kappa} - 1)))^2$ . Otherwise, there is no smooth strategy that

solves the decision making problem of player  $i$ . The full solution is therefore

$$r(a_i|\theta) = R_i(a_i) \frac{1 + \gamma(\bar{\kappa} - 1)}{\sqrt{\pi\mu}} \sqrt{2\pi}\sigma \exp\left(\frac{(\theta - \theta_0)^2}{2\sigma^2}\right) \exp\left(-\frac{(a_i - (a_0 + \bar{\kappa}\theta))^2}{\mu}\right).$$

## C.2 Omitted Proofs

### C.2.1 Proof of Proposition 4.3

Consider variations of the information acquisition strategy of player  $i$ . These variations will be of the type  $\tilde{r} = r + \varepsilon\eta$  for some  $\varepsilon > 0$ . These variations should still be feasible. That is, for all  $\theta$ , it is required that  $r(\theta) + \varepsilon\eta(\theta)$  is a probability distribution over  $A_i$ . It is required, thus, that for all  $\theta$ ,  $\int_{A_i} r(a_i|\theta) + \varepsilon\eta(a_i|\theta) da_i = 1$  which leads to the condition that for all  $\theta$ ,  $\int_{A_i} \eta(a_i|\theta) da_i = 0$ . It also has to be that  $r(a_i|\theta) + \varepsilon\eta(a_i|\theta) \geq 0$  and so  $\eta(a_i|\theta) \geq -r(a_i|\theta)/\varepsilon$  for all  $a_i$  and  $\theta$ . From the above equations, the following is calculated:<sup>1</sup>

$$\begin{aligned} U(r_i + \varepsilon\eta, \mathbf{r}_{-i}) &= \bar{u} - (1 - \gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2 (r_i(a_i|\theta) + \varepsilon\eta(a_i|\theta)) p(\theta) da_i d\theta - \\ &\quad - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2 (r_i(a_i|\theta) + \varepsilon\eta(a_i|\theta)) p(\theta) da_i d\theta. \end{aligned} \quad (\text{C.1})$$

And the derivatives:

$$\begin{aligned} \left. \frac{dU(r + \varepsilon\eta, \mathbf{r}_{-i})}{d\varepsilon} \right|_{\varepsilon=0} &= -(1 - \gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2 \eta(a_i|\theta) p(\theta) da_i d\theta - \\ &\quad - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2 \eta(a_i|\theta) p(\theta) da_i d\theta \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} \left. \frac{dI(r + \varepsilon\eta)}{d\varepsilon} \right|_{\varepsilon=0} &= \int_{\Theta} \int_{A_i} \log(r(a_i|\theta)) \eta(a_i|\theta) p(\theta) da_i d\theta - \\ &\quad - \int_{A_i} \log(R_i(a_i)) H(a_i) da_i \end{aligned} \quad (\text{C.3})$$

<sup>1</sup>The effect of the other players' strategies is incorporated in  $\bar{a}(\theta)$ .

with  $H(a_i) = \int_{\Theta} \eta(a_i|\theta)p(\theta)d\theta$ .

Since the perturbations considered have to be feasible, player  $i$  has to solve the following constrained optimization problem:

$$\begin{aligned} & \max_{r_i \in L^1(\Theta, p)} U(r_i, \mathbf{r}_{-i}) - \mu I(r_i) \\ \text{s.t. } & \int_{A_i} r_i(a_i|\theta) da_i = 1 \quad \text{for all } \theta \in \Theta. \end{aligned}$$

So, the Lagrangian for player  $i$ 's decision problem will be

$$\mathcal{L}(r_i, k(\theta)) = V(r_i, \mathbf{r}_{-i}) - \int_{\Theta} k(\theta) \left( \int_{A_i} r(a_i|\theta) da_i - 1 \right) p(\theta) d\theta$$

where  $k(\theta)$  is the Lagrange multiplier for the  $\theta$ -constraint.

Therefore, for any given  $\theta \in \Theta$  and all possible perturbations  $\eta$ , an optimal strategy  $r$  should satisfy the following first order conditions:

$$\begin{aligned} & \left. \frac{d\mathcal{L}(r_i + \varepsilon\eta, k(\theta))}{d\varepsilon} \right|_{\varepsilon=0} = 0 \Rightarrow \\ & \int_{\Theta} \int_{A_i} \left[ u_i(a_i, \theta) - \mu \log \left( \frac{r(a_i|\theta)}{\log(R_i(a_i))} \right) - k(\theta) \right] \eta(a_i|\theta) p(\theta) da_i d\theta = 0 \end{aligned} \quad (\text{C.4})$$

$$\text{and } \int_{A_i} r_i(a_i|\theta) da_i = 1 \quad \text{for all } \theta \in \Theta. \quad (\text{C.5})$$

Where  $u_i(a_i, \theta) = -(1-\gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a}(\theta))^2$ .

Since condition (C.4) has to be satisfied *for all*  $\eta$ , it has to be the case that

$$-(1-\gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a}(\theta))^2 - \mu [\log(r_i(a_i|\theta)) - \log(R_i(a_i))] = k(\theta) \text{ for all } \theta \in \Theta.$$

So  $r(a_i|\theta)$  has to be:

$$r(a_i|\theta) = R_i(a_i) \exp\left(-\frac{k(\theta)}{\mu}\right) \exp\left(\frac{u_i(a_i, \theta)}{\mu}\right). \quad (\text{C.6})$$

So, (C.6) can be rewritten as

$$r(a_i|\theta) = R_i(a_i) K(\theta) \exp\left(\frac{u_i(a_i, \theta)}{\mu}\right). \quad (\text{C.7})$$

where  $K(\theta) = \exp\left(-\frac{k(\theta)}{\mu}\right)$ . All that remains to be done is to determine the functions  $K(\cdot)$  and  $R_i(\cdot)$ .

Now, from the definition of  $R_i(a_i)$ :

$$R_i(a_i) = \int_{\theta} r(a_i|\theta)p(\theta) d\theta \Rightarrow \int_{\Theta} \frac{r(a_i|\theta)}{R_i(a_i)} p(\theta) d\theta = 1.$$

And using the fact that player  $i$ 's opponents are using a monotone full-support strategy profile, one gets

$$\int_{-\infty}^{+\infty} K(\theta) \exp\left(-\frac{(a_i - a^*(\theta))^2}{\mu}\right) \exp\left(-\frac{\gamma(1-\gamma)}{\mu}(\theta - \bar{a}(\theta))^2\right) p(\theta) d\theta = 1. \quad (\text{C.8})$$

In the above,  $a^*(\theta) = (1-\gamma)\theta + \gamma\bar{a}(\theta)$ . By assumption (monotone full-support strategy profile),  $a^*$  is invertible with  $\phi$  being the inverse of  $a^*$ . With a change of the variable of integration from  $\theta$  to  $x = a^*(\theta)$ , taking into account assumption 3, and by defining  $G(\cdot)$  as

$$G(x) = \frac{K(\phi(x)) \exp\left(-\frac{\gamma(1-\gamma)}{\mu}(\phi(x) - \bar{a}(\phi(x)))^2\right) p(\phi(x))}{(1-\gamma) + \gamma\bar{a}'(\phi(x))} \quad (\text{C.9})$$

condition (C.8) can be rewritten as

$$\int_{-\infty}^{+\infty} G(x) \exp\left(-\frac{1}{\mu}(a_i - x)^2\right) dx = 1. \quad (\text{C.10})$$

Notice that the above condition has to hold for all  $a_i$ . This can only happen if  $G(x) = 1/\sqrt{\pi\mu}$ .

**Proof.** Notice that the left-hand side of equation (C.10) is the convolution of  $g$  and  $f : f(x) = \exp(-\mu^{-1}x^2)$ . Now, take the Fourier transform on both sides and use the convolution theorem:

$$\begin{aligned} \mathcal{F}_x[(g * f)(x)](\xi) &= \mathcal{F}_x[1](\xi) \Rightarrow \mathcal{F}_x[G(x)](\xi) \cdot \mathcal{F}_x[f(x)](\xi) = \delta(\xi) \Rightarrow \\ \mathcal{F}_x[G(x)](\xi) &= \frac{\delta(\xi)}{\sqrt{\pi\mu} \exp(-\mu\pi^2\xi^2)} = \frac{1}{\sqrt{\pi\mu}} \delta(\xi). \end{aligned}$$

Where  $\delta(\cdot)$  is Dirac's delta function. By taking the inverse Fourier transform on both

sides, the statement is proven:

$$G(x) = \mathcal{F}_\xi^{-1} \left[ \frac{1}{\sqrt{\pi\mu}} \delta(\xi) \right] (x) = \frac{1}{\sqrt{\pi\mu}}.$$

□

So now  $K(\theta)$  can be calculated.

$$K(\theta) = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \exp\left(\frac{\gamma(1-\gamma)}{\mu}(\theta - \bar{a}(\theta))^2\right) \quad (\text{C.11})$$

Using (C.11) in (C.7) yields

$$r(a_i|\theta) = R_i(a_i) \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - a^*(\theta))^2}{\mu}\right). \quad (\text{C.12})$$

The solution has to also satisfy  $\int_{-\infty}^{+\infty} r(a_i|\theta) da_i = 1$  for all  $\theta$ . Again, changing the variable from  $\theta$  to  $x = a^*(\theta)$ , this condition yields

$$\int_{-\infty}^{+\infty} R_i(a_i) \exp\left(-\frac{(x - a_i)^2}{\mu}\right) da_i = \sqrt{\pi\mu} p(\phi(x)) \phi'(x). \quad (\text{C.13})$$

Notice that the left-hand side of equation (C.13) is the convolution of  $R_i$  and  $f$ . Now, take the Fourier transform on both sides and use the convolution theorem

$$\begin{aligned} \mathcal{F}_{a_i}[R_i(a_i)](\xi) \cdot \mathcal{F}_x[f(x)](\xi) &= \sqrt{\pi\mu} \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi) \Rightarrow \\ \mathcal{F}_{a_i}[R_i(a_i)](\xi) &= \exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi) \Rightarrow \end{aligned} \quad (\text{C.14})$$

$$R_i(a_i) = \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)](a_i) \quad (\text{C.15})$$

Now notice that  $g(x) = p(\phi(x))\phi'(x)$  is the PDF of the best action  $x = \phi(\theta)$ . From equation (C.14) one obtains

$$\mathcal{F}_{a_i}[R_i(a_i)](\xi) = \exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[g(x)](\xi). \quad (\text{C.16})$$

It follows from the definition of the Fourier transform (4.14) that for any function  $f$ ,  $\mathcal{F}_x[f(x)](0) = \int_{-\infty}^{+\infty} f(x) dx$ . So, as  $g$  is a PDE,  $\mathcal{F}_x[g(x)](0) = 1$ . More than that, the

mean of a random variable  $X$  with a probability PDF  $p$  is given by

$$\mathbb{E}(X) = \frac{1}{-2\pi\iota} (\mathcal{F}_x[p(x)])''(0) \quad (\text{C.17})$$

and its variance given by

$$\text{Var}(X) = \sigma_X^2 = \left( \frac{1}{-2\pi\iota} \right)^2 (\mathcal{F}_x[p(x)])''(0) - (\mathbb{E}(X))^2. \quad (\text{C.18})$$

So, taking the first derivative on both sides of equation (C.16) at  $\xi = 0$  and multiplying by  $(-2\pi\iota)^{-1}$  results in

$$\mathbb{E}(a_i) = \mathbb{E}(a^*) \quad (\text{C.19})$$

and taking the second derivative on both sides of equation (C.16) at  $\xi = 0$ , multiplying by  $(-2\pi\iota)^{-2}$  and taking into account that  $\mathbb{E}(a_i) = \mathbb{E}(a^*)$  results in

$$\text{Var}(a_i) = -\frac{\mu}{2} + \sigma_{a^*}^2. \quad (\text{C.20})$$

Notice that as by assumption  $\sigma_{a^*} > \mu/2$ ,  $a_i$  has a well-defined (positive) variance and  $R$  given by equation (C.15) is the PDF of the marginal of  $a_i$ .

So, the expression for  $r$  given by equation (C.7) is well-defined. Thus, the best reply is given by the following formula:

$$r_i(a_i|\theta) = \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)](a_i) \frac{a^{*'}(\theta)}{p(\theta)\sqrt{\pi\mu}} \exp\left(-\frac{(a_i - a^*(\theta))^2}{\mu}\right) \quad (\text{C.21})$$

This solution is *unique*.

Now, for the “only if” part, if  $\sigma_{a^*} < \mu/2$ ,  $R$  would need to have a negative variance. As this is impossible, the expression on the right-hand side of equation (C.15) would not be describing the Fourier transform of a PDF and thus, its Fourier transform would not exist. This means that player  $i$  would not have a smooth best reply to  $\mathbf{r}_{-i}$ .  $\square$

## C.2.2 Proof of Proposition 4.6

Denote by  $r$  the strategy that all players use in an SMFE (see Proposition 4.5). Since all players use the same strategy (almost surely), the average action function in an

SMFE is given by  $\bar{a}(\theta) = \int_{-\infty}^{+\infty} a r(a|\theta) da$ .

As each player should be best responding to  $\bar{a}$ , each player's strategy should be (almost surely) equal to

$$r(a|\theta) = R(a) \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \exp\left(-\frac{(a - (1-\gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right)$$

with

$$R(a) = \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)](a).$$

Using the property of the Fourier transform (see equation (C.17)), one obtains for the average action function:

$$\bar{a}(\theta) = -\frac{1}{2\pi i} (\mathcal{F}_a[r(a|\theta)])'(0).$$

So the following calculations are obtained:

$$\begin{aligned} r(a|\theta) &= \mathcal{F}_\xi^{-1}[\exp(\mu\pi^2\xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)](a) \times \\ &\times \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \exp\left(-\frac{(a - (1-\gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) \end{aligned} \quad (\text{C.22})$$

Using the convolution theorem and the definitions of the operation of convolution and the Fourier transform:

$$\begin{aligned} &\mathcal{F}_a[r(a|\theta)](\xi) = \\ &= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} (\exp(\mu\pi^2\xi^2) \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)) * \left( \mathcal{F}_a \left[ \exp\left(-\frac{(a - (1-\gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) \right] (\xi) \right) = \\ &= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x))\phi'(x) dx \right) \times \\ &\times \left( \int_{-\infty}^{+\infty} \exp(-2\pi i a(\xi - t)) \exp\left(-\frac{(a - (1-\gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) da \right) dt \end{aligned}$$

And taking the first derivative with respect to  $\xi$ :

$$\begin{aligned} & \frac{d}{d\xi} \mathcal{F}_a[r(a|\theta)](\xi) = \\ & = -2\pi i a \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \left( (-2\pi i a) \exp(-2\pi i a(\xi - t)) \exp\left(-\frac{(a - (1 - \gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) da \right) dt \end{aligned}$$

And so

$$\begin{aligned} & \frac{1}{-2\pi i} \frac{d}{d\xi} \mathcal{F}_a[r(a|\theta)](\xi) \Big|_{\xi=0} = \\ & = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \left( \exp(2\pi i t a) a \exp\left(-\frac{(a - (1 - \gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) da \right) dt = \\ & = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \mathcal{F}_a^{-1} \left[ a \exp\left(-\frac{(a - (1 - \gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) \right] (t) dt = \\ & = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \frac{1}{2\pi i} \frac{d}{dt} \mathcal{F}_a^{-1} \left[ \exp\left(-\frac{(a - (1 - \gamma)\theta - \gamma\bar{a}(\theta))^2}{\mu}\right) \right] (t) dt = \\ & = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi\mu}} \int_{-\infty}^{+\infty} \exp(\mu\pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \frac{1}{2\pi i} \frac{d}{dt} \left( \exp(2\pi i t((1 - \gamma)\theta + \gamma\bar{a}(\theta))) \sqrt{\pi\mu} \exp(-\pi^2 \mu t^2) \right) dt = \\ & = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) \times \\ & \quad \times \frac{\exp(2\pi i t((1 - \gamma)\theta + \gamma\bar{a}(\theta)))}{2\pi i} (2\pi i((1 - \gamma)\theta + \gamma\bar{a}(\theta)) - 2\mu\pi^2 t) dt = \end{aligned}$$

$$\begin{aligned}
&= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t(x - (1 - \gamma)\theta - \gamma\bar{a}(\theta))) p(\phi(x)) \phi'(x) dx \right) \times \\
&\quad \times ((1 - \gamma)\theta + \gamma\bar{a}(\theta) + \mu\pi t i) dt = \\
&= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \left( ((1 - \gamma)\theta + \gamma\bar{a}(\theta)) \int_{-\infty}^{+\infty} \exp(2\pi i t((1 - \gamma)\theta - \gamma\bar{a}(\theta))) \times \right. \\
&\quad \times \left. \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) dt + \right. \\
&+ \mu\pi i \int_{-\infty}^{+\infty} t \exp(2\pi i t((1 - \gamma)\theta + \gamma\bar{a}(\theta))) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) dx \right) dt \Big) = \\
&= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \left( ((1 - \gamma)\theta + \gamma\bar{a}(\theta)) \mathcal{F}_t^{-1} [\mathcal{F}_x [p(\phi(x)) \phi'(x)](t)] ((1 - \gamma)\theta - \gamma\bar{a}(\theta)) + \right. \\
&\quad + \mu\pi i \mathcal{F}_t^{-1} [t \mathcal{F}_x [p(\phi(x)) \phi'(x)](t)] ((1 - \gamma)\theta + \gamma\bar{a}(\theta)) \Big) = \\
&= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \left( ((1 - \gamma)\theta + \gamma\bar{a}(\theta)) \frac{p(\theta)}{1 - \gamma + \gamma\bar{a}'(\theta)} + \right. \\
&\quad \left. + \frac{\mu}{2} \mathcal{F}_t^{-1} [2\pi i t \mathcal{F}_x [p(\phi(x)) \phi'(x)](t)] ((1 - \gamma)\theta + \gamma\bar{a}(\theta)) \right) = \\
&= (1 - \gamma)\theta + \gamma\bar{a}(\theta) - \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \frac{\mu}{2} \frac{d}{dx} (p(\phi(x)) \phi'(x)) \Big|_{x=(1-\gamma)\theta + \gamma\bar{a}(\theta)} = \\
&= (1 - \gamma)\theta + \gamma\bar{a}(\theta) - \frac{\mu}{2} \frac{p'(\theta)}{p(\theta)(1 + \gamma(\bar{a}'(\theta) - 1))} + \frac{\mu}{2} \frac{\gamma\bar{a}''(\theta)}{(1 + \gamma(\bar{a}'(\theta) - 1))^2}
\end{aligned}$$

For the last equality, use was made of the following:

$$x = (1 - \gamma)\phi(x) + \gamma\bar{a}(\phi(x)) \quad \phi'(x) = \frac{1}{1 + \gamma(\bar{a}'(\phi(x)) - 1)} \quad \phi''(x) = -\frac{\gamma\bar{a}''(\phi(x))}{(1 + \gamma(\bar{a}'(\phi(x)) - 1))^3}$$

Thus, as in equilibrium it has to be that  $\bar{a}(\theta) = \frac{1}{-2\pi i} \frac{d}{d\xi} \mathcal{F}_a[r(a|\theta)](\xi) \Big|_{\xi=0}$ , the following equilibrium condition is obtained:

$$\bar{a}(\theta) = \theta + \frac{\mu}{2(1 - \gamma)(1 + \gamma(\bar{a}'(\theta) - 1))} \left( \frac{p'(\theta)}{p(\theta)} - \frac{\gamma\bar{a}''(\theta)}{1 + \gamma(\bar{a}'(\theta) - 1)} \right).$$

And as  $a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta)$ , this is equivalent to

$$a^*(\theta) = \theta + \frac{\mu\gamma}{2(1 - \gamma)} \frac{1}{a^*(\theta)} \frac{d}{d\theta} \left( \log \left( \frac{p(\theta)}{a^*(\theta)} \right) \right).$$

and with a change of variable back to  $x$ , one obtains that

$$\phi(x) = x - \frac{\mu\gamma}{2(1-\gamma)} \frac{d}{dx} \log(p(\phi(x))\phi'(x)). \quad \square$$

### C.2.3 Proof of Proposition 4.8

From Lemma 4.7, if an equilibrium without information acquisition were to exist, then all players would follow the strategy  $r_i(a_i|\theta) = \delta(a_i - \bar{\theta})$  i.e. they would play  $\bar{\theta}$  with probability one. So, the best action would be  $a^*(\theta) = (1-\gamma)\theta + \gamma\bar{\theta}$  and its inverse,  $\phi(x) = (x - \gamma\bar{\theta})/(1-\gamma)$  with  $\phi'(x) = 1/(1-\gamma)$ . So, the best action  $x$  is distributed as  $p(\phi(x))\phi'(x)$  i.e.

$$g(x) = \frac{1}{1-\gamma} p\left(\frac{x - \gamma\bar{\theta}}{1-\gamma}\right).$$

If player  $i$  had a smooth best response to  $a^*(\theta) = (1-\gamma)\theta + \gamma\bar{\theta}$ , then the marginal of her action  $a_i$  would be given by the expression

$$R_i(a_i) = \mathcal{F}_\xi^{-1} \left[ \exp(\mu\pi^2 \xi^2) \cdot \mathcal{F}_x \left[ \frac{1}{1-\gamma} p\left(\frac{x - \gamma\bar{\theta}}{1-\gamma}\right) \right](\xi) \right](a_i).$$

From the definition of the Fourier transform, it is easy to confirm that

$$\mathcal{F}_x \left[ \frac{1}{1-\gamma} p\left(\frac{x - \gamma\bar{\theta}}{1-\gamma}\right) \right](\xi) = \exp(-2\pi i \gamma \bar{\theta} \xi) \cdot \mathcal{F}_x[p(x)]((1-\gamma)\xi)$$

and so

$$\mathcal{F}_{a_i}[R_i(a_i)](\xi) = \exp(\mu\pi^2 \xi^2 - 2\pi i \gamma \bar{\theta} \xi) \cdot \mathcal{F}_x[p(x)]((1-\gamma)\xi). \quad (\text{C.23})$$

As the mean of the best action is  $\bar{\theta}$ , the mean of  $a_i$  will be  $\bar{\theta}$  as well (see equation (C.19)). From the properties of the Fourier transform, one obtains (equation (C.18))

$$\begin{aligned} \text{Var}(a_i) &= \left(\frac{1}{-2\pi i}\right)^2 \mathcal{F}_{a_i}[R(a_i)]''(0) - \bar{\theta}^2 \\ \text{Var}(a^*) &= \left(\frac{1}{-2\pi i}\right)^2 \mathcal{F}_x[g(x)]''(0) - \bar{\theta}^2 \\ \text{Var}(\theta) &= \left(\frac{1}{-2\pi i}\right)^2 \mathcal{F}_\theta[p(\theta)]''(0) - \bar{\theta}^2. \end{aligned}$$

Along with equation (C.23) and after calculation, it is obtained that the variance of  $a_i$  in a best response to  $a^*(\theta)$  is

$$\text{Var}(a_i) = -\frac{\mu}{2} + (1-\gamma)^2 \text{Var}(\theta). \quad (\text{C.24})$$

In an equilibrium without information acquisition, it has to be that the best response of a player to everyone else playing  $\bar{\theta}$  with probability one is to do the same. That is, player  $i$  should not have an incentive to acquire information when others don't. *i.e.* it must be the case that the right-hand side of expression (C.24) calculates to a non-positive number (and so, cannot be the variance of a distribution):

$$-\frac{\mu}{2} + (1-\gamma)^2 \text{Var}(\theta) \leq 0$$

and the expression in the statement of the proposition is obtained

$$\mu \geq 2(1-\gamma)^2 \sigma_\theta^2.$$

For the “if” part of the proposition, notice that if  $\mu \geq 2(1-\gamma)^2 \sigma_\theta^2$  and all other players except for player  $i$  play  $\bar{\theta}$  with probability one, following the previous analysis, player  $i$ 's best response is to also play  $\bar{\theta}$  with probability one and thus an equilibrium without information acquisition is established.  $\square$

### C.2.4 Proof of Proposition 4.9

First of all, from the definition of  $a^*$  it is clear that

$$\int_{-\infty}^{+\infty} a^*(\theta) p(\theta) d\theta = \bar{\theta} \Leftrightarrow \int_{-\infty}^{+\infty} \bar{a}(\theta) p(\theta) d\theta = \bar{\theta}.$$

Begin by integrating condition (4.18).

$$\int_{-\infty}^{+\infty} a^*(\theta) p(\theta) d\theta = \int_{-\infty}^{+\infty} \theta p(\theta) d\theta + \frac{\mu\gamma}{2(1-\gamma)} \int_{-\infty}^{+\infty} \frac{1}{a^*(\theta)} \frac{d}{d\theta} \left( \log \left( \frac{p(\theta)}{a^*(\theta)} \right) \right) p(\theta) d\theta$$

The above expression is well-defined due to assumption 1 of definition 4.1: As for all players  $i$  their action  $a_i$  has a well-defined mean and variance, so does their action conditional on  $\theta$ ,  $a_i|\theta$ . Thus,  $\bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r(a_j|\theta) da_j dj$  will also have a well-

defined mean as an integral of a variable with a well-defined mean over a compact space (the interval  $[0, 1]$ ). Therefore, so will  $a^*$ . What remains to be shown is that

$$\int_{-\infty}^{+\infty} \frac{p(\theta)}{a^{*\prime}(\theta)} \frac{d}{d\theta} \left( \log \left( \frac{p(\theta)}{a^{*\prime}(\theta)} \right) \right) d\theta = 0.$$

One obtains that

$$\int_{-\infty}^{+\infty} \frac{p(\theta)}{a^{*\prime}(\theta)} \frac{d}{d\theta} \left( \log \left( \frac{p(\theta)}{a^{*\prime}(\theta)} \right) \right) d\theta = \int_{-\infty}^{+\infty} \frac{d}{d\theta} \left( \frac{p(\theta)}{a^{*\prime}(\theta)} \right) d\theta = \left[ \frac{p(\theta)}{a^{*\prime}(\theta)} \right]_{\theta=-\infty}^{\theta=+\infty}.$$

As  $p$  is the PDF of a probability distribution, it has to be that  $\lim_{\theta \rightarrow \pm\infty} p(\theta) = 0$ . Now focus on  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta)$ . There are three possible cases:

(i)  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = +\infty$ .

As  $\lim_{\theta \rightarrow +\infty} p(\theta) = 0$ , it has to be that  $\lim_{\theta \rightarrow +\infty} a^{*\prime}(\theta) = 0$ . But then, there exists a  $\theta'$  such that  $a^*(\theta) < \theta$  for all  $\theta > \theta'$ . So, from equation (4.18), it has to be that  $p(\theta)/a^{*\prime}(\theta)$  is decreasing for all  $\theta > \theta'$ .

This contradicts  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = +\infty$ .

(ii)  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = l > 0$ .

In this case, there exists a  $\theta''$  such that  $p(\theta)/a^{*\prime}(\theta) \geq l/2$  for all  $\theta \geq \theta''$ . Since  $a^{*\prime} > 0$ ,  $a^*$  is strictly increasing and thus  $\lim_{\theta \rightarrow +\infty} a^*(\theta)$  is well-defined (possibly infinite). So, for  $\theta \geq \theta''$  it has to be that  $a^{*\prime}(\theta) \leq (2/l)p(\theta)$  and integrating this gives  $\int_{\theta''}^{\theta} a^{*\prime}(x) dx \leq (2/l) \int_{\theta''}^{\theta} p(x) dx \leq 2/l$  and so  $a^*(\theta) - a^*(\theta'') \leq 2/l$ . So,  $\lim_{\theta \rightarrow \infty} a^*(\theta) \leq 2/l + a^*(\theta'') < +\infty$ .

This contradicts condition 3 of definition 4.1.

(iii)  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = 0$ .

Since the other two cases lead to contradictions, it has to be that this is the case.

A similar argument can be made for the case where  $\theta \rightarrow -\infty$ .

So,  $\lim_{\theta \rightarrow +\infty} p(\theta)/a^{*\prime}(\theta) = \lim_{\theta \rightarrow -\infty} p(\theta)/a^{*\prime}(\theta) = 0$  and

$$\int_{-\infty}^{+\infty} a^*(\theta) p(\theta) d\theta = \int_{-\infty}^{+\infty} \bar{a}(\theta) p(\theta) d\theta = \int_{-\infty}^{+\infty} \theta p(\theta) d\theta$$

□

### C.2.5 Proof of Proposition 4.11

From player  $i$ 's point of view, and given that she knows the function  $a^*(\theta)$ , there are two random variables:  $\theta$  and  $a_i$ . One can define more random variables, namely  $y = \mathbb{E}(a_i|\theta)$  which is the (equilibrium) average action given  $\theta$  and  $x = (1-\gamma)\theta + \gamma y$ , which is the best action given  $\theta$ .

Using the variance decomposition formula for  $a_i$ , one obtains

$$\text{Var}(a_i) = \mathbb{E}(\text{Var}(a_i|\theta)) + \text{Var}(\mathbb{E}(a_i|\theta)) = \mathbb{E}(\text{Var}(a_i|\theta)) + \text{Var}(y)$$

Using this and from equation (C.20) (in the proof of Proposition 4.3), one gets

$$\text{Var}(x) = \frac{\mu}{2} + \text{Var}(a_i) = \frac{\mu}{2} + \mathbb{E}(\text{Var}(a_i|\theta)) + \text{Var}(y). \quad (\text{C.25})$$

As  $y = x/\gamma + (1-\gamma)\theta/\gamma$ ,

$$\text{Var}(y) = \left(\frac{1}{\gamma}\right)^2 \text{Var}(x) + \left(\frac{1-\gamma}{\gamma}\right)^2 \text{Var}(\theta) - \frac{2(1-\gamma)}{\gamma^2} \text{Cov}(x, \theta). \quad (\text{C.26})$$

Substituting (C.26) into equation (C.25) and after calculations, one gets

$$\gamma(\text{Var}(\theta) - \text{Var}(x)) = \frac{\mu\gamma^2}{2(1-\gamma)} + \frac{\gamma^2}{1-\gamma} \mathbb{E}(\text{Var}(a_i|\theta)) + \text{Var}(\theta) + \text{Var}(x) - 2 \text{Cov}(x, \theta).$$

Now, notice that

$$\text{Var}(\theta) + \text{Var}(x) - 2 \text{Cov}(x, \theta) = \text{Var}(\theta - x)$$

and thus

$$\sigma_\theta^2 - \sigma_{a^*}^2 = \frac{\mu\gamma}{2(1-\gamma)} + \frac{\gamma}{1-\gamma} \mathbb{E}(\text{Var}(a_i|\theta)) + \frac{1}{\gamma} \text{Var}(\theta - a^*)$$

where  $\sigma_\theta^2 = \text{Var}(\theta)$  and  $\sigma_{a^*}^2 = \text{Var}(x)$ . □

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