The p- and hp- Finite Element Method Applied to a Class of Non-linear Elliptic Partial Differential Equations

A Thesis submitted to the University of Leicester

March 1997

By

David Kay

Department of Mathematics

To Mum and Dad

Abstract

The analysis of the p- and hp-versions of the finite element methods has been studied in much detail for the Hilbert spaces $W^{1,2}(\Omega)$. The following work extends the previous approximation theory to that of general Sobolev spaces $W^{1,q}(\Omega)$, $q \in [1, \infty]$. This extension is essential when considering the use of the p and hp methods to the non-linear α-Laplacian problem.

Firstly, approximation theoretic results are obtained for approximation using continuous piecewise polynomials of degree p on meshes of triangular and quadrilateral elements. Estimates for the rate of convergence in Sobolev spaces $W^{1,q}(\Omega)$ are given. This analysis shows that the traditional view of avoiding the use of high order polynomial finite element methods is incorrect, and that the rate of convergence of the $p-$version is always at least that of the $h-$version (measured in terms of number of degrees of freedom). It is also shown that, if the solution has certain types of singularity, the rate of convergence of the $p-$version is twice that of the $h-$version. Numerical results are given, confirming the results given by the approximation theory.

The p-version approximation theory is then used to obtain the hp approximation theory. The results obtained allow both non-uniform p refinements to be used, and the h refinements only have to be locally quasiuniform. It is then shown that even when the solution has singularities, exponential rates of convergence can be achieved when using the hp-version, which would not be possible for the h- and p-versions.

Acknowledgements

I would like to thank my supervisor Dr Mark Ainsworth for his advice, guidance and encouragement over the last three years. I would also like to take this oppurtunity to thank the Mathematics Department of the University of Leicester for all their help throughout my undergraduate and postgraduate studies. I would also like to thank the Engineering and Physical Science Research Council for supporting me through a research studentship.

Last, but by no means least I would like to thank my family and Sharon and her family, for supporting me in my personal and academic endeavours. Thankyou.

Contents

List of Figures

Chapter 1

Introduction

1.1 Basic Notation

The following basic notation will be used. Throughout, C will be used to denote positive constants that are independent of other quantities appearing in the same relation, and whose values need not be the same in any two places. The notation $a \approx b$ means that there exist positive constants C_1, C_2 such that $C_1 a \leq b \leq C_2 a$.

Let $\mathbb{R}^2$ be the usual Euclidean space with $\mathrm{x} = (x_1, x_2) \in \mathbb{R}^2$. Let Ω be a polygonal domain in $\mathbb{R}^2$ with vertices A_i, $i = 0, \dots M$, $A_0 = A_M$. The boundary $\Gamma = \sum_{i=1}^{M} \Gamma_i$ where Γ_i are open straight lines with end points A_{i-1}, A_i. The internal angle of Γ_i and Γ_{i+1} is denoted by ω_i, $i = 1 \dots M$, $0 < \omega \leq 2\pi$. Let $\mathcal{D}$ and $\mathcal{N}$ be two given sets of integers satisfying $\mathcal{D} \cap \mathcal{N} = \emptyset$ and $\mathcal{D} \cup \mathcal{N} = \{1, 2, \dots, M\}$. Let $\Gamma^D = \sum_{i \in \mathcal{D}} \Gamma_i$ and $\Gamma^N = \Gamma - \Gamma^D = \sum_{i \in \mathcal{N}} \Gamma_i$, Γ^D is called the Dirichlet boundary and Γ^N the Neumann boundary. See Figure 1.1.

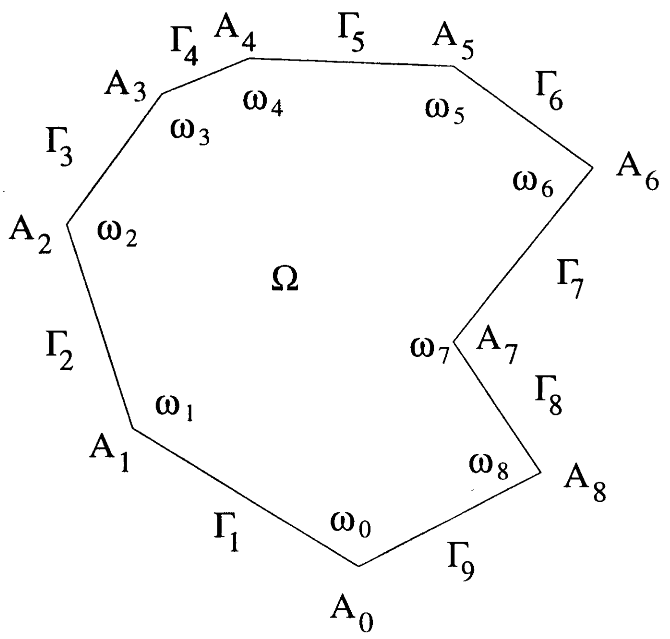

Figure 1.1: An example of a domain Ω

For $q \in [1, \infty]$ the space $L^q(\Omega)$ is defined to be the usual space of classes of functions for which the norm

$$\|f\|_{L^q(\Omega)} = \begin{cases} (\int_\Omega |f|^q \, \mathrm{dx})^{1/q}, & q < \infty \\ \mathrm{ess\ sup}_{\mathbf{x} \in \Omega} |f|, & q = \infty \end{cases} \tag{1.1}$$

is finite. For integer values of s, the Sobolev spaces $W^{s,q}(\Omega)$ are equipped with the norms

$$\|f\|_{W^{s,q}(\Omega)} = \begin{cases} \{\sum_{|\alpha| \le s} \|D^\alpha f\|_{L^q(\Omega)}\}^{1/q}, & q < \infty \\ \max_{|\alpha| \le s} \|D^\alpha f\|_{L^\infty(\Omega)}, & q = \infty \end{cases} \tag{1.2}$$

where, for any index $\alpha = (\alpha_1, \alpha_2)$, with $|\alpha| = \alpha_1 + \alpha_2$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}. \tag{1.3}$$

It may also be convenient sometimes, to use the notation

$$f^{(\alpha_1, \alpha_2)} = D^\alpha f. \tag{1.4}$$

For non-integer values of s, the Sobolev spaces $W^{s,q}(\Omega)$ are defined using the K-method of interpolation [19]. Thus, writing $s = m + \sigma$ where m is an integer and $\sigma \in (0, 1)$, the space $W^{s,q}(\Omega)$ is obtained by interpolating between the spaces $W^{m,q}(\Omega)$ and $W^{m+1,q}(\Omega)$. This process is indicated using the notation

$$W^{s,q}(\Omega) = [W^{m,q}(\Omega), W^{m+1,q}(\Omega)]_{\sigma,q}. \tag{1.5}$$

The subspaces $W_0^{s,q}(\Omega)$ are defined in the usual manner [1].

Equally well, Sobolev spaces can be defined on an interval $I = (a, b)$ and on curves γ.

Let $S(\rho)$, $\rho > 0$, be the square

$$S(\rho) := \{(x_1, x_2) : |x_1| < \rho, |x_2| < \rho\}, \tag{1.6}$$

and by $W_{PER}^{k,q}(S(\rho)) \subset W^{k,q}(S(\rho))$ we denote the space of all periodic functions with period 2ρ.

A partition $\mathcal{P}$ of the domain Ω consists of a finite number of open sub-domains (or elements), such that:

1. each element is either a triangle or a convex quadrilateral,
2. for any distinct pair of elements K and J, the intersection $\overline{K} \cap \overline{J}$ is either empty, a single common edge or a single common vertex,
3. diam $K = h_K$, $\forall K \in \mathcal{P}$,
4. $\rho_K = \sup\{\text{diam(s)} \ : s \text{ is a ball contained in } K\}$, is such that $\rho_K \geq Ch_K$,

5. for every $K \in \mathcal{P}$ the set

$$\Omega_K = \mathrm{int}\{\cup \overline{J} : J \in \mathcal{P} \text{ and } \overline{J} \cap \overline{K} \neq \emptyset\}, \tag{1.7}$$

is such that for each $J \subset \Omega_K$, $\mathrm{diam}(J) \approx h_K$.

Note that, the above properties of the refinement allow regions of the domain to be more refined than others. This type of refinement will be called "locally quasiuniform".

Associated with each type of element is a reference domain given in the case of quadrilateral elements by

$$S(1) = \{(x, y) : -1 \leq x \leq 1; -1 \leq y \leq 1\},$$

or, in the case of triangular elements, by

$$T(1) = \{(x, y) : -1 \leq x \leq 1; -1 \leq y \leq x\}.$$

Polynomial spaces of degree $p \in \mathbb{N}$ are defined on the quadrilateral and triangular reference domains by

$$\widehat{Q}(p) = \text{span } \{x^j y^k : 0 \leq j, k \leq p\}$$

and

$$\widehat{P}(p) = \text{span } \{x^j y^k : 0 \leq j + k \leq p\}$$

respectively.

There exists an invertible mapping $F_K : \widehat{K} \to K$ that is affine for triangular elements and bilinear for quadrilateral elements, where $\widehat{K}$ denotes the reference domain $S(1)$ when K is a quadrilateral element and $T(1)$ when K is a triangular element. A polynomial space P_K is taken to be either $\widehat{Q}(p_K)$ or $\widehat{P}(p_K)$ as appropriate, for each type of element. The space X_{hp} is constructed using the partition $\mathcal{P}$ in such a way that

$$X_{hp} = \{v \in C(\overline{\Omega}) : v|_K = \widehat{v} \circ F_K^{-1} \text{ for some } \widehat{v} \in P_K, \text{ for all } K \in \mathcal{P}\}, \tag{1.8}$$

these spaces will be referred to as being a space of piecewise continuous polynomials.

1.2 The Model Problem

The class of problems to be considered is given by:

Find u such that

$$-\nabla \cdot \{|\nabla u|^{\alpha-2}\nabla u\} = f \quad \text{in } \Omega, \tag{1.9}$$

$$u = g \quad \text{on } \Gamma^D, \tag{1.10}$$

$$(|\nabla u|^{\alpha-2}\nabla u) \cdot \underline{n} = h \quad \text{on } \Gamma^N, \tag{1.11}$$

where $\alpha \in (1, \infty)$, f, g and h are given data and $\underline{n}$ is the outward normal to the boundary Γ_N. This equation is known as the $\alpha-$Laplacian.

It is known that, even when the given data is smooth the solution u of (1.9) may be singular. For example, suppose the domain Ω has a corner of the form shown in Figure 1.2, where ω denotes the internal angle and (r, θ), $\theta \in [0, \omega]$ denote the polar coordinate system with origin at the vertex. Then it has been

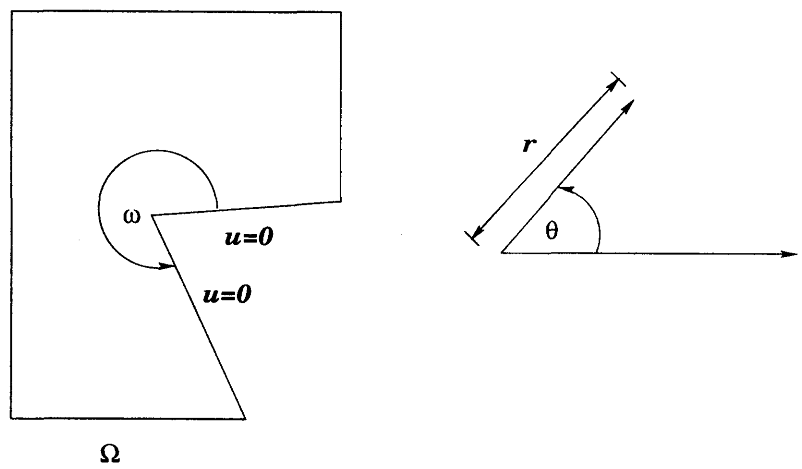

Figure 1.2: Domain With Corner Singularity

shown, see Dobrowolski [25] and also Tolksdorf [36], that the true solution in the neighbourhood of the corner has the form

$$u(\mathbf{x}) = cr^{\lambda}\Theta(\theta) + o(r^{\lambda}), \tag{1.12}$$

where $c \in \mathbb{R}$ is a smooth function with $\Theta(0) = \Theta(\omega) = 0$,

$$\lambda = \begin{cases} s + \sqrt{s^2 + 1/\beta}, & \text{if } 0 < \omega \leq \pi \\ s - \sqrt{s^2 + 1/\beta}, & \text{if } \pi \leq \omega < 2\pi \\ (\alpha - 1)/\alpha, & \text{if } \omega = 2\pi \end{cases} \tag{1.13}$$

with

$$\beta = (\omega/\pi - 1)^2 - 1 \tag{1.14}$$

and

$$s = \frac{(\beta - 1)\alpha - 2\beta}{2\beta(\alpha - 1)}. \tag{1.15}$$

To obtain an approximation to the solution of (1.9) using the finite element method, it must firstly be stated as a variational problem: Find $u \in W^{1,\alpha}(\Omega)$

such that $u = g$ on Γ_D and

$$\int_\Omega |\nabla u|^{\alpha-2}\nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega f v \, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} h v \, \mathrm{ds}, \tag{1.16}$$

for all $v \in W_D^{1,\alpha}(\Omega)$, where $W_D^{1,\alpha}(\Omega) = \{v : v \in W^{1,\alpha}(\Omega) : v = 0 \text{ on } \Gamma_D\}$. For this problem to have any meaning it is required that $f \in W^{-1,\alpha^*}(\Omega)$ and $h \in W^{-1+1/\alpha,\alpha^*}(\Gamma_N)$ where $\alpha* = \alpha/(\alpha-1)$, see [1]. Note that no assumption has been made on g; this is because the natural assumption $g \in W^{1-1/\alpha,\alpha}(\Omega)$ will be seen to be insufficient when applying the p-version finite element method, see chapter 2.

Note that the space of admissible functions for the solution of the variational problem (1.16) is larger than the natural choice of space for the original problem (1.9), which would typically be $C^2(\Omega) \cap C_{\Gamma_D}(\overline{\Omega})$, where $C_{\Gamma_D}(\overline{\Omega}) = \{v : v \in C(\Omega), v = g \text{ on } \Gamma_D\}$. This enlarged space is essential to the application of the finite element method. If u is a solution of (1.9) then u is a solution of (1.16) and conversely if the solution u of (1.16) is sufficiently smooth then it is also a solution of (1.9). The solution of the original boundary value problem is known as the classical solution.

From here on, it will be assumed that the solution of (1.9) can be written in the form:

$$u \;=\; u_1 + u_2 + \sum_{i=1}^{M} u_3^i \tag{1.17}$$

where

$$u_1 \in W_D^{m,\alpha}(\Omega) := W^{m,\alpha}(\Omega) \cap W_D^{1,\alpha}(\Omega), \; m > 1, \tag{1.18}$$

$$u_2 \in W^{k,\alpha}(\Omega),\ u_2 = g \text{ on } \Gamma_D,\ k > 1 + 1/\alpha, \tag{1.19}$$

and

$$u_3^i = c_i \zeta_i(r_i) r_i^{\lambda_i} g_i(|\log r_i|)\Theta_i(\theta_i) \in W_D^{1,\alpha}(\Omega) \tag{1.20}$$

where each g_i is a smooth C^∞ function and each ζ_i are smooth C^∞ functions such that, for some ρ_i, $\zeta_i = 1$ for $0 < r_i < \rho_i$, $\zeta_i = 0$ for $r_i > 2\rho_i$, note that ρ_i may depend on the partition. The polar coordinates (r_i, θ_i) have origins at the vertices A_i of the polygonal domain Ω. It can be seen that the functions u_1 and u_2 relate to the homogeneous and non homogeneous boundary conditions respectively and the u_3^i relate to the singularities that arise from the corners of the domain Ω.

1.3 The Finite Element Method

In this section X will be used to denote the space X_{hp}. For such a space X there exists a finite basis $\{\phi_i\}_{i=1}^N$ and the finite element approximation u_{FE}, for a given space X, has the form $u_{FE} = \sum_{i=1}^N \alpha_i \phi_i$ with $u_{FE} = g_{FE}$ on Γ_D, where g_{FE} is an approximation to the boundary data g which will be looked at in more detail in the following chapter.

For a given space X the discrete form for the variational problem (1.16) is given by: Find $u_{FE} \in X$ such that $u_{FE} = g_{FE}$ on Γ_D and

$$\int_\Omega |\nabla u_{FE}|^{\alpha-2} \nabla u_{FE} \cdot \nabla v \, \mathrm{dx} = \int_\Omega f v \, \mathrm{dx} + \int_{\Gamma_N} h v \, \mathrm{ds}, \tag{1.21}$$

for all $v \in X_D$, where $X_D = \{v : v \in X, v = 0 \text{ on } \Gamma_D\}$. At this stage it will be assumed that the solutions of (1.16) and (1.21) exist and are unique. This assumption will be shown to be true in chapter 5.

Since every function in X is a linear combination of the basis functions the above problem may be written as : find the α_i, $i = 1, 2, \ldots, N$ such that

$$\int_\Omega |\sum_{i=1}^{N}(\alpha_i \nabla \phi_i)|^{\alpha-2} \sum_{i=1}^{N}(\alpha_i \nabla \phi_i) \cdot \nabla \phi_j \, \mathbf{dx} = \int_\Omega f\phi_j \, \mathbf{dx} + \int_{\Gamma_N} h\phi_j \, \mathrm{ds}, \qquad (1.22)$$

for all $j = 1, 2, \ldots, N$.

Using the techniques shown in chapter 5, this problem is reduced to a linear system of equations

$$A\alpha = \mathbf{b}, \qquad (1.23)$$

where A is an $N \times N$ matrix and α and $\mathbf{b}$ are N dimensional vectors. The solution of this system gives the finite element approximation to the true solution of (1.16).

The general method of creating a discrete problem is known as the Galerkin method. The finite element method is a special case of the Galerkin method where the basis functions are defined over a given partition $\mathcal{P}$. To make the final matrix A as sparse as possible the basis functions are chosen so that their support is small, typically over a patch of elements sharing a common vertex or even on a single element, see Figure 1.3.

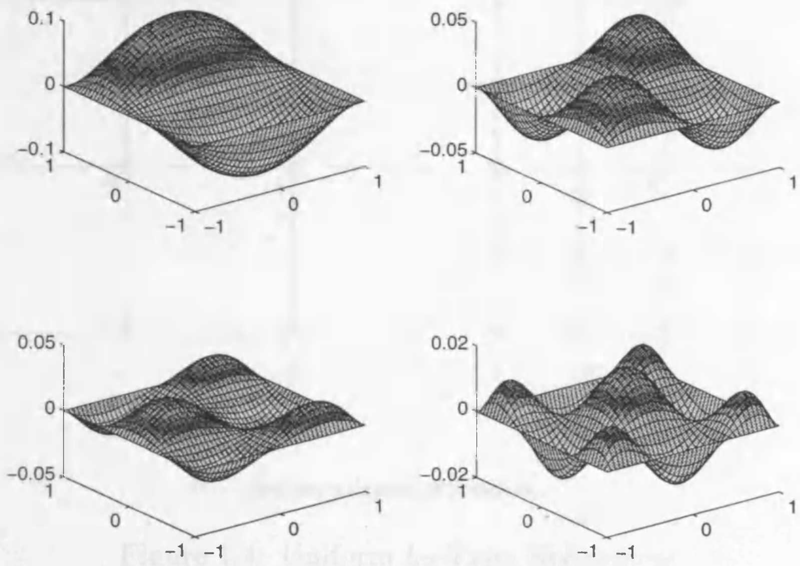

Figure 1.3: Examples of Interior Basis Functions

1.4 Refinements

The finite element approximation u_{FE} that was obtained using partition $\mathcal{P}$ and polynomial degree p_K on each $K \in \mathcal{P}$ may be improved by refining the partition in different ways. In this section the three main types of refinement will be looked at.

1.4.1 $h-$type refinement

This is the standard type of refinement used in the finite element method. The basic idea, when requiring a more accurate approximation, is to reduce the size of each element K and have the same fixed polynomial degree, which is usually very low, one, two or three at most, on each element.

In a uniform $h-$version the elements are such that $h_K \approx h_J$ for all $K, J \in \mathcal{P}$.

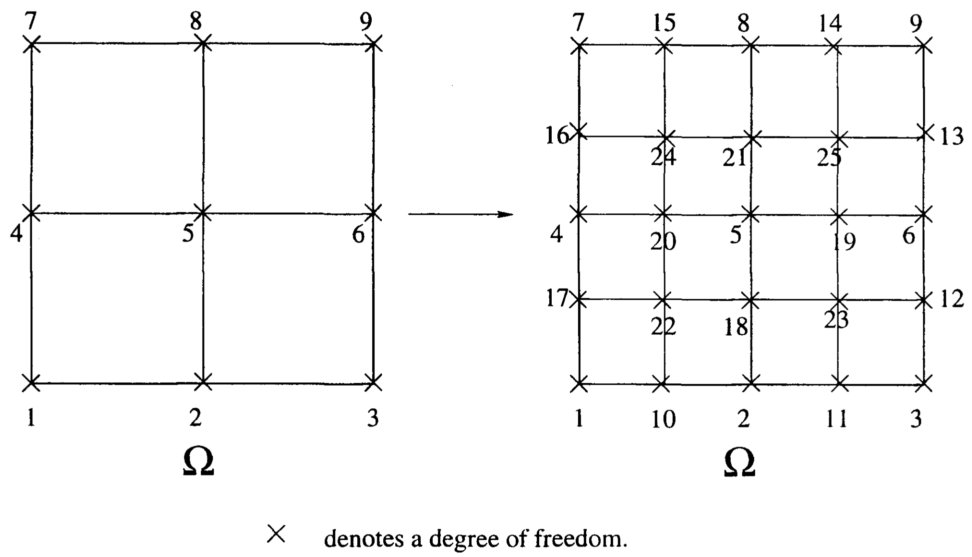

Figure 1.4: Uniform $h-$Type Refinement

The maximum size of an element $K \in \mathcal{P}$ is denoted by h and therefore this method is known as the $h-$version finite element method. For an example of a uniform refinement see Figure 1.4. The finite dimensional space for the $h-$version will be denoted by X_h and the finite element solution by u_h.

If the true solution of the problem is not so smooth in certain areas of the domain then uniform refinements may not be the most efficient type of refinement, instead it may be better to refine more intensely around the areas where the solution is not so smooth, giving a non-uniform mesh.

1.4.2 $p-$type refinement

In this method, the refinement is to increase the polynomial degree of the local basis functions and leave the partition as it is. In a uniform method, i.e. p_K is the same for each element; we denote by p the polynomial degree used and

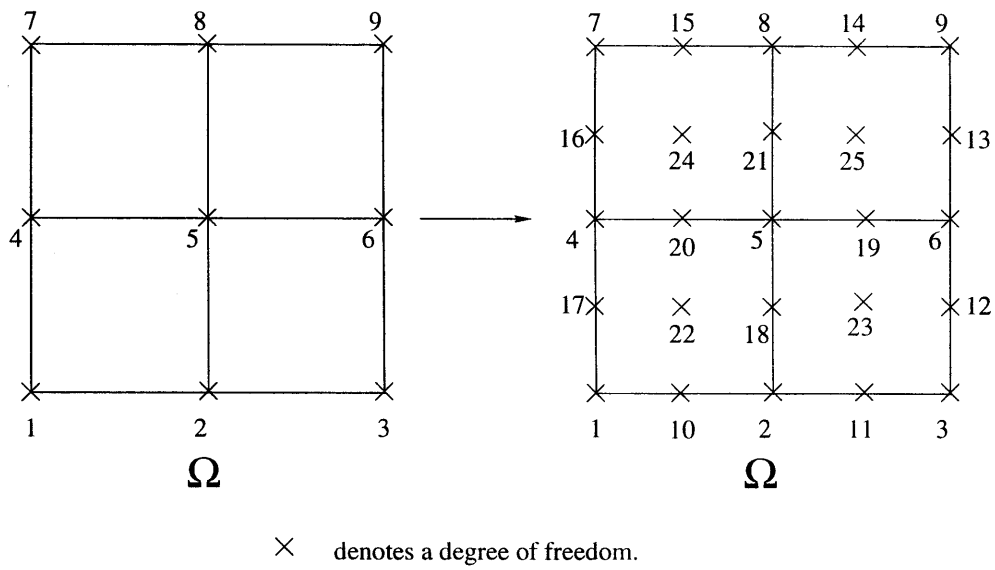

Figure 1.5: Uniform $p-$Type Refinement

thus we call this the $p-$version finite element method. For an example of a uniform $p-$version refinement see Figure 1.5. The finite dimensional space for the $p-$version will be denoted by X_p and the finite element solution by u_p.

1.4.3 $hp-$type refinements

The hp finite element method combines the $p-$version and the $h-$version so that the best properties of both methods may be implemented, hopefully giving exponential rates of convergence. The finite dimensional space for the $hp-$version will be denoted by X_{hp} and the finite element solution by u_{hp}.

When using the $hp-$version constrained nodes occur when either adjacent elements have different polynomial degrees of approximation, or a vertex of an element is located on the edge and not at the vertex of a neighbouring element, or both. In Figure 1.6, it can be seen that two linear elements share a common

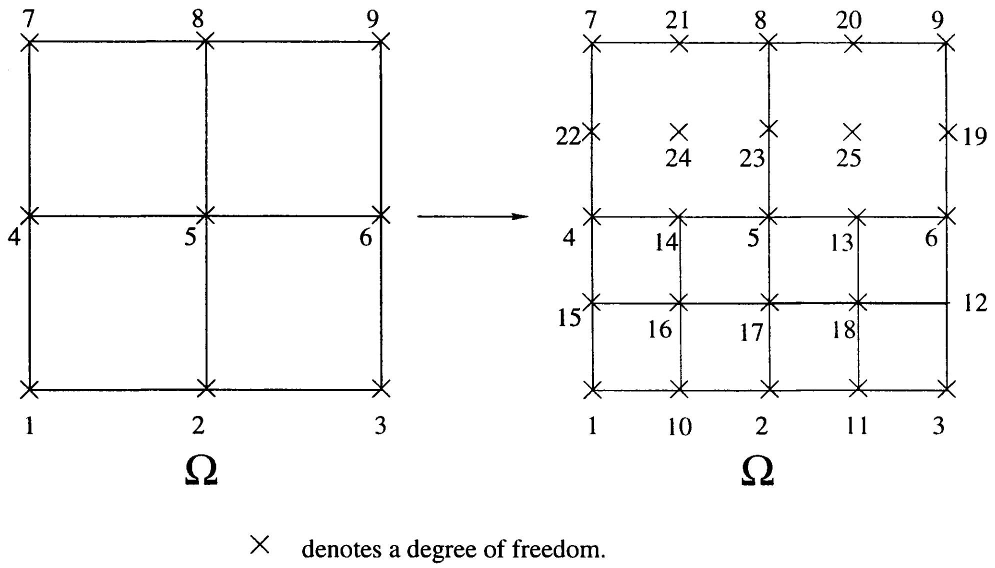

Figure 1.6: An $hp-$Type Refinement

boundary with one quadratic element. These constrained nodes are not trivially dealt with, since a piecewise continuous polynomial is required over the domain Ω. For the constrained nodes given by Figure 1.6 continuity may only be obtained if the approximation across the interelement boundaries is linear. Therefore, the choice of basis functions to be used is very important and must include functions which are supported on the interior of the elements, as in Figure 1.3.

1.5 A priori error estimates for linear elliptic equations

In this section some results for the linear case of (1.9) i.e. $\alpha = 2$, will be given. In this case the problem is given by: find $u \in W^{1,2}(\Omega)$ such that $u = g$ on Γ_D

and

$$B(u,v) \quad = \quad F(v), \text{ for all } v \in W_D^{1,2}(\Omega), \tag{1.24}$$

where

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} \qquad F(v) = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} h v \mathrm{ds}. \tag{1.25}$$

The following properties hold:

1. $B(\cdot,\cdot)$ is bilinear and symmetric,

2. $B(\cdot,\cdot)$ and $F(\cdot)$ are continuous, i.e. there exist constants M, $m > 0$ such that

$$|B(v,w)| \leq M \, \|v\|_{W^{1,2}(\Omega)} \, \|w\|_{W^{1,2}(\Omega)} \text{ for all } v,\, w \in W^{1,2}(\Omega), \tag{1.26}$$

   and

$$|F(v)| \leq m \, \|v\|_{W^{1,2}(\Omega)} \text{ for all } v \in W^{1,2}(\Omega). \tag{1.27}$$

3. $B(\cdot,\cdot)$ is elliptic i.e. there exists a constant $\gamma > 0$ such that

$$|B(v,v)| \geq \gamma \, \|v\|^2_{W^{1,2}(\Omega)} \text{ for all } v \in W^{1,2}(\Omega). \tag{1.28}$$

This is why the problem (1.24) is called linear elliptic. In chapter 5, the case $\alpha \neq 2$ will be considered and it will be shown that the general non-linear problem satisfies a similar elliptic property to that of the linear case.

It is well known (Lax-Milgram lemma, [24]) that under these circumstances, although symmetry is not a necessity, (1.24) has a unique solution $u \in W^{1,2}(\Omega)$, and the respective finite element problem (1.21) with $\alpha = 2$ has a unique solution, u_{FE}.

It is also known (Cea's Lemma, [24]) that there exists a constant C independent of the subspace X such that,

$$\|u - u_{FE}\|_{W^{1,2}(\Omega)} \leq C \inf_{v \in X} \|u - v\|_{W^{1,2}(\Omega)} \tag{1.29}$$

Using the above abstract error estimate, results for the different types of finite element method can now be obtained.

1.5.1 A uniform $h-$version estimate

From [24] there exists a piecewise polynomial approximation $\pi_p v \in X_h$, of degree no more than p in each element $K \in \mathcal{P}$ such that for all $v \in W^{k,q}(\Omega)$, $k > 1$, $q \in [1, \infty]$

$$\|v - \pi_p v\|_{W^{1,q}(\Omega)} \leq C(p) h^{\mu} \|v\|_{W^{k,q}(\Omega)}, \tag{1.30}$$

where $\mu = \min(p, k-1)$.

When the true solution of (1.24) is such that $u \in W^{k,2}(\Omega)$ i.e. $u_3 = 0$, choosing $q = 2$ in (1.30) and combining with (1.29) gives,

$$\|u - u_h\|_{W^{1,2}(\Omega)} \leq C(p) h^{\mu} \|u\|_{W^{k,2}(\Omega)}, \tag{1.31}$$

where $\mu = \min(p, k-1)$.

1.5.2 A uniform p−version estimate

The h−version estimate suggests that there is no need to use high degree approximation when k is small or $p > k-1$, since the rate of convergence is restricted by the smoothness and increasing p will not improve the estimate, however this is not the case. It was shown, firstly by Babuska, Szabo and Katz in [14], that when a uniform p−version is applied and $u \in W^{k,2}(\Omega)$ then u_p is such that, for any $\epsilon > 0$

$$\|u - u_p\|_{W^{1,2}(\Omega)} \leq C(h,\epsilon)p^{-(\mu-\epsilon)} \|u\|_{W^{k,2}(\Omega)}, \tag{1.32}$$

where $\mu = k-1$. Later, Babuska and Suri in [12] removed the ϵ to give

$$\|u - u_p\|_{W^{1,2}(\Omega)} \leq C(h)p^{-\mu} \|u\|_{W^{k,2}(\Omega)}, \tag{1.33}$$

This improvement was quite significant, since the analysis for the first result suggested that the term $C(h,\epsilon) \to \infty$ as $\epsilon \to 0$.

To compare these two methods the number of degrees of freedom N will be used, since this is closely related to the work at each stage. When uniform h− and p−type refinements are used, the following relationship holds for N,

$$p^2 \propto N \quad \text{and} \quad h^{-2} \propto N. \tag{1.34}$$

Using this relationship and the above two estimates, it is clear that the p−version will always converge as fast as the h−version. It can also be seen that, when the true solution is quite smooth then the p−version will exploit this smoothness to the full, while the h−version is restricted by the low polynomial degree it is using.

Suppose now that the true solution of (1.24) is given by (1.17) and $u_3^i \neq 0$ for a given i; then, see [12, Theorem 5.1], when the corresponding corner A_i is at a vertex of an element in the partition, which is not an unreasonable assumption,

$$\|u - u_p\|_{W^{1,2}(\Omega)} \leq Cp^{-2\lambda_i}. \tag{1.35}$$

Now, for any $\epsilon > 0$, $u \in W^{\lambda_i+1-\epsilon,2}(\Omega)$ when such a singularity exists. Therefore, the $p-$version convergence rate is twice the rate of the $h-$version for such singularities.

1.5.3 The $hp-$version

It is not a simple matter to deduce estimates for a general $hp-$version. For the one dimensional case, a thorough investigation was done by Babuska and Gui, see [8, 9, 10]. In the two dimensional case the basic a priori estimates for quasiuniform meshes were given by Babuska and Suri, see [11]. In chapter 4, error estimates for the $hp-$version will be looked at in more detail, and results will be obtained for locally quasiuniform meshes in general Sobolev spaces.

Chapter 2

The p-version Approximation Theory for Smooth Functions

2.1 Introduction

In this chapter approximation theory for functions in Sobolev spaces $W^{k,q}(\Omega)$, $k > 1$, $q \in [1, \infty]$, by functions in the spaces X_p on a fixed partition $\mathcal{P}$ of the domain Ω, will be looked at. The results obtained will extend the results shown in chapter 1 to the spaces $W^{1,q}(\Omega)$ for functions $u \equiv u_1 + u_2$, $\alpha = q$.

Much work has been done on spectral and high order polynomial approximation, for example the one dimensional results established by Quarteroni for the spaces $L^q(-1,1)$, seen in [30], and the work of Bernardi and Maday, see for example [20, 21] where results are obtained for polynomial approximation on weighted Hilbert spaces. However the most relevant works are those of Babuska, Szabo and Katz [14], and Babuska and Suri [12] which considered the p-version approximation theory for the case $q = 2$ as seen in chapter 1. The work in this chapter follows the methods given by Babuska and Suri, [12, section 3] which gave, for $u \in W^{m,2}(\Omega)$ a sequence of polynomials $u_p \in X_p$ such that

$$\|u - u_p\|_{W^{1,2}(\Omega)} \leq Cp^{-(m-1)} \|u\|_{W^{m,2}(\Omega)} . \tag{2.1}$$

The results in this section extend previous results to general Sobolev spaces $W^{1,q}(\Omega)$, $q \in [1, \infty]$. These results will be essential for the application of the p-version finite element method to the α-Laplacian.

The first section deals with trigonometric polynomial approximation, which will then be used in the following section to obtain the algebraic polynomial approximation theory.

2.2 Approximation using trigonometric polynomials

The Fourier series expansion of a sufficiently smooth function $f \in W^{k,q}(\Omega)$ on the square $S(\pi)$ is given by

$$f(x_1, x_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} e^{i(mx_1 + nx_2)} \tag{2.2}$$

where A_{mn} are the Fourier coefficients given by

$$A_{mn} = \frac{4}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t) e^{-ims} e^{-int} ds dt. \tag{2.3}$$

The partial sum of the Fourier series is denoted by

$$s_N(f) = \sum_{|m|<N} \sum_{|n|<N} A_{mn} e^{i(mx_1+nx_2)}. \tag{2.4}$$

For numbers $N \in \mathbb{N}$ and $r \in \mathbb{Z}^+$ let

$$C_{N,r} = \frac{4}{\pi^2} \frac{1+\ln N}{N^r}. \tag{2.5}$$

The following results will be useful: from [29, Theorem 4.3.1] for any $r \in \mathbb{Z}^+$

$$\begin{aligned} \sum_{|m|>N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t) e^{im(x_1-s)} dsdt &= \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathcal{D}_{N,r}(x_1-s) f^{(r,0)}(s,t) dsdt \end{aligned} \tag{2.6}$$

where

$$\mathcal{D}_{N,r}(t) = \sum_{|m|>N} \frac{1}{m^r} \cos{(mt - \frac{\pi r}{2})}. \tag{2.7}$$

For any fixed s

$$\int_{-\pi}^{\pi} |\mathcal{D}_{N,r}(t-s)| ds = C_{N,r} + O(N^{-r}) \tag{2.8}$$

and if $r > 1$

$$\|\mathcal{D}_{N,r}\|_{L_\infty(-\pi,\pi)} \leq \sum_{|m|>N} \frac{1}{m^r} \leq CN^{1-r}. \tag{2.9}$$

Furthermore, from [37, equation 4.1]

$$\begin{aligned} \frac{1}{2\pi} \sum_{|n|\leq N} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{in(x_2-t)} f(s,t) dsdt \\ = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, x_2+t) \frac{\sin{(N+\frac{1}{2})t}}{2\sin{\frac{1}{2}t}} dsdt \end{aligned} \tag{2.10}$$

and finally from [29, Theorem 2.2.1]

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \sum_{|n|\le N} \int_{-\pi}^{\pi} e^{in(x_2-t)} f(s,t)dt \right\|_{L^\infty(S(\pi))} \\
&\qquad\qquad \le \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\,(N+\frac{1}{2})t}{2\sin\frac{1}{2}t} \right| dt\, \|f\|_{L^\infty(S(\pi))} \\
&\qquad\qquad \le C C_{N,0} \|f\|_{L^\infty(S(\pi))} \,. \qquad (2.11)
\end{aligned}$$

The following Lemma looks at L^∞ estimates for such functions f.

Lemma 1 *If $f \in W^{l,\infty}_{PER}(S(\pi))$ then for $0 \le k \le l$*

$$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \le C(1+\ln N)^2 N^{-(l-k)} \|f\|_{W^{l,\infty}(S(\pi))} \,. \qquad (2.12)$$

Proof. Choose $\beta_1,\, \beta_2 \in \mathbb{Z}^+$ such that $\beta_1 + \beta_2 \le k$ then

$$\begin{aligned}
&D^{(\beta_1,\beta_2)}(f - s_N(f)) \\
&\quad = \left(\sum_{|m|>N} \sum_{|n|>N} + \sum_{|m|>N} \sum_{|n|\le N} + \sum_{|m|\le N} \sum_{|n|>N} \right) A_{mn}(im)^{\beta_1}(in)^{\beta_2} e^{i(mx_1+nx_2)} \\
&\quad := \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (2.13)
\end{aligned}$$

Now fix n and consider the term

$$\begin{aligned}
&\sum_{|m|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2} e^{i(mx_1+nx_2)} \\
&\quad = \sum_{|m|>N} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t)(im)^{\beta_1}(in)^{\beta_2} e^{-i(ms+nt)} e^{i(mx_1+nx_2)} ds dt \qquad (2.14)
\end{aligned}$$

since $f \in W^{l,\infty}_{PER}(S(\pi))$,

$$\begin{aligned}
&\sum_{|m|>N} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t)(im)^{\beta_1}(in)^{\beta_2} e^{-i(ms+nt)} e^{i(mx_1+nx_2)} ds dt \quad = \qquad (2.15) \\
&\quad = \sum_{|m|>N} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^{(\beta_1,\beta_2)}(s,t) e^{-i(ms+nt)} e^{i(mx_1+nx_2)} ds dt \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(x_2-t)} dt \sum_{|m|>N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(\beta_1,\beta_2)}(s,t) e^{im(x_1-s)} ds. \qquad (2.16)
\end{aligned}$$

Using (2.6) for $\alpha_1 \in \mathbb{Z}^+$ gives

$$\begin{aligned}\sum_{|m|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2}e^{i(mx_1+nx_2)} &= \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(x_2-t)}\frac{1}{\pi}\int_{-\pi}^{\pi} &\mathcal{D}_{N,\alpha_1}(x_1-s)f^{(\alpha_1+\beta_1,\beta_2)}(s,t)dsdt. \qquad (2.17)\end{aligned}$$

Now summing (2.17) over $n : |n| > N$, for $\alpha_2 \in \mathbb{Z}^+$ gives

$$\begin{aligned}\sum_{|m|>N}\sum_{|n|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2}e^{i(mx_1+nx_2)} &= \\ \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1-s)\frac{1}{\pi}\int_{-\pi}^{\pi} &\mathcal{D}_{N,\alpha_2}(x_2-t)f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(s,t)dtds. \qquad (2.18)\end{aligned}$$

Hence, for any $\alpha_1,\ \alpha_2 \in \mathbb{Z}^+ : \alpha_1+\alpha_2+\beta_1+\beta_2 = l$, (2.8) gives

$$\begin{aligned}|\mathrm{I}| &\leq CC_{N,\alpha_1}C_{N,\alpha_2}\left\|f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}\right\|_{L^\infty(S(\pi))} \\ &\leq C(1+\ln N)^2N^{-(l-\beta_1-\beta_2)}\,|f|_{W^{l,\infty}(S(\pi))}\,. \qquad (2.19)\end{aligned}$$

Summing (2.17) over $n : |n| \leq N$, for $r \in \mathbb{Z}^+$ gives

$$\begin{aligned}\mathrm{II} &= \frac{1}{\pi}\int_{-\pi}^{\pi}\mathcal{D}_{N,r}(x_1-s)\frac{1}{2\pi}\sum_{|n|\leq N}\int_{-\pi}^{\pi} e^{in(x_2-t)}f^{(r+\beta_1,\beta_2)}(s,t)dtds \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|\mathcal{D}_{N,r}(x_1-s)|\,\frac{1}{2\pi}\left|\sum_{|n|\leq N}\int_{-\pi}^{\pi} e^{in(x_2-t)}f^{(r+\beta_1,\beta_2)}(s,t)dt\right|ds \qquad (2.20)\end{aligned}$$

Hence, using (2.8)

$$|\mathrm{II}| \leq C_{N,r}\left\|\frac{1}{2\pi}\sum_{|n|\leq N}\int_{t=0}^{2\pi} e^{in(x_2-t)}f^{(r+\beta_1,\beta_2)}(s,t)dt\right\|_{L^\infty(S(\pi))}. \qquad (2.21)$$

Using (2.10) and (2.11), for $r+\beta_1+\beta_2 = l$ gives

$$\begin{aligned}|\mathrm{II}| &\leq CC_{N,0}C_{N,r}\left\|f^{(r+\beta_1,\beta_2)}\right\|_{L^\infty(S(\pi))} \\ &\leq C(1+\ln N)^2N^{-(l-\beta_1-\beta_2)}\,|f|_{W^{l,\infty}(S(\pi))}\,. \qquad (2.22)\end{aligned}$$

The third term III is dealt with similarly. Therefore, combining (2.19) and (2.22), for $s \in \mathbb{Z}^+ : s + \beta_1 + \beta_2 = l$ gives

$$\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^\infty(S(\pi))} \leq C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,\infty}(S(\pi))} \tag{2.23}$$

Summing over $\beta_1, \beta_2 : \beta_1 + \beta_2 \leq k$

$$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \leq C(1+\ln N)^2 N^{-(l-k)} \|f\|_{W^{l,\infty}(S(\pi))} \tag{2.24}$$

as required. ■

The following Lemma gives error estimates in the $W^{m,1}(\Omega)$ spaces.

Lemma 2 *If $f \in W^{l,1}_{PER}(S(\pi))$ then*

1. *for $0 \leq k \leq l$*

$$\|f - s_N(f)\|_{W^{k,1}(S(\pi))} \leq C(1+\ln N)^2 N^{-(l-k)} \|f\|_{W^{l,1}(S(\pi))}, \tag{2.25}$$

2. *for $0 \leq k+1 < l$*

$$\|f - s_N(f)\|_{W^{k,1}(\gamma)} \leq C(1+\ln N) N^{-(l-1-k)} \|f\|_{W^{l,1}(S(\pi))}, \tag{2.26}$$

where γ is a line contained in $S(\pi)$ on which x_2 is constant or $x_1 = \pm x_2$.

3. *for any $\mathbf{x} \in \overline{S(\pi)}$ and for $0 \leq k+2 < l$*

$$\|f(\mathbf{x}) - s_N(f(\mathbf{x}))\|_{W^{k,\infty}(S(\pi))} \leq C N^{-(l-k-2)} \|f\|_{W^{l,1}(S(\pi))}. \tag{2.27}$$

Proof.

1. As before, choose $\beta_1,\ \beta_2 \in \mathbb{Z}^+$ such that $\beta_1+\beta_2 \le k$ then,

$$\begin{aligned}
&\left\| D^{(\beta_1,\beta_2)}(f-s_N(f))\right\|_{L^1(S(\pi))} \le \\
&\left\| \sum_{|m|>N}\sum_{|n|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2}e^{i(mx_1+nx_2)}\right\|_{L^1(S(\pi))} \\
&+\left\| \sum_{|m|>N}\sum_{|n|\le N} A_{mn}(im)^{\beta_1}(in)^{\beta_2}e^{i(mx_1+nx_2)}\right\|_{L^1(S(\pi))} \\
&+\left\| \sum_{|m|\le N}\sum_{|n|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2}e^{i(mx_1+nx_2)}\right\|_{L^1(S(\pi))} \\
&:= \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned} \tag{2.28}$$

Use (2.6) to obtain for $\alpha_1,\ \alpha_2 \in \mathbb{Z}^+ : \alpha_1+\alpha_2+\beta_1+\beta_2 = l$

$$\begin{aligned}
\mathrm{I} &= \int_{x_1=-\pi}^{\pi}\int_{x_2=-\pi}^{\pi}\left|\frac{1}{\pi}\int_{s=-\pi}^{\pi}\mathcal{D}_{N,\alpha_1}(x_1-s)\right. \\
&\quad \left.\frac{1}{\pi}\int_{t=-\pi}^{\pi}\mathcal{D}_{N,\alpha_2}(x_2-t)f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(s,t)dtds\right|dx_1dx_2 \\
&\le \frac{1}{\pi}\int_{x_1=-\pi}^{\pi}|\mathcal{D}_{N,\alpha_1}(x_1-s)|dx_1\frac{1}{\pi}\int_{x_2=-\pi}^{\pi}|\mathcal{D}_{N,\alpha_2}(x_2-t)|dx_2 \\
&\quad \int_{t=-\pi}^{\pi}\int_{s=-\pi}^{\pi}|f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(s,t)|dtds.
\end{aligned} \tag{2.29}$$

Since $\mathcal{D}_{N,r}$ and f are both periodic with period 2π and $\alpha_1+\alpha_2+\beta_1+\beta_2 = l$, (2.8) gives

$$\begin{aligned}
\mathrm{I} &\le CC_{N,\alpha_1}C_{N,\alpha_2}\left\|f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}\right\|_{L^1(S(\pi))} \\
&\le C(1+\ln N)^2N^{-(l-\beta_1-\beta_2)}\left|f\right|_{W^{l,1}(S(\pi))}.
\end{aligned} \tag{2.30}$$

Similarly, using (2.6) and (2.10) yields, for any $r \in \mathbb{Z}^+ : \beta_1 + \beta_2 + r = l$,

$$\mathrm{II} = \tag{2.31}$$

$$\left\| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \mathcal{D}_{N,r}(x_1 - s) ds \frac{1}{\pi} \int_{t=-\pi}^{\pi} \frac{\sin{(N+\frac{1}{2})t}}{2\sin\frac{1}{2}t} f^{(r+\beta_1,\beta_2)}(s, x_2+t) \right\|_{L^1(S(\pi))}$$

$$\leq \int_{x_1=-\pi}^{\pi} \int_{x_2=-\pi}^{\pi} \frac{1}{\pi} \int_{s=-\pi}^{\pi} |\mathcal{D}_{N,r}(x_1 - s)|$$

$$|f^{(r+\beta_1,\beta_2)}(s, x_2+t)| ds \int_{t=-\pi}^{\pi} \left| \frac{\sin{(N+\frac{1}{2})t}}{2\sin\frac{1}{2}t} \right| dt dx_1 dx_2. \tag{2.32}$$

Since $\mathcal{D}_{N,r}$ and f are both periodic with period 2π, using (2.8) and (2.11) gives

$$\begin{aligned} \mathrm{II} &\leq CC_{N,r} \int_{t=-\pi}^{\pi} \left\| f^{(\beta_1+r,\beta_2)} \right\|_{L^1(S(\pi))} \left| \frac{\sin{(N+\frac{1}{2})t}}{2\sin\frac{1}{2}t} \right| dt \\ &\leq CC_{N,r}C_{N,0} \left\| f^{(r+\beta_1,\beta_2)} \right\|_{L^1(S(\pi))} \\ &\leq C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} \left| f \right|_{W^{l,1}(S(\pi))}. \end{aligned} \tag{2.33}$$

The third term, III, is treated similarly. Consequently, for $s \in \mathbb{Z}^+ : \beta_1 + \beta_2 + s = l$

$$\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^1(S(\pi))} \leq C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} \left| f \right|_{W^{l,1}(S(\pi))} \tag{2.34}$$

and summing $\beta_1, \beta_2 : \beta_1 + \beta_2 \leq k$ gives (2.25).

2. Let γ be the line contained in $S(\pi)$ on which x_2 is constant and let $\beta \in \mathbb{Z}^+ : \beta \leq k$ then

$$\begin{aligned} \left\| D^{(\beta,0)}(f - s_N(f)) \right\|_{L^1(\gamma)} &\leq \left\| \sum_{|m|>N} \sum_{|n|>N} A_{mn}(im)^\beta e^{i(mx_1+nx_2)} \right\|_{L^1(\gamma)} \\ &+ \left\| \sum_{|m|>N} \sum_{|n|\leq N} A_{mn}(im)^\beta e^{i(mx_1+nx_2)} \right\|_{L^1(\gamma)} \end{aligned}$$

$$
\begin{aligned}
&+ \left\| \sum_{|m|\leq N} \sum_{|n|>N} A_{mn}(im)^\beta e^{i(mx_1+nx_2)} \right\|_{L^1(\gamma)} \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (2.35)
\end{aligned}
$$

Using (2.6) gives for $\alpha_1, \alpha_2 \in \mathbb{Z}^+ : \alpha_1 + \alpha_2 + \beta = l$ and $\alpha_2 > 1$,

$$
\begin{aligned}
\mathrm{I} &= \left\| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1 - s) \int_{t=-\pi}^{\pi} \mathcal{D}_{N,\alpha_2}(x_2 - t) f^{(\alpha_1+\beta,\alpha_2)}(s,t)\,dt\,ds \right\|_{L^1(\gamma)} \\
&\leq C C_{N,\alpha_1} \left\| \frac{1}{\pi} \mathcal{D}_{N,\alpha_2}(x_2 - \cdot) \right\|_{L^\infty(\gamma)} \left\| f^{(\alpha_1+\beta,\alpha_2)} \right\|_{L^1(S(\pi))}. \qquad (2.36)
\end{aligned}
$$

Recalling (2.9), gives for all $\beta \leq k$ gives

$$
\begin{aligned}
\mathrm{I} &\leq C C_{N,\alpha_1} N^{1-\alpha_2} \left\| f^{(\alpha_1+\beta,\alpha_2)} \right\|_{L^1(S(\pi))} \\
&\leq C(1 + \ln N) N^{-(l-1-\beta)} |f|_{W^{l,1}(S(\pi))}. \qquad (2.37)
\end{aligned}
$$

Equally well, (2.6) and (2.8) give, for any $\nu \in \mathbb{Z}^+ : \nu = l - \beta$,

$$
\begin{aligned}
\mathrm{II} &= \left\| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \mathcal{D}_{N,\nu}(x_1 - s) \sum_{|n|\leq N} \frac{1}{2\pi} \int_{t=-\pi}^{\pi} e^{in(x_2-t)} f^{(l,0)}(s,t)\,ds\,dt \right\|_{L^1(\gamma)} \\
&\leq C C_{N,\nu} \int_{t=-\pi}^{\pi} \int_{s=-\pi}^{\pi} \frac{1}{2\pi} \sum_{|n|\leq N} \left| e^{in(x_2-t)} \right| \left| f^{(l,0)}(s,t) \right| ds\,dt \\
&\leq C N C_{N,\nu} \left\| f^{(l,0)} \right\|_{L^1(S(\pi))} \\
&\leq C(1 + \ln N) N^{-(l-1-\beta)} |f|_{W^{l,1}(S(\pi))}. \qquad (2.38)
\end{aligned}
$$

Using (2.6), (2.9) and (2.11) gives for $\sigma \in \mathbb{Z}^+ : \sigma = l - \beta$,

$$
\begin{aligned}
\mathrm{III} &= \left\| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \int_{t=-\pi}^{\pi} \frac{\sin(N + \frac{1}{2})s}{2 \sin \frac{1}{2}s} \mathcal{D}_{N,\sigma}(x_2 - t) f^{(\beta,\sigma)}(s,t)\,ds\,dt \right\|_{L^1(\gamma)} \\
&\leq \left\| \frac{1}{\pi} \mathcal{D}_{N,\alpha_2}(x_2 - \cdot) \right\|_{L^\infty(-\pi,\pi)} \int_{s=-\pi}^{\pi} \frac{1}{\pi} \left| \frac{\sin(N + \frac{1}{2})s}{2 \sin \frac{1}{2}s} \right| ds \left\| f^{(\beta,\sigma)} \right\|_{L^1(S(\pi))} \\
&\leq C N^{1+\beta-l} C_{N,0} \left\| f^{(\beta,\sigma)} \right\|_{L^1(S(\pi))} \\
&= C(1 + \ln N) N^{-(l-1-\beta)} |f|_{W^{l,1}(S(\pi))}. \qquad (2.39)
\end{aligned}
$$

Combining (2.37), (2.38) and (2.39) and summing over $\beta \leq k$ gives (2.26) for the case x_2 is constant.

Now let γ be the line contained in $S(\pi)$ given by $x_1 = x_2 = \tau$ and let $\beta \in \mathbb{Z}^+ : \beta \leq k$. Then

$$\begin{aligned}
\left\| \left(\frac{\partial}{\partial \tau}\right)^\beta (f - s_N(f)) \right\|_{L^1(\gamma)} &\leq \left\| \sum_{|m|>N} \sum_{|n|>N} A_{mn}[i(m+n)]^\beta e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&+ \left\| \sum_{|m|>N} \sum_{|n|\leq N} A_{mn}[i(m+n)]^\beta e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&+ \left\| \sum_{|m|\leq N} \sum_{|n|>N} A_{mn}[i(m+n)]^\beta e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&= \left\| \sum_{|m|>N} \sum_{|n|>N} A_{mn} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} m^j n^{\beta-j} e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&+ \left\| \sum_{|m|>N} \sum_{|n|\leq N} A_{mn} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} m^j n^{\beta-j} e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&+ \left\| \sum_{|m|\leq N} \sum_{|n|>N} A_{mn} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} m^j n^{\beta-j} e^{i(m+n)\tau} \right\|_{L^1(\gamma)} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (2.40)
\end{aligned}$$

Since f is a periodic function, using (2.6), (2.7) and (2.37)

$$\begin{aligned}
\mathrm{I} &= \int_{\tau=-\pi}^{\pi} \left| \frac{1}{4\pi^2} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} \sum_{|m|>N} \sum_{|n|>N} \right. \\
&\qquad \left. \int_{t=-\pi}^{\pi} \int_{s=-\pi}^{\pi} f^{(j,\beta-j)}(s,t) e^{-i(ms+nt)} e^{i(m+n)\tau} ds dt \right| d\tau \\
&\leq 2^\beta C(1 + \ln N) N^{-(l-1-\beta)} \, |f|_{W^{l,1}(S(\pi))} \, . \qquad (2.41)
\end{aligned}$$

Using (2.6), (2.7) and (2.38) gives

$$
\begin{aligned}
\mathrm{II} &= \int_{\tau=-\pi}^{\pi} \left| \frac{1}{4\pi^2} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} \sum_{|m|>N} \sum_{|n|\leq N} \right. \\
&\quad \left. \int_{t=-\pi}^{\pi} \int_{s=-\pi}^{\pi} f^{(j,\beta-j)}(s,t) e^{-i(ms+nt)} e^{i(m+n)\tau} ds dt \right| d\tau \\
&\leq 2^{\beta} C(1+\ln N) N^{-(l-1-\beta)} |f|_{W^{l,1}(S(\pi))} . \qquad (2.42)
\end{aligned}
$$

Using (2.6), (2.7) and (2.39) gives

$$
\begin{aligned}
\mathrm{III} &= \int_{\tau=-\pi}^{\pi} \left| \frac{1}{4\pi^2} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} \sum_{|m|\leq N} \sum_{|n|>N} \right. \\
&\quad \left. \int_{t=-\pi}^{\pi} \int_{s=-\pi}^{\pi} f^{(j,\beta-j)}(s,t) e^{-i(ms+nt)} e^{i(m+n)\tau} ds dt \right| d\tau \\
&\leq 2^{\beta} C(1+\ln N) N^{-(l-1-\beta)} |f|_{W^{l,1}(S(\pi))} . \qquad (2.43)
\end{aligned}
$$

Combining (2.41), (2.42) and (2.43) and summing over $\beta \leq k$ gives the result for $x_1 = x_2$, the result for $x_1 = -x_2$ follows immediately.

3. Choosing β_1, $\beta_2 \in \mathbb{Z}^+ : \beta_1 + \beta_2 \leq k$, then for $\mathbf{x} = (x_1, x_2) \in S(\pi)$

$$
\begin{aligned}
\left| D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1,x_2) \right| &\leq \left| \sum_{|m|>N} \sum_{|n|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2} e^{i(mx_1+nx_2)} \right| \\
&+ \left| \sum_{|m|>N} \sum_{|n|\leq N} A_{mn}(im)^{\beta_1}(in)^{\beta_2} e^{i(mx_1+nx_2)} \right| \\
&+ \left| \sum_{|m|\leq N} \sum_{|n|>N} A_{mn}(im)^{\beta_1}(in)^{\beta_2} e^{i(mx_1+nx_2)} \right| \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (2.44)
\end{aligned}
$$

Letting α_1, $\alpha_2 \in \mathbb{Z}^+ : \alpha_1 > 1$, $\alpha_2 > 1$ and $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = l$. Then using

(2.6) and (2.9) gives

$$\begin{aligned}
\mathrm{I} &= \left|\frac{1}{\pi}\int_{s=-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1-s)ds\frac{1}{\pi}\int_{t=-\pi}^{\pi} \mathcal{D}_{N,\alpha_2}(x_2-t)f^{(\alpha_1,\alpha_2)}(s,t)dt\right| \\
&\leq CN^{1-\alpha_1}N^{1-\alpha_2}\left\|f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}\right\|_{L^1(S(\pi))} \\
&\leq CN^{2+\beta_1+\beta_2-l}\,|f|_{W^{l,1}(S(\pi))}\,. \qquad (2.45)
\end{aligned}$$

Letting $\nu \in \mathbb{Z}^+ : \nu+\beta_1+\beta_2 = l$ and noting that $\nu \geq l-k > 2$,

$$\begin{aligned}
\mathrm{II} &= \left|\frac{1}{\pi}\int_{s=-\pi}^{\pi} \mathcal{D}_{N,\nu}(x_1-s)ds\frac{1}{2\pi}\sum_{|n|\leq N}\int_{t=-\pi}^{\pi} e^{in(x_2-t)}f^{(\nu,0)}(s,t)dt\right| \\
&\leq CNN^{1-\nu}\left\|f^{(\nu+\beta_1,\beta_2)}\right\|_{L^1(S(\pi))} \\
&\leq CN^{2+\beta_1+\beta_2-l}\,|f|_{W^{l,1}(S(\pi))}\,, \qquad (2.46)
\end{aligned}$$

where (2.6) and (2.9) have been used.

The third term III is dealt with similarly. Gathering these estimates gives

$$\left\|D^{(\beta_1,\beta_2)}(f-s_N(f))\right\|_{L^\infty(S(\pi))} \leq CN^{-(l-2-\beta_1-\beta_2)}\,|f|_{W^{l,1}(S(\pi))} \qquad (2.47)$$

and taking the maximum over β_1, $\beta_2 : \beta_1+\beta_2 \leq k$ gives the result claimed.

■

The results given in Lemmas 1 and 2 can now be used to obtain estimates in the general Sobolev spaces $W^{k,q}(\Omega)$.

Lemma 3 *Let $f \in W^{l,q}_{PER}(S(\pi))$, $q \in [1,\infty]$. Then*

1. *for $0 \leq k \leq l$*

$$\|f-s_N(f)\|_{W^{k,q}(S(\pi))} \leq CN^{-(l-k)}(1+\ln N)^{2\left|\frac{2}{q}-1\right|}\,\|f\|_{W^{l,q}(S(\pi))}\,, \qquad (2.48)$$

2. for $0 \le k < l + \frac{1}{q}$

$$\|f - s_N(f)\|_{W^{k,q}(\gamma)} \le$$
$$CN^{-(l-k-\frac{1}{q})} \|f\|_{W^{l,q}(S(\pi))} \begin{cases} (1+\ln N)^{(\frac{2}{q}-1)}, & q \in [1,2] \\ (1+\ln N)^{2(1-\frac{2}{q})}, & q \in [2,\infty] \end{cases}, \quad (2.49)$$

where γ *is a line contained in* $S(\pi)$ *on which* x_2 *is constant or* $x_1 = \pm x_2$.

3. for $0 \le k < l + \frac{2}{q}$

$$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \le$$
$$CN^{-(l-k-\frac{2}{q})} \|f\|_{W^{l,q}(S(\pi))} \begin{cases} 1, & q \in [1,2] \\ (1+\ln N)^{2(1-\frac{2}{q})}, & q \in [2,\infty] \end{cases} . \quad (2.50)$$

Proof.

1. Using standard arguments, [14], gives

$$\|f - s_N(f)\|_{W^{k,2}(S(\pi))} \le CN^{-(l-k)} \|f\|_{W^{l,2}(S(\pi))} \quad (2.51)$$

Combining this individually with (2.12) and (2.25) and applying a standard interpolation argument gives (2.48), for $q \in [2,\infty]$ and $q \in [1,2]$ respectively.

2. From [14, equation 3.19], for $m > \frac{1}{2}$

$$\|f - s_N(f)\|_{L^2(\gamma)} \le CN^{-(m-\frac{1}{2})} \|f\|_{W^{m,2}(S(\pi))} . \quad (2.52)$$

Since f is periodic, for any β_1, $\beta_2 \in \mathbb{Z}^+$

$$D^{(\beta_1,\beta_2)} s_N(f) = s_N\left(D^{(\beta_1,\beta_2)} f\right) . \quad (2.53)$$

Using (2.52) and (2.53), for $m > \frac{1}{2}$ and $\beta_1, \beta_2 \in \mathbb{Z}^+ : \beta_1 + \beta_2 \leq k$

$$\begin{aligned}\left\|D^{(\beta_1,\beta_2)}(f - s_N f)\right\|_{L^2(\gamma)} &= \left\|D^{(\beta_1,\beta_2)}f - s_N D^{(\beta_1,\beta_2)}f\right\|_{L^2(\gamma)} \\ &\leq CN^{-(m-\frac{1}{2})}\left\|f\right\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))}. \quad (2.54)\end{aligned}$$

Choosing $m + \beta_1 + \beta_2 = l$ and summing over all $\beta_1, \beta_2 : \beta_1 + \beta_2 \leq k$ gives

$$\|f - s_N(f)\|_{W^{k,2}(\gamma)} \leq CN^{-(l-k-\frac{1}{2})}\|f\|_{W^{l,2}(S(\pi))}. \quad (2.55)$$

Combining (2.55) with (2.12) and (2.26) and applying a standard interpolation argument gives (2.49), for $q \in [2, \infty]$ and $q \in [1, 2]$ respectively.

3. From [14, equation 3.29], for $m > 1$ and $(x_1, x_2) \in S(\pi)$

$$|(f - s_N(f))(x_1, x_2)| \leq CN^{-(m-1)}\|f\|_{W^{m,2}(S(\pi))}. \quad (2.56)$$

Using (2.53) and (2.56), for any $\beta_1, \beta_2 \in \mathbb{Z}^+$ and $m > 1$,

$$|D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1, x_2)| \leq CN^{-(m-1)}\|f\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))}. \quad (2.57)$$

Choosing $m + \beta_1 + \beta_2 = l$ and summing over all $\beta_1, \beta_2 : \beta_1 + \beta_2 \leq k$ gives

$$|D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1, x_2)| \leq CN^{-(l-\beta_1-\beta_2-1)}\|f\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))}. \quad (2.58)$$

Combining (2.58) with (2.12) and (2.27) and applying a standard interpolation argument gives (2.49), for $q \in [2, \infty]$ and $q \in [1, 2]$ respectively. ∎

2.3 Algebraic polynomial approximation on $S(1)$ and $T(1)$

The results from the previous section will be used to deduce the approximation properties for sequences $\{\phi_p(u)\}$ of algebraic polynomials to a function $u \in W^{m,q}(S(1))$ of the form $u \equiv u_1$ in (1.17). The methods used to obtain the following Lemmas are based on the ideas of Babuska, Katz and Szabo, see [14], where the function u must be put in the right form, so that the change of variables in the Fourier series will create an algebraic polynomial.

Algebraic polynomial approximation is firstly considered on the quadrilateral reference element $S(1)$.

Lemma 4 *Let $u \in W^{l,q}(S(1))$, $q \in [1,\infty]$ and let γ be either a side or diagonal of $S(1)$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{Q}(p)$, $p \in I\!N$, which are independent of q, such that,*

1. *for any $0 \leq k \leq l$, $q \in [1,\infty]$*

$$\|u - \phi_p(u)\|_{W^{k,q}(S(1))} \leq C p^{-(l-k)} \|u\|_{W^{l,q}(S(1))} (1 + \ln p)^{2|\frac{2}{q}-1|}, \tag{2.59}$$

2. *for $l > k + \frac{1}{q}$*

$$\|u - \phi_p(u)\|_{W^{k,q}(\gamma)} \leq C p^{-(l-k-\frac{1}{q})} \|u\|_{W^{l,q}(S(1))} \begin{cases} (1 + \ln p)^{(\frac{2}{q}-1)}, & q \in [1,2] \\ (1 + \ln p)^{2(1-\frac{2}{q})}, & q \in [2,\infty] \end{cases} \tag{2.60}$$

where γ is an edge or diagonal of $S(1)$.

3. for $l > k + \frac{2}{q}$

$$\|u - \phi_p(u)\|_{W^{k,\infty}(S(1))} \leq$$
$$Cp^{-(l-k-\frac{2}{q})} \|u\|_{W^{l,q}(S(1))} \begin{cases} 1, & q \in [1,2] \\ (1+\ln p)^{2(1-\frac{2}{q})}, & q \in [2,\infty] \end{cases} \tag{2.61}$$

Proof. Let $\rho > 1$, therefore $\overline{S(1)} \subset S(\rho)$. Since S(1) is convex, from Stein [34, Theorem 5] there exists an extension U of the function u onto the square $S(2\rho)$ such that

$$\text{supp}(U) \subset S(\frac{3\rho}{2}) \tag{2.62}$$

and $U \in W^{m,q}(S(2\rho))$ with

$$\|U\|_{W^{m,q}(S(2\rho))} \leq C\,\|u\|_{W^{m,q}(S(1))}\,. \tag{2.63}$$

Let $\Phi : S(\frac{\pi}{2}) \to S(2\rho)$ be the mapping

$$\mathbf{x} = \Phi(\hat{\mathbf{x}}) = 2\rho(\sin\hat{x}_1, \sin\hat{x}_2). \tag{2.64}$$

Clearly Φ is bijective. Furthermore, define $V \in W^{m,q}(S(\frac{\pi}{2}))$ by

$$V(\hat{\mathbf{x}}) = (U \circ \Phi)(\hat{\mathbf{x}}) \tag{2.65}$$

and observe that $\text{supp}(V) \subset S(\arcsin\frac{3}{4})$. Hence, V may be smoothly extended to $S(\pi)$ so that it is symmetric across the lines $\hat{x}_i = \pm\frac{\pi}{2}$. From (2.64) it is clear that $V \in W^{m,q}_{PER}(S(\pi))$. Moreover,

$$\|V\|_{W^{m,q}(S(\pi))} \leq C\,\|u\|_{W^{m,q}(S(1))} \tag{2.66}$$

Let s_p denote the p-th partial sum of the Fourier series expansion for the function V on $S(\pi)$. Each s_p has the same symmetry properties as V. Therefore,

$$s_p(\hat{\mathbf{x}}) = (\phi_p(u) \circ \Phi)(\hat{\mathbf{x}}) \tag{2.67}$$

where $\phi_p(u)$ is now an algebraic polynomial of degree at most p.

1. By (2.48), for $q \in [1,2]$, $0 \leq k \leq l$ one has,

$$\begin{aligned} \|u - \phi_p(u)\|_{W^{k,q}(S(1))} &\leq C\,\|V - s_p\|_{W^{k,q}(S(\pi))} \\ &\leq Cp^{-(l-k)}(1+\ln p)^{2(\frac{2}{q}-1)}\,\|V\|_{W^{l,q}(S(\pi))} \\ &\leq Cp^{-(l-k)}(1+\ln p)^{2(\frac{2}{q}-1)}\,\|U\|_{W^{l,q}(S(1))} \\ &\leq Cp^{-(l-k)}(1+\ln p)^{2(\frac{2}{q}-1)}\,\|u\|_{W^{l,q}(S(1))} \end{aligned} \tag{2.68}$$

   where (2.66) has been used. The result for $q \in [2,\infty]$ follows in a similar manner.

2. Denote $\hat{\gamma} = \Phi^{-1}(\gamma)$. Then by (2.49), for $q \in [1,2]$ and $0 \leq k + \frac{1}{q} < l$

$$\begin{aligned} \|u - \phi_p(u)\|_{W^{k,q}(\gamma)} &\leq C\,\|V - s_p\|_{W^{k,q}(\hat{\gamma})} \\ &\leq Cp^{-(l-k-\frac{1}{q})}(1+\ln p)^{(\frac{2}{q}-1)}\,\|V\|_{W^{l,q}(S(\pi))} \\ &\leq Cp^{-(l-k-\frac{1}{q})}(1+\ln p)^{(\frac{2}{q}-1)}\,\|U\|_{W^{l,q}(S(1))} \\ &\leq Cp^{-(l-k-\frac{1}{q})}(1+\ln p)^{(\frac{2}{q}-1)}\,\|u\|_{W^{l,q}(S(1))} \end{aligned} \tag{2.69}$$

   and the result for $q \in [2,\infty]$ follows similarly.

3. By (2.50), for $q \in [1,2]$ and $0 \leq k + \frac{2}{q} < l$

$$\|u - \phi_p(u)\|_{W^{k,\infty}(S(1))} \leq C\,\|V - s_p\|_{W^{k,\infty}(S(\pi))}$$

$$\leq Cp^{-(l-k-\frac{2}{q})}\,\|V\|_{W^{l,q}(S(\pi))}$$
$$\leq Cp^{-(l-k-\frac{2}{q})}\,\|u\|_{W^{l,q}(S(1))} \qquad (2.70)$$

and the result for $q \in [2,\infty]$ follows similarly. ∎

Algebraic polynomial approximation will now be considered for the triangular reference element $T(1)$.

Lemma 5 *Let $u \in W^{l,q}(T(1))$, $q \in [1,\infty]$ and let γ be either a side or diagonal of $S(1)$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{P}(p)$, $p \in \mathbb{N}$, which are independent of q, such that,*

1. *for any $0 \leq k \leq l$, $q \in [1,\infty]$*

$$\|u-\phi_p(u)\|_{W^{k,q}(T(1))} \leq Cp^{-(l-k)}\,\|u\|_{W^{l,q}(T(1))}\,(1+\ln p)^{2|\frac{2}{q}-1|}, \qquad (2.71)$$

2. *for $l > k+\frac{1}{q}$*

$$\|u-\phi_p(u)\|_{W^{k,q}(\gamma)} \leq Cp^{-(l-k-\frac{1}{q})}\,\|u\|_{W^{l,q}(T(1))}\begin{cases}(1+\ln p)^{(\frac{2}{q}-1)}, & q\in[1,2]\\(1+\ln p)^{2(1-\frac{2}{q})}, & q\in[2,\infty]\end{cases} \qquad (2.72)$$

where γ is an edge or diagonal of $T(1)$.

3. *for $l > k+\frac{2}{q}$*

$$\|u-\phi_p(u)\|_{W^{k,\infty}(S(1))} \leq Cp^{-(l-k-\frac{2}{q})}\,\|u\|_{W^{l,q}(S(1))}\begin{cases}1, & q\in[1,2]\\(1+\ln p)^{2(1-\frac{2}{q})}, & q\in[2,\infty]\end{cases} \qquad (2.73)$$

Proof. Let $u \in W^{l,q}(T(1))$ be given. By [34, Theorem 5] there exists an extension U of the function u to the square $S(1)$ satisfying

$$\|U\|_{W^{l,q}(S(1))} \leq C\,\|u\|_{W^{l,q}(T(1))} \tag{2.74}$$

By Lemma 4 there exists a sequence $U_p \in \widehat{Q}(p)$ such that for any $0 \leq k \leq l$

$$\|U - U_p\|_{W^{k,q}(S(1))} \leq Cp^{-(l-k)}(1 + 1 + \ln p)^{2|1-\frac{2}{q}|}\,\|U\|_{W^{l,q}(S(1))}\,. \tag{2.75}$$

Now $\widehat{Q}(p) \subset \widehat{P}(2p)$ and therefore the required sequence is defined to be the sequence $\phi_{2p}(u) = U_p$ and $\phi_{2p+1}(u) = \phi_{2p}(u)$. Observing

$$\begin{aligned}\|U - \phi_{2p+1}\|_{W^{k,q}(T(1))} &= \|U - \phi_{2p}\|_{W^{k,q}(T(1))}\\ &= \|U - U_p\|_{W^{k,q}(T(1))}\\ &\leq \|U - U_p\|_{W^{k,q}(S(1))}\,,\end{aligned} \tag{2.76}$$

the result then follows from (2.74) and (2.75). The remaining cases are similar. ■

Lemma 4 and Lemma 5 can now be generalised to the case when the norms on each side are based on different L^q type spaces:

Theorem 6 *Let $u \in W^{m,r}(S(1))$ where $r \in [1,\infty]$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{Q}(p)$, $p \in \mathbb{N}$ such that for $1 \leq q \leq r$ and $0 \leq l \leq m + 2/r - 2/q$*

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \leq Cp^{-(m-l+\frac{2}{r}-\frac{2}{q})}(1 + \ln p)^{2|\frac{1}{r}+\frac{1}{q}-1|}\,\|u\|_{W^{m,q}(S(1))}\,. \tag{2.77}$$

Moreover analogous results hold for approximation on the triangle.

Proof. Using (2.59), with $q = 1$ gives

$$\|u - \phi_p(u)\|_{W^{l,1}(S(1))} \leq Cp^{-(m-l)}(1 + \ln p)^2 \, \|u\|_{W^{m,1}(S(1))} \tag{2.78}$$

and using (2.61) with $q = 1$ gives

$$\|u - \phi_p(u)\|_{W^{l,\infty}(S(1))} \leq Cp^{-(m-l-2)} \, \|u\|_{W^{m,1}(S(1))} . \tag{2.79}$$

Therefore, it follows by standard interpolation for any $r \in [1, \infty]$ that

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \leq Cp^{-(m-l-2+\frac{2}{r})}(1 + \ln p)^{\frac{2}{r}} \, \|u\|_{W^{m,1}(S(1))} . \tag{2.80}$$

It can be seen from (2.59), with $q = r$, $r \in [1, 2]$, that

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \leq Cp^{-(m-l)}(1 + \ln p)^{2(\frac{2}{r}-1)} \, \|u\|_{W^{m,r}(S(1))} \tag{2.81}$$

and with $q = r$, $r \in [2, \infty]$, that

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \leq Cp^{-(m-l)}(1 + \ln p)^{2(1-\frac{2}{r})} \, \|u\|_{W^{m,r}(S(1))} \tag{2.82}$$

Using (2.80), with (2.81), and interpolation gives (2.77) for the case $r \in [1, 2]$ and the case $r \in [2, \infty]$ follows similarly. ∎

Remark. For the case $1 \leq r \leq q \leq \infty$ the following bound can be seen immediately,

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \quad \leq \quad Cp^{-(m-l)} \, \|u\|_{W^{m,q}(S(1))} \, (1 + \ln p)^{2|\frac{1}{r}+\frac{1}{q}-1|}, \tag{2.83}$$

although this may not be the optimal bound for this case.

2.4 Continuous piecewise polynomial approximation

The results of the previous section will now be used to obtain approximation properties for the spaces X_p. This is not a trivial matter, due to the fact that X_p is a space of continuous functions. To obtain continuity the individual element approximations must be "glued" together. The technique used to glue the element approximations together follows that of Babuska and Suri [12].

It will be assumed in this and the proceeding chapter, that the spaces X_p are such that $p_K = p$ for all $K \in \mathcal{P}$, i.e. the uniform p-version will be looked at, this restriction will be removed in chapter 4, when the hp-version is considered.

The first Theorem in this section deals with the case $u \in W^{m,q}(\Omega)$ where $m > 1 + \frac{1}{q}$ and homogeneous Dirichlet boundary conditions are considered. The restriction on q will be dealt with to give a global estimate in Theorem 8 and then is improved to a local estimate in chapter 4. The generalisation to non-homogeneous boundary conditions will be dealt with in Theorem 10.

Theorem 7 *Let $u \in W^{m,q}(\Omega)$, $q \in [1,\infty]$, $m > 1 + \frac{1}{q}$ and assume $\Gamma^D = \emptyset$. Then there exists a sequence u_p, $p \in I\!N$ of continuous piecewise polynomials $u_p \in X_p$ such that for all $K \in \mathcal{P}$, $q \in [1,\infty]$*

$$\left\| u - u_p^{(K)} \right\|_{W^{1,q}(K)} \leq C p^{-(m-1)} \sum_{J \in \mathcal{P}: \overline{K} \cap \overline{J} \neq \emptyset} \|u\|_{W^{m,q}(J)} \left(1 + \ln p\right)^{2\left|\frac{2}{q}-1\right|}. \tag{2.84}$$

Moreover, the following global estimates are valid

$$\|u-u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-(m-1)}\,\|u\|_{W^{m,q}(\Omega)}\,(1+\ln p)^{2|\frac{2}{q}-1|}. \tag{2.85}$$

Proof. Firstly assume that K in the partition $\mathcal{P}$ is a quadrilateral element; a sequence $\{u_{K,p}\}_{p\in\mathbb{N}}$ of polynomial approximations to $u\mid_K$ is constructed as follows:

Since K is a quadrilateral, it is the image of the square reference element $S(1)$ under the mapping F_K; so define $\widehat{u}_K = u\mid_K \circ F_K$. Let $\{\widehat{\omega}_{K,p}\}$ be a sequence of approximations to $\widehat{u}_K$ as in Lemma 4, and define $\omega_{K,p} = \widehat{\omega}_K \circ F_K^{-1}$

Transforming the estimates of Lemma 4 to the element K leads to analogous estimates on the error $e_{K,p} := u - \omega_{K,p}$ on K. In general, if elements K and J share a common edge $\gamma = \overline{K}\cap\overline{J}$ then the approximations $\omega_{K,p}$ and $\omega_{J,p}$ will be discontinuous on the interface. It will now be shown how $\omega_{K,p}$ and $\omega_{J,p}$ may be adjusted so that continuity is obtained while still retaining the accuracy of the approximation. To do this, the polynomial $\psi_p : [-1,1]\to\mathbb{R}$ given by

$$\psi_p(s) = \left(\frac{1-s}{2}\right)^p \tag{2.86}$$

will be required, and note that for any $q\in[1,\infty]$

1. $\|\psi_p\|_{L^q(-1,1)} \leq Cp^{-\frac{1}{q}}$,

2. $|\psi_p|_{W^{1,q}(-1,1)} \leq Cp^{1-\frac{1}{q}}$,

3. $\psi_p(-1) = 1$; $\psi_p(1) = 0$.

Denote, with F_K as before, $\widehat{e}_{K,p} = e_{K,p} \circ F_k$. Now the function $\widehat{\omega}_{K,p}$ is adjusted on each edge to obtain a new polynomial $\widehat{v}_{K,p}$ that interpolates $\widehat{u}\mid_K$ at the vertices of the reference element. For instance, the adjustment at the vertex

$$\widehat{A}_1 = (-1,-1) \tag{2.87}$$

is given by

$$\widehat{\alpha}^{(1)}_{K,p}(x_1,x_2) = \widehat{e}_{K,p}(-1,-1)\psi_p(x_1)\psi_p(x_2). \tag{2.88}$$

It is easily checked using the properties of ψ_p that

$$\begin{aligned}\left\|\widehat{\alpha}^{(1)}_{K,p}\right\|_{W^{1,q}(S(1))} &\leq C\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}\left\|\psi_p\right\|_{L^q(-1,1)}\left|\psi_p\right|_{W^{1,q}(-1,1)} \\ &\leq Cp^{1-\frac{2}{q}}\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}.\end{aligned} \tag{2.89}$$

Construct similar functions for each vertex and define

$$\widehat{v}_{K,p} = \widehat{\omega}_{K,p} + \sum_{i=1}^{4}\widehat{\alpha}^{(\widehat{A}_i)}_{K,p}. \tag{2.90}$$

It is clear that, the polynomial $\widehat{v}_{K,p}$ agrees with $\widehat{u}_K$ at the vertices. Moreover,

$$\left\|\widehat{u}-\widehat{v}_{K,p}\right\|_{W^{1,q}(S(1))} \leq \left|\widehat{e}_{K,p}\right|_{W^{1,q}(S(1))} + Cp^{1-\frac{2}{q}}\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}. \tag{2.91}$$

Therefore, defining $v_{K,p} = \widehat{e}_{K,p} \circ F_K^{-1}$ and mapping back to the element K gives

$$\left\|u - v_{K,p}\right\|_{W^{1,q}(K)} \leq \left|e_{K,p}\right|_{W^{1,q}(K)} + Cp^{1-\frac{2}{q}}\left\|e_{K,p}\right\|_{L^\infty(K)}. \tag{2.92}$$

Proceeding in a similar manner on the element J gives the functions $v_{K,p}$ and $v_{J,p}$ which agree at the end points of the edge γ.

Define the function $\epsilon^{(\gamma)} : \gamma \to \mathbb{R}$ to be the restriction of $v_{K,p}$ and $v_{J,p}$ to the edge γ i.e.

$$\epsilon^{(\gamma)} = (v_{K,p} - v_{J,p})\,|_\gamma\,. \tag{2.93}$$

This function is then extended onto the element K as follows. Suppose, without loss of generality, $\gamma = F_K(\widehat{\gamma})$ and define $\widehat{\epsilon}^{(\widehat{\gamma})}$ on $\widehat{\gamma}$ by $\widehat{\epsilon}^{(\widehat{\gamma})} = \epsilon^{(\gamma)} \circ F_K$. Evidently, $\widehat{\epsilon}^{(\widehat{\gamma})}$ is a polynomial on the edge $\widehat{\gamma}$. Define $\widehat{\beta}^{(\widehat{\gamma})}$ on $S(1)$ by the rule

$$\widehat{\beta}^{(\widehat{\gamma})} = \widehat{\epsilon}^{(\widehat{\gamma})}(x_1)\psi_p(x_2). \tag{2.94}$$

Notice that $\widehat{\beta}^{(\widehat{\gamma})}$ is an extension of $\widehat{\epsilon}^{(\widehat{\gamma})}$ vanishing on the remaining edges of $S(1)$. Furthermore,

$$\begin{aligned} \left|\widehat{\beta}^{(\widehat{\gamma})}\right|_{W^{1,q}(S(1))} &\le C\,\|\widehat{e}_{K,p}\|_{L^q(\widehat{\gamma})}\,|\psi_p|_{W^{1,q}(-1,1)} + C\,|\widehat{e}_{K,p}|_{W^{1,q}(\widehat{\gamma})}\,\|\psi_p\|_{L^q(-1,1)} \\ &\le C\left(p^{1-\frac{1}{q}}\,\|\widehat{e}_{K,p}\|_{L^q(\widehat{\gamma})} + p^{-\frac{1}{q}}\,|\widehat{e}_{K,p}|_{W^{1,q}(\widehat{\gamma})}\right) \end{aligned} \tag{2.95}$$

Translating back to K gives $\beta^{(\gamma)} = \widehat{\beta}^{(\widehat{\gamma})} \circ F_K^{-1}$ with

$$\left|\beta^{(\gamma)}\right|_{W^{1,q}(K)} \le C\left(p^{1-\frac{1}{q}}\,\|e_{K,p}\|_{L^q(\gamma)} + p^{-\frac{1}{q}}\,|e_{K,p}|_{W^{1,q}(\gamma)}\right) \tag{2.96}$$

and for $j = 0, 1$

$$\left\|\epsilon^{(\gamma)}\right\|_{W^{j,q}(\gamma)} \le \|e_{K,p}\|_{W^{j,q}(\gamma)} + \|e_{J,p}\|_{W^{j,q}(\gamma)}\,. \tag{2.97}$$

It is unnecessary to extend $\epsilon^{(\gamma)}$ onto J. The process is repeated for every edge of the element K and the function $u_{K,p}$ is defined by

$$u_{K,p} = v_{K,p} - \sum_\gamma \beta^{(\gamma)}. \tag{2.98}$$

It easily verified that $u_{K,p}$ agrees with $u_{J,p}$ on the edge γ. Consequently, after dealing with all inter-element edges, we may define a function $u_p \in X_p$ whose restriction to any element K is $u_{K,p}$. Furthermore,

$$\|u - u_{K,p}\|_{W^{1,q}(K)} \leq \|u - v_{K,p}\|_{W^{1,q}(K)} + C\sum_{\gamma} \left\|\beta^{(\gamma)}\right\|_{W^{1,q}(K)} \tag{2.99}$$

and hence using (2.92), (2.96), (2.97), (2.99) and all the properties of Lemma 4, the result follows as claimed, for the case of quadrilateral elements.

The treatment of a triangular element K is similar, except that the corrections at the vertices and edges are slightly different. Construct $w_{k,p}$ as in the case of quadrilaterals using instead Lemma 5. The correction at the vertex $\widehat{A}_1 = (-1,-1)$ is given by

$$\widehat{\alpha}^{(1)}_{K,p}(x_1,x_2) = \frac{1}{2}\widehat{e}_{K,p}(-1,-1)\psi_s(x_1)\psi_s(x_2)(1-x_1) \tag{2.100}$$

where $s = [(p-1)/2]$ and the extension $\widehat{\beta}$ associated with the edge $\widehat{\gamma} = \{(x_1,-1) : -1 \leq x_1 \leq 1\}$ is

$$\widehat{\beta}^{(\gamma)} = \frac{1}{2}\psi_1(x_2)\{(x_1-x_2)\widehat{\epsilon}^{\widehat{\gamma}}(x_1) + (1-x_1)\widehat{\epsilon}^{\widehat{\gamma}}(x_1-x_2-1)\}. \tag{2.101}$$

The remaining cases are similar. It is easily verified that the functions have the required properties. ∎

Theorem 8 *Let $u \in W^{m,q}(\Omega)$, $q \in [1,\infty]$, $m > 1$, and assume $\Gamma^D = \emptyset$. Then there exists a sequence u_p, $p \in \mathbb{N}$ of continuous piecewise polynomials $u_p \in X_p$ such that*

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-(m-1)}(1+\ln p)^{2|2/q-1|}\|u\|_{W^{m,q}(\Omega)}. \tag{2.102}$$

Proof. Because of Theorem 7, only the case $m \in (1, 1+1/q]$ need be considered. Firstly, from Bergh and Lofstrom [19, Theorem 6.4.5, equation (4) and Theorem 6.2.4, equation (9)] it can be seen for $\theta = m-1$ that

$$W^{m,q}(\Omega) \subset B^m_{q\infty}(\Omega) = (W^{1,q}(\Omega), W^{2,q}(\Omega))_{\theta,\infty} \tag{2.103}$$

where $B^m_{q\infty}(\Omega)$ is the Besov space defined in [19].

Therefore, using [19, page 49, Top] and [19, Theorem 3.5.2] u may be expressed in the following form, for any $t > 0$

$$u = v_1(t) + v_2(t) \tag{2.104}$$

where $v_1 \in W^{1,q}(\Omega)$ and $v_2 \in W^{2,q}(\Omega)$, such that

$$\|v_1\|_{W^{1,q}(\Omega)} \leq Ct^{m-1} \|u\|_{W^{m,q}(\Omega)} \tag{2.105}$$

$$\|v_2\|_{W^{2,q}(\Omega)} \leq Ct^{m-2} \|u\|_{W^{m,q}(\Omega)} \tag{2.106}$$

where C is independent of u. Then by Theorem 7 there exists a continuous piecewise polynomial u_p such that

$$\|v_2 - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-1} \|v_2\|_{W^{2,q}(\Omega)} \leq Cp^{-1}t^{m-2} \|u\|_{W^{m,q}(\Omega)}. \tag{2.107}$$

Choosing $t = \frac{1}{p}$ and using the triangle inequality gives

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq p^{-(m-1)} \|u\|_{W^{m,q}(\Omega)}. \tag{2.108}$$

■

2.4.1 Non-homogeneous Dirichlet boundary data

So far the functions u that have been looked at have had no Dirichlet boundary conditions imposed, i.e. $\Gamma_D = \emptyset$ or $u \equiv u_1$ in (1.17). If the final approximation results are going to be applicable to the problem (1.9), it is clear that the case of non-homogeneous boundary conditions must be considered. Therefore, it will now be assumed that the function $u \equiv u_1 + u_2$ in (1.17). It is clear that, unless the function g in (1.17) is a polynomial of degree no more than p on each element boundary, then it is necessary to approximate the boundary data.

The Dirichlet data g must be approximated by a polynomial g_p such that $g_p|_{\overline{K}\cap\Gamma_D}$ is a polynomial of degree no more than p and is easily constructed on a machine. It would be nice to use the approximations $\phi_p(u)$ given in the previous section, since these approximations will not produce any degradation in the approximation. However, this would not be a practical method since the polynomials $\phi_p(u)$ are not easily constructed on a machine.

Once the function g_p has been constructed, the problem is then to estimate the accuracy that may be obtained by approximating u with piecewise polynomials $u_p \in X_p$, such that $u_p = g_p$ on the Dirichlet boundary.

Denote the Dirichlet boundary for an element K by $\gamma = \Gamma_D \cap \overline{K}$; without loss of generality, assume that $\gamma = (-1, 1)$. The Dirichlet data for the element K is constructed as follows.

The $p-$th partial sum of the Chebyshev series is given by

$$\sigma_p(g;t) = \sum_{k=0}^{p} A_k T_k(t) \tag{2.109}$$

where T_k is the $k-$th degree Chebyshev polynomial and the coefficients are given by

$$A_k = \frac{2}{\pi}\int_{-1}^{1} g(t)T_k(t)\frac{\mathrm{dt}}{\sqrt{1-t^2}}. \tag{2.110}$$

Bounds for the rate of convergence of the partial sums of the Chebyshev series are now considered.

Lemma 9 *Let $g \in W^{l,q}(-1,1)$ where $q \in [1,\infty]$. Then for $l \geq 0$*

$$\|g - \sigma_p(g)\|_{L^q(-1,1)} \leq C(1+\ln p)p^{-l}\,\|g\|_{W^{l,q}(-1,1)} \tag{2.111}$$

and for $l > 2 - 1/q$

$$\|g - \sigma_p(g)\|_{W^{1,q}(-1,1)} \leq C(1+\ln p)p^{-(l-2+\frac{1}{q})}\,\|g\|_{W^{l,q}(-1,1)} \tag{2.112}$$

Proof.

1. Following [33, Theorem 3.3] for $m > 0$ and $q \in [1,\infty]$

$$\begin{aligned}\|g - \sigma_p(g)\|_{W^{m,q}(-1,1)} &= \|g - \phi_p + \phi_p - \sigma_p(g)\|_{W^{m,q}(-1,1)} \\ &= \|g - \phi_p + \sigma_p(\phi_p - g)\|_{W^{m,q}(-1,1)} \\ &\leq \|g - \phi_p\|_{W^{m,q}(-1,1)} + \|\sigma_p(\phi_p - g)\|_{W^{m,q}(-1,1)}.\end{aligned} \tag{2.113}$$

Choosing $m = 0$ and $q = \infty$ in (2.113), and using [33, equation 3.27, page 133] gives

$$\begin{aligned}\|g - \phi_p\|_{L^\infty(-1,1)} &\leq \|g - \phi_p\|_{L^\infty(-1,1)}\left(1 + \frac{1}{\pi}\int_0^\pi \left|\frac{\sin((2p+1)/2)\theta}{\sin(\theta/2)}\right| d\theta\right) \\ &\leq (1 + C_{p,0})\,\|g - \phi_p\|_{L^\infty(-1,1)},\end{aligned} \tag{2.114}$$

where in the last inequality (2.11) has been used. For a $q \in [1, \infty]$ there exists a polynomial, ϕ_p, of degree no more than p such that

$$\|g - \phi_p\|_{W^{k,q}(-1,1)} \leq Cp^{-(l-k)} \|g\|_{W^{l,q}(-1,1)} \tag{2.115}$$

for all $0 \leq k \leq l$. Choosing $k = 0$ and $q = \infty$ in (2.115) and combining this with (2.114) gives

$$\|g - \sigma_p(g)\|_{L^\infty(-1,1)} \leq C(1 + \ln p)p^{-l} \|g\|_{W^{l,\infty}(-1,1)} . \tag{2.116}$$

2. Let $\theta = \arccos x$, $x \in (-1, 1)$. Then

$$\frac{d}{dx}\sigma_p(g; x) = \frac{1}{\sin\theta}\frac{d}{d\theta}\sigma_p(g; \cos\theta). \tag{2.117}$$

Hence choosing $m = 1$, $q = \infty$ in (2.113) and choosing $k = 1$, $q = \infty$ in (2.115) only the following need be considered,

$$\begin{aligned}
&\left|\frac{\partial}{\partial x}[\sigma_p(\phi_p - g)(x)]\right| = \left|\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\{\sigma_p(\phi_p - g)(\cos\alpha)\}\right| \\
&= \left|\frac{1}{2\pi}\int_0^\pi \frac{\sin((2p+1)/2)\theta}{\sin\alpha\sin(\theta/2)}\right. \\
&\qquad \left.\frac{\partial}{\partial\alpha}[(\phi_p - g)(\cos(\theta+\alpha)) + (\phi_p - g)(\cos(\theta-\alpha))]d\theta\right| .
\end{aligned} \tag{2.118}$$

Now

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}[(g - \phi_p)(\cos(\theta+\alpha)) + -(g - \phi_p)(\cos(\theta-\alpha))] \quad = \\
&[(g - \phi_p)'(\cos(\theta+\alpha)) + (g - \phi_p)'(\cos(\theta-\alpha))]\sin\alpha\cos\theta + \\
&[(g - \phi_p)'(\cos(\theta+\alpha)) - (g - \phi_p)'(\cos(\theta-\alpha))]\sin\theta\cos\alpha.
\end{aligned} \tag{2.119}$$

Since $g \in W^{2,\infty}(-1,1)$

$$|(g-\phi_p)'(\cos(\theta+\alpha)) - (g-\phi_p)'(\cos(\theta-\alpha))| \leq$$
$$2\,\|(g-\phi_p)''\|_{L^\infty(-1,1)}\,|\sin\alpha|\,. \quad (2.120)$$

Combining (2.113), (2.115), (2.119) and (2.120) with $l=1$ and 2 gives,

$$\begin{aligned}\left|\frac{\partial}{\partial x}[\sigma_p(\phi_p-g)(x)]\right| &\leq \frac{2}{\pi}\Big[\|(g-\phi_p)''\|_{L^\infty(-1,1)} \\ &\quad + \|(g-\phi_p)'\|_{L^\infty(-1,1)}\Big]\left\{\int_0^\pi\left|\frac{\sin((2p+1)/2)\theta}{\sin(\theta/2)}\right|d\theta\right\} \\ &\leq C_{p,0}p^{-(m-2)}\,\|g\|_{W^{m,\infty}(-1,1)}\,, \end{aligned} \quad (2.121)$$

where (2.11) has been used in the last inequality.

3. Observe that

$$\|\sigma_p(g)\|_{L^1(-1,1)} \leq \|D_p\|_{L^1(-1,1)}\,\|g\|_{L^1(-1,1)}\,, \quad (2.122)$$

where

$$D_p(\alpha) = \frac{\sin(p+1/2)\alpha}{\sin\alpha/2} \quad (2.123)$$

and thus, as in the L^∞ case,

$$\|g-\sigma_p(g)\|_{L^1(-1,1)} \leq C(1+\ln p)p^{-l}\,\|g\|_{W^{l,1}(-1,1)}\,, \quad (2.124)$$

4. Choosing $m=1$, $q=1$ in (2.113) and choosing $k=1$, $q=1$ in (2.115) the following need only be considered

$$\|\sigma_p(\phi_p-g)\|_{W^{1,1}(-1,1)} = \int_{-1}^1\left|\frac{1}{2\pi}\int_0^\pi\frac{\sin((2p+1)/2)\theta}{\sin\alpha\sin(\theta/2)}\times\right.$$

$$
\begin{aligned}
&\left.\frac{\partial}{\partial\alpha}[(\phi_p - g)(\cos(\theta+\alpha)) + (\phi_p - g)(\cos(\theta-\alpha))]d\theta\right| dx \\
&\leq \int_0^\pi \frac{1}{2\pi}\left|\frac{\sin((2p+1)/2)\theta}{\sin(\theta/2)}\right| \times \\
&\left\{\int_0^\pi (|\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\theta+\alpha))| + |\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\theta-\alpha))|)d\alpha\right\} d\theta. \qquad (2.125)
\end{aligned}
$$

Since $\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\theta-\alpha))$ and $\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\theta+\alpha))$ are 2π periodic and the kernel is 2π periodic

$$
\begin{aligned}
\|\sigma_p(\phi_p - g)\|_{W^{1,1}(-1,1)} &\leq C\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin((2p+1)/2)\theta}{\sin(\theta/2)}\right| d\theta \\
&\quad \int_{-\pi}^{\pi} |\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\alpha))|d\alpha \\
&\leq CC_{p,0}\int_0^\pi |\frac{\partial}{\partial\alpha}(\phi_p - g)(\cos(\alpha))|d\alpha \\
&\leq CC_{p,0}\,\|\phi_p - g\|_{W^{1,1}(-1,1)} \\
&\leq CC_{p,0}p^{-(l-1)}\,\|g\|_{W^{l,1}(-1,1)} \qquad (2.126)
\end{aligned}
$$

where (2.11) and (2.115) have been used.

The claimed results then follow by using interpolation on the four above results. ■

Comparing the results of the previous section and the results for Chebyshev approximation, it can be seen that the p−th partial Chebyshev expansion of the boundary data g does not give optimal rates of convergence. The expected rate would be $O(p^{-(l-1)})$ for approximation in the space $W^{1,\infty}(-1,1)$ when the function $g \in W^{l,\infty}(-1,1)$. At this moment, there does not seem to be any practical method of constructing a polynomial approximation which will give optimal rates of convergence in both L^q and $W^{1,q}$ norms for all values of q. The treatment of non-homogeneous boundary conditions is a non-trivial matter even in the case $q = 2$, see Babuska and Suri [13].

The actual approximation g_p to g on γ is taken to be

$$g_p(t) = \{g(-1) - \sigma_p(g; -1)\}\psi_p(t) + \{g(1) - \sigma_p(g; 1)\}\psi_p(g; 1) + \sigma_p(g; t). \quad (2.127)$$

Constructing similar polynomials on each boundary $\gamma_K = \{\mathbf{x} : \mathbf{x} \in \overline{K} \cap \Gamma_D\}$ gives a continuous piecewise polynomial approximation to the Dirichlet boundary data.

From the above Lemma and Theorem 8 the following Theorem may be obtained.

Theorem 10 *Let $u = u_1 + u_2$ be given by (1.17) and assume $g \in W^{m+1-\frac{2}{q},q}(\Gamma_D)$ where $q \in [1, \infty]$, $m > 1$ and g is the trace of u on Γ_D. Then there exists a sequence $u_p \in X_p$ of continuous piecewise polynomials such that $u_p = g_p$ on the Dirichlet boundary Γ_D. Moreover, the following estimate holds*

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-(m-1)}(1 + \ln p)^{2|1-\frac{2}{q}|}\{\|u\|_{W^{m,q}(\Omega)} + \|g\|_{W^{m+1-\frac{2}{q},q}(\Gamma_D)}\}. \quad (2.128)$$

Proof. Let $\tilde{u}_p$ be a sequence of approximations to u as in Theorem 8. Let K be any element having an edge on the Dirichlet boundary. Without loss of generality we may assume that K is the reference element and the Dirichlet data is on the boundary $\gamma = \{(x_1, -1) : -1 \leq x_1 \leq 1\}$. Let v_p be the polynomial

$$v_p(x_1, x_2) = \tilde{u}_p(x_1, x_2) + (\sigma_p(g; x_1) - \tilde{u}_p(x_1, -1))\psi_p(x_2). \quad (2.129)$$

Following steps similar to those in the proof Theorem 8 and using Lemma 9 leads to the estimate

$$\|u - v_p\|_{W^{1,q}(K)} \leq C \, \|u - \tilde{u}_p\|_{W^{1,q}(K)} + C(1+\ln p)p^{-(m-1)} \, \|g\|_{W^{m+1-\frac{2}{q},q}(\gamma)} . \quad (2.130)$$

The proof is completed by applying exactly the same procedure used in the proof of Theorem 8 to adjust v_p and obtain a continuous piecewise polynomial. ∎

Chapter 3

The p-version Approximation Theory for Singular Functions

For the main part of this chapter the functions to be approximated using piecewise continuous polynomials will be of the form $u \equiv u_3$, where u_3 is given by (1.20). It was shown in chapter 1, that when the linear elliptic problem was considered the rate of convergence for the p-version, for functions given by u_3, was twice that of the h-version, see (1.35). The main aim of this chapter is to generalise this result, to the Sobolev spaces $W^{1,q}(\Omega)$, $q \in [2, \infty)$. The method of proof used will follow closely the analysis given by Babuska, Szabo and Katz [14]. They obtained an estimate for the error e of the form

$$\|e\|_{W^{1,2}(\Omega)} \leq C(\epsilon)p^{-2\lambda+\epsilon} \tag{3.1}$$

where $\epsilon > 0$ is arbitrary. As was the case for the result given in chapter 1, the presence of $C(\epsilon)$ is of some concern, since the analysis suggests that it could blow up as $\epsilon \longrightarrow 0$. The ϵ was later removed in analysis by Babuska and Suri [11, Chapter 5] involving the use of orthogonal polynomials, in the Hilbert spaces $W^{1,2}(\Omega)$. Unfortunately, this method is of little use for the analysis in the spaces $W^{1,q}(\Omega)$, $q \in [2, \infty)$ since, the orthogonality of the polynomials is lost. The following work will extend the results given in the previously mentioned works, to the spaces $W^{1,q}(\Omega)$, $q \in [2, \infty)$. However, the final result will involve ϵ since, the method of proof will follow that of Babuska, Szabo and Katz [14].

The first section will deal with the regularisation of the true solution. In the second section these regularised approximations will be approximated using piecewise continuous polynomials. In the final section the general uniform p-version estimate will be given for all functions described by (1.17).

Throughout the following two sections only one singular function of u_3 will be considered, i.e. $M = 1$ in (1.17). Therefore, the function to be approximated will be given by,

$$u = c\zeta(r)r^{\lambda}g(|\log r|)\Theta(\theta), \tag{3.2}$$

where g and ζ are given by (1.17) with the subscript i dropped and Θ is a smooth function satisfying $\Theta(\theta) = 0$ for $\theta < \theta_0$ and for $\theta > \frac{\pi}{2} - \theta_0$, where $\theta_0 \in (0, \frac{\pi}{4})$. It will also be assumed from here on that, $g(|\log r|)$ satisfies

$$\int_0^R |g(|\log r|)| r^{-\mu}\,\mathrm{dr} \leq Cg(|\log R|)R^{1-\mu}, \tag{3.3}$$

for any $\mu < 1$ and $R > 0$.

3.1 Regularised approximations to singular functions

Following [14], firstly create a family of regularised approximations $\{u^\Delta : 0 < \Delta < 1/2\}$, that approach the singular function u as $\Delta \to 0$. Then Δ will be chosen to be a specific term dependent on p such that the rate at which u^Δ approaches u is sufficiently fast.

For $\rho > 0$, let $\tilde{S}(\rho)$ be the square

$$\tilde{S}(\rho) = \{(x_1, x_2) : 0 < x_1 < \rho; 0 < x_2 < \rho\} \tag{3.4}$$

and let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth (C^∞) cut off function satisfying

$$\chi(r) = \begin{cases} 0, & r < 1/2 \\ 1, & r > 1 \end{cases} . \tag{3.5}$$

The family then consists of functions of the form

$$u^\Delta(\mathbf{x}) = \chi(|\mathbf{x}|/\Delta)u(\mathbf{x}) \tag{3.6}$$

and has the following key properties

1. $u^\Delta \in C^\infty(\tilde{S}(1))$ and $u^\Delta(\mathbf{A}) = 0$, where $\mathbf{A}$ is the position vector of the corner.

2. for some $\kappa > 2$

$$\text{supp } u^\Delta \subset R_\kappa \cap \tilde{S}(1/4), \tag{3.7}$$

where

$$R_\kappa = \{\mathbf{x} : x_1/\kappa < x_2 < \kappa x_1\} \tag{3.8}$$

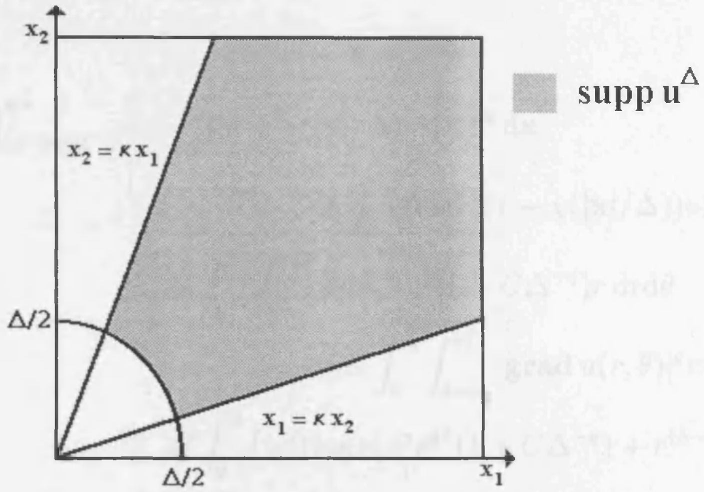

Figure 3.1: The support of u^{Δ}

see Figure 3.1.

3. there exists a non-decreasing, non-negative function $C(\cdot)$ such that for all $\mathbf{x} \in \tilde{S}(1)$

$$|D^{\alpha}u^{\Delta}(\mathbf{x})| \leq C(|\alpha|)g(|\log\Delta|)\max\{\min(x_1, x_2), \Delta\}^{-\langle|\alpha|-\lambda\rangle} \tag{3.9}$$

for all $0 < \Delta < 1/2$ and all $\mathbf{x} \in \tilde{S}(1)$ where $\langle\alpha\rangle = \max(\alpha, 0)$.

Elements of the family $\{u^{\Delta}\}$ approach the singular function u in the following sense:

Lemma 11 *Suppose that u is given by (3.2). Then*

$$\left\|u - u^{\Delta}\right\|_{W^{1,q}(\Omega)} \leq Cg(|\log\Delta|)\Delta^{\lambda-(1-2/q)} \tag{3.10}$$

Proof.

$$\begin{aligned}
\left\|u - u^{\Delta}\right\|^{q}_{W^{1,q}(\Omega)} &= \int_{\Omega} |(1-\chi(|\mathbf{x}|/\Delta))u(\mathbf{x})|^{q}\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\Omega} |\mathbf{grad}\,((1-\chi(|\mathbf{x}|/\Delta))u(\mathbf{x}))|^{q}\,\mathrm{d}\mathbf{x} \\
&\leq C\int_{0}^{\Delta}\int_{\theta=0}^{\pi/2} |u(r,\theta)|^{q}(1+C\Delta^{-q})r\,\mathrm{d}r\mathrm{d}\theta \\
&\qquad + C\int_{0}^{\Delta}\int_{\theta=0}^{\pi/2} |\mathbf{grad}\,u(r,\theta)|^{q} r\,\mathrm{d}r\mathrm{d}\theta \\
&\leq C\int_{0}^{\Delta}\{|g(|\log r|)|^{q} r^{q\lambda}(1+C\Delta^{-q}) + r^{(\lambda-1)q}\} r\,\mathrm{d}r.
\end{aligned}$$

Therefore, (3.3) gives

$$\left\|u - u^{\Delta}\right\|^{q}_{W^{1,q}(\Omega)} \leq C|g(|\log \Delta|)|^{q}\Delta^{\lambda q-(q-2)},$$

taking the $q-$th root on both sides of the above inequality gives the required estimate. ■

3.2 Polynomial approximation to u^{Δ}

In this section it will be shown how well piecewise continuous polynomials of degree at most p in each element approximate the regularised function u^{Δ}. To do this the following lemmas will be required.

Let $\Phi : (0,\frac{\pi}{2})\times(0,\frac{\pi}{2})\rightarrow(0,1)\times(0,1)$ be the bijective map given by

$$\Phi(\widehat{x}_1,\widehat{x}_2) = (\sin^2\widehat{x}_1, \sin^2\widehat{x}_2). \tag{3.11}$$

Define $\widehat{\Delta} = \arcsin\sqrt{\Delta/2}$ and note that $\widehat{\Delta} \leq C\Delta^{\frac{1}{2}}$, since $\Delta \in (0,\frac{1}{2})$ and

$$\widehat{\Delta} = \arcsin(\Delta^{\frac{1}{2}}) = \int_{0}^{\Delta^{\frac{1}{2}}} \frac{ds}{\sqrt{1-s^2}}$$

$$\leq \frac{\Delta^{\frac{1}{2}}}{\sqrt{1-\Delta}} \leq \sqrt{2}\Delta^{\frac{1}{2}}.$$

Lemma 12 *Let* $v \in W^{2/q,q}(T)$, $q \in [2,\infty)$ *where* $T \subset R_\kappa \cap \check{S}(1/3)$ *and define* $\widehat{v} = v \circ \Phi : \widehat{T} \to I\!R$ *and* $T^\Delta = T \cap \check{S}(\frac{\Delta}{2})$. *Then* $\widehat{v} \in W^{2/q,q}(\widehat{T})$ *with*

$$\|v\|_{W^{2/q,q}(T)} \approx \|\widehat{v}\|_{W^{2/q,q}(\widehat{T})} \tag{3.12}$$

and

$$\|v\|_{W^{1,q}(T/T^\Delta)} \leq C\Delta^{-\frac{1}{2}(1-\frac{2}{q})} \|\widehat{v}\|_{W^{1,q}(\widehat{T}/\widehat{T^\Delta})}. \tag{3.13}$$

Proof. Firstly consider (3.12); from [theorem 7.48][1] for $q \in [2,\infty)$

$$\|v\|_{W^{2/q,q}(T)} \approx \left\{ \|v\|^q_{L^q(T)} + \int_T\int_T \frac{|v(\mathbf{x}) - v(\mathbf{y})|^q}{|\mathbf{x}-\mathbf{y}|^4}\,d\mathbf{x}\,d\mathbf{y} \right\}^{1/q} \tag{3.14}$$

Firstly consider bounding the norm $\|v\|_{L^q(T)}$; using Hölder's inequality, with $\frac{1}{t} + \frac{1}{t'} = 1, t = q+1$, gives

$$\begin{aligned} \int_{\widehat{T}} |\widehat{v}|^q\,d\widehat{\mathbf{x}} &= \int_T |v|^q \frac{1}{|\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|}\,d\mathbf{x} \\ &\leq \left(\int_T |v|^{qt}\,d\mathbf{x}\right)^{\frac{1}{t}} \left(\int_T |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|^{-t'}\,d\mathbf{x}\right)^{\frac{1}{t'}}. \end{aligned} \tag{3.15}$$

The second term need only be considered, for the case $\widehat{x}_1$ and $\widehat{x}_2$ are small,

$$\int_T |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|^{-t'}\,d\mathbf{x} \approx \int_T x_1^{\frac{-t'}{2}} x_2^{\frac{-t'}{2}}\,d\mathbf{x}. \tag{3.16}$$

Since $t' = \frac{q+1}{q} < 2$,

$$\int_T |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|^{-t'}\,d\mathbf{x} \approx C. \tag{3.17}$$

Consequently

$$\int_{\widehat{T}} |\widehat{v}|^q\,d\widehat{\mathbf{x}} \leq C\left(\int_T |v|^{tq}\,d\mathbf{x}\right)^{\frac{1}{t}} \leq C\,\|v\|^q_{L^{q(q+1)}(T)}. \tag{3.18}$$

Using the Sobolev Embedding Theorem, see [1, theorem 5.4,equation 6] gives

$$W^{2/q,q}(T) \hookrightarrow L^{q(q+1)} \tag{3.19}$$

and so

$$\|v\|_{L^{q(q+1)}(T)} \leq C \|v\|_{W^{2/q,q}(T)}. \tag{3.20}$$

Hence,

$$\int_{\widehat{T}} |\widehat{v}|^q \, d\widehat{\mathbf{x}} \leq C \|v\|^q_{L^{q(q+1)}(T)} \leq C \|v\|^q_{W^{2/q,q}(T)}. \tag{3.21}$$

Furthermore from (3.15)

$$\|\widehat{v}\|_{L^q(\widehat{T})} \geq \|v\|_{L^q(T)}. \tag{3.22}$$

Next consider

$$\int_T \int_T \frac{|v(\mathbf{x}) - v(\mathbf{y})|^q}{|\mathbf{x} - \mathbf{y}|^4} \, d\mathbf{x} \, d\mathbf{y} = \tag{3.23}$$

$$\int_{\widehat{T}} \int_{\widehat{T}} \frac{|\widehat{v}(\widehat{\mathbf{x}}) - \widehat{v}(\widehat{\mathbf{y}})|^q \sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \sin 2\widehat{y}_1 \sin 2\widehat{y}_2}{((\sin^2 \widehat{x}_1 - \sin^2 \widehat{y}_1)^2 + (\sin^2 \widehat{x}_2 - \sin^2 \widehat{y}_2)^2)^2} \, d\widehat{x} \, d\widehat{y}. \tag{3.24}$$

Thus it will suffice to show that

$$\frac{1}{|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^4} \approx \frac{\sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \sin 2\widehat{y}_1 \sin 2\widehat{y}_2}{((\sin^2 \widehat{x}_1 - \sin^2 \widehat{y}_1)^2 + (\sin^2 \widehat{x}_2 - \sin^2 \widehat{y}_2)^2)^2}. \tag{3.25}$$

Firstly note that, for all $(\widehat{x}_1, \widehat{x}_2) \in \widehat{T}$

$$\frac{\sin \widehat{x}_1}{\sin \widehat{x}_2} \approx 1. \tag{3.26}$$

To show that (3.25) is true, consider these two cases,

1. Let $\widehat{\mathbf{y}} \longrightarrow \widehat{\mathbf{x}} \neq 0$. Denote $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2)$ and $\widehat{\mathbf{y}} = (\widehat{x}_1 + \eta_1, \widehat{x}_2 + \eta_2)$. Using (3.26) gives

$$\lim_{\widehat{\mathbf{y}} \to \widehat{\mathbf{x}}} \frac{|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^4 \sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \sin 2\widehat{y}_1 \sin 2\widehat{y}_2}{((\sin^2 \widehat{x}_1 - \sin^2 \widehat{y}_1)^2 + (\sin^2 \widehat{x}_2 - \sin^2 \widehat{y}_2)^2)^2}$$

$$\approx \lim_{\eta_1\to 0,\eta_2\to 0} \frac{(\eta_1^2+\eta_2^2)^2}{(\eta_1^2(1+\eta_1)^2+\eta_2^2(1+\eta_2)^2)^2}$$
$$\approx \lim_{\eta_1\to 0,\eta_2\to 0} \frac{(\eta_1^2+\eta_1^2)^2}{\eta_1^4+\eta_2^4} \approx C. \tag{3.27}$$

2. Now let $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}} \longrightarrow 0$. Using (3.26) gives

$$\lim_{\widehat{\mathbf{x}},\widehat{\mathbf{y}}\to 0} \frac{|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^4 \sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \sin 2\widehat{y}_1 \sin 2\widehat{y}_2}{((\sin^2\widehat{x}_1-\sin^2\widehat{y}_1)^2+(\sin^2\widehat{x}_2-\sin^2\widehat{y}_2)^2)^2}$$
$$\approx \lim_{\widehat{\mathbf{x}},\widehat{\mathbf{y}}\to 0} \frac{((\widehat{x}_1-\widehat{y}_1)^2+(\widehat{x}_2-\widehat{y}_2)^2)^2\widehat{x}_1\widehat{x}_2\widehat{y}_1\widehat{y}_2}{((\widehat{x}_1^2-\widehat{y}_1^2)^2+(\widehat{x}_2^2-\widehat{y}_2^2)^2)^2}$$
$$\approx \lim_{\widehat{\mathbf{x}},\widehat{\mathbf{y}}\to 0} \frac{((\widehat{x}_1-\widehat{y}_1)^2+(\widehat{x}_2-\widehat{y}_2)^2)\widehat{x}_1\widehat{y}_1}{(\widehat{x}_1^2-\widehat{y}_1^2)^2+(\widehat{x}_2^2-\widehat{y}_2^2)^2}$$
$$\approx \lim_{\widehat{\mathbf{x}},\widehat{\mathbf{y}}\to 0} \frac{(\widehat{x}_1-\widehat{y}_1)^2\widehat{x}_1\widehat{y}_1}{(\widehat{x}_1-\widehat{y}_1)^2(\widehat{x}_1+\widehat{y}_1)^2}$$
$$\approx \lim_{\widehat{\mathbf{x}},\widehat{\mathbf{y}}\to 0} \frac{\widehat{x}_1\widehat{y}_1}{(\widehat{x}_1+\widehat{y}_1)^2} \approx C \tag{3.28}$$

Hence, the first result follows.

For any $v \in W^{1,q}(T)$, $q \in [2,\infty)$

$$\int_{T/T^\Delta} |v|^q \, d\mathbf{x} = \int_{\widehat{T}/\widehat{T}^{\widehat{\Delta}}} |\widehat{v}|^q \sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \, d\widehat{\mathbf{x}} \tag{3.29}$$

and for $i = 1, 2$

$$\int_{T/T^\Delta} \left|\frac{\partial v}{\partial x_i}\right|^q d\mathbf{x} = \int_{\widehat{T}/\widehat{T}^{\widehat{\Delta}}} \left|\frac{\partial \widehat{v}}{\partial \widehat{x}_i}\right|^q \frac{\sin 2\widehat{x}_1 \sin 2\widehat{x}_2}{\sin 2\widehat{x}_i{}^q} \, d\widehat{\mathbf{x}} \tag{3.30}$$

Combining these results with (3.26) and the fact that $\widehat{\Delta} \equiv \Delta^{\frac{1}{2}}$ gives (3.13). ∎

Lemma 13 *For any $k \geq 0$, define $\widehat{u}^\Delta = u^\Delta \circ \Phi$. Then $\widehat{u}^\Delta \in W^{k,q}(\tilde{S}(\pi/2))$ and*

$$\left\|\widehat{u}^\Delta\right\|_{W^{k,q}(\tilde{S})} \leq C(k) g(|\log\Delta|)\Delta^{-\langle\langle k/2-\lambda\rangle-1/q\rangle} \tag{3.31}$$

Proof. Let $\Delta \in (0, 1/2)$ be fixed. Set $\hat{\Delta} = \arcsin\sqrt{\Delta/\kappa}$. From [27, page 19, section 0.43] for any index $k = (k_1, k_2)$

$$\frac{\partial^{|k|}\hat{u}^{\Delta}(\hat{\mathbf{x}})}{\partial \hat{x}_1^{k_1} \partial \hat{x}_1^{k_1}} = \sum_{j=1}^{k_1}\sum_{l=1}^{k_2} W_j^{k_1}(\hat{x}_1) W_l^{k_2}(\hat{x}_2) \frac{1}{j!l!} \frac{\partial^{j+l} u^{\Delta}}{\partial x_1^j \partial x_2^l}(\Phi(\hat{\mathbf{x}})), \tag{3.32}$$

where

$$W_m^n(t) := \sum_{j=1}^{m} (-1)^{m-j} \begin{pmatrix} k \\ l \end{pmatrix} \sin^{2(m-j)} t \frac{d^n}{dt^n} \sin^{2j} t. \tag{3.33}$$

Then from [12, Lemma 4.3] for $\min(\hat{x}_1, \hat{x}_2) \geq \hat{\rho}$ and property 3 of u^{Δ},

$$\left| \frac{\partial^{|m|}\hat{u}^{\Delta}}{\partial \hat{x}_1^{m_1} \partial \hat{x}_2^{m_2}} \right| \leq C(|m|) g(|\log \Delta|) \min(\hat{x}_1, \hat{x}_2)^{-(|m|-2\lambda)} \tag{3.34}$$

and for $\hat{\mathbf{x}} \in \tilde{\mathbf{S}}(\boldsymbol{\Delta})$

$$\left| \frac{\partial^{|m|}\hat{u}^{\Delta}\hat{\mathbf{x}}}{\partial \hat{x}_1^{m_1} \partial \hat{x}_2^{m_2}} \right|^q \leq C(m_1, m_2) \sum_{j=1}^{m_1}\sum_{l=1}^{m_2} g(|\log \Delta|) \hat{x}_1^{q(2j-m_1)} \hat{x}_2^{q(2l-m_2)} \Delta^{-q(j+l-\lambda)}. \tag{3.35}$$

The function $\hat{u}^{\Delta}$ is supported on the set

$$\hat{R} \cap (\tilde{S})(\pi/6) \subset G_1 \cup G_2 \cup G_3 \tag{3.36}$$

where

$$G_1 = \hat{R} \cap (\tilde{S})(\hat{\Delta}), \tag{3.37}$$

$$G_2 = \{(\hat{x}_1, \hat{x}_2) : \hat{\Delta} \leq \hat{x}_1 \leq \pi/6, \hat{\rho} \leq \hat{x}_2 \leq \hat{x}_1\} \tag{3.38}$$

and

$$G_3 = \{(\hat{x}_1, \hat{x}_2) : \hat{\Delta} \leq \hat{x}_2 \leq \pi/6, \hat{\rho} \leq \hat{x}_1 \leq \hat{x}_2\}. \tag{3.39}$$

The contributions from each subset will be considered individually. The fact that $\hat{\Delta} \leq C\Delta^{\frac{1}{2}}$ will be used.

1. On subset G_1 using (3.35) for any $|m| \leq k$ gives

$$\begin{aligned}
&\int_{G_1} \left| \frac{\partial^{|m|}\widehat{u}^{\Delta}}{\partial \widehat{x}_1^{m_1} \partial \widehat{x}_2^{m_2}} \right|^q d\widehat{x}_1 d\widehat{x}_2 \\
&\leq C(m_1, m_2) g(|\log \Delta|) \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} \Delta^{-q\langle j+l-\lambda\rangle} \int_0^{\widehat{\Delta}} \widehat{x}_1^{q\langle 2j-m_1\rangle} d\widehat{x}_1 \int_0^{\widehat{\Delta}} \widehat{x}_2^{q\langle 2l-m_2\rangle} d\widehat{x}_2 \\
&\leq C(m_1, m_2) \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} g(|\log \Delta|) \Delta^{q\langle j+l-\lambda\rangle + \frac{q}{2}\langle 2j-m_1\rangle + \frac{q}{2}\langle 2l-m_2\rangle + 1}
\end{aligned}$$

and

$$-\langle j+l-\lambda\rangle + \frac{1}{2}\langle 2j-m_1\rangle + \frac{1}{2}\langle 2l-m_2\rangle \geq -\langle \frac{k}{2} - \lambda\rangle \tag{3.40}$$

gives

$$\left\| \widehat{u}^{\Delta} \right\|_{W^{k,q}(G_1)} \leq C g(|\log \Delta|) \Delta^{\frac{1}{2}(2\lambda-k)_- + 1/q} \tag{3.41}$$

2. On the subset G_2

$$|D^m \widehat{u}^{\Delta}| \leq C(m) g(|\log \Delta|) \widehat{x}_2^{-\langle |m|-2\lambda\rangle} \tag{3.42}$$

and so

$$\begin{aligned}
\left\| D^m \widehat{u}^{\Delta} \right\|_{L^q(G_2)}^q &\leq C(m) g(|\log \Delta|) \int_{\widehat{\Delta}}^{\pi/6} \int_{\widehat{\rho}}^{\widehat{x}_1} \widehat{x}_2^{-\langle |m|-2\lambda\rangle} d\widehat{x}_1 d\widehat{x}_2 \\
&\leq C\widehat{\Delta}^{-q\langle\langle |m|-2\lambda\rangle - 2/q\rangle}
\end{aligned}$$

giving

$$\left\| \widehat{u}^{\Delta} \right\|_{W^{k,q}(G_2)} \leq C(k) \Delta^{-\langle\langle k/2-\lambda\rangle - 1/q\rangle}. \tag{3.43}$$

3. The treatment of G_3 is essentially the same as G_2.

Summing contributions from each of the subsets completes the proof. ∎

The following Lemma can now be stated

Lemma 14 *Let u^Δ be a regularised singular function given by (3.6) and suppose $k \geq 2\lambda + 2q$, $q \in [2, \infty)$, and that $u^\Delta = 0$ on the lines $x_1 = \kappa x_2/2$ and $x_2 = \kappa x_1/2$. Let K be an element with one vertex at $\mathbf{A}$, and two of its sides lying on the lines $x_1 = \kappa x_2/2$ and $x_2 = \kappa x_1/2$ and satisfying $K \subset R_\kappa \cap \tilde{S}(1)$. Then there exists a sequence of polynomials u_p^Δ of degree p, with $u_p^\Delta = 0$ on the edges of the element K and*

$$\left\| u^\Delta - u_p^\Delta \right\|_{W^{1,q}(K)} \leq C(k)(1 + \ln p)^{2(1-2/q)} g(|\log \Delta|) \Delta^{-(\frac{k}{2} - \lambda - 1/q)}$$
$$\left[p^{-(k-1)} \Delta^{-(1/2-1/q)} + p^{-(k-2+2/q)} \Delta^{-(1-2/q)} + p^{-(k-2)} \right] \quad (3.44)$$

Proof. Firstly note that, by definition, supp $u^\Delta \subset R_\kappa$. Let K_κ be an open polygonal domain such that $K \subset K_\kappa \subset \tilde{S}(1)$ and supp $u^\Delta \subset K_\kappa$

Extend the function $\widehat{u}^\Delta$ to the square square $S(\pi)$ as an even periodic function so that the extended function is symmetric on the lines $\widehat{x}_i = 0, \pm\pi$ for $i = 1, 2$. Let $s_p(\widehat{u}^\Delta)$ be $p-$th partial sum of the Fourier series expansion of $\widehat{u}^\Delta$. Then by Lemma 3 and Lemma 13, for any $0 \leq m \leq k$, $k \geq 2\lambda + 2/q$

$$\begin{aligned} &\left\| \widehat{u}^\Delta - s_p(\widehat{u}^\Delta) \right\|_{W^{m,q}(\tilde{S}(\frac{\pi}{2}))} \\ &\leq C(k)(1 + \ln p)^{2(1-2/q)} p^{-(k-m)} \left\| \widehat{u}^\Delta \right\|_{W^{k,q}(\tilde{S}(\frac{\pi}{2}))} \\ &\leq C(k)(1 + \ln p)^{2(1-2/q)} p^{-(k-m)} g(|\log \Delta|) \Delta^{-(\frac{k}{2} - \lambda - 1/q)}. \end{aligned} \quad (3.45)$$

By Lemma 3

$$\left\| \widehat{u}^\Delta - s_p(\widehat{u}^\Delta) \right\|_{L^\infty(\tilde{S}(\frac{\pi}{2}))} \leq C(k)(1 + \ln p)^{2(1-2/q)} p^{-(k-2/q)} g(|\log \Delta|) \Delta^{-(\frac{k}{2} - \lambda - 1/q)} \quad (3.46)$$

From Schmidt's inequality, Markov's inequality and using interpolation the following inverse estimate holds: let v be a polynomial of degree p and Ω be an open domain with $\mathrm{diam}(\Omega) = h$, then for $q \in [2, \infty]$ and $k \geq m \geq 0$

$$\|v\|_{W^{k,q}(\Omega)} \leq Ch^{-(k-m)} p^{2(k-m)} \|v\|_{W^{m,q}(\Omega)} . \tag{3.47}$$

Let $v_p^\Delta = s_p(\widehat{u}^\Delta) \circ \Phi^{-1}$ which, by symmetry is an algebraic polynomial, the restriction of v_p^Δ to the element K also denoted by v_p^Δ satisfies

$$\left\|u^\Delta - v_p^\Delta\right\|_{W^{1,q}(K)}^q \leq \left\|u^\Delta - v_p^\Delta\right\|_{W^{1,q}(K^\Delta)}^q + \left\|u^\Delta - v_p^\Delta\right\|_{W^{1,q}(K/K^\Delta)}^q . \tag{3.48}$$

Each term on the right hand side of (3.48) will be considered individually. Since, $u^\Delta \equiv 0$ on K^Δ, using the inverse estimate (3.47) with $h = \Delta$, Lemma 12 and (3.45) gives

$$\begin{aligned}
\left\|u^\Delta - v_p^\Delta\right\|_{W^{1,q}(K^\Delta)} &\leq \left\|v_p^\Delta\right\|_{W^{1,q}(K^\Delta)} \\
&\leq Cp^{2(1-2/q)} \Delta^{-(1-2/q)} \left\|u^\Delta - v_p^\Delta\right\|_{W^{2/q,q}(K)} \\
&\leq Cp^{2(1-2/q)} \Delta^{-(1-2/q)} \left\|\widehat{u}^\Delta - s_p(\widehat{u}^\Delta)\right\|_{W^{2/q,q}(\tilde{S}(\frac{\pi}{2}))} \\
&\leq C(k) p^{-(k-2+2/q)} \Delta^{-(1-2/q)} (1 + \ln p)^{2(1-2/q)} \\
&\qquad g(|\log \Delta|) \Delta^{-(\frac{k}{2} - \lambda - 1/q)} .
\end{aligned} \tag{3.49}$$

For the second term of (3.48), using Lemma 12 and (3.45) gives

$$\begin{aligned}
\left\|u^\Delta - v_p^\Delta\right\|_{W^{1,q}(K/K^\Delta)} &\leq C\Delta^{-(1/2-1/q)} \left\|\widehat{u}^\Delta - s_p(\widehat{u}^\Delta)\right\|_{W^{1,q}(\tilde{S}(\frac{\pi}{2}))} \\
&\leq C(k) p^{-(k-1)} \Delta^{-(1/2-1/q)} (1 + \ln p)^{2(1-2/q)} \\
&\qquad g(|\log \Delta|) \Delta^{-(\frac{k}{2} - \lambda - 1/q)} .
\end{aligned} \tag{3.50}$$

Combining (3.48), (3.49) and (3.50) gives

$$\begin{aligned}\left\|u^{\Delta}-v_p^{\Delta}\right\|_{W^{1,q}(K)} \leq\ & C(k)(1+\ln p)^{2(1-2/q)}g(|\log\Delta|)\Delta^{-(\frac{k}{2}-\lambda-1/q)} \\ & \left[p^{-(k-1)}\Delta^{-(1/2-1/q)}+p^{-(k-2+2/q)}\Delta^{-(1-2/q)}\right].\end{aligned} \tag{3.51}$$

Estimate (3.46) is preserved under the transformation. Therefore,

$$\left\|u^{\Delta}-v_p^{\Delta}\right\|_{L^{\infty}(K)} \leq C(k)(1+\ln p)^{2(1-2/q)}p^{-(k-2/q)}g(|\log\Delta|)\Delta^{-(\frac{k}{2}-\lambda-1/q)}. \tag{3.52}$$

Assume that K is a quadrilateral element; the proof for triangular elements is similar, as in the proof of Theorem 7 construct a bilinear bijective map F_K from the reference element $S(1)$ to the element K such that, for all $u \in W^{m,q}(K)$

$$\|u\|_{W^{m,q}(K)} \approx \|\tilde{u}\|_{W^{m,q}(S(1))} \tag{3.53}$$

and denote the coordinates in the reference element by $(\tilde{x}_1, \tilde{x}_2) = F_K^{-1}(x_1, x_2)$ and $\tilde{u} = F_K^{-1} \circ u$. Now adjust v_p^{Δ} to give a new polynomial w_p^{Δ} which vanishes at the vertices. A typical adjustment, say, at the vertex $(-1,1)$ on the reference element is given by,

$$\alpha_1(\tilde{x}_1,\tilde{x}_2) = v_p^{\Delta}(-1,1)\psi_1(\tilde{x}_1)\psi_1(\tilde{x}_2) \tag{3.54}$$

where ψ_1 is given in the proof of Theorem 7 and the fact that $u^{\Delta} = 0$ on the boundary of K has been used. Denote by α_i, $i = 1, \ldots, 4$ the adjustments at the vertices of K. Therefore, from (3.51) and (3.52)

$$
\begin{aligned}
\left\|u^{\Delta}-w_p^{\Delta}\right\|_{W^{1,q}(K)} &\leq \left\|u^{\Delta}-v_p^{\Delta}\right\|_{W^{1,q}(K_\kappa)}+\sum_{i=1}^{4}\|\alpha_i\|_{W^{1,q}(K)} \\
&\leq \left\|u^{\Delta}-v_p^{\Delta}\right\|_{W^{1,q}(K_\kappa)}+C\sum_{i=1}^{4}\left\|u^{\Delta}-v_p^{\Delta}\right\|_{L^\infty(K)} \\
&\leq C(1+\ln p)^{2(1-2/q)}p^{-(k-2/q)}g(|\log\Delta|)\Delta^{-(\frac{k}{2}-\lambda-1/q)} \\
&\leq C(k)(1+\ln p)^{2(1-2/q)}g(|\log\Delta|)\Delta^{-(\frac{k}{2}-\lambda-1/q)} \\
&\left[p^{-(k-1)}\Delta^{-(1/2-1/q)}+p^{-(k-2+2/q)}\Delta^{-(1-2/q)}+p^{-(k-2/q)}\right]. \quad (3.55)
\end{aligned}
$$

To obtain the desired polynomial, adjustments on the edges of the element must now be done. These adjustments are slightly different to those of Theorem 7. Consider the polynomial $\tilde{\beta}_p^1$ given by

$$
\tilde{\beta}_p^1(\tilde{x}_1,\tilde{x}_2)=\tilde{w}_p^{\Delta}(-1,\tilde{x}_2)\psi_1(\tilde{x}_1). \quad (3.56)
$$

Transforming this polynomial to the element K and denoting $\beta_p^1=F_K\circ\tilde{\beta}_p^1$, it is clear that, on the boundary of K, $\beta_p^1(x_1,\kappa x_1/2)=w_p^{\Delta}(x_1,\kappa x_1/2)$ on one side and is zero on all other sides of the element K. Subtracting this polynomial from w_p^{Δ} gives a polynomial which is zero along the side $2x_1=\kappa x_2$ and

$$
\left\|u^{\Delta}-(w_p^{\Delta}-\beta_p^1)\right\|_{W^{1,q}(K)} \leq \left\|u^{\Delta}-w_p^{\Delta}\right\|_{W^{1,q}(K)}+\left\|\beta_p^1\right\|_{W^{1,q}(K)}. \quad (3.57)
$$

Using the previous results the following estimate holds for β_p^1,

$$
\begin{aligned}
\left\|\beta_p^1\right\|_{W^{1,q}(K)} &\leq C\left\|w_p^{\Delta}\right\|_{W^{1,q}(\partial K)} \\
&\leq C\left\|v_p^{\Delta}\right\|_{W^{1,q}(\partial K)}+\left\|u^{\Delta}-v_p^{\Delta}\right\|_{L^\infty(K)} \\
&\leq Cp^{2(1-1/q)}\left\|v_p^{\Delta}\right\|_{W^{1/q,q}(\partial K)}+\left\|u^{\Delta}-v_p^{\Delta}\right\|_{L^\infty(K)}
\end{aligned}
$$

$$\leq \; Cp^{2(1-1/q)} \left\| u^{\Delta} - v_p^{\Delta} \right\|_{W^{2/q,q}(K_\kappa)} + \left\| u^{\Delta} - v_p^{\Delta} \right\|_{L^\infty(K)}$$

$$\leq \; C(k)(1+\ln p)^{2(1-2/q)} g(|\log \Delta|) \Delta^{-(\frac{k}{2}-\lambda-1/q)}$$

$$\left[p^{-(k-1)} \Delta^{-(1/2-1/q)} + p^{-(k-2+2/q)} \Delta^{-(1-2/q)} + p^{-(k-2)} \right], \quad (3.58)$$

where the fact that $q \geq 1$ has been used.

Hence, denoting by β_p^i, $i = 1, \ldots, 4$, the similar adjustments on the four sides of K and letting $u_p^{\Delta} = w_p^{\Delta} - \sum_{i=1}^{4} \beta_p^i$ gives a polynomial which is zero on all the sides of the element K and combining the above results, satisfies the required estimate (3.44). ∎

The main result of this section can now be stated:

Theorem 15 *Let K be an element with a vertex at the origin and two of its sides on the lines $\theta = \theta_0$ and $\theta = \pi/2 - \theta_0$ where $0 < \theta_0 < \pi/4$. Suppose $q \in [2, \infty)$ and let u be given by (3.2), with $\lambda > 1 - 2/q$ and*

$$u(r, \theta_0) = u(r, \pi/2 - \theta_0) = 0 \quad (3.59)$$

and

$$supp\ u^{\Delta} \subset K_{\theta_0}, \quad (3.60)$$

where $K_{\theta_0} = (K \cup \{(r,\theta) : \theta_0/2 < \theta < \pi/2 - \theta_0/2\}) \cap S(1/4)$.

Then there exists a sequence of polynomials u_p of degree p that vanish on the boundary of the element K and given $\epsilon > 0$ there exists a $k_0 > 0$ such that for all

$k > k_0$

$$\|u - u_p\|_{W^{1,q}(K)} \leq C(k)(1 + \ln p)^{2(1-2/q)} p^{-2(\lambda-1+2/q-\epsilon)}. \tag{3.61}$$

Proof. From Lemma 11 and Lemma 14 there exists a sequence of polynomials u_p^Δ such for all $k \geq 2\lambda + 2/q$

$$\begin{aligned} \left\|u - u_p^\Delta\right\|_{W^{1,q}(K)} &\leq \left\|u - u^\Delta\right\|_{W^{1,q}(K)} + \left\|u^\Delta - u_p^\Delta\right\|_{W^{1,q}(K)} \\ &\leq Cg(|\log\Delta|)\Delta^{\lambda-1+2/q} + C(k)(1+\ln p)^{2(1-2/q)} g(|\log\Delta|)\Delta^{-(\frac{k}{2}-\lambda-1/q)} \\ &\quad \left[p^{-(k-1)}\Delta^{-(1/2-1/q)} + p^{-(k-2+2/q)}\Delta^{-(1-2/q)} + p^{-(k-2)}\right]. \end{aligned} \tag{3.62}$$

Now for given $\epsilon > 0$ there exists a k_0 such that

$$\mu := \frac{k_0 - 2}{k_0/2 - 1/q} \geq 2 - \epsilon. \tag{3.63}$$

Choosing $\Delta = p^{-\mu}$ gives for any $k \geq k_0$

$$\begin{aligned} \|u - u_p\|_{W^{1,q}(K)} &\leq Cg(|\log p|)p^{-2(\lambda-1+2/q-\epsilon)} + \\ &\quad C(k)(1+\ln p)^{2(1-2/q)} g(|\log p|)p^{k-2-2\lambda+\epsilon} \\ &\leq \left[p^{-(k-1)}p^{-2(1/q-1/2)} + p^{-(k-2+2/q)}p^{-2(2/q-1)} + p^{-(k-2)}\right] \\ &\leq C(k)(1+\ln p)^{2(1-2/q)} p^{-2(\lambda-1+2/q-\epsilon)} \end{aligned} \tag{3.64}$$

as required. ∎

Remark 16 *When the element K is the union of two elements K_1 and K_2 then similar results to those of Lemma 14 and Theorem 15 hold. Although the final polynomial is zero on the boundary of K and not necessarily on the interelement* *boundary $\overline{K}_1 \cap \overline{K}_2$. The reason for this is that no adjustments of the initial approximation v_p^{Δ}, given in the proof of Lemma 14, are required on the interelement boundary other than those given by the vertex adjustments, which do not effect the result since $u^{\Delta}(\mathbf{A}) = 0$ for every vertex* **A**, *of the elements K_1 and K_2.*

3.3 The convergence rate

The results of the previous two chapters are now joined together, to give the general uniform p-version estimate for functions u given by (1.17).

Theorem 17 *Let u be of the form described by (1.17), with sufficiently smooth boundary data. If $q \in [2, \infty)$ and $\underline{\lambda} > 1 - 2/q$ where*

$$\underline{\lambda} \quad = \quad \min\{\lambda_1, \ldots, \lambda_M\} \tag{3.65}$$

then given $\epsilon > 0$ there exists a sequence $\{u_p\} \in X_p$ such that

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq C(\epsilon)(1 + \ln p)^{2(1-2/q)} g(|\log p|) p^{-\sigma}, \tag{3.66}$$

where

$$\sigma = \min\{m - 1, 2(\underline{\lambda} - 1 + 2/q - \epsilon)\}. \tag{3.67}$$

Proof. It suffices to consider the case when there is one singular element $u_3 = w$ say, and will be given by (3.2), with exponent $\lambda = \underline{\lambda}$ associated with the corner **A**. Let, for any partition $\mathcal{P}$, the patch of elements surrounding the corner **A** to be given by Figure 3.2. Let the line joining **A** to $\mathbf{B}_j$ have angular coordinate θ_j.

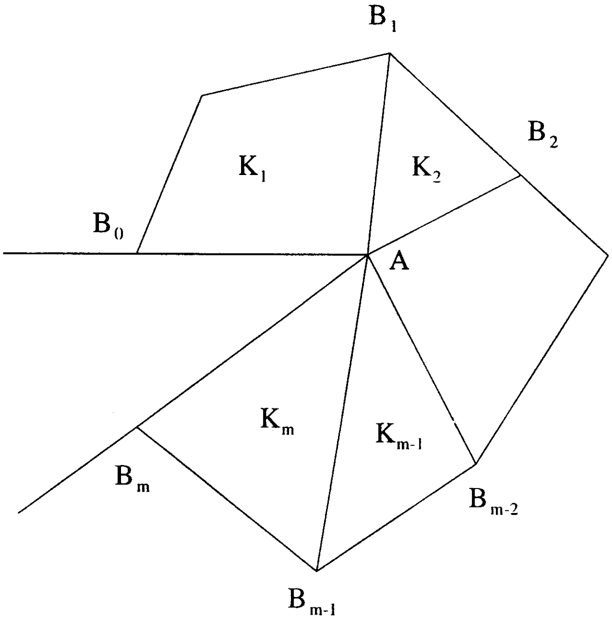

Figure 3.2: patch of elements associated with corner **A**

The function $\Theta(\cdot)$ maybe partitioned into a sum of smooth functions $\Theta^{[j]} \in C^{\infty}[0, 2\pi]$ and are supported on $(\theta_{j-1}, \theta_{j+1})$. Therefore, w may be expressed as,

$$w = \sum_{j=1}^{m-1} w^{[j]} \tag{3.68}$$

where

$$w^{[j]} = \chi(r) r^{\lambda} g(|\log r|) \Theta^{[j]}(\theta). \tag{3.69}$$

Hence, approximation of the functions $w^{[j]}$ must be considered. To do this, two cases must be considered:

1. $\theta_{j+1} - \theta_{j-1} < \pi$:

   In this case, a linear map $\mathcal{F}$ may be applied, so that the two elements are mapped onto a region R_{κ}. Therefore, from Theorem 15 and Remark

16 there exists a continuous piecewise polynomial $w_p^{[j]}$ that is zero on all elements in the partition, except for the two elements K_j and K_{j-1}, and is such that for any $\epsilon > 0$

$$\left\| w^{[j]} - w_p^{[j]} \right\|_{W^{1,q}(\Omega)} \leq C(\epsilon)(1 + \ln p)^{2(1-2/q)} g(|\log p|) p^{-2(\lambda-1+2/q-\epsilon)} \quad (3.70)$$

2. $\theta_{j+1} - \theta_{j-1} > \pi$:

In this case, firstly apply the linear map $\mathcal{G} : K_j \to \tilde{K}_j$ which is such that $\overline{OB}_j$ is mapped onto itself and $\tilde{\theta}_{j+1} - \theta_{j-1} < \pi$, see Figure 3.3 and Figure 3.4.

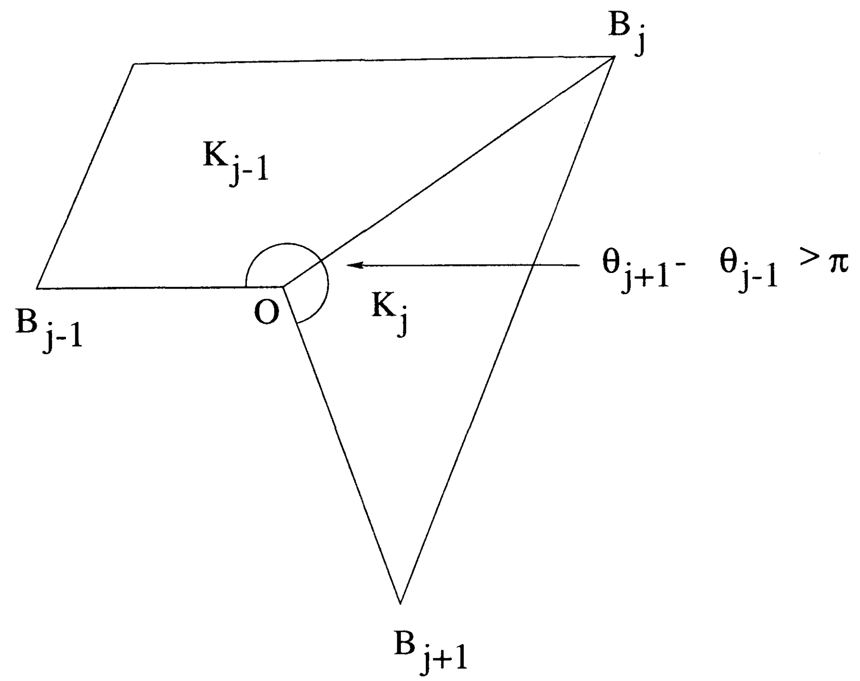

Figure 3.3: Two elements, with $\theta_{i+1} - \theta_{i-1} > \pi$

The function $w^{[j]}|_{\overline{K}_{j-1}}$ may be smoothly extended onto $\tilde{K}_j$, denote the extension by $\tilde{v}^{[j]}$ to give a function that is supported on the interior of $\overline{K}_{j-1} \cup \overline{\tilde{K}}_j$.

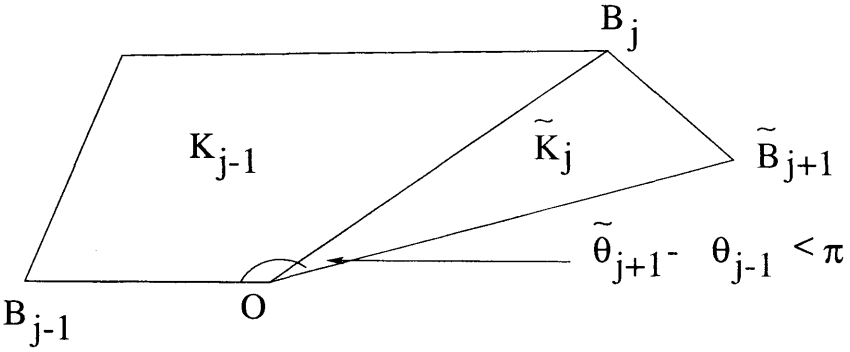

Figure 3.4: Elements after the map $\mathcal{G}$ is Applied

The function $w^{[j]}|_{\overline{K}_{j-1}} + v^{[j]}$, where $v^{[j]} = \hat{v}^{[j]} \circ \mathcal{G}^{-1}$, is approximated by a polynomial $w^{[j]}_{1,p}$ which is given by Theorem 15. Therefore, from Remark 16

$$\left\| w^{[j]} - w^{[j]}_{1,p} \right\|_{W^{1,q}(K_{j-1})} \leq C(\epsilon)(1+\ln p)^{2(1-2/q)} g(|\log p|)p^{-2(\underline{\lambda}-1+2/q-\epsilon)} \quad (3.71)$$

and

$$\left\| v^{[j]} - w^{[j]}_{1,p} \right\|_{W^{1,q}(K_j)} \leq C(\epsilon)(1+\ln p)^{2(1-2/q)} g(|\log p|)p^{-2(\underline{\lambda}-1+2/q-\epsilon)}. \quad (3.72)$$

Note that, the function $w^{[j]}_{1,p}$ is a piecewise continuous polynomial on the two elements.

The function $w^{[j]} - v^{[j]}$ on K_j satisfies the conditions of Theorem 15, that is that the function is zero on the boundary of the element. Hence, there exists a polynomial $w^{[j]}_{2,p}$ defined on K_j, which is zero on the boundary and

$$\left\| (w^{[j]} - v^{[j]}) - w^{[j]}_{2,p} \right\|_{W^{1,q}(K_j)} \leq C(\epsilon)(1+\ln p)^{2(1-2/q)} g(|\log p|)p^{-2(\underline{\lambda}-1+2/q-\epsilon)} \quad (3.73)$$

Combining these two polynomials gives a piecewise continuous polynomial $w_p^{[j]}$ satisfying

$$\left\| w^{[j]} - w_p^{[j]} \right\|_{W^{1,q}(K_j)} \leq \left\| w^{[j]} - w_{1,p}^{[j]} \right\|_{W^{1,q}(K_j)} \tag{3.74}$$

and

$$\begin{aligned} \left\| w^{[j]} - w_p^{[j]} \right\|_{W^{1,q}(K_{j+1})} \leq & \left\| (w^{[j]} - v^{[j]}) - w_{2,p}^{[j]} \right\|_{W^{1,q}(K_{j+1})} \\ & + \left\| v^{[j]} - w_{1,p}^{[j]} \right\|_{W^{1,q}(K_{j+1})}. \end{aligned} \tag{3.75}$$

Therefore, using the above estimates gives a piecewise polynomial satisfying the required result.

That completes the proof. ■

Chapter 4

The hp-version Approximation Theory

The aim of this chapter is to obtain approximation results for the hp-version finite element method. The results given will extend those of Babuska and Suri, [11], to general Sobolev spaces and reduce the restrictions on the meshes, so that local refinements in both h and p may be considered. An example of such refinement is the strong refinement used around a corner singularity.

So far it has been assumed that a partition $\mathcal{P}$ is fixed and high order polynomial approximation was used. This chapter will retain the notation from chapter 1 for the partition and obtain estimates involving both h and p.

In section 1 the approximation of smooth functions will be considered, as in chapter 2. The results obtained not only introduce the element sizes h_K, $K \in \mathcal{P}$, but also remove the restriction of uniform p refinement allowing each element K

to have its individual degree p_K. A local estimate will also be provided for the case $m > 1$ when $u \in W^{m,q}(\Omega)$, which improves, in terms of the p-version, the estimate given in Theorem 8.

In section 2 hp results will be obtained for singular functions of the type described earlier. Combining these results will give the main hp result.

4.1 Continuous Piecewise Polynomial Approximation of Smooth Functions

The following extra notation will be required. For each $K \in \mathcal{P}$, let $m_K > 1$ be such that $u \in W^{m_K,q}(K)$. Now define $\mathbf{m}_{\mathcal{P}}$ to be the set of all such m_K for $K \in \mathcal{P}$ and the space

$$W^{\mathbf{m}_{\mathcal{P}},q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_K \in W^{m_K,q}(K)\}. \tag{4.1}$$

The corresponding boundary space is given by,

$$W^{\mathbf{m}_{\mathcal{P}},q}(\Gamma_D) = \{g \in W^{1+1/q,q}(\Gamma_D) : g|_{\overline{K}\cap\Gamma_D} \in W^{m_K+1-2/q,q}(\overline{K} \cap \Gamma_D)\}. \tag{4.2}$$

The extra smoothness required on the boundary is due to Theorem 10.

For now, it will be assumed that the function u to be approximated is given by (1.17) with $u_2 = u_3 = 0$.

Assume $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$, $q \in [1, \infty]$ and $m_K > 1$ for all $K \in \mathcal{P}$. Define $\widehat{\Omega}_K = H_K^{-1}(\Omega_K)$, where H_K is the affine map given by

$$H_K(\widehat{x}_1, \widehat{x}_2) = (h_K\widehat{x}_1, h_K\widehat{x}_2), \quad \text{for all } (\widehat{x}_1, \widehat{x}_2) \in \widehat{\Omega}_K, \tag{4.3}$$

for each $K \in \mathcal{P}$. The following properties now hold:

- $\operatorname{diam}(\widehat{J}) \approx 1$, for all $\widehat{J} \subset \widehat{\Omega}_K$ where $\widehat{J} = F_K^{-1}(J)$,

- $\rho_{\widehat{J}} \approx 1$, for all $\widehat{J} \subset \widehat{\Omega}_K$,

- for every $u \in W^{\mathbf{m}_p,q}(\Omega)$, denote $m_{\Omega_K} = \min_{J \subset \Omega_K} m_J$ and

$$\widehat{u}_K = u|_{\Omega_K} \circ H_K^{-1}; \tag{4.4}$$

  then $\widehat{u}_K \in W^{m_{\Omega_K},q}(\widehat{\Omega}_K)$,

- for every index α, $|\alpha| \leq m_K$,

$$\|D^\alpha u\|_{L^q(J)} \leq C h_K^{2/q-|\alpha|} \left\| \widehat{D}^\alpha \widehat{u}_K \right\|_{L^q(\widehat{J})} \tag{4.5}$$

  and

$$\left\| \widehat{D}^\alpha \widehat{u}_K \right\|_{L^q(\widehat{J})} \leq C h_K^{|\alpha|-2/q} \|D^\alpha u\|_{L^q(J)} \,. \tag{4.6}$$

For any $m = s + \sigma \in \mathbb{R}^+$, $s \in \mathbb{Z}^+$ and $\sigma \in [0,1)$ define

$$\lfloor m \rfloor = s. \tag{4.7}$$

Finally define $P^p(\Omega_K) = \{v : v \text{ is a polynomial of degree } p \text{ on } \Omega_K\}$.

The following Lemma will be required:

Lemma 18 *Let $u \in W^{k,q}(\Omega_K)$, $k \geq 0$. Then*

$$\inf_{\widehat{v} \in P^p(\widehat{\Omega}_K)} \|\widehat{u} - \widehat{v}\|_{W^{k,q}(\widehat{\Omega}_K)} \leq C(k) h_K^{\mu-2/q} |u|_{W^{k,q}(\Omega_K)}, \tag{4.8}$$

where $\mu = \min(p+1, k)$.

Proof. Firstly assume that k is an integer. The case $k = 0$ follows immediately from the properties of the linear map H_K. Now

$$\inf_{\widehat{v}\in P^p(\widehat{\Omega}_K)} \|\widehat{u}-\widehat{v}\|_{W^{k,q}(\widehat{\Omega}_K)} \le \inf_{\widehat{v}\in P^p(\widehat{\Omega}_K)} \left\{ \|\widehat{u}-\widehat{v}\|_{W^{\mu,q}(\widehat{\Omega}_K)} + \sum_{l=\mu+1}^{k} |\widehat{u}|_{W^{l,q}(\widehat{\Omega}_K)} + \sum_{l=\mu+1}^{k} |\widehat{v}|_{W^{l,q}(\widehat{\Omega}_K)} \right\} \tag{4.9}$$

and $\sum_{l=\mu+1}^{k} |\widehat{v}|_{W^{l,q}(\widehat{\Omega}_K)} \equiv 0$ for $k > \mu+1$. Using [24, Theorem 3.1.1] gives

$$\inf_{\widehat{v}\in P^p(\widehat{\Omega}_K)} \|(\widehat{u}-\widehat{v})\|_{W^{k,q}(\widehat{\Omega}_K)} \le \sum_{l=\mu}^{k} |\widehat{u}|_{W^{l,q}(\widehat{\Omega}_K)} \tag{4.10}$$

Mapping back to the original domain Ω_K gives the result for integer k. The result for general k follows, using standard interpolation. ∎

The first local estimate can now be stated.

Lemma 19 *Let $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$, $q \in [1,\infty]$ and $m_K > 1 + 1/q$ for all $K \in \mathcal{P}$ be such that supp $u \subset \Omega_K$ for a $K \in \mathcal{P}$. Then there exists a sequence $u_{hp} \in X_{hp}$ which is independent of q such that,*

$$\|u-u_{hp}\|_{W^{1,q}(K)} \le C(\mu) p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1+\log p_{\Omega_K})^{2|1-2/q|} \tag{4.11}$$

$$\left[h_K^{\mu-1} |u|_{W^{\mu,q}(\Omega_K)} + \sum_{|\alpha|>\mu}^{\lfloor m_{\Omega_K}\rfloor} h_K^{|\alpha|-1} \|D^\alpha u\|_{L^q(\Omega_K)} + h_K^{m_{\Omega_K}-1} |u|_{W^{m_{\Omega_K},q}(\Omega_K)} \right] \tag{4.12}$$

where $\mu = \min(p_{\Omega_K}+1, m_{\Omega_K})$, $p_{\Omega_K} = \min_{J\in\Omega_K} p_J$ and supp $u_{hp} \subset \Omega_K$.

Proof. First note that the finite dimensional space

$$\tilde{V} = \Big\{ v \in C(\overline{\Omega}) : v|_K = \widehat{v} \circ G_K^{-1} \text{ for some } \widehat{v} \in \widehat{P}(p_{\Omega_K}) \text{ or } \widehat{Q}(p_{\Omega_K}) \text{ for all } K \subset \Omega_K \text{ and } v \equiv 0 \text{ for all } K \subset \Omega/\Omega_K \Big\} \tag{4.13}$$

is a subspace of X_{hp}. Hence, there exists a $u_{hp} \in X_{hp}$ such that

$$\|u - u_{hp}\|_{W^{1,q}(K)} \leq \inf_{v \in \tilde{V}} \|u - v\|_{W^{1,q}(K)} . \tag{4.14}$$

Therefore, it is sufficient to construct an approximation of uniform degree p_{Ω_K}, which satisfies the Theorem. To construct such a piecewise continuous polynomial the ideas in the proof of Theorem 7 are required. That is, construct individual polynomials w_K of degree no more than p_{Ω_K} on each element $K \in \mathcal{P}$. Since $u \equiv 0$ on every $J \subset \Omega/\Omega_K$ it follows that $w_J \equiv 0$ for every such $J \subset \Omega/\Omega_K$. Note that the w_J, $J \subset \Omega/\Omega_K$, need never be adjusted. Denoting $w \in \tilde{V}$ to be the final piecewise continuous polynomial constructed in this manner, it is clear from (4.14) and Theorem 7 that there exists a $u_{hp} \in X_{hp}$ such that

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} &\leq \|u - w\|_{W^{1,q}(K)} \\ &\leq C p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1 + \log p_{\Omega_K})^{2|1-2/q|} \\ &\qquad \sum_{J \in \mathcal{P}: \overline{K} \cap \overline{J} \neq \emptyset} \|u\|_{W^{m_{\Omega_K},q}(J)} . \end{aligned} \tag{4.15}$$

and supp $u_{hp} \subset \Omega_K$.

Let $v \in P^{p_{\Omega_K}}(\Omega_K)$. Then

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} &= \|(u - v) - (u_{hp} - v)\|_{W^{1,q}(K)} \\ &\leq C h_K^{2/q-1} \|(\hat{u} - \hat{v}) - (\hat{u}_{hp} - \hat{v})\|_{W^{1,q}(\hat{K})} \\ &\leq C h_K^{2/q-1} p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1 + \log p_{\Omega_K})^{2|1-2/q|} \\ &\qquad \|(\hat{u} - \hat{v})\|_{W^{m_{\Omega_K},q}(\widehat{\Omega}_K)} \\ &\leq C h_K^{2/q-1} p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1 + \log p_{\Omega_K})^{2|1-2/q|} \end{aligned}$$

$$\left[\|\widehat{u}-\widehat{v}\|_{W^{\mu,q}(\widehat{\Omega}_K)}+\sum_{|\alpha|>\mu}^{\lfloor m_{\Omega_K}\rfloor}\left\|\widehat{\mathrm{D}}^{\alpha}\widehat{u}\right\|_{L^q(\widehat{\Omega}_K)}+|\widehat{u}|_{W^{m_{\Omega_K},q}(\widehat{\Omega}_K)}\right]. \quad (4.16)$$

From Lemma 18, with $p = p_{\Omega_K}$ and $k = \mu$,

$$\inf_{\widehat{v}\in P^{p_{\Omega_K}}(\widehat{\Omega}_K)}\|(\widehat{u}-\widehat{v})\|_{W^{\mu,q}(\widehat{\Omega}_K)}\leq C(\mu)h_K^{\mu-2/q}|u|_{W^{\mu,q}(\widehat{\Omega}_K)}. \quad (4.17)$$

Combining (4.16), (4.17) and mapping back to the original domain gives,

$$\begin{aligned}\|u-u_{hp}\|_{W^{1,q}(K)} &\leq C(\mu)h_K^{2/q-1}p_{\Omega_K}^{-(m_{\Omega_K}-1)}(1+\log p_{\Omega_K})^{2|1-2/q|}\\ &\left[|\widehat{u}|_{W^{\mu,q}(\widehat{\Omega}_K)}+\sum_{|\alpha|>\mu}^{\lfloor m_{\Omega_K}\rfloor}\left\|\widehat{\mathrm{D}}^{\alpha}\widehat{u}\right\|_{L^q(\widehat{\Omega}_K)}+|\widehat{u}|_{W^{m_{\Omega_K},q}(\Omega_K)}\right]\\ &\leq C(\mu)p_{\Omega_K}^{-(m_{\Omega_K}-1)}(1+\log p_{\Omega_K})^{2|1-2/q|}\\ &\left[h_K^{\mu-1}|u|_{W^{\mu,q}(\Omega_K)}+\sum_{|\alpha|>\mu}^{\lfloor m_{\Omega_K}\rfloor}h_K^{|\alpha|-1}\|\mathrm{D}^{\alpha}u\|_{L^q(\Omega_K)}\right.\\ &\left.+h_K^{m_{\Omega_K}-1}|u|_{W^{m_{\Omega_K},q}(\Omega_K)}\right], \end{aligned} \quad (4.18)$$

as required. ∎

The general result for functions $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$ with the minimal smoothness on u and the restriction on the support of u removed, can now be proved.

Theorem 20 *Let $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$, $q \in [1,\infty]$ and $m_K > 1$ for all $K \in \mathcal{P}$. Then there exists a sequence $u_{hp} \in X_{hp}$ which are independent of q such that for $K \in \mathcal{P}$ and $m_{\Omega_K} \in (1, 1+1/q]$*

$$\begin{aligned}\|u-u_{hp}\|_{W^{1,q}(K)} &\leq C(\mu)p_{\Omega_K}^{-(m_{\Omega_K}-1)}(1+\log p_{\Omega_K})^{2|1-2/q|}\\ &\qquad h_K^{\mu-1}\|u\|_{W^{m_{\Omega_K},q}(\Omega_K)}, \end{aligned} \quad (4.19)$$

where $\mu = m_{\Omega_K}$ *and for* $m_{\Omega_K} \in (1 + 1/q, \infty)$

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} &\leq C(\mu) p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1 + \log p_{\Omega_K})^{2|1-2/q|} \qquad (4.20) \\ & \left[h_K^{\mu-1} |u|_{W^{\mu,q}(\Omega_K)} + \sum_{|\alpha|>\mu}^{\lfloor m_{\Omega_K} \rfloor} h_K^{|\alpha|-1} \|D^\alpha u\|_{L^q(\Omega_K)} \right. \\ & \left. + h_K^{m_{\Omega_K}-1} |u|_{W^{m_{\Omega_K},q}(\Omega_K)} \right] \qquad (4.21) \end{aligned}$$

where $\mu = \min(m_{\Omega_K}, p_{\Omega_K} + 1)$ *and* $p_{\Omega_K} = \min_{J \in \Omega_K} p_J$.

Proof. The result will firstly be shown to hold for the case $h_K \approx 1$.

Let e_n, $n = 1, 2, \ldots, N_{\mathcal{P}}$ denote the vertices in the partition $\mathcal{P}$. With each e_n associate a bounded open domain $U_n = \mathrm{int}\{\cup \overline{K} : \overline{K} \cap e_n \neq \emptyset\}$. Note that $U_n \subset \Omega_K$, where K is any element contained in U_n, and $\{\cup U_n : n = 1, 2, \ldots, N_{\mathcal{P}}\} = \Omega$.

Now construct a partition of unity ϕ_n, $n = 1, 2, \ldots N_{\mathcal{P}}$, subordinate to the covering U_n. Write $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$ in the following manner

$$\begin{aligned} u &\equiv \sum_{n=1,2,\ldots,N_{\mathcal{P}}} \phi_n u \\ &:= \sum_{n=1,2,\ldots,N_{\mathcal{P}}} u_n, \end{aligned} \qquad (4.22)$$

with supp $u_n \subset U_n$ and $u_n \in W^{m_K,q}(K)$, see figure. The aim is now to construct polynomial approximations to each u_n.

In the case $m_{\Omega_K} > 1 + 1/q$ for some $K \in \mathcal{P}$, Lemma 19 states that there exists an approximation $u_{n,hp} \in X_{hp}$ to the function u_n, where n is such that $e_n \cap \overline{K} \neq \emptyset$, such that

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} &\leq C(\mu) p_{\Omega_K}^{-(m_{\Omega_K}-1)} \\ & (1 + \log p_{\Omega_K})^{2|1-2/q|} \|u_n\|_{W^{m_{\Omega_K},q}(\Omega_K)} \qquad (4.23) \end{aligned}$$

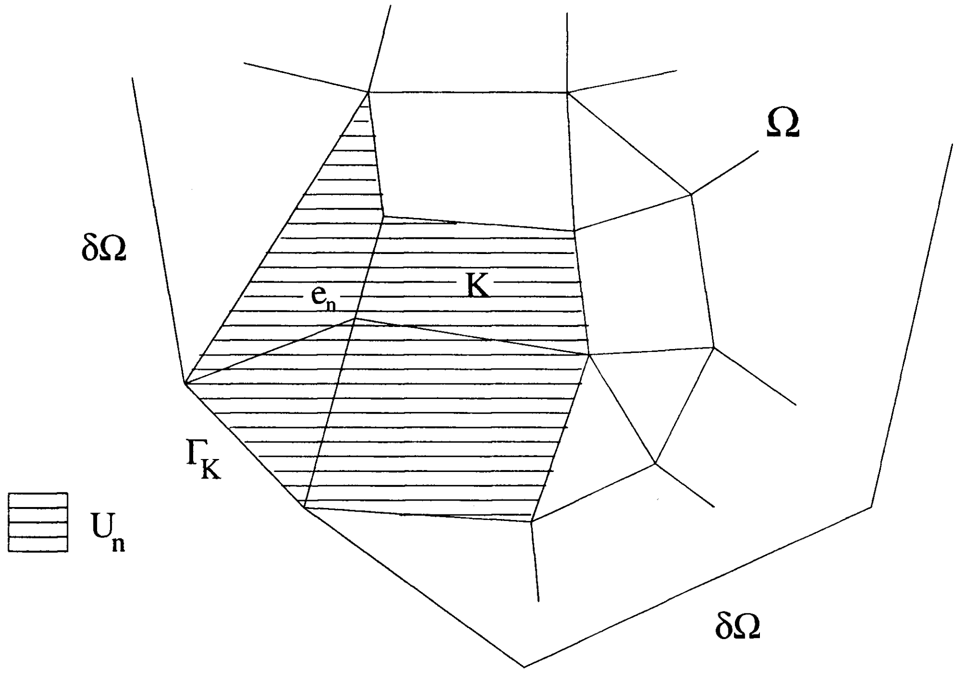

Figure 4.1: A general patch of elements, showing the support of a ϕ_n.

and supp $u_{n,hp} \subset \Omega_K$.

Now consider the case when $m_{\Omega_K} \in (1, 1+1/q]$ for some $K \in \mathcal{P}$. From the characterisation

$$W^{m_{\Omega_K},q}(\Omega_K) = \left(W^{1,q}(\Omega_K), W^{2,q}(\Omega_K)\right)_{\theta,q}, \tag{4.24}$$

where $\theta = m_{\Omega_K} - 1$ and the proof of Theorem 8, it can be seen that for any $t_K > 0$, u_n with n such that supp $u_n \in U_n$ may be decomposed as $u_n = v_{1,n}(t_K) + v_{2,n}(t_K)$ with $v_{1,n} \in W^{1,q}(\Omega_K)$ and $v_{2,n} \in W^{2,q}(\Omega_K)$ satisfying

$$\|v_{1,n}\|_{W^{1,q}(\Omega_K)} \leq C t_K^{m_{\Omega_K}-1} \|u_n\|_{W^{m_{\Omega_K},q}(\Omega_K)} \tag{4.25}$$

$$\|v_{2,n}\|_{W^{2,q}(\Omega_K)} \leq C t_K^{m_{\Omega_K}-2} \|u_n\|_{W^{m_{\Omega_K},q}(\Omega_K)}. \tag{4.26}$$

Using Lemma 19 to construct approximations $v_{2,n,hp}$ to the functions $v_{2,n}$, with supp $v_{2,n,hp} \subset \Omega_K$ and

$$\begin{aligned}\|v_{2,n} - v_{2,n,hp}\|_{W^{1,q}(K)} &\leq Cp_{\Omega_K}^{-1}(1+\log p_{\Omega_K})^{2|1-2/q|}|v_{2,n}|_{W^{2,q}(\Omega_K)} \\ &\leq Cp_{\Omega_K}^{-1} t^{m_{\Omega_K}-2}(1+\log p_{\Omega_K})^{2|1-2/q|}\,\|u_n\|_{W^{m_{\Omega_K},q}(\Omega_K)}\,. \quad (4.27)\end{aligned}$$

Apply similar decompositions for every element $K \in \mathcal{P}$ in which $m_{\Omega_K} \leq 1 + 1/q$.

Choosing $t_K = 1/p_{\Omega_K}$ for every such K and using the triangle inequality gives

$$\begin{aligned}\|u_n - u_{n,hp}\|_{W^{1,q}(K)} &\leq \|v_{2,n} - v_{2,n,hp}\|_{W^{1,q}(K)} + \|v_{1,n}\|_{W^{1,q}(K)} \\ &\leq Cp_{\Omega_K}^{-(m_{\Omega_K}-1)}(1+\log p_{\Omega_K})^{2|1-2/q|}\,\|u_n\|_{W^{\Omega_K,q}(m_{\Omega_K})}\,, \quad (4.28)\end{aligned}$$

where $u_{n,hp} := v_{2,n,hp}$ for all n such that, when $\overline{K} \cap e_n \neq \emptyset$, then $m_{\Omega_K} \leq 1 + 1/q$.

Defining $u_{hp} = \sum_{n=1}^{N_{\mathcal{P}}} u_{n,hp}$ gives

$$\begin{aligned}\|u - u_{hp}\|_{W^{1,q}(K)} &= \left\|\sum_{n=1}^{N_{\mathcal{P}}}\{u - u_{hp}\}\right\|_{W^{1,q}(K)} \\ &\leq \sum_{n:e_n\cap\overline{K}\neq\emptyset} \|u_n - u_{n,hp}\|_{W^{1,q}(K)}\,. \quad (4.29)\end{aligned}$$

Combining (4.23), (4.28), (4.29) and noting, for the case $h_K \approx 1$, that

$$\|u_n\|_{W^{m_{\Omega_K},q}(\Omega_K)} \leq C\,\|u\|_{W^{m_{\Omega_K},q}(\Omega_K)}\,, \quad (4.30)$$

gives a $u_{hp} \in X_{hp}$ such that

$$\|u - u_{hp}\|_{W^{1,q}(K)} = Cp_{\Omega_K}^{-(m_{\Omega_K}-1)}(1+\log p_{\Omega_K})^{2|1-2/q|}\,\|u\|_{W^{\Omega_K,q}(\Omega_K)}\,. \quad (4.31)$$

The general hp results are now obtained by using (4.31) as follows. Let $v \in P^{p_{\Omega_K}}(\Omega_K)$ then from the above there exist a $u_{hp} \in X_{hp}$ such that

$$\begin{aligned}\|u - u_{hp}\|_{W^{1,q}(K)} &\leq Ch_K^{2/q-1} \|(\widehat{u} - \widehat{v}) - (\widehat{u}_{hp} - \widehat{v})\|_{W^{1,q}(\widehat{K})} \\ &\leq Ch_K^{2/q-1} p_{\Omega_K}^{-(m_{\Omega_K}-1)} \\ &\qquad (1 + \log p_{\Omega_K})^{2|1-2/q|} \|\widehat{u} - \widehat{v}\|_{W^{m_{\Omega_K},q}(\widehat{\Omega}_K)} . \qquad (4.32)\end{aligned}$$

Combining the above inequality with Lemma 19 gives the required result. ■

The following global estimate is an immediate consequence of the above Theorem.

Corollary 21 *Let $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$, $q \in [1,\infty]$ and $m_K > 1$ for all $K \in \mathcal{P}$. Then there exists a sequence $u_{hp} \in X_{hp}$ which are independent of q such that,*

$$\begin{aligned}\|u - u_{hp}\|_{W^{1,q}(\Omega)} &\leq Cp^{-(m-1)}(1 + \log p)^{2|1-2/q|} \qquad (4.33) \\ &\qquad \left[C(\mu)h^{\mu-1}|u|_{W^{\mu,q}(\Omega)} + \sum_{|\alpha|>\mu}^{m} h^{|\alpha|-1} \|D^\alpha u\|_{L^q(\Omega)} \right]\end{aligned}$$

where $\mu = \min(p+1, m)$, $h = \max_{J\in\mathcal{P}} h_J$, $p = \min_{J\in\mathcal{P}} p_J$ and $m = \min_{J\in\mathcal{P}} m_J$ i.e. $u \in W^{m,q}(\Omega)$.

Note that, for the case $q = 2$ the above global estimate is the same as that given by Babuska and Suri in [11].

4.2 Non-homogeneous Dirichlet boundary data

Now suppose that the function to be approximated is given by (1.17) with $u_2 \neq 0$ and $u_3 = 0$. Following chapter 2, assume that g given in (1.17) is approximated using Chebyshev polynomials.

Denote the approximate Dirichlet data for an element K having an edge $\gamma_K = \Gamma_D \cap \overline{K}$ by g_K. The approximation to be used will be the p_K-th partial sum of the Chebyshev series expansion of $\widehat{g}_K$, where

$$\widehat{g}_K = g|_K \circ \mathcal{F}_K, \tag{4.34}$$

and $\mathcal{F}_K$ is used to define the restriction of the map F_K to the edge γ_K. The approximation will be denoted by $\sigma_{hp,K}(\widehat{g}_K)$. The resulting global approximation to the Dirichlet data is given by:

$$g_{hp} := \sum_{K \in \mathcal{P}: \overline{K} \cap \Gamma_D \neq \emptyset} \widehat{g}_{hp,K} \circ \mathcal{F}_K^{-1} \tag{4.35}$$

where

$$\begin{aligned} \widehat{g}_{hp,K}(t) &= \{\widehat{g}_K(-1) - \sigma_{hp,K}(\widehat{g}_K; -1)\}\psi_{p_{\Omega_K}}(t) + \\ & \quad \{\widehat{g}_K(1) - \sigma_{hp,K}(\widehat{g}_K; 1)\}\psi_{p_{\Omega_K}}(-t) + \sigma_{hp,K}(\widehat{g}_K; t). \end{aligned} \tag{4.36}$$

Lemma 22 *Let $g \in W^{l,q}(\gamma_K)$, for some $K \in \mathcal{P}$ where $q \in [1, \infty]$. Then for $l > 2/q$*

$$\|g - \sigma_{hp,K}(g)\|_{L^q(\gamma_K)} \leq C(1 + \log p_{\Omega_K}) p_{\Omega_K}^{-l} h_K^{\mu} \|g\|_{W^{l,q}(\gamma_K)} \tag{4.37}$$

and for $l > 2 - 1/q$

$$\|g - \sigma_{hp,K}(g)\|_{W^{1,q}(\gamma_K)} \leq C(1 + \log p_{\Omega_K}) p_{\Omega_K}^{-(l-2+1/q)} h_K^{\mu-1} \|g\|_{W^{l,q}(\gamma_K)} \tag{4.38}$$

where $\mu = \min(p_{\Omega_K} + 1, l)$.

Proof. Mapping γ_K onto $(-1, 1)$, then combining Lemma 9, the equivalent one dimensional result of Lemma 18, and noting that the length of $\gamma_K \approx h_K$ gives the required estimate. ∎

The main result of this section can now be stated:

Theorem 23 *Let $u \in W^{\mathbf{m}_{\mathcal{P}},q}(\Omega)$ be given by (1.17) with $u_3 \equiv 0$ and assume $g \in W^{\mathbf{m}_{\mathcal{P}},q}(\Gamma_D)$ where $q \in [1,\infty]$. Then there exists a sequence of polynomials $u_{hp} \in X_{hp}$ such that $u_{hp} = g_{hp}$ on the Dirichlet boundary Γ_D and the following local estimate holds*

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} \leq\ & C(\mu) p_{\Omega_K}^{-(m_{\Omega_K}-1)} (1+\log p_{\Omega_K})^{2|1-2/q|} h_K^{\mu-1} \\ & \{\|u\|_{W^{m_{\Omega_K},q}(\Omega_K)} + \|g\|_{W^{m_{\Omega_K}+1-2/q,q}(\Gamma_K)}\} \end{aligned} \tag{4.39}$$

where $\mu = \min(p_{\Omega_K}+1, m_{\Omega_K})$, $p_{\Omega_K} = \min_{J\in\Omega_K} p_J$ and $\Gamma_K = \overline{\Omega}_K \cap \Gamma_D$.

Proof. Note that the function is of the form $u = u_1 + u_2$ with u_2 satisfying the boundary conditions and that $u_2 \in W^{m_K,q}(K)$, $m_K > 1 + 1/q$ for all $K \in \mathcal{P}$. For this proof it is sufficient to assume that $u = u_2$.

Let w_{hp} denote the sequence of polynomials constructed in Theorem 20. Assume, without loss of generality, that $\mathcal{F}_K(\gamma_K) = \{(x_1,-1) : -1 \leq x_1 \leq 1\}$ and let $\widehat{v}_{hp,K}$ be the polynomial

$$\widehat{v}_{hp,K} = \widehat{w}_{hp}(x_1,x_2) + (\sigma_{hp,K}(\widehat{g}_K; x_1) - \widehat{w}_{hp}(x_1,-1))\psi_{p_{\Omega_K}}(x_2). \tag{4.40}$$

Due to the construction of w_{hp} it is clear that, see proof of Theorem 7,

$$\begin{aligned} \|\widehat{u} - \widehat{w}_{hp}\|_{W^{j,q}(-1,1)} \leq\ & C p_{\Omega_K}^{-(m_{\Omega_K}-j-1/q)} \\ & (1+\log p_{\Omega_K})^{2(|1-2/q|)} \|u\|_{W^{m_{\Omega_K},q}(\widehat{\Omega_K})} \end{aligned} \tag{4.41}$$

for $j = 0, 1$, since $m_{\Omega_K} > 1 + 1/q$.

Hence, combining Lemma 22, using the triangle inequality and mapping back to the original domain, Ω_K, gives

$$\begin{aligned} \|u - v_{hp}\|_{W^{1,q}(K)} &\leq C\,\|u - w_{hp}\|_{W^{1,q}(K)} + \\ & C(1+\log p_{\Omega_K})^{2|1-2/q|} p_{\Omega_K}^{-(m_{\Omega_K}-1)} h_K^{\mu-1}\, \|g\|_{W^{m_{\Omega_K}+1-2/q,q}(\Gamma_K)}. \end{aligned} \quad (4.42)$$

For every element with a boundary on the Dirichlet boundary, a similar polynomial is constructed. This gives a piecewise continuous polynomial satisfying (4.39). ∎

4.3 Piecewise Polynomial Approximation of Corner Singularities

The aim of this section is to obtain an hp estimate for the rate of convergence of sequences of polynomials approximating functions of the form

$$u(\mathbf{x}) = cr^{\lambda}|\log r|^{\gamma}\Theta(\theta), \quad (4.43)$$

where (r,θ) are polar coordinates with origin at $\mathbf{A}$, a corner of the domain Ω, with Θ assumed to be a smooth $(C^{\infty}(\overline{\Omega}))$ function which vanishes along the edges corresponding to the boundary of the domain Ω. γ integer and $\lambda > 1 - 2/q$ for some $q \in [2, \infty)$.

The function (4.43) is not quite the same as a typical singular function of u_3. Firstly the function of $|\log r|$ is given and secondly the smooth cutoff function

ζ is not included. The latter is due to the fact that ζ is mesh dependent. The following notation will be required.

Denote, for any partition $\mathcal{P}$, the patch of elements surrounding the corner $\mathbf{A}$ by $K_1, K_2, \cdots K_m$, see figure 3.2.

Let $\zeta \in C^\infty(\overline{\Omega})$ be a function of r only and be such that

$$\zeta(r) = \begin{cases} 1, & r < 1/4 \\ 0, & r > 1/2 \end{cases} \tag{4.44}$$

Partition the function $\Theta(\cdot)$ into a sum of smooth functions $\Theta_i(\cdot) \in C^\infty([0, 2\pi])$ supported on $(\omega_{i-1}, \omega_{i+1})$. From the properties (3) and (4) of $\mathcal{P}$ we have

$$\min_{i=0,1,\cdots,m} |AB_i| = c_1 h_{\mathbf{A}}$$

where $h_{\mathbf{A}} \approx h_{K_i}$ for $i = 1, 2, \ldots m$, also denote $\min_{i=1,2,\ldots m} p_{K_i} := p_{\mathbf{A}}$. Now write u in the form

$$\begin{aligned} u(\mathbf{x}) &= \sum_{i=1}^{m-1} \zeta(r/c_1 h_{\mathbf{A}})\Theta_i(\theta) r^\lambda |\log r|^\gamma + \Theta(\theta) r^\lambda |\log r|^\gamma (1 - \zeta(r/c_1 h_{\mathbf{A}})) \\ &:= w_1(\mathbf{x}) + w_2(\mathbf{x}) \end{aligned} \tag{4.45}$$

and note that

1. $w_1 \in W^{\lambda+2/q,q}(\Omega)$

2. $w_2 \in C^\infty(\overline{\Omega})$

3. supp $w_1 \subset B(\mathbf{A}, c_1 h_{\mathbf{A}}/2)$

First look at approximating the function w_1; to do this consider each term in the sum individually. Denote

$$v_i = \zeta(r/c_1 h_{\mathbf{A}})\Theta_i(\theta)r^{\lambda}|\log r|^{\gamma} \tag{4.46}$$

giving supp $v_i \subset K_{i,i+1}$, where $K_{i,i+1} = \text{int}\left(\overline{K}_i \cup \overline{K}_{i+1}\right)$

Theorem 24 *Let v_i be given by (4.46); then given $\epsilon > 0$ there exists a sequence of polynomials $v_{hp,i} \in X_{hp}$ with supp $v_{hp,i} \subset K_{i,i+1}$ which are such that, for any $q \in [2,\infty)$, $j = \{1,2,\ldots,m\}$ and $\lambda > 1-2/q$*

$$\|v_i - v_{hp,i}\|_{W^{1,q}(K_j)} \le C(\epsilon)h_{\mathbf{A}}^{\lambda-1+2/q}C(h_{\mathbf{A}},p_{\mathbf{A}},\gamma)p_{\mathbf{A}}^{-2(\lambda-1+2/q-\epsilon)} \tag{4.47}$$

where the constant is independent of both $h_{\mathbf{A}}$ and $p_{\mathbf{A}}$, but dependent on the function v_i, and

$$C(h_{\mathbf{A}},p_{\mathbf{A}},\gamma) = \max\left(|\log h_{\mathbf{A}}|^{\gamma}, |\log p_{\mathbf{A}}|^{\gamma}\right) \tag{4.48}$$

Thus

$$w_{1,hp} := \sum_{i=1}^{m-1} v_{hp,i} \tag{4.49}$$

is such that

$$\|w_1 - w_{1,hp}\|_{W^{1,q}(K_j)} \le Ch_{\mathbf{A}}^{\lambda-1+2/q}C(h_{\mathbf{A}},p_{\mathbf{A}},\gamma)p_{\mathbf{A}}^{-2(\lambda-1+2/q)} \tag{4.50}$$

Proof. Following the proof of Theorem 17, it may be assumed that the elements $K_{i,i+1}$, can be enclosed inside a square $\tilde{s}(c_2 h_{\mathbf{A}})$ where $\tilde{s}(\rho) = \{(x_1,x_2) : 0 < x_1 < \rho, 0 < x_1 < \rho\}$, and appropriate rotations of global coordinates have been used to obtain the coordinate system seen in the figure below.

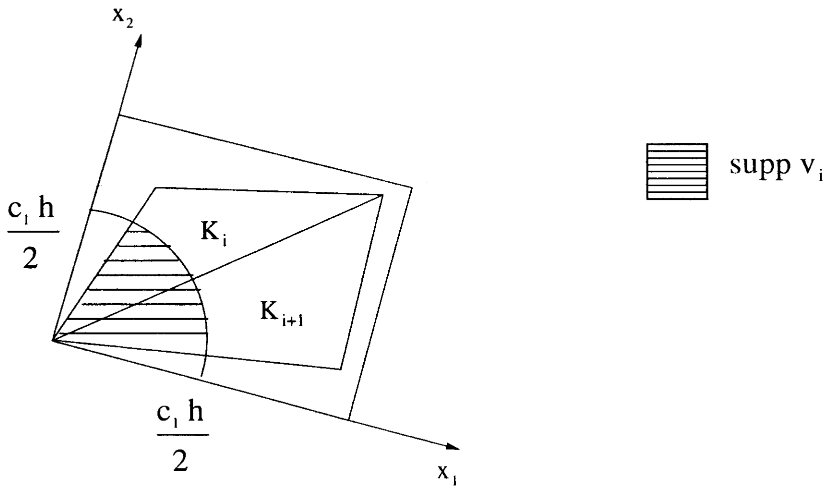

Figure 4.2: New coordinate system

Mapping $K_{i,i+1}$ onto $\widehat{K}_{i,i+1}$, where supp $\widehat{K}_{i,i+1} \subset \mathring{s}(1)$, under the map F_i^{-1}, where

$$F_i(\widehat{\mathbf{x}}) := c_2 h_{\mathbf{A}} \widehat{\mathbf{x}}, \quad \forall \widehat{\mathbf{x}} \in \mathring{s}(1), \tag{4.51}$$

gives

$$\begin{aligned} v_i(\mathbf{x}) &= c h_{\mathbf{A}}^{\lambda} \zeta(c_2 \widehat{r}/c_1) |\log r|^{\gamma} \Theta(\theta) \widehat{r}^{\lambda}, \\ &= \sum_{l=0}^{\gamma} C(l) h_{\mathbf{A}}^{\lambda} \zeta(c_2 \widehat{r}/c_1) \widehat{r}^{\lambda} |\log \widehat{r}|^{\gamma-l} |\log h_{\mathbf{A}}|^{l}, \\ &:= \sum_{l=0}^{\gamma} C(l) h_{\mathbf{A}}^{\lambda} |\log h_{\mathbf{A}}|^{l} \widehat{v}_i^l, \end{aligned} \tag{4.52}$$

where

$$\widehat{r} = (\widehat{x}_1^2 + \widehat{x}_2^2)^{1/2}. \tag{4.53}$$

The form of each $\widehat{v}_i^l$ is such that, given $\epsilon > 0$ there exists a $\widehat{v}_{hp,i}^l \in \widehat{Q}(p_{\mathbf{A}})$, see Theorem 17, such that

$$\left\| \widehat{v}_i^l - \widehat{v}_{hp,i}^l \right\|_{W^{1,q}(\widehat{K})} \leq C(\epsilon) p_{\mathbf{A}}^{-2(\lambda - 1 + 2/q - \epsilon)} |\log p_{\mathbf{A}}|^{\gamma - l}, \tag{4.54}$$

and the constant C is independent of both $h_{\mathbf{A}}$ and $p_{\mathbf{A}}$. Defining

$$v_{hp,i} := \sum_{l=0}^{\gamma} C(l) h_{\mathbf{A}}^{\lambda} |\log h_{\mathbf{A}}|^{l} (\widehat{v}_{hp,i}^{l} \circ F_i), \tag{4.55}$$

and then combining (4.52) and (4.54) gives

$$\begin{aligned} \|v_i - v_{hp,i}\|_{W^{1,q}(K)} &\leq C(\epsilon) h_{\mathbf{A}}^{\lambda-1} h_{\mathbf{A}}^{2/q} C(h_{\mathbf{A}}, p_{\mathbf{A}}, \gamma) \|\widehat{v}_i - \widehat{v}_{hp,i}\|_{W^{1,q}(\widehat{K})} \\ &\leq CC(h_{\mathbf{A}}, p_{\mathbf{A}}, \gamma) h_{\mathbf{A}}^{\lambda-1+2/q} p_{\mathbf{A}}^{-2(\lambda-1+2/q-\epsilon)} \qquad (4.56) \end{aligned}$$

and the constant is independent of both $h_{\mathbf{A}}$ and $p_{\mathbf{A}}$, but dependent on the function v_i. Estimate (4.50) follows using the triangle inequality. ∎

Now consider approximating the smooth part w_2, this must be done since this function is dependent on $h_{\mathbf{A}}$. Firstly note that $w_2 \in C^{\infty}(\Omega)$ and that when the function is outside the patch surrounding the corner $\mathbf{A}$, it has no $h_{\mathbf{A}}$ dependence, since the function $\zeta(r/c_2 h_{\mathbf{A}}) \equiv 0$. Therefore, only approximation on the patch of elements surrounding the corner $\mathbf{A}$ need be considered.

Theorem 25 *Let w_2 be of the form given in (4.45); then there exists a sequence of polynomials $w_{2,hp} \in X_{hp}$ such that for any $i = 1, 2, \ldots, m$, $k \in (1, \infty)$ and $q \in [1, \infty]$*

$$\begin{aligned} \|w_2 - w_{2,hp}\|_{W^{1,q}(K_i)} &\leq C(\Theta, \alpha) C(h_{\mathbf{A}}, p_{\mathbf{A}}, \gamma) p_{\mathbf{A}}^{-(k-1)} \\ &\qquad (1 + \log p_{\mathbf{A}})^{2|1-2/q|} h_{\mathbf{A}}^{\lambda-1+2/q} \qquad (4.57) \end{aligned}$$

where $C(\Theta, \alpha)$ is a constant dependent on Θ restricted to Ω_{K_i} and $C(h_{\mathbf{A}}, p_{\mathbf{A}}, \gamma)$ is defined in the previous theorem.

Proof. From Theorem 20 for any $q \in [1,\infty]$ there exists a $w_{2,hp} \in X_{hp}$ independent of q such that for any $k \in (1,\infty)$

$$\begin{aligned}
\|w_2 - w_{hp}\|_{W^{1,q}(K_i)} \leq\ & C(\mu)p_{\mathbf{A}}^{-(k-1)}(1+\log p_{\mathbf{A}})^{2|1-2/q|} \\
& \left[h_{\mathbf{A}}^{\mu-1}|w_2|_{W^{\mu,q}(\Omega_{K_i})} + \sum_{\alpha>\mu}^{\lfloor k \rfloor} h_{\mathbf{A}}^{|\alpha|-1} \|\mathrm{D}^{\alpha} w_2\|_{L^q(\Omega_{K_i})} \right. \\
& \left. h_{\mathbf{A}}^{k-1}|w_2|_{W^{k,q}(\Omega_{K_i})} \right]
\end{aligned} \tag{4.58}$$

where $\mu = \min(p_{\mathbf{A}}+1,k)$ and $C(\mu)$ independent of both $h_{\mathbf{A}}$ and $p_{\mathbf{A}}$, for $i = 1,2,\ldots,m$.

Since $\zeta(r/c_2 h_{\mathbf{A}}) \equiv 0$ on the elements away from the corner, the above norms of w_2 may be restricted to the elements K_j, for $j = i-1, 1, 1+1$. Also for any index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2$ the following hold:

1. $w_2 \equiv 0$ for $r < c_1 h_{\mathbf{A}}/4$,

2.
$$|D^{\alpha} w_2| \leq C(\Theta,\alpha) r^{\lambda-|\alpha|} |\log r|^{\lambda}. \tag{4.59}$$

Therefore, for any $i = 1,2,\ldots,m$

$$\begin{aligned}
\|D^{\alpha} w_2\|^q_{L^q(K_i)} &= \int_{K_i} |D^{\alpha} w_2|^q \, \mathrm{d}\mathbf{x} \\
&\leq C(\Theta,\alpha) \int_{r=h_{\mathbf{A}}c_1/4}^{ch_{\mathbf{A}}} [r^{\lambda-|\alpha|} |\log r|^{\lambda}]^q r \, \mathrm{d}r \\
&\leq C(\Theta,\alpha) h_{\mathbf{A}}^{q\lambda - q|\alpha| + 2} |\log h_{\mathbf{A}}|^{q\lambda}.
\end{aligned} \tag{4.60}$$

Combining (4.58) and (4.60) for any $k \in (1,\infty)$,

$$\begin{aligned}
\|w_2 - w_{hp}\|_{W^{1,q}(K_i)} \leq\ & C(\Theta,\alpha) p_{\mathbf{A}}^{-(k-1)} (1+\log p_{\mathbf{A}})^{2|1-2/q|} \\
& |\log h_{\mathbf{A}}|^{\lambda} h_{\mathbf{A}}^{\lambda-1+2/q}
\end{aligned} \tag{4.61}$$

which is the required result. ∎

4.4 The Main Result

The main approximation result can now be stated. This result will be used in the proceeding chapter, in the application of the finite element method to non-linear elliptic problems.

Theorem 26 *Let u be of the form described in (1.17) with each u_3^j to be given by:*

$$u_3^j(\mathbf{x}) = c_j r_j^{\lambda_j} |\log r_j|^{\gamma_j} \Theta_j(\theta_j), \tag{4.62}$$

where (r_j, θ_j) are the polar coordinates of $\mathbf{x}$ relative to the point $\mathbf{A}_j$, the Θ_j are sufficiently smooth functions and γ_j are non negative integers.

If $\underline{\lambda} > 1 - 2/q$, where

$$\underline{\lambda} = \min\{\lambda_1, \cdots, \lambda_n\} \tag{4.63}$$

and the Dirichlet data g is sufficiently smooth, then given $\epsilon > 0$ there exists a sequence of piecewise continuous polynomials $u_{hp} \in X_{hp}$ such that

$$\begin{aligned} \|u - u_{hp}\|_{W^{1,q}(K)} \leq\; & C(u|_{\Omega_K}, \epsilon) C(h_K, p_{\Omega_K}, \beta) \\ & p_{\Omega_K}^{-\sigma} h_K^{\mu} (\log p_{\Omega_K} + 1)^{2(1-2/q)}, \end{aligned} \tag{4.64}$$

where

$$\sigma = \begin{cases} \min_j\{m_{\Omega_K} - 1, 2(\lambda_j - 1 + 2/q - \epsilon)\} & \overline{\Omega_K} \cap A_j \neq \emptyset \\ m_{\Omega_K} - 1 & \overline{\Omega_K} \cap A_j = \emptyset \end{cases} \tag{4.65}$$

$$\mu = \begin{cases} \min_j\{m_{\Omega_K} - 1, p_{\Omega_K}, \lambda_j - 1 + 2/q\} & \overline{\Omega_K} \cap A_j \neq \emptyset \\ \min\{m_{\Omega_K} - 1, p_{\Omega_K}\} & \overline{\Omega_K} \cap A_j = \emptyset \end{cases} \tag{4.66}$$

$$\beta = \begin{cases} \min_j \gamma_j & \overline{\Omega_K} \cap A_j \neq \emptyset \\ 0 & \overline{\Omega_K} \cap A_j = \emptyset \end{cases} \tag{4.67}$$

Proof. By considering the approximation of each function given by (1.17) individually and using Theorem 23, Theorem 24 and Theorem 25, along with the triangle inequality gives the required result. ■

Chapter 5

Application to Finite Element Approximation of Non-linear Elliptic Problems

5.1 A priori Estimates

The approximation results from the previous three chapters will now be used in the application of the finite element method to the model problem, the α-Laplacian, given by (1.9), which is recalled, in its weak form: Find $u \in W^{1,\alpha}(\Omega)$ such that $u = g$ on Γ_D and

$$\int_\Omega |\nabla u|^{\alpha-2}\nabla u \cdot \nabla v \,\mathrm{dx} \;=\; \int_\Omega fv\,\mathrm{dx} + \int_{\Gamma_N} hv\,\mathrm{ds}, \tag{5.1}$$

for all $v \in W_D^{1,\alpha}(\Omega)$, where $W_D^{1,\alpha}(\Omega) = \{v : v \in W^{1,\alpha}(\Omega) : v = 0 \text{ on } \Gamma_D\}$. In chapter 1, it was assumed that this problem and the equivalent finite element problem were well posed. This assumption is now justified in the following Theorem for the case $\Gamma_N = \emptyset$ and $g = 0$, the extension to the general problem follows immediately.

Theorem 27 *The variational problems (1.16) and (1.21) both have unique solutions. Furthermore the solutions $u \in W^{1,\alpha}(\Omega)$ and $u_{hp} \in X_{hp}$ of (1.16) and (1.21) respectively, are also the unique solutions of the minimisation problems: find $u \in V$ such that*

$$J(u) \leq J(v), \qquad \textit{for all } v \in V, \tag{5.2}$$

and

$$J(u_{hp}) \leq J(v), \qquad \textit{for all } v \in X_{hp}, \tag{5.3}$$

respectively, where

$$J(v) = \frac{1}{\alpha}\int_\Omega |\nabla v|^\alpha \, \mathrm{d}\mathbf{x} - \int_\Omega f v \, \mathrm{d}\mathbf{x}. \tag{5.4}$$

Finally,

$$\|u\|_{W^{1,\alpha}(\Omega)}, \|u_{hp}\|_{W^{1,\alpha}(\Omega)} \leq C \begin{cases} \max(1, \|f\|^*)^{\frac{1}{\alpha-1}}, & 1 < \alpha \leq 2 \\ \|f\|^*, & 2 \leq \alpha < \infty \end{cases} \tag{5.5}$$

where $\|\cdot\|^*$ is the dual norm of $W^{1,\alpha}(\Omega)$.

Proof. When the h-version of the finite element method is considered, since Ω is a polygonal domain, the result follows from [24, Theorem 5.3.1].

Following the proof of [24, Theorem 5.3.1] and using the results obtained in chapter 2, in particular the use of Theorem 8 to make the last equation on [24, page 316] hold, it is clear that the existence and uniqueness for the p- and hp-versions holds. The other results follow immediately from [24] as in the case of the h-version.

The final bound is given by Chow [23, equation (32)]. ∎

To use the results of the last three chapters an abstract error estimate of the form given by Cea's Lemma (1.29), will be required. The following result provides such an estimate.

The finite element approximation u_{hp}, to the true solution u, see Chow [23], is such that

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \leq C \begin{cases} \inf_{v \in X_{hp}} \|u - v\|^{\alpha/2}_{W^{1,\alpha}(\Omega)}, & \text{if } \alpha \in (1, 2] \\ \inf_{v \in X_{hp}} (\|u\|_{W^{1,\alpha}(\Omega)} + \|v\|_{W^{1,\alpha}(\Omega)})^{(\alpha-2)/\alpha} \|u - v\|^{2/\alpha}_{W^{1,\alpha}(\Omega)}, & \text{if } \alpha \in [2, \infty) \end{cases} \tag{5.6}$$

This result is also given by Barrett and Liu [15]. Looking at this estimate it is clear that the exponent on the right hand side reduces the rate of convergence. For the h-version this has been looked at in much detail by Barrett and Liu, see [15, 16, 17], and under certain extra regularity assumptions on u the rate of convergence for the h-version, for linear elements, can be improved. However, in what follows the above abstract estimate will be used.

To obtain estimates for the finite element approximation the piecewise continuous polynomial given by Theorem 26 will be used and will be denoted by v_{hp}. For the case $\alpha \in [2, \infty)$, it is clear that an upper bound for v_{hp} independent of

h_K and p_K will be required. But from Theorem 26 and the triangle inequality:

$$\|v_{hp}\|_{W^{1,\alpha}(\Omega)} \leq C(\|u\|_{W^{1,\alpha}(\Omega)} + C(h,p)\,\|u\|_{W^{k,\alpha}(\Omega)}), \tag{5.7}$$

where $k = \min_{K\in\mathcal{P}} m_K$ and the constant $C(h,p)$ is monotonically decreasing with respect to h and p. Hence, (5.6) reduces to:

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \;\leq\; C \begin{cases} \|u - v_{hp}\|^{\alpha/2}_{W^{1,\alpha}(\Omega)}, & \text{if } \alpha \in (1,2] \\ C\,\|u\|^{(\alpha-2)/\alpha}_{W^{k,\alpha}(\Omega)}\,\|u - v_{hp}\|^{2/\alpha}_{W^{1,\alpha}(\Omega)}, & \text{if } \alpha \in [2,\infty) \end{cases} \tag{5.8}$$

Since the *log* and ϵ terms of p are relatively small in comparison to the other terms of p, from here on they will be ignored.

5.1.1 Uniform Refinements for Smooth Functions

In this section uniform refinements will be considered for approximation of u, where u is given by (1.17) with $u_3 \equiv 0$. Therefore, using (5.8) and Corollary 21 the following a priori estimates for the uniform h- and p-versions can be obtained:

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \;\leq\; C(p) \begin{cases} h^{\mu\alpha/2}\,\|u\|^{\alpha/2}_{W^{k,q}(\Omega)}, & \text{if } \alpha \in (1,2] \\ Ch^{2\mu/\alpha}\,\|u\|^{(\alpha-2)/\alpha}_{W^{k,\alpha}(\Omega)}\,\|u\|^{2/\alpha}_{W^{k,\alpha}(\Omega)}, & \text{if } \alpha \in [2,\infty) \end{cases} \tag{5.9}$$

where $\mu = \min(p, k-1)$ and

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \;\leq\; C(h) \begin{cases} p^{-\sigma\alpha/2}\,\|u\|^{\alpha/2}_{W^{k,q}(\Omega)}, & \text{if } \alpha \in (1,2] \\ Cp^{-2\sigma/\alpha}\,\|u\|^{(\alpha-2)/\alpha}_{W^{k,\alpha}(\Omega)}\,\|u\|^{2/\alpha}_{W^{k,\alpha}(\Omega)}, & \text{if } \alpha \in [2,\infty) \end{cases} \tag{5.10}$$

where $\sigma = k - 1$.

As in chapter 1, the number of degrees of freedom N will be used to compare the uniform h- and p-versions. Therefore, as in the linear case, it is clear that the p-version is always as good as the h-version and when the solution is very smooth, then exponential rates of convergence can be expected for the p-version.

5.1.2 Uniform Refinements for Singular Functions

Now assume that the true solution is given (4.43) and that $\alpha \in [2, \infty)$. In this case using (5.8) and Theorem 26 it can be seen that

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \leq C(p, u)h^{2\mu/\alpha}, \tag{5.11}$$

where $\mu = \lambda - 1 + 2/\alpha$ and

$$\|u - u_{hp}\|_{W^{1,\alpha}(\Omega)} \leq C(h, u)p^{-2\sigma/\alpha} \tag{5.12}$$

where $\sigma = 2(\lambda - 1 + 2/\alpha)$. Therefore, a similar result holds for the non-linear problem as in the linear case, that is when such singularities exist, the rate of convergence for the p-version is twice the rate of the h-version.

5.1.3 hp-type Refinements

The approximation theory from the previous chapter can now give more insight into how to refine the mesh for the hp-version when the solution is given by (1.17), with $g_i(|\log r_i|) = |\log r_i|^{\gamma_i}$ for all $i = 1, \ldots M$ and $\gamma_i \in \mathbb{Z}$. Let $\alpha \in [2, \infty)$; from (5.8) and Theorem 26 it can be seen that

$$\begin{aligned}\|u - u_{hp}\|^{\alpha/2}_{W^{1,q}(K)} \leq\ & C(u|_{\Omega_K})C(h_K, p_{\Omega_K}, \beta) \\ & p^{-\sigma}_{\Omega_K} h^{\mu}_K (\log p_{\Omega_K} + 1)^{2(1-2/q)},\end{aligned} \tag{5.13}$$

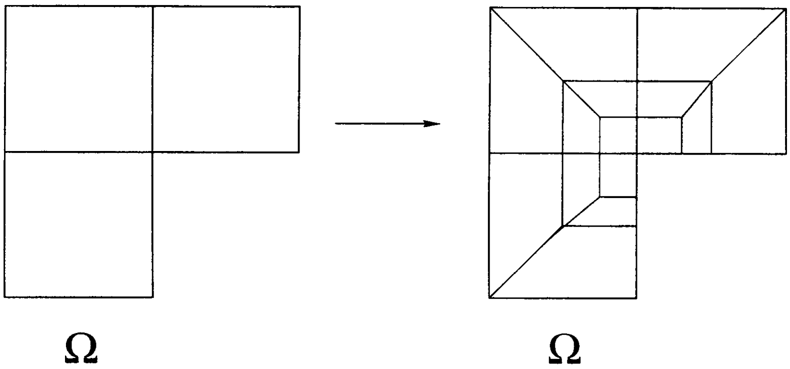

Figure 5.1: An example of geometric h-type refinements

where the β, σ and μ are given in Theorem 26. It is clear that around corner singularities the convergence rate is dramatically reduced and in general only algebraic rates of convergence may be produced. To overcome this and to try to obtain exponential rates of convergence some non-uniform refinements must be implemented. The most common h-type refinement to overcome the degradation, is to strongly refine around the corner. The type of strong refinement used for the hp-version, see Babuska and Gui [8, 9, 10], leads to what is known as geometric meshes, with mesh parameter γ, see Figure 5.6. However, this geometric h-type refinement will not, in general, lead to exponential rates of convergence. This is due to the fact that any type of h-version method can not exploit the smoothness of the true solution away from such singularities, due to the fact that the h-version convergence is always being restricted by the polynomial degree of approximation being used and thus, only algebraic rates of convergence can be obtained.

When the true solution is very smooth, for example C^∞ which could be the case in the elements away from the corners, then the above estimate suggests that p-type refinements will give exponential rates of convergence in elements that are not immediately adjacent to corner elements. It is also clear from the above estimate that having large degree jumps between adjacent elements is not a good thing, since the minimum degree of the two elements would be used in the above estimate.

From these observations, it is quite natural to choose a mesh refinement strategy around a corner that uses a geometric h-type refinement and increases the polynomial element degree as the elements move away from the singularity. This can be thought of as ignoring the singularity and getting the most out of the smooth part of the function. These observations were made by Ainsworth and Senior [6], who also give an adaptive hp algorithm and obtain exponential rates of convergence.

Finally, the rate of increase in polynomial degree must be considered. In Babuska and Gui [8, 9, 10], it was shown for the one dimensional case, that a linear growth away from the corner combined with geometric h-type refinements, with mesh parameter 0.15, leads to an overall optimal exponential rate of convergence. This method of refinement will be adopted for the two dimensional case. For an example of such refinement see Figure 5.2. This refinement strategy was also considered by Babuska and Suri in [11]. It is clear that this type of refinement fits with the observations made from the a priori estimate. In the next section it will be shown that this method of refinement leads to exponential rates of convergence.

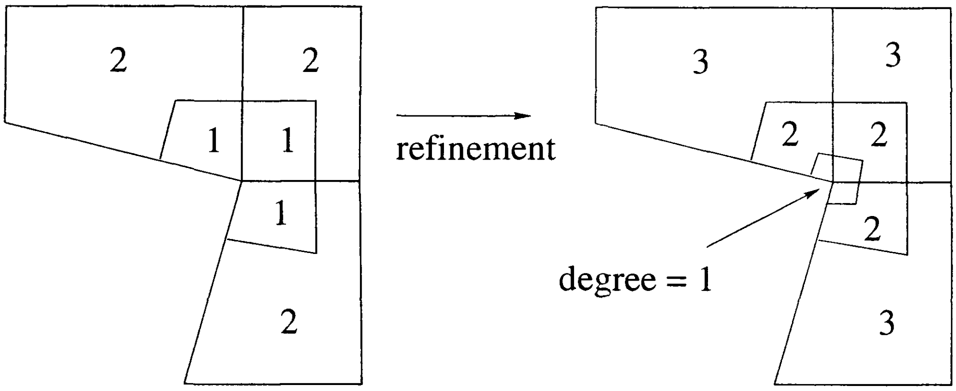

Figure 5.2: An example of hp-type refinement around a corner singularity

5.2 Numerical Results

Since the results shown are of asymptotic character, it is essential to show that the results hold for realistic values of p_K and h_K i.e. values that are acceptable for a machine. The numerical results will verify that the estimates given, in the first chapter for the linear case and in the previous section for the non-linear case, hold for realistic values.

For simplicity, it will be assumed that all Dirichlet boundary conditions are homogeneous from here on.

5.2.1 Linearisation of the problem

When solving the problem numerically, a method is required to reduce the problem to a linear system or a set of linear systems. The first step towards achieving this is to add in a pseudo time step to the initial problem (1.9) which leads to the problem: Find u such that

$$\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u|^{\alpha-2}\nabla u) = f \text{ in } \Omega, \tag{5.14}$$

with the same boundary conditions imposed. The finite element method is then given by: Find $u_{hp} \in X_{hp}$ such that

$$\int_\Omega \left[\frac{\partial u_{hp}}{\partial t} v + |\nabla u_{hp}|^{\alpha-2}\nabla u_{hp} \cdot \nabla v\right] \mathrm{d}\mathbf{x} = \int_\Omega f v \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} g v \,\mathrm{d}s, \tag{5.15}$$

for all $v \in X$. The aim is now to construct a sequence $\{u_n\}_{n\in\mathbb{N}}$ of approximations to the solution u_{hp} of (5.15) using the following technique.

Let

$$\frac{\partial u_n}{\partial t} = \frac{u_n - u_{n-1}}{\Delta t}, \tag{5.16}$$

where $\Delta t \in (0, T)$ for some $T \in \mathbb{R}^+$. Now solve the linear problem: Find $u_n \in X$ such that

$$B(u_{n-1}; u_n, v_X) = F(u_{n-1}; v_X), \tag{5.17}$$

for all $v_X \in X$, where

$$B(u; v, w) = \Delta t \int_\Omega |\nabla u|^{\alpha-2}\nabla v \cdot \nabla w \,\mathrm{d}\mathbf{x} + \int_\Omega v w \,\mathrm{d}\mathbf{x} \tag{5.18}$$

and

$$F(u; v) = \Delta t \left[\int_\Omega f v \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} g v \,\mathrm{d}s\right] + \int_\Omega u v \,\mathrm{d}\mathbf{x}, \tag{5.19}$$

and u_0 is a given initial function. The non-linearity of the problem has been dealt with by using the previous approximation u_{n-1}.

This method as it stands at the moment is very unstable, i.e. convergence to u_X is not guaranteed. The instability gives rise to inefficiency in the method, since when the problem is unstable Δt must be reduced and in doing so the convergence rate of $u_n \to u_{hp}$ is dramatically reduced. The stability and efficiency will now be increased by implementing a fourth order Runge-Kutta method, see [22], as follows: Let u_0 be a given initial solution, define u_j, $j = 1, 2, \ldots$ as follows

1. Let $\widehat{k}_1 \in X$ be such that

$$B(u_j; \widehat{k}_1, v) = F(u_j; v) \quad \text{for all } v \in X. \tag{5.20}$$

   Define

$$k_1 = (\widehat{k}_1 - u_j)\Delta t. \tag{5.21}$$

2. Let $\widehat{k}_2 \in X$ be such that

$$B(u_j + k_1/2; \widehat{k}_2, v) = F(u_j + k_1/2; v) \quad \text{for all } v \in X. \tag{5.22}$$

   Define

$$k_2 = (\widehat{k}_2 - u_j)\Delta t. \tag{5.23}$$

3. Let $\widehat{k}_3 \in X$ be such that

$$B(u_j + k_2/2; \widehat{k}_3, v) = F(u_j + k_2/2; v) \quad \text{for all } v \in X. \tag{5.24}$$

   Define

$$k_3 = (\widehat{k}_3 - u_j)\Delta t. \tag{5.25}$$

4. Let $\widehat{k}_4 \in X$ be such that

$$B(u_j + k_3; \widehat{k}_4, v) = F(u_j + k_3; v) \quad \text{for all } v \in X. \tag{5.26}$$

Define

$$k_4 = (\widehat{k}_4 - u_j)\Delta t. \tag{5.27}$$

Then define

$$u_{j+1} = u_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.28}$$

This method is much more stable and therefore T is much larger, this causes a faster convergence rate for $u_n \to u_{hp}$ and on the whole leads to a more efficient method, even though we are required to do four assemble and solves for each iteration of u_j.

5.2.2 Linear problems

Before looking at numerical results for the general α-Laplacian some linear elliptic examples will be considered.

A smooth linear problem

Firstly look at the rates of convergence, for linear elliptic problems with smooth a solution i.e. $u \in C^\infty(\Omega)$. Consider the problem: Find u such that

$$-\Delta u = f, \tag{5.29}$$

with true solution $u = \exp(x + y)$ on the domain $\Omega = (0,1) \times (0,1)$.

This problem is now solved using both the $h-$version and $p-$version of the finite element method. From the estimates seen in chapter 1, it is expected that the $h-$version will converge at a rate $O(N^{-p/2})$ where p is the maximum polynomial degree used in the subspace X_h, while the $p-$version should produce an exponential rate of convergence, since for all values k the $p-$version will be faster than $O(N^{-(k-1)/2})$. The results shown in Figure 5.3 confirms this.

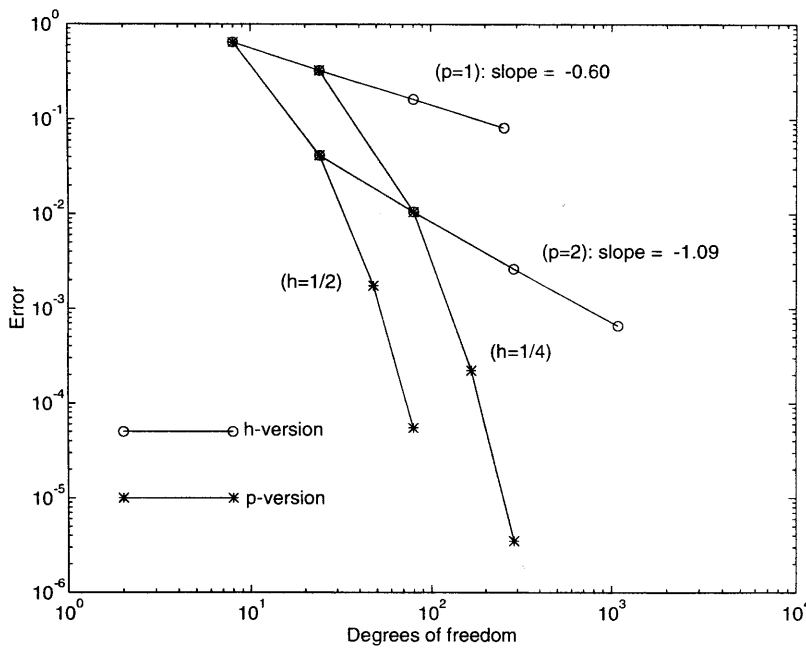

Figure 5.3: Rate of convergence for a linear problem with smooth solution

A singular problem

Now consider the linear elliptic problem on the domain Ω given by Figure 5.4. It is known that the true solution of this problem, in polar coordinates with origin at the corner, is given by $u = r^{2/3}\sin(2/3)\theta$. Therefore, $u \in W^{5/3,2}(\Omega)$.

The $h-$version and $p-$version are firstly implemented with uniform refinements with an initial partition of three elements and linear basis functions. From estimates (1.30) and (1.35), the expected convergence rate for the $p-$version

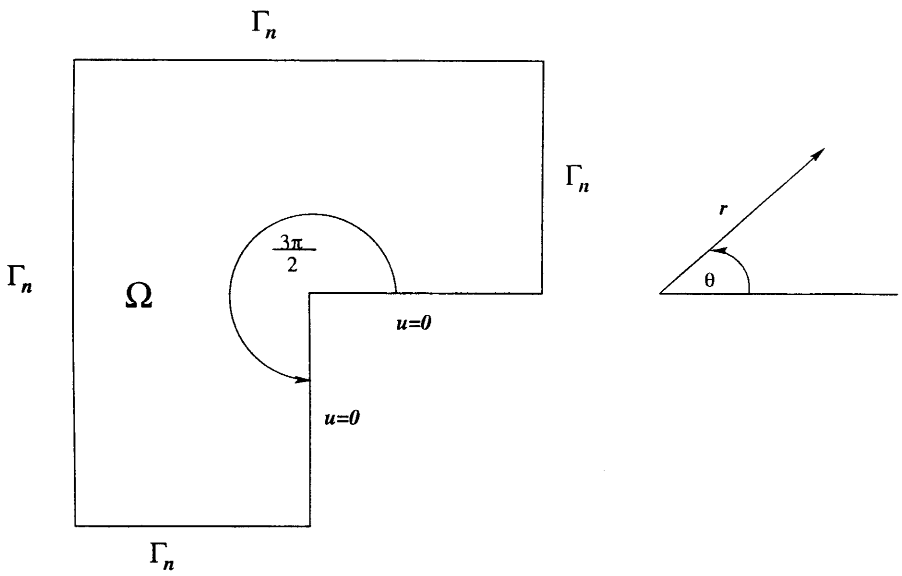

Figure 5.4: The singular problem

should be twice that of the $h-$version which should itself converge at a rate $O(N^{2/3})$. From Figure 5.5 this is confirmed.

Now consider using geometric $h-$type refinements on the corner domain, these refinements are given by Figure 5.6. On each of these $h-$type refinements a uniform $p-$version is then applied.

From Figure 5.7 it can be seen that, by choosing suitable steps of refinement at each level it is possible to create an exponential rate of convergence even when degrees of freedom are wasted from using a uniform p refinement. These refinements would be both $h-$type and $p-$type and thus the exponential rate would be caused by implementing an $hp-$version of the finite element method. This exponential rate can be seen in Figure 5.8. This observation suggests that when the suggested hp refinements from the previous section are implemented i.e. non uniform p, then even faster rates are to be expected.

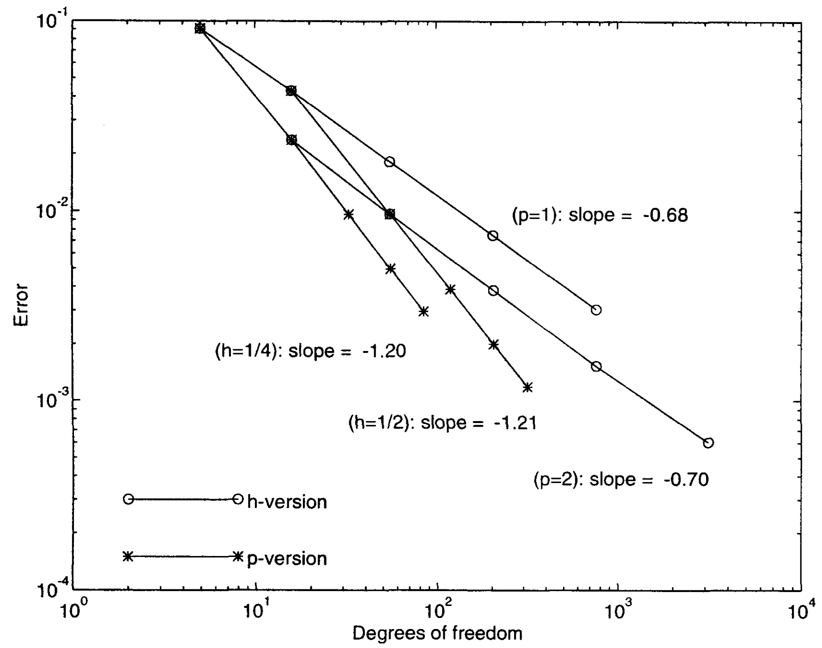

Figure 5.5: Convergence rates for uniform refinements

5.2.3 Non-linear Problems

Examples will now be shown for the general α-Laplacian.

A smooth problem

Consider the α-Laplacian problem with $u = \exp(x + y) \in C^\infty(\Omega)$ where Ω is the unit square and $\alpha = 3/2$. The problem is solved using both uniform h- and p-refinements. From (5.9) and (5.10) the h-version is expected to converge at a rate of at least $O(N^{-3/4})$ for linear elements and $O(N^{-6/4})$ for quadratic elements, although Figure 5.9 suggests that these estimates may not be optimal and that h-version of the finite element method converges at optimal rates for linear and quadratic elements. Since the true solution is infinitely smooth the p-version is

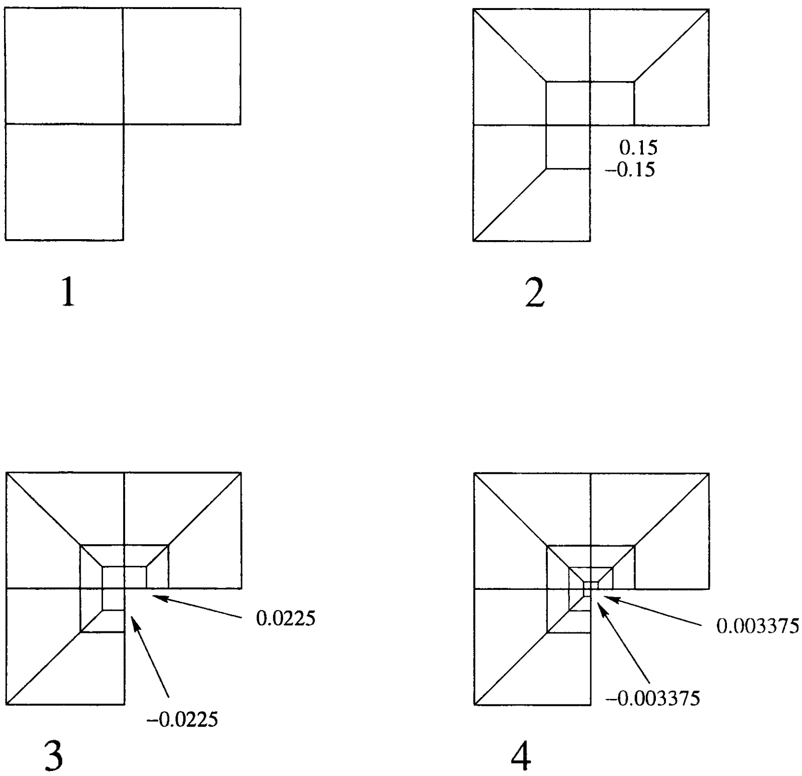

Figure 5.6: Geometric h-type refinements used on the L-shaped domain

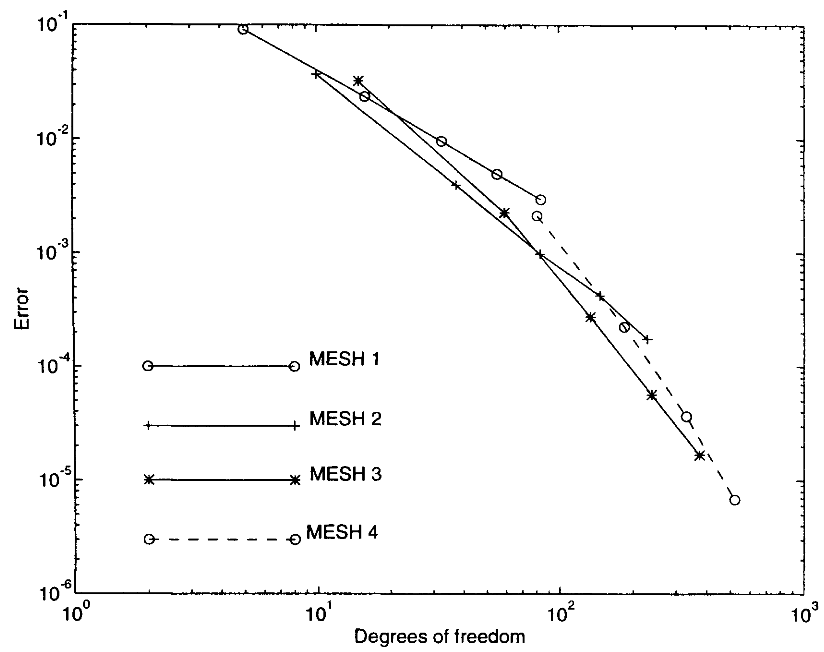

Figure 5.7: Convergence rates for the p-version at each step of h refinement

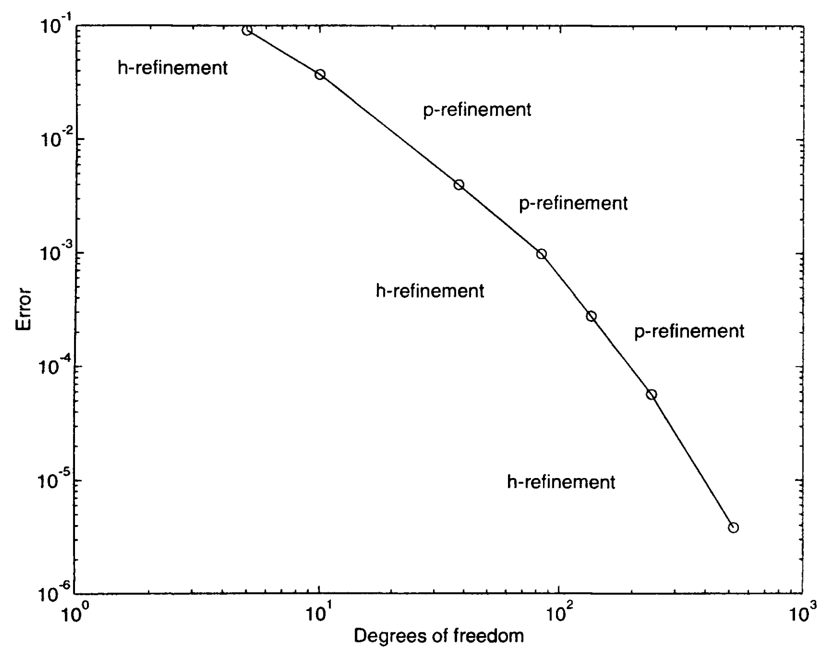

Figure 5.8: Convergence rates for an simple hp-method for the singular problem

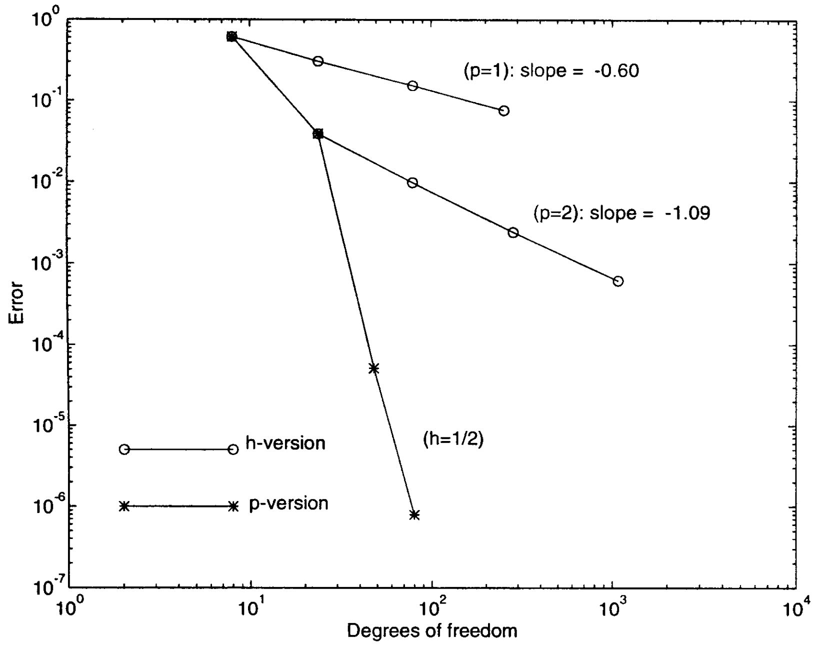

Figure 5.9: Convergence rates for non-linear problem with smooth solution

expected to converge at an exponential rate. This can also be seen in Figure 5.9.

A one dimensional singular problem

Before looking at a two dimensional non-linear problem with a singularity, a one dimensional example will be given. In this case $\Omega = (0, 1)$ and $\alpha = 2.7$. The true solution is given by $u = x^{27/17} \in W^{1.959-\epsilon, 2.7}(0, 1)$, where $\epsilon > 0$ is arbitrarily small and x is the distance measured from the origin.

The estimate given for the p-version around a corner singularity in two dimensions, seems to also hold for the equivalent one dimensional problem, see Figure 5.10, that is, the p-version is at least twice as effective as the h-version

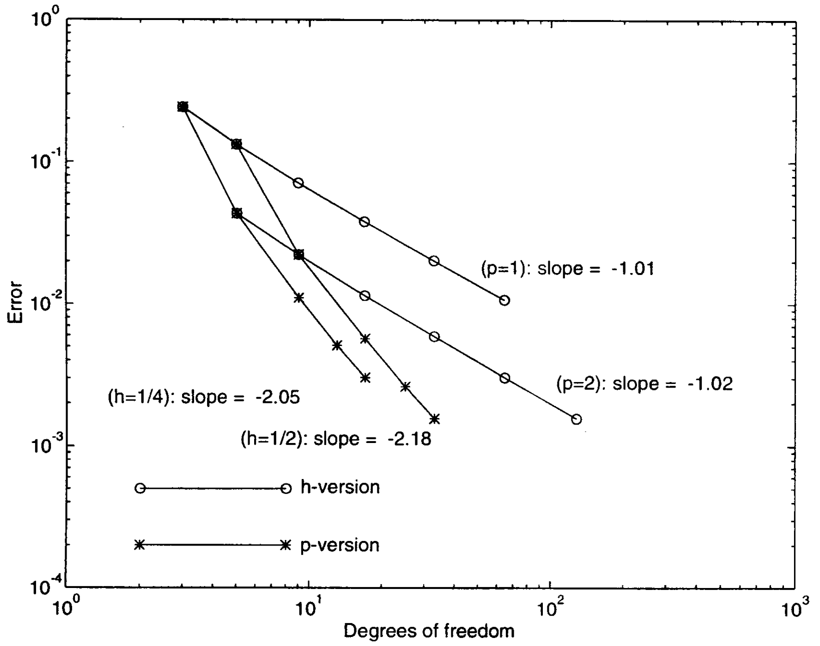

Figure 5.10: Convergence rates for non-linear one dimensional problem with singular solution

when a singularity occurs at a vertex of an element. Also, from Figure 5.10 it can be seen that the rates of convergence obtained from (5.9) and (5.12) are again suboptimal.

5.2.4 Two-dimensional singular problems

In the following two dimensional singular problems the true solution will be a function of r only on the domain $\Omega = (0,1) \times (0,1)$.

Example 1 Let $\alpha = 3$ and the true solution $u = r^{3/4}$. This function belongs to the space $W^{13/6-\epsilon,3}(\Omega)$. Both uniform h and p refinements where used on the domain. For the h-version the expected rate of convergence, using (5.9),

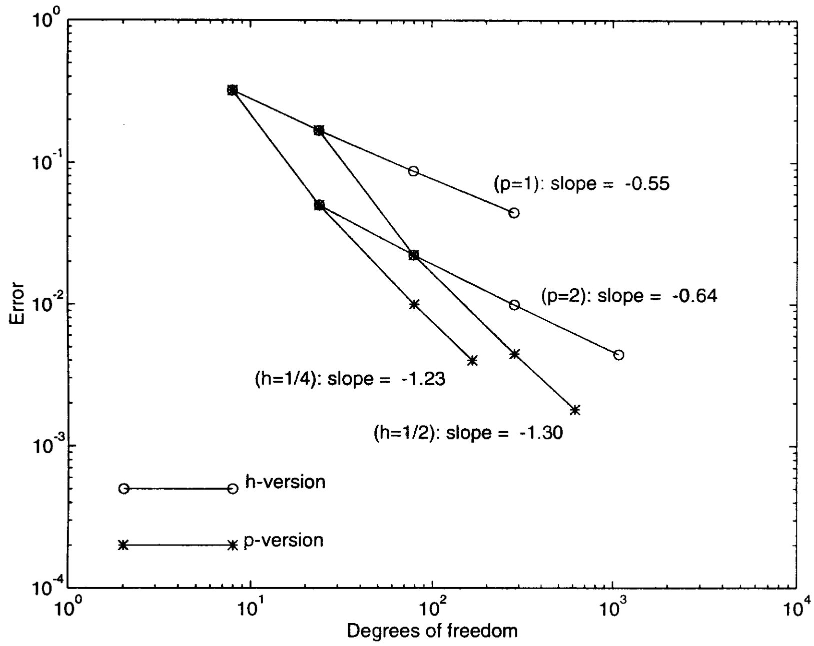

Figure 5.11: Convergence rates for Example 1

is $O(N^{-2\mu/3})$ where $\mu = \min(7/6 - \epsilon, p)/2$. Therefore, when the fixed degree $p > 1$ the rate of convergence is affected and the maximum rate expected would be $O(N^{-(2/3)(7/6)})$, while for a p-version the expected rate is twice this rate, $O(N^{-(2/3)(7/6)})$, always. The results shown in Figure 5.11 show that both methods converge one and a half times faster than expected; this suggests that the abstract a priori bound (5.8), is suboptimal for these type of functions.

Example 2 Let $\alpha = 4$ and the true solution $u = r^{4/3}$. The function belongs to the space $W^{11/6-\epsilon,4}(\Omega)$. In this example both types of uniform refinement are considered and a simple hp-version is considered, using the ideas given in the previous section. An example of a mesh obtained from using the suggested

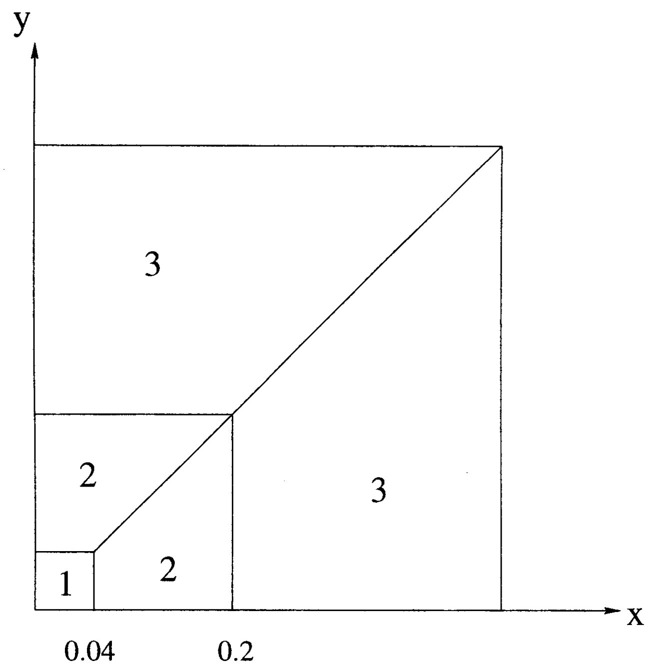

Figure 5.12: The third level of refinement for the hp-version

hp-strategy can be seen in Figure 5.12.

Looking at the uniform convergence rates, see Figure 5.13, it is clear that the a priori results (5.9) and (5.12) are confirmed although, as before, the rates seem to be without the degradation caused by initial abstract error (5.8).

It can also be seen, quite clearly, that the adopted hp method produces an exponential rate of convergence. It can also be seen that the hp method is giving the best refinement at each level and when compared to the h-version is far superior.

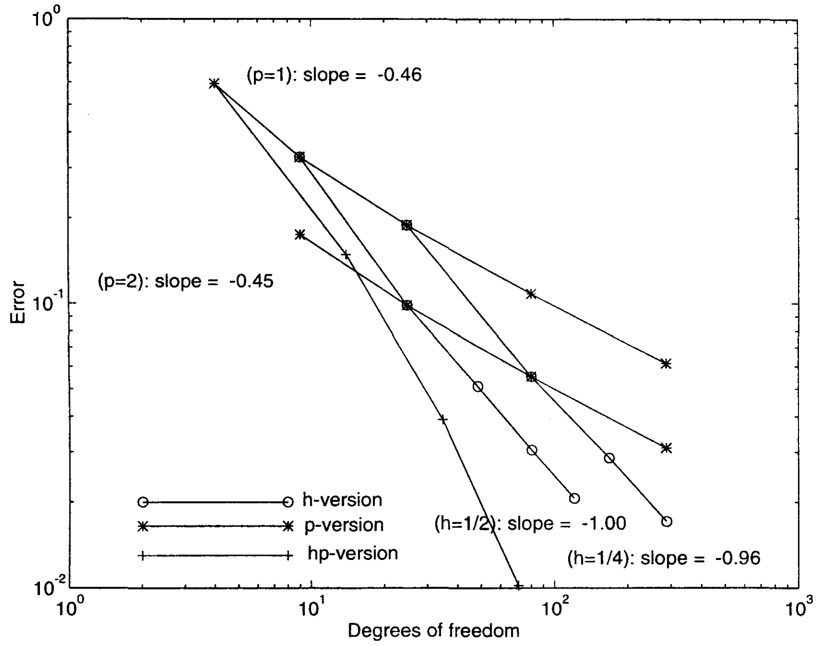

Figure 5.13: A comparison of the uniform h, p and hp-versions for Example 2

5.3 Further Comments

Although the above results where obtained for the $\alpha-$Laplacian, it is quite clear that the results given hold for any functions whose derivatives behave in a similar manner. Such allowable functions satisfy problems: Find u such that

$$-\nabla \cdot \{K_1 + K_2|\nabla u|^{\alpha-2}\nabla u\} = f \quad \text{in } \Omega, \tag{5.30}$$

for $K_1, K_2 > 0$, along with boundary conditions.

It was also assumed throughout, that the elements were only allowed to be polygonal. This is not the case, these results may be extended to curvilinear triangles and quadrilaterals which are such that there exists a sufficiently smooth one-to-one mapping from these elements to the reference elements.

Finally, different types of singularities may arise from these α-Laplacian problems. In particular singularities which occur in the interior will cause degradation in the finite element method. A look at the approximation theory for the *hp*-version for such singularities is being considered and will be given in Ainsworth and Kay [5].

Bibliography

[1] R. A. ADAMS, *Sobolev Spaces*, Academic Press, (1978).

[2] M. AINSWORTH AND D. KAY, *The Approximation Theory for the p-version Finite Element Method and Application to Non-linear Elliptic Partial Differential Equations*, In Review..

[3] M. AINSWORTH AND D. KAY, *The Analysis of the hp-version Finite Element Method on Locally Refined Meshes Applied to Non-linear Elliptic Partial Differential Equations*, In prep.

[4] M. AINSWORTH AND D. KAY, *The Rate of Convergence of the p-version Finite Element Method for the Non-linear Laplacian*, Prague Mathematical Conference, Proceedings, To appear..

[5] M. AINSWORTH AND D. KAY, *Approximation and Treatment of Interior Singularities of Partial Differential Equations Using the hp-version Finite Element Method*, In prep.

[6] M. Ainsworth and B. Senior *An Adaptive Refinement Strategy For hp Finite Element Computations* Technical Report 96/29, Leicester University.

[7] O. Axelsson and V. A. Barker, *Finite Element Solution of Boundary Value Problems*, Academic Press, Inc., (1984).

[8] I. Babuska and W. Gui, *The h, p and h-p versions of the finite element method for one dimensional problem. Part I: The error analysis of the p-version*, Numer. Math, pp. 577-612, **49**, (1986).

[9] I. Babuska and W. Gui, *The h, p and h-p versions of the finite element method for one dimensional problem. Part II: The error analysis of the h and h-p versions*, Numer. Math, pp. 613-657, **49**, (1986).

[10] I. Babuska and W. Gui, *The h, p and h-p versions of the finite element method for one dimensional problem. Part III: The adaptive h-p version*, Numer. Math, pp. 658-683, **49**, (1986).

[11] I. Babuska and Manil Suri, *The $h-p$ version of the Finite Element Method with Quasiuniform Meshes*, Math. Mod. and Num. Anal. , pp. 199–238, **21**, (1987).

[12] I. Babuska and M. Suri, *The optimal convergence rate of the p version of the finite element method*, SIAM J. Numer. Anal., pp. 750–776, **24**, (1987).

[13] I. BABUSKA AND M. SURI, *The treatment of non-homogeneous Dirichlet boundary conditions by the p−version of the finite element method*, Numer. Math, pp. 97–121, **55**, (1989).

[14] I. BABUSKA, I.N. KATZ AND B. SZABO, *The p version of the finite element method*, SIAM J. Numer. Anal., pp. 515–545, **18**, (1981).

[15] J. W. BARRETT AND W. B. LIU, *Finite Element Approximation of the p-Laplacian*, Math. Comp., pp. 523–537, **61**, (1993).

[16] J. W. BARRETT AND W. B. LIU, *A Remark on the Regularity of the Solutions of the p-Laplacian and its Applications to their Finite Element Approximation*, J. Math. Anal. Appl., pp. 470–487, **178**, (1993).

[17] J. W. BARRETT AND W. B. LIU, *Quasi-Norm error Bounds for the Finite Element Approximation of some Degenerate Quasilinear Elliptic Equations and Variational Inequalities*, Math. Mod. Num. Anal., pp. 725–744, **28, no.6**, (1994).

[18] R. BELLMAN, *A Note on an inequality of E. Schimdt*, Bull. Amer. Math. soc, pp. 734-737, **50**, (1944).

[19] J. BERGH AND J. LOFSTROM, *Interpolation Spaces: an introduction*, Springer-Verlag, (1976).

[20] C. BERNARDI AND Y. MADAY, *Properties of some Weighted Sobolev Spaces and Application to Spectral Approximations*, SIAM J. Numer. Anal., pp. 769–829, **26, No.4**, (1989).

[21] C. BERNARDI AND Y. MADAY, *Polynomial Interpolation Results in Sobolev Spaces*, J. comp. Appl. Math., pp. 53–80, **43**, (1992).

[22] R. L. BURDEN AND J. D. FAIRES, *Numerical Analysis, Fourth Edition*, PWS Kent, (1989).

[23] S.-S.CHOW, *Finite element method error estimates for non-linear elliptic equations of monotone type* , Numer. math., pp. 373–393, **54**, (1989).

[24] P. G. CIARLET, *The Finite Element Method For Elliptic Problems*, North-Holland, (1980).

[25] M. DOBROWOLSKI, *On quasilinear elliptic equations in domains with conical boundary points*, J.reine angew. math, pp. 186–195, **394**, (1989).

[26] T. DUPONT AND R. SCOTT, *Polynomial Approximation of Functions in Sobolev spaces*, Math.Comp., pp. 441–463, **34**, (1980).

[27] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, (1980).

[28] A. KOLMOGOROFF, *Zur Grössenordnung des restgliedes Fourierscher reihen differenzierbarer Funktionen*, Ann. math, pp. 521–526, **36**, (1935).

[29] N. KORNEICHUK, *Exact Constants in Approximation Theory*, Cambridge University Press, (1991).

[30] A. QUARTERONI, *Some Results of Bernstein and Jackson Type for Polynomial Approximation in L^p Spaces*, Japan J. Appl. Math., pp. 173–181, **1**, (1984).

[31] W. RACHOWICZ, *An Anisotropic $h-$Type Mesh-Refinement Strategy*, Cracow University of Technology, Section of Applied Mathematics. Report No. 1, 1993, pp. 257–270, **29**, (1993).

[32] S. RIPPA, *Long Thin Triangles Can Be Good For Linear Interpolation*, SIAM J. Numer. Anal., pp. 257–270, **29**, (1992).

[33] T. J. RIVLIN, *The Chebyshev polynomials*, Wiley, (1974).

[34] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, (1970).

[35] G. SZEGO, *Orthogonal Polynomials*, American Mathematical Society, (1975).

[36] P. TOLKSDORF, *Invariance properties and special structures near conical boundary points. In W. Wendland P. Grisvard and J. R. Whiteman, editors*, Springer Lecture Notes in Mathematics, pp. pages 308–318, **volume 1121**, (1985).

[37] G. TOLSTOV, *Fourier Series*, Prentice-Hall, (1962).